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Instructional decisions in a mathematics course for elementary education majors

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INSTRUCTIONAL DECISIONS IN A MATHEMATICS COURSE
FOR ELEMENTARY EDUCATION MAJORS

by

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DEDICATION

I dedicate this work to my father. It has been a long road since I first told you that I was accepted to graduate school. Thank you for believing in me. Miss you, and love you.
ACKNOWLEDGMENTS

This work would have been impossible without the assistance and support of my committee, professors, fellow graduate students, colleagues, family, and friends. I have not the words to express the many ways in which you helped, or the many kindnesses that I received. My gratitude is boundless.
INSTRUCTIONAL DECISIONS IN A MATHEMATICS COURSE FOR ELEMENTARY EDUCATION MAJORS

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ABSTRACT

Although it recommended that pre-service elementary teachers be provided with opportunities to develop mathematical understanding through engagement in experiences where they reason, explain, justify and generalize about mathematics, much still remains to be learned about how a mathematics teacher educator can support pre-service teachers in developing understanding during these experiences. This study investigated the instructional decisions of an experienced instructor in an undergraduate mathematics course for pre-service elementary teachers as he supported developing understanding around geometric measurement topics. Two lessons on the geometric measurement topic of area formulas were considered by the researcher. Multiple interviews were conducted with the instructor including a pre-interview session, four video-stimulated recall sessions, and one post-interview. All observed lessons and interviews were recorded and transcribed. Lastly, participants completed a Pre-test and Post-test on area formula.

Analysis of the instructor’s descriptions of his teaching enabled the researcher to construct a description of the intended implementation of the two area formula lessons. Video-stimulated recall sessions along with the classroom observations and interviews were used to analyze the instructor’s decisions during teaching. The instructor’s actions,
decisions, and strategies during whole-class discussion were mapped to the Math-Talk Learning Community Framework (Hufferd-Ackles, Fuson & Sherin, 2004) in order to provide a description of how the instructor actually supported the development of participants’ mathematical understanding. Three levels of instructional decisions emerged. High-level decisions included the instructor’s choice of curriculum and his use of discussion as the primary instructional methodology. Mid-level decisions included the instructor’s decisions around the social and academic norms created in the classroom. For instance, the instructor provided few explicit mathematical statements so that participants were the source of mathematical ideas. Additionally, the instructor would not accept partial mathematical justifications from participants. Also, to engage the class in discussion, the instructor reminded participants of their roles as future teachers and their responsibility to ask questions of each other. Micro-level decisions included the instructor’s choice of when and how to use talk moves (Chapin, O’Connor & Anderson, 2013) and his selection of discussion participants. There was evidence that participants’ understanding improved as shown by significant change in achievement on the Area Formula Pre- and Post-test. Overall the instructor’s intended instructional decisions and enacted instructional decisions were aligned.
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CHAPTER 1: THE RESEARCH PROBLEM

Section I: Introduction

In recent years, the goal of engaging students in mathematical reasoning and justification, as a strategy to help students develop understanding, has moved to the forefront of mathematics education. Providing learners opportunities to reason about mathematical ideas, either formally or informally, as a method for developing more thorough and longer lasting understanding is recommended by both researchers and organizations (Ball, Hoyles, Jahnke, Hans and Movshovitz-Hadar, 2002; National Council for Teachers of Mathematics [NCTM], 2000; National Governors Association & Council for Chief School Officers [NGA & CCSSO], 2010). The formal justification of theorems and procedures is already a well-known practice in upper-level mathematics courses where students produce formal justifications or proofs (Stylianides, 2007). The informal justification of mathematical ideas is also being recommended for students in lower grades (Ball & Bass, 2003; Hoyles, 1997). Research has found that expecting informal reasoning about mathematics is reasonable at the elementary level (Yackel, 2000).

In June 2010, as part of the Common Core State Standards Initiative, standards for both mathematics and English language arts were released for implementation on a national scale to provide a coherent and consistent vision of mathematics education for American students. The Common Core State Standards for Mathematics (CCSSM) provide both standards for mathematical practice and content for all students from kindergarten through grade 12 (NGA & CCSSO, 2010). As of June, 2014, the Common
Core State Standards for Mathematics (CCSSM) had been adopted by 43 states, four territories and the District of Columbia. At the state level, individual states have begun the process of altering their mathematical standards and educational frameworks to align with the CCSSM standards. Districts and schools have also started to expect teachers to implement the standards for mathematical practice in their classrooms (Council of Chief State School Officers, 2012). Within these practices is the expectation that all students should be able to explain ideas, construct mathematical arguments, and justify the statements that they make.

Implementing the CCSSM practices with their focus on reasoning and justification will necessitate a shift in classroom practice for many teachers. Teachers will now be expected to incorporate instruction in which their students construct and critique arguments. Teachers without experience in constructing arguments will need professional development to support this shift in instruction. This instructional adjustment may be especially challenging for elementary school teachers since elementary educators, as a group, tend to have incomplete mathematical knowledge (Ball, 1990; Carpenter, Fennema, Peterson & Carey, 1988; Ma, 1999). Furthermore, mathematics is often taught by demonstrating procedures rather than having students reason about problems (Hiebert & Stigler, 2000). Since learners and educators become accustomed to the way a discipline is taught (Jarvis-Selinger, Collins & Pratt, 2007), this makes change especially difficult. Educator preparation programs will need to address these challenges in order to shift the focus of instruction in the classroom.

An immediate concern is the preparation of pre-service teachers (PSTs) in
educator programs. As it currently stands, the mathematics preparation of undergraduate PSTs is insufficient to prepare them to teach to current standards with fidelity and high quality (Science and Mathematics Teacher Imperative & The Leadership Collaborative [SMTI & TLC], 2011). The recently released *Mathematical Education of Teachers II* (*MET II*) calls for teacher preparation programs to expand their requirements in mathematics in order to provide teachers with mathematical experiences similar to those suggested in the *Common Core State Standards for Mathematics* (Conference Board of the Mathematical Sciences, 2012). Teacher preparation courses must be revised so that teachers experience coursework that aligns with and builds upon the CCSSM standards for content and practice (Conference Board of the Mathematical Sciences, 2012; SMTI & TLC, 2011). Pre-service teachers will themselves need opportunities to experience the standards with coursework that includes high quality instruction (Conference Board of the Mathematical Sciences, 2012; SMTI & TLC, 2011). However, more research is needed that focuses on mathematics education of pre-service teachers. Specifically, what are the instructional strategies that can be used to prepare high quality educators with a deep understanding of mathematics (SMTI & TLC, 2011)?

This study contributes to the literature on preparing elementary pre-service teachers to teach mathematics. Through studying the decisions and actions of one instructor in a pre-service mathematics course designed to foster reasoning, the researcher provided descriptions of the instructors’ thoughts and decisions as he worked to foster understanding through discussion. With better knowledge of the instructional techniques that support pre-service teachers in discussing and reasoning about mathematics, faculty
members and instructors of mathematics teacher content courses can better prepare these young people for their future roles.

**Section II: Statement of the Problem**

The purpose of this dissertation was to investigate the delivery of a mathematics education curriculum for elementary teachers designed to develop pre-service teachers' understanding of mathematics. Specifically, the researcher examined the decisions of the instructor during whole-class discussion where pre-service teachers were expected to explain, justify, and/or generalize mathematical ideas. The goal of this research was to provide rich descriptions of the implementation of lessons on geometric measurement, as guided by the decisions of an experienced and knowledgeable instructor, in order to help math teacher educators better understand how to support the development of mathematics knowledge. This study was designed to answer the following question:

What are the instructional decisions made by a knowledgeable instructor of a mathematics course as he supports pre-service elementary teachers in developing their understanding of key instructional content?

**Section III: Justification for the Study**

Sense-making as a social interaction with peers can potentially have a great impact on learning since discussion can compel learners to make connections, develop concepts, and broaden their understanding (Kiernan, 2001; Krummheuer, 2000, 2007; Sfard, 2001; Vygotsky, 1978). For the greatest impact, peers need to listen to each other, respect each other, make decisions on the soundness of each other's reasoning, and work together towards understanding (Kiernan, 2001; Krummheuer, 2000, 2007; Sfard, 2001;
Vygotsky, 1978). Organizations central to the teaching and learning of mathematics have based K-12 standards and recommendations on developing understanding through reasoning around mathematical ideas (NGA & CCSSO, 2010; NCTM, 1989, 2000; Mathematics Association of America, 2014). In the *Mathematical Education of Teachers II (MET II)* report, recommendations for the education of pre-service teachers were presented that emphasized providing PSTs their own opportunities to make sense of mathematics by explaining, justifying and generalizing about mathematical topics (CBMS, 2012, p.17). This report recommended that PSTs be provided opportunities to engage in the mathematical practices required of their own future students; specifically PSTs “should have time and opportunity to reason abstractly and quantitatively, to construct viable arguments, to listen carefully to other people’s reasoning, and to discuss and critique it” (CBMS, 2012, p.33).

The MET II report’s focus on providing opportunities for the PSTS to reason around mathematics is, in part, a response to the need for instruction to stress reasoning and understanding. Researchers involved in the 1999 *Trends in International Mathematics and Science Study* (TIMSS) researchers found that U.S. students spent less than 1% of their time exploring and discussing math problems (Stigler & Hiebert, 2004). This was despite the fact that 70% of TIMSS teachers claimed that the recorded lessons were at least a *fair amount* in accord with current math ideas (Hiebert & Stigler, 2000, p. 5) which emphasized the exploration and discussion of mathematics problems (National Council of Teachers of Mathematics, 1989, 1991). If educators are to going be able to teach using the CCSSM mathematical practices, they need experiences using those same
practices when they learn mathematics such as constructing arguments and critiquing the arguments of others (Conference Board of the Mathematical Sciences, 2012). Prior to implementing the CCSSM practices, educators need experiences explaining, justifying, and generalizing in mathematics classes to assist them in developing mathematical understanding.

More research on how to prepare elementary teachers to implement the mathematical practices, especially those focused on reasoning in their future classrooms, is necessary. Previous research has considered the roles K-12 teachers play in guiding student reasoning (Krussel, Edwards, & Springer, 2004; Sfard, 2007) and how informal argumentation or justification unfolds for K-12 students of mathematics (Inglis, Mejia-Ramos, & Simpson, 2007; Krummheuer, 1995, 2000, 2007; Tall & Mejia-Ramos, 2006; Yackel 2001). There are very few studies that address how to assist pre-service elementary teachers in developing their own abilities to construct arguments that justify mathematical ideas.

As it stands, we do not know enough about instructional strategies for preparing high quality educators (SMTI & TLC, 2011). Researchers have looked at mathematical argumentation at the K-12 and university level (Inglis, Mejia-Ramos, & Simpson, 2007; Krummheuer, 1995, 2000, 2007; Tall & Mejia-Ramos, 2006; Yackel 2001). Research studies have also considered the questions and discourse moves used to guide argumentation by both K-12 educators (Boaler & Brodie, 2004; Boerst, Sleep, Ball & Bass, 2011; Henning, McKeny, Foley & Balong, 2012; Lampert, 2001; Perry, Vanderstoep & Yu, 1993), and pre-service teachers (Boerst, Sleep, Ball & Bass, 2011;
Magiera, van den Kieboom & Moyer, 2013). However, the literature is lacking on what constructing reasoning around mathematical ideas looks like in a mathematics course for pre-service elementary teachers. Specifically, more research is needed on the strategies math teacher educators can use to elicit reasoning about mathematics from pre-service elementary teachers. This study examined one undergraduate instructor’s decisions and actions as he supported pre-service teachers’ in developing understanding around the geometric measurement topic of area formulas.

This chapter presented the problem and provided a justification for this study. Chapter II presents a review of the relevant literature around teaching geometry and previous research on supporting reasoning in the classroom. Chapter III describes the study’s research design, including the study sample, data collection, and procedures for data analysis. The first part of Chapter IV presents the results of the pre-/post-test data analysis. The second part of Chapter IV provides a description of the intended instruction. The third part of Chapter IV provides a description of class instruction using the results of analysis of the whole-class discussion data, and the collected interview data. Chapter V summarizes the results of the study, presents limitations, and provides suggestions for future research.
CHAPTER 2: THE LITERATURE REVIEW

In 2012 the Conference Board of the Mathematical Sciences (CBMS) published the *Mathematical Education of Teachers II* (MET II) which made recommendations for the mathematics preparation of elementary educators. The 2012 MET II report and its 2001 predecessor *Mathematical Education of Teachers I* (MET I) recommended mathematics coursework allow PSTs to “develop a solid understanding of the mathematics they will teach” (CBMS, 2012, p.17). The CBMS explained that PSTs need to have their own experiences “reasoning, explaining, and making sense of the mathematics” (CBMS, 2012, p.17). Not only were PSTs expected to gain a mastery of the mathematics they were expected to teach, but they also needed to develop an understanding of the mathematics topics that occurred in adjacent grade levels. As part of the 2012 recommendations, the MET II report stressed the need for elementary teachers to have a strong knowledge of both the K–8 content standards and standards for mathematical practice presented within the 2010 *Common Core State Standards for Mathematics* (CBMS, 2012). The MET recommendations tasked teacher preparation programs with preparing elementary PSTs to know and teach mathematical content and to be prepared to implement mathematical practices.

2.1 The Preparation of Teachers: Geometric Content

One topic that pre-service teachers (PSTs) will need to understand is geometric measurement, which includes being able to justify area formulas. Developing reasoning around area formulas corresponds with the recommendations in the 2001 MET I and the updated 2012 MET II reports. Within the MET I the CBMS suggested that PSTs be
afforded opportunities to develop competence in knowing the properties of basic shapes, the process of measurement, the visualization of two dimensional objects (including shape manipulation and decomposition), area, and “devising area formulas for basic shapes” (CBMS, 2001, p. 21). Specifically relevant is the development of area formulas of which PSTs “can build from an understanding of rectangles” (CBMS, 2001, p. 22).

The MET I report recommended that elementary mathematics teachers be given experiences “devising area formulas for triangles, parallelograms, and trapezoids; and knowing the formula for the area of a circle” (CBMS, 2001, p. 85).

Building on these ideas, the MET II report speaks to the clear vision provided by the CCSSM and recommends “enabling teachers to teach that mathematics” (CBMS, 2012, p. xii) found within the CCSSM document. Since elementary pre-service teachers will eventually be expected to lay the foundation for understanding area for students, it is vital that they personally have a thorough understanding of these concepts. In fact, the MET II report recommends that elementary PSTs not only know the K-5 content standards in depth, but they should also understand the connections to pre-kindergarten and grade 6-8 mathematics (CBMS, 2012, p. 23).

Being able to justify area formulas requires understanding of the standards that begin in kindergarten and progress through middle school. For instance, it is in the early grades that students begin to gain experience in decomposing and recomposing shapes. In kindergarten, students are introduced to combining shapes to form another shape through a trial-and-error process. At the first-grade level, students should start to informally recognize that certain shapes combine into other shapes. They should also start to gain
experience partitioning shapes. By the time students are in the second grade, they should be fairly competent in composing and decomposing basic shapes. The partitioning of an object into smaller sub-shapes is a relevant connection to the spatial structuring necessary for understanding and justifying area formulas. Specifically, PSTs will need to understand how to decompose and reconstruct shapes as part of the justification for area formulas.

In the second and third grades, students also start to practice skills that provide context for the meaning of area measurement. In the geometry domain, it is at the second-grade level that students are introduced to counting the number of same-sized squares in rectangles and in other shapes. This introduction is not formally represented as learning area, but it does present ideas necessary for understanding area when it is formally taught in grade three. The understanding of area as a covering measured using equal-sized square units is the foundation for all area formulas. In the third grade, students are introduced to area as an attribute of plane figures. Students learn that area is measured in square units and must cover a 2D shape without holes or gaps. Students then progress to multiplying length and width to determine the number of square units in an array, which is a key content component for understanding the area of a rectangle. As stated in the MET I report, it is from the area formula for a rectangle that the area formulas for other shapes are justified.

In grade four, students should be provided opportunities to build upon their knowledge and use the rectangle area formula as they solve real-world problems. Area formulas become a focus again in grade six, where students work on decomposing triangles and special quadrilaterals, allowing them to develop justifications for the area
formula for a parallelogram and triangle by connecting to their knowledge of the area of rectangles. By the end of grade seven, the CCSSM expect that students should know and be able to apply the area formula for a circle. Thus, in order to prepare their own students to justify area formulas as provided in the CCSSM, the MET II report urges that PSTs be given opportunities to think about these ideas and how they develop during their teacher preparation programs.

In summary, according to the MET II report, teacher preparation programs will need to ensure that future educators have the appropriate conceptual knowledge to implement the CCSSM standards with their own elementary students. Elementary pre-service teachers will need to understand the mathematics at the level they will be teaching, and also the mathematics that comes before and after that level. The understanding of mathematical concepts by pre-service teachers will need to be deeper than that of their students so that they can guide students in developing their own conceptual understanding of the topic. For example, teachers need to be able to understand students' explanations and help students to correct mistakes and errors.

2.2 Geometry Knowledge of Pre-service Teachers

Geometry is a fundamental domain of mathematics, with many real world applications (Marchis, 2012). The ability to measure space, including qualities such as area, is fundamental to understanding structure (Lehrer, Jaslow, & Curtis, 2003); geometry allows students to understand their physical environment (National Council of Teachers of Mathematics, 2000). Additionally, both geometry and measurement are necessary for scientific reasoning (Lehrer, Jaslow, & Curtis, 2003; National Council of
Teachers of Mathematics, 2000). Pre-service teachers will be expected to educate their future students on these important topics, and therefore require a deep understanding themselves.

Since students are now expected have a solid understanding of geometric concepts, it is important for their teachers to be prepared to teach this content with understanding. In fact, teachers should have more content knowledge than their students (Sinclair, Pimm, Skelin, & Zbiek, 2012). If making geometric connections to measurement formulas is essential knowledge for students, then it should also be essential knowledge for elementary teachers. However, research has shown that PSTs do not have a deep understanding of geometric concepts. Batrovo and Nason (1996) interviewed 13 pre-service teachers in the first year of an education program as they completed a series of eight measurement tasks. The researchers found that the subjects' knowledge of area measurement seemed to be rule-dominated, without connection to experience. The PSTs interviewed tried to recall disjointed facts to create meaning in the provided problems, and saw little connection to real-life situations. In particular, the first-year pre-service teachers in the study were unable to connect the area formulas they knew, such as between a triangle and rectangle.

Having a rule-dominated understanding of mathematics topics means that one is dependent on memory alone for recall, which may lead to difficulties in retrieving formulas (Hiebert & Carpenter, 1992). In a 2001 study, three pre-service K–8 teachers were asked to determine the area for a parallelogram (Ward & Anhalt, 2002). None of the three teachers were able to recall the area formula for a parallelogram initially. When
given a drawing of a parallelogram, subjects eventually placed the provided drawing within a rectangle and subtracted the areas (triangles) that were not part of the original parallelogram shape. However, these same subjects had great difficulty justifying the area formulas for either a rectangle or triangle. Although they were able to state and use the area formulas, the PSTs were unable to explain why the area formulas for a rectangle and a triangle worked. These results were echoed in a 2010 study by Yew, Zamri, and Lian. In their study, eight pre-service teachers were interviewed as they constructed area formulas for a rectangle, parallelogram, triangle, and trapezoid. None of the teachers were able to justify the formula for the area of a rectangle; however, five of the subjects were able to justify the formula for a parallelogram (by recomposing the figure into a rectangle). Of the eight teachers studied, only two of the subjects were able to develop a formula for the area of a triangle, even when seven knew the formula. Lastly, only three of the eight subjects were able to explain the derivation of the formula for the area of a trapezoid. Taken together these studies suggest participants had a rule-dominated knowledge of area measurement and while PSTs might be able to provide area formulas, they have difficulty justifying formulas.

Although the geometric measurement knowledge of pre-service elementary teachers reflects a deficiency, professional development or coursework has been found to increase the geometry knowledge of pre-service and in-service elementary teachers. For instance, one study used pre-/post-test data of van Hiele levels to consider growth of geometric knowledge during a four-week professional development course conducted for 49 in-service middle school teachers (Swafford, Jones & Thornton, 1997). The van Hiele
levels are five hierarchical levels based on a person’s geometric thinking and reasoning (van Hiele, 1986). The expectation is that by middle school students should be thinking at third or possibly the fourth hierarchical level; high schools students and college students should at the 4th and 5th hierarchical level (Malloy, 2002). Swafford and colleagues who conducted the study found a positive change in overall geometric thinking of the in-service teachers through coursework as measured by van Hiele levels (Swafford, Jones, & Thornton, 1997).

The van Hiele levels have also been used to consider the geometry knowledge of pre-service teachers. Bulut and Bulut administered a pre- and post-test that measured the van Hiele geometric thinking level of 27 first-grade pre-service elementary teachers in an undergraduate mathematics course. The researchers found an average van Hiele level of 1.81 before the course, and an average van Hiele level of 3 after the course (Bulut & Bulut, 2012). In another study of pre-service teachers, Knight (2006) evaluated the pre-test and post-test scores of more than 40 PSTs enrolled in a mathematics geometry course for K–8 teachers. Knight found that the van Hiele test level scores of teachers did improve with course participation. However, Knight expressed concern that most of the subjects entering the course were not at van Hiele level 3, which is informal deduction. The informal deduction level is the expected level for students graduating grade eight (Malloy, 2002). By taking the course, most of the teachers were able to reach level three, but they were not able to pass it (Knight, 2006). These research studies suggest that teachers can increase their level of geometric thinking through professional development or teacher preparation experiences. However it is important to note that van Hiele levels
do not correspond to whether or not a person can specifically justify an area formula.

Though, presumably having reasoning abilities as a greater van Hiele level would suggest a greater likelihood of being able to justify area formulas.

Researchers have also found growth in the geometric understanding of PSTs in similar courses using pre-/post-testing unrelated to van Hiele levels. Matthews, Rech, and Grandgenett (2010) examined the content knowledge of two groups of undergraduate pre-service teachers. The experimental group was composed of PSTs that had enrolled in one or both of two mathematics content courses for elementary teachers (one on number topics and the second on geometry topics). The control group was composed of subjects who did not take part in either content course. Researchers found that PSTs who participated in the content courses performed significantly better on the content assessment (which included geometry concepts). In another study, Senk and colleagues also found content courses had a positive effect on teacher content knowledge. The researchers provided pre- and post-tests on geometry topics to 450 elementary PSTs enrolled in a mathematics course (Senk, Park, Demir & Crespo, 2009). Researchers found that the PSTs scored better on the post-test than the pre-test, almost doubling their initial score. However, in a further examination of the questions, researchers found that although subjects’ understanding on topics such as area improved, they still struggled with fully explaining their ideas. Overall, this research suggests that coursework can increase the geometric knowledge of PSTs. However, PSTs can benefit from further development of geometric concepts.
2.3 The Preparation of Teachers: Practice

The MET I and MET II reports not only address the expected content that PSTs should learn, but also lists expectations for the opportunities that PSTs should experience within their mathematics coursework. The reports recommend that coursework provide experiences to “develop the habits of mind of a mathematical thinker and problem-solver, such as reasoning and explaining, modeling, seeing structure, and generalizing” (CBMS, 2012, p. 19). The 2012 MET II specifically recommends that PSTs have their own experiences implementing the standards for mathematical practice within the CCSSM which present a way for thinking about and interacting with mathematical ideas in the classroom. The CCSSM practices move away from viewing mathematics exclusively as a fixed discipline. Teacher preparation programs need to provide PSTs the time and opportunity to make sense of mathematics problems, reason about problems, and consider the reasoning of others. PSTs “must have classroom experiences in which they become reasoners, conjecturers, and problem solvers” (CBMS, 2001, p. 56).

It is the instructor that structures the learning environment and facilitates the classroom experience for the learner. Through his use of moves to facilitate talk, and his choice of questions, the instructor is crucial in the establishment and maintenance of a discussion-based community of learners. Through discussion the instructor can provide PSTs the classroom experiences in reasoning and justification recommended by the MET I and MET II reports. The learning environment orchestrated by the instructor and the strategies that the instructor implements influence the development of reasoning abilities in the classroom.
2.4 Learning Environment

In *How People Learn*, the National Research Council considered the design characteristics of effective classrooms and school environments. According to the council, based on a synthesis of the research, effective teaching and learning depend on maintaining a balance of four different environments: learner-centered, knowledge-centered, assessment-centered, and community-centered (National Research Council, 2005a, 2005b). These four types of environments are equally important in creating an effective classroom for student learning.

In a learner-centered environment the background and prior knowledge of the learner is attended to when developing ideas (National Research Council, 2005a, p. 134-136). The instructor does not merely tell students information but builds understanding using the foundation that is already there. This means that students need to be provided opportunities to think about and respond to ideas in order to construct knowledge based on what they know. In a learner-centered classroom there is a focus on fully developed thinking. The learner needs to be considered in how knowledge is constructed. The *why?*, *how?*, and *what?* of the knowledge are central themes in effective classrooms.

The knowledge-centered classroom considers the knowledge that the learner needs, how that knowledge is important, and what the best methods are for mastery (National Research Council, 2005a, p. 136-139). In creating the knowledge-centered classroom, the importance of both procedural knowledge and conceptual knowledge are recognized and the need to balance both in terms of instruction is emphasized. Furthermore, connections need to be made between mathematical ideas so that learners
do not believe knowledge is a set of isolated topics. Following the development of knowledge, the learners’ understanding of the topics needs to be assessed. Assessment of understanding is a crucial component of effective classrooms.

The third environment discussed in *How People Learn* is the assessment-centered environment. The assessment-centered environment uses both formal and informal assessments to determine what knowledge has been gained (National Research Council, 2005b, p. 16-17). The instructor can then adapt the knowledge-centered environment in response to assessments. Assessment not only provides an opportunity for the instructor to adapt instruction to the needs of the learners, but it also allows learners to reflect upon their own learning. Metacognition is a crucial component of self-assessment. Learners need to consider their own understanding, what they know, and what they need to do in order to find answers.

The fourth environment critical in teaching and learning is the community-centered classroom environment (National Research Council, 2005b, p. 17-20). The norms that are created determine participation and subsequently learning. Furthermore, creating a community-centered environment is greatly dependent on the first three environments. The community-centered environment is reliant upon the norms around talk, who can contribute, the source of ideas, and who can ask questions. In a community-centered classroom environment, learners need to feel comfortable expressing knowledge and reasoning around ideas. The focus is not on being right, but on the reasoning provided. Having the learners feel comfortable asking questions and responding to each other are components of this environment.
The four classroom environments are all important and influence each other. A learner’s background knowledge is very relevant to the knowledge they need. A student’s cultural background may also influence whether a student feels comfortable presenting and contributing to classroom discussion. The learners, the knowledge being considered, and the use of assessment all contribute to the community-centered learning classroom environment. In bridging these various environments, discussion becomes a crucial component to facilitate learning. In a community-centered classroom, talk allows the understanding of the learner to become clearer (National Research Council, 2005b, p. 586). Classroom discussion allows student understanding to be challenged, and knowledge to grow (National Research Council, 2005b, p. 586). In creating a community-centered environment all members of the class need to feel comfortable contributing and reasoning around ideas.

A focus on discussion or talk in the classroom, as a component of creating and nurturing effective classroom environments, is a theme that is consistently seen throughout the 2004 How People Learn and the 2005 How People Learn Mathematics. The council asserts that “One important way to make students’ thinking visible is through math talk—talking about mathematical thinking” (National Research Council, 2005b, p. 228). Math talk allows “teachers to draw out and work with the preconceptions students bring with them to the classroom and then helps students learn how to do this sort of work for themselves and for others” (National Research Council, 2005b, p. 228); it also helps “teachers become more learner focused and make stronger connections with each of their students” (National Research Council, 2005b, p. 228). However, just talking
is not enough. Discussions need to be productive and the instructor needs to be able to respond dynamically to student contributions. The components of a class where mathematical ideas are developed through talk needs to be considered.

2.5 Math-Talk Learning Community

Researchers from Northwestern University coined the term Math-Talk Learning Community to describe a “community in which individuals assist one another’s learning of mathematics by engaging in meaningful mathematics discourse” (Hufferd-Ackles, Fuson and Sherin, 2004, p. 81). As part of a year-long research study of an urban elementary teacher, Hufferd-Ackles and colleagues created a framework to explain the trajectory of the students and instructor interactions in the development of a Math-Talk Learning Community. Using the classroom transcripts and developed themes of “evidence of a mathematics community, teacher actions, and student actions,” (Hufferd-Ackles, Fuson & Sherin, 2004, p. 87) the researchers created a framework that described levels of math-talk in the classroom. The framework is based on growth in four categories: (1) questioning, (2) explaining mathematical thinking, (3) sources of mathematical ideas, and (4) the responsibility for learning. The developmental trajectory for each is considered.

