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Promoting uncertainty to support preservice teachers’ reasoning about the tangent relationship

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Promoting Uncertainty to Support Preservice Teachers’ Reasoning about the Tangent Relationship
Abstract

Including opportunities for students to experience uncertainty in solving mathematical tasks can prompt learners to resolve the uncertainty, leading to mathematical understanding. In this article, we examine how preservice secondary mathematics teachers’ thinking about a trigonometric relationship was impacted by a series of tasks that prompted uncertainty. Using dynamic geometry software, we asked preservice teachers to compare angle measures of lines on a coordinate grid to their slope values, beginning by investigating lines whose angle measures were in a near-linear relationship to their slopes. After encountering and resolving the uncertainty of the exact relationship between the values, preservice teachers connected what they learned to the tangent relationship and demonstrated new ways of thinking that entail quantitative and covariational reasoning about this trigonometric relationship. We argue that strategically using uncertainty can be an effective way for promoting preservice teachers’ reasoning about the tangent relationship.

*Keywords*: quantitative reasoning, tangent, trigonometry, uncertainty
Promoting Uncertainty to Support Preservice Teachers’ Reasoning about the Tangent Relationship

Trigonometry is an essential mathematical topic for school mathematics, but many students are taught, and subsequently learn, a procedural approach to trigonometry, consisting of memorized formulas about the trigonometric ratios [1-3]. This approach does not lend itself to coherent ways of thinking about trigonometry as the study of angle measures, ratios and values on the unit circle, and relations within right triangles. Unfortunately, incoherent thinking about trigonometry is often shown by secondary school teachers, which can hinder their students’ quantitative and covariational reasoning with these topics [4, 5].

One way to support learners’ coherent ways of thinking about trigonometry is by attending to quantitative and covariational reasoning. As Madison, Carlson, Oehrtman, and Tallman [6, p. 56] summarize, ‘quantitative reasoning refers to the mental actions that an individual engages in to conceptualize measurable attributes of some object or situation.’ Solving mathematics tasks are one way to provide opportunities for students to engage in and develop their quantitative reasoning. Moore and LaForest [2] identified the value of tasks in promoting learners’ coherent understanding of trigonometry through: (a) attention to relevant quantities such as angles and measuring arcs, (b) supporting covariational reasoning of the quantities through graphing and real-world problems, (c) measuring quantities in radii of the unit circle, and (d) connecting circle trigonometry to right triangle trigonometry, specifically by having ‘students understand that an angle measured in right-triangle settings generates a family of circles centered at the angle’s vertex’ [p. 621]. Quantitative reasoning plays an essential role in developing students’ understanding of angle measure, the unit circle, and relating quantities through trigonometric functions [2, 7].
Covariational reasoning refers to the mental actions involved in determining how two varying quantities change together. Covariational reasoning is essential for learners’ thinking about patterns of change, dynamic events, functions, and major concepts of pre-calculus and calculus, yet students at all levels of education, and even teachers, often have difficulty engaging in covariational reasoning [8-10]. Teachers, both novice and experienced, should have a deep understanding of covariational reasoning to support their learners and should themselves engage in tasks to support their own development and understanding. Although many authors [11-14] have published tasks on the sine and cosine trigonometric relationships that provide preservice teachers (PSTs) opportunities to ‘engage in quantitative reasoning in ways that can perturb and advance their understandings’ [5, p. 143], little research exists on how to develop PSTs’ quantitative and covariational reasoning about the tangent relationship.

In this article, we investigated how PSTs’ thinking about the tangent relationship developed in terms of the quantitative and covariational reasoning expressed as they engaged in a series of tasks that purposefully incorporated uncertainty. The 3-day series of tasks incorporated uncertainty by providing opportunities for PSTs to consider unknown solution paths, competing claims, and non-readily verifiable outcomes. This approach builds on Zaslavsky’s [15] uncertainty framework and is similar to Moore’s [16] work of providing PSTs perturbations to advance their reasoning about the sine and cosine functions, though in this study PSTs investigated the tangent relationship by using technology, specifically dynamic geometry software, to explore how an angle results in a ray with a slope that can be measured on a coordinate grid. By having tasks that purposefully focused on specific lines whose changing slopes seemed to be proportional to their changing angles, PSTs were challenged to resolve conflicting arguments through later questions and class discussions. Our research question was:
How does uncertainty impact PSTs’ quantitative and covariational reasoning about the tangent relationship?

**Theoretical Framework**

Zaslavsky’s [15] uncertainty framework is rooted within a constructivist perspective [17], drawing on the notion of that uncertainty is a catalyst for learning through intellectual needs for certainty [17, 18], social interactions, and reflective thinking [19] that give rise to learning and the construction of knowledge. To identify types of uncertainty, Zaslavsky [15] conducted longitudinal studies to examine the design and implementation of mathematical tasks that enhance the learning of a diverse set of learners, including PSTs and in-service teachers. She found that tasks which evoke uncertainty for the learner add value in terms of the mathematical and pedagogical knowledge gained. This work highlights the importance of teacher educators and teachers becoming aware of the role of uncertainty as related to specific tasks and the possible benefits that may arise from intentionally providing opportunities for PSTs to experience and resolve perturbations about mathematical concepts. However, Zaslavsky argues that the task does not always create the uncertainty; instead, ‘social interactions may play a central role in creating uncertainty surrounding certain mathematical tasks as well as in leading to resolving uncertainties’ [p. 300].

Provoking uncertainty with tasks in the classroom, whether in teacher education courses or K–12 classrooms, provides opportunities for learners to encounter pertubations and confront their own misconceptions. This approach uses tasks that challenge learners to reconsider previous conceptions, realize misconceptions, and ultimately decide on improved ways of thinking [20, 21]. In this way, tasks can be purposely implemented to support learners with evidence that contradicts their prior thoughts or beliefs. Using uncertainty to generate conflict
has been found to improve PST thinking about mathematical concepts. For example, Hadas, Hershkowitz, and Schwarz [22] used uncertainty and contradiction within dynamic geometry environments to advance students’ thinking about angles and triangles. Moore et al. [5] have developed PSTs’ quantitative reasoning about the sine and cosine functions by evoking uncertainty to challenge their previously held conventions. The importance of uncertainty in generating perturbations and mathematics learning is a key point in Harel’s [17] duality, necessity, and the repeated-reasoning principle (DNR). However, this is only one theory of learning and some researchers do not subscribe to this theory [23, 24].

Zaslavsky [15] describes three types of uncertainty: (1) competing claims, (2) non-readily verifiable outcomes, and (3) unknown path or questionable conclusion. Competing claims includes outcomes, definitions, beliefs, and assertions that lend different points of view to a single idea, including ‘contradicting statements with which the learner is confronted or an outcome that contradicts a well-known (to the learner) mathematical truth’ [p. 299]. Non-readily verifiable outcomes is defined as learners’ low confidence about the validity of an outcome due to a lack of verification methods, occurring when the learner does not know methods to verify a solution, or if the methods the learner could use to verify are not available. Unknown path or questionable conclusion is when learners engage in open-ended tasks requiring the exploration of previously unknown patterns and relationships, such as when a student uses a dynamic geometry environment to explore a proposed geometry conjecture. Zaslavsky [15] notes these three kinds of uncertainty are interrelated: ‘for example, uncertainty regarding a questionable conclusion (third type) could (though need not) be enhanced by the lack of verification tools (second type)’ [p. 305]. We consider these types of uncertainty in the present study, as they relate to PSTs’ interactions with tasks around covariational reasoning about the tangent relationship.
In this study, we use the quantitative reasoning framework and covariational reasoning framework described by Thompson and colleagues [9, 12, 13]. These frameworks allowed us to identify how PSTs described the tangent relationship and make comparisons about the amount of covariational reasoning in their thinking before, during, and after the series of tasks.

**Quantitative Reasoning Framework**

Thompson’s [13] quantitative reasoning framework defines quantitative reasoning using quantities and quantitative relationships. A quantity is a conceptualization of: (1) an object, (2) a measurable attribute of the object, (3) a unit of measurement for the attribute, and (4) a conceivable numerical value, or values, associated through a proportional relationship with the unit of measurement. Thompson describes a quantitative relationship as a situation in which a person conceives of two quantities being joined through a quantitative operation to create a third quantity. Quantitative reasoning is attending to and identifying quantities, constructing new quantities, and identifying and representing quantitative relationships [7].

