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PROPAGATION OF SONIC BOOMS THROUGH A REAL, STRATIFIED ATMOSPHERE

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PROPAGATION OF SONIC BOOMS THROUGH A REAL, STRATIFIED ATMOSPHERE

by

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PROPAGATION OF SONIC BOOMS THROUGH A REAL, STRATIFIED ATMOSPHERE

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by

Robin Olav Cleveland

1995
This dissertation is dedicated to my mum Hæge and the memory of my dad Roger.
ACKNOWLEDGEMENTS

The two major protagonists in the development of this dissertation have been my supervisor “DTB” and Mark Hamilton, who has been a supervisor in all but name. They have both enthused me with their love of acoustics in the courses they taught me and the large amount interaction they have allowed me to have with them. They have made the process of research a pleasurable and rewarding experience. I am grateful to NASA Langley for funding this project and to the director of Applied Research Labs, Dr. Pestorius, for allowing me to do my research in such a relaxed environment. It is at ARL that one finds Blackstock’s graduate students—the Nonlinear Acoustics Division. First and foremost, I thank Mike Bailey who arrived in Texas at the same time as I. He has been a great friend with whom I discuss not just acoustics but anything else that comes up. Also thanks to those NADs that made me feel so welcome when I arrived and have now moved on: the Jims, Bart, Chuck, Pingwah and Leick. And thanks to those who have arrived and so easily slipped into the group: Larry, Rob and Penia. Finally, thanks to the woman who held us all together our erstwhile secretary Becky Ellis. The two people whom I worked closely with on the NASA sonic boom exercise are Kathy Needleman from NASA Langley and Jim Chambers from Ole Miss—who pointed out a bug in my code. My family, though geographically far away, have been a support I can always count on. My mother and father both instilled in me a great desire to read, question, and learn which I can not thank them enough for. My step-parents Rolf and Heather have both become both friends and wise council. My brother John has just managed his fifth letter in four years, a remarkable feat for him. Finally, I send my love to the most important person in my life, my fiancé Christine Cotton, with whom I have shared the best times of my life and will continue to do so.

May, 1995
Sonic boom propagation in a quiet, stratified, lossy atmosphere is the subject of this dissertation. Two questions are considered in detail: (1) Does waveform freezing occur? (2) Are sonic booms shocks in steady state? Both assumptions have been invoked in the past to predict sonic boom waveforms at the ground. A very general form of the Burgers equation is derived and used as the model for the problem. The derivation begins with the basic conservation equations. The effects of nonlinearity, attenuation and dispersion due to multiple relaxations, viscosity, and heat conduction, geometrical spreading, and stratification of the medium are included. When the absorption and dispersion terms are neglected, an analytical solution is available. The analytical solution is used to answer the first question. Geometrical spreading and stratification of the medium are found to slow down the nonlinear distortion of finite-amplitude waves. In certain cases the distortion reaches an absolute limit, a phenomenon called
waveform freezing. Judging by the maturity of the distortion mechanism, sonic booms generated by aircraft at 18 km altitude are not frozen when they reach the ground. On the other hand, judging by the approach of the waveform to its asymptotic shape, N waves generated by aircraft at 18 km altitude are frozen when they reach the ground.

To answer the second question we solve the full Burgers equation and for this purpose develop a new computer code, THOR. The code is based on an algorithm by Lee and Hamilton (J. Acoust. Soc. Am. 97, 906–917, 1995) and has the novel feature that all its calculations are done in the time domain, including absorption and dispersion. Results from the code compare very well with analytical solutions. In a NASA exercise to compare sonic boom computer programs, THOR gave results that agree well with those of other participants and ran faster. We show that sonic booms are not steady state waves because they travel through a varying medium, suffer spreading, and fail to approximate step shocks closely enough. Although developed to predict sonic boom propagation, THOR can solve other problems for which the extended Burgers equation is a good propagation model.
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Chapter 1

Introduction

The United States is considering the development of a new supersonic passenger aircraft. The proposed aircraft is currently expected to have a cruising altitude of 17 km (about 55 000 ft)* at a speed of Mach 2.0 to Mach 2.5 (about 2 000 km/h or 1 250 mph). An important concern is the annoyance of the sonic boom that is generated by the aircraft once it is in supersonic flight. A number of issues related to the prediction of sonic boom waveforms on the ground are addressed in this work.

Figure 1.1 illustrates the important aspects of sonic boom propagation through the atmosphere. A supersonic aircraft creates an acoustical disturbance in a conical region behind the aircraft. Ray theory is used to predict the path of the sonic boom from the aircraft to the ground. The waveform near the aircraft can be quite complicated, containing many shocks. As the boom propagates downward, nonlinear distortion simplifies the waveform so that it tends towards an N shape by the time it reaches the ground. However, the turbulent boundary layer near the ground often distorts the signal so that it no longer resembles the classic N wave. The loudness of the boom at the ground is related to the peak overpressure and the rise time of the shocks (at least for observers outdoors). The overall duration of the boom does not appear to be so important.

Not only is the boom intense enough that finite-amplitude effects need to be considered but, the atmosphere is not homogeneous: the acoustical properties are stratified. Stratification, normally regarded as a deterministic inhomogeneity of the atmosphere, causes large scale refraction or bending of the sound rays. Refraction determines the shape of the primary sonic boom carpet on the ground, produces the secondary carpet, and can cause focusing (Darden et al. 1989). Stratification also generally weakens the effect of nonlinear distortion on the propagating boom. Indeed, so-called “freezing” of

\*In this work thousands are indicated by a space rather than a comma.
Waveform near the aircraft
Density increases
Sound-speed increases
Waveform near the ground before turbulence
Waveform at the ground after turbulence.

Figure 1.1: Sonic boom propagation through the atmosphere.

the sonic boom signature has been considered possible by some (Hayes et al. 1969).
Waveform freezing refers to the absolute limit on nonlinear distortion of a waveform imposed by the increase of sound speed and density along the downward ray path, in combination with geometric spreading. Waveform freezing of sonic booms is analyzed in this work.

Stratification also affects absorption. Absorption depends strongly on humidity and also on the temperature and pressure. All three quantities vary with altitude and the associated variation in absorption can be quite marked. Atmospheric absorption plays both simple and subtle roles in sonic boom propagation. The simple role is to attenuate the boom by frequency-dependent dissipation. The more subtle role is at the shocks
where, in competition with nonlinear distortion, it determines the profile and hence the rise time (nonlinearity tends to steepen shocks while absorption tends to diffuse them). It has been postulated that shocks in a sonic boom waveform immediately compensate to changes in absorption with altitude (Pierce and Kang 1990), that is, the shocks are always in steady-state. This claim too is investigated.

Finally, the waveform can be massively distorted by passage through turbulent regions. The turbulent boundary layer near the earth is up to 1 km thick and the medium is randomly inhomogeneous because of temperature, density, and velocity fluctuations. The turbulence can produce significant scattering of acoustic energy. Wavefronts can be strongly focussed and defocussed (Darden 1989). The statistical nature of turbulence means that sonic booms from the same aircraft recorded just a few hundred meters apart can be significantly different (see, for example, Maglieri 1966, Lee and Downing 1991).

This work was motivated by an attempt to answer two questions related to sonic boom propagation. First, does waveform freezing occur for sonic booms? Two, are sonic boom shocks always in steady state? An effort was also made to perform experiments to study the interaction of nonlinear distortion and turbulence, but this was an unsuccessful enterprise.

**Dissertation Flight Path**

In Chapter Two an equation is derived to model the propagation of sonic booms in the atmosphere. It is referred to as the extended Burgers equation. The effects of nonlinear distortion, absorption and dispersion due to thermoviscosity and multiple relaxation processes, geometrical spreading, and stratification of all ambient properties are included.

The lossless version of the extended Burgers equation has an analytical solution and is used in Chapter Three to examine the phenomenon of waveform freezing. It is shown that nonlinear distortion is reduced if a finite-amplitude waveform suffers spreading or travels into a medium with increasing impedance. In certain cases there is a finite limit on the amount of distortion that can occur—waveform freezing. For sonic booms
propagating downward in the atmosphere, spreading and increasing impedance couple to slow down the effect of nonlinear distortion and there is the potential for waveform freezing. A realistic stratified atmosphere is used to determine whether sonic booms do indeed freeze. We show that the occurrence of freezing, for sonic booms at least, depends on how one defines waveform freezing.

A new computer code is described in Chapter Four to solve the extended Burgers equation. The innovation of the code is that it remains in the time domain to solve all effects including absorption and dispersion. The algorithms used are analyzed using Von Neumann (frequency domain) analysis. Output from the code is tested against a number of analytical results. Although the code has been implemented using a uniform time grid, a method for using a nonuniform time grid, which adaptively alters as the waveform distorts, is also described. The code is named THOR after the Norse god who generates thunder and lightning.*

We test THOR’s ability to propagate sonic booms in the atmosphere in Chapter Five. First, the calculation of the absorption and dispersion of sound in air is discussed. Then, results from a NASA exercise to compare sonic boom computer codes are presented. Ground waveforms predicted by THOR are compared against those predicted by weak shock theory and two other computer codes. The agreement in the results is excellent. We demonstrate that weak shock theory tends to overpredict the peak pressures of sonic boom waveforms. Finally, we show THOR agrees well with the analytical solutions of Chapter Three.

In Chapter Six THOR is used to check whether sonic boom shocks are in steady

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*The following excerpt is from Of Gods and Giants - Norse Mythology by Harald Hverberg, translated by Pat Shaw Iverson. (Tanums 1969).

Tor is the son of Odin [The highest of the Æsir (gods)] and Jord (Earth). He is the strongest of the gods. His home is called Trudvang, or the “field of strength”. There stands his hall, Bilkirne (the flashing), which has 540 rooms.

Tor is often called “The Charioteer”, because he drives across the heavens in a chariot drawn by his two billygoats, Tanngnjost (the gnasher) and Tanngrisne (the gaper). When he drives, mountain tremble and crack, and the earth is scorched beneath his chariot. Then a mighty booming is heard which people call tordinn (or thunder).
state at the ground. First, we examine how far a step shock must propagate to adjust to a change in relative humidity. Then the effects of spreading and waveform shape on the shock are investigated. Propagation through realistic atmospheres is used to demonstrate how close to steady state sonic boom shocks are at the ground. In the final section, the sensitivity of predicted ground waveforms to fluctuations in the measured atmospheric data is demonstrated.

Chapter Seven summarizes the results and lists other problems still to be addressed. Two aspects of sonic boom propagation that are not touched in this research are wind and turbulence. A method to include wind in the propagation model is discussed. We also explain how THOR could be generalized to a multidimensional model to investigate the propagation of sonic booms through a turbulent medium. However, use of THOR is not restricted to the problem of sonic boom propagation. THOR can be used in any problem where the extended Burgers equation is a valid model. It is noted that the promise of the nonuniform time grid algorithm has not yet been realized.

Two unsuccessful experiments were considered in this research. Both were to use spark generated N waves as scaled sonic booms. The first experiment was designed to test waveform freezing predictions. A stratified medium was to be set up by creating a temperature gradient within an insulated box. However, because significant freezing was predicted to occur only for temperature gradients of the order 1000 K/m (see Chapter Three), the experiment was never initiated. The second laboratory experiment was an attempt to measure the propagation of N waves within a turbulent field. The construction of microphones small enough to make the required measurements was not successful (see Appendix A). This experiment too was abandoned.

The sonic boom computer codes were run on IBM RISC 6000 workstations supplied by the Mechanical Engineering Department of The University of Texas at Austin. All other analysis was done using MATLAB 3.5 from Mathworks Inc. on a Macintosh Quadra 800. No Intel Pentium chips were used.
Chapter 2

Theory: Variations on a Theme by Burgers

2.1 Introduction

An equation is sought to model the propagation of sonic booms through the atmosphere. In this chapter a Burgers type equation is developed for one dimensional propagation of finite amplitude waves in an inhomogeneous, relaxing fluid. First a brief overview of the Burgers equation and some of its variations is presented. The fluid dynamics equations are then laid out in a form appropriate for acoustic propagation in an inhomogeneous medium. Next, a ranking system based on the amplitude of the wave, the dissipation coefficients and the spatial variation of the inhomogeneities is described. The ranking system is used to determine which terms in the fluid dynamics equations can be neglected. Finally, we assume progressive wave motion and combine the equations into a Burgers type equation for a variety of different wave fields and fluid properties.

2.2 Background

The Burgers equation is a standard model for the propagation of progressive finite amplitude waves in a lossy medium. The classical Burgers equation (see, for example, Lighthill 1956, Blackstock 1964), which models plane finite-amplitude waves in a thermoviscous medium is

\[
\frac{\partial p'}{\partial x} - \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial t'} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2}.
\]  

(2.1)

Here \(p'\) is acoustic pressure, \(x\) distance, \(t' = t - x/c_0\) retarded time, \(c_0\) small-signal sound speed, \(\rho_0\) ambient density, \(\beta\) the coefficient of nonlinearity and \(b\) a combination of the coefficients of viscosity and thermal conduction (see Sec. 2.6).
The effects of geometrical spreading (see, for example, Lighthill 1956) and propagation through an inhomogeneous medium have been included (see, for example, Carlton and Blackstock 1974, Fridman 1976) in what is commonly known as the generalized form of the Burgers equation

\[
\frac{\partial p'}{\partial s} + \frac{\partial}{\partial s} \left( \frac{S}{2S} p' \right) - \frac{\partial}{\partial s} \left( \frac{\rho_0 c_0}{2\rho_0 c_0} p' \right) = \frac{\beta}{2\rho_0 c_0^2} \frac{\partial^2 p'}{\partial t'^2} + \frac{b}{2\rho_0 c_0^2} \frac{\partial^2 p'}{\partial t'^2},
\]

(2.2)

where propagation is along ray tubes, \( s \) is distance along the ray tube, \( S \) is ray tube area, and retarded time is now given by \( t' = t - f_{ds} \) . The effect of a motion of the medium has also been considered (Robinson 1991).

Pierce (1981) introduced what he called the augmented Burgers equation* where the effects of relaxation processes, which can be important mechanisms in the attenuation and dispersion of sound, are accounted for. Each relaxation process \( \nu \) is characterized by a relaxation time \( \tau_\nu \) and a change in small-signal sound speed \( (\Delta c)_\nu \) due to the relaxation. In operator notation Pierce’s Burgers equation may be written

\[
\frac{\partial p'}{\partial s} - \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial t'^2} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2} + \sum_{\nu} \frac{(\Delta c)_\nu}{c_0^2} \frac{\tau_\nu}{1 + \tau_\nu} \frac{\partial^2 p'}{\partial t'^2},
\]

(2.3)

where the operator may be expressed as an integral:

\[
\frac{\tau_\nu}{1 + \tau_\nu} f(t') = \int_{-\infty}^{t'} e^{(\tau - \tau')/\tau_\nu} f(\tau) \, d\tau.
\]

2.3 Fluid Dynamics

The fundamental conservation equations for a viscous, heat conducting fluid are the starting point in our analysis. The equations presented here come from a number of sources (Landau and Lifshitz 1959; Pierce 1981, Chap. 1; Panton 1984, Chap. 5; Blackstock 1996, Chap. 2). The variables used in the equations are

\[ \rho = \text{density}, \]
\[ p = \text{pressure}, \]

---

*The terminology is not consistent in the literature, for example, Blackstock (1985) developed a Burgers equation that has an arbitrary linear absorption and dispersion operator (examples include a single relaxation process and boundary layer effects) which he calls a generalized Burgers equation.
\( \mathbf{u} = \) particle velocity, 
\( \mathbf{B} = \) body force, 
\( T = \) temperature, 
\( \chi = \) entropy (per unit mass), 
\( \lambda = \) dilatational viscosity coefficient, 
\( \mu = \) shear viscosity coefficient, 
\( \kappa = \) thermal conductivity coefficient.

The continuity equation (see, for example, Landau and Lifshitz 1959, Eq. 1.2) is

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.4}
\]

The equation for conservation of momentum is typically written for a fluid with constant coefficients of viscosity. In the atmosphere the coefficients of viscosity are not constant; they vary with altitude. In addition, the coefficients can vary due to temperature fluctuations associated with an acoustic wave. In this work momentum equation that accounts for the variation of the viscosity coefficients is required. Hunt (1955) gives the appropriate equation for the conservation of momentum as follows:

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \rho \mathbf{B} + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} \\
+ (\nabla \lambda) \nabla \cdot \mathbf{u} + 2 (\nabla \mu) \cdot \nabla \mathbf{u} + (\nabla \mu) \times (\nabla \times \mathbf{u}).
\]

For most acoustic fields the flow can be considered to be irrotational (Thompson 1984 gives a nice discussion in Chap. 2.4). In an irrotational flow \( \nabla \times \mathbf{u} = 0 \) and the momentum equation reduces to

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \rho \mathbf{B} + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) + (\nabla \lambda) \nabla \cdot \mathbf{u} + 2 (\nabla \mu) \cdot \nabla \mathbf{u}. \tag{2.5}
\]

If \( \lambda \) and \( \mu \) are constants the more familiar form of the momentum equation is recovered (see, for example, Blackstock 1996, Eq. 2.A.44)

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \rho \mathbf{B} + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}).
\]

The energy equation for a viscous, heat conducting fluid (Panton 1984, Eq. 5.10.2) is

\[
\frac{\partial}{\partial t} (\rho e) + \partial_i (\rho u_i e) = -p \partial_i u_i + \tau_{ij} \partial_j u_i - \partial_i q_i,
\]
where $e$ is the thermodynamic internal energy and $q_i$ is the heat flux. For our purposes it is more convenient to write the energy equation in terms of entropy (Panton 1984, Eq. 5.12.3): 

$$\rho T \frac{DX}{Dt} = \nabla \cdot (\kappa \nabla T) + \tau_{ij} \partial_j u_i,$$

where Fourier's law of heat conduction (Panton 1984, Eq. 6.5.4) is used to model the heat flux and the material derivative $D/Dt$ is defined as $\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$. For irrotational flow $\tau_{ij} = \lambda \partial_u u_k \delta_{ij} + 2\mu \partial_t u_j$, and so

$$\rho T \frac{DX}{Dt} = \nabla \cdot (\kappa \nabla T) + \left(\lambda \partial_u u_k \delta_{ij} + 2\mu \partial_j u_i\right) \partial_t u_j,$$

$$= \nabla \cdot (\kappa \nabla T) + \lambda \left(\partial_u u_k \right)^2 + 2\mu \left(\partial_t u_j \partial_j u_i\right). \quad (2.6)$$

Pierce refers to this as the Kirchhoff-Fourier equation (Pierce 1981, Eq. 10-1.15).

To close the system of equations an equation of state is needed for the fluid. In acoustics it is common to assume that pressure is a function of density and entropy

$$p = p(\rho, \chi). \quad (2.7)$$

However, when relaxation processes are included we will assume further that the equation of state is also a function of a number of internal coordinates; see Sec. 2.5.

### 2.4 Ranking System for Acoustic Waves

The fluid dynamics equations need to be simplified before solutions can be attempted. In acoustics the approach is to assume that the acoustic field can be characterized as a perturbation from an ambient field (see, for example, Pierce 1981, Chap. 1-5). That is, the fluid variables can be expressed as: $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, $u = u$, $T = T_0 + T'$ and $\chi = \chi_0 + \chi'$, where the subscript "0" designates the ambient value and the prime ' denotes the excess or acoustic variation. Although the ambient values are static and do not vary with time, they can vary in space.

In this section a system to rank the importance of the terms in the fluid dynamics equations is introduced. Four types of dimensionless small quantities are identified:

$$\varepsilon = \frac{u_0}{c_0} \text{ a measure of the amplitude of the acoustic wave}, \quad (2.8)$$
The quantity $u_0$ is a characteristic particle velocity of the waveform, e.g., the peak particle velocity at the source, $l_e$ is a characteristic length of the waveform, e.g., the fundamental wavelength for a time harmonic signal, $l_i$ is the characteristic length scale for variations in the ambient properties, and $l_h$ is a length scale associated with the change in ray tube area (geometric spreading).

Notice that the only loss effect mentioned in the ranking system is viscosity. Strictly, the other dissipative effects, thermal conduction and relaxation, should be characterized by their own small parameter. Thermal conduction is typically the same order or smaller than $\delta$. Relaxation processes on the other hand are frequency dependent: at high frequencies they are much less than $\delta$, and at low frequencies may be $10^3$ or greater than $\delta$. In the interests of convenience we use just one measure of dissipation $\delta$ and then relate the other loss terms to $\delta$. Incidentally $\mu/\rho_0 c_0 l_e$ is a Reynolds number based on the small signal sound speed as the characteristic velocity and $l_e$ as the characteristic length scale (Lighthill 1956, Sec. 10).

Terms in the conservation equations that are order $\varepsilon$ only are called first order terms. Retention of just first order terms leads to the linear, lossless wave equation. Terms that are of the order of the product of two small quantities, that is, $\varepsilon^2$, $\varepsilon \delta$, $\varepsilon \zeta$, or $\varepsilon \nu$ are called second order. Although second order terms are much smaller than first order terms, second order terms have cumulative effects that can become important over long propagation distances. All of the rest of the terms are the order of the product of three small quantities or more, e.g., $\varepsilon^3$, $\varepsilon^2 \delta$, and $\varepsilon^2 \delta \zeta$ and are referred to as third order or higher order terms. The underlying assumption in most studies of nonlinear acoustics, including the one presented here, is that terms that are third order and higher are negligible and can be discarded.