In the Math-Talk Learning Community framework there are four levels of math talk. Each level describes the trajectory of growth in each of the four areas mentioned above. The first area is the use of questioning as explained in the framework. Asking questions is important because it allows for the development of ideas, and can push students to further clarify the reasoning they are providing for the mathematical ideas.
The researchers attributed different levels to the questioning category depending on the type of question asked and according to who was asking the questions (Hufferd-Ackles, Fuson & Sherin, 2004, p. 92-96). In the framework, as discussions in the classroom shift from that of a traditional, teacher-centered classroom to one with more student talk, the questions asked change, from being explicit with precise answers, to questions that are more open-ended, requiring students to explain their thinking. Furthermore, as the questioning evolves through the levels, the expectation is that the instructor is not the sole source of questions for the community of learners. Although the teacher may still guide the direction of the discussion, student-to-student questions are also expected. As a natural progression from asking questions, the next category within the Math-Talk Learning Community framework is explaining mathematical thinking.

In a Math-Talk Learning Community, learners work together to explain mathematical thinking. As students become accustomed to participating with community of learners, their mathematical explanations and justifications becomes more descriptive, with fuller justifications, and with less required teacher assistance (Hufferd-Ackles, Fuson & Sherin, 2004, p. 96-102). Participants grow accustomed to knowing that their justifications will be questioned, and begin to address possible inquiries in their initial explanations. As a Math-Talk Learning Community develops, participants grow better at communicating their ideas and the quality of provided explanations improves. As the type of questions and the quality of students’ justifications increase, the sources of the mathematics in the classroom also adjusts.
The next category in the Math-Talk Learning Community framework created by Hufferd-Ackles and colleagues is the source of mathematical ideas. In a teacher-centric classroom, it is the teacher that provides the majority of mathematical ideas to the students. Namely, the teacher is the sole source of knowledge. At the upper level in the Math-Talk Learning Community hierarchy, the students take on more of an active role in generating knowledge. Student knowledge and student strategies form the foundation for the knowledge that is agreed upon in the classroom (Hufferd-Ackles, Fuson & Sherin, 2004, p. 102-106). As the source of the mathematical ideas shifts within the classroom, the responsibility for learning also changes.

In a Math-Talk Learning Community (upper level) the responsibility for learning is such that both the instructor and the students are responsible for the learning that occurs. As students take on a greater role in classroom discussions they also take on a greater responsibility in making sure that they and their fellow classmates understand the content. This means that the students need to be active contributors and not just passive observers (Hufferd-Ackles, Fuson & Sherin, 2004, p. 106-110). Additionally, students need to interact with each other in developing ideas and supporting each other when there is confusion. This means that the students need to listen to each other's ideas, and push for clarification when needed.

The Math-Talk Learning Community framework addresses the changing role of both the students and the instructor in developing understanding in a class grounded in talking and discussing ideas. At the highest level, students have a greater responsibility for their own understanding and the understanding of everyone in the class. It is expected
that students will contribute to the mathematical ideas in the class; students will not only provide more thorough justifications, but they will also ask open-ended questions of each other in order to develop deeper understanding. This does not mean that the instructor can just step back and allow the class to simply talk. It is the instructor that needs to make sure that all of the students participate, that explanations and justifications are thorough, and that the expected mathematical trajectory is still maintained. But how does instructor facilitate the conversation so that a Math-Talk Learning Community is established and maintained? One way involves the moves that the instructor uses to facilitate student talk. These instructor behaviors or “talk moves” will be considered next.

2.6 Talk Moves

As participants in a Math-Talk Learning Community, both the instructor and the students have responsibilities that are different from what is typically found in a mathematics class. All members of the class are responsible for developing mathematical reasoning, asking questions, and supplying mathematical ideas. Students need to take an active role as participants in the classroom discussions and justifications. The instructor needs to take on the role of both facilitator of discussions and supporter of developing reasoning around mathematical ideas. A Math-Talk Learning Community requires that the instructor supports participants as they clarify their initial thoughts and create thorough justifications. Furthermore, to construct an actual community of learners using talk, students need to be oriented to the thoughts and reasoning of the other members in the community. Chapin, O’Connor, and Anderson (2003, 2009, 2013) have been working together since 1998 on approaches for developing students’ mathematics reasoning using
talk in the mathematics classroom. Over time the researchers have articulated a set of moves that instructors can implement to facilitate productive talk during classroom discussions.

The list of talk moves and talk tools composed by Chapin, O'Connor and Anderson were developed through their own research and through the contributions of other researchers. Chapin and colleagues describe four goals that they consider essential for the instructor to support productive talk. These steps are: (1) helping the student clarify their own thinking, (2) helping the student orient to the thinking of others, (3) helping the student deepen their own reasoning around the mathematics, and (4) helping the student engage with the reasoning of others (Chapin, O'Connor & Anderson, 2013, p. 10-11). It is important to note that all of the talk moves and talk tools can be used to support any of these goals. Before delving into the various talk moves, the four instructor goals for supporting productive talk will be considered.

In supporting productive talk, the first and third teacher goal focus on facilitating student’s engagement with their own ideas. The first goal emphasizes the instructor task of helping students to clarify their own thinking. When students initially start to express themselves orally they often struggle to communicate clearly. In fact, they might not even have a clear understanding of what it is that they are trying to communicate! Through talk moves and talk tools the instructor can help the student to fully articulate their thinking so that both the instructor and the class have an opportunity to hear and understand the student’s thoughts. The third instructor goal to implement productive talk, presented by Chapin and colleagues, is working with students to deepen their reasoning about
mathematical concepts (Chapin, O’Connor & Anderson, 2013). With this goal students are asked to re-engage with the mathematical explanations or explanations they have provided in order to provide more thorough reasoning, a more comprehensive explanation, or a more complete justification. Together these two instructor goals set the expectation for the instructor to not simply accept the initial answers from students, but to push students to present well clarified explanations, and then to push those students to extend their thinking.

In addition to clarifying and deepening their own thinking, students need to also consider the reasoning of others. As part of establishing a community of learners implementing productive discussion, Chapin and colleagues (2013) provide two more instructor goals; the instructor needs to help orient students to the thinking of others (second goal), and instructor needs to help students to engage with the reasoning of other students (fourth goal). If students are going to work on creating explanations together, then they need to be listening to each other. Furthermore, students need to be able to reflect on and engage with the reasoning of their peers in their class. These goals imply that students need to assume a certain amount of responsibility for developing their own understanding and that they need to contribute to developing the reasoning of others. These talk moves help establish that the responsibility for learning is on all of the participants, which is crucial component of the Math-Talk Learning Community framework discussed earlier (Hufferd-Ackles, Fuson & Sherin, 2004).

Chapin and colleagues described a number of talk moves and two talk tools that the instructor could use to help maintain productive talk in a community of learners
(Chapin, O'Connor & Anderson, 2013). The first two that will be discussed here are talk tools that allow the students in the class more processing time. These talk tools can be used in association with any of the four instructor goals previously discussed. The first talk tool presented by the Chapin and colleagues is “wait time.” When using “wait time” the instructor provides the student additional time to think about his responses. In a synthesis of the research, Rowe (1986) found that using “wait time” can increase the length of responses, increase the quality of responses and foster student-student exchanges. “Wait time” allows both the speaker and other participants’ time to process the talk that has occurred. Chapin & colleagues mentioned that “wait time” pauses can last four or more seconds in length (Chapin, O’Connor & Anderson, 2013). Similar to “wait time,” the talk tool of “stop and jot” allows students more time to process the mathematics that is being discussed. With “stop and jot” students are encouraged to stop, reflect and then write down what they are thinking. Writing down explanations and ideas allows more thinking time, and can also allow more reticent students the opportunity to later contribute using their written notes as support. Both of these talk tools allow students more time to process their own thinking.

The next talk tool is called “turn and talk.” When the instructor uses “turn and talk” he directs the students to turn and talk about the reasoning they used with another student (also known as partner talk) or group; this allows students additional time to process and consider ideas (Chapin, O’Connor & Anderson, 2013). With “turn and talk,” the participant has the opportunity to run their thinking by another member of the class and to practice saying aloud their thoughts. Students still working through the
mathematical reasoning of a problem get time to think about ideas and to practice articulating their thoughts. Furthermore, students that may feel initially uncomfortable contributing to a whole-class discussion may feel more comfortable about participating after talking through their reasoning with another class member. Implementing the talk move of “turn and talk” can be used for all four of the instructor goals since the reasoning that the students are being asked to verbalize may be about their own work or of another class participant. Additionally, students are asked to not only articulate their own mathematics reasoning, but they need to listen to the mathematics reasoning of their partner. This partner-to-partner exchange allows both students the opportunity to clarify their reasoning about the mathematics, orient to the thinking of others and deepen their understanding through engaging with the mathematical reasoning of a partner.

Sometimes in a discussion the provided explanation may be too short, unclear, or even wrong. Chapin and colleagues (2013) provided the “say more” talk move for the instructor to use to elicit further thinking and reasoning from the students. When the instructor asks participants to say more, he is asking them to clarify or expand on the explanation that was provided. Implementing the “say more” talk move also provides the class another opportunity to hear more on the topic under consideration. “Say more” can be directed towards the initial student presenting or the instructor can also press other participants to say more about the work of someone else (building on the reasoning already provided). To be able to say more and add on to the contributions of another classmate, a student needs to have an understanding of the provided explanations so that they can elaborate and build on it. This talk move can be used to support the developing
understanding of a participant, regarding their own ideas or the ideas of another participant, as they create comprehensive explanations.

The next two talk moves involve the revoicing or restating of provided mathematics reasoning. Chapin and colleagues found that the “revoice” or “restate” talk moves were especially effective in allowing students the opportunity to orient to the thinking of others, or sometimes even their own mathematics reasoning (Chapin, O’Connor & Anderson, 2013). “Revoicing” is a talk move where the instructor articulates what a student said using the student’s words, phrases and logic. A component of the “revoicing” talk move is that the teacher checks with the student that the revoiced information was accurate. “Revoicing” can be used to help the class understand a complex contribution made by one participant. It can also be employed to help a student work through an inarticulate contribution or justification by forcing them to reflect on what they just said. Often students restate their ideas more clearly following a teacher's revoicing. Finally, as the instructor revoices the work of a student, he is providing credit for ideas to the student (O’Connor & Michaels, 1993). A key component of a Math-Talk Learning Community is that the instructor is not perceived as the source of all mathematical ideas (Hufferd-Ackles, Fuson & Sherin, 2004). Building on instructor revoicing, the instructor may ask the participants to restate what has been said. When students are expected to restate in their own words the statements, ideas, or justifications of other students, they need to be oriented to the thinking of others in their class. Additionally, having a justification restated means that everyone in the class has another opportunity to hear the provided reasoning. Implementing these talk moves also reminds
the class that the initial student presenter is the source of the ideas and not the teacher, which helps shift perceived authority from teacher to students.

In a community of learners that is productively implementing talk, the class should strive to provide as clear of an explanation as possible, and to address all components of a justification. Initial ideas, reasons, and justifications communicated by students are often unclear or incomplete. This means that the instructor often needs to push students to examine their own ideas more thoroughly. The final talk move presented by Chapin, O’Connor, and Anderson (2013) is “press for reasoning.” When implementing the “press for reasoning” move the instructor is asking the students to provide additional reasoning about mathematical ideas, as well as, evidence of justifications for their statements. In order to have students provide more thorough explanations, the instructor can push, probe or press for more reasoning. Pushing for reasoning need not be a verbatim request, and can take many forms such as asking the student to elaborate, asking the student for their evidence, or asking the student to justify their work. The key is that students are expected to re-engage with their own reasoning by providing more explanation or more evidence. The instructor can also press participants for reasoning about the work of the other participants; having participants engage with the mathematics reasoning of others is an essential instructor goal for supporting productive talk. For example, the instructor may ask a participant if they agree or disagree with the work of another participant, and why. In implementing this form of the “press for reasoning” talk move, the instructor is not only asking students to be oriented to the mathematics reasoning of a classmate, but he is also asking them to
respond to that reasoning and to support their response with evidence. This means that students need to not only hear, but they need to understand the work of their fellow classmates so that they can respond.

Teachers have a critical role in the Math-Talk Learning Community, they need to push students to thoroughly develop their mathematics reasoning, they need make sure that all students are involved in the discussion process, and they need to make sure that the discussion is one where the community of learners works to negotiate understanding around topics. The talk moves provided by Chapin, O’Connor, and Anderson (2013) provide teachers with a set of moves or behaviors that they can use in order to help students develop their own reasoning about the mathematics. Furthermore, the talk moves can be used to support students as they negotiate and work with the provided mathematical reasoning of other students in the class. However, in creating a Math-Talk Learning Community where students are engaged in productive talk, there are other things that an instructor needs to do to make sure that the overall mathematics reasoning remains at a high level and follows a productive path.

To recap, the National Research Council provided descriptions of four school environments associated with effective classrooms that allow for student learning (National Research Council, 2005a, 2005b). The fourth environment presented was the community-centered classroom where a classroom culture is established based on valuing ideas, productive exchange and collaborative thinking (National Research Council, 2005b, p. 242). In a community-centered classroom, the community of learners work together to develop mathematical ideas; there is a “culture of questioning, respect, and
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risk taking” (National Research Council, 2005b, p. 13).

In implementing a discussion-based classroom, the goal is for understanding to be
developed by the community of learners. Researchers have offered strategies that the
instructor can implement to develop student reasoning. For instance, Chapin and
colleagues provided instructor goals that include helping students to clarify and deepen
their own understanding, and helping students to orient and engage with the mathematics
reasoning of others (Chapin, O’Connor & Anderson, 2013). In addition to instructor
goals, Chapin and colleagues also provided a series of strategies, called talk moves, to
allow the community of learners more thinking time, more time to talk through their
ideas, and more opportunities to re-hear provided reasoning. Using talk moves the
instructor can support participants in maintaining a learning community focused on
talking about mathematical ideas. The Math-Talk Learning Community framework
provided by (Hufferd-Ackles, Fuson & Sherin, 2004) provides the structure needed for
this community-centered environment. The instructor goals, talk moves, and strategies
provide the recommendations to obtain and maintain it.

2.7 Teacher Questions

One way to improve the quality of discussion— helping students to develop
student understanding and build connections—is through teacher questions. Historically,
questions have not always been used effectively in developing understanding. The classic
K–12 classroom discussion has been teacher-centered where the teacher provided most
concepts and ideas, and the students were expected to retain them. When classroom
discussion did occur it tended to follow an Initiation-Response-Evaluation (IRE) pattern.
The IRE pattern of communication is one where the teacher initiates a question, the student responds, and the teacher evaluates the response (Cazden, 2001).

Initiation: Teacher asks, “What is the area of the square?”
Response: Student answers, “The area is five square meters.”
Evaluation: Teachers replies, “Correct.”

The teacher has engaged the student, the student has responded, and so in the teacher’s mind discussion has been achieved. However, this exchange does not require any understanding from the student. In the Math-Talk Learning Community the goal is to move away from IRE exchanges, and move towards talk as an exchange of ideas. First, in a Math-Talk Learning Community it is expected that more conceptual questions are asked by both the instructor and the students. Also, when answering questions students are expected to provide answers that thoroughly address the reasoning that they applied to the solution. Additionally, in a Math-Talk Learning Community, when an answer is provided without explanation or a complete justification, it is expected that either the instructor or another student will press for more reasoning. In creating a Math-Talk Learning Community, and moving away from an IRE teaching pattern, a teacher can facilitate discussion through his use of talk moves and crafted teacher questions.

Teacher questions set the tone and the expectations for the classroom. Previous studies have used teacher questions to contrast the expectations in a classroom where a great deal of discussion was used, to a classroom where little was used. For example, to study discourse, Imm and Stylianou (2012) collected data from five urban middle school classrooms in the form of teacher interviews, classroom observations, classroom videos,
student interviews, and student artifacts. The researchers considered the types of discourse, questions asked, and associated patterns. Questions within the transcript were coded as *procedural* or *conceptual*. Procedural questions fell into an IRE (teacher initiation, student response, and teacher evaluation) pattern (Cazden, 2001) and were found to be dominant in the classrooms with little talk. Patterns within the classes with much discussion did not adhere to Cazden’s IRE pattern. The instructors in these classrooms disrupted the default pattern by inviting questions, contrasting strategies, and building upon ideas. Instead of evaluating, as in the classic IRE pattern, “teachers continually filtered, or re-directed, student contributions back to them” (Imm & Stylianou, 2012, p. 139). The researchers found that teacher questions took on many purposes. The teacher questions directed students to engage with one another, ensured student participation, asked for justification, focused students on making connections, encouraged alternative reasoning, and had students focus on the process of solving (instead of just determining the answers). In the classes with high levels of discussion, students were able to work with one another and with the teacher to develop knowledge. The teacher was not the only authority in the room and students were expected to explain, justify, and support their ideas to peers.

The flow of talk as orchestrated by the questions a teacher asks, such as probing questions, can provide an opportunity for students to further reflect upon ideas, subsequently providing better explanations (Franke, Webb, Chan, Ing, Freund, & Battey, 2009). For instance, researchers working on the Cognitively Guided Instruction professional development program examined how teachers probed students’ ideas
through questioning (Franke, Webb, Chan, Ing, Freund, & Battey, 2009). The researchers considered the work of three elementary educators as they used questions to engage students in algebraic reasoning. Specifically, the researchers considered the types of follow-up questions asked by teachers after initial student explanations. Teacher questions were coded as: probing, general question, specific question, leading question, other question, or no question (Franke, Webb, Chan, Ing, Freund, & Battey, 2009, p. 384). Franke and colleagues found that probing questions led to further student explanation and in some cases led students to correct previously incorrect work.

In another study, Martino and Maher (1999) also found that the types of follow-up questions that teachers asked affected the thoroughness of student justification and provided reasoning. Martino and Maher looked at the role of the classroom teacher in third- and fourth-grade classrooms as the teacher created a culture of reasoning around mathematics, where students would listen to one another and exchange ideas. Videotaped instructor-led interview sessions were conducted with students and transcribed for analysis. Researchers found a strong relationship between the teacher's questions as the student progressed through the solution process, and the teacher's ability to have the student connect to prior learning, revise current understanding, or move towards better understanding (Martino & Maher, 1999).

The use and implementation of teacher questions is a critical component of classroom discussion and can influence student achievement. Using teacher questions, Hiebert and Wearne (1993) compared student achievement and methods of curriculum implementation in six second-grade classrooms. Four classrooms used a traditional
curriculum that involved little discussion and two classrooms used curriculum that was more student-centered. Classes were observed over the course of the year during 12 weeks of asynchronous instruction on place value, and addition/subtraction of multi-digit numbers. Students were provided written assessments at the beginning, middle, and end of the school year. Transcripts of observed lessons were coded for problem number, time spent, problem type, and materials available. Discourse was analyzed by speaker, length of talk, and type of questions. Researchers found that not only did teachers in the student-centered classrooms ask more explain and analysis questions, but the students also talked for longer periods of time. Greater student gains were found for those students in the classrooms where fewer problems were discussed, but students spent more time talking about ideas. The types of questions that the teacher asked had a positive effect on student achievement.

The questions that teachers use to elicit information from students can take many forms. In a longitudinal study that looked at 1,000 students at three different schools, Boaler and Brodie (2004) examined how the types of questions asked by teachers influenced the flow of discussion and the cognitive opportunities for students in different classrooms. The researchers considered six lessons from seven teachers using a mixture of curricular reform approaches at three schools. Lessons were coded at 30-second intervals as to whether: students were in groups, students were working individually, the teacher was questioning or explaining, or there was a student focus. Nine categories of teacher questions were derived: gathering information, inserting terminology, exploring meaning/relationships, probing, generating discussion, linking/applying, extending
thinking, orienting and focusing, and establishing context (Boaler & Brodie, 2004, p. 776). The researchers found that more than 95% of the questions asked by teachers using a traditional curriculum, with little discussion, were gathering information questions, compared with 60%–75% for the classroom with much discussion. Teachers implementing discussion asked questions that allowed students different experiences or opportunities to explore. More importantly, the researchers found that the types of questions teachers asked shaped classroom discourse and hence the cognitive opportunities available to students. For instance, if the teacher asked probing and conceptual questions, the students would also start to ask probing and conceptual questions as the observation progressed. This last piece is especially relevant as the students within a Math-Talk Learning Community need to share the responsibility for questioning and developing understanding.

Research has also considered the function of the teacher questions during discussion. Henning, McKeny, Foley, and Balong (2012) looked at the discourse in a classroom of 21 seventh-grade students working on area and volume lessons over the course of nine 45-minute classroom sessions. Data were collected in the form of lesson plans, student work, student examples, teacher reflections, and observations. Researchers coded teachers' moves and found four overarching categories of teacher move function that included: eliciting student responses, confirming responses, teacher-guided follow-up moves, and non-instructional discourse moves (Henning, McKeny, Foley & Balong, 2012, p. 463). Eliciting student responses accounted for 30% to 44% of the total discourse moves and was the predominant move used. The segments of class where
conceptual discussions occurred had the lowest percentage of confirming responses, but the highest volume of teacher-guided follow-up. This research suggests a connection between the teacher questions, and the type of discussion.

As part of their work to prepare pre-service teachers to lead whole-class discussions, Boerst and colleagues (2011) also considered the function or purpose of a teacher’s questions. The researchers, with input from teacher educators, created a framework for the purpose of teacher questions that included: “initial eliciting of student thinking,” “probing students’ answers,” “focusing students to listen and respond to other’s ideas,” “supporting students to make connections,” “guiding students to reason mathematically,” and “extending students’ current thinking” (Boerst, Sleep, Ball & Bass, 2011). Initial elicit questions were those questions where the instructor asked students for their solutions, approaches, ideas, or thoughts. Probing questions pushed for clarification, confirmed understanding, or checked student thinking (Boerst, Sleep, Ball & Bass, 2011). With focusing questions, the instructor asked students to explain, interpret, agree or disagree, or add on to another student’s ideas. The fourth question type, supporting connections, was used when an instructor asked students to compare or contrast two different things. Questions that guided students to reason mathematically pushed students to justify, generalize and provide proof. Lastly, when an instructor asked an extending question, he was looking for students to think beyond the problem structures and concepts that they had previously worked with. The goal of the researchers was to provide a description of the steps they implemented to develop the discussion skills of PSTs. The researchers employed the teacher educator created framework of teacher
question types as a tool for PSTs to develop their skills reflecting on the purpose of
teacher questions while watching video of an experienced educator. Additionally, the
researchers had the PSTs practice utilizing questions, with their own student, in a
recorded session that the PSTs later re-watch and reflected upon. The researcher’s
focus on the use of questions alludes to the importance of questions as a tool for
implementing discussion.

Prior research helps to provide an understanding of how academics have used
components of teacher questions in analyzing classroom talk. Discussion within a Math-
Talk Learning Community is facilitated by the instructor through his use of strategies, his
implementation of practices and his use of questions. Researchers have associated certain
question types with high discourse classrooms, types of student questions, and student
achievement. For instance, researchers have found that in classrooms where the instructor
asked more conceptual questions there were was a higher level of discussion and
reasoning around the mathematics (Imm & Stylianou, 2012). Researchers have also
found that probing questions (Franke, Webb, Chan, Ing, Freund & Battey, 2009) and
follow-up questions (Martino & Maher, 1999) can lead to better student explanations and
greater understanding. Probing, open-ended teacher questions are an expectation in a
Math-Talk Learning Community (Hufferd-Ackles, Fuson & Sherin, 2004). In fact,
student-centered classrooms where the instructor asked more explain and analysis
questions, associated with more talk, also reflected a positive effect on student
achievement (Hiebert & Wearne, 1999). Additionally, researchers have found that
teacher questions influence student questions; in classes where the instructor asks more
probing and conceptual questions, students also ask more probing and conceptual questions (Boaler & Brodie, 2004). This is important since high quality questions from both the students and the instructor are an expectation in a Math-Talk Learning Community. The questions asked by the instructor play a role in the creation of the Math-Talk Learning Community as facilitated by the instructor.

2.8 Summary

The MET I and MET II reports recommend pre-service educators (PSTs) have experience with both geometric content, and with “reasoning, explaining, and making sense of the mathematics” (CBMS, 2012, p.17). One way that teacher preparation programs can meet these recommendations is to provide PSTs the experience of being a student in a student-centered, discussion based, Math-Talk Learning Community. In this community, the PSTs will need to provide logical and systematic explanations for their responses. Ideally the goal is for PSTs to respond to one another logically, coherently, and with responses that are supported by evidence. The instructor should not be the sole provider of mathematical knowledge, but should function as a facilitator that guides the path of the discussion in the classroom as PSTs work with and justify mathematical concepts. In facilitating the Math-Talk Learning Community, the instructor can implement various practices, use talk moves, and ask purposeful questions in order to guide the PSTs toward meaningful and productive discussions. This may mean that classroom discussions do not proceed in a straightforward manner. PSTs may argue and debate ideas. However, the hope is that in arguing about mathematical ideas, reasoning improves, explanations become clearer, and deeper understanding is developed.
The current research has predominantly focused on developing student’s understanding within the K-12 classroom. Previous researchers have looked at patterns of talk, questions types and teacher strategies that can be used to support student’s understanding. The MET II report recommends that pre-service elementary teachers be provided opportunities to develop mathematical understanding through reasoning about mathematical ideas, constructing mathematical arguments and engaging with the arguments of others. Unfortunately, there is limited research on supporting developing understanding of mathematics in undergraduate mathematics classes for PSTs. In order to contribute to the literature, and support the teacher education of PSTs, this research provides a description of an undergraduate mathematics class that engages PSTs with mathematics concepts through discussion. Specifically, this research considers the instructor decisions as he supports PSTs in developing understanding of area formulas.
CHAPTER 3: METHODS & PROCEDURES

The purpose of this study was to explore the actions and decisions of a knowledgeable instructor as he guided undergraduate pre-service teachers to reason about mathematics. Specifically, the research considered how the instructor implemented two lessons on geometric measurement from an inquiry-based undergraduate mathematics curriculum designed to develop a deep understanding of mathematics. Two lessons on area formulas were observed and videotaped, and interactions were transcribed for analysis. Instructor interviews were conducted prior to and following the lessons. Video-stimulated recall interview sessions were conducted following the lessons and allowed the researcher to directly ask the instructor what he was thinking when he made particular decisions in the class. Pre- and post-tests on area formulas were administered to the participants before and after the lesson sequence in order to gain an understanding of their knowledge of this topic. In Section I of this chapter a description of the sample and the implementation of instruction are described. In Section II, the data sources are provided. In Section III, the methods for data analyses are described.

Section I: The Sample and Implementation of Instruction

3.1.1 Study Context and Sample

This research study took place at a major private institution in the northeastern United States within a mathematics course offered for pre-service elementary teachers. The course was the second course in a two-semester sequence on mathematics for elementary education majors, offered through the mathematics department. The course is
required for all undergraduates who are working towards educator licensure in elementary education, special education, or deaf studies in the institution’s School of Education. The lessons in the course sequence focus on algebra, measurement, geometry, and statistics. At the time of the study, there were 27 planned class sessions in the course schedule, plus one scheduled final. Class sessions were 110 minutes in length. Approximately 40% of the course, or six weeks, was dedicated to geometry topics including geometric measurement. Geometric measurement lessons covered topics such as perimeter, area, surface area, volume, and the associated geometry formulas. The geometric measurement lessons comprised about 20% of the course material (five lessons, or 9 hours of instruction). This research focused on two lessons on area formulas within the geometric measurement portion of the course. These lessons spanned approximately two and a half regular class sessions.

The course was chosen as the setting for this study because of the instructional materials used by the department, and the knowledge and experience of the course instructor. The instructional materials used in this course consisted of lessons developed as part of the Elementary Pre-service Teachers Mathematics Project (EMP) (NSF 2009-2011, 2013-2015, PI Chapin). The goal of the EMP was to design instructional materials that developed future teacher’s specialized content knowledge (Chapin et al., 2011). The materials used discussions as the primary pedagogy.

The instructor examined in this research was chosen for a number of reasons. First, his scholarship focused on the mathematical knowledge of pre-service teachers who are preparing to become elementary school teachers. Additionally, the instructor had
extensive experience teaching mathematics content to pre-service teachers. For example, the instructor had been working with pre-service teachers for five years at the time of the study. He taught this course or the other course in the sequence (MA 108 and MA 107, respectively) 8 times over the past four years, and he was a teaching assistant for both courses for one year prior to this. In addition, the instructor possessed extensive knowledge of mathematics and mathematical knowledge for teaching; he had a doctoral degree in mathematics education, a master’s degree in mathematics education and a bachelor’s degree in economics and mathematics.