According to research, PSTs tend to struggle to reason quantitatively with trigonometric relationships. For example, Greer [25] summarized how students worldwide tend to apply linear thinking to non-linear settings in mathematical concepts ranging from kindergarten to university level. Esteley, Villarreal, and Alagia [26] detailed this phenomenon with university-level students applying linear thinking to trigonometric relationships, specifically the cosine function. Understanding the nature of trigonometric relationships requires quantitative reasoning to explain why these relationships are non-linear [5]. As Greer [25] notes, ‘relatively little research has been done on intervention studies’ that improve learners’ understanding of linear and non-
linear relationships, which should occur in part by ‘giving problems to students [that] don’t make
the use of superficial problem characteristics undeservedly successful’ [p. 113-114].

While this kind of reasoning about trigonometric relationships is difficult for learners, it
is nevertheless vital. Moore [16] and colleagues [17] found that quantitative reasoning was
essential to constructing a coherent system of trigonometric thoughts. For example, in terms of
the sine function, students and teachers, including PSTs, need to develop understanding of angle
measure as the input quantity of a function and then relate that input to the vertical distance,
measured from the circle’s horizontal diameter in units of radius length, as the output quantity.

**Covariational Reasoning Framework**

Covariational reasoning is ‘the cognitive activities involved in coordinating two varying
quantities while attending to the ways in which they change in relation to each other’ [9, p. 4].
This type of reasoning occurs when a person considers ‘how the output values of a function are
changing while imagining changes in a function’s input values’ [28, p. 115]. Carlson developed a
framework to study students’ covariational reasoning that researchers can use to categorize
students’ covariational reasoning and use hierarchical levels of mental actions. Carlson et al. [9,
p. 8] describes that this framework:

> provides a lens for analyzing and reporting students’ covariational reasoning abilities
> when responding to dynamic function tasks. The framework describes five developmental
> levels of images of covariation that are successively more sophisticated and complex. The
> five developmental levels are described in terms of the mental actions or operations that
> each image supports. [Appendix A] provides a description of the five mental actions and
> associated behaviors that have previously been identified in students.

Detailed in the Methods section, we used Carlson et al.’s Mental Actions as a means to describe
the change in PSTs’ covariational reasoning about the tangent relationship.
Context: Integrating the Uncertainty Framework with the Covariational Reasoning Frameworks

Our approach to investigating the tangent relationship aligns with Moore’s recommendation of having PSTs engage in quantitative and covariational reasoning to challenge their previous superficial conventions about trigonometry as well as Zaslavsky’s [15] uncertainty framework for improving teachers’ mathematical thinking. Integrating the uncertainty framework, rooted in social constructivist notions, with the quantitative and covariational frameworks, rooted in cognitive traditions, presents both opportunities and challenges. We frame this paper with the theoretical notion that uncertainty enhances mathematical and pedagogical learning and purport that the notion of uncertainty promotes learning because of intellectual and psychological needs for certainty [17] which, under the proper conditions, could motivate an individual to resolve these conflicts and persevere through challenges. This process often occurs within the context of social interactions within a mathematics classroom. Therefore, we situate uncertainty as a mediating process, through which tasks may provoke uncertainty leading to social interactions, or in which social interactions in a learning environment are the catalyst for uncertainty. We operationalize Zaslavsky’s [15] uncertainty framework by identifying the instances where the task incorporates competing claims, non-readily verifiable outcomes, and unknown path or questionable conclusion. Then, using the data sources, we determined the opportunities resulting from each type of uncertainty and how PSTs reacted to these opportunities to (re)consider the tangent relationship.

In addition to the uncertainty framework, we used the quantitative and covariational reasoning frameworks to identify the statements made by the PSTs before, during, and after the
task, to identify shifts in PSTs’ reasoning about the tangent relationship\(^1\). When the uncertainty presented an opportunity for PSTs to reason about the tangent relationship, we claim the appearance of uncertainty impacted PSTs’ reasoning when shifts in reasoning were present. In other words, the uncertainty (any of the 3 types) created a space or a context which allowed the PST to (re)consider the role of tangent and thus an opportunity to discuss this relationship in ways that provided evidence of quantitative and covariational reasoning. Integrating these frameworks required an inference that the PSTs’ verbal and written statements reflect what occurs mentally. When the uncertainty in a task was documented to provide an opportunity for the PSTs to reason about the tangent relationship, we also inferred that an increase in the sophistication of the Mental Actions using the covariational framework corresponded with the uncertainty. Both of these inferences stem from using social cues (e.g. PSTs’ written and verbal statements) as indicators of mental actions (e.g. PSTs’ mental constructions of quantities and relationships between quantities). The data analysis process is further described in the methods section.

**Methods**

This study explores the research question: How does uncertainty impact PSTs’ quantitative and covariational reasoning about the tangent relationship? We applied the uncertainty framework while developing and implementing tasks that challenged PSTs to consider the tangent relationship in ways that supported quantitative and covariational reasoning. This section provides details on the 18 participants, key features of the tasks that took place over a three-day span, sources of data collection, and our analysis procedures.

\(^1\) We reserve the word ‘function’ to refer to rule assigning every input value exactly one output value, while ‘relationship’ entails flexibly thinking about how two quantities vary without regards to one being an independent or dependent variable.
Participants

Participants for the present study were 18 preservice secondary teachers\(^2\) in an upper-level undergraduate content course in a larger university located in the United States of America. The 3-credit hour course focused on deepening PSTs’ content knowledge of trigonometry, polar coordinates, and complex numbers through attention to quantities and quantitative reasoning. The 12 women and 6 men in the semester-long course had a mean age of 28.8 years; 17 of the 18 were working on a Bachelor of Science degree in Mathematics Education, while the remaining PST’s degree program was Mathematics. The instructor of this course was one of the authors of this paper and had taught this undergraduate course twice before.

Before engaging in the tasks described in this study, PSTs were familiar with the collaborative learning that took place in the course, where they often worked on tasks in groups of 3 or 4, presented solutions to the whole class, and were expected to explain and justify their reasoning. In the same manner as described in Moore [16], PSTs had learned how to quantify an angle using degree and radian measurements by connecting angle measure to measuring arcs and conceiving the radius as a unit of measure. PSTs relate the angle quantity to other quantities and used this understanding to define the sine and cosine functions. They had demonstrated covariational reasoning about the sine and cosine functions and used these functions to calculate the horizontal and vertical position of a point given an angle measurement. As we detail in the data analysis section, PSTs demonstrated only a weak procedural, if any, understanding of tangent prior to engaging in this study’s tasks.

\(^2\) Henceforth we call the participants Pre-Service Teachers (PSTs) for clarity.
Tasks

We adapted a series of tasks from the Core Connections Geometry textbook [30] to attend to quantitative and covariational reasoning about the tangent relationship. The tasks were designed to prompt learners to consider the relationship between two quantities: (1) the measure of an angle of a line on a coordinate grid measured from the 3 o’clock position counterclockwise rounded to the nearest whole degree, and (2) the slope quantity of that line measured as a ratio between the values of two quantities in a composed unit, specifically a number or y-units to the number of x-units. We selected tasks that prompted learners to consider how these quantities varied covariationally, specifically by asking learners to answer what occurs to the slope as the angle measure increases from 0 to 90 degrees (the resulting slope of the line varies by increasing as the angle increases, but not linearly). Note that the resulting relationship between these quantities is the tangent relationship, and if the input is considered to be the angle measure and the output the resulting slope, this is one way of defining the tangent function (another way would be measuring the angle in radians).

The series of tasks occurred over two 75-minute class meetings and the first half of the third class meeting, for a total of three class days. The tasks were designed to promote uncertainty in three ways that aligned with Zaslavsky’s [15] uncertainty framework. We encouraged uncertainty by ensuring the instructor and/or tasks did not use the word ‘tangent’. This decision was made because we wished to avoid inciting any mnemonics that would bypass reasoning about the tangent relationship. Had the instructor or tasks mentioned the tangent relationship, the PSTs may have tried to apply previously memorized information to solve the problem, and thus not engage in reasoning. Thus in this article, we frequently reference the

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3 For details on how learners can conceptualize these quantities, see Moore [16] for the angle quantity and Lobato and Siebert [35] and Castillo-Garsow [36] for the slope quantity.
relationship as the ‘tangent relationship’ for reading clarity, but note that during class, this term was not used until introduced by the PSTs.