The various terms and quantities in the fluid dynamics equations are now ranked. For sake of simplicity a homogeneous fluid is considered first, and the rank of the

\[
\delta = \frac{\mu}{\rho_0 c_0 l_e} \text{ a measure of the viscous dissipation,} \quad (2.9)
\]

\[
\zeta = \frac{l_e}{l_i} \text{ a measure of the gradients in the static properties.} \quad (2.10)
\]

\[
v = \frac{l_e}{l_h} \text{ a measure of the geometric spreading.} \quad (2.11)
\]
acoustical variables and dissipation terms is determined. The effect of the temperature
dependence of the dissipation coefficients is considered. Then an inhomogeneous fluid is
investigated and we rank the gradients of the static values, gradients of loss coefficients,
and the body force.

2.4.1 Acoustical Variables

In what follows the order of the acoustical variables, that is, \( u, \rho', p', \) and \( T' \), are
determined from the linearized field equations for a homogeneous fluid. The decom­
position of field values into ambient components and acoustic perturbations allows the
continuity equation for a homogeneous fluid to be expressed as

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot u + \nabla \cdot (\rho' u) = 0.
\] (2.12)

Before the terms in this equation can be ranked, dimensionless time and space coor­
dinates are defined. A characteristic time scale \( t_c \) and length scale \( l_c \) for the acoustic
wave are identified. If we introduce the dimensionless time \( t' = t/t_c \) and distance
\( x' = x/l_c \), Eq. 2.12 becomes

\[
\frac{1}{t_c} \frac{\partial \rho'}{\partial t'} + \frac{\rho_0}{l_c} \nabla' \cdot u + \frac{l_c}{t_c} \nabla'(\rho' u) = 0,
\]

where \( \nabla' \) is the dimensionless gradient operator, i.e., \( \nabla' = l_c \nabla \). For acoustical prob­
lems it is appropriate to take \( l_c/t_c = c_0 \) thus the dimensionless continuity equation
is

\[
\frac{\partial \rho'}{\partial t'} \rho_0 + \nabla' \cdot \frac{u}{c_0} + \nabla' \cdot \left( \frac{\rho'}{\rho_0} \frac{u}{c_0} \right) = 0.
\] (2.13)

We now linearize the continuity equation and then later come back to check whether
it was appropriate to do so. The linearized version of Eq. 2.13 is

\[
\frac{\partial \rho'}{\partial t'} \rho_0 + \nabla' \cdot \frac{u}{c_0} = 0,
\]

Since \( u/c_0 \) is of order \( \varepsilon \), this equation shows that \( \rho'/\rho_0 \) is also of order \( \varepsilon \). Consequently
the nonlinear term in Eq. 2.13 is of order \( \varepsilon^2 \). The linearization step is therefore
appropriate under the condition that $\varepsilon \ll 1$. We now adopt the assumption $\varepsilon \ll 1$ as the first step in the ranking process.

The momentum equation is used to determine the rank of the acoustic pressure. The decomposition into ambient and perturbed quantities yields the following equation:

$$\rho_0 \frac{\partial u}{\partial t} + \rho' \frac{\partial u}{\partial t} + \rho_0 \nabla \cdot u + \rho' \nabla \cdot u = -\nabla p' + (\lambda + 2\mu) \nabla \nabla \cdot u + (\nabla \lambda) \nabla \cdot u + 2(\nabla \mu) \nabla u.$$

When the dimensionless coordinate system is introduced one finds that the underlined terms on the left-hand side are of order $\varepsilon^2$ or $\varepsilon^3$. It is shown below that the underlined terms on the right-hand side are of order $\delta \varepsilon$ or $\delta \varepsilon^2$. If we assume $\delta$ is also much less than one then the linearized momentum equation is

$$\frac{\partial}{\partial t} \frac{u}{c_0} + \nabla \left( \frac{p'}{\rho_0 c_0^2} \right) = 0.$$

Therefore $p' / \rho_0 c_0^2$ must be of order $\varepsilon$ and the acoustic pressure is also a first order term.

The equation of state may be used to determine the order of the temperature fluctuation $T'$. If temperature is assumed to be a function of density and entropy, then the first terms in the Taylor series are

$$T' = \left. \frac{\partial T}{\partial \rho} \right|_{\rho_0} \rho' + \left. \frac{\partial T}{\partial x} \right|_{x_0} \chi',$$

or

$$\frac{T'}{T_0} = \left. \frac{\rho_0}{T_0} \frac{\partial T}{\partial \rho} \right|_{\rho_0} \rho' \left. + \frac{\rho_0}{T_0} \frac{C_p}{T_0} \frac{\partial T}{\partial x} \right|_{x_0} \chi'. $$

It was shown above that $\rho' / \rho_0$ is of order $\varepsilon$ and it is shown below that $\chi' / C_p$ is of order $\delta \varepsilon$. Since the coefficients of the $\rho'$ and $\chi'$ are both of order one,* $T'/T_0$ must be of the same order as the larger quantity $\rho' / \rho_0$, that is, $T'/T_0$ is of order $\varepsilon$.

The smallness assumption is valid even for quite intense acoustic waves, for example, the threshold of pain is about 140 dB (Kinsler, Frey, et al. 1982, p. 263) which corresponds to $\varepsilon = 2 \times 10^{-3}$ for a sine wave, where $u_0$ is the peak particle velocity.

*For example, in an ideal gas $\left. \frac{\alpha_x \frac{\partial T}{\partial x}}{T_0} \right|_{\rho_0} = \gamma - 1$ and $\left. \frac{C_p}{T_0} \frac{\partial T}{\partial x} \right|_{x_0} = \gamma.$
2.4.2 Dissipation Terms

The order of the loss terms associated with viscosity and heat conduction are now established. With order $\varepsilon^3$ terms omitted the momentum equation is

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho \frac{\partial \mathbf{u} \cdot \mathbf{u}}{\partial t} + \rho_0 \mathbf{u} \nabla \cdot \mathbf{u} = -\nabla p' + (\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} + (\nabla \lambda) \nabla \cdot \mathbf{u} + 2(\nabla \mu) \cdot \nabla \mathbf{u}.$$

To simplify the analysis we shall assume one dimensional flow, $\nabla = \delta_x \frac{\partial}{\partial x}$. The terms $(\nabla \lambda) \nabla \cdot \mathbf{u}$ and $2(\nabla \mu) \cdot \nabla \mathbf{u}$ can now be combined into one term $\frac{\partial (\lambda + 2\mu)}{\partial x} \frac{\partial \mathbf{u}}{\partial x}$. The coefficients of viscosity can be rewritten as

$$\lambda + 2\mu = \mu \tilde{V},$$

where $\tilde{V}$ is the viscosity number and is order one for most fluids. The shear viscosity varies primarily with temperature (Pierce 1981, p. 513–4). We model the viscosity to first order by

$$\mu = \mu|_{T_0} + T' \frac{\partial \mu}{\partial T}|_{T_0}.$$

The shear viscosity is written in the following shorthand

$$\mu = \mu_0 + \mu'T',$$

where $\mu_0 = \mu|_{T_0}$ and $\mu' = \frac{\partial \mu}{\partial T}|_{T_0}$. The one-dimensional momentum equation becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho \frac{\partial \mathbf{u}}{\partial t} + \rho_0 \frac{\partial \mathbf{u}}{\partial x} = -\frac{\partial p'}{\partial x} + \mu_0 \tilde{V} \frac{\partial^2 \mathbf{u}}{\partial x^2} + \mu' T' \frac{\partial \mathbf{u}}{\partial x} \frac{\partial \mathbf{u}}{\partial x},$$

or in dimensionless form

$$\frac{\partial \mathbf{u}}{\partial t' + \frac{\partial \mathbf{u}}{\partial t'} + \frac{\partial \mathbf{u}}{\partial x'} + \mu' \frac{\partial \mathbf{u}}{\partial x'} = -\frac{\partial p'}{\partial x'} + \frac{\mu_0 \tilde{V}}{\rho_0 c_0 T_0} \frac{\partial^2 \mathbf{u}}{\partial x'^2} + \frac{\mu'T_0 \tilde{V} T'}{\rho_0 c_0} \frac{\partial \mathbf{u}}{\partial x'} \frac{\partial \mathbf{u}}{\partial x'},$$

(2.14)

Recall (Eq. 2.9) that $\mu_0/\rho_0 c_0 l_c = \delta$ the nondimensional viscosity coefficient. It is assumed that $\delta$ is a small quantity. The term $\frac{\mu_0 \tilde{V}}{\rho_0 c_0 T_0} \frac{\partial^2 \mathbf{u}}{\partial x'^2}$ is therefore order $\delta\varepsilon$ and is classified as a second order term. The quantity $\mu'T_0/\rho_0 c_0 l_c$ is also assumed to be order

*Although we assume both $\varepsilon$ and $\delta$ are small quantities we do not assume $\varepsilon$ is the same order as $\delta$. 
The two terms which contain the temperature dependence of $\mu$ are thus order $\delta \varepsilon^2$ and can be neglected.

The assumption of smallness, $\delta \ll 1$, is reasonable for most fluids even in the ultrasound spectrum. For example, in air $\mu_0 \approx 20 \times 10^{-6} \text{Nm/s}^2$ and $\delta < 10^{-3}$ for frequencies less than 7 MHz. In water $\mu_0 \approx 10^{-3} \text{Nm/s}^2$ and $\delta < 10^{-3}$ for frequencies less than 2 GHz.

The ranking system is now applied to the entropy equation, Eq. 2.23, which allows us to rank thermal conduction and entropy. If in the interest of simplicity terms that are obviously third order, such as the viscous dissipation terms, are thrown away, the entropy equation becomes

$$\rho T \frac{D\chi'}{Dt} = \kappa \nabla^2 \chi' + (\nabla \kappa' \cdot (\nabla T')).$$

The variation of $\kappa$ with temperature is modelled in the same way as the temperature dependence of the viscous coefficient,

$$\kappa = \kappa_0 + \kappa'T',$$

where $\kappa_0 = \kappa|_{T_0}$ and $\kappa' = \frac{\partial \kappa}{\partial T}|_{T_0}$. The entropy equation for a homogeneous fluid is

$$\rho_0 T_0 \frac{\partial \chi'}{\partial t} + \rho_0 T_0 \frac{\partial \chi'}{\partial t} + \rho_0 T_0 \frac{\partial \chi'}{\partial t} + \rho_0 T_0 \frac{\partial \chi'}{\partial t} = \kappa_0 \nabla^2 T' + \kappa' T' \nabla^2 T' + (\nabla \kappa' T') \cdot (\nabla T').$$

The same dimensionless scales are used and the entropy equation becomes

$$\frac{\partial \chi'}{\partial t'} + \rho_0 \frac{\partial \chi'}{\partial t'} + \rho_0 \frac{\partial \chi'}{\partial t'} + \rho_0 \frac{\partial \chi'}{\partial t'} = \frac{\kappa_0}{C_p \rho_0 c_0 l_c} \frac{\nabla^2 T'}{T_0} + \frac{\kappa' T_0}{C_p \rho_0 c_0 l_c} \frac{T'}{T_0} \nabla^2 T' + (\nabla \kappa' T_0) \cdot (\nabla T').$$

The dimensionless conduction coefficient $\kappa_0/C_p \rho_0 c_0 l_c = \delta/\Pr$, where $\Pr = C_p \mu_0 / \kappa_0$ is the Prandtl number. Since the Prandtl number is order one for gases and order 10 for most liquids, it is clear that the primary heat conduction term is of order $\varepsilon \delta$. The quantity $\kappa' T_0$ is taken to be the same order as $\kappa_0$.* Therefore the two terms on the

\footnote{It is reasonable to expect that variations in $\mu$ are the same order as $\mu$, for example, using Pierce's expression one finds for air $\mu' T_0 = 0.7969 \mu_0$ and for water $\mu' T_0 = 7.3 \mu_0$.}

\footnote{For example, in water the expression from Pierce (1981, Eq. 10-1.16b) yields $\kappa' T_0 = 0.835 \kappa_0$.}
right-hand side which contain the temperature dependence of $\kappa$ are third order and can be neglected.

The lowest order form of the entropy equation, strictly speaking there are no linear (first order) terms in the equation, is

$$\frac{\partial \chi'}{\partial t'} = \kappa_0 \frac{\nabla^2 T'}{T_0}.$$  

It follows that the acoustic entropy $\chi'$ is of order $\delta \varepsilon$.

The fluid dynamics equations for a homogeneous fluid, correct to second order, are

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = -\nabla \cdot (\rho' \mathbf{u}),$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' = -\rho' \frac{\partial \mathbf{u}}{\partial t} - \rho_0 \mathbf{u} \cdot \nabla \mathbf{u} + (\lambda + 2 \mu) \nabla (\nabla \cdot \mathbf{u}),$$

$$\rho_0 T_0 \frac{\partial T'}{\partial t'} = \kappa \nabla^2 T'.$$

The loss terms $\lambda$, $\mu$, and $\kappa$ are evaluated at the ambient conditions.

2.4.3 Inhomogeneous Fluid

The ranking system in an inhomogeneous fluid is now considered. For ambient properties that vary slowly it turns out that the inhomogeneity adds second order terms. The variation of any ambient quantity is assumed to occur over a characteristic inhomogeneous length scale $l_i$. The appropriate dimensionless gradient operator for ambient properties is $\nabla'' = l_i \nabla'$, that is, $\frac{1}{p_0} \nabla'' p_0$ is of order 1. We assume $l_c/l_i = \zeta$ is a small quantity, that is, the length scale of the acoustic wave is much less than the length scale of the inhomogeneity.

The continuity equation becomes

$$\frac{1}{l_c} \frac{\partial \rho'}{\partial t'} + \frac{\rho_0}{l_c} \nabla' \cdot \mathbf{u} + \frac{\mathbf{u}}{l_i} \cdot \nabla'' \rho_0 + \frac{1}{l_c} \nabla' \cdot (\rho' \mathbf{u}) = 0.$$  

Because the ambient quantities vary in space we must use the chain rule to move them through the gradient operator, for example, $\nabla' \cdot (\mathbf{u}/c_0) = \frac{1}{c_0} \nabla' \cdot \mathbf{u} + \frac{\mathbf{u}}{c_0} \cdot (\frac{\zeta}{c_0} \nabla'' c_0)$. The extra term introduced by the chain rule is of order $\varepsilon \zeta$ so it can be neglected when
manipulating second order terms. Note that dimensionless quantities are relative to the local ambient properties. The dimensionless continuity equation is

\[ \frac{\partial \rho'}{\partial t} + \nabla' \cdot \frac{u}{c_0} + \frac{\zeta}{c_0} \nabla'' \rho_0 + \nabla' \left( \frac{\rho' u}{\rho_0 c_0} \right) = 0. \] (2.16)

The order is indicated below each term.

The momentum equation for an inhomogeneous fluid with a body force is

\[ \rho_0 \frac{\partial u}{\partial t} + \rho' \frac{\partial u}{\partial t} + \rho_0 u \cdot \nabla u = -\nabla p' - \nabla p_0 + \rho_0 B + \rho' B + (\lambda_0 + 2\mu_0) \nabla (\nabla \cdot u) + (\nabla \lambda_0) \nabla \cdot u + 2(\nabla \mu_0) \cdot \nabla u. \]

By inspection the last two terms are third order because of the gradient operation on the (ambient) viscosity coefficients. For example, the term \((\nabla \mu_0) \cdot \nabla u\) becomes the following dimensionless term \(\zeta (\nabla''_{\mu_0 \rho_0 c_0} \nabla' u_{c_0})\). The spatial variation in the viscous coefficients is of order \(\zeta \delta e\) and can be neglected.

If the static problem (no acoustic wave) is solved one finds that the ambient density and pressure are related to the body force by \(\frac{1}{\rho_0} \nabla p_0 = B\). The ambient pressure varies on the inhomogeneous length scale \(l_i\). It follows then that

\[ \frac{B}{l_i \rho_0 c_0^2} = O(\zeta). \] (2.17)

In the atmosphere gravity is the body force, \(B = -g \hat{e}_z\) and \(|B| \approx 10 \text{ m/s}^2\). The small signal sound speed is \(c_0 \approx 333 \text{ m/s}\), which means for \(\frac{|B|}{l_i \rho_0 c_0^2} < 0.001\) then \(t_c < 3 \text{ s}\) or \(l_c < 1 \text{ km}\) which is quite reasonable for most acoustic waves. The body force can be replaced with the pressure gradient and the dimensionless momentum equation becomes

\[ \frac{\partial u}{\partial t} + \frac{\rho'}{\rho_0} \frac{\partial u}{\partial t} + \frac{u}{c_0} \nabla' \frac{u}{c_0} = \frac{-1}{\rho_0 c_0^2} \frac{\partial p'}{\partial x'} + \frac{\rho'}{\rho_0 \rho_0 c_0^2} \frac{\partial u}{\partial x'} \nabla'' \rho_0 + \frac{\mu_0 \nabla}{\delta e \rho_0 c_0^2} \frac{u}{c_0}. \]

Finally the ranking system is applied to the entropy equation for an inhomogeneous fluid. From the analysis of a homogeneous fluid a number of terms can be thrown away and the entropy equation can be written as

\[ \rho_0 T_0 \left( \frac{\partial \chi'}{\partial t} + u \cdot \nabla \chi_0 \right) = \kappa_0 (\nabla^2 T' + \nabla^2 T_0) + (\nabla \kappa_0) \cdot (\nabla T' + \nabla T_0). \]
By inspection the last term involving the spatial variation of \( \kappa_0 \) (that is, \( \nabla \kappa_0 \)) is third order. When the dimensionless time and space scales are introduced the entropy equation can be written in dimensionless form

\[
\frac{\partial \phi}{\partial \tau} + \frac{u}{\epsilon_0} \cdot \nabla \phi = \frac{\kappa_0}{C_p \rho_0 \alpha l c} \nabla^2 T' + \frac{\kappa_0}{C_p \rho_0 \alpha l c} \frac{\xi^2}{\epsilon_0} \nabla^4 l^2 T_0, \tag{2.18}
\]

where \( \kappa_0 / (C_p \rho_0 \alpha l c) = \delta/Pr \) is order \( \delta \). The term involving spatial derivative of \( T_0 \) is third order.

### 2.4.4 Summary

The fluid dynamics equations, correct to second order, for an inhomogeneous fluid are

\[
\begin{align*}
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} &= -\mathbf{u} \cdot \nabla \rho_0 - \nabla \cdot (\rho' \mathbf{u}), \\
\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' - \rho' \nabla \rho_0 &= -\rho' \frac{\partial \mathbf{u}}{\partial t} - \rho_0 \mathbf{u} \nabla \cdot \mathbf{u} + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}), \\
\rho_0 T_0 \left( \frac{\partial \phi'}{\partial \tau'} + \mathbf{u} \cdot \chi_0 \right) &= \kappa_0 \nabla^2 T'.
\end{align*}
\]

By retaining certain quantities in these equations different types of acoustics problems can be addressed. For example, linear lossless acoustics is where only terms of order \( \epsilon \) are kept, all second order terms are neglected. In Chapter Three we analyze the problem of finite-amplitude lossless acoustics—terms of \( \delta \epsilon \) are neglected but all others are kept. The extended Burgers equation that is developed in this Chapter, and solved numerically in Chapter Four, keeps all terms up to second order. All third order terms and higher are neglected. Note this assumption requires that third order terms be much smaller than second order terms, it does not require that first order terms be similar in size, i.e., \( \delta \neq \epsilon \). Therefore the equations are not restricted to problems where, for example, absorption and nonlinearity are balancing effects.

An invaluable corollary of using the ranking system is that second order terms can be manipulated with first order relations since the error this introduces is third order. For example, the continuity equation yields the first order relation

\[
\frac{\partial \rho'}{\partial t} = -\nabla \cdot \mathbf{u} + O(\epsilon^2).
\]
The second order term \( \rho' \nabla \cdot \mathbf{u} \) can therefore be rewritten as

\[
\rho' \nabla \cdot \mathbf{u} = -\rho' \frac{\partial \rho'}{\partial t} + O(\varepsilon^3),
\]

where the third order correction can be neglected.

### 2.5 Thermodynamic Properties

The equation of state needs to be written in terms of acoustic variables and known constants so that it can be combined with the other fluid dynamics equations. The ranking system described in the previous section is used to expand the equation of state \( p = p(\rho, \chi) \) in a Taylor series around the ambient conditions:

\[
p_0 + p' = p(p_0, x_0) + \rho' \frac{\partial p}{\partial \rho \mid_{p_0,x_0}} + \frac{\rho'^2}{2} \frac{\partial^2 p}{\partial \rho^2 \mid_{p_0,x_0}} + \chi' \frac{\partial p}{\partial \chi \mid_{p_0,x_0}} + \cdots.
\]

Since the ambient pressure is \( p_0 = p(\rho_0, x_0) \), the equation of state reduces to

\[
p' = \rho' \frac{\partial p}{\partial \rho \mid_{p_0,x_0}} + \frac{\rho'^2}{2} \frac{\partial^2 p}{\partial \rho^2 \mid_{p_0,x_0}} + \chi' \frac{\partial p}{\partial \chi \mid_{p_0,x_0}} + O(\varepsilon^3, \delta \varepsilon^2).
\]

This is the basic form of the equation of state used in the derivation of subsequent wave equations. In the rest of this section the coefficients of the density terms are addressed, then the entropy term for a thermally conducting fluid is evaluated and finally the effect of relaxation processes are added to the equation of state.