The instructor was also one of the authors of the curricular material used in the study. This means that he had an in-depth knowledge of the curriculum being implemented. This mathematics curriculum was specifically created to foster inquiry and reasoning in the mathematics classroom with PSTs. Although studies have shown that the implementation of curriculum can vary from that which was intended (Eisenmann & Even, 2009; Stein & Kim, 2009), past observations of the instructor indicated that this was not the case. At the time of the study, the instructor was involved in editing and revising the EMP curricular materials. Furthermore, as part of his participation in the EMP, the instructor had been videotaped while teaching and demonstrated comfortableness with this method of data collection.

The sample for this study was all pre-service teachers enrolled in the elementary mathematics course in the spring of 2013. Course enrollment was 24 students. The course used for this study contained one freshman, six juniors, and 17 sophomores. The enrollment for this course was predominantly female, with only two male students.
Twenty of the 24 students enrolled were in an undergraduate concentration of either special education or elementary education. Four students in the section had not taken the companion course. Study participation was voluntary, and participants were recruited according to IRB standards.

### 3.1.2 Confidentiality

All participants were provided with a Study Participation/Video Filming Consent Form to complete and sign prior to the collection of data. This consent form asked participants if they agreed to be videotaped, and/or if the participants agreed to have their pre- and post-test data used for the research study. All participants agreed to take the Area Formula Pre- and Post-Test, and to have the data analyzed. One participant declined to have his image recorded on video, but did agree to have his oral contributions recorded. This participant was placed in an area of the classroom that was off camera.

To ensure confidentiality, each participant received a unique non-identifying subject number, ranging from S1-S24, which was randomly assigned. This unique participant number was used to replace subject identifiers (names) on transcriptions and collected artifacts. Additionally, in order to ensure anonymity, masculine pronouns were used within the document regardless of the sex of the participant. The master code with subject identifying data (complete names) was stored on the researcher's computer and on a secured portable hard-drive. Classroom video data were transcribed by the researcher to ensure anonymity. The master code and all identifying participant data were destroyed at the end of the research study.
3.1.3 Implementation of Instruction

The chosen lessons for this study were titled *Area Formulas I* and *Area Formulas II*. The first task, *Area Formulas I*, occurred during day 18 and day 19 of the 27 regularly scheduled class sessions. The second task, *Area Formulas II*, occurred during day 19 and day 20. The selected lessons were designed so that through a series of activities and carefully posed questions participants developed an understanding of certain area formulas. Participants worked on these activities and questions in groups of 3 or 4. During the lesson there were a series of instructional discussion points, provided in the curricular materials, where small groups were expected to pause and the whole class reconvened to discuss ideas. Focus questions for these whole-class discussions were provided within the student materials and labeled as “Group Discussion Questions.” (See Appendix A to review the Area Formula tasks).

This research focused on the whole-class discussions. Key content points to be covered during the whole-class discussions were provided within the Instructor’s Guide. The instructor’s goal during the lessons was to help navigate whole-class discussions towards an agreed upon understanding around these key points. This research study focused on four instructional topics: the area formula for a parallelogram, the area formula for a triangle, the area formula for a trapezoid, and the area formula for a circle.

*Area Formulas I*

In the *Area Formulas I* lesson, participants justified the area formulas for both parallelograms and triangles. Participants began the lesson by constructing a general definition for the term area, discussing how to find two-dimensional area, and reflecting
on the meaning of the area formula for a rectangle. As the lesson progressed, participants constructed a formula for the area for a parallelogram based on shape decomposition and their previous knowledge of the area formula for a rectangle. In the last part of the lesson, participants derived the formula for the area for a triangle. In doing so, they were directed to use various types of triangles to create parallelograms composed of two congruent triangles. Participants then used the area formula for a parallelogram to justify the area formula for a triangle.

Within this first lesson, there were three places where whole-class discussions were indicated and questions were provided within the student materials to help initiate those discussions. The Instructor's Guide provided instructional suggestions regarding the focus and facilitation of the whole-class discussions. Each of the three whole-class discussion topics is summarized below.

**Area Formulas I, Whole-Class Discussion 1:** Whole-class discussion 1 occurred after the participants had worked through Question 1 in their small groups. For Question 1, participants developed a strategy for measuring the area of an irregular shape. During the whole-class discussion, the participants were expected to discuss methods for measuring the area of irregular shapes and the units of measure needed for determining the area of a shape (e.g., square units).

The Whole-Class Discussion 1 questions presented within the student materials were:

- *Describe your strategy for measuring the area of the shape. What unit of measure did you employ? Why did you choose this specific unit of measure?*
Area Formulas I, Whole-Class Discussion 2: The second whole-class discussion occurred after small groups had worked on Questions 2 and 3. Question 2 had participants consider the area of two identical shapes, each covered with a different square grid. In Question 3, participants considered the area formula for a rectangle \((\text{length} \times \text{width})\) and how the units chosen for both of the dimensions needed to be equivalent. For the whole-class discussion it was expected that participants would discuss the meaning of area, as well as, issues regarding the choice of units for calculating or measuring the area of shapes.

The Whole-Class Discussion 2 questions presented within the student materials were:

- **Define area.**
- **How does the size of a square unit affect area measurements?**

Area Formulas I, Whole-Class Discussion 3: The third whole-class discussion occurred after the participants had worked on Questions 4 and 5 in their small groups. In Question 4, participants divided and recomposed a parallelogram to form a rectangle. Since the area of the constructed rectangle and the original parallelogram were the same, participants were expected to connect the area formula for their constructed rectangle back to that of their original parallelogram. In Question 5, participants used sets of congruent triangles to construct parallelograms. They then used the area formula for a parallelogram, along with their constructed shapes, to derive the area formula for a triangle. During the whole-class discussions, participants were expected to explain why the formula for a parallelogram was \(\text{Area} = \text{base} \times \text{height}\) based on their knowledge of
rectangles. The justification of the area formula for a parallelogram was a key instructional focus topic and a justification topic on the Area Formula Pre- and Post-Test. Participants also were expected to justify the area formula for a triangle. This justification was built upon the previously discussed area formula for a parallelogram. The justification of the area formula for a triangle was an instructional focus topic and a justification topic on the Area Formula Pre- and Post-Test.

The Whole-Class Discussion 3 questions presented within the student materials were:

- Provide a convincing argument for why the area of all parallelograms is the product of base length and height.

- Explain why your formula for the area of a triangle is correct for all triangles.

Area Formulas II

In the second lesson, Area Formulas II, the participants justified the area formulas for a trapezoid and for a circle. These justifications were key instructional topics for this research. The lesson began with participants working in small groups. The participants used sets of congruent trapezoids to construct parallelograms. Using the formula for the area for a parallelogram, a general area formula for a trapezoid was constructed. Participants then worked on decomposing trapezoids into shapes (e.g. rectangles, parallelograms, or triangles), determining the areas of the individual shapes, and finding the total numerical area of the original trapezoid by summing the calculated areas of the individual shapes. The class then came together to discuss how a generalized formula for a trapezoid could be created. In the second half of class, the participants worked with
circles. Using a circle deconstructed into an even number of sectors, participants created a parallelogram-like shape. Using the area formula for a parallelogram, participants found that the area of the constructed parallelogram approximated that of the original circle. As the original circle was divided into more and more sectors, the area of the parallelogram approximated more closely the area of the original circle. The lesson concluded with the idea that a parallelogram constructed from an infinite number of circle sectors should have the same area as the original circle. This second lesson had two whole-class discussions embedded within the instructional materials.

**Area Formulas II, Whole-Class Discussion 1:** The first whole-class discussion occurred after participants worked through Questions 1 and 2 in their small groups. In Question 1, participants were asked to use two congruent trapezoids to form a parallelogram, and then use the parallelogram formed to derive an area formula for a trapezoid. Since two trapezoids were used to form the parallelogram, the area of the trapezoid was half the area of the parallelogram. In Question 2, participants were asked to determine the area for a trapezoid by dividing the trapezoid into known shapes and summing the individual areas. In Whole-Class Discussion 1, participants were asked to state and explain the area formula for a trapezoid. It was anticipated that participants would explain the area formula, for a trapezoid, either by using sets of congruent trapezoids, or dividing the trapezoid into known shapes (two triangles, or two triangles and a rectangle).

The justification of the area formula for a trapezoid was an instructional focus topic and a justification topic on the Area Formula Pre- and Post-Test.
The Whole-Class Discussion 1 question presented within the student materials was:

- **What are some ways to find the area of the trapezoid below?**

![Trapezoid Image](image)

*Figure 3.1.1 Area Formulas II, Whole-class discussion 1, trapezoid image*

**Area Formulas II, Whole-Class Discussion 2**: The second whole-class discussion occurred after participants had worked on Question 3. In Question 3, participants were asked to take a circle decomposed into 16 sectors and reconfigure them to construct a parallelogram. The base of the parallelogram measured about $\pi r$, or half the circumference of the original circle. The height of the parallelogram measured $r$, or the radius of the original circle. As the number of sectors in which the whole circle was divided approached infinity, the shape of the reconstructed parallelogram approached the shape of an actual parallelogram with height of $r$ and a base of $\pi r$.

The justification of the area formula for a circle was an instructional focus topic and a justification topic on the Area Formula Pre- and Post-Test.

The Whole-Class Discussion 2 question presented within the student materials was:

- **Why is the area formula for a circle $A = \pi r^2$?**
Section II: Data Sources

3.2.1 Area Formula Pre-test and Post-test

The Area Formula Pre- and Post-Tests (AF) were five-question paper and pencil assessments on geometric measurement. The AF Pre- and Post-tests were identical. The tests consisted of four area formula questions and one question on surface area.

Participants were asked to separately justify the area formulas for a triangle, parallelogram, trapezoid, and circle. A fifth question required participants to justify the surface area formula for a rectangular prism, but this was not included in the final analyses. The geometric measurement formulas that were the foci of this research are listed below.

Table 3.2.1 Geometric Measurement Formulas by Lesson and Lesson Question(s)

<table>
<thead>
<tr>
<th>Geometric Measurement Formulas</th>
<th>Lesson Alignment</th>
<th>Related Question(s) in Lesson</th>
</tr>
</thead>
<tbody>
<tr>
<td>The area formula for a parallelogram is $A = bh$.</td>
<td>Area Formulas I</td>
<td>4</td>
</tr>
<tr>
<td>The area formula for a triangle is $A = \frac{1}{2}bh$.</td>
<td>Area Formulas I</td>
<td>5</td>
</tr>
<tr>
<td>The area formula for a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$.</td>
<td>Area Formulas II</td>
<td>1, 2</td>
</tr>
<tr>
<td>The area formula for a circle is $A = \pi r^2$.</td>
<td>Area Formulas II</td>
<td>3</td>
</tr>
</tbody>
</table>

Questions 1 through 4 on the AF consisted of three parts that followed the same format. The first part of each test question asked participants to draw three examples of the shape under investigation (e.g., parallelogram, triangle, trapezoid or circle). The
second part asked participants to provide the formula to find the area of the shape.

Finally, the third part of each question asked participants to provide a convincing argument for the area formula that they had stated. Participants were not allowed to use calculators. The format for all four questions on the AF is summarized in the following table.

<table>
<thead>
<tr>
<th>Question Number</th>
<th>Shape</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Parallelogram</td>
<td>1a. Draw three different parallelograms.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1b. Provide the formula for the area of any parallelogram.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1c. Provide a convincing argument for the area formula that you stated.</td>
</tr>
<tr>
<td>2</td>
<td>Triangle</td>
<td>2a. Draw three different triangles.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2b. Provide the formula for the area of any triangle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2c. Provide a convincing argument for the area formula that you stated.</td>
</tr>
<tr>
<td>3</td>
<td>Trapezoid</td>
<td>3a. Draw three different trapezoids.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3b. Provide the formula for the area of any trapezoid.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3c. Provide a convincing argument for the area formula that you stated.</td>
</tr>
<tr>
<td>4</td>
<td>Circle</td>
<td>4a. Draw three different circles.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4b. Provide the formula for the area of any circle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4c. Provide a convincing argument for the area formula that you stated.</td>
</tr>
</tbody>
</table>

The AF Pre-test administration took place during class time a week prior to the presentation of the instructional content. During the first administration, prior to being provided the pre-test, participants were advised that the test was for research purposes
and that the pre-test scores would not be used for an in-class grade. Participants were instructed to complete the test to the best of their ability. During the second administration, the AF Post-test was provided as part of a larger assessment on geometric measurement and counted towards a course grade. Both sessions were not timed.

3.2.2 Video of Classroom Instruction

The second source of collected data for this study was video recordings of the three class sessions of the two area formula lessons. Two cameras were used to record whole-class discussions during class. One camera remained stationary on a tripod at the back of the room overlooking the room and was directed at the whiteboard where classwork was displayed. The second camera was hand held by the researcher, and was either directed at the speaker or at the whiteboard where classwork was displayed.

3.2.3 Instructor Interviews

The third source of data were six instructor interviews conducted by the researcher. There was one interview conducted before the class sessions were recorded (referred to as the pre-interview), one interview that was conducted after the class sessions were recorded (referred to as the post-interview), and four video-stimulated recall (VSR) interviews (Meade & McMeniman, 1992; Muir, 2010; Schepens, Aelterman & Van Keer, 2007) that occurred within a 24 hour period of the three observed class sessions. During the VSR interview sessions, the instructor was asked to reflect on participants' provided mathematical reasoning, the lessons in general, and his instruction. Each interview was videotaped. (See Appendix C to review interview questions.)
The pre-interview was a semi-scripted interview held a week prior to the observed classroom sessions. During this interview, the instructor had access to both the student instructional materials and the Instructor’s Guide for each of the lessons. To begin the interview, the researcher explained that the goal of this study was to gain a deeper understanding of the pedagogical decisions made by an instructor in a mathematics course for pre-service elementary teachers; the researcher was trying to understand how a knowledgeable instructor guided participants to explain, justify and generalize about key mathematical content.

The first part of the pre-interview focused on the instructor’s background, his experiences, his teaching style, how his teaching style was expressed in the course, and his use of instructional strategies to elicit reasoning. Since the research study was focused on the pedagogy of an instructor it was important to gain a general understanding of how the instructor viewed his own teaching. In the second part of the pre-interview, the instructor was asked to reflect on the lessons that would be taught. Specifically, the instructor was asked to describe each of the lessons, the parts of the lesson where he thought it would be easiest to elicit reasoning, and parts where he thought participants might struggle.

Video-stimulated recall (VSR) interviews were conducted and recorded within 24 hours of each class session. Four VSR interviews, two for each lesson, were conducted. The goal of the video-stimulated recall sessions was to allow the instructor the opportunity to reflect and to explain the decisions and choices that he made. However, in case the instructor was not forthcoming with his thoughts, the researcher created a list of
pause points associated with times in the videoed instruction where a possible
instructional decision was being made. The list of pause points, where the instructor was
directly asked to reflect, included: transitions to and from the small group setting,
transitions to and from whole-class discussion, when the instructor elicited reasoning
from a new participant, when the instructor asked a question to the community of
learners, or when the instructor wrote something on the whiteboard. The researcher also
included particularly interesting moments within the class, not defined in the previous
list. The researcher selected a question to pose to the instructor for each pause point.
These questions, shown in the table below, were adapted from the work of Henry and
Fetters (2012).

Table 3.2.3 Video-Stimulated Recall Session Interview Questions

<table>
<thead>
<tr>
<th>Video-Stimulated Recall Session Interview Questions*</th>
</tr>
</thead>
<tbody>
<tr>
<td>What were you doing / trying to do at this point in the lesson?</td>
</tr>
<tr>
<td>What were you doing / trying to do at this point in the discussion?</td>
</tr>
<tr>
<td>What were you noticing / hearing at this point?</td>
</tr>
<tr>
<td>What were you thinking about at this point?</td>
</tr>
<tr>
<td>Why did you make that statement?</td>
</tr>
<tr>
<td>What do you notice about your actions at this point?</td>
</tr>
<tr>
<td>Why did you do … at this point in the video?</td>
</tr>
<tr>
<td>Why did you ask that question?</td>
</tr>
</tbody>
</table>

* Henry & Fetters, 2012

All VSR sessions began with the researcher reading a pre-written statement
informing the instructor that the purpose of the research was to consider the pedagogical
decisions of instructor as he guided class participants to explain, justify and generalize
mathematical ideas. To begin the VSR interview session, the instructor was asked to
elaborate on the lesson as a whole, if he reached his goal of getting participants to
explain, justify and generalize, and what he might do differently. At this point the researcher and the instructor watched the recorded videos of the whole-group class discussions. During the video viewing, either the instructor or the researcher could pause the video for discussion (Henderson, Grant, Henderson & Huang, 2010, p. 10). Primarily, the instructor was encouraged to identify points in the lesson where he believed he was making an instructional decision and explain his thoughts. However, the researcher also paused the video using the list of pause points as a guide, and asked the instructor an open-ended question regarding his actions and/or thinking during that point in the lesson.

One thing to note is that it is possible that the VSR interview sessions may have influenced the instructor’s teaching in the recorded class lessons that followed the first VSR session. In considering this possibility, it is important to mention that the instructor was chosen because he was an experienced classroom instructor with well-developed patterns of instruction. Although significant instructional changes were not noted in the recorded lessons that followed the VSR interview sessions, the possibility that the instructor may have altered his instruction should be mentioned.

After the pre-interview and the four VSR interviews, there was one post-interview session. The post-interview allowed the instructor to reflect on the lessons in the curriculum and his instructional decisions. The post-interview was conducted by the researcher using the questions in the table below. The questions focused on the instructor’s teaching style, compelling teaching moments, and the pedagogical practices and strategies he used. To conclude the interview, the instructor was asked if he had any
final comments. The post-interview occurred within one week of the final recorded class lesson.

Table 3.2.4 Post-interview Questions

<table>
<thead>
<tr>
<th>Post-interview Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>During the first interview, you said that you felt that you embodied a …teaching style. After watching the videos, would you respond the same or differently? Explain.</td>
</tr>
<tr>
<td>After teaching both lessons, were there any moments that you recall as being particularly striking when reflecting on pedagogical practices?</td>
</tr>
<tr>
<td>After watching the video sessions, what pedagogical practices do you consider essential to get students to explain, justify, and generalize?</td>
</tr>
<tr>
<td>Which pedagogical strategies would you like to include more of to get students to explain, justify, and generalize?</td>
</tr>
<tr>
<td>Do you have any additional thoughts or comments on the lessons as presented?</td>
</tr>
<tr>
<td>Do you have any additional thoughts or comments on pedagogical practices and the work of getting students to explain, justify, or generalize?</td>
</tr>
</tbody>
</table>

Section III: Data Analyses

3.3.1 Area Formula Pre-test and Post-test

The Area Formula Pre- and Post-tests (AF) were used to measure growth in achievement and were scored according to a rubric constructed by the researcher based on the instructional content expectations. Instructional content was developed as part of the Elementary Pre-service Teachers Mathematics Project (EMP). The scoring rubric was created based on input from the principal investigator from the EMP, and from the course
instructor who taught the lessons and was a contributor on the EMP. (See Appendix B to review the detailed scoring rubric.)

The maximum score on both the pre- and post-test was 27 points. The general scoring outline for each problem was similar. One point was given for providing three different drawings of the shape being considered (parallelogram, triangle, trapezoid, or circle). A second point was granted for providing the correct formula for the shape being considered. Four to five points were assigned for the provided justification depending on the rubric. The scoring per question is shown in the table below.

Table 3.3.1 Area Formula Pre-test and Post-test Basic Question Scoring

<table>
<thead>
<tr>
<th>Question Number</th>
<th>Shape</th>
<th>Points Possible</th>
<th>Necessary Answer Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Parallelogram</td>
<td>1</td>
<td>Drawing three different parallelograms. Providing the correct area formula for a parallelogram.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1 Providing a convincing argument for the area formula for a parallelogram.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>Triangle</td>
<td>1</td>
<td>Drawing three different triangles. Providing the correct area formula for a triangle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1 Providing a convincing argument for the area formula for a triangle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>Trapezoid</td>
<td>1</td>
<td>Drawing three different trapezoids. Providing the correct area formula for a trapezoid.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1 Providing a convincing argument for the area formula for a trapezoid.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>Circle</td>
<td>1</td>
<td>Drawing three different circles. Providing the correct area formula for a circle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1 Providing a convincing argument for the area formula for a circle.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>
The first AF question focused on the justification for the area formula for a parallelogram. The scoring rubric, based on the justification in the *Area Formulas I* task, allowed a maximum of seven points to be assigned to problem solutions. A point was given for providing a drawing of three different parallelograms. A second point was granted for providing the correct formula. Additional points were assigned for showing/stating that the shape could be decomposed, showing/stating how the new shape could be reconstructed into a rectangle, explaining why the new shape was a rectangle, connecting the dimensions of the newly created rectangle to the original parallelogram, and lastly, connecting the area formula for a rectangle to the area formula for a parallelogram.

**Table 3.3.2 Scoring Rubric for Area Formula Pre-test and Post-test Question 1 on the Area Formula Justification for a Parallelogram**

<table>
<thead>
<tr>
<th>Points Possible</th>
<th>Explanation Requirement for AF Question 1 on the Area Formula for a Parallelogram</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing three different drawings of parallelograms.</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a parallelogram is ( A = bh ).</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a parallelogram can be decomposed.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that the decomposed parts of a parallelogram can be recomposed into a rectangle.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining why the new recomposed shape is a rectangle.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that one measure of the newly constructed rectangle is equivalent to the height of the original parallelogram, and that the base of the rectangle is the same as the base of the original parallelogram.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that since the recomposed shape is a rectangle, the area formula for a rectangle can be used to find the area of the original shape. OR Explaining how the area of a rectangle is determined by using the length and width to find the number of square units in the shape.</td>
</tr>
</tbody>
</table>
The second AF question focused on the justification for the area formula for a triangle. The scoring rubric, based on the justification in the Area Formulas I task, allowed a maximum of seven points to be assigned to problem solutions. A point was granted for providing a drawing of three different triangles. A second point was granted for providing the correct formula. Additional points were assigned for showing/stating that an identical triangle could be drawn, showing how a new shape could be constructed from the two congruent triangles, explaining why the newly constructed shape was a parallelogram, connecting the dimensions of the new shape to the original triangle, and using the area formula for a parallelogram to justify the area formula for a triangle.

<table>
<thead>
<tr>
<th>Points Possible</th>
<th>Explanation Requirement for AF Question 2 on the Area Formula for a Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing three different drawings of a triangle.</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a triangle is $A = \frac{1}{2}bh$.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that an identical, congruent triangle can be created.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating how the two congruent triangles can be reformed.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that the two congruent triangles are formed into a parallelogram.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that one measure of the parallelogram is determined by the triangle height, and the other is from the triangle base.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that since the recomposed shape is a parallelogram, composed from two congruent triangles, half the area formula for a parallelogram can be used to find the area of one triangle.</td>
</tr>
</tbody>
</table>

The third AF question focused on the justification for the area formula for a trapezoid. Within the task, three different methods for determining the area formula for a trapezoid were presented. Each method was separately addressed in the scoring rubric. Regardless of the method chosen, the scoring rubric for the trapezoid area formula
justification allowed for a maximum of six points to be assigned to problem solutions. For all three methods, a point was granted for providing a drawing of three different trapezoids; and a second point was granted for providing the correct formula. For method 1, four additional points were assigned for showing/stating that an identical, congruent trapezoid could be created, showing/stating that two congruent trapezoids could be used to form a parallelogram and explaining why the newly constructed shape was a parallelogram, connecting the dimensions of the new shape to the original trapezoid, and using the area formula for a parallelogram to justify the area formula for a trapezoid. For method 2, four additional points were assigned for decomposing the shape into two triangles, connecting the dimensions of the triangles to the original trapezoid, showing or explaining that individual areas could be summed to find the original area, and then simplifying the expression to that of the area formula for a trapezoid. For method 3, four additional points were assigned for decomposing the shape into a rectangle and two triangles, connecting the dimensions of the original trapezoid to the decomposed shapes, showing or explaining that individual areas could be summed to find the original area, and then simplifying the expression to that of the area formula for a trapezoid.
Table 3.3.4 Scoring Rubric for Area Formula Pre-test and Post-test Question 3 on the Area Formula Justification for a Trapezoid - Method 1

<table>
<thead>
<tr>
<th>Points Possible</th>
<th>Explanation Requirement for AF Question 3 on the Area Formula for a Trapezoid - Method 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing three different drawings of a trapezoid.</td>
</tr>
<tr>
<td>1</td>
<td>Stating the area formula for a trapezoid is ( A = \frac{1}{2} (b_1 + b_2)h ).</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that an identical, congruent trapezoid can be created.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that the two trapezoids can be reformed into a parallelogram.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining how the dimensions of the parallelogram relate to the dimension of the original trapezoid.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that since the recomposed shape is a parallelogram, composed from two congruent trapezoids, half the area formula for a parallelogram can be used to find the area of one trapezoid.</td>
</tr>
</tbody>
</table>

Table 3.3.5 Scoring Rubric for Area Formula Pre-test and Post-test Question 3 on the Area Formula Justification for a Trapezoid - Method 2

<table>
<thead>
<tr>
<th>Points Possible</th>
<th>Explanation Requirement for AF Question 3 on the Area Formula for a Trapezoid - Method 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing three different drawings of a trapezoid.</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a trapezoid is ( A = \frac{1}{2} (b_1 + b_2)h ).</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a trapezoid can be decomposed into two triangles with ( b_1 ) and ( b_2 ) as bases.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining how to determine the dimensions of each triangle.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that the area of the original trapezoid is the sum of the areas of the two triangles.</td>
</tr>
<tr>
<td>1</td>
<td>Showing the mathematics that correctly simplifies that trapezoid formula to ( A = \frac{1}{2} (b_1 + b_2)h ).</td>
</tr>
</tbody>
</table>
Table 3.3.6 Scoring Rubric for Area Formula Pre-test and Post-test Question 3 on the Area Formula Justification for a Trapezoid - Method 3

<table>
<thead>
<tr>
<th>Points Possible</th>
<th>Explanation Requirement for AF Question 3 on the Area Formula for a Trapezoid – Method 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing three different drawings of a trapezoid.</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a trapezoid is ( A = \frac{1}{2}(b_1 + b_2)h ).</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a trapezoid can be decomposed into two triangles and a rectangle.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining how to determine the dimensions of each shape.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that the area of the trapezoid is the sum of the areas of all the individual shapes.</td>
</tr>
<tr>
<td>1</td>
<td>Showing the mathematics that correctly simplifies the trapezoid formula to ( A = \frac{1}{2}(b_1 + b_2)h ).</td>
</tr>
</tbody>
</table>

The fourth AF question focused on the justification for the area formula for a circle. The scoring rubric, based on the justification in the Area Formulas II task, allowed a maximum of seven points to be assigned to problem solutions. A point was granted for providing a drawing of three different circles. A second point was granted for providing the correct formula. Additional points were assigned for showing/stating the shape could be decomposed into sectors; showing how the sectors could be reconstructed into a shape resembling a parallelogram; showing/or stating that as the number of sectors used increased the shape became more like a parallelogram (rectangle); connecting the dimensions of the original circle to the decomposed circle; and using the area formula for a parallelogram to justify the area formula for a circle.
Table 3.3.7 Scoring Rubric for Area Formula Pre-test and Post-test Question 4 on the Area Formula Justification for a Circle

<table>
<thead>
<tr>
<th>Points Possible</th>
<th>Explanation Requirement for AF Question 4 on the Area Formula for a Circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing three different drawings of a circle.</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a circle is $A = \pi r^2$.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a circle can be decomposed into sectors.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that the sectors can be recomposed into a shape that is similar to parallelogram.</td>
</tr>
<tr>
<td>1</td>
<td>Stating or showing that as a circle is cut into a greater number of sectors the constructed shape becomes more like a parallelogram (or rectangle).</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that one measure of the newly constructed parallelogram (or rectangle) is equivalent to the radius of the original circle, and that the base is the same as half the circumference of a circle or $\pi r$.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining that since the recomposed shape is a parallelogram (or rectangle), the area formula for a parallelogram can be used to find the area of the original shape.</td>
</tr>
</tbody>
</table>

All AF Pre-tests and Post-tests were scored by the researcher according to the rubric. All AF Pre-test and Post-test questions were separately coded by another mathematics educator in order to ensure reliability. In cases where initial scores did not align, the researcher and second scorer discussed and negotiated final scores. There was 100% agreement on final scores.