Second, we intentionally set parameters around tool use related to the task in a way that would afford PSTs opportunities to explore uncertainty. PSTs were permitted – and at times prompted – to use protractors, string, graph paper, and technology such as Desmos, GeoGebra, and Geometer’s Sketchpad as well as graphing calculators. These tools allowed the PSTs to observe how changing an angle value impacted the vertical position, horizontal position, and slope value of the slope triangles formed by that angle. At the same time, PSTs were instructed not to use ‘trigonometry buttons’ on any digital technology; in other words, PSTs were restricted from using calculators or computers to compute values of sine, cosine, or other trigonometric relationships. Restricting students from using technology encouraged PSTs to reason through the task using triangles formed by various angle measures rather than immediately receiving an answer from their calculator. As noted in Author [31], technology use in certain situations can hinder learning, so use should be purposeful and restricted in cases where learning without the technology may be advantageous. The tasks used in this study are best described by what occurred each class day, called Day 1, Day 2, and Day 3, and in the homework assignments between each class.

**Day 1 Task**

Using the Day 1 task, we encouraged PSTs to explore the relationship between two quantities: an angle measure (in whole degrees counterclockwise from the 3 o’clock position) and the resulting slope of a ray formed by the x-axis, the terminal ray of the angle, and any vertical segment in quadrant I; these three sides form a right triangle which was called the slope triangle associated with a particular angle measure. With the task, PSTs were asked to consider
the relationship between the angle and slope value in two ways. First, given a single angle measurement, PSTs were asked to construct multiple slope triangles and discuss the relationship between various parts of these slope triangles. Figure 1 shows how PSTs were prompted to draw, then consider, features of the slope triangles using the line that had an 11 degree angle when measured against the horizontal on a coordinate grid. To avoid misconceptions about the angle [32], we note this is a unit grid where each axis measured in the same units. Given this first part of the Day 1 task, we aimed to have PSTs reason about what quantities changed as different slope triangles were considered within the same angle line. The central question was: What is the relationship between the slope of two different slope triangles which share the same line? This first part of the Day 1 task encouraged PSTs to reason covariationally by considering how the slope changes with the size of the slope triangle (note: it does not).

[Insert Figure 1]

Based on the second part of the Day 1 task, we had PSTs investigate what happened to the slope when the angle measure varied. Later questions in this part of the task also had PSTs graph a line with a specific slope of $\frac{3}{5}$ and then measure the resulting angle created (which appeared to be $22^\circ$ when rounded to the nearest degree). Thus, the remaining part of the Day 1 task was designed to support PSTs covariational reasoning by having them consider how the angle (quantity 2) was impacted by doubling the slope (quantity 1). Because lines with small angle measures (e.g. under 22 degrees) and small slopes (e.g. between 0 and 1) were investigated, and angles were measured to the nearest degree, a near-linear relationship seemed to appear between the quantities (i.e. a doubling of the angle doubles the slope), and thus allowed for future uncertainty from PSTs about the nature of the relationship between these two quantities.
At the end of Day 1, PSTs were given homework due on Day 2, which asked them to use technology to investigate more deeply the relationship between angles and slopes:

Use Geogebra (geogebra.org), Desmos (desmos.com), or Geometers’ Sketchpad (http://www.dynamicgeometry.com/) to create two files. These files should be dynamic in that you can click and drag a point to get different slope triangles. Make sure the side length and height is displayed. Your first file should contain the 11 degree angle similar to graph shown on the first page of the in-class activity [from Day 1]. Your second file should contain a graph of the line $y=(2/5)x$ with many different slope triangles, similar to problem 4a [from Day 1]. Your sketch should include an explanation of what angle is formed between this line and the 3 o’clock position of the x-axis.

Day 2 Task

From the Day 2 task, we asked PSTs to use the dynamic geometry sketches they had created in their homework to make conclusions about the relationship between angles and the resulting slope of the lines. The first 20 minutes of class on Day 2 were spent having PSTs present their technology files to peers and having discussions about how this information relates to the essential question: How does the slope of an 11 degree angle line relate to the slope of a 22 degree angle line? In this task, PSTs were asked to create a table of values relating specific angle measures and the corresponding slope, using the idea of a slope triangle to compute the slope (Figure 2). This table, termed the Trig Table Toolkit, provided the PSTs information to begin discussing the essential question: What kind of relationship exists between the angle and corresponding slope of lines, and how would you characterize this relationship? Covariational reasoning was encouraged by having PSTs complete the table, consider what happened as the angle value increased, and make conclusions about the slopes of lines whose angle measures were not in the table.

[Insert Figure 2 Here]

At the end of Day 2, the PSTs presented evidence to the whole group about why this relationship was non-linear. For the Day 2 homework, we asked PSTs to complete the Day 2
tasks and make another dynamic sketch, this time of a triangle with a slope angle measuring 55 degrees and solve for the missing length in the slope triangle formed with a height of 8 (Figure 3). They were also given similar questions to Figure 3 with differing angle measures or triangle side lengths, requiring the use of protractors, graph paper, or using the dynamic geometry sketches PSTs produced.

[Inert Figure 3 Here]

**Day 3 Task**

For the Day 3 task, we again had PSTs use the dynamic geometry sketches to make conclusions about the relationship between an angle and the resulting slope of a line. This part of the task was designed to have PSTs present their solutions to the Trig Table Toolkit (Figure 2), justify their responses using technology and discuss the graph of the relationship between angle measure and slope. At this point, the PSTs concluded that this relationship, around which the tasks centered, could be defined as the tangent relationship.

**Data Collection and Analysis**

Data sources included (1) PSTs’ initial survey responses about their definition of trigonometry collected at the beginning of the semester, (2) PSTs’ in-class written work from completing the tasks each of three days synchronized with the audio comments at that exact moment, (3) PSTs’ dynamic geometry sketches from each of the homework assignments, (4) audio recordings of whole-class discussions during these three days, (5) audio recordings from three of the five small groups during the three days (termed Group 1, Group 2, and Group 3 in findings), (6) PSTs’ responses from a midterm question asking about the nature of the tangent relationship, and (7) PSTs’ responses from the final course reflection asking what they learned in the course. This type of data collection recorded PSTs’ utterances with evolutions in their written
work and inscription to gather information about PSTs’ reasoning. Note that in data source (3), the three small groups had 3-4 PSTs in each, meaning 11 of the 18 were recorded during all class conversations each day. We consider this group work to be a social context within which the PSTs operated as they worked in class daily to solve the series of tasks. The setting in which the PSTs solved the tasks was social within the course, with much of the audio data capturing PST interaction. All audio data from these group conversations were transcribed prior to analysis.

After transcribing the audio data (data sources 3 and 4), we positioned the transcript with the corresponding group’s written work from that day’s task to consider PSTs’ verbal and written statements made each day. Data source (6) was collected three weeks after Day 3 by having PSTs take a midterm assessment where they were asked ‘How can slope ratios be used to explain and estimate the tangent function? Your explanation should include at least two worked out examples with a visual representation (graph, table, picture, etc.).’ Data source (7) was an open-ended final prompt at the end of the semester that asked PSTs how their understanding of mathematical topics has changed.

We analyzed the audio and written data sources 2 through 5 by identifying statements relating to the three main constructs of Zaslavsky’s [15] uncertainty framework: (a) unknown solution paths, (b) initial non-readily verifiable outcomes, and (c) competing claims. For each of the identified statements, we recorded the source of the uncertainty and how the uncertainty was resolved. We also coded statements from all data sources referencing how quantities were related. This coding used Carlson et al.’s [9] covariational reasoning framework (Appendix A) as a means to measure progress in the sophistication of students’ reasoning. As the course did not focus on average or instantaneous rates of change in the tangent relationship (Mental Actions 4 & 5), coding was limited to Mental Action 1 (coordinating the value of one variable with
changes in the other), Mental Action 2 (coordinating the direction of change of one variable with changes in the other variable), and Mental Action 3 (coordinating the amount of change of one variable with changes in the other variable).

We applied these codes to the context of the tangent relationship in the following way: Mental Action 1 was applied if a PST’s statement identified the two quantities involved in the tangent relationship (i.e. angle, slope) and indicated a change in one quantity impacts a change in the other quantity. Mental Action 2 was applied if a PST’s statement indicated how the slope varied with respect to a change in the angle (e.g. as the angle increases, so does the resulting slope). Mental Action 3 was applied if a PST’s statement indicated how much the slope varied with respect to a change in the angle (e.g. justifying a non-linear relationship between the two quantities or explaining the relationship of the change of more than two angle measurements and resulting slopes). Mental Action 0 was applied if a PST’s statement referenced quantities but not do so correctly or did not relate the quantities. Within each Mental Action, the PSTs did not need to recognize this relationship as the tangent relationship but could use other ways of describing the ways in which the quantities changed together.