For fluids the first two derivatives of pressure with respect to density are

\[
\frac{\partial p}{\partial \rho \mid_{p_0,x_0}} \triangleq c_0^2,
\]

where \( c_0 \) turns out to be the small signal sound speed,\(^*\) and

\[
\frac{\partial^2 p}{\partial \rho^2 \mid_{p_0,x_0}} \triangleq \frac{c_0^2 B}{\rho_0 A}.
\]

\(^*\)When relaxation processes are accounted for \( c_0 \) becomes the low frequency limit to the sound speed and a slightly different definition is appropriate.
For an ideal gas $B/A = \gamma - 1$, where $\gamma = C_p/C_v$ is the ratio of specific heats. For isentropic flow, $\chi' = 0$, and the equation of state is

$$p' = c_0^2 \rho' + \frac{c_0^2}{\rho_0} B \rho'^2.$$

(2.22)

For flow for which the entropy is not constant the term $\chi' \frac{\partial \rho}{\partial X} \mid_{\rho_0, \chi_0}$ needs to be evaluated. For the sake of simplicity the analysis is carried out for a perfect gas; the final result, however, is valid for an arbitrary fluid (Hamilton 1993). For a perfect gas the equation of state is

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma e^{\frac{x-x_0}{C_v}},$$

where $C_v$ is the heat capacity at constant volume. The temperature is given by

$$T = \frac{p}{\rho R},$$

where $R = C_p - C_v$ is the universal gas constant and $C_p$ is the heat capacity at constant pressure.

It is desirable to write the equation of state in the form $p' = f(\rho')$, therefore the entropy needs to be expressed in terms of density. The entropy equation can be used to accomplish this task. For a thermally conducting fluid the Kirchhoff-Fourier equation, Eq. 2.18, can be written correct to second order as

$$\frac{\partial \chi'}{\partial t} + u \cdot \nabla \chi_0 = -\frac{\kappa}{\rho_0 T_0} \nabla^2 T'.$$

In what follows that acoustic temperature $T'$ is expressed to first order as a function of $\rho'$ so that the entropy can be expressed as a function of density.

The temperature is assumed to be a function of density and entropy, $T(\rho, \chi)$. The first order Taylor series is

$$T' = \rho' \frac{\partial T}{\partial \rho} \mid_{\rho_0, \chi_0} + O(\epsilon \delta, \epsilon^2),$$

and

$$\nabla^2 T' = \frac{\partial T}{\partial \rho} \mid_{\rho_0, \chi_0} \nabla^2 \rho'.$$

The terms $A$ and $B$ come from the Taylor series expansion of the isentropic equation of state $p' = A \rho' + \frac{B}{\rho_0} (\rho')^2$ (see, for example, Beyer 1960).
For an ideal gas

\[ \frac{\partial T}{\partial \rho} \bigg|_{\rho_0, \chi_0} = (\gamma - 1) \frac{T_0}{\rho_0}, \]

and the entropy equation can now be written

\[ \frac{\partial \chi'}{\partial t} + \mathbf{u} \cdot \nabla \chi_0 = \frac{\kappa}{\rho_0^2} (\gamma - 1) \nabla^2 \rho'. \]

A result from Sec. 2.6, \( \nabla^2 \rho' = \frac{1}{c_0^2} \frac{\partial^2 \rho'}{\partial x^2} + O(\varepsilon^2) \), (Eq. 2.43) can be used to remove the spatial derivative. The entropy equation becomes

\[ \frac{\partial \chi'}{\partial t} + \mathbf{u} \cdot \nabla \chi_0 = \frac{\kappa}{c_0^2 \rho_0^2} (\gamma - 1) \frac{\partial^2 \rho'}{\partial t^2} + O(\varepsilon^2). \quad (2.23) \]

For a homogeneous fluid \( \nabla \chi_0 = 0 \) and Eq. 2.23 can be integrated once with respect to time to yield an expression for \( \chi' \). The whole term involving entropy in Eq. 2.19 becomes

\[ \chi' \frac{\partial \rho}{\partial \chi} \bigg|_{\rho_0, \chi_0} = \frac{p_0}{C_0} \frac{\kappa}{c_0^2 \rho_0^2} (\gamma - 1) \frac{\partial \rho'}{\partial t} + O(\varepsilon^2, \varepsilon^3), \]

\[ = \frac{\kappa}{C_p} (\gamma - 1) \frac{\partial \rho'}{\partial t}, \]

where the identities \( c_0^2 = \gamma \rho_0 / \rho_0 \) and \( \frac{\partial \rho}{\partial \chi} \bigg|_{\rho_0, \chi_0} = \frac{p_0}{C_v} \) were used. The equation of state for a homogeneous thermally conducting fluid is therefore,

\[ p' = c_0^2 \rho' + \frac{\kappa}{C_p} (\gamma - 1) \frac{\partial \rho'}{\partial t} + \frac{c_0^2}{\rho_0} \frac{B}{2A} \rho'^2. \quad (2.24) \]

For an inhomogeneous fluid \( \nabla \chi_0 \neq 0 \). If we take the partial derivative of Eq. 2.19 with respect to time and substitute Eq. 2.23 for the acoustic entropy yields

\[ \frac{\partial p'}{\partial t} = c_0^2 \frac{\partial \rho'}{\partial t} + \frac{\kappa}{C_p} (\gamma - 1) \frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial p}{\partial \chi_0} \bigg|_0 \mathbf{u} \cdot \nabla \chi_0 + \frac{c_0^2}{\rho_0} \frac{B}{2A} \frac{\partial \rho'^2}{\partial t} \]

The gradient of the equation of state \( p = p(\rho, \chi) \) yields

\[ \nabla p_0 = \frac{\partial p}{\partial \rho} \bigg|_{0} \nabla \rho_0 + \frac{\partial p}{\partial \chi} \bigg|_{0} \nabla \chi_0. \]

*The integration constant is zero because in the absence of sound \( \chi' = 0 \) and \( \rho' = 0 \).
The entropy gradient can therefore be expressed in terms of the gradient of the density and the gradient of the pressure or the body force

\[ \frac{\partial p}{\partial \chi} \bigg|_0 \nabla \chi_0 = \nabla p_0 - \frac{\partial p}{\partial \rho_0} \nabla \rho_0 , \]

\[ = \rho_0 B - c_0^2 \frac{\partial \rho_0}{\partial x} . \]

The equation of state for an inhomogeneous fluid can now be written:

\[ \frac{\partial p'}{\partial t} = c_0^2 \frac{\partial \rho'}{\partial t} + \frac{\kappa}{C_p} (\gamma - 1) \frac{\partial^2 \rho'}{\partial t^2} + c_0^2 \frac{B}{\rho_0} \frac{\partial ^2}{\partial t} - \mathbf{u} \cdot (\nabla p_0 - c_0^2 \nabla \rho_0) . \quad (2.25) \]

As mentioned above although the result was obtained for an ideal gas it is valid for any fluid (see, for example, Hamilton 1993). Note for liquids \( C_p \) and \( C_v \) are nearly identical, \( \gamma \approx 1 \), and there is very little heat conduction; classical absorption is nearly entirely due to viscosity. In gases both thermal conduction and viscosity are important. However, in many fluids other loss mechanisms can dominate thermoviscous absorption.

2.5.1 Relaxing Fluid

The internal energy of molecules in many fluids is not just determined by the translational energy. There exists a large number of internal degrees of freedom. For example, in air the rotational and vibrational states of \( \text{N}_2 \) and \( \text{O}_2 \) molecules can store energy. The manner in which energy is transferred in and out of internal states is called a relaxation process. Relaxation processes are equilibrium phenomena and the presence of an acoustic wave can prevent the process from being in equilibrium. This is because after being disturbed a relaxation process requires a finite time, the relaxation time, to reach the new equilibrium condition. The frequency of an acoustic wave plays a major role in determining how close to equilibrium a process can get. The rotational and vibrational states of \( \text{N}_2 \) and \( \text{O}_2 \) molecules are important relaxing processes in air and dominate the absorption of sound waves in the audio range. In sea water it is the relaxation processes related to the dissociation of Boric acid and magnesium sulphate into their respective ions that are important in determining the absorption of sound at low frequencies.
A model for relaxation processes needs to be incorporated into the equation of state. Each relaxation process is represented by a new internal coordinate. The new variables can be accommodated in the energy equation, as Pierce (1981, Chap 10-7) does for air. Alternatively, the equation of state can be explicitly written in terms of the new thermodynamic variables (see, for example, Rudenko and Soluyan 1977, Chap. 4). In this section the latter route is followed.

The new thermodynamic variable associated with each relaxation process is \( \xi_\nu \). The equation of state is now \( p = p(p, \chi, \xi_1, \xi_2, \cdots) \) and is expanded around the equilibrium value not the ambient conditions. Each \( \xi_\nu \) is broken up into an equilibrium value \( \xi_{\nu,q} \) and the deviation \( \xi'_\nu \) from equilibrium, that is,

\[
\xi_\nu = \xi_{\nu,q} + \xi'_\nu .
\]  

The Taylor series of the equation of state in this case is

\[
p' = \rho' \left. \frac{\partial p}{\partial \rho} \right|_q + \frac{\rho'^2}{2} \left. \frac{\partial^2 p}{\partial \rho^2} \right|_q + \chi' \left. \frac{\partial p}{\partial \chi} \right|_q + \sum_\nu \xi'_\nu \left. \frac{\partial p}{\partial \xi_\nu} \right|_q + \cdots .
\]

The summation is over all the relaxation processes and it is assumed that each relaxation term is second order. Note the subscript \( q \) denotes that the term is evaluated at the equilibrium conditions as opposed to the subscript \( 0 \) which is evaluated at the ambient conditions.

The small signal sound speed \( c_0 \) is defined as the equilibrium sound speed (Pierce 1981, p. 558), that is,

\[
c_0 \equiv \left. \frac{\partial p}{\partial \rho} \right|_q .
\]  

However, the previous definitions for \( \frac{\partial^2 p}{\partial \rho^2} \) and \( \frac{\partial p}{\partial \chi} \) are at the ambient conditions. It is assumed that the equilibrium value is a small order \( \varepsilon \) fluctuation about the ambient condition, that is,

\[
\left. \frac{\partial^2 p}{\partial \rho^2} \right|_q = \left. \frac{\partial^2 p}{\partial \rho^2} \right|_0 + O(\varepsilon) .
\]

The quantity \( \frac{\partial^2 p}{\partial \rho^2} \) is involved in a second order term so it is valid to replace it with a first order equivalent, that is,

\[
\frac{\rho'^2}{2} \left. \frac{\partial^2 p}{\partial \rho^2} \right|_q = \frac{\rho'^2}{2} \left. \frac{\partial^2 p}{\partial \rho^2} \right|_0 + O(\varepsilon^3) .
\]
Similarly $\chi' \frac{\partial p}{\partial \chi_q}$ can be replaced with $\chi' \frac{\partial p}{\partial \chi_{0q}}$.

The summation over the relaxation processes now needs to be addressed. It is assumed that each process tends towards its equilibrium value according to the following law (see, for example, Rudenko and Soluyan 1977, Eq. 4.1.13; Pierce 1981, p. 549),

$$\frac{\partial \xi_{\nu}}{\partial t} = -\frac{\xi_{\nu} - \xi_{\nu,q}}{\tau_{\nu}}, \quad (2.28)$$

where $\tau_{\nu}$ is the relaxation time. The left-hand side can be expanded as follows

$$\frac{\partial \xi_{\nu}}{\partial t} = \frac{\partial \xi'_{\nu}}{\partial t} + \frac{\partial \xi_{\nu,q}}{\partial t}. \quad (2.29)$$

Beware that the equilibrium value is time dependent — it varies with the passage of an acoustic wave. For example, in air during the compression phase of a travelling wave the gas is hotter and the equilibrium condition has more molecules in an excited vibrational state than during the cooler, rarefraction phase of the wave. To first order the equilibrium value can be written as a function of the acoustic density

$$\xi_{\nu,q} = \xi_{\nu,0} + \rho' \frac{\partial \xi_{\nu,q}}{\partial \rho'} \bigg|_0 + O(\varepsilon^2, \delta \varepsilon), \quad (2.30)$$

where $\xi_{\nu,0}$ is the ambient equilibrium state. Note it is assumed that all the relaxation processes are independent to second order.

When the expressions for $\frac{\partial \xi_{\nu}}{\partial t}$ and $\xi_{\nu,q}$ (Eqs. 2.28 and 2.30) are substituted into Eq. 2.29 one obtains

$$\frac{\xi'_{\nu}}{\tau_{\nu}} = \frac{\partial \xi'_{\nu}}{\partial t} + \frac{\partial \xi_{\nu,q}}{\partial \rho'} \bigg|_0 \frac{\partial \rho'}{\partial t},$$

$$\xi'_{\nu} + \tau_{\nu} \frac{\partial \xi'_{\nu}}{\partial t} = -\tau_{\nu} \frac{\partial \xi_{\nu,q}}{\partial \rho'} \bigg|_0 \frac{\partial \rho'}{\partial t},$$

$$\xi'_{\nu} = -\frac{\partial \xi_{\nu,q}}{\partial \rho'} \bigg|_0 \frac{\tau_{\nu}}{1 + \frac{\tau_{\nu} \rho'}{\partial t}}. \quad (2.31)$$

The operator may be expressed as the following integral:

$$\tau_{\nu} \int_0^t \frac{f(t)}{1 + \frac{\tau_{\nu} \rho'}{\partial t}} dt = \int_{-\infty}^t e^{(t-t')/\tau_{\nu}} f(t') dt'.$$

The equation of state can now be written in a form that expresses the acoustic pressure in terms of the acoustic density alone

$$p' = c_0^2 \rho' + \frac{c_0^2}{\rho_0} B \rho'^2 + \frac{\kappa}{C_p} \gamma (\gamma - 1) \frac{\partial \rho'}{\partial t} - \sum_{\nu} \frac{\partial p}{\partial \xi_{\nu,q}} \bigg|_0 \frac{\partial \xi_{\nu,q}}{\partial \rho'} \bigg|_0 \frac{\tau_{\nu} \rho'}{1 + \frac{\tau_{\nu} \rho'}{\partial t}}.$$
Rudenko and Soluyan (1977, Eq. 4.1.16) identify

\[
\frac{\partial p}{\partial \xi_{\nu q}} \bigg|_0 \frac{\partial \xi_{\nu q}}{\partial p} \bigg|_0 = c_0^2 - c_\infty^2,
\]

where \( c_0^2 \) is the equilibrium sound speed and \( c_\infty^2 \) the frozen sound speed — these terms are discussed below. The dispersion parameter \( m_\nu \) is introduced

\[
m_\nu = \frac{c_\infty^2 - c_0^2}{c_0^2},
\]

\[
= -\frac{1}{c_0^2} \frac{\partial p}{\partial \xi_{\nu q}} \bigg|_0 \frac{\partial \xi_{\nu q}}{\partial p} \bigg|_0.
\]

(2.32)

The equation of state may now be written in the following form:

\[
p' = c_0^2 \rho' + \frac{c_0^2}{\rho_0} B \rho'^2 + \frac{\kappa}{C_p} (\gamma - 1) \frac{\partial \rho'}{\partial t} + c_0^2 \sum_{\nu} m_{\nu} \tau_{\nu} \frac{\partial \rho'}{\partial t}.
\]

Recall that it was assumed that each of the relaxation terms is second order. The assumption requires that \( m_\nu \) be small parameter. For air \( m_\nu \) is indeed small, \( 6.4 \times 10^{-4} \) and \( 1.3 \times 10^{-4} \) at \( 20^\circ C \) for \( \text{O}_2 \) and \( \text{N}_2 \) respectively. To avoid generating a large number of small parameters we shall consider each \( m_\nu \) to be the same order as \( \delta \), even though relaxation processes are not related to the viscous dissipation.

The physical meaning of \( m_\nu \) can be observed by the following simple analysis. Assume only one relaxation process exists and that the nonlinearity and thermal conduction can be ignored, the equation of state reduces to

\[
p' = c_0^2 \rho' + \frac{c_0^2}{\rho_0} m \tau \frac{\partial \rho'}{\partial t}.
\]

The first order the relationship between \( p' \) and \( \rho' \) yields the small signal sound speed. In the low frequency limit \( \tau \frac{\partial \rho'}{\partial t} \sim 0 \) and \( p' = c_0^2 \rho' \). The small signal sound speed is \( c_0 \) which is also called the equilibrium sound speed because the fluctuations of the acoustic wave occur so slowly that the relaxation process always has enough time to adjust and stay in equilibrium. At high frequencies,

\[
\frac{m \tau \frac{\partial \rho'}{\partial t}}{1 + \tau \frac{\partial \rho'}{\partial t}} \sim \frac{m \tau j\omega}{\tau j\omega} = m,
\]

and so

\[
p' = (c_0^2 + mc_0^2) \rho'.
\]
The small signal sound speed is now slightly higher, \( c_\infty = c_0 \sqrt{1 + m} \). This is called the \textit{frozen sound speed} as the acoustic fluctuations are so rapid that the relaxation process has no time to react and therefore remains frozen in the ambient equilibrium state. The parameter \( m = (c^2_\infty - c^2_0) / c^2_0 \) characterizes the dispersion of the relaxation process.*

The equation of state for a homogeneous fluid can now be written in the form

\[
p' = c^2_0 \rho' + \mathcal{K} \frac{\partial \rho'}{\partial t} + \frac{c^2_0 B}{\rho_0 2 A} \rho'^2.
\]

where \( \mathcal{K} \) is a linear operator which includes both relaxation and conduction effects

\[
\mathcal{K} = \frac{\kappa}{\rho_0 C_p} (\gamma - 1) + c^2_0 \sum_{\nu} \frac{m_{\nu} \tau_{\nu}}{1 + \tau_{\nu} \frac{\partial}{\partial t}}.
\]

It is common to group thermal and viscous coefficients into a single thermoviscous coefficient \( b = \lambda + 2\mu + \frac{\kappa}{C_p} (\gamma - 1) \). It is convenient therefore to define the relaxation "coefficient"

\[
\mathcal{R} = \rho_0 c^2_0 \sum_{\nu} \frac{m_{\nu} \tau_{\nu}}{1 + \tau_{\nu} \frac{\partial}{\partial t}}.
\]

For an inhomogeneous relaxing fluid the equation of state is

\[
\frac{\partial p'}{\partial t} = c^2_0 \frac{\partial \rho'}{\partial t} + \mathcal{K} \frac{\partial^2 \rho'}{\partial t^2} + \frac{c^2_0 B}{\rho_0 2 A} \frac{\partial \rho'^2}{\partial t} - u \cdot (\nabla p_0 - c^2_0 \nabla \rho_0).
\]

### 2.6 Westervelt Type Equation

In this section a full three dimensional wave equation for finite-amplitude waves is derived. This equation is akin to the Westervelt equation but includes relaxation processes. In the next section it is shown that the Burgers equation is a limiting case of the Westervelt equation. The derivation in this section follows that presented by Hamilton (1993, Chap. 4).