Descriptive statistics were first used to calculate the mean score, minimum score, maximum score, and standard deviation for the participant pre-test and post-test scores total scores as a group. The mean gain between the pre-test to post-test, as well as, the standard deviation were also calculated. Additionally, descriptive statistics were used to calculate the mean score, minimum score, maximum score, and standard deviation for the pre-test and post-test scores per question. Furthermore, the pre-test to post-test gains between the mean scores and the associated standard deviation were calculated for each question. Lastly, a two-tailed, paired sample t-test was conducted to see if there was a
statistical difference between the AF Pre-test and AF Post-test class cumulative scores and the pre-test/post-test scores for each question.

3.3.2 Video Data Analyses

Both interview and lesson video data were transcribed. Video transcriptions were completed using the Transcription Translation Manual from the TIMSS 1999 Video Study (U.S. Department of Education, National Center for Education Statistics, 2003). The one deviation to note is that for purposes of clarity, the transcriptions did not necessarily always include stuttering, restarts, and word repeats. In all transcripts, transcriber notes were placed within brackets []; transcriber notes included items such as participant movement notes (such as moving to the whiteboard, sitting, standing, or pointing), clarification of changes to written work on the whiteboard, notes on classroom transitions, descriptions of noises or tone (such as laughing, murmuring, sounding unsure) or dialogue pauses. Additionally, whiteboard work was included as images within the transcription. The four video-stimulated recall (VSR) interviews following the lessons were later arranged adjacent to the lesson transcriptions so that comments made about the lesson videos were aligned with the appropriate timing within the lesson.

The three videoed class sessions were organized into two complete whole-class discussions transcripts by lesson (Area Formulas I and Area Formulas II). Using video time stamps, each lesson was analyzed to determine the time lengths spent in various formats. The duration of time spent was established using video time stamps for three formats: the whole-class discussion format, the small group work format, or in a third format termed logistics, which accounted for the periods when participants were not
engaged in mathematics discussion. Additionally, video time stamps were used to
determine the amount of time the instructor or the participants held the floor during
whole-class discussions. In order to determine this amount of time, time stamps were
associated with each talk turn by speaker, and the amount of time for each of these turns
was determined. A talk turn was considered an uninterrupted utterance by a speaker. For
each pause in excess of three seconds a new turn was started. This means that pauses in
excess of 3 seconds were not included within the speaker time length calculations. Using
the amount of time for each turn during whole-class discussions, the rates and overall
percentages of participation for the instructor and the participants were provided.

Lesson and interview transcripts were read and then reread for implementation
themes specifically relevant to supporting participants in developing understanding.
Using the instructor's description of his class and his intended strategies to develop
participant understanding, a series of prevalent ideas or themes emerged. The themes
included the instructor's role as a facilitator who developed understanding, the roles of
the participant's as future teachers, and the use of teacher discourse moves such as “press
for reasoning,” “revoicing” and “wait time” (Chapin, O'Connor & Anderson, 2013).
From these emerging themes a series of codes based on implemented teacher strategies
was devised and used to code transcripts.

The instructor pre-interview, VSR interviews and the post-interview were used to
provide an additional layer of understanding of the instructor’s thoughts regarding his
own instruction. The pre-interview was analyzed for themes and key components
summarized to provide an understanding of the instructor's reflections on his intended
instruction. The VSR interviews and the post-interview were analyzed in tandem with the instruction to understand the instruction as perceived, as coordinated with the instructor’s own reflections on his instruction.

Using the created codes all instructor turns were coded by the researcher according to a coding rubric. (As previously stated, a turn was defined as a new uninterrupted utterance by a speaker and established when there was a change in speaker or when there was a pause in excess of 3 seconds.) All instructor turns were separately coded by another mathematics educator in order to ensure reliability. In cases where initial codes did not align, the researcher and second coder discussed and negotiated final codes. There was 100% agreement on final codes.

In researching how an instructor facilitated the developing understanding of the participants in his class, the questions that he asked were very important. One of the key talk strategies discussed by the instructor and found in the prevailing literature was the “press for reasoning” talk move. When the instructor asks a press for reasoning question, he is looking for participants to provide further explanation or clarification (Chapin, O’Connor & Anderson, 2013). Two different types of press for reasoning questions were coded. In the first type of press for reasoning question, the instructor asked a single participant to provide additional mathematics justification in regards to an explanation or statement that he had provided. In the second type of press for reasoning question, participants were pressed to provide evidence or justification for an explanation or statement that they had not provided. This means participants in the community of learners were asked to consider and respond to the mathematical reasoning provided by
others. Instructor turns only received a press for reasoning question code if participants responded in some way to the question that was being asked. This response could be verbal or physical.
Table 3.3.8 Lesson Transcript Codes Description — Press for Reasoning

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instructor presses for reasoning from a single participant</td>
<td>Questions that push for clarification, evidence, justification, elaboration or for more information from a single participant following a contribution by that participant. These questions are helpful for pushing for more detail, confirming understanding, or checking thinking. Questions that initially elicit information were not coded.</td>
<td>You now have this rectangle, I am going to highlight it in black. And S11, why did you-why did you choose to turn it into a rectangle? What's so special about a rectangle?</td>
<td>How do you know they are congruent?</td>
</tr>
<tr>
<td>Instructor presses for reasoning from participants about work from another participant</td>
<td>Questions that push for clarification, evidence, justification, elaboration or for more information from the whole class. These questions are helpful for pushing for more detail, confirming understanding, or checking thinking. Questions that initially elicit information were not coded.</td>
<td>Oh, so what do you think about what S7 did, because he's saying, that should be height. S13?</td>
<td>Okay. What shapes do you think are effective in finding area? Cause we had-S16, you used triangles and rectangles. S22 used circles and rectangles. S14, you used just squares. What shapes can we use to find area? What would be efficient?</td>
</tr>
</tbody>
</table>

The next set of transcript codes, based on instructor interviews and the prevailing literature on instructor moves (Chapin, O’Connor & Anderson, 2013), were also
constructed using the set of teacher strategies referred to as talk moves. The talk move “revoicing” was coded. “Revoicing” is a talk move where the instructor restates a contribution of a participant using the words of the participant as much as possible (O’Connor & Michaels, 1993; Chapin, O’Connor & Anderson, 2013). The instructor then checks with the speaker to confirm that the restatement was correct. Additionally, although not a talk move, the instructor may also repeat a component of the participant’s mathematical statements during the lesson, but not confirm accuracy. These strategies, revoking and instructor repeat, were placed under a merged code designating the mentioning of previous work by the instructor. This code did not differentiate as to whether the instructor repeated an utterance exactly, whether the instructor repeated the utterance in his own words, or whether or not the instructor checked in with the participant about the validity of the statement that they had made.

Another set of codes was used to track the instructor’s use of participant’s names to attribute revolved or restated work. In the coded transcript, during each instructor talk turn, a count was generated the first time the instructor mentioned a participant’s name and associated that name with a specific idea. This means that if the instructor mentioned two different participants and their work in a single talk turn then a code of two was generated. This was the only code employed in this way.
Table 3.3.9 Lesson Transcript Codes Description – Revoice/Restate & Participant Name Use

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instructor restates or revoices mathematical concepts, ideas or statements that had been previously presented.</td>
<td>Instructor restates or revoices mathematical concepts, ideas or statements that had been previously presented.</td>
<td>So you used a big rectangle, then a bunch of triangles around it, to try and get all that other area?</td>
<td>That's what we are trying to prove. That was the whole point. Because S7, you came up there and you said this whole thing is a parallelogram, so we know its area is base times height. Right?</td>
</tr>
<tr>
<td>The instructor mentions a participant by name in reference to a mathematical idea that they presented. [This is a count generated per name.]</td>
<td>The instructor mentions a participant by name in reference to a mathematical idea that they presented. [This is a count generated per name.]</td>
<td>Wow. That’s a lot of stuff. [3 sec]. Does anyone think they can re-explain what S17 just went through?</td>
<td>You now have this rectangle, I am going to highlight it in black. And S11, why did you- why did you choose to turn it into a rectangle? What's so special about a rectangle?</td>
</tr>
</tbody>
</table>

The next code was created based on a theme that was articulated during the interview sessions. The participants within the class were learning concepts germane to their future profession as educators. As future teachers, the PSTs not only needed to have a solid understanding of the mathematics discussed thus far, but they also needed to consider the questions that their own students might ask. The necessity for deep understanding relative to their future careers and livelihood was unique to this mathematics class. In reviewing instructor interviews, the role of the PSTs as future
teachers and the necessity for them to fully know and understand the mathematics being discussed was a theme that was repeatedly addressed. Furthermore, as this research focused on the understanding of PSTs in an undergraduate mathematics course, the mentioning of their position and the importance of developing a complete understanding of the mathematics being developed seemed especially relevant.

Table 3.3.10 *Lesson Transcript Codes Description – PSTs Role as Future Teachers*

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instructor mentions being a future teacher or having future students</td>
<td>The instructor mentions the PSTs role as future teachers.</td>
<td>So it's kinda, I want you guys to kinda put yourselves a little bit further than just kind of giving kinda broad answers about, &quot;well it kinda looks like it&quot; or I can move or use the properties of the, of the shapes that we do know to make a case. Ok? Are there any other questions about this? [3s] Things that either you are unclear about, or things that you think your students might be unclear about?</td>
<td>Ok, a couple (of groups). Now S24, before I let you go, um, looking at your method here, now we are going to be teachers, so we have to think about our students. Based on that method, what do you think might be some questions that your students might have about what you did, if there is anything they'd want clarified? Or something you think that they might not immediately grasp.</td>
</tr>
</tbody>
</table>

In analyzing the coded lesson transcript data, and the emergent themes found within the instructor interviews, a correlation to a previously devised framework within the literature became apparent. Building on the work of Hufferd-Ackles, Fuson & Sherin (2004), the instructor decisions and strategies to develop participant understanding were
mapped to the Math-Talk Learning Community framework. The framework addressed the developmental trajectory for four components of a Math-Talk Learning Community: questions, explaining mathematical thinking, sources of mathematical ideas, and responsibility for learning. Each component of the Math-Talk Learning Community framework was considered and an account of the instructor’s implementation was provided using relevant descriptive data, coded transcript results, transcript samples and instructor explanations.

In order to fully address all of components presented within the Math-Talk Learning Community framework two additional codes were created and used to analyze the transcript data. The first code was used to mark those talk turns in the transcript where the instructor presented a mathematical statement that had not previously been provided during the lessons. The third component of the Math-Talk Learning Community framework referred to the source of mathematical ideas in the classroom. The instructor’s role as a provider of mathematical truths was relevant to this framework trajectory.

Building off of this concept a second code was created to mark those talk turns where the instructor responded to a participant’s contribution with a statement that could be perceived as an endorsement of correctness. This would be the same as making an evaluative statement as that found in the IRE pattern of communication (Cazden, 2001). To gain an understanding of how the instructor was responding to participant contributions this last code was devised.
<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instructor states new mathematical information</td>
<td>Instructor presents a new mathematical idea, concept or justification.</td>
<td>And you can kinda think theoretically, 'what if I were to cut this into an infinite number of slices?' It's going to be barely noticeable as an arc. It's going to start to straighten out.</td>
<td>And now that (you guys) look, that's the same- the formula that S18's group has at the end, that's the same formula. I just noticed that S24's group has kind of... right below the last picture. Do you guys see that? So S18 you were- I think you were right if y- unless I am noticing it wrong. This is their final formula and I think that's the same as S24's formula here, right?</td>
</tr>
<tr>
<td>Instructor makes a confirming statement of correctness (or inaccuracy)</td>
<td>Instructor makes a statement that could be perceived as a clear confirmation of accuracy or inaccuracy.</td>
<td>Right.</td>
<td>T: What are the units on something like this? If I say this is three units and this is one unit [labels rectangle], what would be the units on the area?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>S12: Three square units.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>T: It would still be three square units, but S12 is on to something.</td>
</tr>
</tbody>
</table>

In conclusion, themes that emerged in the instructor interviews and classroom interactions along with codes based on the Math-Talk Learning Community framework resulted in coding the transcripts seven different codes. These codes, which focused on instructor actions, were:

- Instructor presses for reasoning with a single participant.
• Instructor presses for reasoning from participants about work from another participant.
• Instructor restates or revoices previous mathematical work from the community of learners.
• Instructor mentions a participant by name (number of times).
• Instructor mentions being a future teacher or having future students.
• Instructor states new mathematical information.
• Instructor makes a confirming statement of correctness (or incorrectness).

Using instructor descriptions of his class, coded lesson transcripts overlaid with instructor VSR interviews, and transcript time data an account of the methods the instructor implemented was developed. This account was mapped to the Math-Talk Learning Community Framework (Hufferd-Ackles, Fuson & Sherin, 2004) in order to provide a full description of the actions and strategies the instructor implemented in supporting a community of learners develop understanding of mathematical ideas.

3.3.3 Summary

The intention of this research is to describe the strategies that a knowledgeable instructor implemented to support the developing understanding of pre-service teachers in an undergraduate mathematics course. First pre- and post-test scores were utilized to establish baseline knowledge and growth data. Next the coded lesson transcripts and instructor interviews were used to construct a description of the decisions, actions and strategies the instructor intended to implement and implemented in developing participant understanding. Descriptions of the instructor's intentions, decisions and strategies were
mapped to the four components found within the Math-Talk Learning Community framework (Hufferd-Ackles, Fuson & Sherin, 2004) in order to gain an understanding of the role of instructor as he supported the developing understanding of PSTS. The instructor’s statements, instructor’s requests, instructor’s questions and the use of class time, as well as, the resulting participant contributions were used to provide an account of the instructor’s decisions as he supported participants’ developing understanding though the facilitation of discussion about the mathematics.
CHAPTER 4: DESCRIPTION

This study investigated how a knowledgeable instructor supported the developing understanding of pre-service teachers in an undergraduate mathematics class. Both quantitative and qualitative data from a series of undergraduate mathematics lessons for pre-service teachers were collected. The first section of this chapter presents the results of participants’ responses on the Area Formula (AF) Pre-test and Post-test, which were administered before and after the observed instructional content lessons. Video data collected as part of the lessons and interviews are discussed in the second and third section of this chapter.

Section I: Area Formula Pre-test and Post-test Analysis

In order to provide summary information about participants’ understanding of the area formulas under investigation in this study, the Area Formula (AF) Pre-test and Post-test were administered. The AF Pre-test was taken by twenty-three participants a week prior to the observed lessons. The AF Post-test was taken by twenty-four participants a week following the observed lessons. One participant did not take the Pre-test. Since the AF tests were used to measure initial understanding and growth, data were only considered for those participants that had both pre-test and post-test scores.

The AF Pre-test and Post-test required that participants justified the area formulas for a parallelogram, triangle, trapezoid and circle. The same questions were used on both tests. As part of establishing content validity two mathematics educators reviewed the test questions, and scoring rubric, to determine whether or not the questions provided a valid assessment of the instructional objectives for justifying area formulas. The Area Formula
test was reviewed by both the instructor of the course (a mathematics educator), and the main developer of the EMP instructional materials that were used in the course (a mathematics educator).

Participants’ mean AF Pre-test score was 7.17 out of 27 points, with a standard deviation of 2.24. The mean AF Post-test score was 19.17 out of 27 points, with a standard deviation of 2.68. The lowest gain between the pre-test and post-test was 4 points. The greatest gain between the pre-test and post-test was 18 points.

**Table 4.1.1 Score Results of Area Formula Pre-test & Post-test**

<table>
<thead>
<tr>
<th></th>
<th>Minimum Score</th>
<th>Maximum Score</th>
<th>Mean Score</th>
<th>Standard Deviation</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Area Formula</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Pre-test</strong></td>
<td>5</td>
<td>15</td>
<td>7.17</td>
<td>2.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Maximum Score 27 points</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Post-test</strong></td>
<td>15</td>
<td>25</td>
<td>19.17</td>
<td>2.68</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Difference in scores</strong></td>
<td>12</td>
<td>3.18</td>
<td>t(22)=12.6</td>
<td>p &lt; 0.0001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A paired t-test was conducted on participants’ AF Pre-test and Post-test mean scores. The results of the paired t-tests showed that there was a significant difference between the AF Pre-test and AF Post-test scores; t(22)=17.7, p < 0.0001. This significant change between AF Pre-test and Post-test scores provides evidence that participants achieved knowledge of the geometric measurement concepts that were a focus of instruction (area formulas) between the AF Pre-test and Post-test that was not due to chance.

It is important to note that change between the AF Pre-test and Post-test does not provide clear evidence of growth of understanding on the geometric measurement topic.
of area formulas. However, the significant growth gain between the AF Pre-rest and Post-test does suggest evidence of a gain in understanding.

4.1.1 Area Formula Analysis by Question

In order to investigate if any of the area formulas were more challenging than others for participants, mean scores for each question were computed. Question 1, Question 2 and Question 4 were out of 7 points; Question 3 was out of 6 points. For each question, participants received one point for correctly providing the area formula for the shape, and one point for providing three different drawn representations of the shape. The rest of the points were assigned, according to a rubric, based on the components of the participants’ justifications for each area formula. Question 1 was on parallelograms and had mean AF Pre-test and Post-test scores that ranged from 1.87 to 4.7. The second question, on triangles, had a mean AF Pre-test score of 2.48 points, and a mean post-test score of 4.52 points. Question 3 on trapezoids was the only question scored out of 6 points. The participants’ mean AF Pre-test score on Question 3 was 1.39, and the mean AF Post-test score was 4.61 points. On the fourth and final question the mean scores for the AF Pre-test and Post-test were 1.43 and 5.35 points respectively. All tests had at least one participant who scored a perfect score on the post-test as designated by the rubric. Participants’ AF Pre-test and Post-test score results by question are presented in Table 4.1.2.
Table 4.1.2 Results of Area Formula Pre-test & Post-test by Question

<table>
<thead>
<tr>
<th>Question 1: Area Formula for a Parallelogram</th>
<th>Minimum Score</th>
<th>Maximum Score</th>
<th>Mean Score</th>
<th>Standard Deviation</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test Question 1</td>
<td>1</td>
<td>4</td>
<td>1.87</td>
<td>0.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post-test Question 1</td>
<td>3</td>
<td>7</td>
<td>4.7</td>
<td>1.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Difference in scores</td>
<td></td>
<td></td>
<td>2.83</td>
<td>1.05</td>
<td>t(22)=12.6</td>
<td>p &lt; 0.0001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Question 2: Area Formula for a Triangle</th>
<th>Minimum Score</th>
<th>Maximum Score</th>
<th>Mean Score</th>
<th>Standard Deviation</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test Question 2</td>
<td>1</td>
<td>4</td>
<td>2.48</td>
<td>0.71</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post-test Question 2</td>
<td>3</td>
<td>7</td>
<td>4.52</td>
<td>1.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Difference in scores</td>
<td></td>
<td></td>
<td>2.04</td>
<td>1.49</td>
<td>t(22)=6.4</td>
<td>p &lt; 0.0001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Question 3: Area Formula for a Trapezoid</th>
<th>Minimum Score</th>
<th>Maximum Score</th>
<th>Mean Score</th>
<th>Standard Deviation</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test Question 3</td>
<td>0</td>
<td>5</td>
<td>1.39</td>
<td>0.97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post-test Question 3</td>
<td>2</td>
<td>6</td>
<td>4.61</td>
<td>0.97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Difference in scores</td>
<td></td>
<td></td>
<td>3.22</td>
<td>1.18</td>
<td>t(22)=12.6</td>
<td>p &lt; 0.0001</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Question 4: Area Formula for a Circle</th>
<th>Minimum Score</th>
<th>Maximum Score</th>
<th>Mean Score</th>
<th>Standard Deviation</th>
<th>t-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-test Question 4</td>
<td>0</td>
<td>2</td>
<td>1.43</td>
<td>0.58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post-test Question 4</td>
<td>4</td>
<td>7</td>
<td>5.35</td>
<td>0.87</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Difference in scores</td>
<td></td>
<td></td>
<td>3.91</td>
<td>1.06</td>
<td>t(22)=17.3</td>
<td>p &lt; 0.0001</td>
</tr>
</tbody>
</table>

Question 1 focused on the justification of the area formula for a parallelogram.

From the Pre-test and Post-test, participants demonstrated a mean gain of 2.83 points.
with a standard deviation of 1.05 points. There was a significant difference between the AF Pre-test and AF Post-test mean scores on Question 1; \( t(22)=12.6, p < 0.0001 \).

Question 2, on triangles, had the highest overall pre-test score of 2.48 points, but also the lowest growth overall of 2.04 points. The lower growth as compared to the other questions could be attributed to the higher pre-test scores for this question. On Question 2, there was also a significant difference between the AF Pre-test and AF Post-test mean scores; \( t(22)=6.4, p < 0.0001 \).

Further analysis revealed that on Question 1 on the pre-test, most participants were able to provide the area formula for a parallelogram. Additionally, on Question 2 on the pre-test, most participants were able to provide the area formula for a triangle. Furthermore, on the pre-test most participants made three different, but accurate drawings of those shapes. Overall Questions 1 and 2 had higher mean pre-test scores than Questions 3 and 4, suggesting that participants were more familiar with parallelograms and triangles.

Question 3, on trapezoids, was out of six points. Participants demonstrated a mean growth of 3.22 (standard deviation of 1.18) between the pre-test and post-test. There was a significant difference between mean scores on Question 3; \( (22)=12.8, p < 0.0001 \). The AF Pre-test scores for Question 3 revealed the greatest range of responses with a low score of 0 points and a high score of 5 points. This means that there were participants who were not able to provide the area formula for a trapezoid or draw three different trapezoids on the pre-test. This is in contrast to other participants that scored 5 points out of the available 6 points on the pre-test. This difference in pre-test scoring speaks to the
varied backgrounds of PSTs. Interestingly, Question 3 also had the highest mean post-test score of 4.61 points.

The mean post-test score on Question 4, on circles, was also high. Question 4 participants had a mean post-test score of 5.35 points out of 7 points (standard deviation of 0.87). There was a significant difference between the AF Pre-test and AF Post-test mean scores on Question 4; $t(22)=17.3$, $p < 0.0001$. Question 4 had at least one participant that scored a 0 on the pre-test which means they were unable to draw three different circles, or provide the area formula for a circle. The maximum score on the AF Pre-test for Question 4 was only 2 points. None of the participants were able to provide any component of the justification for the area formula for a circle until after the observed lessons. With the low pre-test scores, it is unsurprising that the greatest gain between the AF Pre-test and Post-test was found for Question 4 with a mean gain of 3.91 points (standard deviation of 1.06).

Overall participants' pre-test scores were lower on Questions 3 and 4. On the pre-test, most participants were able to draw three different trapezoids (Question 3), and three different circles (Question 4). Yet, on the pre-test only half of the class was able to provide the area formula for a circle from memory, and only one quarter of the class was able to provide the area formula for a trapezoid. Both Questions 3 and 4 had lower pre-test scores, suggesting that participants were less comfortable with trapezoids and circles. Additionally, both Questions 3 and 4 had a participant that scored 0 points on the pre-test. However, both of these questions also had the highest growth between the pre-test and post-test suggesting evidence for an increase in understanding.
In conclusion, there was a significant difference between the overall mean AF Pre-test score and mean AF Post-test scores ($p < 0.0001$). Scores were significantly higher on the post-test following instruction, providing evidence that the instruction made a significant difference in participants’ achievement. Furthermore, there was a significant difference between each of the individual AF Pre-test and Post-test questions ($p < 0.0001$ for all questions). Following instruction participants were not only able to draw the shapes and supply formulas, but they were also able to provide more thorough justifications for the area formulas. This analysis suggests evidence of a significant change in the level of participants’ understanding in the area of geometric measurement.

Section II: Instruction

This research considered the instructional decisions of an instructor of an undergraduate mathematics course for elementary education pre-service teachers (PSTs). The goal of this research was to provide a comprehensive description of how the instructor supported PSTs in developing understanding around geometric measurement topics. Using instructor interviews and transcript data, a description was provided of how the instructor intended to develop the understanding of the PSTs in the class, how his instruction was enacted, and his explanation of the decisions that he made.

Classroom lessons were recorded and instructor interviews were conducted to gain insight into his thoughts and instructional approaches around developing PSTs’ understanding. To understand the instructor’s intentions, a pre-interview with the instructor was conducted prior to the recording of the observed lessons, and the video-stimulated recall interview sessions. During this pre-interview the instructor was asked to
reflect on his experiences as an instructor of PSTs, his style of teaching, how he believed that his style of teaching was expressed, and the lessons that were to be observed. These initial responses provided insight into the intended instructional decisions of the instructor.

4.2.1 Pre-interview: Reflection on Teaching

During the pre-interview the instructor first elaborated on his teaching style, and the strategies he implemented to get PSTs to explain, justify and generalize about the mathematics. The instructor explained that the "pedagogical lens through which I teach is discourse-based instruction" [pre-interview]. The instructor further explained that he tried "to get the class to interact with each other and with me around some of the key mathematical ideas through small group and whole-class discussions" [pre-interview]. This suggests that the instructor expected the PSTs to talk about their mathematical ideas. The instructor viewed his role as one where he was "to facilitate discussion and to have students be the ones to bring up certain ideas and to grapple with certain concepts" [pre-interview]. In those circumstances where there was a disagreement he expected the PSTs "to debate" [pre-interview] mathematical ideas and answers.

It was the instructor's expectation "that there is always a discussion, whether it is them grappling with ideas or problems in small groups and they are talking about it, or we summarize and connect some of these ideas in whole-class discussion" [pre-interview]. This is suggestive of a class where the format is not that of lecture-based instruction, but where the focus is on discussing ideas in small groups, or as a whole class. Within his intended discussion-based instruction, the instructor not only saw his
role as one where he was “pushing them [PSTs] to justify why they’re saying what they are saying” [pre-interview], but he always wanted the PSTs to “not only acknowledge, but also address what other people are saying” [pre-interview]. This would imply that the instructor expected to not only support PSTs in deepening their own understanding, but he also intended to support the PSTs in orienting to and engaging with the reasoning of others.

When the instructor was asked to elaborate on his style of teaching, the instructor disclosed that due to the rigorous nature of the materials, he tried to keep things “as informal as [he] possibly [could] without diminishing what we are working on in the mathematics” [pre-interview]. He articulated that the tasks in the instructional materials were complex and that “small groups are a really intense time” [pre-interview]. In describing the lessons the instructor said, “there is a lot in every task, and so in every class most of them are really working very intensely on it and they’re talking to each other” [pre-interview]. During small group time, both he and the teaching assistant would move from small group to small group “working with students, and asking questions to get them to think about things” [pre-interview]. PSTs would transition from small group to the whole-class discussion to talk about ideas. During whole-class discussion, the instructor explained he would be “asking questions, having students comment on each other’s ideas or on their work” [pre-interview] or he would “call students up to the board and have them present what they have done, and then I move off to the side” [pre-interview]. These statements suggested that the instructor planned to constantly push
PSTs to engage with the materials, and each other, in both the small group and the whole-class discussion format.

During the interview sessions, the instructor was asked how he intended to elicit mathematical reasoning from the PSTs. The instructor explained that he viewed himself as a facilitator in the classroom, guiding PSTs to reason about ideas. As the facilitator, he expected to use various strategies in the form of “talk moves” to have PSTs talk about mathematical ideas. For the instructor, talk moves were things that he did or said “to students to get them to talk about what they are thinking” [pre-interview]. The talk moves discussed by the instructor included: “revoicing,” “add on,” “restating,” “wait time” and “turn and talk.” He explained that the use of the talk moves provided additional opportunities and time for PSTs to develop their understanding of the concepts under discussion.

The instructor elucidated that he might use the talk move “revoicing” if a PST has said something “incoherent, or I feel other students, or the student who said it, doesn’t really quite understand exactly what was said” [pre-interview]. The instructor would revoice the statement and ask the PST if that is what they actually meant to have said. The expectation was that the PSTs would then confirm or disagree with the statement. According to the instructor, through revoicing other PSTs and the original participant would gain a better understanding of what has been said. “Restating” was another move that the instructor described. When using restating the instructor would ask a PST to restate the work of another PST. The instructor explained that the “restate move” presented the individual who was restating an opportunity to “make sense” [pre-
interview] of the idea from the original presenter. It also allowed the original presenter an opportunity to “rehear what they just said” [pre-interview]. Furthermore, restating presented an opening for “all the other students in the room to … think about what’s going on” [pre-interview].