After coding for covariational reasoning, the constant comparative method [33] was used to determine changes in Mental Actions by single PSTs between the pre-assessment (data source 1), tasks (data sources 2-6), midterm assessment (data source 7) and final examination (data source 8). Analyzing the responses and utterances of a PST across these data sources allowed us to make conclusions about how PSTs’ thinking about the tangent relationship changed during the tasks. In this process, we defined an increase in the Mental Action (e.g. initial coding of MA0 to a later coding of MA1 for an individual) as evidence of increased covariational reasoning. When a PST made a statement coded as Mental Action higher than the Mental Action from the pre-
assessment due to an opportunity created by uncertainty, we claim the PSTs’ reasoning was positively impacted by presence of uncertainty.

When analyzing the transcript and observation notes, we examined plausible causes for the increase in Mental Action; as we detail in the findings section, each shift followed an episode of uncertainty, and in these cases attribute uncertainty as a likely impact on the mental shift. Next, we conducted a holistic analysis [34] to corroborate findings across the various types of data. This holistic approach was conducted in accordance with the theoretical framing described by Zaslavsky [15]. We then considered our findings across all data items to triangulate findings and arrive at final themes within the three constructs of the framing and structured the findings section to present evidence of how each type of uncertainty contributed to the changes in Mental Actions seen in the PSTs. From the data analysis, we were able to evaluate (a) how the task incorporated uncertainty, (b) how PSTs responded to the uncertainty, and (c) how PSTs’ responses evidenced their quantitative and covariational reasoning about the tangent relationship.

**Findings and Discussion**

From the analysis, we found (1) the task was effective for promoting covariational reasoning about the tangent relationship, (2) the task was successful in inducing Zaslavsky's [15] three categories of uncertainty, and (3) the uncertainty in the task presented opportunities that contributed to PSTs’ development of covariational reasoning. This section first provides evidence for the first claim by detailing the increase of Mental Actions in the PSTs’ statements as they engaged in the task and the following weeks. Second, we provide evidence for claims two and three by describing how each type of uncertainty was incorporated during the task. These opportunities resulted from the uncertainty for the PSTs to (re)consider the tangent relationship
and shifts in Mental Actions indicating increased covariational reasoning because of these opportunities.  

**Increased Reasoning about the Tangent Relationship**

When completing the tasks, the PSTs began incorporating quantitative and covariational reasoning to describe more coherently the tangent relationship. In the initial pre-survey, the PSTs were asked to explain trigonometric relationships. Seventeen of the 18 PSTs did not specify the quantities involved in any trigonometric relationships and thus were coded as not attending to any of the Mental Actions of the covariational framework. Eight of these PSTs did not, or stated they could not, describe any trigonometric relationship. For example, Sarah (all names pseudonyms) stated they could not describe any trigonometric relationship ‘off my current memory.’ Nine of these PSTs gave descriptions of trigonometric relationships using the pneumonic ‘SOA-CAH-TOA’. One PST, Addie, gave the unique response when defining the sine function: ‘The sine function is one of the trigonometry function where the value of sine is the y coordinate of a point moving in a unit circle.’ This response indicated Addie could identify the output quantity of the sine function but did not give a clear indication to the other quantity that was changing (e.g. was it time, the central angle, and so on). Thus Addie’s pre-survey response was coded as Mental Action 0.

Of the 18 PSTs, only Judith was coded as Mental Action 1 (coordinating the value of one variable with changes in the other) of the covariational framework because of she identified the input and output quantities of a trigonometric relationship and indicated they changed together. In her pre-survey, Judith wrote:

The sine function, of a respective angle (radians), represents the ratio between the leg (opposite of respective angle) and the hypotenuse. Trigonometric is translated as ‘triangle-measuring’. So a trigonometric function is triangle measuring function. Specifically, the sine function can find the measure of angles of a right triangle, by
describing the function with respect to a certain angle (radian) as the ration [sic] between the opposite leg and the hypotenuse.

Here Judith identified the input quantity (angle, in radians) and output quantity (ratio of opposite side of the triangle to hypotenuse of the triangle formed by the angle) of the sine function. She also indicated the sine function is not a single ratio, but in fact a function that can compute the ratio for multiple angles. Additionally, she suggested the sine function can be used to determine an angle measurement given a ratio of triangle formed by that angle. Although Judith demonstrated Mental Action 1 about sine by acknowledging that one quantity depends on the other quantity, she did not attend to the direction of change that occurred as one quantity varied with the other, and thus were not coded as Mental Action 2 (coordinating the direction of change of one variable with changes in the other variable).

As they completed the series of tasks, all PSTs were observed to make statements about the tangent relationship in ways attending to all three Mental Actions of the covariational framework. For example, in Group 2 all three PSTs made verbal and written comments in the Day 2 task describing how a change in one quantity (angle, measured from the 3 o’clock position counterclockwise, ranging between 0 and 90 degrees) influences a change in the other quantity (slope of the terminal ray). Each PST in Group 2 then wrote down the relationship between the angle measure and resulting slope, which they label as ‘ratio’ in their work (Figure 4). Note at this point in this activity they are focused on angle measures in the first quadrant. These statements were coded as evidence of PSTs attending to the tangent relationship using covariational reasoning Mental Action 2 (coordinating the direction of change of one variable with changes in the other variable). Similar statements were made by PSTs in other groups, such as when Moises said to his group, ‘This angle increases with the slope increase, is the ratio’, and a peer responded, ‘Together…as the angle increases, the slope will change.’
Evidence of PSTs engaging in Mental Action 3 (coordinating the amount of change of one variable with changes in the other variable) can be found later during the Day 2 activity. For example, the following conversation took place in the whole-group discussion summarizing the activity:

Robert (Group 4): The slope ratios did not form a linear, like their relationship is not a linear relationship, but instead it’s kind of like tan[gent]. Show them the graph. Sarah (Group 4): [shows graph of plotted Trig Table points] On the x axis is all of the degrees that we have and then on the y is the slopes of those degrees, which makes sense because it’s tangent in there. Robert: It kind of makes sense that [we previously thought relationship was linear] because [these measurements] are approximations and we know when you start a curve like this, a lot of times it doubles or triples would be, you know, they are going to be close, but once it hits that point where it start to rise, at an amazingly increasing rate, that is when you will be like, uh oh, this should be double but this is ten times as big.

Robert and Sarah make statements referring to how changes in the slope vary in relation to the angle increasing in the first quadrant. Although Robert could be clearer in his explanation, we interpreted his statement as saying if the angle measure doubles (or triples), then the resulting slope value would be expected to also double (or triple). Robert and Sarah justified the amount of change in one quantity (angle) impacts the changes of the other quantity (slope), and thus their statements were coded as Mental Action 3. Additional statements were made during this whole-group discussion that showed other small groups had discussed these same ideas and were building off their classmate’s conversation. For example, the following contribution was made a few minutes after Sarah presented the graph:

Judith (Group 3): We talked about how it wasn’t changing proportionally, but I feel that, that graph, I couldn’t really visualize what was happening, so…it, like took all of our word explanation and gave a visual explanation to it.

Group 3 had similar conversations justifying how the amount of change in slope was not proportionally related to the change in angle measure. In this way, all the PSTs at some point
made a verbal or written comment that was coded as Mental Action 3 because they described the amount of the change between the two quantities involved. Overall this evidence shows all PSTs began making statements about the tangent relationship that attended to more advanced Mental Actions of covariational reasoning during the series of tasks.

Several weeks after this series of activities, we posed a related question on the midterm and asked PSTs to describe the tangent relationship. In their responses, PSTs continued to incorporate aspects of covariational reasoning after the tasks (Table 1). All PSTs gave responses that were coded as Mental Action 1 by identifying the input and output quantities and indicating how the tangent function’s output (slope) depended on the function’s input (angle). The majority (13 of 18) of PSTs also gave responses attending to how one quantity changed in relation to the other (Mental Action 2). Six of the 18 PSTs also described the nature of this relationship (Mental Action 3). For example, Willie’s response (Figure 5) attended to Mental Action 3 because he described the nature of the change occurring as the angle measure increased and the impact it had on the resulting slope value.

With the final examination question, PSTs were provided the opportunity to summarize in their own words any changes in thinking that occurred during the course. The prompt was open ended to allow PSTs to discuss any topic they chose and was not limited to trigonometry concepts. Four of the PSTs described how their thinking about the tangent function developed and made statements attending to Mental Action 1 of the covariational framework. For example, Addie wrote about how her new perspective on the tangent has changed how she plans to instruct his future high school students:
I learned that an approach to trigonometry that involves quantitative reasoning begins with supporting students' understanding of the quantitative relationships between angle measures and the arc that subtends the angle...I learned that it is necessary to teach students the slope ratio before teach[ing] them that the tangent of an angle in right triangle equal to the measure of the side opposite over the measure of the adjacent side...[and] can explain to a secondary student if I give him a slope in order to get the slope angle he has to draw a unit circle and then draw a vertical distance equal to in from the 3 o'clock position and then connect the origin of the circle and the top of the vertical distance...All in-class activities and article I read about trigonometry change[d] my plan on how to teach these topic [sic] in the future.