The fluid dynamic equations, correct to second order, for irrotational flow in a homogenous fluid are repeated here. They are the continuity equation (Eq. 2.4),

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot u = -\rho' \nabla \cdot u - u \cdot \nabla \rho',
\]

*Pierce (1981, p. 561) describes the dispersion in terms of a change in small signal sound speed \( (\Delta c)_\nu, m_\nu \approx 2(\Delta c)_\nu / c_0. \)
conservation of momentum (Eq. 2.5),
\[ \rho_0 \frac{\partial u}{\partial t} + \nabla p' = (\lambda + 2\mu) \nabla (\nabla \cdot u) - \frac{\rho_0}{2} \nabla u^2 - \rho \frac{\partial u}{\partial t}, \]  
(2.38)

where \( u^2 = u \cdot u \), and the equation of state, (Eq. 2.7),
\[ p' = c_0^2 p' + k \frac{\partial p'}{\partial t} + \frac{c_0^2}{\rho_0} \frac{B}{2} \rho'^2. \]  
(2.39)

Recall that the ranking system allows first order relations to be used to manipulate second order terms, since the error this introduces is third order. The first order relations in this case are:
\[ \frac{\partial p'}{\partial t} = -\rho_0 \nabla \cdot u, \]  
(2.40)
\[ \frac{\partial u}{\partial t} = -\nabla p', \]  
(2.41)
\[ p' = c_0^2 p'. \]  
(2.42)

Note the linear acoustic wave equation may be recovered by subtracting the divergence of Eq. 2.41 from the time derivative of Eq. 2.40,
\[ \frac{\partial^2 p'}{\partial t^2} = \nabla^2 p'. \]

Equation 2.42 can be used to eliminate either the pressure or density, for example, the following relation was used in Sec. 2.5:
\[ \nabla^2 p' = \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} + O(\varepsilon^2). \]  
(2.43)

The first order equations are now used to manipulate the fluid dynamics equations. For example, the two terms on the right-hand side of the continuity equation, Eq. 2.37, which are of second order, can be manipulated using the first order relations as follows:
\[ -p' \nabla \cdot u - u \cdot \nabla p' = \frac{1}{2\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} + \frac{\rho_0}{2c_0^2} \frac{\partial u^2}{\partial t}. \]

Because an equation is sought in terms of the acoustic pressure \( p' \) it is sensible to express second order terms as functions of \( p' \). Unfortunately it is not possible to
eliminate all the non-pressure terms. What is left over is a quantity know as the Lagrangian density (Aanonsen et al. 1984)

$$\mathcal{L} = \frac{\rho_0 u^2}{2} - \frac{p'^2}{2\rho_0 c_0^2},$$

(2.44)

for example,

$$-\rho' \nabla \cdot u - u \cdot \nabla \rho' = \frac{1}{\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} + \frac{1}{c_0^2} \frac{\partial \mathcal{L}}{\partial t}.$$ 

It turns out the Lagrangian density can be neglected except in situations where local effects are significant (Naze Tjøtta and Tjøtta 1981). In general local effects are important: close to a source, near boundaries, at the edges of beams, in waveguides, at focal regions and where standing waves exist (Hamilton 1993). In the case of progressive plane waves, which are of interest in this work, $p' = \rho_0 c_0 u + \mathcal{O}(\epsilon^2)$ and the Langrangian density is zero, in which case

$$-\rho' \nabla \cdot u - u \cdot \nabla \rho' = \frac{1}{\rho_0 c_0^2} \frac{\partial p'^2}{\partial t}.$$ 

The Lagrangian density is retained in this section but will be set to zero in subsequent sections where progressive wave motion is assumed. The equation of continuity can now be written as

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot u = \frac{1}{\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} + \frac{1}{c_0^2} \frac{\partial \mathcal{L}}{\partial t}. \quad (2.45)$$

The momentum equation can be expressed as

$$\rho_0 \frac{\partial u}{\partial t} + \nabla p' = (\lambda + 2\mu) \nabla (\nabla \cdot u) - \nabla \mathcal{L}. \quad (2.46)$$

Note that Eq. 2.46 becomes linear if $\mathcal{L} = 0$ as the order $\epsilon^2$ terms in the momentum equation cancel. The nonlinear behavior observed in finite-amplitude acoustics appears because of convection in the continuity equation and nonlinearity in the equation of state, not because of nonlinearity in the momentum equation (Hamilton and Blackstock 1988, 1990, Tarkenton 1990).

For a homogeneous medium, subtracting the time derivative of Eq. 2.45 from the divergence of Eq. 2.46 yields

$$\nabla^2 p' - \frac{\partial^2 \rho'}{\partial t^2} - (\lambda + 2\mu) \nabla^2 (\nabla \cdot u) = -\frac{1}{\rho_0 c_0^2} \frac{\partial^2 p'^2}{\partial t^2} - \left( \nabla^2 + \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \mathcal{L}. \quad (2.47)$$
The viscous term is second order and can be manipulated using first order substitutions (Eqs. 2.40, 2.43 and 2.42)
\[
\nabla^2(\nabla \cdot u) = -\frac{1}{\rho_0} \frac{\partial}{\partial t} \nabla^2 \rho',
\]
\[
= -\frac{1}{\rho_0 c_0^2} \frac{\partial^3 \rho'}{\partial t^3},
\]
\[
= -\frac{1}{\rho_0 c_0^2} \frac{\partial^3 \rho'}{\partial t^3} + O(\varepsilon^2).
\]
Equation 2.47, can now be written as
\[
\nabla^2 p' - \frac{\partial^2 \rho'}{\partial t^2} + \frac{(\lambda + 2\mu)}{\rho_0 c_0^4} \frac{\partial^3 p'}{\partial t^3} = -\frac{1}{\rho_0 c_0^4} \frac{\partial^2 p'^2}{\partial t^2} - \left( \nabla^2 + \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \mathcal{L}.
\] (2.48)
All that is left to do is remove the density.

When first order substitutions are applied to the equation of state, Eq. 2.39, one obtains:
\[
\rho' = \frac{p'}{c_0^2} - \frac{\kappa}{c_0^2} \frac{\partial p'}{\partial t} - \frac{1}{\rho_0 c_0^4} \frac{B}{2A} p'^2.
\] (2.49)
Equation 2.49 can be used to remove the density from Eq. 2.48, and produce a wave equation correct to second order in terms of $p'$ and $\mathcal{L}$ only
\[
\left( \nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p' + \frac{(b + R)}{\rho_0 c_0^4} \frac{\partial^3 p'}{\partial t^3} + \frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p'^2}{\partial t^2} = -\left( \nabla^2 + \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \mathcal{L},
\] (2.50)
where
\[
b = \lambda + 2\mu + \frac{\kappa}{C_p} (\gamma - 1),
\] (2.51)
accounts for both thermal and viscous coefficients and $R$ is the coefficient for relaxation absorption, Eq. 2.35. Note that $b + R = \lambda + 2\mu + \rho_0 \kappa$.

The left-hand side of Eq. 2.50 is a version of Westervelt’s equation (Westervelt 1963), extended to include relaxation processes (Aanonsen et al. 1984). The right-hand side describes local effects and as mentioned above can often be neglected. Various limits of this equation lead to many of the major equations used for modeling finite-amplitude acoustics. In the following section the classical Burgers equation is obtained. Naze Tjøtta and Tjøtta (1981) explain how the KZ equation (Zabolotskaya and Khokhlov 1969) and the KZK equation (Kuznetsov 1971), both of which include diffraction effects, are also special cases of Eq. 2.50.
2.7 The Classical Burgers Equation

The classical Burgers equation models the propagation of progressive, plane waves in thermoviscous fluids. As mentioned in the previous section the Lagrangian density is zero for progressive plane waves. We use this fact along with a coordinate transformation to turn the Westervelt equation into Burgers’ equation.

For one dimensional waves $\nabla = \frac{\partial}{\partial z}$ and the Westervelt equation becomes

$$
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p' + \frac{b}{\rho_0 c_0^4} \frac{\partial^3 p'}{\partial t^3} = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p^2}{\partial t^2}.
$$

(2.52)

A suitable coordinate transformation is sought so that a progressive wave equation can be achieved. The transformation can be gleaned by looking at the effects of absorption and nonlinearity independently (see, for example, Hamilton 1993).

If the nonlinear term is dropped in Eq. 2.52, i.e., $\beta = 0$, the linear wave equation for a lossy fluid is recovered. The time harmonic solution for a progressive wave is

$$
p'(z, \tau) = A e^{i\omega \tau - \frac{\omega^2 k}{2\rho_0 c_0^2} z},
$$

(2.53)

where $A$ is the amplitude at the source and $\tau = t - z/c_0$ is the time frame moving at the small signal sound speed.

On the other hand, if the losses are neglected, i.e., $b = 0$, an approximate first integral of Eq. 2.52 yields a progressive wave equation (see, Appendix of Hamilton and Blackstock 1988)

$$
\frac{\partial p'}{\partial t} + (c_0 + \beta u) \frac{\partial p'}{\partial z} = 0.
$$

Given a source condition $f(t) = p'(z = 0, t)$ the Poisson solution (see, for example, Blackstock 1972) is

$$
p'(z, t) = f \left( \tau + \frac{\beta}{c_0} \left( \frac{p'}{\rho_0 c_0^2 z} \right) \right).
$$

(2.54)

From inspection of Eqs. 2.53 and 2.54 it is apparent that in the frame of reference moving at the small signal sound speed $t - \frac{z}{c_0}$, both solutions vary on a slow length scale. The dissipative, linear solution varies as $\delta z$ and the lossless nonlinear solution varies as $\varepsilon z$. The Westervelt equation can be transformed to a progressive wave equation using a retarded time frame and slow length scale.
A new coordinate system is introduced with a retarded time frame $\tau$ and a slow range variable $x$:

$$\tau = t - z/c_0,$$

$$x = \varepsilon z.$$

It is common in the literature to use $\varepsilon$ for the slow scale. However, any slow scale is valid, for example, $\delta$ or $\zeta$ could also have been used. For sonic booms at least, nonlinearity effects generally have the shortest length scale, therefore all other effects should occur slowly even on this length scale. The derivatives in the old coordinate system can be expressed as

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial x}{\partial t} \frac{\partial}{\partial x},$$

$$= \frac{\partial \tau}{\partial t},$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z} \frac{\partial}{\partial x} + \frac{\partial \tau}{\partial z} \frac{\partial}{\partial \tau},$$

$$= \frac{\partial}{\partial x} - \frac{1}{c_0} \frac{\partial}{\partial \tau},$$

$$\frac{\partial^2}{\partial x^2} = \varepsilon \frac{\partial^2}{\partial x^2} - \frac{2}{c_0} \varepsilon \frac{\partial}{\partial x} \frac{\partial}{\partial \tau} + \frac{1}{c_0^2} \frac{\partial}{\partial \tau^2}.$$  

When the transformations are applied to Eq. 2.52 one obtains, to second order,

$$-\frac{2}{c_0} \varepsilon \frac{\partial^2 p'}{\partial x \partial \tau} + \frac{(b + \mathcal{R})}{\rho_0 c_0^4} \frac{\partial^3 p'}{\partial \tau^3} = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2 p'^2}{\partial \tau^2}.$$

This equation can be integrated once with respect to $\tau$

$$\varepsilon \frac{\partial p'}{\partial x} - \frac{(b + \mathcal{R})}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial \tau^2} = \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial \tau}.$$  

The constant of integration is set to zero on the assumption that at $\tau = \pm \infty$ there is no acoustical signal.

It is possible to return to the usual length scale, but keep the retarded time frame, with the transformations $t' = \tau$ and $z = \frac{1}{\varepsilon} x$. The derivatives become $\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial z} = \frac{1}{\varepsilon} \frac{\partial}{\partial x}$. For the case of no relaxation, $\mathcal{R} = 0$, the classical Burgers equation, Eq. 2.1, is obtained

$$\frac{\partial p'}{\partial z} - \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial t'} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2}.$$
If relaxation processes are present a form of the augmented Burgers equation, Eq. 2.3, is recovered

\[ \frac{\partial p'}{\partial z} - \frac{\beta}{2\rho_0 c_0^2} \frac{\partial p'^2}{\partial t'} = \frac{(b + R)}{2\rho_0 c_0^2} \frac{\partial^2 p'}{\partial t'^2}. \]

**2.8 Propagation in a Relaxing Fluid**

In this section the augmented Burgers equation is rederived. However, rather than going through the Westervelt equation, the retarded time and slow length scale transformations, introduced in the previous section, are applied directly to the fluid dynamics equations. The method produces the same result as going through the full wave equation. The technique is used in subsequent sections to obtain a Burgers type equation for cases where a Westervelt type equation is awkward to derive.

The equations of continuity and conservation of momentum, in one dimension, follow from Eqs. 2.37 and 2.38, (where \( \nabla \) is replaced with \( \frac{\partial}{\partial z} \)),

\[ \frac{\partial p'}{\partial t} + \rho_0 \frac{\partial u}{\partial z} = -\frac{\partial (p' u)}{\partial z}, \quad (2.55) \]

\[ \rho_0 \frac{\partial u}{\partial t} + \frac{\partial p'}{\partial z} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial z^2} - \frac{\rho_0}{2} \frac{\partial u^2}{\partial z} - \rho' \frac{\partial u}{\partial t}. \quad (2.56) \]

The equation of state for a fluid with multiple relaxation processes, (Eq 2.7), is

\[ p' = c_0^2 \rho' + K \frac{\partial p'}{\partial t} + \frac{c_0^2 B}{\rho_0} \rho'^2. \quad (2.57) \]

Recall that the transformations of the independent variables to a retarded time frame and slow range variable are \( \tau = t - \frac{z}{c_0} \) and \( x = \epsilon z \). The derivatives in the old coordinate system become \( \frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial \tau} - \frac{1}{c_0} \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}. \)

When the transformations are applied to the continuity equation one obtains:

\[ \frac{\partial p'}{\partial \tau} + \rho_0 \left( \epsilon \frac{\partial u}{\partial x} - \frac{1}{c_0} \frac{\partial u}{\partial \tau} \right) = -\epsilon \frac{\partial p' u}{\partial x} + \frac{1}{c_0} \frac{\partial p' u}{\partial \tau}. \]

The term \( \epsilon \rho' u \) is third order and can be neglected, whence

\[ \frac{\partial p'}{\partial \tau} - \frac{\rho_0}{c_0} \frac{\partial u}{\partial \tau} = -\rho_0 \epsilon \frac{\partial u}{\partial x} + \frac{1}{c_0} \frac{\partial p' u}{\partial \tau}. \quad (2.58) \]
The momentum equation becomes

\[ \rho_0 \frac{\partial u}{\partial t} + \left( \varepsilon \frac{\partial p'}{\partial x} - \frac{1}{c_0} \frac{\partial p'}{\partial t} \right) = (\lambda + 2\mu) \left( \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{2\varepsilon}{c_0} \frac{\partial^2 u}{\partial x \partial t} + \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} \right) \]

\[ - \frac{\rho_0}{2} \left( \varepsilon \frac{\partial u}{\partial x} + \frac{1}{c_0} \frac{\partial u}{\partial t} \right) - \rho' \frac{\partial u}{\partial t}, \]

\[ \rho_0 \frac{\partial u}{\partial t} - \frac{1}{c_0} \frac{\partial p'}{\partial t} = -\varepsilon \frac{\partial p'}{\partial x} - \frac{(\lambda + 2\mu)}{c_0^2} \frac{\partial^2 u}{\partial t^2} - \frac{\rho_0}{2c_0} \frac{\partial^2 u}{\partial t^2} - \rho' \frac{\partial u}{\partial t} . \quad (2.59) \]

The equation of state remains

\[ p' = c_0^2 \rho' + K \frac{\partial \rho'}{\partial t} + \frac{\rho_0}{\rho_0} \frac{B}{2A} \rho'^2 . \quad (2.60) \]

The first order relations from Eqs. 2.58-2.60 are: \( \rho' = \rho_0 u/c_0, \ p' = \rho_0 c_0 u \) and \( p' = c_0^2 \rho' \). We manipulate the second order terms to become functions of acoustic pressure \( p' \). The continuity equation becomes

\[ \frac{\partial p'}{\partial t} - \frac{\rho_0}{c_0} \frac{\partial u}{\partial t} = -\varepsilon \frac{\partial p'}{\partial x} + \frac{1}{\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} . \]

The momentum equation becomes

\[ \rho_0 \frac{\partial u}{\partial t} - \frac{1}{c_0} \frac{\partial p'}{\partial t} = -\varepsilon \frac{\partial p'}{\partial x} + \frac{(\lambda + 2\mu)}{c_0^2} \frac{\partial^2 p'}{\partial t^2} + \frac{1}{2\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} - \frac{1}{2\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} , \]

\[ = -\varepsilon \frac{\partial p'}{\partial x} + \frac{(\lambda + 2\mu)}{\rho_0 c_0^2} \frac{\partial^2 p'}{\partial t^2} . \]

Note that at second order \( u^2 \) and \( \rho' u \) cancel each other just as occurred when \( \mathcal{L} = 0 \) in the previous section. The equation of state is

\[ p' = c_0^2 \rho' + K \frac{\rho_0}{c_0^2} \frac{B}{2A} \rho'^2 . \]

The equation of state can be used to remove the density from the continuity equation, whence

\[ \frac{1}{c_0^2} \frac{\partial p'}{\partial t} - \frac{K}{c_0^4} \frac{\partial^2 p'}{\partial t^2} - \frac{1}{\rho_0 c_0^2} \frac{B}{2A} \frac{\partial^2 p'}{\partial t^2} - \frac{\rho_0}{c_0} \frac{\partial u}{\partial t} = -\varepsilon \frac{\partial p'}{\partial x} + \frac{1}{\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} , \]

\[ \frac{1}{c_0} \frac{\partial p'}{\partial t} - \frac{\rho_0}{c_0} \frac{\partial u}{\partial t} = -\varepsilon \frac{\partial p'}{\partial x} + \frac{K}{c_0^3} \frac{\partial^2 p'}{\partial t^2} + \frac{\beta}{\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} . \quad (2.61) \]
Equation 2.61 is added to the momentum equation to eliminate the term with the particle velocity \( \frac{\partial v}{\partial t} \) (\( \frac{\partial v}{\partial t} \) is also eliminated). This yields what turns out to be a progressive wave equation in terms of \( p' \) only

\[
0 = -2 \epsilon \frac{\partial p'}{\partial x} + \frac{(\lambda + 2\mu + \rho_0 K)}{\rho_0 c_0^3} \frac{\partial^2 p'}{\partial \tau^2} + \frac{\beta}{\rho_0 c_0^3} \frac{\partial p'^2}{\partial \tau}.
\]

The derivation is completed by returning from the slow range variable to the physical range variable. The retarded time frame is maintained. The new coordinates are, \( z = x/e \) and \( t' = \tau \) and the derivatives are \( \frac{\partial}{\partial x} = \frac{1}{e} \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t'} \). The augmented Burgers equation, Eq. 2.3, is rederived

\[
\frac{\partial p'}{\partial z} - \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial t'} = \frac{b + \mathcal{R}}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2}.
\]

The technique of transforming the fluid dynamics equations to a moving time frame and slow length scale can now be applied to the more complex cases of ray tube spreading and an inhomogeneous medium.

Note that if the relaxation operator is expanded the augmented Burgers equation can be written

\[
\frac{\partial p'}{\partial z} - \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial t'} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2} + \frac{1}{2c_0} \sum\nu \frac{m\nu \tau v\partial^2 p'}{1 + \tau v \partial p'}.
\]

If the internal pressure \( p_\nu \) is defined as \( p_\nu = \tau_\nu \frac{\partial p'}{\partial t'} / (1 + \tau_\nu \frac{\partial}{\partial t'}) \) then a form of Pierce's coupled equations (Pierce 1981, Eqs. 11.6.5 and 11.6.3b) can be recovered

\[
\frac{\partial p'}{\partial z} - \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial t'} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2} + \frac{1}{c_0^2} \sum\nu (\Delta c)_\nu \frac{\partial p_\nu}{\partial t'},
\]

\[
\left(1 + \tau_\nu \frac{\partial}{\partial t'}\right) p_\nu = \tau_\nu \frac{\partial p'}{\partial t'}.
\]

This form alleviates the need for an integral operator.

2.9 Propagation along Ray Tubes and Horns

The propagation of sound down a ray tube is identical to that of propagation down a horn. In both cases it is assumed that the flow is one dimensional, that is, there is no
particle velocity perpendicular to the propagation direction. In ray theory this means that there is no interaction between neighbouring ray tubes, and hence no diffraction can occur. The most common place for the one dimensional assumption to fail in ray theory is at caustics and foci. In horns the one dimensional assumption fails if the flare is rapid or the mouth is large compared to a characteristic length of the wave (Post 1994).

Sonic boom propagation in the atmosphere is typically modeled using linear geometrical acoustics to predict ray paths and ray tube areas. Finite-amplitude effects only enter into the propagation problem along the ray path. The effects of self refraction, where finite-amplitude effects distort the wavefront, are neglected (Rudenko and Soluyan 1977, Chap. 9.5). For cases where diffraction and self refraction are important, such as high intensity ultrasonic beams, the KZK equation may be an appropriate propagation model.

In this section finite-amplitude propagation down a horn or ray tube in a homogeneous fluid is considered. The generalized Burgers equation derived in this section is most useful for the propagation of cylindrically and spherically spreading waves, which are common spreading behaviour in an isothermal fluid. In the next section an inhomogeneous fluid is considered where refraction can occur and more complicated ray tube areas are possible.

For the one dimensional assumption to hold it is necessary that the area of the ray tube or horn $S$ changes slowly with respect to a characteristic length $L_c$ associated with the wave. The length scale for the ray tube or horn is

$$l_h \sim \left( \frac{1}{S} \frac{dS}{dz} \right)^{-1}. \quad (2.63)$$

It is assumed that $\nu = l_c/l_h$ is a small parameter. The consequence of this restriction is discussed for the case of cylindrical or spherical spreading later in this section.

The fluid dynamics equations for sound propagation in a horn are (see for example, Pierce 1981, Chap. 7-8; Morse 1981, Sec VI.24; Kinsler, Frey, et al. 1982, Chap. 14.7; Blackstock 1996, Chap. 7)

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u}{\partial z} + \rho_0 u \frac{\partial \ln S}{\partial z} = -\rho' \frac{\partial \ln S}{\partial z} - \frac{\partial \rho' u}{\partial z}, \quad (2.64)$$
Note that only the continuity equation is changed by the inclusion of the effect of the horn or ray tube area.

We have assumed that the spreading occurs on a slow scale. A generalized Burgers equation can therefore be obtained using the same retarded time frame and slow range variable transformations on the fluid dynamics equations. Recall the transformations are $\tau = t - z/c_0$ and $x = \varepsilon z$ and the derivatives become $\frac{\partial}{\partial \tau} = \varepsilon \frac{\partial}{\partial x} - \frac{1}{c_0} \frac{\partial}{\partial \tau}$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}$.