During the pre-interview the instructor also mentioned the talk move “add on.” The “add on” talk move was to be used following an explanation. The instructor explained that he would ask PSTs, “Does anyone have anything else they want to add on?” [pre-interview]. The instructor indicated that he would use this move “to give some of the other students an opportunity to come in to the conversation” [pre-interview]. Another move that the instructor presented, to afford opportunities for PSTs to join in with the conversation, was the “turn and talk” move. When using this move, the instructor would direct PSTs to turn to a partner to discuss what had been said, and determine whether they “agree or disagree with that, and why” [pre-interview]. The instructor explained that he would use this talk move “every so often” [pre-interview] to provide participants “kind of a safe space, that’s with just one other person, to grapple with whatever kind of idea I think is really important” [pre-interview].

The last talk move the instructor focused on during the pre-interview was the talk move “wait time.” When using “wait time” the instructor explained that when asking a question, he wouldn’t “always jump to the first student who raises their hand” [pre-interview] and that he would “wait, maybe five or ten seconds” [pre-interview] before calling on someone. He considered the use of the “wait time” move another way to provide additional opportunities for PSTs to join the conversation.
In summary the instructor was looking to create a class were PSTs would talk about and develop understanding around mathematical ideas. First, PSTs would work in small groups and talk about ideas and then the PSTs would come together to have a whole-class discussion about the mathematics they had been working on. The purpose of the whole-class discussions were to build consensus around key mathematical ideas. The instructor intended to implement various talk moves to support PSTs in explaining, justifying and generalizing mathematical ideas. Due to the density of the tasks and the intensity of working on the mathematics, the instructor tried to keep the class interactions as informal as the mathematics would allow. The next section will explore the instructor’s description of the curriculum, the topics to be covered, and the questions that he might ask to further develop understanding.

4.2.2 Pre-interview: Reflection on Curricular Materials

Through discussion and the provided questions in the lessons, the instructor expected the PSTs to work together towards constructing an understanding of key mathematical topics. As explained by the instructor, my “goal through discussion and through specific questions … [is] to have some of these key ideas and procedures come up, and have us construct them as we go” [pre-interview]. PSTs were guided towards particular mathematical understandings through instructor questions, and through tasks and questions provided within the curricular materials. The curricular materials were those developed as part of the Elementary Pre-service Teachers Mathematics Project (EMP) of which the instructor was a contributor. This research focused on lessons from the geometric measurement unit on area formulas.
The expected enactment of the two Area Formula geometric measurement lessons, Area Formulas I (AF1) and Area Formulas II (AF2), both followed a pattern of small group work time followed by whole-class discussion time. During small group work, the PSTs were asked to work on a question(s) or task(s) within their small groups. When groups finished their small group work, they were directed to reflect on the Group Discussion Question(s) found in the instructional materials. The expectation was that the entire class would reconvene for whole-class discussion. During whole-class discussion the class would review the small group work and focus on answering the Group Discussion Question(s). The instructor explained that, “for each subset of questions, we have these follow-up questions called Group Discussion Questions which are supposed to encapsulate the key mathematical ideas within that subset that they had just worked on. And so, those are fodder for whole-class discussion” [pre-interview]. Whole-class discussion time was where the instructor expected the PSTs to work together to make sense of specific mathematical ideas.

There were five key focal components of instructional content that the instructor expected the whole-class discussions to concentrate on during the AF1 and AF2 lessons. In AF1, the instructor explained that the three foci were to: (1) “articulate what area actually means” [pre-interview], (2) justify and generalize a method for finding the area of a parallelogram, and (3) justify and generalize a method for finding the area of a triangle. According to the instructor the foci for AF2 built off of the foci for AF1. The first foci in AF2 was to (4) derive, justify and generalize the area formulas for a
trapezoid. The second foci was to (5) derive, justify and generalize the area formulas for a circle.

The first instructional content foci was: “What is area?” The instructor explained that he believed that the area “is something that a lot of pre-service teachers, and I think people don’t often think about” [pre-interview]. The instructor further explained, “In their small groups, I will ask them something like ... what do you think area means? ... And I'll start the conversation that way, just to elicit different responses, and I'll ask multiple people, what they think area is. And in the past, it doesn’t always immediately come up, that the area of a two-dimensional figure is the number of square units that can cover the figure completely” [pre-interview]. Once the instructor had the PSTs consider the multiple ways to articulate area in their small groups, “then as a group, the goal is to decide as a group on how we should, as a group, define area” [pre-interview]. The instructor’s statements suggested that he was not going to tell the PSTs the definition for area, but he was going to have them develop a definition as a group guided by instructor questions and the scaffolding found within the curricular materials.

Related to developing a definition of area was the role of units in measuring area. The instructor mentioned that PSTs often struggled to quantify area and define an appropriate unit. He explained, “Many students use different shapes to cover the silhouette. They will use a mixture of trapezoids and triangles, and squares, and rectangles” [pre-interview]. Eventually the group “kind of gets to the point where square tiles are the most efficient shapes to iterate” [pre-interview]. This leads them to make sense of the idea that “the size of the unit you use will affect the numerical value of area,
but not the actual area itself, not the area of the figure” [pre-interview]. The instructor indicated that the concept is one “that they either do not understand well, or do not explicitly think about” [pre-interview]. His goal was to have PSTs discuss the topic and come to an understanding during the whole-class discussion.

The second instructional content foci was for PSTs to provide a convincing argument for why the area of a parallelogram was the product of its base length and height. As part of the scaffolding found in the instructional materials, PSTs were given a grid on which they were expected to draw different parallelograms with the same base and height. During the pre-interview, the instructor stated that he expected that most of the PSTs would decompose the “parallelogram into a right triangle and a trapezoid” [pre-interview] and that they would “move that right triangle over to the other side of the trapezoid” [pre-interview] to form a rectangle. The instructor further explained that because of decomposition, re-composition, and the conservation of area, the PSTs would create a rectangle with the same length and width of the original parallelogram. However, the instructor indicated that he wouldn’t accept a basic explanation from the PSTs of simply moving the triangular piece.

In order to justify the area formula for a parallelogram, PSTs were expected to explain the mathematical reasoning behind each of the statements that they made. For the instructor, having the PSTs just state that a triangle piece could be shifted, from one side of a parallelogram to another, to create a rectangle was “not sufficient, I would typically follow up, and say something like, convince us, how do you know that you know a parallelogram has the same dimensions as a rectangle?” [pre-interview]. From there he
intended to call on “different, multiple students to come up to the board, and show how that decomposition and re-composition works” [pre-interview]. He intended to push the PSTs to explain how they knew that the decomposed shapes could be moved around and recomposed to form a rectangle. The instructor asserted that he wanted the PSTs “generalize beyond just specific examples” [pre-interview] during the whole-class discussion.

Providing a full and complete explanation was a theme expressed throughout each of the instructor’s descriptions of the instructional foci topics. For the third instructional foci topic, PSTs were asked to justify the area formula for a triangle. While PSTs were knowledgeable of the formula \(A = \frac{1}{2} bh\), most had not engaged with reasoning about its derivation. PSTs were given two copies of different types of triangles and instructed to use the sets of congruent triangles to form parallelograms. However, just saying that the area of a triangle is half that of a parallelogram was not considered a complete justification. For instance, the instructor expected to push PSTs to explain how they knew that the triangles actually fit against each other to form a parallelogram. He intended to have the PSTs to generalize beyond using a “specific pair of triangles” [pre-interview] and “attend to the properties of those figures” [pre-interview]. Similar to his description of the second instructional foci topic, for the justification of the area for a parallelogram, the instructor intended to have many students “articulate their reasoning” [pre-interview]. The instructor explained that he thought doing so “not only helps each of those students make sense of their own strategy, but it helps the whole class make sense of all the different strategies” [pre-interviews]. The instructor’s descriptions suggested that he
intended to push the PSTs to provide full and complete justifications during whole-class discussion.

Deriving, justifying and generalizing the area formula for a trapezoid was the fourth instructional foci of the two lessons. There are multiple ways to justify the area formula of a trapezoid. The instructor described how the curriculum developers had recently decided to provide less scaffolding for the question about the area formula for a trapezoid in order to allow for flexibility in strategies, and so that the various justifications could be discussed in whole-class discussion. In the pre-interview, the instructor mentioned that he expected PSTs to take advantage of some of the previous strategies they had discussed as part of their justifications for the area formulas of parallelograms and triangles, and that he expected the group to discuss the multiple strategies that PSTs used during whole-class discussion. For instance, the PSTs could recompose two identical trapezoids to form a parallelogram, or they might use decomposition and separate their trapezoid "into two right triangles and a rectangle" [pre-interview], or they could even take their trapezoid and "cut a diagonal through it" [pre-interview] creating two triangles. For these different methods he expected to push the PSTs to provide full explanations. For the instructor, "again the focus is, can they explain their methods? Can they justify that their methods work? Can they generalize that? Does this work for all trapezoids?" [pre-interview]. For the area formula of a trapezoid, the instructor expected the PSTs to present and discuss multiple justifications.

In order to prepare for the group discussion about the area formula for a trapezoid, the instructor said he planned to walk between the groups and monitor the strategies that
PSTs were using. The instructor intended to use this time to see what strategies were being discussed in the groups and from those observations he could “select which of those I want to use … [and] to figure out what order I want to do that” [pre-interview]. After selecting participants to present, the instructor was prepared to ask PSTs questions about their methods and the various strategies the group discussed so that the PSTs could contrast methods. The instructor offered the following selection of questions that he might provide to PSTs as they compared strategies:

“Ask them does this make sense to you? Why? Why not? … How does this compare to your method? What’s the same about it? What’s different about it? … Does this work for all trapezoids? We have two wonderful methods here. We justified them. Can I use these methods for all trapezoids? Will this formula always work? What are the properties of trapezoids? Do those properties always exist?” [pre-interview].

The provided questions indicate that the instructor wanted PSTs to not only consider the method or strategy that they used, but all of the strategies provided by the other PSTs in the class. These questions imply that the instructor intended to push PSTs to orient to the thinking of others in the classroom, and to be able to comment on that thinking. Furthermore, he expected that PSTs would compare and contrast the methods that were presented in whole-class discussion. Lastly, the instructor’s comments suggested that he was going to push the PSTs to provide formulas that could be generalized for any trapezoid, and then to justify that generalization.
The fifth and last instructional foci of the whole-class discussions was on the area formula of a circle. This content was difficult for the PSTs and due to this, the instructional materials were more tightly scaffolded. During the activity, PSTs were to divide a circle into an even amount of sectors and then use those sectors to construct a new shape. The instructor explained, “The goal is to have them be able to explain this method for finding [the area formula for a circle] using these decomposed slices and recomposing them. Again, the theme is conservation of area ... And again, just having them generalize. How do you know this works for all circles?” [pre-interview]. The PSTs were likely to justify the area formula of a circle ($A = \pi r^2$) using the dimensions of their constructed parallelogram, some algebra, and the fact that area was conserved. To help push the PSTs to fully justify the instructor intended to ask them questions such as “Can you determine the new shapes dimension? How do you know that these are its dimensions? What’s your new shape’s area?” [pre-interview].

The instructor expected PSTs to struggle with the fact that although a shape constructed from 16 sector slices of a circle may resemble a parallelogram, it is not exactly a parallelogram due to the curving or scalloping at the bases. The instructor explained, “the struggles they will have is either understanding, or being convinced, that the recomposed shape is actually a parallelogram. Cause, it’s really not. Right, because of the curves, they’re curved, they’re not- they’re not lines” [pre-interview]. To help the PSTs develop understanding about the mathematics he intended to have “different people articulate the relationship between the base of that reconfigured shape, to a parallelogram” [pre-interview]. He expected that he would have to spend some time
assisting the PSTs, and discussing how the number of sectors used would affect the shape of the constructed parallelogram. Specifically, as the number of sectors that the circle was divided into increased, the reassembled shape began to resemble more closely a parallelogram. In addressing this issues he might ask the PSTs, “If I decompose the circle into a hundred or a thousand slices, what does that do to my derivation? Does it make it more accurate? Less accurate? Why? Why not?” [pre-interview]. He felt that by asking questions, and having knowledgeable PSTs provide explanations, he could meet this goal.

There were five instructional foci for the two area formula lessons that the PSTs were expected to work on. The first foci concentrated on a general understanding of area. The last four foci focused on providing, justifying and generalizing area formulas for a parallelogram, triangle, trapezoid and circle. For the instructor, the “overarching theme ... is to construct mathematical arguments, to construct reasoning, for these area formulas” [pre-interview]. During the pre-interview the instructor focused on the need for PSTs to provide complete justifications that could be generalized for any parallelogram, triangle, trapezoid or circle. Specifically when working with mathematical concepts the instructor wanted the PSTs to, “think about why, and understand how to justify... and that these kind of concepts apply across the board” [pre-interview].

4.2.3 Pre-interview: Reflection on Pre-Service Teachers

When describing his own experiences working with PSTs, the instructor expressed the fact that the PSTs most likely had prior experiences using area formulas. However, “if they are going to teach elementary school students these concepts, and they don’t have a good grounding in why the concepts make sense, then they are going to
perpetuate the kind of superficial teaching that they experienced in their childhood” [pre-interview]. An important component of the construction of the classroom culture with PSTs was the instructor’s focus on the future work of the PSTs. He believed that PSTs needed to work towards developing a deeper understanding of the mathematics so that they would then be fully prepared to teach future students. The instructor’s comments suggested that he intended to support norms where PSTs were expected to explain and justify the mathematics being explored.

In describing the lessons with PSTs, the instructor elucidated “the way in which I approach these is having them kind of explain. And having students kind of ask questions, if there are questions to be asked. And then me asking follow-up questions” [pre-interview]. The instructor wanted PSTs to guide the conversation and his role was to ensure that ideas were fully explained. The instructor expressed concern that PSTs sometimes “make claims, that they don’t immediately see the need to justify” [pre-interview]. The instructor expected to watch for these moments so that when he “see(s) a claim, hear(s) a claim being made that I believe needs to be justified mathematically, that is where I come in and I ask a probing question. Why does that work? How do you know that?” [pre-interview]. The instructor stated he might ask “probing questions that get them to think more deeply about what it is that they are understanding and not understanding, ... so they are engaging with each other” [pre-interview]. This suggests the instructor was working to create norms where mathematical ideas were fully explored, where PSTs were engaged with each other in the exploration of mathematical ideas, and where the instructor pushed PSTs to develop ideas further when needed.
One of the classroom norms established in this class involved the PSTs role in discussion. The instructor explained, “my goal is, across the board, to have students articulate their own understanding, their own methodologies, their own ways of thinking about it and having others respond to that” [pre-interview]. This indicated that PSTs would need to be able to explain their ideas, and be ready to respond to questions about their ideas. Furthermore, the PSTs would also need to orient to the mathematics reasoning of others, and that be able to respond to that reasoning. The instructor articulated that it was important for PSTs to address each other’s reasoning about the mathematics because, “by addressing someone else’s reasoning, you oftentimes can start to make sense of your own confusions. And, as a teacher that's a really important skill to have, to be able to make sense of someone else. And that's the entire job of being a teacher” [pre-interview]. Throughout the pre-interview the instructor’s description of his class suggested that he intended promote norms where interaction and dialogue about the mathematics was the norm for the PSTs. As future educators, PSTs need to know how to engage in discussion about the mathematics, and they need to have a thorough understanding of the mathematics being considered.

4.2.4 Pre-interview: Summary

The pre-interview focused on the instructor’s reflections on his style of teaching, his methods to elicit understanding, his reflections on the lessons to be taught and his experiences working with PSTs. The instructor’s comments suggested that he was trying to create a culture of collaboration where final conclusions about the mathematics were constructed by the community of learners through whole-class discussion sessions. The
instructor spoke of his intended use of talk moves in order to facilitate discussion about the mathematics between PSTs.

When discussing the intended class instruction, the instructor focused on the need for the PSTs to provide complete and thorough explanations, and to be able to understand the explanations provided by others. The instructor explained that, as future educators, the PSTs needed to have more than a superficial understanding of mathematics topics. The PSTs needed to not only know the area formulas and know how to use them, but they also needed to know why they worked. Furthermore, the PSTs needed to be able to respond to each other’s reasoning. As future teachers, the PSTs need to be able to listen to and respond to the reasoning of their own students. Classroom discussions where PSTs were to orient and engage with each other’s reasoning are important steps in preparing PSTs for their future role as educators.

**Section III: Classroom and Interview Data Analysis**

Two lessons were recorded as part of this research. The first lesson, Area Formulas 1 (AF1), took place over 125 minutes of class time. The second lesson, Area Formulas 2 (AF2), was shorter at 107 minutes. The instructor used two main learning formats during these classes: small group work and whole-class discussion. In addition a small amount of time (20 minutes) was spent on logistics such as logistical transition talk at the beginning and the end of class, instructions, and/or social interactions unrelated to the lesson. In the two area formula lessons (232 minutes total), SA1 and SA2, approximately 42% of the overall class lesson time (96 minutes total) was spent on small group work; during small group time participants worked together to answer questions
about the content. Many of the questions they answered required them to explain and justify their responses. About 50% of the overall class lesson time (116 minutes) was spent in whole-class discussion; where participants discussed specific problems and shared their reasoning about the mathematics with the entire group. In both lessons there were more than one whole-class discussion; the whole-class discussions ranged in length from 1 minute to 35 minutes. This research focused on the 116 minutes of the class where whole-class discussion occurred.

Table 4.3.1 Time Spent (Minutes) in Small Group work or in Whole-Class Discussion

<table>
<thead>
<tr>
<th>Area Formulas I</th>
<th>Area Formulas II</th>
<th>Both Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Small Group Work</strong></td>
<td><strong>Whole-Class Discussion</strong></td>
<td><strong>Logistics</strong></td>
</tr>
<tr>
<td>53 min</td>
<td>42%</td>
<td>43 min</td>
</tr>
<tr>
<td>65 min</td>
<td>52%</td>
<td>51 min</td>
</tr>
<tr>
<td>8 min</td>
<td>6%</td>
<td>12 min</td>
</tr>
<tr>
<td>Total</td>
<td>125 min</td>
<td>100%</td>
</tr>
</tbody>
</table>

A majority of both lessons were conducted in either the small group work format, or in the whole-class discussion format, where participants were expected to discuss mathematical ideas. There were also sections of time where the participants were not working on developing understanding and were discussing other matters. These time intervals were labelled logistics. Logistic intervals tended to occur during transition periods in the class, such as starting a lesson, ending the class, or transitioning between the small group work or whole-class discussion format.

The first lesson, Area Formulas I, took place during two class sessions. On the
first day participants worked on determining the area of an irregular shape, and then created a definition for area. Participants spent 50 minutes working on Area Formulas I during day 1. First participants worked in their small groups to develop an approximate value for the area of an irregular shape. The whole-class was then called together, and participants were called on to explain the methods they used in approximating the area. Participants then went back into their small groups to discuss the choice of unit for area, and how one might define area. The participants then reconvened in the whole-class discussion setting, first to discuss the choice of units, and then to work together on developing a definition for area.

During day 2, participants spent an additional 73 minutes working on the Area Formulas I task. First participants discussed the previous day’s work, and then provided area formulas that they thought they could recall from memory. Participants then worked in their small groups to create justifications for the area formulas of a parallelogram and a triangle. After 27 minutes, participants came back together in the whole-class discussion format to develop a justification for the area formula for a parallelogram. When participants struggled to provide a generalized area formula justification for any parallelogram, the instructor directed the class to go back into their small groups to discuss a generalized justification. The class ended with a whole-class discussion on the justification for the area formula for any parallelogram, followed by the justification for the area formula for any triangle. The breakdown of time intervals for Area Formulas I is provided in Figure 4.3.1.
Figure 4.3.1 Time intervals during Area Formulas I lesson
The second lesson, Area Formulas II, took place on day 2 and day 3 of the observed class sessions. The Area Formulas II lesson was 107 minutes in length, with 9 minutes occurring on day 2 of the observed sessions and 98 minutes occurring on day 3 of the observed sessions. During Area Formulas II, the participants first worked on justifying the area formula for a trapezoid in the small group work setting. Eventually, each of the five small working groups was directed by the instructor to prepare a justification on chart paper to present to the class. During the whole-class discussion period that followed, a member from each group was asked to present their group’s justification (with support from their group members). Four unique area formula justifications for a trapezoid were provided. The participants then shifted back to small group work where they worked on the second part of the Area Formulas II task to create an informal justification for the area formula for a circle. The participants came back together as a whole class to discuss this justification. The breakdown of time intervals for Area Formulas II is provided in Figure 4.3.2.
Figure 4.3.2 Time intervals during Area Formulas II lesson
In addition to the observed class sessions, multiple interviews were conducted with the instructor to gain insight into his thoughts and instructional approaches around developing participants’ understanding. Within these interviews, the instructor reflected both on his view of the classroom culture and his pedagogical approach to teaching. In discussing classroom culture and strategies, the instructor’s descriptions suggested a community of learners where the learners were creators and justifiers of mathematical ideas. As described by the instructor, mathematical ideas were to be created by the participants. Idea development occurred during discussions where the community of learners engaged in reasoning around mathematics. The instructor saw his role as a facilitator, supporting participants in generating, explaining and questioning ideas where there was a focus on reasoning about mathematical ideas.

This second part of this chapter explores how the instructor supported the development of mathematical understanding with the community of learners. Hufferd-Ackles and colleagues used the term Math-Talk Learning Community to describe a community of learners where participants work together to develop understanding through discussion (Hufferd-Ackles, Fuson & Sherin, 2004, p. 81). Although the instructor studied did not use the phrase Math-Talk Learning Community in describing his class, the created community of learners as described does suggest one. Hufferd-Ackles and colleagues (2004) described four components that contribute to the creation of a Math-Talk Learning Community: questions, explanations of mathematical ideas, sources of mathematical ideas, and responsibility for learning. The researchers created a framework to describe the trajectory for each of these areas as the community of learners
progressed to that of a Math-Talk Learning Community. The following chapter uses the Math-Talk Learning Community framework to explore the instructor’s work as he facilitated the development of participants’ mathematical understanding.

4.3.1 Math-Talk Learning Community – Questions

The first component of a Math-Talk Learning Community to be explored is the use of questions. In the Math-Talk Learning Community framework, at the lowest framework level, the instructor is the source of questions (Hufferd-Ackles, Fuson & Sherin, 2004). If the instructor even bothers to ask questions they are predominantly procedural, or they are questions that require short-answers. As the class progresses to that of a higher level Math-Talk Learning Community, the instructor begins to ask more open-ended, probing questions. Eventually, through instructor facilitation and through self-initiation, the participants begin to ask open-ended questions of each other.

When asked about the strategies that he used to get participants to explain, justify and generalize, the instructor specifically mentioned using talk moves as his primary tool. The instructor’s implementation of talk moves, as a tool for developing understanding, was found throughout the two lessons and is examined throughout this chapter. The talk move most relevant to the question developmental trajectory in the Math-Talk Learning Community framework, is the “press for reasoning” move.

In accordance with the instructor’s explained intentions during the pre-interview, the instructor implemented talk moves and talk tools throughout the two lessons. The “press for reasoning” talk move was utilized by the instructor in both lessons. When the instructor asked participants a press for reasoning question, he was asking them to clarify,
explain or extend their explanations. Using press for reasoning questions, the instructor could clarify the specifics of an explanation (“Is that what you're saying? ... The heights are the same?” [AF lessons]), clarify a general explanation (“What do you mean?” [AF lessons]), engage the community of learners in talking about mathematical ideas (“Does everyone understand the predicament here? What do you think?” [AF lessons]), push for unstated mathematical reasoning (“But how do we know that?” [AF lessons]), ask participants to take a stance (“Do people agree with that? That's the height? Does anyone disagree with that? Why do you disagree?” [AF lessons]), or push the participants to extend their thinking by making connections (“What do all of these methods have in common? They all have something in common.” [AF lessons]). This non-exhaustive list provides just some of the examples of questions that the instructor posed which pressed participants to reason about the topics under discussion.

During whole-class discussions, the instructor asked a press for reasoning question in 48% of his overall uninterrupted, talk turns (152 out of 346 turns) and received a response. During 26% of his overall talk turns, the instructor directed his questions to a specific participant and asked about a statement that the participant had just provided. In 22% of his overall talk turns, the instructor asked a press for reasoning question to a participant which asked him to respond to the mathematics that another participant had provided. Although not all of the press for reasoning questions required lengthy explanations, these questions did push participants to provide better explanations and justifications. Since the questioning component of the Math-Talk Learning Community framework did not focus on the answers to questions, and only the types of
In addition to the types of questions asked, who asks the question is also important in the Math-Talk Learning Community framework. At the highest level of the framework, norms are established so that the participants ask questions of each other. The instructor had the difficult task of initiating and maintaining these norms. In implementing this component of the Math-Talk Learning Community framework the instructor would at times just ask the group outright if they had questions about the mathematical reasoning provided by their fellow PSTs. In the following example a participant, S5, has just finished justifying the area formula for a trapezoid,

T Any questions for S5?

S6 How do you know that the two um, trapezoids make a parallelogram?

T Excellent question. Love that question.

The instructor explained, during the VSR interview session, that the mathematical reasoning that S5 had provided was sound, but S5’s articulation was not particularly clear so he felt it was important to ask the class if they had any questions. As shown above, one of the other participants (S6) immediately came forward. The instructor explained “although my class tends to be fairly quiet, when certain people don’t understand
something, usually someone does speak up. And I should say, when people know they don’t understand something, someone usually asks a question,” [VSR interview]. This suggests that the instructor had established classroom norms such that the community of participants felt comfortable asking questions of each other when explanations were unclear.

In the previous transcript example, S6 posed a question to participant S5. It is important to note that the instructor not only had PSTs ask questions, but he also expected that they should be prepared to reflect on and respond to the answers that were provided back to them. For instance, following S5’s response to S6 the instructor made a point to ask S6, “Is that okay with you S6?” Here the instructor purposely directed the conversation back to S6, who had asked the original question, to react to the response that S5 had provided. During the VSR interview session the instructor explained, that “when someone asks a question in class, we want to … let people know, ‘Okay, you know, there is value in asking questions.’ But really just to let people know that it is not enough to give an answer, we want to confirm. You know it is our responsibility to confirm that, that answer wasn’t just given, that it was adequate that it helped.” [VSR interview]. The actions of the instructor suggested that he supports classroom norms where the participants are important contributors in the question asking process. However, just having the PSTs ask question isn’t enough. In a community that talks about and exchanges ideas, the questions, the responses, and establishing understanding are important.

During both the pre-interview and the VSR interviews, the instructor explained
that the participants' future roles as teachers depended on their ability to make sense of the work of someone else and to ask questions. He expected that participants would ask questions of each other. To help maintain this expectation the instructor not only requested questions, but reminded participants that they would be required to ask questions of their future students. During the lessons, there were 11 different occasions where the instructor reminded participants of their role as a teacher.

The following section of transcript shows the instructor reminding participants of their roles as future teachers, and how they would someday need to push their own students in providing full and precise explanations by asking questions. The following transcript section followed a long justification by a participant on the area formula for a parallelogram. The instructor had felt that the justification that had been provided was not entirely clear. In order to engage the participants to think about questions to clarify the justification, the instructor asked participants to consider their roles as future teachers and what difficulties their students might have with the content. Next a participant volunteered a question for the community of learners to consider.

T Any questions about that for S11? Did that make sense to people?

T [Six seconds of wait time had occurred so a new turn was started.] Thanks, S11. That was really nice. Now, we're going to be teachers, right? So, we all get it, right? Cause everyone has been working on this for the last (fifty minutes). We all get it, but if you look at the drawings that S11 made, and the explanations he just gave for moving that triangle over. How can we explain it precisely, so that your future students will get it? [Two seconds of wait time.] And in doing that
you kinda have to anticipate, what kind of questions might they [elementary students] have about this drawing, and about S11’s explanation?

T [Five seconds of wait time had occurred so a new turn was started.] Yeah, S17?

S17 Um, one of the things we thought about was, how do we know that those two triangles are the same? So, like how do we know that we can take that and move it over? So, S19 came up with the idea of looking at similarity.

At the highest framework level of a Math-Talk Learning Community, the teacher not only asks questions, but also expects the participants to ask each other questions. During the VSR session that followed the instructor reflected on the exchange and explained how he tried to engage the PSTs in asking questions by having them reflect on “what kind of questions are they going to have to answer from their own future students. And thinking about their obligation to their future students, hopefully get them to now think a little bit more deeply about, ‘Okay, what’s involved in explaining why this is true?’ ” [VSR interview]. This prodding by the instructor prompted S17 to provide a question to the class that his group had reflected upon during the small group work period. The class then proceeded to spend 4 minutes addressing the question provided by S17.