Addie’s response identifies the series of tasks as being influential on her covariation reasoning of the tangent function.

Overall, using the covariational framework to identify the ways in which PSTs made statements about the tangent relationship allowed documentation of a shift in PSTs’ thinking. We found initially PSTs were not describing tangent, or other trigonometric relationships, in ways that attended to covariational reasoning. During the tasks, all PSTs began engaging in all three levels of covariational reasoning. All the PSTs continued describing the tangent relationship by attending to at least Mental Action 1 of the covariational reasoning framework weeks later on the midterm question. These comments, along with evidence from the final examination question, support the notion that PSTs’ thinking about tangent changed to incorporate covariational reasoning.

Evidence of Increased Reasoning Resulting from Uncertainty

From the analysis, we identified multiple ways uncertainty was promoted in the task, how PSTs resolved the uncertainty, and how this resolution prompted PSTs to reason differently about the tangent relationship. We structure this section by discussing how the PSTs revealed the three types of uncertainty (competing claims, non-readily verifiable outcomes, and unknown solution paths), how the uncertainty was resolved, and the observed impact on PSTs’ reasoning, as evidenced by changes in the Mental Actions of their written and verbal statements about the
tangent relationship. In accordance with Zaslavsky’s work, we found that these three categories of uncertainty were interrelated, and thus some evidence corroborates more than one type of uncertainty. We also found stronger evidence for the first and second type of uncertainty (i.e. competing claims, non-readily verifiable outcomes) as compared to the third (i.e. unknown solution paths).

**Increased Reasoning Resulting from Competing Claims**

First, we utilized tasks that prompted PSTs to consider the tangent relationship in ways that were contradictory to their implicit assumptions or to reach conclusions that contradicted each other. In both cases, we called these patterns competing claims, which we found caused the PSTs to be uncertain about the nature of the tangent relationship. For example, PSTs developed competing claims stemming from their implicit assumptions during the Day 1 task. When working on the questions in Figure 1, which asks PSTs to draw three new slope triangles on a line, with each being a different size, all three groups of PSTs said the slopes between similar triangles were proportional, such as Group 3’s response in Figure 6. At least one PST in each group initially indicated the slopes were proportional.

[Insert Figure 6 Here]

PSTs were prompted to generate a competing claim when asked to write the slope for each triangle as a fraction to confirm their earlier conjecture. As Group 3’s work below shows (Figure 7), the slopes are more than proportional, they are in fact equal, and thus their answers competed/contradicted their previous claim. Even though PSTs in each group initially thought the slopes were proportional, the following question provided contradictory evidence to their claim.

[Insert Figure 7 Here]
With the instructor’s support, the task incorporated the competing claims to prompt PSTs to explain which claim was correct and ultimately overcome their initial uncertainty about relating the angle measure and the slopes. PSTs realized the slopes were not (non-trivially) proportional, but instead equal for similar triangles and began justifying their conclusion. For example, Group 2 began with an assumption that the slopes were proportional, giving a response similar to Group 3 (Figure 6). A few minutes later, after completing work similar to Group 3’s (Figure 7), the following discussion occurred:

Charlotte: So, because our angle is maintained, the proportion must…the proportion making up the slope, which is tangent, must stay the same. You get what I’m saying? So, the angle, the proportion has to stay the same.

Patrick: Ok, that makes sense. So since the proportion is the same, you can know that the angle is the same. In the proportionality of each triangle is really the same, because right here you’ve got 1/5, 2/10, that’s 1/5, 3/15, that’s 1/5. So that makes the slope still the same right here, since we’re basing it off the slope line4. Yes. No?

Charlotte: Well…I guess you can’t do that. Well, you can. But, yeah, it’ll still be one over five...The ratios that make up the slope remain the same...the ratio remains the same no matter what size the triangle is. K, boom.

In this exchange Charlotte and Patrick resolve the uncertainty surrounding the nature of relationship between a single angle value and the slope of multiple triangles formed by the terminal ray of that angle.

In addition to Patrick and Charlotte, other PSTs similarly resolved these competing claims during a whole-group discussion near the end of Day 1. In this discussion the PSTs connected their justification to the algebraic representations of their conclusion:

[Insert Figure 8 Here]

Their discussion included comments about how the radius quantity ‘cancels out’, or put more mathematically, is invariant within the tangent relationship. This conversation indicated PSTs’

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4 Here we believe the PST is referring to the slope of the terminal ray formed by this angle.
reasoning about the tangent relationship was deepened, particularly in that when considering an angle, the size of the radius does not influence the tangent of the angle.

Another way we used the tasks to promote competing claims occurred in Day 2 when PSTs were asked, ‘How does the slope of an 11 degree angle relate to the slope of a 22 degree angle?’ Recall that from Day 1, PSTs knew an 11 degree angle generated a slope of (approximately) $\frac{1}{5}$. At least one PST from every group brought up the notion that then a 22 degree generated a slope of (approximately) $\frac{2}{5}$. Figures 7 and 9 contain examples of various group responses to this problem.

A completing claim occurred when PSTs completed the Trig Table Toolkit (Figure 2), particularly calculating the slope created by a 55 degree angle. Group 1 discussion demonstrates them grappling with the competing claims:

Willie: I doubled [the $\frac{2}{5}$ slope of the 22 degree angle] to $\frac{4}{5}$ to get approximately 45 degree angle, but that’s, we know our ratio for 45 is one. So, then I was like, well, that doesn’t work now, you know.
Instructor: Figure out why. Use technology--Nicole: Figure out why... [trails off]. Why 55 should be 45? Basically, something like that?
Willie: No. Doubling 11 degrees, 22, 33, 44 degrees was approximately a $\frac{4}{5}$ our ratio, but we know that 45 degrees is, should have a slope of one. It should be $\frac{4}{4}$.
Priscilla: Basically our conjecture was wrong, that double the angle will result in double the y value.

Group 1 acknowledges competing claims about the nature of the relationship between angle values and resulting slopes. They recognize that an angle of 45 degrees would have a slope of $\frac{4}{4}$, which competes with their initial conjecture of a $\frac{4}{5}$ ratio.

The PSTs resolved the uncertainty from these competing claims mainly by reasoning with the aid of technology, particularly their GeoGebra and Geometers Sketchpad files showing the 11
and 22 degree angle measurements. For example, in the following excerpt from Group 2, Nicole references her GeoGebra file to discover the slopes are actually approximations:

Nicole: If everything’s a, an approximation. So, we said eleven would have the equation of 1/5, but 1/5 doesn’t have...it is probably just an approximation.
Priscilla: I know, you aren’t supposed to use it, but just to check, eleven is point one nine.
    [uses tangent button on graphing calculator].
Willie: Which is approximately 1/5.
Nicole: But, even if you are doing approximation it is like way off. So, I am saying, like if you keep doubling it every time, if you keep doubling it, it is going to mess it up. Because we are not doing, like, eleven isn’t really 1/5.
Priscilla: True, but this is very close approximation. So, this should match up. You know what I mean?
Nicole: But 1/5 isn’t 11 degrees.
Priscilla: True, but you are not using, let’s not talk about the 1/5, but this is 11 degrees. So, if we double it to 22 degrees, it should double. You know what I mean? 11 degrees times two is 2/5, which is way off from this, so doubling the angle doesn’t give us the double y value. Do you know what I am saying?

Nicole and Priscilla state three claims: (1) there is a proportional relationship between angles and resulting slopes because if you double the 11 degree angle this doubles the slope of 1/5, coinciding with a 2/5 slope for the 22 degree angle; (2) if there is a proportional relationship between angles and resulting slopes, then multiplying the 11 degree angle and slope of 1/5 by 5 results in a slope ratio of 1 for a 55 degree angle; and (3) a 45 degree angle has a slope of 1. Nicole and Priscilla recognized claims 2 and 3 are competing in that they cannot both be true. These PSTs reason why claim 3 is undoubtedly correct and then conclude that claim 2 must consequently be incorrect. Since claim 1 and claim 2 both rely on the same premise, that there is a proportional relationship between angles and resulting slopes, these PSTs move to determine that claim 1 is also incorrect. Furthermore, the PSTs then move to discuss why claim 1 is false. They realize there is in fact a near-linear relationship for these angle measurements that can be explained when accounting for small rounding errors. Although Priscilla’s example is noteworthy because she disavowed the rules of the task and used the trig buttons of their calculator to verify some of their conjectures, the other PSTs used their dynamic geometry
sketches to reason that claim 1 and claim 2 are false. These competing claims prompted the PSTs to consider the amount of change in slope when the angle increased from 0 to 90 degrees, which is Mental Action 3 of the covariational reasoning framework.