The continuity equation in the new coordinate system is

$$\frac{\partial \rho'}{\partial \tau} + \rho_0 \left( \varepsilon \frac{\partial u}{\partial x} - \frac{1}{c_0} \frac{\partial u}{\partial \tau} \right) + \rho_0 u \left( \varepsilon \frac{\partial \ln S}{\partial x} - \frac{1}{c_0} \frac{\partial \ln S}{\partial \tau} \right) = \rho' u \left( -\varepsilon \frac{\partial \ln S}{\partial x} - \frac{1}{c_0} \frac{\partial \ln S}{\partial \tau} \right) - \varepsilon \frac{\partial \rho' u}{\partial x} + \frac{1}{c_0} \frac{\partial \rho' u}{\partial \tau}.$$  \hspace{1cm} (2.67)

Note that $\frac{\partial S}{\partial \tau} = 0$ and $\varepsilon \rho' u$ is third order so the continuity equation correct to second order is

$$\frac{\partial \rho'}{\partial \tau} - \frac{\rho_0 \varepsilon u}{c_0} \frac{\partial \ln S}{\partial x} = \frac{\rho_0}{c_0} \frac{\partial u}{\partial \tau} + \frac{1}{c_0} \frac{\partial \rho' u}{\partial \tau}.$$  \hspace{1cm} (2.67)

The momentum equation is identical to that in the previous section (Eq. 2.56)

$$\frac{\rho_0}{c_0} \frac{\partial u}{\partial \tau} - \frac{1}{c_0} \frac{\partial \rho'}{\partial \tau} = -\varepsilon \frac{\partial \rho'}{\partial x} + \left( \lambda + 2 \mu \right) \frac{\partial^2 u}{\partial \tau^2} - \frac{\rho_0}{2c_0} \frac{\partial u}{\partial \tau} - \rho' \frac{\partial u}{\partial \tau}.$$  \hspace{1cm} (2.67)

The equation of state is also the same (Eq. 2.57)

$$\rho' = c_0^2 \rho' + \kappa \frac{\partial \rho'}{\partial \tau} + \frac{c_0^3}{\rho_0} \frac{B}{2A} \rho'^2.$$  \hspace{1cm} (2.67)

The first order relations are $\rho' = \rho_0 u/c_0$, $\rho' = \rho_0 c_0 u$ and $\rho' = c_0^2 \rho'$. Again these expression are used to manipulate second order terms so that they can be expressed in terms of $\rho'$.

The continuity equation can now be written

$$\frac{\partial \rho'}{\partial \tau} - \frac{\rho_0}{c_0} \frac{\partial u}{\partial \tau} = -\varepsilon \frac{\partial \rho'}{\partial x} + \frac{\rho_0}{c_0} \frac{\partial u}{\partial \tau} - \rho' \frac{\partial u}{\partial \tau} + \frac{1}{c_0} \frac{\partial \rho'^2}{\partial \tau}.$$  \hspace{1cm} (2.68)
The momentum equation is manipulated in the same way as the previous section. The nonlinear acoustical terms cancel, and one obtains
\[ \rho_0 \frac{\partial u}{\partial t} - \frac{1}{c_0^2} \frac{\partial p'}{\partial t} = \frac{\varepsilon}{c_0} \frac{\partial p'}{\partial x} + \frac{(\lambda + 2\mu)}{\rho_0 c_0^3} \frac{\partial^2 p'}{\partial r^2}. \]

The equation of state is
\[ p' = \rho_0 c_0^2 \ell + \frac{K}{c_0^3} \frac{\partial p'}{\partial r} + \frac{1}{\rho_0 c_0^2} \frac{B}{2A} p'^2. \]

The equation of state can be used to remove the density from the continuity equation,
\[ \frac{1}{c_0^2} \frac{\partial p'}{\partial t} - \frac{K}{c_0^3} \frac{\partial^2 p'}{\partial r^2} - \frac{1}{\rho_0 c_0^3} \frac{B}{2A} \frac{\partial p'}{\partial t} - \rho_0 \frac{\partial u}{\partial t} = -\varepsilon \frac{\partial p'}{\partial x} - \frac{\rho_0}{c_0} \frac{\varepsilon}{c_0} \frac{\partial \ln S}{\partial x} + \frac{1}{\rho_0 c_0^3} \frac{\partial p'}{\partial r}. \]

This equation can be added to the momentum equation to eliminate the particle velocity \( \frac{\partial u}{\partial t} \). We obtain a wave equation in terms of \( p' \) only:
\[ 0 = -2\varepsilon \frac{\partial p'}{\partial x} - \frac{p'}{c_0} \frac{\partial \ln S}{\partial x} + \frac{(b + R)}{\rho_0 c_0^3} \frac{\partial^2 p'}{\partial r^2} + \frac{\beta}{\rho_0 c_0^3} \frac{\partial p'^2}{\partial r}. \]

The range variable is returned to its ordinary scale again via the transformations \( z = x/\varepsilon \) and \( t' = t \), the derivatives are \( \frac{\partial}{\partial x} = \frac{1}{\varepsilon} \frac{\partial}{\partial z} \), and \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} \). The generalized Burgers equation for propagation in a horn or a ray tube is
\[ \frac{\partial p'}{\partial z} + \frac{1}{2} \frac{\partial \ln S}{\partial z} p' = \frac{(b + R)}{2 \rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2} + \frac{\beta}{2 \rho_0 c_0^3} \frac{\partial p'^2}{\partial t'}. \] (2.69)

Cylindrically and Spherically Spreading Waves.

In the case of cylindrically or spherically spreading waves the ray tube area varies as
\[ \frac{S}{S_0} = \left( \frac{r}{r_0} \right)^a, \]
where \( S \) is the ray tube area at radial distance \( r \) (\( S_0 \) is a reference ray tube area at the source radius \( r_0 \)) and \( a = 1 \) for cylindrical waves or \( a = 2 \) for spherical waves. The generalized Burgers equation is
\[ \frac{\partial p'}{\partial r} + \frac{a p'}{2} \frac{1}{r} \frac{\partial p'^2}{\partial t'} = \frac{(b + R)}{2 \rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2}. \] (2.70)
The length scale for changes in the ray tube area is

\[
L_h = \left( \frac{1}{\frac{dS}{dr}} \right)^{-1},
\]
\[
= \frac{r}{a}.
\]

(2.71)

For outward propagation the smallest length scale is \( L_h = r_0/a \). The restriction on the variation of the ray tube area is that \( L_h \ll L_c \). For time harmonic waves, where \( L_c \) is the wavelength, the restriction can be recast as

\[
k r_0 \gg 1,
\]

where \( k = 2\pi/L_c \) is the wavenumber. This is not a strong restriction, as \( k r_0 \) needs to be much greater than one for the source to radiate efficiently in the first place.

2.10 Propagation in an Inhomogeneous Medium

Propagation of sound in the atmosphere (and the ocean) can often be modelled as propagation through a stratified medium with slowly varying ambient properties; the notable exception being regions of turbulence. It was assumed in the section on ranking that the length scale for the variation in ambient properties \( l_i \) is long with respect to the length scale of the wave \( L_c \). For sonic boom propagation this assumption is quite reasonable. For example, in the lower 20 km of the ISO 9613-1 (1991) or U.S. Standard Atmosphere (1962) the variation in sound speed has a length scale \( l_i \approx 46 \text{ km} \) and for the variation in density \( l_i \approx 8 \text{ km} \). Therefore, even for long sonic booms \( L_c \approx 160 \text{ m} \) (a duration of approximately 0.5 s), \( L_c/l_i < 0.005 \). This restriction is also one of two sufficient conditions for geometrical acoustics (i.e., ray theory) to be valid (Kinsler, Frey, et al. 1982, p. 118). The other is that the amplitude does not change appreciably over distances comparable to the characteristic length of the wave \( L_c \), that is, diffraction is negligible. The second restriction is also valid for sonic booms except at caustics. In this dissertation ray theory is used to describe the propagation of sonic booms through the atmosphere. Although it is possible to accommodate the passage of sonic booms through caustics (see, for example, Robinson 1991) this issue is not addressed in this work.
The derivation in this section follows the same path as in the previous two sections. First, the fluid dynamics equations are laid out. Second, coordinate transformations to a retarded time frame and slow length scale are applied; in this case ray theory is needed to perform the transformations. Finally the equations are manipulated using first order relations to obtain a wave equation.

The continuity equation (Eq. 2.4) is
\[
\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 u) + \nabla \cdot (\rho' u) = 0.
\] (2.72)

The momentum equation (Eq. 2.5) correct to second order, for the case of irrotational flow, is
\[
\rho_0 \frac{\partial u}{\partial t} + \rho' \frac{\partial u}{\partial t} + \rho_0 u \cdot \nabla u = -\nabla p + \rho B + (\lambda + 2\mu) \nabla (\nabla \cdot u),
\]
where the body force is retained. In a quiet medium, with no acoustic field present, the static condition exists:
\[
\nabla p_0 = \rho_0 B,
\]
\[
B = \frac{1}{\rho_0} \nabla p_0.
\] (2.73)

The body force \( B \) can be eliminated from the momentum equation and, after throwing away third order terms,
\[
\rho_0 \frac{\partial u}{\partial t} + \nabla p' = \frac{\rho'}{\rho_0} \nabla p_0 + (\lambda + 2\mu) \nabla (\nabla \cdot u) - \rho_0 u \cdot \nabla u - \rho' \frac{\partial u}{\partial t}.
\] (2.74)

The equation of state for an inhomogeneous medium (Eq. 2.36) is
\[
\frac{\partial p'}{\partial t} = c_0^2 \frac{\partial p'}{\partial t} + \frac{\rho_0}{\rho_0} \frac{B}{2A} \frac{\partial \rho'}{\partial t} + \kappa \frac{\partial^2 p'}{\partial t^2} + u \cdot (c_0^2 \nabla \rho_0 - \nabla p_0) + u \cdot (c_0^2 \nabla \rho' - \nabla p').
\] (2.75)

Note in an inhomogeneous medium it is common to expand the equation of state in terms of the material derivative (Blackstock 1996, Chap. 8), that is,
\[
\frac{Dp}{Dt} = \frac{\partial p}{\partial \rho} \bigg|_{\rho_0} \frac{D\rho}{Dt} + \frac{\partial p}{\partial \chi} \bigg|_{\rho_0} \frac{D\chi}{Dt}.
\]
The advantage with this method is that for isentropic flow in an inhomogeneous medium \( D\chi/Dt = 0 \). The gradients of \( \rho_0 \) and \( \rho_0 \) enter through the material derivatives. In Sec. 2.5 the standard expansion of the equation of state for an inhomogeneous
fluid was developed and the gradients of $p_0$ and $\rho_0$ enter through the entropy term (Pierce 1981, Chap. 8-6).

A progressive wave solution is sought; the retarded time frame and slow range variable transformations are applied once more to the fluid dynamics equations. To do this it is necessary to assume that the waveform travels along rays, that is, to invoke geometrical acoustics. The retarded time frame along a known ray is

$$\tau = t - \int_0^s \frac{1}{c_0} \, ds,$$

where $s$ is the path length and $c_0$ the small signal sound speed can vary with location. The *eikonal* (Cotaras 1985) is defined as $\Psi(r) = \int_0^s \frac{1}{c_0} \, ds$. The eikonal $\Psi(r)$ is the travel time, or phase, between the source and some other point $r$ in space. The surfaces described by constant $\Psi(r)$ (that is, equal travel time) are called the wavefronts. The normal to the wavefronts is in the same direction as $\nabla \Psi(r)$. From the definition of the eikonal it follows that the unit normal is

$$n = c_0 \nabla \Psi(r).$$

In geometrical acoustics it is assumed that ray paths are normal to the wavefront (this must be amended in the case of a moving medium (Uginčius 1965; Pierce 1981, p. 371; Robinson 1991, Chap. 2.2)). Because all the energy is traveling along the ray tube, i.e., normal to the wavefront, the particle velocity can be written

$$u = un.$$

There is no particle velocity perpendicular to the direction of propagation.

The transformations to the retarded time frame and slow range variable are

$$\tau = t - \Psi, \quad r' = \varepsilon r.$$

The derivatives in the old coordinate system are

$$\nabla = \varepsilon \nabla' - \nabla \Psi \frac{\partial}{\partial \tau} = \varepsilon \nabla' - \frac{n}{c_0} \frac{\partial}{\partial \tau},$$
where $\nabla'$ is the gradient operator for the slow range scale, and
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}.
\]

Three new useful identities are used in what follows. First, the length of the unit normal is one, $n \cdot n = 1$. Second, $n \cdot (\nabla' f) = \frac{\partial f}{\partial s'}$, the derivative along the ray path on the slow scale (Cotaras 1985, Chap. 3). The final identity is obtained via the following manipulation
\[
\nabla' \cdot n = \nabla' \cdot (c_0 \nabla' \psi),
\]
\[
= c_0 \nabla'^2 \psi + \nabla' \psi \cdot \nabla' c_0.
\]

Cotaras (1985, Appendix A) shows that $\nabla'^2 \psi = \frac{1}{S} \frac{\partial}{\partial s'} \left( \frac{S}{c_0} \right)$ where $S$ is the ray tube area, and so
\[
\nabla' \cdot n = \frac{c_0}{S} \left( \frac{1}{c_0} \frac{\partial S}{\partial s'} - S \frac{\partial c_0}{\partial s'} \right) + \frac{1}{c_0} \frac{\partial c_0}{\partial s'},
\]
\[
= \frac{1}{S} \frac{\partial S}{\partial s'}.
\]

This is the third identity.

The continuity equation becomes
\[
\frac{\partial \rho'}{\partial \tau} + \left( \varepsilon \nabla' \cdot (\rho_0 u n) - \frac{n}{c_0} \frac{\partial (\rho_0 u n)}{\partial \tau} \right) = -\varepsilon \nabla' \cdot (\rho' u n) + \frac{n}{c_0} \frac{\partial (\rho' u n)}{\partial \tau},
\]
\[
\frac{\partial \rho'}{\partial \tau} = -\rho_0 u \varepsilon \nabla' \cdot n - \varepsilon n \cdot \nabla' (\rho_0 u) + \frac{n}{c_0} \frac{\partial (\rho' u)}{\partial \tau}.
\]

Use of the identities allow the continuity equation to be written as
\[
\frac{\partial \rho'}{\partial \tau} - \frac{\rho_0}{c_0} \frac{\partial u}{\partial \tau} = -\rho_0 u \varepsilon \frac{\partial S}{S} \frac{\partial s'}{\partial \tau} - \varepsilon \frac{\partial (\rho_0 u)}{\partial s'} + \frac{1}{c_0} \frac{\partial (\rho' u)}{\partial \tau}.
\]
This has a similar form to the continuity equation for horns, Eq. 2.67.

The momentum equation becomes
\[
\rho_0 \frac{\partial (u n)}{\partial \tau} + \varepsilon \nabla' p' - \frac{n}{c_0} \frac{\partial p'}{\partial \tau} = \rho_0 \left( \varepsilon \nabla' p_0 - \frac{n}{c_0} \frac{\partial p_0}{\partial \tau} \right) - \rho_0 u \varepsilon \left( \varepsilon \nabla' (u n) - \frac{n}{c_0} \frac{\partial (u n)}{\partial \tau} \right)
\]
\[
+ \left( \lambda + 2\mu \right) \left( \varepsilon^2 \nabla'^2 (u n) + \frac{2\varepsilon}{c_0} \nabla' \frac{\partial u}{\partial \tau} - \frac{n}{c_0^2} \frac{\partial^2 u}{\partial \tau^2} \right) - \rho' \frac{\partial (u n)}{\partial \tau}.
\]
The component of the momentum equation along the ray path can be obtained by dotting the momentum equation with \( \mathbf{n} \). After throwing away higher order terms the momentum equation in the \( \mathbf{n} \) direction is

\[
\frac{\partial u}{\partial t} - \frac{1}{c_0} \frac{\partial p'}{\partial t} = -\frac{\varepsilon}{\partial s'} + \rho' \frac{\varepsilon}{\rho_0} \frac{\partial p_0}{\partial s'} + \frac{(\lambda + 2\mu)}{\rho_0} \frac{\partial^2 u}{\partial t^2} + \frac{\lambda}{\rho_0} \frac{\partial u}{\partial t} - \frac{\partial p'}{\partial t}. \tag{2.79}
\]

Note that the ray tube area \( S \) does not enter into the momentum equation, as was observed in the section on horns.

The equation of state can be written to second order as

\[
\frac{\partial p'}{\partial t} = \frac{\rho_0}{c_0} \frac{\partial p'}{\partial t} + \frac{C_0}{\rho_0} \frac{\partial^2 p'}{\partial t^2} + \frac{\partial^2 p'}{\partial t^2} + \frac{\varepsilon}{\rho_0} \frac{\partial p_0}{\partial s'} \left( \frac{\partial^2 p_0}{\partial s'^2} - \frac{\partial^2 p'}{\partial s'^2} \right) + \frac{\varepsilon}{\rho_0} \frac{\partial p'_0}{\partial s'} - \frac{\varepsilon}{\rho_0} \frac{\partial^3 p'_0}{\partial s'^3} \cdot \tag{2.80}
\]

The first order relations are \( p' = \rho_0 u/c_0, p'_0 = \rho_0 c_0 u, \) and \( p' = \rho_0 c_0 u' \). These relations are used to express all second order terms as functions of acoustic pressure \( p' \). The three equations are: continuity:

\[
\frac{\partial p'}{\partial t} = -\frac{\varepsilon}{\rho_0 c_0} \frac{\partial u}{\partial s'} - \frac{\varepsilon}{\rho_0 c_0} \frac{\partial p'_0}{\partial s'} + \frac{1}{\rho_0 c_0} \frac{\partial p'^2}{\partial t}. \tag{2.81}
\]

Momentum:

\[
\frac{\rho_0}{c_0} \frac{\partial u}{\partial t} - \frac{1}{c_0} \frac{\partial p'}{\partial t} = -\frac{\varepsilon}{\rho_0 c_0} \frac{\partial p_0}{\partial s'} + \rho' \frac{\varepsilon}{\rho_0 c_0} \frac{\partial p_0}{\partial s'} + \frac{(\lambda + 2\mu)}{\rho_0} \frac{\partial^2 p'}{\partial t^2} + \frac{\lambda}{\rho_0} \frac{\partial p'}{\partial t} - \frac{1}{\rho_0} \frac{\partial^3 p'_0}{\partial s'^3} \cdot \tag{2.81}
\]

State:

\[
\frac{\partial p'}{\partial t} = \frac{\partial^2 p'}{\partial t^2} + \frac{1}{\rho_0 c_0} \frac{\partial^2 p_0}{\partial s'^2} - \frac{\partial^2 p'_0}{\partial s'^2}. \tag{2.82}
\]

The equation of state can be used to remove the excess density from the continuity equation,

\[
\frac{1}{c_0} \frac{\partial p'}{\partial t} + \frac{1}{\rho_0 c_0} \left( \frac{\partial^2 p_0}{\partial s'^2} - \frac{\partial^2 p'_0}{\partial s'^2} \right) = -\frac{\varepsilon}{\partial s'} + \frac{1}{\rho_0 c_0} \frac{B}{2A} \frac{\partial^2 p'}{\partial t^2} - \frac{1}{\rho_0 c_0} \frac{\partial^3 p'_0}{\partial s'^3} + \frac{1}{\rho_0 c_0} \frac{\partial p'^2}{\partial t} \tag{2.82}
\]

\[
\frac{1}{c_0} \frac{\partial u}{\partial t} = \frac{1}{\rho_0} \left( \frac{\partial p_0}{\partial s'} - \frac{\partial^2 p'_0}{\partial s'^2} \right) - \frac{\partial^2 p'}{\partial s'^2} - \frac{\partial p'_0}{\partial s'} + \rho' \frac{\varepsilon}{\partial s'} + \frac{\partial p'_0}{\partial s'} \tag{2.82}
\]

\[
+ \frac{\beta}{\rho_0 c_0} \frac{\partial^2 p'_0}{\partial s'^2} + \frac{\partial^2 p'_0}{\partial s'^2}. \tag{2.82}
\]
When Eq. 2.82 is combined with the conservation of momentum, Eq. 2.81, the result is:

\[
0 = -\varepsilon \frac{\partial p'}{\partial s'} + p' \varepsilon \frac{\partial \rho_0}{\partial s'} - p' \frac{\varepsilon S}{\rho_0 c_0^3} \frac{\partial S}{\partial s'} + \frac{(\lambda + 2\mu)}{\rho_0 c_0^3} \frac{\partial^2 p'}{\partial \tau^2} - p' \left( \varepsilon \frac{\partial \rho_0}{\partial s'} - \frac{\lambda}{\rho_0 c_0^3} \frac{\partial \rho_0}{\partial s'} \right) \\
- \varepsilon \frac{\partial p'}{\partial s'} + p' \frac{\varepsilon c_0}{\rho_0} \frac{\partial \rho_0}{\partial s'} + \frac{\beta}{\rho_0 c_0^3} \frac{\partial p'^2}{\partial \tau} + \frac{K}{c_0^4} \frac{\partial^2 p'}{\partial \tau^2},
\]

\[
= -2\varepsilon \frac{\partial p'}{\partial s'} - p' \frac{\varepsilon S}{s' \partial s'} + p' \frac{\varepsilon \partial \rho_0}{\rho_0} \frac{\partial S}{\partial s'} + p' \frac{\varepsilon \partial c_0}{c_0} \frac{\partial S}{\partial s'} + \frac{\beta}{\rho_0 c_0^3} \frac{\partial p'^2}{\partial \tau} + \frac{(b + R)}{\rho_0 c_0^3} \frac{\partial^2 p'}{\partial \tau^2}. \tag{2.83}
\]

To complete the derivation the range variable is transformed from the slow scale back to the physical path length scale, \(s = s' / \varepsilon\). The retarded time frame is maintained \(t' = \tau\). This yields what is referred to in this work as the extended Burgers equation

\[
\frac{\partial p'}{\partial s} + \frac{1}{2S} \frac{\partial S}{\partial s} p' - \frac{1}{2\rho_0} \frac{\partial \rho_0}{\partial s} p' - \frac{1}{2c_0} \frac{\partial c_0}{\partial s} p' = \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial \tau} + \frac{(b + R)}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial \tau^2}. \tag{2.84}
\]

It is similar to the generalized Burgers equation, Eq. 2.2, for propagation of finite amplitude waves in an inhomogeneous medium, but the effect of multiple relaxation processes has been included.