This instructional move of asking the whole class to respond to the work of another participant and to ask questions of that participant was repeatedly seen during the AF1 and AF2 lessons. For example, after S24 provided a justification for the area of trapezoid, the instructor had the PSTs reflect on whether they had questions. When no
one responded, the instructor had the PSTs reflect back on the previous work that they had completed in their groups to contrast their methods with the provided example. He then asked the PSTs to think about what questions their future students might have.

T Any questions for S24?

T [Four seconds of wait time had occurred so a new turn was started.] How many of your guys did that method in your groups? Let me get some hands. [T puts up left hand.] Now S24, before I let you go, looking at your method here, now we are going to be teachers, so we have to think about our students. Based on that method, what do you think might be some questions that your students might have about what you did, if there is anything they’d want clarified? Or something you think that they might not immediately grasp?

S24 Um, hm.

T It’s kind of a hard question. We have to kind of get into the minds of our students. If anyone in the audience has a question they think their students might ask- feel free- to also help.

S24 Of course, S17.

S17 How do we know that we can smoosh the two triangles together to make a big triangle?

During the VSR interview session the instructor spoke to focusing on the PSTs' future role as an educator when asking the PSTs to contribute questions. First, “as future teachers they’re going to need to start of think about what students think about when they do this kind of mathematics.” [VSR interview]. Teachers need to able to anticipate the
difficulties that their own students might have. Secondly, if a PST was struggling with the materials being presented, a chance to ask questions “their student might ask” [AF lesson] could allow him an opportunity to clarify his own confusion. Wording it as such, the instructor might “make them [PSTs] feel more comfortable to ask a question, just kind of framing as ‘well my student might ask’ when it’s really them asking” [VSR interview]. A PST who may have felt embarrassed to ask a question about something they themselves did not completely understand, may feel more comfortable asking that same question as an example of something that their future students might ask. This would still allow PSTs an opportunity to have their questions addressed, but without the discomfort sometimes associated with admitting one’s own confusion. In supporting classroom norms where the PSTs are contributors to the questions asked within the class, PSTs are provided opportunities to clarify confusion, which leads to better understanding. Additionally by considering questions that their own students might ask the PSTs are provided an opportunity to reflect on the confusion of their future students and how that can be addressed. PST provided questions allow PSTs to prepare for their future role as educators.

At the highest framework level of a Math-Talk Learning Community, the expectation is that participants should be asked probing questions, and that both the instructor and the participants should take responsibility for asking questions. This occurred in these observations. By asking “press for reasoning” questions, the instructor compelled participants to provide mathematical explanations that were detailed and thorough. Furthermore, the instructor supported the participants in asking questions by
reminding them of their future roles as educators. As an extension of asking questions, the next section reflects on providing complete explanations of mathematical ideas.

4.3.2 Math-Talk Learning Community – Explanation of Mathematical Idea

Another key component of a Math-Talk Learning Community involves the explanation of mathematical ideas. As a class proceeds along the framework trajectory to that of a Math-Talk Learning Community, the intention is that participants will develop their abilities to provide complete and thorough mathematical explanations (Hufferd-Ackles, Fuson & Sherin, 2004). At the lowest level of the framework the teacher expects short, simple answers from the participants and/or the teacher provides all explanations. At the highest level, participants need to fully defend and justify their answers, predicting and addressing possible questions without prompting (Hufferd-Ackles, Fuson & Sherin, 2004, p.90). To ensure thorough explanations, the instructor must pay close attention and support participants in thinking deeply about the mathematical reasoning they are providing. Thus, questioning of participants is closely connected to the explanations that participants provide. Namely, the types of questions that the instructor asks can lead to more thorough reasoning and better explanations of mathematical ideas.

In the previous section the instructor’s use of questions was presented as a means to push or press participants for further reasoning about the mathematics. One of the most crucial objectives of questions is to guide participants to provide enough reasoning about the mathematics so that a thorough justification is constructed. Talk moves that fall under the general category of press for reasoning (Chapin, O’Connor & Anderson, 2013) were often used by the instructor in this study to accomplish this goal. If the instructor pressed
for reasoning, and the participants were confused, the confusion was identified and addressed. If a participant was able to immediately and coherently respond to a question, his response allowed other participants another opportunity to hear a clear explanation. Furthermore, when the instructor pushed participants to reflect, clarify, and address missing components of a justification, the community of learners as a whole began to gain a clearer picture of the many components of a full mathematical justification.

In the two situations that follow the instructor pressed for a more thorough explanation of a mathematical idea by asking questions that required participants to provide more backing for their ideas. In the first example, a participant, S11, was justifying the area formula for a parallelogram. His explanation had included information about decomposing a parallelogram into a triangle and trapezoid, and then recombining the pieces to form a rectangle. Next another participant, S14, reiterated S11's explanation. The example begins with the instructor briefly restating that the parallelogram had been decomposed into shapes, and then confirming the shapes had been recombined to form a rectangle. The instructor then directed the conversation back to the original presenter and asked why it was that he deconstructed the parallelogram to form a rectangle in the first place.

T Thanks, S14. Good job. So it seems like we can take a parallelogram and we can decompose, that's kind of a word that are going to be using a lot. We can decompose or break apart the parallelogram into this piece, which is a trapezoid and this triangle. And then like S11 said, we can move this triangle over here.
[Gestures to right triangle position.] And put it here. And now, what's the result?

Once we move this, what kind of shape do you now have?

S14 A rectangle.

T You now have this rectangle, I am going to highlight it in black. And S11, why did you choose to turn it into a rectangle? What's so special about a rectangle?

S11 Because it's easier to understand the base times height in a rectangle, cause it's divided into little squares and you can see like a certain number of rows and columns together creates an area, instead of like half-way cut off squares.

In this situation, the instructor specifically asked a presenter to explain why he chose to reconstruct a decomposed shape into a rectangle. In discussion, sometimes, key components of a justification are not stated. The presenter may have thought that a point was so obvious that it didn’t need to be stated, or it is also possible that the presenter did not realize that an important part of the justification was missing. It is the instructor who was tasked with making sure that a full and complete explanation was provided. As future teachers, participants need to be able to know these concepts and explain them fully.

In the next example the instructor use the “press for reasoning” talk move and the participant’s uncertainty was revealed. The instructor had asked participants to create a definition for area and wrote their statements on the whiteboard. So far, the instructor had written on the whiteboard: the amount of units squared within the sides of the two-D
The instructor then asked the class to provide a concise definition. Participant S8 raised his hand to volunteer his thoughts.

S8  I think it's important. The units squared is important.

T  Why's that? Why do you say that?

S8  Well, we had a big debate over whether it has to be squared or if could be other shapes? And it came to that it has to be squares.

T  Why is that?

S8  I don't really know why, but I know that's important.

In this example, a volunteering participant raised his hand to express to the class that using square units was an important component of the definition for area. However, when asked to provide backing for why he thought squares were important, the participant couldn’t explain. He only knew that his group had debated it and that they had decided it was important. This exchange highlights the importance of probing participants’ thinking. Without pressing for information it is easy to assume that S8 knew and understood why area was measured in square units. By pressing for explanations and reasoning around the mathematics, the instructor was able to ascertain that S8’s understanding was lacking.

During the video-stimulated recall (VSR) interviews that followed the class, the instructor expressed the importance he placed on knowing and understanding the area definition. For the instructor, just stating that area should be measured using squares was not enough; “we need to have some kind of justification, some kind of explanation here, for why squares are important” [VSR interview]. S8’s inability to provide support for his ideas informed the instructor that participants did not fully understand why area was
measured using square units. Following this exchange, the class then proceeded to spend more than five minutes addressing this one issue in whole-class discussion. In this case, the instructor did not simply provide the community of learners with a reason for why square units were the appropriate unit. Instead the community of learners worked together to develop and justify this point with guidance from the instructor.

Another way the instructor supported participants in providing complete mathematical explanations was through his use of the talk moves of “restating” and “revoicing.” The full development of a mathematical explanation can take a while. Furthermore, sometimes new concepts build upon previously introduced topics and ideas. When the instructor restated information, he would provide a mathematical statement, idea, or concept that had been provided earlier in the lesson. Through restating the instructor reminded participants of the work that had previously been completed to help develop understanding. When the instructor used revoicing, he would voice the statements of a participant, and then he would ask “[them] to actually acknowledge whether that was what they said or not” [pre-interview]. The instructor explained that revoicing provided an avenue for clarifying a participant’s explanation that was incoherent, both for the speaker and the listeners. By “revoicing” or restating what was just said, the instructor can provide another opportunity for participants to make sense of the mathematical reasoning provided by others.

Within the transcripts, the instructor sometimes would revoice immediately following a participant’s utterance. The instructor also would restate or revoice a participant’s statements when he wanted to remind the community of learners what that
participant had said. During the two lessons, the instructor restated or revoiced a mathematical idea in 40% of his talk turns, or 127 out of 346 of his turns. These moves allowed the instructor to remind participants of statements previously provided by others in the class and was used to support participants in developing thorough mathematical explanations.

With all of the reasoning around the mathematics expected of participants, the instructor mentioned talk tools that he used to provide the participants with a moment to think. The instructor stated in an interview that sometimes questions were asked that “required a lot to think, a lot of thought, couldn’t be answered just in a split second” [post-interview]. He explained that providing “wait time allows other people to process what was being asked and maybe to construct an argument” [post-interview]. The instructor saw “wait time” as a tool to “try to and give students an opportunity” [pre-interview] to reflect and contribute. Similar to “wait time,” the instructor also mentioned instructing participants to take a moment to turn a partner (or their group) and explain their thinking. Both “wait time” and partner talk were tools that allowed participants extra time to develop their explanations.

“Wait time” was used extensively by the instructor throughout his lessons. In pushing participants to construct complete explanations and justifications, the instructor constantly provided “wait time,” allowing participants time to think about ideas (Rowe, 1986). In some instances, the use of “wait time” was quite dramatic. The next example demonstrates how important “wait time” pauses can be during discussion. This transcription section starts after participants had already spent 8.5 minutes talking about
the area formula for a parallelogram. The participants had initially been asked to “generalize this to all parallelograms, not just the examples that you have on your sheet” [AF lessons]. However, although participants were tasked with providing an area formula generalized for all parallelograms, the community of learners had been discussing a specific parallelogram with a base of four units and a height of three units in their explanations. At this point in the discussion, the participants had talked about decomposing the shape into a triangle and trapezoid, and then recombining the two shapes to form a rectangle. Participants had explained why a rectangle could be created by moving a triangle over.

Why is the area of any parallelogram, \( A=bh \)?

Figure 4.3.3 Image reproduction of board work – Area formula justification for a parallelogram

Furthermore, participants had already explained why they chose to recombine the shapes into a rectangle. Lastly, participants had discussed how and why the triangle that was moved fit against the remaining shape, which was now a trapezoid. In the following
example, the instructor again asked the community of learners to provide a general justification showing that the area of a parallelogram is the product of the base and the height.

T Are there any questions about that?

T [The instructor waits 4 seconds so a new turn was started.] One thing I want to make clear as we go through this, is that I was going around and a lot of people were talking about how, "a parallelogram will be base times height because it has the same area as the rectangle" cause "really the rectangle is just kinda slanted over," or "its skewed." But this explanation is a little more precise right? Because we are using exactly what we know about parallelograms in order to say exactly why you can move these things around. And why this area, the resulting area, will be the same.

So it's kinda, I want you guys to kinda put yourselves a little bit further than just kind of giving kinda broad answers about, "well it kinda looks like it" or "I can move or use the properties of the shapes that we do know" to make a case. Ok? Are there any other questions about this? [The instructor waits 3 seconds.] Things that either you are unclear about, or things that you think your students might be unclear about?

T [The instructor waited 8 seconds so a new turn was started.] So remember I said we need to prove this for any parallelogram. This is a parallelogram with a base of four and a height of three. What if I give you a parallelogram that looks
like this? And I just say, this has a base of $b$ and a height of $h$. Can you convince me that the area of this is still going to be base times height?

Board work

![Figure 4.3.4 Image reproduction of board work – Drawing of a parallelogram](image)

T [The instructor waited 7 seconds so a new turn was started.] Who thinks they can give a convincing argument, based on now, what we have talked about?

T [The instructor waited 8 seconds so a new turn was started.] Let's do this, let's take about two minutes, I want each group to use this picture and come up with a justification for why this area is base [times] height. Come up with one justification for... Okay? So, no more numbers.

T [The instructor calls the group back together after allowing the participants to talk about ideas in small groups for 4 minutes and 15 seconds.] Alright guys, lets come back together. Who thinks- Let me just ask you- I just realized while I was looking at this, this is not a good looking parallelogram. [The instructor redraws sides of the parallelogram on the whiteboard.] Um- who thinks they can come up the whiteboard and give us a general justification for why this
parallelogram's area is base [times] height. [The instructor points to newly drawn parallelogram on the whiteboard.]

T [The instructor waited 11 seconds so a new turn was started.] I am waiting for hands.

T [The instructor waited 6 seconds so a new turn was started.] Alright S22, do you want come on up?

In the preceding example, the instructor had first tried to engage participants by asking for questions and then by asking them if there was anything about which they were unclear. None of the participants volunteered a response. The instructor then asked the group if anyone could provide a convincing argument using a general parallelogram with a base of \( b \) and a height of \( h \), because he “want[ed] them to generalize” [VSR interview]. Despite all of the prior discussion very few participants in the class raised their hands to volunteer. At this point, instead of pushing forward as a whole class, the instructor had all of the participants discuss creating a general justification in their small groups. The instructor explained that he “believe[d] at this point [was] too big of a leap. I had to plod them along through the [previous] example” [VSR interview]. The instructor had them talk with their small groups in order to give the participants an “opportunity to kind of grapple with this” [VSR interview]. After giving the participants over four minutes to discuss within their small groups, the instructor called the class back together and asked for a volunteer to present his justification. The instructor patiently waited for volunteers and then selected someone to present. After calling on S22, the group spent
over 10 more minutes discussing the generalization of the formula for the area of a parallelogram.

This example was chosen for a number of reasons. First and foremost was the use of "wait time." When removing the interval of small group work, the instructor held the floor for a total of four minutes and 10 seconds. Some of that time was associated with talk, and some of that time was associated with whiteboard work. However, 45 seconds of that time was attributed to the use of "wait time" by the instructor. During the example, there were seven instances where the instructor waited for three seconds or longer. In the first two instances, the instructor waited four seconds and three seconds, respectively, after asking the participants if they had any questions. After receiving no response, the instructor asked what questions they or their students might have and waited eight seconds. After receiving no response, the instructor reminded the participants that they were supposed to be providing a justification for a general parallelogram. He asked if anyone could do so and waited eight seconds. The instructor then asked who thought that they could provide an argument based on what they talked about. The instructor waited eight seconds for hands to go up, and after few participants volunteered, he decided to give the participants more time to discuss in small groups.

When the participants returned from small groups, the instructor once again asked who could justify. After waiting 11 seconds, he still did not have many volunteers. The instructor stated that he was waiting for more hands and then waited another six seconds before calling on someone. In using "wait time," the instructor made it clear that it was
the participants who had to come up with a response to the question and provide an explanation.

In addition to “wait time,” this example also illustrates the importance of the instructor in developing thorough understanding. To the instructor, the lack of volunteers indicated participant uncertainty when moving from a justification with specific measurements to providing a generalized justification for all parallelograms. This was an important extension for participants in the class. It might seem like moving to a generalization would be an easy transition after the work that had already been done, but when the instructor asked for volunteers he didn’t get many. During the VSR interview sessions, the instructor mentioned that even after he allowed the participants’ time to work in small groups to talk about ideas, he felt he still did not get a lot of participant volunteers. By using “wait time,” the instructor hoped to increase the participation rate.

As the facilitator, the instructor needed to ensure that the participants were given the time needed to fully develop understanding. Sometimes more time needs to be spent on a topic and it is the instructor’s role to be aware of this and to respond. By pushing participants to create a justification, which would work for all parallelograms, the instructor was ensuring that participants provided a complete justification.

In pushing the participants to provide thorough explanations, the instructor pressed for reasoning, restated pertinent mathematical ideas, and provided opportunities for the participants to have thinking time. As members of the Math-Talk Learning Community, it was expected that the participants would be active contributors who asked questions and provided thorough explanations. As more active contributors, participants’
strategies, ideas and methods also provided the basis for the mathematical lesson. This
leads us to the third component in a Math-Talk Learning Community — the source of
mathematical ideas.

4.3.3 Math-Talk Learning Community • Source of Mathematical Ideas

The third component that Hufferd-Ackles and colleagues associated with a Math-
Talk Learning Community involves the source of mathematical ideas (Hufferd-Ackles,
Fuson & Sherin, 2004). As a class progresses along the Math-Talk Learning Community
framework trajectory, it is expected that a shift occurs from a class where the instructor is
the source of all mathematical ideas to where there is a community of learners who are all
active contributors to the ideas that form the lesson. Participants feel free to interject
ideas and the instructor uses those ideas to guide the development of the lesson.

The instructor’s perspective was that reasoning and justifications were to emanate
from the participants. As stated previously, the instructor wanted to “have students be the
ones to bring up certain ideas and to grapple with certain concepts” [pre-interview].
When introducing ideas, the instructor wanted participants to “articulate their own
understanding, their own methodologies, [and] their own ways of thinking about it” [pre-
interview]. In examining the transcripts, the participants were predominantly the source
of newly stated mathematical ideas. A newly stated mathematical idea was considered
any new mathematical fact, specifically relevant to the lesson that was not previously
stated verbally during whole-class discussion, or had not previously been provided within
the instructional material. During the whole-class discussion sessions, which totaled one
hour and 56 minutes in length, the instructor rarely stated new mathematical facts that
had not previously been provided by a participant, or were not found in the written instructional materials. The participants supplied the majority of the new mathematical statements provided in whole-class discussion. These statements were developed further by the participants during discussion, guided by the instructor.

Overall, the instructor strived to make the PSTs be the source of mathematical ideas. He did this by letting the class know they were going to come up with ideas.

T What I want to do is I want to get some definitions up for area. And then kind of decide together, in the time we have left, on what is the definition for area that we want to use. So, um, S5 can you start us off? I was talking to your group about a definition. What did you guys come up with?

When the instructor asked S5 to provide a definition for area, the instructor specifically mentioned that he had already talked to S5’s group about a definition. By monitoring small group work the instructor could later select PSTs to present, or the instructor could sequence the presentation of PST ideas, in order to ensure certain mathematical ideas were introduced. So, although the instructor was not the verbal source of knowledge, in this instance he selected a PST to verbally present their knowledge to the whole class based on the ideas he heard presented in small group.

The instructor explained that this pattern was not unusual:

“So typically when we start a whole-class discussion I like to get an idea up on the board, a student generated idea about the group discussion questions. And then I typically, unless I have a specific agenda or not, I like to kind of open up the floor and ask if anyone has any general comments they want to make about what we’ve just put up on the board, what’s been discussed” [VSR interview].
Here the instructor specifically states that he intended to use participant generated ideas to seed the group discussion. He specifically said that he wanted to "open up the floor" so that PSTs could comment on the original PST provided idea. These actions are indicative of a class where participants are supported as the source of mathematical ideas. Instead of just telling the participants key mathematical ideas, the instructor created situations where PSTs presented their mathematical ideas based on the work done in small groups. Using the PSTs suggested ideas, the instructor then worked to "generate either debate or consensus" [VSR interview] from the PSTs. In lieu of presenting ideas himself, the instructor constantly supported PSTs as the source of comments, questions and ideas throughout the class. He supported the participants in being the source of knowledge by giving them tools (materials, time and verbal clues) so that they could provide information during whole-class discussion.

Additionally, in supporting participants as the source of mathematical ideas, the instructor actively tracked contributions and continuously recognized participants as the source of developing ideas. The instructor often made a point of stating the name of the participants that had originally presented an idea. For example, "So we have S16 made a big rectangle and a bunch of little triangles around it. S22 made a big circle and then he made rectangles using a grid to try to get the smaller areas" [AF lessons]. Likewise, the instructor would often asked PSTs to reflect on the work of another specific PST, "Do people agree or disagree with S5?" [AF lessons]. By stating participants' names in association with ideas, the instructor reminded the class that they were the source of ideas.
During the two lessons, there were 68 occasions where the instructor mentioned the name of a participant in the association with a mathematical idea that they had provided to the class. Of the 24 participants in the class, the instructor specifically stated the names of 19 participants.

![Instructor Mentions Participant's Contribution](image)

**Figure 4.3.5 Participant contributions mentioned by name**

On average a participant's name was mentioned 2.8 times in association with a mathematical idea, with a mode of 2. Five participants were not mentioned by name in association with a mathematical idea. The maximum number of times a participant's name was mentioned in association with an idea was participant S14 (9 times).

Additionally, during the two lessons, there were very few occasions where the instructor explicitly provided mathematical ideas. However, by the questions he asked and the threads of discussion he chose to pursue, the instructor did give information to the class about which mathematical ideas were most valued. Here, two occasions where he did provide information are explored.

During one occasion, groups had been presenting posters that they had created
justifying the area formula for a trapezoid. In this example, the instructor called attention to the fact that although two groups had approached the justification for the area formula for a trapezoid differently, the two groups had an identical formula for area on their posters. The instructor’s statement was made after S24 explained his group’s poster, and after S18 had finished explaining his group’s poster. A participant, S16, had been trying to explain to the class that S24 and S18 had similar ideas. It was at that point that the instructor was looking at the posters and stated, “And now that (you guys) look, that’s the same formula that S18’s group has at the end. That’s the same formula. I just noticed that S24’s group’s- S24’s group has kind- right below the last picture. Do you guys see that?” [AF Lessons]. Here the instructor attempted to engage the participants in making a connection between the two area formula justifications that had been created by participants. The instructor highlighted the connection that the posters contained the same written formula, but the participants actually were the source of the information that was placed on the posters. In this example, the instructor clearly stated the mathematical connection aloud instead of working with the PSTs to have them verbally provide the connection. The instructor later explained one of things that he wanted to do was “to acknowledge the fact that there are kind of different ways to interpret the same formula, so people can start making connections” [VSR interview].

Another mathematical statement made by the instructor occurred at the end of the last class, less than two minutes before the conclusion of the lesson. The instructor had already spent 13 minutes engaging the class in informally justifying the area formula for a circle in whole-class discussion. (This was after the small groups were given 21
minutes to talk about this same topic.) At this point the class had discussed the fact that if a circle is decomposed into sectors, the sectors can be reconstructed into a shape that looks like a parallelogram. However, the entire class was not convinced the shape was a parallelogram, so the instructor explained that if the circle is cut into an infinite number of sectors and repositioned, the shape would become a parallelogram.

This base that looks curvy would start to look straighter and straighter, because now we're cutting it into more finely sliced... the more slices you cut your circle into the straighter and straighter this thing is eventually going to look. And you can kinda think theoretically, 'what if I were to cut this into an infinite number of slices?' It's going to be barely noticeable as an arc. It's going to start to straighten out.

In this case, the instructor built upon the ideas that participants had provided, but clarified a point to bring closure to the discussion about the area formula of a circle. The instructor later explained that he made the statement because he wanted to focus on:

"the issue of the idea of cutting into more and more slices. And at that point, you know, I am watching the clock, I am thinking about that I've got to give them their midterm back. And I want to talk about their midterm. And I felt like continuing the discussion, I just didn't think it was going to add that much more value. I wasn't going to now have them cut their slices into more and more slices, maybe we would do that later, maybe not, into next year. So I felt like I wanted to just talk to them about the notion of getting kind of infinitely closer to [a] parallelogram through cutting. And so I was just telling, you know we need to take it, (kind of keep moving things on)" [VSR interview].

Here the instructor clearly provided some mathematical information. However, he explained that it was a conscious decision due to time restrictions. Sometimes during
instruction, an instructor needs to make a judgment call on how to spend the time. In this instance, the instructor opted to clearly provide information that the class had not yet fully articulated.

In summary, the majority of the initial mathematical statements, and the subsequent mathematical backing for these statements, were participant generated. However, the instructor was actively selecting and emphasizing which statements to pursue and elaborate. New mathematical statements included initial answers to task questions, definitions, and justifications from which the participants used as a base when constructing understanding. Additionally, as the discussions progressed new mathematical ideas or concepts also arose. It is important to note that the PSTs did not provide new mathematical statements by chance. The tasks that the PSTs were using were developed to assist them in constructing knowledge of area formulas. The instructor also supported the participants by asking questions and selecting participants to contribute based on what he had seen during small group work. Furthermore, the instructor also supported the PSTs as the source of mathematical ideas by associating mathematical ideas with the participants who had originally provided them. The final component that is part of Hufferd-Ackles and colleagues’ Math-Talk Learning Community is the responsibility for learning. This is closely related to this section on the source of mathematical ideas.

4.3.4 Math-Talk Learning Community □ Responsibility for Learning

The responsibility for learning in the classroom is the fourth and final component of the Math-Talk Learning Community framework (Hufferd-Ackles, Fuson & Sherin,
At the highest level, the instructor expects participants in a Math-Talk Learning Community to be active contributors who listen, ask questions, and help their fellow classmates. As explained in the framework, the trajectory progresses from the lowest level where the instructor takes on all the responsibility, to where the instructor encourages participants to engage with the ideas of others, and finally to the highest level where the instructor supported participants as needed while the participants initiate engagement with the work of others. At the highest framework level, the responsibility for learning should be on all members in the community. This means that all participants should be engaged, and all the participants should contribute to the development of understanding by offering ideas or asking questions.

The instructor explained that his goals, for the PSTs, were “to push them to think deeply about the mathematics, to push them to explain and justify and generalize, and to get as many people involved as I can” [VSR interview]. His intentions, as described, suggest a community-centered classroom environment (National Research Council, 2005a, 2005b) where the community of learners works together to construct complete justifications with many opportunities for participants to reflect on what they do and do not understand. In order to provide a rich description of the classroom environment and the responsibility for learning within it, the following section first examines how the class time was utilized.

During the 116 minutes of whole-class discussion, the instructor and the participants made verbal contributions. To gain an understanding of who was talking, video time stamps were used to determine the total number of minutes both the instructor
and the participants spent talking during the class. In AF1, the instructor and the
participants contributed evenly to whole-class discussion. In the AF2 lesson, a larger
portion of the whole-class discussion was associated with participant talk. In AF2, 60%
of talk time was associated with the participants and only 40% with the instructor.
Overall, 46% of the whole-class discussion time involved instructor talk, and 54% of
whole-class discussion time involved participant talk.

Table 4.3.2 Breakdown of Talk during Whole-Class Discussion

<table>
<thead>
<tr>
<th></th>
<th>AF1 Lesson</th>
<th>AF2 Lesson</th>
<th>Both Lessons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole-Class Teacher Time (seconds)</td>
<td>1945</td>
<td>1246</td>
<td>3191</td>
</tr>
<tr>
<td>Whole-Class Participant Time (seconds)</td>
<td>1954</td>
<td>1850</td>
<td>3804</td>
</tr>
<tr>
<td>Teacher Time % of Total</td>
<td>50%</td>
<td>40%</td>
<td>46%</td>
</tr>
<tr>
<td>Participant Time % of Total</td>
<td>50%</td>
<td>60%</td>
<td>54%</td>
</tr>
</tbody>
</table>

In most undergraduate mathematics classrooms, lessons are taught in a lecture format led
by the instructor (Bergsten, 2007) where participants assume a passive role (Fritze &
Nordkvelle, 2003). Yet, during interviews the instructor described his role as facilitator
not lecturer. His role as a facilitator of developing understanding for the community of
learners was supported by the fact that that over 50% of talk turn time in the whole-class
setting was attributed to participant talk. The participants were responsible for and were
expected to provide explanations and to justify ideas and concepts; it makes sense that the
percentage of time participants talked was high.
Another way the roles of the instructor and the participants in these lessons were examined was by contrasting the length of talk turns. As previously stated, a turn was defined as a new uninterrupted utterance by a speaker, and established when there was a change in speaker or when there was a pause in excess of 3 seconds. During whole-class discussion, the longest instructor talk turn was 62 seconds. Interestingly, the maximum participant talk turn was more than two minutes at 128 seconds. This means that, at one point, a participant was allowed two minutes of uninterrupted class time to provide an explanation/justification to his fellow participants. In a Math-Talk Learning Community, participants are significant contributors to the learning. These descriptive statistics suggested that both participants and the instructor were important contributors to the whole-class discussions.