Following the resolution of these competing claims, the PSTs began considering the tangent relationship in new ways. Each group of PSTs made comments suggesting they were thinking about this relationship in terms of how the angle related to the resulting slope. For example, Group 3 stated:

James: The relationship is that...it’s a...what do you call it? It’s a non...nonsensical relationship. Like a sine relationship you can’t describe it as anything. It’s not exponential, it’s not rational, it's not linear. It’s kinda weird.
Judith: Well this is ultimately giving us a graph that is not proportional.
Darlene: If you double the 22, and then 44 would that be \( \frac{3}{5}\)?
James: That’s just happens to be like that. That just is lucky.
Darlene: Yeah, that’s what I was...it’s not like a pattern.

This passage explains why the PSTs initially had competing claims, because the graph had a seemingly linear relationship between angle measures of 11 and 22 degrees. The relationship between angles and slopes become more clearly non-linear for larger angles. PSTs made this connection in Day 1. For example, James from Group 1 said:

From forty-five degrees to ninety degrees...our slope is going to start rapidly increasing. Because eventually, when it gets to, to ninety, our slope will be undefined. So, from like eighty-nine to ninety degrees, our slope is going to go from a million to like even more and more and more.

This conversation was revisited in Day 3 during a class discussion about why the double angle formula for tangent is not \( \tan(2x)=2\tan(x) \). These conversations indicate the tasks prompted PSTs to engage in Mental Action 3 by considering the amount of change in the slope that resulted from a change in the angle.

The PSTs continued engaging in covariational reasoning during Day 2 using the dynamic geometry sketches they created for homework. In the whole-class discussion, the PSTs built on
each other’s representations and explanations to demonstrate a coherent understanding of the tangent relationship. For example, the following exchange shows Robert and Sarah sharing the graph in Figure 10 with the whole class:

Robert (Group 4): The slope ratios did not form a linear— their relationship is not a linear relationship, but instead it’s kind of like tan, and then she graphed…
Sarah (Group 4): On the x axis is all of the degrees that we have and then on the y is the slopes of those degrees, which makes sense because its tangent.
Robert: And, we just said it’s just a coincidence that [the apparent linear relationship] happened to occur. It kind of makes sense that it happened because [these measurements] are approximations and we know when you start a parabola like this, there are times like doubles or triples would be, you know, they are going to be close, but once it hits that point where it start to rise, or uh, amazingly increasing rate that is when you will be like, Uh oh, this should be double this is ten times as big.
Charlotte (Group 2): Can I point on that circle? Okay. We were saying the same thing with the slope but were more so related it to the [unit] circle. So, with the circle if you trace [a point on the circle], we know [the change in horizontal position] is not a constant rate of change - in the beginning [when theta increases from 0, the change is] kind of faster…Between 30 and 60 degrees is when everything [the change in horizontal position] becomes very similar [linear] because if you think about the regular ratios that we know already, 30 degrees, 45, and 60…you do the cosine of all three of them, they are kind of close [to a linear relationship]. Like $\cos(30) = \frac{\sqrt{3}}{2}$, $\cos(45) = \frac{\sqrt{2}}{2}$, $\cos(60) = \frac{1}{2}$…But then somewhere up here is way different, so right in this middle area [points to angle measurements between 30 and 60 degrees] is where everything becomes the same [linear] and that’s why between technically, between 22 and 45 degrees, 45 degrees would be the five over five because it’s one. And technically, between those, that’s why it’s such a small gap where at the beginning you could just double it, so we went from 11 all the way to 22. So were are thinking, so let’s double it again, or let’s add another 11 and now we are thirty-three - It doesn’t work like that.
Judith (Group 3): We talked about how it wasn’t changing proportionally, but I feel that, that graph, I couldn’t really visualize what was happening, so I like, she, it, like took all of our word explanation and gave a visual explanation to it.

Notice Sarah’s graph shows discrete points on the tangent function graph alongside the initial linear assumption of the relationship between the angle quantity (on the horizontal axis) and slope quantity (on the vertical axis). Charlotte’s explanation attempts to use the near-linearity of the cosine function for angles between 30 and 60 to justify the claim that there was a near-linear
relationship between the angle and slope values, particularly between the 11 and 22 degree angles and resulting slopes. As Charlotte explained these ideas, she pointed to these values on the graph and how they overlapped the tangent values on the graph shown to the class (Figure 10). These comments, along with others who made similar points, such as Judith and all of Group 3, provide evidence of how competing claims prompted PSTs to consider the amount of change of one variable as the other quantity changes, thus Mental Action 3.

These competing claims resulted in a need for PSTs to justify why this relationship was not linear. This resulted in PSTs thinking covariationally about the tangent relationship and making connections between their written justification of the tangent relationship, the algebraic representation of the tangent function, and the graphical representation of the tangent function. Thus, the presence of competing claims was also productive in terms of getting a PST to engage in Mental Action 3 when reasoning covariationally about the tangent function.

**Increased Reasoning Resulting from Non-Readily Verifiable Outcomes**

We engaged the PSTs in tasks where they came to conclusions about mathematical relationships in which the validity could not initially be determined. These conclusions, also called non-readily verifiable outcomes, caused PSTs to be uncertain about the mathematical correctness when describing the tangent relationship, creating opportunities for PSTs to (re)consider the nature of the tangent relationship and ultimately contributed to increased reasoning about this relationship.

A main way we evoked consideration of non-readily verifiable outcomes was when the PSTs investigated the slope of the 11 degree angle in comparison to the 22 degree angle. As detailed in the previous section, these questions occurring in Day 1 resulted in PSTs producing competing claims. PSTs resolved the competing claims using technology, showing their claim
about angles and slopes being proportional was incorrect. However, even after the PSTs reached this conclusion, they did not demonstrate why their conclusion was correct. Specifically, during Day 1 the PSTs showed examples of how a proportional relationship would not continue for larger angle measures (e.g. 11 versus 44 degree), but did not justify how angles and slopes could be determined from one another without using trigonometry buttons on their calculators. For example Group 3 had the following discussion at the end of Day 1 about the task question ‘different lines will have different slope angles and different slope ratios’:

Addie: We can’t figure out the angle without any calculator or anything.
Judith: I don’t think you can.
James: Nope.
Darlene: I personally cannot.
Addie: I don’t know how [the instructor] is seeing it.
Judith: Maybe there’s like something to come. Like a follow-up...I think I could get, given the slope, I could approximate the angle, just using reasoning but I couldn’t get the exact, necessarily, with every slope, any slope I’ve been given.
Darlene: Use can use the unit circle and say the slope between this…
Judith: You can reason like okay it’s in quadrant one and so it has to be whatever, and then you get that greater than twenty-five, you know, you kind of narrow it down, but I don’t know about approximate.

In this discussion, the four PSTs were not able to verify the outcome of the question about how angles and slopes varied together; these types of discussions occurred in the other groups at the end of Day 1. This evidence shows the Day 1 tasks were difficult for PSTs to verify and they produced uncertainty about the relationship between angle measures and slopes.

Resolution to this uncertainty occurred when we asked the PSTs to use dynamic geometry sketches in the Day 1 homework and present them in Day 2. The Day 1 homework asked PSTs to create two dynamic geometry sketches: a sketch of an 11 degree angle and a sketch of the line $y=(2/5)x$ with many different slope triangles. These sketches gave PSTs the tools to investigate the differences between the 11 and 22 degree cases. Through the dynamic nature of the sketches, PSTs could then adjust the angle or slope and calculate or verify the other
quantity. For example, Figures 11 and 12 show how the PSTs created sketches that demonstrated the relationship between angle and slope, specifically that this was a non-linear relationship. Thus, this type of uncertainty created an opportunity for PSTs to engage in Mental Action 3 of the covariational reasoning framework.

Additional opportunities for PSTs to engage in Mental Action 3 occurred during the whole class discussion when PSTs used the sketches to justify conclusions about the angle and slope relationship. For example, the following exchange shows PSTs verifying why the 11 degree angle corresponds to a slope slightly less than $\frac{1}{5}=0.2$.

Moises (Group 5): I said [the 11 degree angle] is about $\frac{1}{5}$ because you're using an approximation. I don't remember exactly what the numbers were but when you were doing the ratio you got, let's say, 11.3-degrees. Was your $\frac{1}{5}$ ratio, I don't remember exactly what the number was ... If that was 11.3. Well, obviously, if you do your approximation on this then it's going to be about because it's not exactly that 11 degrees. That make sense? When you convert it back. Anybody else do it differently or have a different opinion on that?