Equation 2.84 can also be written as

\[
\frac{\partial p'}{\partial s} - \frac{\partial}{\partial s} \left( \ln \sqrt{\frac{\rho_0 c_0}{S}} \right) = \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial \tau} + \frac{(b + R)}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial \tau^2}.
\]

If the right-hand side is zero, that is, reducing to the case of linear, lossless acoustics, the pressure is given by

\[
p'(s, t') = \sqrt{\frac{S(0)(\rho_0 c_0)^2}{S(s)(\rho_0 c_0)^2}} p'(0, t').
\]

The acoustic intensity \(p'^2 S / \rho_0 c_0\) is constant along the ray tube.

The extended Burgers equation is used in this dissertation to model the propagation of sonic booms through the atmosphere. The description of the ray path and ray tube area for sonic booms is given in Appendices B and C.
Chapter 3

Analytical Solution: Lossless Propagation

3.1 Introduction

The purpose of this chapter is to investigate waveform freezing and its relevance to sonic booms. For this investigation the effects of nonlinearity, spreading and stratification are needed in the model equation. It is not necessary to include the loss terms in the extended Burgers equation, that is, we use a lossless Burgers equation. The lossless Burgers equation has an exact analytical solution and some of the classical solutions are introduced. Because the loss terms are neglected, the analytical solution predicts multivalued waveforms. A method referred to as weak shock theory is used to recover single valued waveforms and is reviewed here. We show how spreading and inhomogeneities can slow down the amount of nonlinear distortion a finite-amplitude wave suffers. An effective coefficient of nonlinearity is defined to explain the weakening of nonlinear distortion. The condition necessary for finite distortion (waveform freezing) to occur is given. The possibility of waveform freezing for sonic booms in a real atmosphere is analyzed. Finally, it is shown that the effective coefficient of nonlinearity is not a good measure of the distortion of an N wave, therefore we introduce a better measure of distortion for a sonic boom.

3.2 Background

There are no analytical solutions of the generalized form of the Burgers equation,* that is, where spreading and inhomogeneities are modeled. However, if losses are neglected it is possible to obtain an implicit, analytical solution. In this chapter, the absorption terms in the Burgers equation are neglected to analyze the phenomenon of waveform

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*With one exception: a similarity solution exists for cylindrically spreading waves in a homogeneous atmosphere (Rudenko and Soluyan 1977, Chap. 3.4).
freezing of sonic booms. Neglect of the loss terms leaves what is commonly referred to as the lossless Burgers equation.

The lack of energy dissipation in the model leads to a major problem for the propagation of finite-amplitude sound. As a finite-amplitude wave propagates, the waveform translates much like a small-signal wave but it also distorts. When losses are neglected the distortion usually leads to multivalued waveforms. The solution is then non-physical. A physically meaningful waveform can be recovered by using weak shock theory. Pure shocks are inserted into the waveform to remove the multivaluedness (Blackstock 1966). The Rankine-Hugoniot relations (see, for example, Pierce 1981, Chap. 11-3) are applied to ascertain the location and amplitude of shocks in the waveform and correctly account for energy dissipation at the shock. A shock inserted by this method is a discontinuity and no information about the structure of the shock is given. This is the penalty paid for not including absorption explicitly in the propagation equations. Weak shock theory is therefore unsatisfactory if, for example, the loudness of the sonic boom is of interest, because the loudness is strongly dependent on the profile of the shocks. Weak shock theory is also inaccurate when the wave becomes so weak that losses are not well modeled by only accounting for energy dissipation at the shocks. On the other hand, weak shock theory allows one to obtain analytical solutions. The computational price of weak shock theory is minimal compared to solutions of the Burgers equation. A wide variety of important propagation problems can be analyzed using the analytical solution.

For plane waves the amount of distortion is linearly dependent on the propagation distance. However, nonlinear distortion effects can be reduced by spreading or the effects of stratification of the medium. Waveform freezing occurs when the amount of distortion possible is limited. Lighthill (1956, Sec. 9.3) alluded to waveform freezing when he postulated that distortion could “cease for sound propagating in a horn whose cross-sectional area increases faster than spherical spreading.” A lot of early work into the slowing of distortion, because of spreading, was done in the analysis of blast waves in the ocean (Carlton and Blackstock 1974, Fridman 1976, Morfey 1984, Cotaras 1985). Stratification in the ocean is so small that waveform freezing is unlikely.

For downward sonic boom propagation in the atmosphere both spreading and strat-
ification (due to gravity) lead to a slowing down of nonlinear distortion.* It is not clear whether waveform freezing occurs. This work was initiated because it was commonly believed that waveform freezing does occur for sonic booms in the atmosphere (Hayes et al. 1969; Hayes coined the term “waveform freezing”). However, results from the computer code ZEPHYRUS (Robinson 1991) cast doubt on this belief.

3.3 Solution of the Lossless Equation

For plane waves in a homogeneous medium Eq. 2.1 becomes the lossless Burgers equation

\[ \frac{\partial p'}{\partial x} = \frac{\beta}{2\rho_0 c_0^3} \frac{\partial p'^2}{\partial t'} . \]  

(3.1)

An implicit solution of this equation can be obtained using a variant of the Poisson solution (see, for example, Blackstock 1972). If the source excitation is \( p'(0, t) = f(t) \), the solution of Eq. 3.1, is

\[ p'(x, t') = f \left( t' + \frac{\beta x p'}{\rho_0 c_0^3} \right) . \]  

(3.2)

The argument \( t' + \beta x p'/\rho_0 c_0^3 \) may be thought of as the phase of the wave. The first term moves the reference frame along at the small signal sound speed (the small signal solution is \( p'(x, t') = f(t') \)). The second term describes the nonlinear distortion the waveform suffers as it propagates. The distortion term can be rewritten as \( \beta x u/c_0^3 \). The factor \( \beta x/c_0 \) has dimensions of time and is a simple form of what Hayes calls the age variable (Hayes et al. 1969).

As mentioned above this solution can generate multivalued waveforms. Just before a multivalued waveform is generated the waveform develops an infinite slope—a shock—at some point. The distance to the first occurrence of a shock is the shock formation distance \( \bar{x} \). Mathematically, the shock formation distance is the location where \( \frac{\partial p'}{\partial t'} \) first becomes infinite. From Eq. 3.2, one obtains

\[ \frac{\partial p'}{\partial t'} = \frac{\partial f(t)}{\partial t} \left( 1 + \frac{\beta x}{\rho_0 c_0^3} \frac{\partial p'}{\partial t'} \right) , \]

*For upward propagating finite-amplitude waves the effect of stratification is to speed up nonlinear distortion, see, for example, Cook 1965.
The expression for the slope becomes infinite when the denominator is zero, that is, when \( \frac{\beta x}{\rho_0 c_0^2} \frac{\partial f(t)}{\partial t} = 1 \). The shock formation distance is therefore

\[
\bar{x} = \frac{\rho_0 c_0^3}{\beta \max \left( \frac{\partial f(t)}{\partial t} \right)}.
\]  (3.4)

To demonstrate distortion consider a sinusoidal source with the boundary value \( f(t) = \hat{p} \sin(2\pi f t) \), the Poisson solution is

\[
p'(x, t') = \hat{p} \sin \left( \omega t' + \frac{\beta x p'}{\rho_0 c_0^2} \right),
\]  (3.5)

where \( \omega = 2\pi f \). The shock formation distance is

\[
\bar{x} = \frac{1}{\beta \varepsilon k},
\]  (3.6)

where \( \varepsilon = \hat{p} / \rho_0 c_0^2 \) and \( k = \omega / c_0 \). For distances greater than the shock formation distance the solution Eq. 3.5 predicts a multivalued waveform. Figure 3.1 illustrates the effect of distortion on an initially sinusoidal waveform. The solution, in the retarded time frame, is shown at a number of ranges. At \( x = \bar{x} \) an infinite slope has developed at \( t' = 0 \). At \( x = 3\bar{x} \) and \( x = 5\bar{x} \) the multivalued waveform predicted by the Poisson solution is clearly seen.

Figure 3.1: The propagation and distortion of a finite-amplitude wave, that is initially a sine wave. The solid line is the waveform given by weak shock theory; the dotted line by the lossless theory.

For \( x > \bar{x} \) weak shock theory is needed to keep the waveform well behaved. One method of achieving this is the so called equal area rule (see, for example, Rudenko
and Soluyan 1977, Chap. 1.4). A vertical line is drawn so that the areas between the vertical line and the curve are equal (see the shaded areas in Fig. 3.1 for $x = 3\bar{x}$ and $x = 5\bar{x}$); the location of this line is where the shock lies.

For $x > 3\bar{x}$ the sine wave becomes effectively a sawtooth wave (Blackstock 1966),* and the solution is approximately

$$p' = \begin{cases} \rho \frac{\pi - \omega' t'}{1 + x/\bar{x}} & 0 < t' < \pi \\ \rho \frac{\omega' t' - \pi}{1 + x/\bar{x}} & -\pi < t' < 0 \end{cases}$$

As the wave propagates the distortion remains a sawtooth wave, but the losses at the shock mean the amplitude reduces with distance.

If the loss terms are retained in the Burgers equation, then at very large propagation distances (many absorption lengths) the solution eventually returns to its original sinusoidal shape. The amplitude is significantly attenuated though. This is referred to as the *old age region* (Blackstock 1964).

### 3.4 Weak Shock Theory

The method of weak shock theory requires the waveform to be divided into two parts: the continuous sections of the wave and the shocks.

The continuous parts of the waveform are described by the simple Poisson or Earnshaw (see, for example, Pierce 1981) solutions. For the continuous segments the approximate Earnshaw solution (in terms of the acoustic pressure) is (see, for example, Blackstock 1972,)

$$p' = f(\phi),$$

$$\phi = t - \frac{x}{c_0} + \frac{\beta x f(\phi)}{\rho_0 c_0^3},$$

$$= t' + \frac{\beta x f(\phi)}{\rho_0 c_0^3},$$

where $\phi$ is referred to as the *phase* of the wave. The phase $\phi$ is identical to the argument in Eq. 3.2. Equations 3.7 and 3.9 can be combined to produce the Poisson solution, Eq. 3.2.

---

*In general any periodic finite-amplitude waveform turns into a sawtooth wave.

†The effect of source displacement is neglected.
The equal area rule is one way to correct multivalued waveforms predicted by the implicit solution. The formal approach is to keep track of the shocks and solve the Rankine-Hugoniot conditions across the discontinuity (see, for example, Pierce 1981, Chap. 11-3). The Rankine-Hugoniot relations require that each shock moves with the velocity

\[ v_s = c_0 + \frac{\beta}{2} (u_a + u_b), \]

where \( u_a \) is the particle velocity just ahead of the shock, \( u_b \) is the particle velocity just behind the shock, and \( c_0 \) is the small-signal sound speed, that is, where \( u = 0 \). In terms of pressure the shock speed can be rewritten to \( O(\epsilon^2) \) as

\[ v_s = c_0 + \frac{\beta}{2\rho_0 c_0} (p'_a + p'_b). \]

(3.10)

The equal area rule can be deduced from weak shock theory (Rudenko and Soluyan 1977, Chap. 1.5). For many waveforms it is easier to apply the equal area rule than to explicitly apply weak shock theory at every shock. However for one important case, N waves, a result using Eqs. 3.7-3.10 is easily obtained.

3.4.1 Propagation of N waves

The propagation of N waves, using weak shock theory, is discussed here. The N wave is a simple, but realistic, model for a sonic boom.* The derivation here follows that of Blackstock (1983), and the results are used later in this work for benchmark tests.

The source condition for an N wave is

\[ p'(0, t) = \begin{cases} \rho_0 \gamma \frac{c}{T_{h0}} & -T_{h0} < t < T_{h0} \\ 0 & \text{elsewhere} \end{cases} \]

where \( T_{h0} \) is the intial half duration of the N wave, and \( \rho_0 \) is the initial peak pressure. The waveform is shown in Fig 3.2(a); there are shocks at \( t = -T_{h0} \) and \( t = T_{h0} \). Although application of the equal area rule is valid for N waves it is possible to obtain an explicit solution for the peak pressure \( \hat{p} \) and half duration \( T_h \) of an N wave as a function of distance. First a solution for the continuous part of the N wave is obtained.

*In general sonic boom waveforms tend towards an N wave shape.
using the Earnshaw solution. Then weak shock theory is applied at the shocks to
determine the shock location and hence N wave duration.

The Earnshaw solution for the continuous segment of an N wave is

\[ f(\phi) = -\frac{\phi}{T_{h_0}}. \]

The phase is

\[ \phi = t' + \frac{\beta x f(\phi)}{c_0^2}, \]

\[ = t' - \frac{\beta \rho_0 x}{\rho_0 c_0^3 T_{h_0}} \phi. \]

Therefore,

\[ \phi = \frac{t'}{1 + ax}, \]

where \( a = \beta \rho_0 / \rho_0 c_0^3 T_{h_0}. \)

The solution for the continuous section is

\[ p'(x, t') = -\frac{\rho_0}{T_{h_0}} \frac{t'}{1 + ax} \quad -T_h < t' < T_h. \quad (3.11) \]

The half duration \( T_h \) is not yet known but we know that it is controlled by the location
of the shocks at the head and the tail of the N wave. Weak shock theory, Eq. 3.10,
can be used to predict the location of the shocks.

Equation 3.10 requires the shock move at speed \( \frac{dx}{dt} \bigg|_s = c_0 + \frac{\beta}{2 \rho_0 c_0^2} (p'_a + p'_b). \) We can invert this equation, correct to order \( \varepsilon, \)

\[ \frac{dt}{dx} \bigg|_s = \frac{1}{c_0} \left( 1 - \frac{\beta}{2 \rho_0 c_0^2} (p'_a + p'_b) \right). \]

The coordinate system can be transformed from \( (t, x) \) to the retarded time frame \( (t', x') \)

where \( t' = t - x/c_0 \) and \( x' = x. \) The result is

\[ \frac{dt'}{dx'} \bigg|_s = \frac{1}{c_0} \left( 1 - \frac{\beta}{2 \rho_0 c_0^2} (p'_a + p'_b) \right) - \frac{1}{c_0} \frac{dx}{dx'} \bigg|_s, \]

\[ = -\frac{\beta}{2 \rho_0 c_0^3} (p'_a + p'_b). \]

*The quantity \( 1/a \) is somewhat akin shock formation distance, for example, \( 1/(\pi a) \) is the shock
formation distance for a sine wave with amplitude \( \rho_0 \) and period \( 2T_{h_0}. \)
The head shock location is $t'|_s = -T_h$. The pressures just ahead of and behind the shock are $p'_a = 0$ and $p'_b = \hat{p}$. The half duration $T_h$ therefore varies as

$$\frac{dT_h}{dx} = -\frac{dt'}{dx},$$

$$= \frac{\beta}{2\rho_0 c_0^2} \left( 0 + \frac{\hat{p}}{T_h / (1 + ax)} \right),$$

$$= \frac{\beta \hat{p}}{2\rho_0 c_0^2 T_h (1 + ax)}.$$

This equation is separable

$$\int_{T_h_0}^{T_h} \frac{dT_h}{T_h} = \int_0^x \frac{a}{2(1 + ax)} \, dx,$$

and integration yields

$$T_h = T_h_0 \sqrt{1 + ax}. \quad (3.12)$$

This result combined with the result for the continuous section, Eq. 3.11, is the solution for the propagation of an N wave. Note the overpressure of the head shock varies as

$$\hat{p}(x) = \frac{\hat{p}_0}{\sqrt{1 + ax}}. \quad (3.13)$$

The distortion of an N wave is shown in Fig. 3.2.

### 3.5 Solution to the Generalized Lossless Equation

A solution for the lossless generalized Burgers also exists (Whitham 1956). The generalized form of the Burgers equation, Eq. 2.2, includes the effect of geometrical spreading and propagation through an inhomogeneous medium (that is, $\rho_0$, $c_0$ and $S$ vary with the distance $s$ along the ray path) and is appropriate for the propagation of sonic booms in the atmosphere. The lossless Burgers equation for this case is

$$\frac{\partial p'}{\partial s} + \frac{\partial}{\partial x} \left( \frac{S S'}{2} \right) - \frac{\partial (\rho_0 c_0)}{\partial x} p' = \frac{\beta}{2\rho_0 c_0^2} \frac{\partial p'^2}{\partial t'}. \quad (3.14)$$

A solution can be obtained by transforming the equation into one which has the same form as the plane wave equation, Eq. 3.1. Two transformations are required. First, a new dependent variable $q$ (a scaled pressure) is defined,

$$q = \sqrt{\frac{\rho_0 c_0 S}{\rho_0 c_0 S}} p'. \quad (3.15)$$
Figure 3.2: The propagation of an N wave. (a) The initial N wave. (b) The N wave after propagating a distance $x = 1/a$. The dotted line shows the initial waveform. The dashed line shows the waveform predicted by the Earnshaw solution without weak shock theory.

where, an overbar is used to denote a value at the source.* This transformation may be deduced from the fact that in a ray tube the energy flow, which is proportional to $S \rho \psi^2 / \rho_0 c_0$, is constant†. That is, the transformation compensates for any amplification or attenuation to which a small-signal, lossless wave would be subject. The lossless wave equation becomes

$$\frac{\partial q}{\partial x} = \frac{\beta}{2\rho_0 c_0^3} \sqrt{\frac{S}{S \rho_0 c_0^5}} \frac{\partial q^2}{\partial \psi^2}. \quad (3.16)$$

It has a form similar to that of Eq. 3.1, except the coefficient of the nonlinear term is not constant.

The second transformation turns the spatially dependent coefficient into a constant. A new independent variable $\tilde{x}$ (a scaled distance) is introduced,

$$\tilde{x} = \int_0^s \sqrt{\frac{S \rho_0 c_0^5}{S \rho_0 c_0^5}} \, ds'. \quad (3.17)$$

* Other authors may use the ground as a reference point.

† If the atmosphere has a steady flow, for example, when wind is present, the Blokhintsev invariant must be used and a different independent variable is appropriate (Hayes et al. 1969, Robinson 1991).
The transformed wave equation is

\[ \frac{\partial q}{\partial \bar{x}} = \frac{\beta}{2\rho_0 c_0^2} \frac{\partial^2 q}{\partial t'} . \]  
(3.18)

Its form is identical to that of Eq. 3.1.

In terms of the new variables, the Poisson solution of Eq. 3.18 is

\[ q(\bar{x}, t') = f(t' + \frac{\beta \bar{x} q}{\rho_0 c_0^2}) . \]  
(3.19)

This solution is similar to Eq. 3.2 except that the distance in the distortion part of the phase term is replaced by the scaled distance \( \bar{x} \). The scaled distance can be considered to be a distortion distance in the sense that \( \bar{x} \) plays the same role in determining the amount of distortion in this case as the true distance \( x \) does in the plane wave case.

### 3.6 Waveform Freezing

The above analysis is now used to investigate the phenomenon of waveform freezing. We consider the distortion of waves that spread or travel through a stratified medium. The role of distance on the distortion is seen by examining the solution for the propagation of plane finite-amplitude waves (Eq. 3.2),

\[ p' = f \left( t' + \frac{\beta \bar{x} p'}{\rho_0 c_0^3} \right) . \]

Recall the argument \( t' + \beta \bar{x} p'/\rho_0 c_0^3 \) is referred to as the phase of the wave. In the case of small-signal waves there is no distortion, the second (distortion) term is negligible and \( p' = f(t') \). For finite-amplitude waves, however, the shape of the wave is distorted as described by the second phase term, \( \beta \bar{x} p'/\rho_0 c_0^3 \). The distortion term is linearly dependent on \( x \). The waveform becomes more distorted as it propagates. Because the distortion continues without end, no waveform freezing of plane waves occurs in a homogeneous medium. However, in the solution of the generalized equation the dependence of \( \bar{x} \) on \( s \) opens up the possibility that the distortion may occur more slowly than for plane waves. If \( \bar{x} \) has a finite limit as \( s \to \infty \), the distortion is limited and waveform freezing is said to occur.
3.6.1 Spherical and Cylindrical Spreading

Consider a spherically spreading wave in a homogeneous medium. In this case \( \rho_0 \) and \( c_0 \) are constant (overbars for these quantities are therefore omitted), the rays are straight lines, and the ray tube area is proportional to \( r^2 \) (distance \( s \) along the ray tube is the radial distance \( r - r_0 \) where \( r_0 \) is the source radius). Equations 3.15 and 3.17 become

\[
q = \frac{r}{r_0} p',
\]

\[
\bar{x} = r_0 \ln (r/r_0).
\]

The first relation shows that \( q \) is the acoustic pressure, scaled to compensate for spherical spreading. When the second relation is combined with Eq. 3.19, the result is

\[
q(r, t') = f \left( t' + \frac{\beta r_0 \ln (r/r_0) q}{\rho_0 c_0^2} \right).
\]

The distortion grows as \( r_0 \ln (r/r_0) \) (in place of the factor \( x \) that appears in Eq. 3.2). For spherical waves the distortion develops more gradually than for plane waves. Note, however, that although the distortion grows ever more slowly as distance increases, the growth never ceases altogether, i.e., waveform freezing does not occur.