As part of maintaining the participants’ responsibility for learning, within the class, the instructor constantly pushed participants to reflect on and respond to the reasoning of others. During the lessons, the instructor asked questions facilitating opportunities for participants to pause, reflect, and respond in some way to presented ideas. To maintain a community of learners discussing mathematical ideas, it is essential that everyone is listening and trying to make sense of what is being said. As stated previously, if members of the community of learners know that they can be called on at any point, they are also more likely to focus on the reasoning provided by others in the class. In its simplest implementation, the instructor might ask a participant to restate or revoice. Having participant’s rephrase a peer’s contributions provides them an opportunity to present ideas in their own words. Participant restating allows the original
presenter to "rehear what they just said" [pre-interview] and it allows the class an opportunity to "think about what's going on" [pre-interview]. During the implementation of the lessons, the instructor asked participants to respond to the thoughts of another participants in 22% of his overall talk turns. In asking the participants to restate, explain, or expand upon an idea, the instructor guided the class to focus on ideas that were important.

Another aspect of the responsibility for learning component is that the community must be responsible for coming to conclusions about the mathematics together. In the observed lessons, the instructor created a Math-Talk Learning Community where participants felt comfortable disagreeing. In the following example S17 explained that bigger shapes are better for measuring area because there will be less of them. Another participant, S5, explained that he thought that smaller shapes were easier to use because with large shapes, it was hard to tell how much of the image was covered.

S17 Also, the bigger that they are the easier it is to measure. Cause there's less than you have to like count.

T So, the bigger the unit you're using.

T Why do you say it's easier to measure?

S17 Because there's less that account for the area. So if the units are really small then there's going to be more of them. If they're bigger there is going to be less of them. So, it just makes it easier.

T Oh, okay, so the smaller the unit [writes on whiteboard].
T The more of it will be used to measure area.

T So what shapes- Oh yeah, sorry S5.

S5 Um, I kinda disagree with that. I think the smaller the unit would be more accurate. Cause if you’re using say like bigger rectangles, or bigger squares. Like how can you judge like a half or an eighth or something like- I think it would be harder to judge, what fraction of a larger unit is used as opposed to a smaller.

T For example, in this picture of what you guys are doing [points to image drawn on the whiteboard], so you’re saying, if I use a big square or a big rectangle.

S5 It will be less accurate, because you won’t be able to tell exactly how much is not covered by those rectangles or squares.

T So, if I do something like this [draws large squares on top of image on the whiteboard], this would be less accurate than maybe doing something like this [starts to draw smaller squares on another congruent image].

T Something like that

S5 Yeah.

T Do people agree or disagree with S5?

T How many of you guys agree? [Raises left hand.] How many disagree? [Raises right hand.] Yeah, S7?
This example illustrates a participant initiating an idea, and the instructor engaging the entire community in considering it. Participants needed to select a stance by raising their left or right hand. The instructor then selected a new participant from the class to add to the discussion. The conversation did not end at the conclusion of the transcribed session. The participants spent five more minutes talking about the best shapes for measuring area before moving on to a follow-up task. This example typifies the role of the instructor as a facilitator focused on helping participants take responsibility for their learning.

The previous example also illustrates the instructor creating an environment where the group is the authority. The instructor did not view himself, nor did he want the class to view him, as the sole authority presenting and confirming mathematical truth in the classroom. The responsibility for learning was to be placed on all members in the community. As explained previously the instructor worked to create a perception he was “not the final authority and that they [the participants] shouldn’t just look to me to give the stamp of approval on whatever idea or work that they are doing” [pre-interview]. The instructor expected participants to reflect on the accuracy of each other’s statements, by asking them if they agreed or disagreed, as opposed to the instructor just confirming, okay that’s wrong, or that’s right, and giving the answer. In fact, the instructor only made seventeen statements, during less than 5% of instructor talk turns, which could be perceived as clear evaluative statements of correctness. In summary, the instructor’s intent was that mathematical understanding be developed by the class and ideas be agreed upon through whole-class discussion.
During the observed lessons, the entire community of learners was responsible for contributing to the developing mathematical understanding. As the facilitator, the instructor guided participants in developing mathematical ideas through his questions, his support, and his expectations for the community of learners. However, it is important to note that facilitating the class, and holding participants accountable as responsible members can be difficult. As part of maintaining the responsibility for learning, the instructor had to work at maintaining the involvement of the participants when the lessons might be “a bit on a slower pace … because [he, the instructor.] was pushing people to try to justify things and to provide their own explanations. And … that that just takes time” [VSR interview]. Unfortunately, participants don’t always push themselves to thoroughly understand the topics when given the time to do so. Participant need to make sure that they really understand the mathematics, and the instructor must constantly push the PSTs to meet this challenge.

The instructor indicated his intention was to create a classroom environment where participants were responsible for their own learning and the learning of their peers. Within the class both the instructor and the participants were equal contributors to discussion. The instructor, revoiced and attributed contributions from participants, pushed participants to engage with the mathematics and provided time for participants to consider the mathematical ideas that had been provided. Both the instructor and participants challenged the mathematical ideas of others. The responsibility for learning was on the entire community of learners, with the instructor facilitating engagement, contact, and thoroughness of explanations.
4.3.5 Summary

This chapter considered AF Pre-test and Post-test scores, class video data, and instructor interviews in order to provide a description of how a knowledgeable instructor supported the developing understanding of pre-service teachers in an undergraduate mathematics class. Both quantitative and qualitative data from a series of undergraduate mathematics lessons for pre-service teachers were considered.

In the first section of this chapter, participants’ responses on the Area Formula (AF) Pre-test and Post-tests were analyzed. There was a significant difference between the overall mean AF Pre-test score and mean AF Post-test score ($p < 0.0001$) suggesting evidence that the instruction significantly influenced participants’ understanding.

In the second section of this chapter the instructor’s instruction as intended was summarized. The instructor intended to create a class where participants constructed understanding about the key instructional topics through whole-class discussion, and where all participants were expected to contribute and to support each other. In discussing strategies, the instructor specifically cited “talk moves” as a key approach to facilitating discussion and developing participant understanding. Lastly, the instructor focused on the PSTs future role as educators and the need for the PSTs to not only fully develop their own justifications, but to also orient to and engage with the reasoning of others.

In the last section of this chapter, the four developmental trajectories of a Math-Talk Learning Community were used as the framework to describe the instructor’s decisions as he supported the trajectory areas of: questioning, explaining mathematical
ideas, sources of mathematical ideas, and responsibility for learning (Hufferd-Ackles, Fuson and Sherin, 2004).

- In accordance with the upper level trajectory of questioning in the Math-Talk Learning Community framework, the instructor asked participants’ questions that pressed participants for further reasoning about the mathematics in almost half of his talk turns. He also supported participants in providing questions by reminding them of their future work as educators.

- The instructor pushed participants to provide full explanations by allowing the participants “wait time,” and through reminding participants of the previous mathematical work that had been discussed.

- In association with the upper level framework trajectory of source of mathematical ideas, during interviews the instructor expressed that he wanted ideas to be developed by the community of learners; in accordance with this the majority of new mathematical statements were participant provided (with support from the instructional materials, and instructor).

- The Math-Talk Learning Community framework trajectory of responsibility for learning, with responsibility for learning on all participants, harmonized with the instructor description of his role in the class as a facilitator supporting participants as they discussed ideas. The instructor's role of facilitator was also supported through the analysis of quantities of talk time during whole-class discussion where both the instructor and the participants provided comparable amounts of talk.
This chapter provided a description of the instructor’s decisions as he successfully supported the developing understanding of PSTs in a Math-Talk Learning Community where the instructor facilitated the development of understanding through talk. Specifically, this chapter provided examples of instructor decisions that facilitated components of a Math-Talk Learning Community, and how those decisions played out in the classroom.
CHAPTER 5: CONCLUSION

The purpose of this study was to examine the instructional decisions and actions of a knowledgeable instructor as he supported pre-service teachers (PSTs) in developing understanding in an undergraduate mathematics class. This research specifically looked at the instructor’s decisions during two lessons on geometric measurement that focused on the justification of the area formulas for four shapes: parallelograms, triangles, trapezoids, and circles. The lessons, Area Formulas I and Area Formulas II, occurred across three observed class sessions in the spring of 2013.

The sample for this research consisted of 24 undergraduate elementary PSTs participating in a mathematics course at a private institution in the Northeastern United States. The course, part of a two-course sequence, was selected due to the instructor’s experience working with PSTs, his experience teaching the class, and his experience with the instructional materials used in the course. The materials were developed as part of the Elementary Pre-service Teachers Mathematics Project (EMP) (NSF 2009-2011, 2013-2015, PI Chapin) and were designed to develop future teachers’ specialized content knowledge of mathematics (Chapin et al., 2011) using discussions as the primary pedagogy.

The study was a mixed-methods study with two main sources of data. The first source of data were pre-/post-tests on area formulas used to provide summary information about participants’ understanding of area formulas. The Area Formulas (AF) Test, administered before the observed lessons, and one week after the observed lessons, consisted of four questions on parallelograms, triangles, trapezoids, and circles.
Participants were asked to draw examples of the shape, provide the formula for the shape, and then justify the formula for the shape. The AF Pre-test and Post-test were scored according to a rubric designed with input from the instructor, and the main developer of the EMP instructional materials that were used in the course.

The second source of data were video data collected as part of the observed lessons and instructor interviews. Videos of the two observed lessons were recorded and transcribed by the researcher. The transcription focused on the whole-class discussions that were conducted. During whole-class discussions, the participants constructed a definition for area, discussed the choice of units for area, and justified the area formula for a parallelogram, triangle, trapezoid, and circle. The justifications for the area formulas for triangles and circles made use of the area formula for a parallelogram. The justifications for the area formula for trapezoids referenced the area formulas for parallelograms and triangles.

In addition to the observed lessons, six instructor interviews were conducted and recorded. One interview was conducted before the observed lessons. During this pre-interview, the instructor was asked to reflect on his teaching, on how he elicited explanations or justifications from participants, and on his expectations for the observed lessons. Following each lesson, video-stimulated recall interview sessions were conducted where the instructor watched video of himself facilitating whole-class discussions. First, the instructor was asked to reflect upon the lesson, if he reached his goal of eliciting explanations, and what he might do differently. Then, the instructor watched the lesson and was specifically asked to pause the video and reflect upon any
points within the class where he believed that an instructional decision was being made. The researcher had created a list of possible instructional decision points prior to the interview, which were also used as a guide for when to pause the video and have the instructor reflect. A post-interview was conducted with the instructor where he was asked to comment on his teaching, including his motivation for using particular strategies and talk moves.

Data were examined in a number of ways. First, AF Pre-test and Post-test cumulative scores, and individual question scores, were compared using t-tests and descriptive statistics. Next, using video time stamps, the observed lessons were analyzed for time usage by format and by community member (instructor or participants). Then, using the emergent themes from the instructor interviews and related literature, a description was constructed of the instructor’s decisions as he supported developing participant understanding. The research question that was addressed in this study was:

What are the instructional decisions made by a knowledgeable instructor of a mathematics course as he supports pre-service elementary teachers in developing their understanding of key instructional content?

Section I of this chapter offers an explanation of the study’s findings. In Section II, the limitations of this study are provided. Lastly, Section III addresses suggestions for future research.
Section I: Study Findings

The 2012 *Mathematical Education of Teachers II* (MET II) report released by the Conference Board of the Mathematical Sciences (CBMS) recommended that all educator programs provide experiences for pre-service teachers (PSTs) to reason about mathematical ideas. However, there is a dearth of research on how instructors help PSTs to develop mathematical understanding. Prior studies have looked at the use of curricula, teacher moves, teacher strategies, and teacher questions in developing understanding about mathematical ideas with K-12 students (Boaler & Brodie, 2004; Franke, Webb, Chan, Ing, Freund & Battey, 2009; Hiebert & Wearne, 1993; Imm & Stylianou, 2012; Martino & Maher, 1999). Additionally, research has considered questions as a mechanism for PSTs to reflect on their own developing instruction (Boerst, Sleep, Ball & Bass, 2011). The research on how an instructor can support PSTs in developing reasoning around mathematics ideas, and how this support is enacted in undergraduate classrooms, is lacking.

The instructor in this course was able to increase PSTs’ mathematics understanding of area formulas through his instructional decisions. Gain scores between the Area Formula Pre-test and Post-test were used as a proxy to measure increased understanding. Participant achievement on the AF Post-test provided evidence of increased understanding suggesting the instructor’s efforts were successful. Through instructional decisions, such as choice of curricular materials, implementing discussion-based instruction, facilitating discussions focused on providing complete justifications, and using talk moves, evidence suggests that the instructor was able to support PSTs in
developing understanding of area formulas. To understand the decisions the instructor made to support participant understanding varying levels of instructional decisions are explored.

The instructor constantly made instructional decisions to support the PSTs in developing understanding. In order to highlight these decisions, three levels of instructional decisions are considered: high-level, mid-level and micro-level. High-level instructional decisions are those decisions that the instructor decided before entering the classroom. Mid-level decisions are those decisions that the instructor made which determined the in-class flow of instruction, maintained classroom norms and ensured that mathematical learning goals were met. Mid-level instructional decisions were often maintained through the accumulation of micro-level instructional decisions. Micro-level instructional decisions were those individual and distinct decisions made at a specific moment in time and include the explicit questions, statements, or choices made by the instructor to support PST understanding.

5.1.1 High-level Instructor Decisions

There were different levels of instructional decisions made by the instructor in order to support the educational and social goals of developing participant understanding around key instructional topics. At the highest level were decisions the instructor decided before instruction had begun. The instructor made an academic instructional decision in his choice of curriculum. The curriculum that was chosen required participants to develop understanding through scaffolded activities focused on explaining and justifying
geometric measurement topics. In addition, discussion points were built into the curriculum to ensure that participant discussion focused on key instructional content.

The instructor made a social instructional decision to have ideas developed through participant discussion. As previously stated, the instructor explained, "my goal is, across the board, to have students articulate their own understanding, their own methodologies, their own ways of thinking about it and having others respond to that" [pre-interview]. The instructor’s comments suggested that participant talk would be an important part of the intended classroom instruction. The instructor’s comments allude to the creation of norms where the entire community of PSTs would engage in talk around mathematical ideas as they worked to develop understanding. This indicated that PSTs would need to be able to explain their ideas, and be ready to respond to questions about their ideas. The creation of these norms and the maintaining of this talk community compose the next level of instructor decisions: mid-level decisions.

5.1.2 Mid-level Instructor Decisions

The mid-level instructional decisions are those overarching decisions that maintain the overall flow of the classroom environment. For this instructor, his mid-level decisions, as implemented, extended from those at the highest level. Academically, he pushed participants to provide thorough and complete justifications for key instructional topics. Furthermore, the instructor utilized the curriculum in alignment with the implementation proposed in the curricular materials. Socially, the instructor continuously worked to include all participants in the sense making process and to sustain norms of community responsibility. These various mid-level decisions are explored here.
One of the key mid-level decisions, implemented by the instructor, was his focus on discussion-based instruction. As previously stated, during the pre-interview, the instructor explained that his goal was to get the class to interact with “key mathematical ideas through small group and whole-class discussions” [pre-interview]. As a testament to this, the class spent 212 minutes, or 91% of class time, discussing the two area formula lessons, in small group and whole-class discussion, developing a definition for area and justifying the area formula for parallelograms, triangles, trapezoids, and circles.

Both participants and the instructor were contributors to the discussion-based community. A time analysis of interactions during whole-class discussion indicated that both the instructor and the participants contributed approximately equally to the mathematical discussions. In fact, participants held the floor for slightly more time in whole-class discussion than the instructor (54% to 46%), which suggests that the participants were active contributors to the class discussion. This makes sense; if participants are to discuss ideas, then they are going to require more classroom time to do so. This lesson-time breakdown is in contrast to the classical vision of a mathematics classroom, where the instructor predominantly lectures and the participants are passive recipients (Fritze & Nordkvelle, 2003). The instructor implemented instruction such that there was a focus on participant contributions. In addition, the instructor made sure that discussions focused on participants’ explanations and justifications of key content.

The instructor’s use of discussion-based learning aligns with the current research where there is a focus on using talk or classroom discussion as a mechanism to support learners to develop more thorough understanding (CBMS, 2012; National Research
Council, 2005b). The distribution of discussion times in the class is an important component of discussion-based learning. However, the role that the instructor assumed in the discussion-based community was also an important part of ensuring discussion was productive. In order to support the developing understanding of participants, the instructor needed to assume a different role than that of lecturer found in the classic undergraduate mathematics course (Bergsten, 2007).

The instructor used discussion as the primary methodology, and he made an instructional decision to act as a facilitator of the discussions. In assuming the role of facilitator, the instructor supported participants by guiding the construction of ideas. During instructor interviews, the instructor specifically explained that he did not want the PSTs in his class to view him as the source of stated mathematical ideas. Within the class, PSTs were the predominant source of mathematical ideas (with support from the instructional materials and the instructor). Furthermore, in supporting participants in constructing understanding, the instructor allowed participants to grapple with ideas instead of just confirming provided statements. Overwhelmingly, the instructor required that the participants reason about the mathematics; he rarely provided verification of correctness or answers.

The instructor’s mid-level instructional decisions aligned with current ideology around instruction in mathematics education focused on sense making. For instance, in How People Learn Mathematics, the National Research Council provided characteristics of effective classroom environments. germane to this research was the recommendation to create a community-centered environment where instruction “draws out and builds on
student thinking” (National Research Council, 2005b, p. 242). In addition, using participants’ ideas in instruction was also emphasized within the Math-Talk Learning Community trajectory framework presented by Hufferd-Ackles, Fuson and Sherin (2004). Sense making as a component in developing understanding means that the instructor needed to push participants to provide thorough and complete justifications as a community, and to support each other in this endeavor.

As part of constructing a community-centered environment, the instructor made decisions such that PSTs were required to shoulder a greater responsibility for classroom learning. In order to engage participants in this process, the instructor reminded the participants of their status as future teachers and their responsibility to be active contributors to the learning environment. In guiding participants to question explanations and justifications, there were 11 different occasions, across the two lessons, where the instructor mentioned the PSTs’ future role as educators as a means of pushing the participants to provide questions for each other. Having PSTs consider the mathematics from the perspective of a teacher is a recommendation found within the MET II report (CBMS, 2012). Additionally, as part of reminding participants of their roles as future educators, the instructor asked PSTs to consider, and ask, questions that their future students might ask. Guiding participants to ask questions also aligns with the Math-Talk Learning Community trajectory framework of responsibility for learning which states that both the instructor and the participants should be responsible for asking questions and supporting understanding (Hufferd-Ackles, Fuson & Sherin, 2004).

The mid-level instructional decisions are those overarching decisions maintained
throughout the lesson. The instructor in this study made a number of decisions to support both the academic goals of developing complete justifications, and the social goals of implementing a talk based community. The next level of instructor decisions is: micro-level decisions. The micro-level decisions are those “in the moment” decisions, made by the instructor, that support the constructed norms of community involvement, responsibility for learning, and fully developing understanding.

5.1.3 Micro-level Instructor Decisions

The micro-level instructional decisions are those individual decisions made at any one moment in time. Micro-level decisions could be actions, or strategies used by the instructor throughout the lesson. Micro-level decisions are often implemented to support mid-level and high-level instructional decisions.

The choice of individual talk moves was an example of micro-level instructional decisions. As part of developing participants’ reasoning abilities, the instructor needed to support participants in providing full explanations. In accomplishing this task, the instructor used talk moves such as “wait time,” “turn-and-talk,” and “revoicing” in his instruction (Chapin, O’Connor & Anderson, 2013) to allow participants time to formulate responses, practice providing a response, or hear ideas again. During whole-class instruction, the instructor restated or revoiced previously provided mathematical statements in 40% of his talk turns. Through restating or “revoicing” mathematics statements, the instructor allowed participants more time to identify the key ideas and to focus in on especially relevant contributions to the discussion (Chapin, O’Connor & Anderson, 2013).
During interviews the instructor specifically spoke about using the “press for reasoning” talk move to guide participants to explain ideas. When implementing the “press for reasoning” move, the instructor asked participants to clarify or build upon their ideas, justify their ideas or generalize when appropriate. The instructor implemented this move in 48% of his overall talk turns (139 out of 346) and received either a verbal or physical participant response. In almost half of the times the instructor utilized the “press for reasoning” move, he pressed participants as responsible members of the learning community. The instructor pressed a participant to respond to the mathematics that another participant had provided in 22% of his overall turns. Participants needed to be prepared to not only engage with their own explanations, but also the explanations of others. Previous research had found that probing questions (Franke, Webb, Chan, Ing, Freund & Battey, 2009) and conceptual questions (Imm & Stylianou, 2012) can lead to better understanding.

Other micro-level decisions involved the instructor’s selection and ordering of participant contributions. The instructor selected participants to talk in whole-class discussion for a number of reasons including explanations in small group, participant overall comfort with the content, and to ensure equitable participation. These selections were also sometimes influenced by time constraints. Additionally, the instructor made decisions when he reminded participants of their roles as contributors by emphasizing who provided contributions. These decisions were motivated by the instructor’s desire to remind the participants of their responsibility for learning and recognize participants as the source of mathematical ideas.
In summary, in order to develop participants’ understanding of area formulas, the instructor made high-level instructional decisions about his choice of curriculum and his use of discussion as the primary instructional methodology. As a result of his high-level decisions, the instructor made mid-level instructional decisions to facilitate discussion about mathematics and to push for complete justifications. In facilitating mathematics discussion, the instructor expected that participants would provide mathematical ideas, grapple with the mathematics, and that they would verify and/or establish the truth of the mathematical statements as a class. As a way to motivate participants, the instructor reminded the PSTs of their future role as teachers. Finally, the instructor made micro-level decisions such as when to use talk moves and who to call upon. In combination, all of these level of decision making enabled the instructor to build a community of learners focused on developing understanding about mathematical ideas.

Section II: Limitations of the Study

The findings of this research study must be evaluated in the context of the study as designed. Limitations for this study must be considered. The specific study limitations are described below.

- The study only considered developing understanding in the mathematics area of geometric measurement. Geometry was selected due to its focus on reasoning and proof. Study findings might vary in other areas of mathematics instructional content.
• This research only considered one class taught by the instructor. Different classes have different class dynamics, which could influence studying findings.

• Only two lessons were observed as part of this research. A longer observation period with more class data could help to corroborate or elaborate on the findings in this study.

• In viewing his classroom instruction, as part of the VSR interview sessions, the instructor was asked to reflect on his teaching. This reflection may have caused the instructor to alter the way he taught in the recorded lessons that followed the first VSR interview session.

• The Area Formula Pre-test and Post-test were used as a proxy to measure understanding gained during instruction. The scores showed that participants were able to provide more details for the justification for the area formulas for four shapes. However, it is possible that the gains in growth may be reflective of the participants’ ability to repeat details, and may not necessarily be a growth in participant understanding.

• Although participants were given unlimited in-class time to take the pre-test and post-test, the pre-test results were not used as part of an in-class grade. This factor may have influenced the commitment of the participants in providing full and complete answers on the pre-test.

• The research only looked at instructor decisions that took place during the whole-class discussion period. It is possible that there were relevant instructor decisions that occurred during the small group work that were not represented in this data.
Section III: Recommendations for Future Research

The following are recommendations for future research, which were constructed based on the findings of this study and its limitations.

- This was a small study that only examined the instructor decisions during two lessons on geometric measurement with one class. Future research could study developing understanding for more lessons or with a different group of subjects. Examining different classes, or extending the research to include more lessons in the area of geometric measurement, could contribute to the study’s findings.

- The current study focused on the instructor’s facilitation of explanations and justifications around geometric measurement topics. Future research could consider how an instructor supports reasoning about the mathematics, and maintains a high level of participant contributions, on a mathematical topic other than geometric measurement.

- This research focused on an instructor’s decisions. Additional research that elaborates on PSTs’ responses, as a result of instructor decisions, could contribute to literature on developing the mathematics understanding of elementary PSTs.

- This research only focused on whole-class discussion. Future research could also look at the instructor’s decisions in the small group setting. It is possible that instructor interactions, which occurred in small group, may influence the orchestration of productive discussion in the whole-class setting. Furthermore, there may be strategies that the instructor implements in the small group setting that could contribute to the literature.
Area Formulas I Task
Question 1
1. Come up with a strategy to measure
the area of the provided shape, and
then used that strategy to estimate the
area of the shape.

Group Discussion Questions:
• Describe your strategy for measuring the area of the shape. What unit of measure
did you employ? Why did you choose this specific unit of measure?

Whole-class discussion key points:
• There are various strategies for finding area. Among them are: partitioning a
shape into known shapes that you can find the area of, determining a larger area
and subtracting the area not needed, and counting grid squares.
Question 2

2. Compare the gridded work of two students who used different measures of square units. Explain why the students would think the left shape has a larger area.

![Grids showing different measurements](image1.png)

Question 3

3. A student claims that the area of the rectangle below is 150. Is he correct? Why or why not.

![Rectangle with measurements](image2.png)

Group Discussion Questions:

- Define area.

- How does the size of a square unit affect area measurements?

Important small group content:

- The formula for the area of a rectangle is \( \text{length} \times \text{width} \).
- The units for each dimension should be the same to calculate area.
- Choosing a smaller square unit for dimensions (cm\(^2\) vs. m\(^2\)) will result in a larger numerical number for area, but not a larger value for area.
Whole-class discussion key points:

- Area is the number of equal-sized square units that cover a figure.
- The size of the square unit chosen to determine area will affect the numerical number of squares needed to cover a figure.
- The same square unit should be used for dimensions to determine area.

Question 4

4a. On the provided grid paper, draw 5 different parallelograms with a base of 4 units and a height of 3 units (including one rectangle).

4b. Use these parallelograms to explain why the formula for the area of any parallelogram is $A=bh$, where $b$ is the length of the base and $h$ is the height.
Important small group content:

- Each of the parallelograms, with a base of 4 units and a height of 3 units, can be redrawn to form a rectangle with a base of 4 units and a height of 3 units. (We can redraw each parallelogram into a rectangle by shifting a triangle section. The figure is a rectangle because it is a quadrilateral with opposite sides that are congruent and parallel.) The rectangle has the same base and height as the original parallelogram. The area of the original parallelogram is the same as the recomposed rectangle. Since $Area_{rect} = l \times w = base_{para} \times height_{para}$, then $Area_{para} = base_{para} \times height_{para}$.

If the height is outside the figure, the process may have to be repeated more than once.
Question 5

5. Cut out 2 copies of each of the provided triangles from the end of the packet (2 right triangles, 2 isosceles triangles, 2 scalene triangles, 2 equilateral triangles). Use the pairs to form parallelograms. Derive a formula for the area of a triangle using these parallelograms. Explain.

Group Discussion Questions:

- Provide a convincing argument for why the area of a parallelogram is \( \text{base} \times \text{height} \).
- Explain why your formula for the area of a triangle is correct for all triangles.
Important small group content:

• Every triangle can be matched with a congruent version of itself. Translate one of the triangles in such a way that two congruent sides from each triangle match up to form a quadrilateral. (By matching up congruent corresponding triangle sides, a new shape is formed. The new shape formed is a parallelogram since it is a quadrilateral with congruent opposite angles [from the congruent triangles].) The area of a parallelogram is base × height. Since the parallelogram is formed by two congruent triangles, half of the area of the parallelogram will be the area of one triangle.

The area formula of a triangle is \( \frac{1}{2} \) (triangle base length) × (triangle height).

Whole-class discussion key points:

• The area formula of a parallelogram is base × height. [See important small group content, question 4.]

• The area formula of a triangle is \( \frac{1}{2} \times \text{base} \times \text{height} \). [See important small group content, question 5.]
Area Formulas II Task

Question 1

1. Form a rectangle using copies of congruent trapezoid pairs (2 right trapezoids, 2 isosceles trapezoids and 2 trapezoids). Using the area formula for a parallelogram, derive an area formula for any trapezoid and explain why it makes sense. (Use \( b_1, b_2 \) and \( h \).)

**Important small group content:**
- The area formula of a parallelogram is \( A = bh \). Any trapezoid can be combined with a congruent version of itself and used to form a parallelogram. One set of congruent sides (not the parallel base sides) are matched up. The opposite angles in the newly formed quadrilateral should be congruent. Since opposite angles are congruent, a parallelogram is formed.