Judith (Group 3): I agree with that because I noticed that too. When we were doing our homework that it was just a touch off.

Moises (Group 5): Yeah, when you're doing the GeoGebra it's not exactly ... Nicole (Group 1): It's 0.19.

Moises (Group 5): Yeah, this isn't exactly 22-degrees, it was 21.8 degrees and that's exactly what I got too when I did mine. They both are approximations around it.

The sketches gave PSTs the tools needed to make sense of the relationship between the angle and slope and they were able to recognize slight variations between their original estimated calculations without technology and the outcomes technological tools provided. Throughout the remainder of Day 2, PSTs used the sketches to complete the Trig Table Toolkit (Figure 2). The exchange between Group 3 shows PSTs gravitated towards varying methods of verifying their conclusions, including the sketches as well as physical tools:
Darlene: I’m confused. How should we…other than the ones we already did, how are we supposed to find the others. Are we?
Judith: I’m looking at this chart too [unclear what chart].
Addie: Because we can get this [dynamic sketch] to the appropriate numbers. We can move the slope down to get this [value needed in the table].
James: Protractors.
Judith: He said we can use technology, too
Darlene: Oh, okay.

PSTs used these tools to verify the values for various angles and slopes, replacing their previous methods that had resulted in non-readily verifiable outcomes. Understanding why these verification tools provided the values was an important step that impacted how the PSTs thought about tangent. By generating these values, PSTs were able to create and discuss a graph of the results (Figure 10). PSTs were also able to discuss the tangent function in terms of quantities that covary by comparing the proposed proportional relationship to the actual measurement gathers from the sketches. The whole-group discussion occurring at the end of Day 2 illustrates how the verification methods led to PSTs making statements about the tangent relationship:

Judith (Group 3): We talked about how it wasn’t changing proportionally, but I feel that, that graph, I couldn’t really visualize what was happening, so I like, she, it, like took all of our word explanation and gave a visual explanation to it.
Sarah (Group 5): I was just saying that maybe if you, if you did the degree measures for the second quadrant and you graphed those relationships it would probably get at the negatives, it would probably end up being a tangent graph.

This thinking and discussion about verification methods was revisited in Day 3 with a discussion about the double angle formula for tangent. This evidence suggests that prompting for uncertainty with initially non-verifiable outcomes provided opportunities for PSTs to engage in Mental Action 3 and contributed to PSTs’ increased reasoning about the tangent relationship.

**Increased Reasoning Resulting from Unknown Solution Paths**

We found the tasks prompted PSTs to explore relationships that were unknown from the PSTs’ perspective; however, as previously stated, we found stronger evidence for the first and second type of uncertainty (competing claims, non-readily verifiable outcomes) as compared to
the third (unknown solution paths). The nature of the tasks brought about uncertainty with the solution paths for the PSTs, though their conversations about this uncertainty were not as straightforward as their conversations around competing claims and non-readily verifiable outcomes. Despite the difficulty in showing concrete excerpts of this type of uncertainty, we include a brief section here to illustrate the strongest examples available with respect to unknown solution paths.

As one example, *unknown solution paths* were seen in the following exchange, occurring when PSTs reached a question in the Day 1 task requiring them to create an 18 degree angle QPR on graph paper and then find the approximate slope for the terminal ray. Projected on the front screen was a note telling PSTs they could use any of the following materials: protractor, string, ruler, graph paper, and technology such as Desmos, Geometers’ Sketchpad, or GeoGebra. The following exchange took place in Group 2:

Charlotte: Create QPR? On the graph so that it measures eighteen degrees?
Patrick: So is this just a guestimation? What does the angle look like to me?
Charlotte: I think I actually did it a little bit, and it's twenty-two.
Patrick: Oh did you? Okay. I was like, it says, ‘What does it appear to be?’ I was about to say twenty-five.
Charlotte: Instructor, are we supposed to use a protractor?
Instructor: You can use a protractor.
Charlotte: Wait, or it says technology.
Teacher: Mm-hmm [affirmative] - But not trig buttons.
[a few minutes pass of PSTs attempting unsuccessfully to answer this question]
Charlotte: How do I do this? Oh my gosh.
Patrick: You don't know how to use that?
Charlotte: I feel ancient because I can’t use a calculator ... wait. Let me try [with the protractor].

In this exchange Charlotte and Patrick come across the problem of creating an angle and determining the slope using physical or virtual tools. Both these and the other PSTs resolved the initial uncertainty about how to create and relate the quantities angle and slope ratio.

Resolution to this uncertainty occurred when PSTs considered how a specific angle measurement (11 degrees) had a single slope (approximately 1/5) regardless of the size of the
triangle (or equivalently regardless of the size of the radius) and then comparing the 11 degree angle to the 22 degree angle and discuss the nature of that relationship. In addition to the scaffolding in the tasks, resolution to the unknown solution path occurred in the form of collaboration. As PSTs worked together in the small groups, they built on each other’s ideas, challenged conjectures, and adopted successful solution paths that were proposed. For example, when describing the relationship between angle and slope, Judith (Group 3) benefitted from the graph presented by Charlotte (Group 2), as detailed in the exchange that accompanied with graph in Figure 10. Judith was better able to see and explain the relationship between two quantities due to this whole-class discussion. Furthermore, Sarah (Group 4) then tied in the conversation her group was having about ‘the degree measures for the second quadrant and [if] you graphed those relationships it would probably get at the negatives, it would probably end up being a tangent graph’. These comments indicated the whole-group discussions allowed PSTs to make progress in determining how to solve the essential question of the tasks, ‘what kind of relationship exists between the angle and corresponding slope, and how would you characterize this relationship?’ In this way, having the series of tasks include unknown solution paths encouraged PSTs to resolve the uncertainty in a way that promoted covariational reasoning.

Implications

In this study we used the uncertainty framework and the quantitative and covariational reasoning frameworks to investigate PSTs’ reasoning before, during, and after three days of tasks. We found the tasks promoted uncertainty by having PSTs encounter competing claims, non-readily verifiable outcomes, and unknown solution paths, all while operating with the social context of group work within the methods course. PSTs resolved the uncertainty by testing claims with specific angle measures, using dynamic geometry sketches to verify outcomes,
collaborating in their small groups, and having whole-class discussions. To measure differences
in how PSTs reasoned about this relationship, we defined an increase in the Mental Action (e.g.
initial coding of Mental Action 0 to a later coding of Mental Action 1 for an individual) as
evidence of increased quantitative and covariational reasoning.

Using this coding based on these frameworks, we found PSTs’ reasoning about the
tangent relationship was related to when PSTs encountered and resolved the uncertainty in the
tasks. Initially 18 of the 19 PSTs were coded as Mental Action 0 when describing the tangent
relationship, which they did using only the pneumonic SOH-CAH-TOA for those that indicated
any understanding of the tangent relationship at all. When comparing PSTs’ thinking before,
during, and after the series of tasks, we found the increased reasoning often occurred as a result
of resolving uncertainty in the tasks. All PSTs made statements during and after the tasks
indicating Mental Action 1 (coordinating the value of one variable with changes in the other) and
some PSTs exhibited Mental Actions 2 (coordinating the direction of change of one variable with
changes in the other variable) and Mental Action 3 (coordinating the amount of change of one
variable with changes in the other variable). Even without evidence of these higher level Mental
Actions, we still argue these tasks were effective because a basic level of covariational reasoning
is needed for PSTs to foster a conceptual understanding of trigonometry with their own students.
For example, consider the situation when PSTs distinguish trigonometric relationships from
linear relationships and justify the double angle formula for tangent. Increased Mental Actions
suggests PSTs’ reasoning about the tangent relationship is supported by role of uncertainty in the
tasks and other mathematics teacher educators could repeat this process and find similar results.
Thus, we summarize the opportunities with uncertainty prompted PSTs to consider the quantities
involved in the tangent relationship (the measure of an angle and the resulting slope values of the
extended ray) and covariation occurring as one of these quantities changes (i.e. as the angle increases, the resulting slope increased but not in a non-linear manner). Technology through the use of dynamic geometry software was an important factor in how students explored this relationship.

This work suggests that uncertainty can be an effective way to promote PSTs’ reasoning of the tangent relationship and perhaps could be applied to other mathematics disciplines. This has implications for mathematics teacher educators because there is currently little research on how to foster PSTs’ quantitative and covariational reasoning about the tangent relationship. This study indicates one method of promoting PSTs to consider the quantities and covariation occurring between the quantities in this relationship. Similar to Moore’s [16] work with the sine and cosine function, we invite others to implement these tasks, or a similar series of tasks, within a social context such as a methods course to support PSTs’ thinking about trigonometry in ways that attend to quantitative reasoning.