For cylindrically spreading waves the transformations are

\[
q = \sqrt{\frac{r}{r_0}} p',
\]

\[
\bar{x} = 2\sqrt{r_0}(\sqrt{r} - \sqrt{r_0}).
\]

The solution is

\[
q(r, t') = f \left( t' + \frac{\beta 2\sqrt{r_0}(\sqrt{r} - \sqrt{r_0}) q}{\rho_0 c_0^2} \right).
\]

Again distortion develops more slowly than for plane waves (\( \sqrt{r} \) compared to \( x \)) but faster than for for spherical waves. Cylindrically spreading waves do not freeze either.
3.6.2 Isothermal Atmosphere

The atmosphere is not a homogeneous medium, it is stratified by gravity. To a first approximation we consider a plane wave propagating straight downward through an isothermal atmosphere. Let \( z \) be the altitude \( z_s \), the source altitude, and \( x = z_s - z \) the propagation distance (positive downward), see Fig. 3.3. In an isothermal atmosphere the sound speed does not change with distance (we therefore omit the overbar with \( c_0 \)), but the density does. Recall from Chapter Two that the static condition for the atmosphere (gravity is the body force, \( \mathbf{B} = -g \hat{\mathbf{e}}_x \)), Eq. 2.73, is

\[
\frac{d\rho_0}{dz} = -\rho_0 g .
\]

For an ideal gas \( p = \rho RT \). If \( T = T_0 \), a constant, then

\[
\frac{d\rho_0}{dz} = -\frac{g}{RT_0} \rho_0 .
\]

If the density at the source is \( \rho(z_s) = \rho_0 \) then

\[
\rho_0(z) = \rho_0 e^{(z_s - z)/H} ,
\]

where \( H = RT_0/g \) is the scale height of the atmosphere. For an ambient temperature \( T_0 = 290 \) K, the scale height is \( H = 8.5 \) km. The density increases with propagation distance, \( \rho_0 = \rho_0 e^{x/H} \).

The expressions for \( q \) and \( \bar{x} \) are found to be

\[
q = e^{x/2H} \rho' ,
\]

\[
\bar{x} = 2H(1 - e^{-x/2H}) .
\]

The distortion distance \( \bar{x} \) does not increase indefinitely with propagation distance \( x \) but instead only approaches the asymptotic value \( \bar{x}_{\text{max}} = 2H \). Substitution of Eq. 3.27 into Eq. 3.2 yields

\[
q(x, t') = f \left( t' + \frac{\beta 2H(1 - e^{-x/2H}) q}{\rho_0 c_0^2} \right) .
\]
In this case the distortion of the waveform not only slows down as the wave travels, it has an upper bound. In the limit as $x \to \infty$, the solution is

$$\lim_{x \to \infty} q(x, t') = f \left( t' + \frac{\beta^2 H q}{\rho_0 c_0} \right).$$  

(3.29)

Like a small-signal wave, the phase term has no explicit dependence on $x$. Distortion, although present (as indicated by the dependence of the phase on $q$), no longer changes with distance: the waveform is frozen.

In Fig. 3.4 the distortion distance profile is shown for a plane wave in a homogeneous medium, a spherically spreading wave in a homogeneous medium ($r_0 = 100$ m), and a plane wave in an isothermal atmosphere ($z_s = 17$ km).*

Figure 3.4: The distortion distance $\tilde{x}$ for a plane wave in a homogeneous medium, a spherically spreading wave in a homogeneous medium ($r_0 = 100$ m), and a plane wave in an isothermal atmosphere ($z_s = 17$ km).*

*The scale height used here is $H = 7.0$ km, the value based on $T_0 = 240$ K, which is the average temperature from the ground to an altitude of 17 km in the U.S. Standard Atmosphere.
in an isothermal atmosphere. For the plane wave in a homogeneous medium 20 km of propagation yields 20 km of distortion. For a spherically spreading wave 45 km of propagation is needed to produce the same 20 km of distortion. For a plane wave propagating straight downward in an isothermal atmosphere no more than 14 km of distortion can occur no matter how far the wave travels.

3.6.3 Effective Nonlinearity

A physical explanation for freezing is that the coefficient of nonlinearity appears to decrease as the wave propagates. To see this, it is necessary to inspect the wave equations for which Eqs. 3.2, 3.22, and 3.28 are solutions. For plane waves in a homogeneous medium the equation is

$$\frac{\partial p'}{\partial x} - \frac{\beta}{2\rho_0 c_0^2} \frac{\partial p'}{\partial y} = 0,$$

(3.30)

for spherical waves

$$\frac{\partial q}{\partial r} - \frac{\beta_{ra}}{2\rho_0 c_0^3} \frac{\partial q}{\partial t'} = 0,$$

(3.31)

and for plane waves in an isothermal atmosphere

$$\frac{\partial q}{\partial x} - \frac{\beta e^{-x/2H}}{2\rho_0 c_0^3} \frac{\partial q}{\partial t'} = 0.$$

(3.32)

They all have the same form

$$\frac{\partial q}{\partial x} - \frac{\beta_{eff}}{2\rho_0 c_0^3} \frac{\partial q}{\partial t'} = 0,$$

(3.33)

where $\beta_{eff}$, the effective coefficient of nonlinearity, has been introduced.

One sees that propagation of spherical waves is like propagation of plane waves in a medium having an effective nonlinearity coefficient $\beta_{eff}$ that decreases as $1/r$. Similarly, the isothermal atmosphere resembles a homogeneous medium in which the nonlinearity coefficient decreases exponentially, i.e., $\beta_{eff} = \beta e^{-x/2H}$.

The concept of an effective coefficient of nonlinearity is easily extended to include all changes in cross-sectional area and properties of the medium. Equation 3.16 shows that the general definition of $\beta_{eff}$ should be

$$\beta_{eff} = \beta \sqrt{\frac{S \rho_0 c_0^5}{S \rho_0 c_0^5}}.$$

(3.34)
Recall that Eq. 3.16 was solved by introducing the distortion distance transformation to replace the varying coefficient of nonlinearity with a constant $\beta$, the actual coefficient of nonlinearity. The relationship between $\bar{x}$ and $\beta_{\text{eff}}$ is

$$\int_0^\infty \beta_{\text{eff}} \, ds = \beta \, \bar{x}$$

(3.35)

Note that $\beta_{\text{eff}}$ is independent of the source waveform or amplitude. Whether the waveform freezes then depends on whether the infinite integral of $\beta_{\text{eff}}$ (or equivalently the distortion distance) is bounded. The integral is proportional to the age variable introduced by Hayes et al. (1969). In the case of spherical waves, the integral is proportional to $\ln(r/r_0)$, which implies that distortion, while slowing down as propagation distance increases, never comes to a full stop. For waves travelling downward in an isothermal atmosphere, however, the integral approaches a finite value as $x \to \infty$. In this case the waveform freezes.

It should be noted that, as appropriate for the lower atmosphere, $\beta$ has been treated as a constant in this analysis. For a medium in which $\beta$ varies, such as the ocean, the variation may be accounted for by including the factor $(\bar{\beta}/\beta)^2$ inside the square root of Eq. 3.34 (Morfey 1984), where the overbar denotes a reference value at the source.

### 3.7 Application to the Atmosphere

The foregoing analysis is now applied to the atmosphere. For simplicity a quiet medium is assumed. Since the cruising altitude of the high speed civil transport aircraft is expected to be about 17 km (roughly 55,000 ft), attention is restricted to the atmosphere below this height. In this region the ISO Atmosphere (ISO 9613-1 1993) and the U. S. Standard Atmosphere (1962) may be modeled as having a bilinear temperature profile. The profile is shown in Fig. 3.5. The temperature at the ground is 15°C ($T_0 = 288.15$ K). There is a linear decrease at a rate of 6.5°C/km up to an altitude of 11 km. The rest of the atmosphere (up to 20 km) is at constant temperature.

![Figure 3.5: Standard atmosphere temperature profile.](image)
$-56.5^\circ C \left(T_b = 216.65 \text{ K}\right)$. The altitude $z_k = 11 \text{ km}$ is called the knee of the profile and $T_k = 216.65 \text{ K}$. The altitude of the aircraft is assumed to be $z_a = 17 \text{ km}$. Initial sonic boom signatures are given at a distance $r_s$, a few body lengths beneath the aircraft.

### 3.7.1 Isothermal Atmosphere

Previously (Sec. 3.6.2) it was shown that in an isothermal atmosphere, $T_0$ and $c_0$ are constant and $\rho_0 = \rho_0(0)e^{-z/H}$. The analysis in Sec. 3.6.2 was for plane waves propagating straight down to the ground. Sonic booms spread cylindrically in an isothermal atmosphere, and the rays do not propagate straight downwards.

The ray paths in an isothermal atmosphere are described in Appendix B. The grazing angle of a ray coming off a sonic cone is $\cos \theta_0 = \cos \psi \cdot \cos \phi$, where $\psi$ is the angle of the sonic cone and $\phi$ the azimuthal launch angle; see Fig. B.4. The grazing angle $\theta_0$ is negative for a downward propagating ray. A ray that has propagated a distance $s$ from the source, suffers a change of altitude $\Delta z = s \sin \theta_0$ (Eq. B.1). The initial vertical distance for the ray depends on the azimuthal angle $z_s = z_a - r_s \cos \phi$.

The relationship between path length and altitude is

$$z(s) = z_a - r_s \cos \phi + s \sin \theta_0.$$

The ray tube area for cylindrical spreading is given by

$$\frac{S}{S_0} = \frac{s + s_0}{s_0},$$

where $s_0$ is the initial path length and is the apparent distance the ray traveled from the aircraft to the source location. From the geometry of the sonic cone it follows that $s_0 = r_s \cos \psi$; see Fig. B.4.

The expressions for $q$ and $\tilde{x}$ are:

$$q = \sqrt{\frac{s + s_0}{s_0} e^{-s \sin \theta_0/2H}} p', \quad (3.36)$$

$$\tilde{x} = \int_0^s \sqrt{s' + s_0} \sqrt{\frac{\rho_0}{\rho_0 e^{-s' \sin \theta_0/2H}}} \, ds'.$$
To evaluate the integral let $\xi^2 = (s' + s_0) \sin(-\theta_0)/2H$, therefore $\frac{d\xi}{ds'} = \frac{\sin(-\theta_0)}{8H}(s' + s_0)^{-\frac{1}{2}}$, and

\[
\bar{x} = \sqrt{\frac{8Hs_0}{\sin(-\theta_0)}} \int_{s_0 \sin(-\theta_0)/2H}^{s + s_0 \sin(-\theta_0)/2H} e^{-\xi^2} e^{s_0 \sin(-\theta_0)/2H} d\xi.
\]

By definition (see, Appendix B) $\theta_0$ is negative for downward propagating rays so the minus sign inside the square root is acceptable.

For straight downward ($\theta_0 = -\pi/2$) cylindrical propagation, where $r_0$ is the source radius, the transformations are

\[
q = \sqrt{\frac{r}{r_0}} e^{-(r-r_0)/2H} p',
\]

\[
\bar{x} = \sqrt{2\pi Hr_0} e^{r_0/2H} \left( \text{erf} \sqrt{r/2H} - \text{erf} \sqrt{r_0/2H} \right).
\]

### 3.7.2 Fully Isothermal Atmosphere

Although only the region of the atmosphere from 11 km to 20 km is close to being isothermal, it is instructive to approximate the whole atmosphere as isothermal because a closed form expression for the distortion is available. The average temperature of the atmosphere from the ground to 17 km is $T = 240$ K. The scale height of the atmosphere associated with this temperature is $H = 7015$ km. The distortion distance for sonic booms emanating from an aircraft flying at Mach 2 and at various azimuthal angles is shown in Fig. 3.6. Notice that differing amounts of distortion occur at each angle. At higher azimuthal angles the waveform travels a longer distance to the ground, which means more distortion can occur. It does appear that for each case the distortion distance is tending towards an asymptotic value. But it appears the sonic boom would have to travel well into the “ground” to achieve asymptotic behavior.

Examination of Eq. 3.37 shows that the asymptotic value of the distortion distance ($\lim_{s \to \infty} \text{erf}(\sqrt{s}) = 1$) is

\[
\bar{x}_{\text{max}} = \sqrt{\frac{2\pi Hs_0}{\sin(-\theta_0)}} e^{s_0 \sin(-\theta_0)/2H} \left( 1 - \text{erf} \left\{ \sqrt{\frac{s_0 \sin(-\theta_0)}{2H}} \right\} \right).
\]
Figure 3.6: The distortion distance in an isothermal atmosphere for an aircraft flying at Mach 2.0 and 17 km altitude. Curves are shown for three different azimuthal angles.

If the distortion distance is normalized by $\tilde{x}_{\text{max}}$ then we can see how the distortion of each ray approaches its asymptotic value. The resulting curves are shown in Fig. 3.7. It is now apparent that waveform freezing has not occurred when the waveform reaches the ground — the distortion reaches about 86% of its maximum value. Notice also that although the amount of distortion in Fig. 3.6 varies with azimuthal angle, Fig. 3.7 shows that the approach to waveform freezing is only weakly dependent on the azimuthal angle. In the previous section it was seen that geometrical spreading alone does not lead to waveform freezing but that density variation due to stratification does. This explains why the approach to freezing of the rays in Fig. 3.7 is very similar — they all suffer the same density change. The rays at higher azimuthal angles have traveled further and more spreading occurs but this only slightly enhances the tendency toward waveform freezing.
Figure 3.7: The normalized distortion distance in an isothermal atmosphere for an aircraft flying at Mach 2.0 and 17 km altitude.

The effect of varying the Mach number is shown in Fig. 3.8. The amount of freezing that occurs at the ground is shown for various azimuthal angles $\phi$. Reducing the Mach number increases the distance the ray travels to the ground and so more spreading occurs. However, there is virtually no effect on how close to waveform freezing each ray is, except at very low Mach numbers. The variation with $\phi$ is also small. Again, it is the density change from the source to the ground that controls the waveform freezing.

To get closer to waveform freezing it is necessary to have a greater density variation between the source and the ground, i.e., the aircraft has to fly higher. If the isothermal model of the atmosphere is used for the analysis, then for the distortion to be at 90% of its maximum value, the aircraft must fly at an altitude of 20 km. To obtain 95% an altitude of 27 km is required. However, above 20 km in a real atmosphere, the
temperature gradient changes and the sound speed starts to increase—which works against waveform freezing. For an isothermal atmosphere it is appears that waveform freezing does not occur for sonic booms generated from aircraft flying in the lower 20 km, and probably does not occur even if the aircraft is at an altitude of 30 km or higher.

3.7.3 Linear Profile for Temperature

We now turn our attention to waveform freezing in an atmosphere with a linear temperature profile. The linear temperature profile is expressed as

\[ T_0(z) = T_g(1 + nz) \]

where \( z \) is positive upward and \( T_g = 288.15 \) K is the temperature at the ground. Recall that \( z_k = 11 \) km and \( T_k = 216.65 \) K, so \( n = -22.56 \times 10^{-6} \) m\(^{-1}\).
The static condition for the atmosphere is
\[ \frac{dp_0}{dz} = -\rho_0 g. \]

Use of the ideal gas law yields
\[ RT_0 \frac{dp_0}{dz} + \rho_0 R \frac{dT_0}{dz} = -\rho_0 g, \]
\[ \frac{d\rho_0}{dz} = -\frac{\rho_0 g - \rho_0 R T g}{RT}, \]
\[ \int_{\rho_0(0)}^{\rho(z)} \frac{d\rho_0}{\rho_0} = \left( -\frac{g}{RT_g} - n \right) \int_0^z \frac{dz}{1 + nz}, \]
\[ \ln \left( \frac{\rho_0(z)}{\rho_0(0)} \right) = -\left( \frac{1}{H_g} + n \right)^{-1} \ln (1 + nz), \]
\[ \rho_0(z) = \rho_g (1 + nz)^{-1/nH_g}. \]

where \( \rho_g = \rho_0(0) \) is the ambient density at the ground. The scale height, based on the temperature at the ground, is \( H_g = \frac{RT_g}{g} \). For air, \( R = 287 \text{ J/(kg} \cdot \text{K}) \) and \( g = 9.81 \text{ m/s}^2 \). The scale height is \( H_g = 8.43 \text{ km} \) and the value of the exponent is \( 1 + 1/nH_g = -4.259 \). The sound speed is given by
\[ c_0 = \sqrt{\gamma RT_g (1 + nz)}, \]
\[ = c_g \sqrt{1 + nz}, \]

where \( c_g = \sqrt{\gamma RT_g} \) is the small-signal sound speed at the ground.

First the case of plane waves propagating vertically downward is considered, for which case refraction can be ignored. The distortion distance is
\[ \int_0^s \sqrt{\frac{\rho_0 c_0^5}{\rho_0 c_0^5}} \, dz = \int_{z_s}^z \sqrt{\frac{\rho_0 c_0^5}{\rho_0 c_0^5}} \, \sqrt{\frac{\rho_g c_g^5}{\rho_0 c_0^5}} \, dz, \]
\[ = \sqrt{(1 + nz_s)^{-1/nH_g} + 5/2} \int_{z_s}^z \sqrt{(1 + nz)^{-1/nH_g} + 5/2} \, dz, \]
\[ = (1 + nz_s)^{3/4 - 1/2nH_g} \int_{z_s}^z (1 + nz)^{3/4 - 1/2nH_g} \, dz, \]
\[ = \frac{1}{1/4 + 1/2nH_g} \frac{1}{n} \left[ (1 + nz_s)^{(1/4 + 1/2nH_g)} \right]_{z_s}^z, \]
\[ = 2H_g \frac{1}{1 + nH_g/2} \left( \frac{1 + nz}{1 + nz_s} \right)^{(1/4 + 1/2nH_g)} \, dz. \]
In the limit as $z \to -\infty$ the term $(1 + nz)^{-2.379} \sim (nz)^{-2.379}$ tends towards zero. The maximum distortion distance is

$$\tilde{x}_{\text{max}} = \frac{2H_g}{1 + nH_g/2}(1 + nz).$$

Because the distortion distance is limited, waveform freezing does occur in a linear temperature profile. For the standard atmosphere the maximum distortion distance for a source at $z_s = z_k$ is $\tilde{x}_{\text{max}} = 14.8$ km.

If the analysis is now extended to include spreading, we still expect waveform freezing to occur because the effect of spreading has been seen to slightly enhance freezing. It is not clear however whether a sonic boom at the ground is close to waveform freezing. The fact that the maximum distortion distance possible is 15 km implies that the sonic boom needs to travel much further than this to approach a frozen state and so waveform freezing is unlikely. For example, in Fig. 3.4 a propagation distance of 50 km is necessary to achieve a distortion distance of 14 km.

The case of non-planar waves is now investigated. The sound speed profile can be approximated by a linear sound speed profile for $|nz| \ll 1$. In the atmosphere here this requires $z \ll 44$ km, an acceptable restriction, since the linear profile exists only in the lower 11 km. The sound speed profile is

$$c_0 = c_g + mz,$$

where $m = nc_g/2$. The ray tube area for rays propagating in an atmosphere having a linear sound-speed profile is developed in Appendix C. The transformations for a sonic boom waveform are

$$q = \sqrt[6]{\frac{S}{S}}(1 + nz(s))^{-1/2nH_g+3/4}p',$$

$$\tilde{x} = \int_0^s \sqrt[6]{\frac{S}{S}}(1 + nz(s))^{-1/2nH_g+3/4}ds.'$$

Because these integrals cannot be evaluated analytically, they are solved numerically using the trapezoidal rule.

The results from the isothermal and linear parts of the atmosphere can be patched together to yield the distortion distance in a bilinear atmosphere. Unfortunately because it is not possible to obtain a closed form expression for the distortion distance in
a linear profile we cannot normalize the distortion distance. We use the shape of the
distortion distance curve $\bar{x}$ to estimate whether freezing occurs. If at ground level the
curve seems to be very close to an asymptotic value, freezing is deemed to have oc-
curred. If not, then the waveform is still changing appreciably when the boom reaches
the ground.

Results have been obtained for various Mach numbers and azimuthal angles. Fig-
ure 3.9 shows the distortion at various angles for an aircraft flying at Mach 2 at an
altitude of 17 km. In order to indicate how close the ground value is to an asymptote,
the distortion distance curves have been continued beyond ground level (the atmo-
sphere has been assumed to continue with the same properties, i.e., the sonic boom is
not reflected). For high azimuthal angles the rays often do not make it to the ground
because of refraction.

Figure 3.10 shows the distortion for aircraft at various Mach numbers but with
$\phi = 0$. For all the cases considered, Mach numbers in the range 1.2 to 10 and azimuthal
angle from 0° to 60°, it was found that waveform freezing never occurred at the ground.
At best the distortion distance is estimated to be within 10% of reaching its asymptotic
value.