![Diagram of parallelogram formed from trapezoids]

Since the parallelogram is composed of two trapezoids, half of that area will be the area of one trapezoid. The area of the created parallelogram is: \( A = (b_1 + b_2)h \); half of that area, the area of a trapezoid, is: \( A = \frac{1}{2}(b_1 + b_2)h \).
Question 2

2. How can the trapezoid below, which has a height of 7, be decomposed in order to find its area?
Important small group content:

- A trapezoid can be divided into two triangles. The area of two triangles is the same as the area of the original trapezoid.

\[ \text{Area}_{\text{Trapezoid}} = \text{Area}_{\text{Triangle1}} + \text{Area}_{\text{Triangle2}} \]

If \( \text{Area}_{\text{Triangle1}} + \text{Area}_{\text{Triangle2}} \)
\[= \frac{1}{2} (10 \text{ units})(7 \text{ units}) + \frac{1}{2} (18 \text{ units})(7 \text{ units}) \]
\[= (5 \text{ units})(7 \text{ units}) + (9 \text{ units})(7 \text{ units}) \]
\[= 35 \text{ units sq.} + 63 \text{ units sq.} \]
\[= 98 \text{ units sq.} \]
then,
\[ \text{Area}_{\text{Trapezoid}} = 98 \text{ units sq.} \]

The area of the entire trapezoid is 98 units squared.

OR

A trapezoid can be divided into two triangles and a rectangle. The sum of the areas is the same as the original trapezoid.

\[ \text{Area}_{\text{Trapezoid}} = \text{Area}_{\text{Tri}} + \text{Area}_{\text{Rect}} \]
\[= \frac{1}{2} (18 \text{ units} - 10 \text{ units})(7 \text{ units}) + (10 \text{ units})(7 \text{ units}) \]
\[= \frac{1}{2} (8 \text{ units})(7 \text{ units}) + (10 \text{ units})(7 \text{ units}) \]
\[= \frac{1}{2} (56 \text{ units sq.}) + 70 \text{ units sq.} \]
\[= 98 \text{ units sq.} \]
Group Discussion Question:

• Using the trapezoid below, find a formula for its area and explain why it makes sense.
Whole-class discussion key points:

- The area formula for any trapezoid is \( A = \frac{1}{2}(b_1 + b_2)h \). This formula can be explained using the important small group content found in question 1. However, it is also possible that students will derive the formula by decomposing the parallelogram into triangles as presented, but not explained, in question 2. The area formula of a triangle is \( \frac{1}{2} \times (\text{base}) \times (\text{height}) \). Any trapezoid can be divided into two triangles by drawing a diagonal. The sum of the areas of the two triangles is equal to the area of the original trapezoid.

\[
\text{Area}_{\text{trapezoid}} = \text{Area}_{\text{triangle}1} + \text{Area}_{\text{triangle}2}
\]

\[
= \frac{1}{2} b_1 h + \frac{1}{2} b_2 h
\]

\[
= \frac{1}{2} h (b_1 + b_2)
\]

Then, \( A_{\text{trapezoid}} = \frac{1}{2} (b_1 + b_2)h \)

- It is also true that any trapezoid can be divided into two triangles and a rectangle by introducing two altitudes. The sum of the areas of the individual shapes is equal to the area of the original trapezoid.

\[
\text{Area}_{\text{trap}} = \text{Area}_{\text{tri}} + \text{Area}_{\text{rect}}
\]

\[
= \frac{1}{2} (b_2 - b_1)h + bh
\]

\[
= \frac{1}{2} b_2 h - \frac{1}{2} b_1 h + bh
\]

\[
= \frac{1}{2} b_2 h + \frac{1}{2} b_1 h
\]

\[
= \frac{1}{2} h (b_1 + b_2)
\]
Question 3

3. Using a circle decomposed into 16 sector slices, answer the following questions.

3a. Compose the 16 slices you have been given into a circle. Give different examples of the radius of this circle. Draw a picture of the examples.

3b. Compose the 16 slices into another familiar shape whose area formula you have already derived. Draw the new shape.

3c. Determine the new shape’s dimensions and explain why those dimensions are correct.

3d. What is your new shape’s area?

3e. Explain what the shape would look like if the circle was cut into 25, 40, or 100 slices instead of 16?

Group Discussion Question:

- Why is the area formula for a circle $A = \pi r^2$?
Whole-class discussion key points:

- The area of a parallelogram that is composed of the sectors from a circle, cut into an infinite amount of even-sized sectors, will have an area that is equal to the area of the original circle. This is because both shapes are composed of the same identical sectors.

- A circle can be decomposed into an even number of sectors and recombined to form a parallelogram. If an infinite number of sectors are used, the parallelogram will have bases that approximate a line segment. The base of the parallelogram will measure $\pi r$, or half the circumference. The height of the parallelogram will measure $r$, or the radius of the circle. The area formula for a parallelogram can be used to provide an area that is the same as the total area of the original circle. The area formula of a parallelogram is $A = bh$. Substituting in the dimensions of the created parallelogram, the area of a circle is $A = \pi r = \pi r^2$. 

![Diagram of a circle decomposed into sectors forming a parallelogram](image)
APPENDIX B: AREA FORMULA PRE-/POST-TEST RUBRIC
1. 

a. Draw three different parallelograms.

b. Provide the formula for the area of any parallelogram.

c. Provide a convincing argument for the area formula you stated.

*Complete Scoring Rubric for Pre- and Post-test Question 1 on the Area Formula for a Parallelogram*

<table>
<thead>
<tr>
<th>Pts.</th>
<th>Explanation Requirement</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing drawings of three different parallelograms.</td>
<td>[Images must suggest quadrilaterals with two sets of parallel sides. The image does not need to be labeled to show congruency or parallel sides.]</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a parallelogram is $A = bh$.</td>
<td>The area formula for a parallelogram is $A = bh$. [Note: The answer cannot be $A=l*w$, unless the participant makes a direct connection between the base and height of the shape with length and width either verbally or with an image.]</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a parallelogram can be decomposed.</td>
<td>A parallelogram can be decomposed into a triangle and a trapezoid. OR</td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Diagram of parallelogram decomposed into a triangle and a trapezoid. Note: The image does not need to have a clear right angle marked.]</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that the decomposed parts of a parallelogram can be recomposed into a rectangle.</td>
<td>A triangle from one side of the parallelogram can be shifted to the other side to form a new shape. The hypotenuse of triangle will be matched with its congruent pair on the other side of the parallelogram.</td>
</tr>
</tbody>
</table>
Specifically explaining how the triangle can be moved and reattached.

| 1 | Showing or stating that the decomposed parts of a parallelogram can be recomposed into a rectangle. Note: Participants must specifically state why the new shape is a rectangle. | OR

The constructed shape is a quadrilateral with four right angles, so it is a rectangle.

There were three right angles created from the drawn altitude which are also right angles in the newly constructed rectangle. One right angle is from the shifted right triangle. Two right angles were from inside the non-triangle shape. Because three of the angles of the reconstructed quadrilateral are right angles, the fourth also had to be a right angle and the shape is a rectangle.

OR

The shape is a rectangle.

[Note: The participant needs to show at least three right angles in the constructed rectangle. The participant does not need to show congruency for this step.]
Showing or explaining that one measure of the newly constructed rectangle is equivalent to the height of the original parallelogram, and that the base of the rectangle is the same as the base of the original parallelogram.

The height of the rectangle is the same as the height of the original parallelogram. This means that one dimension of the rectangle is the same as the height of the original parallelogram. The other dimension for the constructed rectangle will be the base measure of the original parallelogram. Although a section is shifted, the base length remain the same.

| 1 | Showing or explaining that since the recomposed shape is a rectangle, the area formula for a rectangle can be used to find the area of the original shape. |
|   | The area of the original parallelogram can be determined by using the area formula for a rectangle or \( \text{length} \times \text{width} \), and substituting in the corresponding known measures of base and height. So, \( \text{Area} = \text{base} \times \text{height} \). |

OR

Explaining how the area of a rectangle is determined by using the length and width to find the number of square units in the rectangle.

\[
\text{Area}_{\text{Rect}} = \text{length} \times \text{width} = \text{base} \times \text{height}
\]

\[
\text{Area}_{\text{Para}} = \text{Area}_{\text{Rect}} = \text{base} \times \text{height}
\]

OR
<table>
<thead>
<tr>
<th>Shape.</th>
<th>The area of a rectangle is length times width, which gives you the number of rows times the number of columns or the number of square units that cover a rectangle.</th>
</tr>
</thead>
<tbody>
<tr>
<td>OR</td>
<td></td>
</tr>
<tr>
<td><img src="image" alt="Diagram" /> width</td>
<td>length</td>
</tr>
</tbody>
</table>
2.

   a. Draw three different triangles.

   b. Provide the formula for the area of any triangle.

   c. Provide a convincing argument for the area formula you stated.

*Complete Scoring Rubric for Pre- and Post-test Question 2 on the Area Formula for a Triangle*

<table>
<thead>
<tr>
<th>Pts.</th>
<th>Explanation Requirement</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing drawings of three different triangles.</td>
<td>[Images must suggest three different triangles. Similar triangles are ok. ]</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a triangle is ( A = \frac{1}{2}bh ).</td>
<td>The area formula for a triangle is ( A = \frac{1}{2}bh ).</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a triangle can be duplicated.</td>
<td>A triangle can be duplicated so that there are two congruent triangles. OR</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[Note: Participant must show or state that the triangles are congruent. ]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>OR</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A triangle can be duplicated so that there are two identical triangles.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating how the two congruent triangles can be reformed.</td>
<td>The two triangle are connected along two sides that are congruent. OR</td>
</tr>
<tr>
<td></td>
<td>Showing or stating that the two congruent triangles are formed into a parallelogram.</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>----------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The new shape is a parallelogram because it is a quadrilateral with opposite sides that are parallel.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>If the alternate interior angles are congruent for a transversal crossing two lines, then the lines are parallel. This can be shown for both sets of parallel sides.</td>
<td></td>
</tr>
<tr>
<td>OR</td>
<td>The new shape is a parallelogram because it is a quadrilateral with congruent opposite angles and at least one set of congruent opposite sides.</td>
<td></td>
</tr>
<tr>
<td>OR</td>
<td>The new shape is a parallelogram because it is a quadrilateral with at least one set of parallel sides, and opposite sides are congruent. (If the alternate interior angles are congruent for a transversal crossing two lines, then the lines are parallel.)</td>
<td></td>
</tr>
<tr>
<td>OR</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 1 | Showing or explaining that one measure of the parallelogram is determined by the triangle height, and the other is from the triangle base.  
Note: A connection must be made between the base and height of an original triangle and the constructed shape. |
|---|---|
| | **The height of the parallelogram is the same as the height of the original triangle. The base of the parallelogram is the same as the original triangle.**  
**OR** |
| 1 | Showing or explaining that since the recomposed shape is a parallelogram, composed from two congruent triangles, half the area formula for a parallelogram can be used to find the area of one triangle. Known dimensions will be substituted into the formula.  
(Note: Participants can still receive this point even if the participant start with a parallelogram and are incorrectly working backwards from that.) |
| | The area of any parallelogram is \( \text{base} \times \text{height} \). The base and height of the constructed parallelogram correspond to the dimensions of base and height in the original triangle.  
\[ \text{Area} = \text{base} \times \text{height} \]  
Since the parallelogram is composed of two congruent triangles, the area of one triangle can be determined by finding half the area of the parallelogram.  
\[ \text{Area}_{\text{tri}} = \frac{1}{2} \times (\text{base} \times \text{height}) = \frac{1}{2}bh \] |
3.

a. Draw three different trapezoids.

a. Provide the formula for the area of any trapezoid.

b. Provide a convincing argument for the area formula you stated.

NOTE: In a case where a participant provides more than one method, each method will be scored separately. The final score will be determined by the method with the most points.

Method 1 - Complete Scoring Rubric for Pre- and Post-test Question 3 on the Area Formula for a Trapezoid

<table>
<thead>
<tr>
<th>Pts.</th>
<th>Explanation Requirement</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing drawings of three different trapezoids.</td>
<td>[Images should suggest three different quadrilaterals with only one set of parallel sides. The images do not need to show congruency or parallel-ness.]</td>
</tr>
<tr>
<td>1</td>
<td>Stating the area formula for a trapezoid is $A = \frac{1}{2} (b_1 + b_2) h$.</td>
<td>The area formula for a trapezoid is $A = \frac{1}{2} (b_1 + b_2) h$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>OR</td>
</tr>
<tr>
<td></td>
<td></td>
<td>The area formula for a trapezoid is $A = \left( \frac{b_1 + b_2}{2} \right) h$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>OR</td>
</tr>
<tr>
<td></td>
<td></td>
<td>The area formula for a trapezoid is $A = \frac{1}{2} (b_2 - b_1) h + b_1 h$.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a trapezoid can be duplicated.</td>
<td>A trapezoid can be duplicated so that there are two congruent trapezoids.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>OR</td>
</tr>
</tbody>
</table>
|   | Showing or stating that the two trapezoids can be reformed into a parallelogram. [Note: The participant must explain why it is a parallelogram.] | The trapezoids are congruent.
[Note: The participant must say that the two trapezoids are congruent or that the same exact trapezoid is used. Or the participant must show congruency in the drawing.] |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The new shape is a parallelogram because it is a quadrilateral with opposite sides that are parallel. If the alternate interior angles are congruent for a transversal crossing two lines, then the lines are parallel. This can be shown for both sets of congruent sides. OR</td>
</tr>
<tr>
<td></td>
<td>The new shape is a parallelogram because it is a quadrilateral with opposite congruent angles and at least one set of congruent opposite sides. OR</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining how the dimensions of the</td>
</tr>
</tbody>
</table>
| parallelogram relate to the dimension of the original trapezoid. [Note: Participants must clearly connect trapezoid base1, trapezoid base2, and trapezoid height to the original parallelogram dimensions.] | of the parallelogram is the sum of $b_1$ from one trapezoid and $b_2$ from the congruent trapezoid. This means that one dimension of the parallelogram is the trapezoid height and the other is $b_1 + b_2$.

**OR**

![Diagram of a parallelogram formed by two congruent trapezoids]

[Note: If the participant uses a right trapezoid, the participant must clearly state that the height is the same in both the original trapezoid and the constructed parallelogram.]

| Showing or explaining that since the recomposed shape is a parallelogram, composed from two congruent trapezoids, half the area formula for a parallelogram can be used to find the area of one triangle. | The area formula for a parallelogram, $A=bh$, can be used to determine the area of the constructed parallelogram. The area of the parallelogram will be

$A=bh = hb = height_{trap} \times (b_1 + b_2)$.

Since the parallelogram is composed of two congruent trapezoids, the area of one trapezoid can be determined by finding half the area of the parallelogram, or

$Area_{T_{trap}} = \frac{1}{2} \times (height_{trap}) \times (b_1 + b_2) = \frac{1}{2} h (b_1 + b_2)$.

| 1 |  |
Method 2 - Complete Scoring Rubric for Pre- and Post-test Question 3 on the Area

Formula for a Trapezoid

<table>
<thead>
<tr>
<th>Pts.</th>
<th>Explanation Requirement</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing drawings of three different trapezoids.</td>
<td>[Images should suggest three different quadrilaterals with only one set of parallel sides. The images do not need to show congruency or parallel-ness.]</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a trapezoid is $A = \frac{1}{2} (b_1 + b_2) h$.</td>
<td>The area formula for a trapezoid is $A = \frac{1}{2} (b_1 + b_2) h$. OR $A = \frac{(b_1 + b_2)}{2} h$. OR The area formula for a trapezoid is $A = \frac{1}{2} (b_2 - b_1) h + b_1 h$.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a trapezoid can be separated so that there are two triangles.</td>
<td>A trapezoid can be decomposed so that there are two triangles. OR</td>
</tr>
<tr>
<td>1</td>
<td>Showing or explaining how to determine the dimensions of each triangle.</td>
<td>The height and base dimensions of one triangle will be the height of the original trapezoid and the measure of $b_2$. The height and base dimensions of the second triangle will be the height of the original trapezoid and the measure of $b_1$. OR</td>
</tr>
</tbody>
</table>
Showing or explaining that the area of the original trapezoid is the sum of the areas of the two triangles.

The area of a triangle is base $\times$ height. The area of the trapezoid will be the sum of the two triangle areas.

$$Area_{\text{trap}} = Area_{\text{Tri}1} + Area_{\text{Tri}2}$$

Showing the mathematical steps to simplify the trapezoid formula to $A = \frac{1}{2} (b_1 + b_2)h$.

[Note: Although the participant does get a point above for stating the simplified formula, the participant should receive another point here for simplifying correctly.]

The area of a triangle is base $\times$ height. The area of the trapezoid will be the sum of the two triangle areas.

$$Area_{\text{trap}} =$$

$$Area_{\text{Triangle}1} + Area_{\text{Triangle}2}$$

$$= \frac{1}{2} b_1 h + \frac{1}{2} b_2 h$$

$$= \frac{1}{2} h (b_1 + b_2)$$

$$= \frac{1}{2} (b_1 + b_2)h$$
Method 3 - Complete Scoring Rubric for Pre- and Post-test Question 3 on the Area Formula for a Trapezoid

<table>
<thead>
<tr>
<th>Pts.</th>
<th>Explanation Requirement</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing drawings of three different trapezoids.</td>
<td>[Images should suggest three different quadrilaterals with only one set of parallel sides. The images do not need to show congruency or parallel-ness.]</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a trapezoid is ( A = \frac{1}{2} (b_1 + b_2)h ).</td>
<td>The area formula for a trapezoid is ( A = \frac{1}{2} (b_1 + b_2)h ). OR ( A = \left( \frac{b_1 + b_2}{2} \right)h ). OR The area formula for a trapezoid is ( A = \frac{1}{2} (b_2 - b_1)h + b_1h ).</td>
</tr>
</tbody>
</table>
| 1    | Showing or stating that a trapezoid can be separated so that there are two triangles and a rectangle. | A trapezoid can be decomposed so that there are two triangles and a rectangle. OR \[
\begin{array}{c}
  \hline
  & \hline
  \end{array}
\]
[Note: No point should be assigned here for an example that only presents a right trapezoid. The example needs to include two triangles and a rectangle.] [Note: No points should be awarded for pointless deconstruction.] |
| 1    | Showing or explaining how to determine the dimensions of each shape.                   | The two triangles can be combined to form one triangle with a height that is the same as the original trapezoid and a base that is equivalent to the difference between the larger trapezoid base and the smaller trapezoid base (the smaller base measure is the removed rectangle length). The rectangle dimensions are the length of the smaller base measure and the height of the original |
Showing or explaining that the area of the trapezoid is the sum of the areas of all the individual shapes.

The area of the trapezoid will be the sum of the individual areas.

OR

\[ A = \frac{1}{2} (b_1 + b_2)h. \]

[Note: Although the participant does get a point above for stating the simplified formula, the participant should receive another point here for simplifying correctly.]
4.

a. Draw three different circles.

b. Provide a formula for the area of any circle.

c. Provide a convincing argument for the area formula you stated.

*Complete Scoring Rubric for Pre- and Post-test Question 4 on the Area Formula for a Circle*

<table>
<thead>
<tr>
<th>Pts.</th>
<th>Explanation Requirement</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Providing drawings of three different circles.</td>
<td>[Images should suggest three different circles.]</td>
</tr>
<tr>
<td>1</td>
<td>Stating that the area formula for a circle is $A = \pi r^2$.</td>
<td>The area formula for a circle is $A = \pi r^2$.&lt;br&gt;OR&lt;br&gt;The area formula for a circle is $A = \pi \left(\frac{d}{2}\right)^2$.&lt;br&gt;OR&lt;br&gt;The area formula for a circle is $A = \pi \frac{d^2}{4}$.</td>
</tr>
<tr>
<td>1</td>
<td>Showing or stating that a circle can be decomposed into sectors.</td>
<td>A circle can be decomposed into sectors.&lt;br&gt;OR&lt;br&gt;<img src="" alt="Circle decomposed into sectors" />&lt;br&gt;OR&lt;br&gt;<img src="" alt="Circle decomposed into sectors" />&lt;br&gt;[Note: The sectors do not need to be connected.]</td>
</tr>
</tbody>
</table>
| 1 | Showing or stating that the sectors can be recomposed into a shape that is similar to a parallelogram. | The sectors from the deconstructed circle can be recomposed into a shape that is similar to a parallelogram.  
[Note: The participants can get this point if they say the shape is “approximate to a parallelogram” “looks like a parallelogram” or “parallelogram-like shape.”]  
OR  
![Image](image1.png)  
[Note: No point if the participant leaves out sector arcs.]  
[Note: No point if the participant just says that the shape is a rectangle unless they mention that the shape is composed of an infinite number of pieces.] |

| 1 | Stating or showing that as circle is cut into a greater number of sectors the constructed shape becomes more like a parallelogram (or rectangle). | As the number of sectors approaches infinity (or increases) the shape becomes to look more like a parallelogram (or rectangle).  
OR  
If the shape is cut into a greater number of pieces the shape looks more and more like a parallelogram (or rectangle).  
![Image](image2.png)  
[Note: The participant can’t just say it “it goes to infinite.” They need to explain the importance of going to infinite.] |

| 1 | Showing or explaining that one measure of the newly constructed parallelogram (or rectangle) is equivalent to the radius of the original circle, and that the base of the rectangle is the same as half the circumference of the original circle. | The base of the shape (or parallelogram or rectangle) has the same dimension as half the circumference. The height of the shape (or parallelogram or rectangle) is the same as the radius of the circle.  
OR  
![Image](image3.png) |
<table>
<thead>
<tr>
<th></th>
<th>Showing or explaining that since the recomposed shape is a parallelogram (or rectangle), the area formula for a parallelogram can be used to find the area of the original shape.</th>
</tr>
</thead>
</table>
| 1 | The area of the original circle can be determined by using the area formula for a parallelogram or \( \text{base} \times \text{height} \) and substituting the corresponding known measures of \( \pi r \) and \( r \).

OR

\[
\text{Area}_{\text{Para}} = \text{base} \times \text{height}
\]

\[
\text{Area}_{\text{Circle}} = \text{Area}_{\text{Para}} = \pi r \times r = \pi r^2
\]

\[
A = \pi r^2
\]

[Note: Participant must say that the area of parallelogram is base multiplied by height.]

[Note: Participant must clearly state that the area of the circle is the same area of the reconstructed parallelogram.]
APPENDIX C: INTERVIEW QUESTIONS
Pre-Lesson Interview Elicitation Questions

Read to instructor: The goal of this study is to gain a deeper understanding of the pedagogical decisions made by an instructor in a mathematics course for pre-service elementary teachers. Specifically, the researcher is trying to understand how a knowledgeable instructor guides students to explain, justify and generalize around key instructional content. During the first part of this interview, I will be asking you about the course, the class, and your teaching methodology. The second part of this interview will focus on the two lessons that will be studied. Feel free to provide as much information as you would like and to add additional information that you feel is relevant. Do you have any questions?

- Please tell me about yourself and your experience working with elementary pre-service educators.
- What kind of teaching style do you feel that you embody?
- How do you feel that your style of teaching is expressed?
- How do you feel that your style of teaching is expressed in using the curriculum materials?
- The focus of this research is on pedagogical strategies used to get students to explain, justify, and generalize. When considering strategies for guiding students to explain, justify and generalize, what comes to mind?
Part II

- Please describe the Area Formulas I lesson.
- What part(s) of the lesson do you think will lead to explaining, justifying and generalizing?
- In this lesson, what area(s) do you feel are problematic for students around the processes of explaining, justifying and generalizing?

- Please describe the Area Formulas II lesson.
- What part(s) of the lesson do you think will lead to explaining, justifying and generalizing?
- In this lesson, what area(s) do you feel are problematic for students around the processes of explaining, justifying and generalizing?
Video-Stimulated Recall Session Interview Questions

Asked before watching the video:

- Tell me about the lesson.
- Did you reach your goal of getting students to explain, justify, and generalize?
- What is the evidence that that happened?
- Would you do anything differently if you had the chance in order to get students to explain, justify, and generalize?

General elicitation questions for video viewing:

- What were you doing / trying to do at this point in the lesson?
- What were you doing / trying to do at this point in the discussion?
- What were you noticing / hearing at this point?
- What were you thinking about at this point?
- Why did you make that statement?
- What do you notice about your actions at this point?
- Why did you do ... at this point in the video?
- Why did you ask that question?
Post-Interview Questions

This last interview is an opportunity to discuss the lessons, tasks, class, or video-stimulated recall sessions that occurred as part of this study. The goal of this study is to gain a deeper understanding of pedagogical decision making that occurs during a mathematics course for pre-service elementary teachers. Specifically, the researcher is trying to understand how a knowledgeable instructor guides students to explain, justify and generalize around areas of instructional focus.

- During the first interview, you said that you felt that you embodied a …teaching style. After watching the videos, would you respond the same or differently? Explain.
- After teaching both lessons, are there any moments that you recall as being particularly striking when reflecting on pedagogical practices?
- After watching the video sessions, what pedagogical practices do you consider essential to get students to explain, justify, and generalize?
- Which pedagogical strategies would you like to include more of to get students to explain, justify, and generalize?
- Do you have any additional thoughts or comments on the lessons as presented?
- Do you have any additional thoughts or comments on pedagogical practices and the work of getting students to explain, justify, or generalize?
BIBLIOGRAPHY


CBMS. See Conference Board of the Mathematical Sciences.


Eisenmann, T. & Even, R. (2009). Similarities and differences in the types of algebraic activities in two classes taught by the same teacher. In J. Remillard, B. Herbel-Eisenmann & G. Lloyd (Eds.), *Teachers’ use of mathematics curriculum materials:*


NGA & CCSSO. See National Governors Association Center for Best Practices & Council of Chief State School Officers.


Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1), 13-57.


SMTI & TLC. See Science and Mathematics Teacher Imperative & The Leadership Collaborative.


CURRICULUM VITAE

Johanna Lynn Bunn

Email: johannabunn@yahoo.com

EDUCATION

Doctor of Education  Curriculum and Teaching  Boston University  2014
Masters of Education  Secondary Education  Salem State College  2003
Bachelor of Science  Engineering  Boston University  2000

PROFESSIONAL EXPERIENCE

Salem Public School System, Salem, MA  September 2013 – Present
Certified Mathematics Teacher
- Create, present and implement creative mathematics lessons in alignment with the Massachusetts curriculum frameworks for 6th grade.

Teacher Educator - National Center for Technological Literacy
- Designed, organized, and conducted high quality, hands-on professional development sessions for teachers on science, technology, engineering and mathematics. Sessions were conducted in AL, GA, MA, ME, NC, ND, NH, TX and online for educators of various backgrounds for both professional development hours and college credit.
- Researched and developed engineering curriculum for the Engineering the Future project the first introductory secondary school technology and engineering course that mapped to MA state standards.

Boston University, Boston, MA  September 2009 – August 2013
Instructor/Teaching Assistant
- Provided lessons, and guidance for pre-service and in-service teachers on topics such as algebra, geometry, measurement and statistics in CAS MA107/108 and SED ME503/ME504. Classes were discourse-based with a focus on generating deep understanding of mathematics through hands-on activities.
Bunker Hill Community College, Boston, MA  January 2006 – May 2007
Adjunct Instructor - Mathematics
- Presented and explained mathematics concepts as aligned with standards determined by the mathematics department associated with MAT090/MAT095/MAT195.

Production Accuracy Verifier/Editor/Presenter
- Presented, wrote, reviewed, edited and verified accuracy of lessons for on-camera presentations, and educational texts of various college mathematics textbooks from Addison-Wesley.

VPG Integrated Media, Boston, MA  September 2004 – June 2007
On-Camera Mathematics Instructor/Editor/Writer
- On-camera instructor for 5 books published by Addison Wesley for college level introductory mathematics.

Writer/Editor
- Wrote and edited teacher edition book chapters and test questions for the Kaplan K-12 program for states including New York, Florida, Texas, Missouri, Georgia, Kansas and California with topics including understanding numeracy, algebra, geometry and trigonometry.

Salem Public School System, Salem, MA  October 2001 – June 2004
Certified Mathematics Teacher
- Created, presented and implemented creative, college preparatory, mathematics lessons in alignment with the Massachusetts curriculum frameworks.

College Prep Plus, Essex County, MA  July 2002 – June 2004
SAT Mathematics Preparatory Instructor
- Presented, and explained problem solving and solution techniques for the Scholastic Aptitude Test (SAT) at high schools across northeast Massachusetts.
SYNERGISTIC ACTIVITIES


- Member of the Massachusetts Board of Education’s Technology/Engineering Advisory Council (2008-2009)


- Member of Salem High School Design team, an organized initiative as part of a Department of Education grant, to develop units of study which effectively integrate technology into instruction (2003-2004)

- Collaborator on the Adolescent Literacy in the Content Areas Project with Brown University (2003-2004)

PUBLICATIONS


CONFERENCES & PRESENTATIONS


Bunn, J. (2011). *Putting the M in STEM.* Central PA STEM Regional Conference, Johnstown, PA.