Findings from the study also support Zaslavsky’s [15] notion that building in opportunities for uncertainty can be a productive way to structure a task that supports learners’ thinking about mathematical concepts. We found that PSTs engaging in tasks that prompted competing claims, non-readily verifiable outcomes, and unknown solution paths promoted their reasoning about a notoriously difficult mathematical relationship. When tasks incorporate uncertainty, PSTs are afforded the opportunity to experience mathematics again from the perspective of learners, and that uncertainty can be leveraged to motivate teachers, PSTs in this case, to expand their understanding of content to incorporate multiple approaches, explanations, representations, and connections. We argue mathematics teacher educators can take aspects of this approach to have learners collaboratively explore relationships beyond tangent, such as
logarithms or the inverse trigonometric relationship. Although using uncertainty to promote mathematical understanding is not a new idea, seeing how uncertainty manifests in a task promoting high school concepts can provide useful insights for how other mathematics teacher educators can promote PSTs’ reasoning. With some consideration, this approach can be extended to teaching secondary students these concepts.

A final implication of this work is sharing our combination of frameworks with other mathematics educator researchers. This study joined the social perspective of the uncertainty framework with the cognitive perspective of the quantitative and covariational reasoning frameworks. By examining what behaviors and responses occurred in the group settings when the PSTs encountered competing claims, difficulty in verifying outcomes, and not initially knowing how to solve a problem, the uncertainty framework provided us the opportunity to identify and categorize when the task presented PSTs with uncertainty, how the uncertainty was resolved, and the statements the PSTs made following this resolution. Framing the study in terms of quantitative and covariational reasoning afforded a way to identify when PSTs’ statements indicated their reasoning was improving, specifically by an increase in the Mental Actions using Carlson et al.’s [9] framework. Together, these approaches provided a means to track how PSTs’ statements about the tangent relationship changed in response to the uncertainty, giving us insight into the mental constructions of quantities and how these quantities varied together. We contend that this approach can be used by other mathematics education researchers to examine how learners reason about a mathematical topic and how this reasoning changes in response to conflicts they encounter in a task. We also believe there may be space for exploration of the use of the uncertainty framework with middle or high school students, as they engage in tasks.
Future research could explore the application of these frameworks beyond the presently studied PST context.

We further acknowledge that making the claim that a PST has a particular quantity or covariational relationship in mind is a strong claim to make [10; 13]. This study has the limitation of not having numerous one-on-one clinical interviews or teaching experiments that continually prompt PSTs to explain their thinking and made a strong pre- and post-comparison of their thinking. Having these data would have allowed us to build developmental model of PSTs’ reasoning, and we recommend future studies improve on this approach in this way. These data would have provided additional details about the extent to which it was the uncertainty that resulted in certain gains, as opposed to other inputs, such as technology, group work, or ad hoc instructional prompts.

**Conclusion**

For mathematics teacher educators, there is a dearth of research on how to foster PSTs’ quantitative and covariational reasoning about the tangent relationship. This study provides one method of prompting PSTs to consider the covariation occurring between the two quantities in this relationship and apply this understanding in the context of teaching students through presenting opportunities to encounter and resolve uncertainty. Even more importantly, within this study, the combining of the social perspective of the uncertainty framework with the cognitive perspective of the quantitative and covariational reasoning frameworks advances approaches in the mathematics education community through this convergence of frameworks. We encourage other mathematics teacher educators to consider a similar approach and contemplate how to incorporate uncertainty when supporting PSTs’ understanding of mathematical content.
References


31. Author, 2015


Appendix A

Mental Actions of the Covariation Framework [9, p. 9].

<table>
<thead>
<tr>
<th>Mental Action</th>
<th>Description of Mental Action</th>
<th>Behaviors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mental Action 1 (MA1)</td>
<td>Coordinating the value of one variable with changes in the other</td>
<td>• labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)</td>
</tr>
<tr>
<td>Mental Action 2 (MA2)</td>
<td>Coordinating the direction of change of one variable with changes in the other variable</td>
<td>• constructing an increasing straight line • verbalizing an awareness of the direction of change of the output while considering changes in the input</td>
</tr>
<tr>
<td>Mental Action 3 (MA3)</td>
<td>Coordinating the amount of change of one variable with changes in the other variable</td>
<td>• plotting points/constructing secant lines • verbalizing an awareness of the amount of change of the output while considering changes in the input</td>
</tr>
<tr>
<td>Mental Action 4 (MA4)</td>
<td>Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.</td>
<td>• constructing contiguous secant lines for the domain • verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input</td>
</tr>
<tr>
<td>Mental Action 5 (MA5)</td>
<td>Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function</td>
<td>• constructing a smooth curve with clear indications of concavity changes • verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct)</td>
</tr>
</tbody>
</table>
Table 1

Summary of the covariation Mental Action present before, during, and after the series of tasks were implemented

<table>
<thead>
<tr>
<th>Covariational Framework</th>
<th>Data before tasks (Pre-Assessment responses)</th>
<th>Data during tasks (audio recordings, written work)</th>
<th>Data after tasks (Midterm question responses)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No covariational reasoning present</td>
<td>94.7% (18 of 19 PSTs)</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Mental Action 1</td>
<td>5.3% (1 of 19 PSTs)</td>
<td>100% (18 of 18 PSTs)</td>
<td>100% (18 of 18 PSTs)</td>
</tr>
<tr>
<td>Mental Action 2</td>
<td>0%</td>
<td>100% (18 of 18 PSTs)</td>
<td>72% (13 of 18 PSTs)</td>
</tr>
<tr>
<td>Mental Action 3</td>
<td>0%</td>
<td>100% (18 of 18 PSTs)</td>
<td>33% (6 of 18 PSTs)</td>
</tr>
</tbody>
</table>
Figure 1. The leading question in Day 1’s task, instructing PSTs to investigate patterns that emerge in the corresponding ratios of the triangles formed from a specific angle measured counterclockwise in degrees from the 3 o’clock position.
Figure 2

Trig Table Toolkit

<table>
<thead>
<tr>
<th>Angle</th>
<th>Slope triangle</th>
<th>Approximate slope ratio as a fraction and a decimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta^\circ$</td>
<td>$\frac{\Delta y}{\Delta x}$</td>
<td>$\frac{\Delta y}{\Delta x}$</td>
</tr>
<tr>
<td>0°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11°</td>
<td><img src="image" alt="Diagram" /></td>
<td>$\frac{1}{5} = 0.2$</td>
</tr>
<tr>
<td>18°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>55°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>68°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>72°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>79°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>83°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>84°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>89°</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Figure 2.* The questions in Day 2’s tasks build up to PSTs being able to complete the table above, called the Trig Table Toolkit.
Figure 3. Part of the Day 2 homework was for PSTs to create a dynamic sketch of a slope angle measuring 55 degrees, then determine the missing vertical leg length using measurement tools.
Figure 4

*Increases:* The ratio increases
*Decreases:* The ratio decreases

*Figure 4.* Evidence of Group 2’s task of Mental Actions 1 and 2 of the covariational framework by showing the direction of change as the angle varies within the first quadrant.
Figure 5. Willie’s response to the midterm question asking PSTs to explain the tangent function, showing evidence of covariational reasoning Mental Action 3.
Figure 6

The slope ratios are proportional.

Figure 6. Group 3’s initial response to how the similar triangles’ slopes are related.
Figure 7

<table>
<thead>
<tr>
<th>Triangle 1</th>
<th>Triangle 2</th>
<th>Triangle 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>3/15</td>
<td>5/25</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure 7. Group 3’s response to ‘confirm your conclusion about how the slope ratios are related by writing the slope ratio for each triangle as a fraction’.
Figure 8

\[ \tan(\theta) \neq r \left( \frac{\sin(\theta)}{\cos(\theta)} \right), \quad \tan(\theta) = \frac{r \sin(\theta)}{r \cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} \]

*Figure 8. Algebraic representation of conclusion*
Figure 9: Group 2’s (top) and Group 1’s (bottom) responses on determining the slope of a 22 degree angle.
Figure 10. A reproduction of Sarah’s graph that she made on Desmos and displayed to the class.
Figure 11. Darlene’s sketch from the homework assignment showing a 22 degree angle does not produce a slope of (exactly) $\frac{2}{5}$. 
Figure 12. Addie’s sketch from the homework 1 assignment showing a slope of (exactly) $\frac{3}{5}$ does not produce an angle measure of 22 degrees.