The general conclusion from this work is that although nonlinear distortion slows
down a great deal, actual waveform freezing does not occur to sonic booms in the
atmosphere for an aircraft flying at 17 km (Cleveland and Blackstock 1992, see also
the phrase *chilling* to describe the fact that although sonic booms suffer a slowing of
nonlinear distortion they do not freeze.

3.8 Waveform Freezing in a Heated Box

A model laboratory experiment to observe waveform freezing was considered early in
this research. A model stratified medium was envisioned within a thermally insulated
box. A linear temperature profile may be achieved within the box by heating the top
of the box and cooling the bottom. Model sonic booms can be created by an electric
spark.
Figure 3.9: The distortion distance $\tilde{z}$ in a bilinear atmosphere for a source at altitude 17 km flying at Mach 2.0, for various azimuthal angles.

This model does not mimic the atmospheric stratification very well. Gravity endows the atmosphere with the curious property of having an inverted temperature gradient, that is, cooler air sits on top of warm air. As a sonic boom travels down towards the earth it encounters both increasing density and sound speed. Both of these effects produce a slowing down of distortion. Instead the gradient is established by cooling the bottom of the box and heating the top (this is the only arrangement that can produce a stable temperature gradient). In the model then, the air is less dense in the hotter part of the box (the top) and more dense in the cooler part. If $N$ waves are generated near the bottom of the box and propagate upward, the increasing temperature slows down waveform distortion but the decreasing density speeds it up — unlike the atmosphere where both the density variation and the temperature variation enhance the development of waveform freezing. The spherical spreading of the spark-
generated N wave as opposed to cylindrical spreading assists the tendency toward waveform freezing. However, the work done so far indicates that variation in spreading has little influence on the conditions for waveform freezing.

Inside the box, assume that the air behaves as an ideal gas, that is, $P = \rho RT_b$. The small-signal sound speed is given by $c_0^2 = \gamma RT_b$. For a linear temperature profile $T_b(z) = T_b + mz$, where $z$ is upward positive and $T_b$ is the temperature of the bottom of the box ($z = 0$), and $m$ the temperature gradient. The sound speed is therefore a function of height

$$c_0(z) = \sqrt{\gamma R(T_b + mz)} .$$

(3.40)

The linear temperature gradient yields and approximately linear sound speed profile
if \( \frac{mz}{\gamma RT_b} \ll 1 \),

\[
c_0(z) = \gamma RT_b \left( 1 + \frac{mz}{\gamma RT_b} \right)^{\frac{1}{2}},
\]

\[
\simeq c_b \left( 1 + \frac{mz}{2c_b} \right),
\]

where \( c_b \) is the small-signal sound speed at the bottom of the box. The temperature gradient in a box of height \( h \), with temperature \( T_b \) at the bottom and \( T_t \) at the top, is

\[
m = \frac{T_t - T_b}{h} = \frac{T_b}{h} \left( \frac{T_t}{T_b} - 1 \right).
\]

The inequality becomes

\[
\frac{(T_t/T_b - 1)z}{\gamma Rh} \ll 1,
\]

or

\[
z \ll \frac{\gamma Rh}{T_t/T_b - 1}.
\]

For a box of height \( h = 1 \text{ m} \), with a temperature ratio \( T_t/T_b = 5 \), the approximation for a linear sound speed profile requires that \( z \ll 100 \text{ m} \), a requirement easily met within the confines of the box.

The sound speed gradient causes the rays to refract or bend. The ray tube area should be derived using a differential ray tube area method as done for sonic booms in Appendix C. However, Morfey (1984, Appendix E) gives the following approximate expression for the ray tube area of a vertically propagating ray in a medium having a linear sound speed profile:

\[
\frac{S_z}{S} = \frac{x^2 c_0}{z^2 c_0},
\]

where barred quantities are evaluated at the source. Recall that spreading has very little effect on whether the freezing occurs. Morfey's approximation should be useful for determining whether waveform freezing is going to occur within the box.

For vertically propagating waves the distortion distance is

\[
\tilde{x} = \int_0^s \sqrt{\frac{4\pi x^2 c_0}{4\pi x^2 c_0}} \sqrt{\frac{\rho_0}{\rho_0 c_0^5}} \, ds.
\]
But because \( P_0 = \rho_0 c_0^2 / \gamma \) stays constant within the confines of the box, the density factor disappears from the integrand. It is in this way that the density decrease within the box counteracts some of the sound speed increase. The integral may now be written

\[
\tilde{x} = \int_0^z \frac{z c_0^2}{z c_0^2} \, dz,
\]

\[
= \int_0^z \frac{1}{z} \frac{1}{z (T_b + m z)} \, dz,
\]

\[
= \int_0^z \left( 1 + \frac{m z}{T_b} \right) \ln \left| \frac{z}{1 + m z / T_b} \right| \, dz,
\]

\[
= \int_0^z \left( 1 + \frac{m z}{T_b} \right) \ln \left| \frac{z}{1 + m z / T_b} \right| \, dz.
\]

It follows from this analysis that the requirement for waveform freezing is \( z \gg T_b / m \) waveform freezing occurs. The maximum distortion is therefore

\[
\tilde{x}_{\text{max}} = \tilde{x}(1 + \frac{m z}{T_0}) \ln \left| \frac{T_0}{m z} + 1 \right|.
\]

Results for the normalized distortion distance for various temperature ratios are shown in Fig. 3.11. For comparison, the U.S. Standard Atmosphere (1962) has a temperature of 288.15 K at the ground and 216.65 K at 17 km, a ratio of 1.33. From Fig. 3.11 it is apparent that the distortion distance only approaches an asymptote within the confines of the box for large temperature gradients. For the distortion distance to achieve asymptotic behavior within the box a temperature ratio of at least 5 is needed. If the bottom is cooled to \(-20^\circ\text{C}\) then this means that the top has to be about \(1000^\circ\text{C}\). First, it would be very challenging to generate such a gradient. Second, to record the waveforms, microphones must be stable over the temperature range. We therefore concluded that it is not practical to observe waveform freezing with this experiment.

A second model experiment was considered, one in which propagation through the atmosphere is modeled by propagation down an exponential horn. It too was considered unsatisfactory (see Appendix D). The hope for a model experiment was therefore abandoned.
Figure 3.11: The normalized distortion distance for various temperature gradients using the heated box.

3.9 Waveform Freezing of Sonic Booms

In the above analysis of waveform freezing we used the distortion distance as a criterion for determining whether waveform freezing occurs. This criterion is convenient because the predictions are independent of waveform shape. We concluded from the behavior of the distortion distance that waveform freezing does not occur for sonic booms in the atmosphere.

Now we investigate whether using a property of the waveform as a criterion for freezing alters the predictions (Cleveland 1995). Sonic boom pressure signatures are calculated in an isothermal atmosphere. The sonic boom is modeled as an N wave, for which the change in the duration of an N wave can be calculated from Eq. 3.12,

\[ T_h = T_{h0}\sqrt{1 + \frac{\Delta T}{T_{h0}}}. \]
Here $\bar{a} = \beta \hat{p}_0 / \hat{p}_0 \bar{c}_0^3 T_{h0}$ is evaluated using conditions at the source and $x$ in Eq. 3.12 is replaced with the distortion distance $\bar{x}$. The increase in half duration can be interpreted as proportional to the distortion of the sonic boom. The half duration (distortion) has a maximum (frozen) value, given by

$$T_{h\text{ max}} = T_{h0} \sqrt{1 + \bar{x}_{\text{max}}},$$

We use the normalized half duration $T_h / T_{h\text{ max}}$ to determine whether waveform freezing has occurred for sonic booms.

$$\frac{T_h}{T_{h\text{ max}}} = \frac{T_{h0} \sqrt{1 + \bar{x}}}{T_{h0} \sqrt{1 + \bar{x}_{\text{max}}}},$$

or

$$\left( \frac{T_h}{T_{h\text{ max}}} \right)^2 = \frac{1 + \bar{x}}{1 + \bar{x}_{\text{max}}}.$$

If $T_h / T_{h\text{ max}} = 0.95$ is chosen as the criterion for waveform freezing, the distortion distance at which the N wave can be considered to be frozen is

$$1 + \bar{x} = (0.95)^2 (1 + \bar{x}_{\text{max}}),$$

$$\bar{x} = 0.9025 \bar{x}_{\text{max}} - 0.0975 / \bar{a},$$

or

$$\bar{x} = 0.9025 \bar{x}_{\text{max}} - \frac{0.0975}{\bar{x}_{\text{max}}}.$$

That is, using the half duration criterion, waveform freezing occurs when the distortion distance is only at 90% of its maximum value. For cases where $\bar{x}_{\text{max}}$ is small, freezing (as determined by the half duration) occurs even sooner.

The early onset of freezing is demonstrated by example. A source N wave of peak amplitude $\hat{p}_0 = 300$ Pa and duration $2T_{h0} = 100$ ms is considered. The aircraft is flying at Mach 2 at an altitude of 17 km, and the reference distance is $r_s = 100$ m. Assuming an isothermal atmosphere, we calculate the maximum distortion distance to be $\bar{x}_{\text{max}} = 2.069$ km. For this N wave $\bar{a} = 1823$ m$^{-1}$, and $\bar{x}_{\text{max}} = 3.772$. The behavior of the normalized half duration $T_h / T_{h\text{ max}}$ is shown in Fig. 3.12(a). Although at the ground $\bar{x} / \bar{x}_{\text{max}} = 0.869$, however, the half duration there is 94.7% of its maximum value. Figure 3.12(b) shows the scaled waveform $q$ at the source, at the ground,
Figure 3.12: The distortion of an N wave in an isothermal atmosphere. (a) The duration of an N wave as a function of altitude, normalized by its maximum value. (b) The waveform at the source and ground. Also shown is the frozen waveform for this condition.

and in the frozen stage. The ground waveform is indeed very close to the frozen waveform. For an aircraft flying at an altitude of 10 km, $T_h / T_{h\text{ max}} = 0.921$ at the ground ($\tilde{x} / \tilde{x}_{\text{max}} = 0.729$). At an altitude of 25 km, $T_h / T_{h\text{ max}} = 0.971$ at the ground ($\tilde{x} / \tilde{x}_{\text{max}} = 0.939$). For an aircraft at an altitude greater than 18 km the duration at the ground is greater than 95% of its frozen value. One concludes that for aircraft at altitudes greater than about 18 km, the sonic booms at the ground can be considered to be frozen.

It can be shown that the spark-generated N waves that were to be used in the heated box experiment behave in a similar manner. Consider a 1 m tall box for which the bottom has been cooled to a temperature $T_b = -20^\circ \text{C}$ and the top heated to a temperature $T_t = 100^\circ \text{C}$. The temperature gradient is 120 K/m, which is still challenging but much more practical than the value 1000 K/m used in the calculation in Sec. 3.8. The source radius is assumed to be $\bar{z} = 5$ cm, the initial peak pressure is $\bar{p}_0 = 500$ Pa, and the initial half duration $T_{h0} = 7\mu s$. Figure 3.13(a) shows the
normalized half duration of an N wave in the box. Figure 3.13(b) shows the scaled pressure waveform at the bottom of the box (the source), the top of the box, and in the frozen condition. This example makes clear the possibility of observing waveform freezing in a laboratory experiment. It is likely that the horn experiment discussed in Appendix D might also demonstrate freezing.

We conclude this chapter by recognizing that the distortion distance (or the age variable) is not necessarily a good criterion for determining when a waveform is close to freezing. The distortion distance does correctly predict whether a finite-amplitude wave will freeze. But if the distortion distance is used to measure how close a waveform is to freezing, the result can be misleading. For the case of finite-amplitude propagation in the atmosphere, analysis of the distortion distance leads one to believe that waveform freezing does not occur to sonic booms. However, for N waves under the same atmospheric conditions we have found that the ground waveforms may be considered to be frozen after all. Analysis of the distortion distance is useful as a guide
to the occurrence of waveform freezing, because the result is independent of waveform shape, but the predictions can be conservative. From a practical point of view waveform freezing does occur for sonic booms in the atmosphere for aircraft at altitudes of 18 km and higher. In addition, the analysis here implies that the model experiments, which were abandoned in the early stages of this research project, may be feasible after all.
Chapter 4

Numerical Solution: THOR

4.1 Introduction

In this chapter a new numerical scheme for solving the extended Burgers equation is developed. This scheme is based upon a code of Lee and Hamilton (1995) that solves the KZK equation. The novel feature of the algorithm is that it remains in the time domain to apply all effects, including absorption and dispersion, but does not pay the computational price of a convolution.

The chapter leads off with an overview of other popular techniques to solve the Burgers equation. This is followed by a description of the time domain algorithm, in particular, the implementation of classical absorption and relaxation. Some frequency domain stability analysis is undertaken to investigate implementation of the nonlinearity, absorption, and relaxation algorithms. The code is tested against a number of analytical solutions to ensure that the algorithms are behaving correctly.

Finally, a scheme for implementing the code using a nonuniform time base is introduced. This new method has the potential for requiring far fewer points to describe a given waveform as the time grid moves dynamically to match the distortion of the waveform.

4.2 Background

The extended Burgers equation is a partial differential equation. Except for very special cases it must be solved numerically. It is common to use some sort of numerical marching scheme. As the code marches along it takes account of nonlinear distortion, absorption and any other effects separately at each step.

Early sonic boom codes, such as those of Hayes et al. (1969) and Thomas (1972), neglected losses. As seen in Chapter Three the lossless equation can be solved ana-
lytically. All that is necessary is to calculate the distortion distance and scaling that occurs to the waveform. A ground waveform is calculated by distorting the initial waveform according the Earnshaw solution and scaling the amplitude. Finally, weak shock theory is used to remove any multivaluedness. The penalty paid is that no information about the structure of the shocks, and hence the rise time, is available.

Pestorius (1973) developed a marching algorithm that takes into account absorption and dispersion as a waveform propagates. The algorithm takes small steps. At each step it first uses lossless theory to distort the waveform. The code then transfers into the frequency domain, using the fast Fourier transform (FFT), and applies absorption and dispersion according to linear theory. Finally, it inverse transforms to the time domain and takes the next step. Pestorius was interested in the propagation of plane waves in a tube. He only included boundary layer absorption and dispersion in the frequency domain. Weak shock theory was used to account for thermoviscous effects, i.e., to stop the waveform becoming multivalued.

Anderson (1974) followed the Pestorius algorithm, but applied thermoviscous absorption in the frequency domain.* By including thermoviscous absorption, Anderson did not need to rely on weak shock theory. At each step the waveform is distorted (steepened) by nonlinear effects and smoothed by thermoviscous absorption. The step size is chosen to ensure that a multivalued waveform is not formed. In Anderson’s code shock profiles are completely characterized rather than being represented by pure shocks. Although Anderson only applied thermoviscous attenuation, he noted that any absorption and dispersion laws could be used. The code has been extended to include relaxation effects to examine both spark generated N waves (Orenstein 1982) and sonic booms (Raspet et al. 1992).

Fenlon (1971) implemented the Burgers equation entirely in the frequency domain. The spectral version of the Burgers equation is (Hamilton 1993)

\[
\frac{dP'(\omega)}{dx} = -\frac{b\omega^2}{2\rho_0 c_0} P'(\omega) - \frac{j\omega \beta}{2\rho_0 c_0^2} P'(\omega) * P'(\omega),
\]

(4.1)

where \( P'(\omega) \) is the Fourier transform of \( p'(t') \). The loss term (which can be an arbitrary absorption and dispersion operator) is a multiplicative coefficient and the nonlinear

*Although he added in absorption ad hoc, Anderson effectively solves the Burgers equation.
term is a convolution. The spectral Burgers equation is an infinite set of coupled ordinary differential equations. Trivett and Van Buren solved these equations using a Runge-Kutta integrator. When implemented on a computer one must use a finite number of harmonics which can lead to two problems. First, a truncated Fourier series yields Gibbs oscillations. Second, the finite number of harmonics can lead to high frequency energy being trapped at the high end of the spectrum. In the physical finite-amplitude waves nonlinear distortion keeps pumping energy into higher and higher harmonics, but in the numerical model, energy cannot be pumped beyond the highest harmonic. The amplitudes of the higher harmonics grow unrealistically as energy being pumped in from lower harmonics cannot be pumped out to still higher harmonics. The energy can end up cascading back down into the lower harmonics with great numerical error.

The computational price paid using the purely frequency domain method is high. If $M$ harmonics are used the computational time is proportional to $M^2$ because of the convolution. On the other hand, codes that implement nonlinear distortion in the time domain (order $M$ operations) and absorption in the frequency domain (order $M$ operations) have a computational time proportional to $M \log M$ because of the FFT. Unfortunately, the FFT introduces some numerical error. This can become a problem for long propagation problems where a large number of FFT calculations are performed.

The computational price for modeling shocks can be high. Typically shock thicknesses are small, compared to a characteristic frequency in the waveform. To properly capture the shocks the sampling rate has to be high enough that there are enough data points within the shock that the profile can be properly modeled. Sampling theory requires that the sampling rate be at least twice the maximum frequency of interest (Oppenheim and Schafer 1989, Chap. 3.2). Unfortunately, the sampling rate required to model shocks must be applied over the whole waveform, as the FFT requires a uniformly sampled signal. It is therefore necessary to sample the whole waveform at a very high rate even if the sampling density is required in only a small fraction of the waveform. Typically for sonic booms the shock thickness is less than 1 ms, whereas the whole waveform may be 200 ms or more. The shock must be sampled at a rate of
20 μs or less, which means at least 10 000 points are required to describe the waveform.

Robinson (1991) tried to avoid the high sampling rate by reverting to the Pestorius approach: a mix of weak shock theory and linear absorption and dispersion. Because he was not characterizing steep shocks he could use large space steps before transforming into the frequency domain. This reduces both the computational cost and the numerical error associated with the FFTs. Robinson used weak shock theory to remove any multivaluedness that occurred between applications of absorption. Robinson included all absorption and dispersion effects in the frequency domain, including thermoviscosity. The approach is open to the criticism of double dipping, as thermoviscous absorption is applied in two ways.

In this chapter a new computer code, THOR, is presented to solve the Burgers equation. The algorithm was developed by Lee and Hamilton (1991, 1995, see also Lee 1993) to model the propagation of pulsed finite-amplitude sound beams. Lee and Hamilton solved the KZK equation, rather than the Burgers equation. The KZK equation includes the effects of diffraction, as well as nonlinearity and thermoviscous absorption. The algorithm remains in the time domain but without having to pay the computational price of a convolution. Absorption is taken into account by recognizing that the equation used to model absorption is a simple diffusion equation:

$$\frac{\partial p'}{\partial t} = \frac{\partial^2 p'}{\partial x^2}.$$

There are many well known finite difference algorithms for solving this equation. They usually entail solving a tridiagonal matrix system, which can be done very efficiently.

Lee and Hamilton (1995) also described a method for including the effects of multiple relaxation phenomena. The equation for a single relaxation process is

$$\left(1 + \tau \frac{\partial}{\partial t'}\right) \frac{\partial p'}{\partial t} = \frac{m \tau}{2c_0} \frac{\partial^2 p'}{\partial t'^2},$$

which is similar to the absorption equation and can also be approximated by finite differences which yield a tridiagonal matrix. As part of this work, the procedure described by Lee and Hamilton for including relaxation has been implemented.

As mentioned in Chapter Two, the geometrical approximation is sufficient for the propagation of sonic booms. The diffraction routine in Lee and Hamilton's code is
replaced with one which accounts for spreading, either simple cylindrical or spherical spreading, or ray tube area spreading. A subroutine is also added to allow the code to handle propagation through a stratified medium, in which all acoustic properties can vary along the ray path.

One advantage with the time domain code is that the waveform need not be periodic. Use of the FFT requires that the signal be made periodic, i.e., the endpoints must match. Also, the frequency domain multiplication is equivalent to a circular convolution in time (Oppenheim and Schafer 1989, Chap. 8.9). For nonperiodic waveforms, e.g., sonic booms, it is important to ensure that the waveform is properly padded with zeros so that the convolution does not interact with waveforms in two periods (Oppenheim and Schafer 1989, Fig. 8.21).

Because it imposes no restrictions on the end points, the time domain algorithm has the nice property that it can easily propagate pulses, e.g., step shocks and N waves. This is particularly advantageous when a steady-state step shock solution is desired. For example, Raspet et al. (1992) used a version of the Anderson code to propagate step shocks. They used the lead shock of a square pulse waveform to model a step shock. However, a square pulse wave has limited propagation range. As the wave propagates the rear edge catches up with the leading shock and the waveform eventually turns into a sawtooth wave, which does not maintain constant amplitude. The length of the pulse must be sufficient that the shock profile is not affected by the presence of the rear of the pulse. The length restriction can demand a very long pulse and hence long computation time. In the time domain code there is no length issue and the distance a true step shock can be propagated is virtually unlimited.

On the other hand, codes that use the FFT have a big advantage for continuous waves. The frequency domain code requires just one cycle of the primary frequency in the time window that it calculates. Although it is possible to model a periodic signal with the finite difference method, the tridiagonal nature of the system is lost. Therefore, the time domain code must use a long pulse of the frequency component of interest. The pulse should be long enough that the center cycle is not affected by the behavior at the edge of the pulse. Five cycles is typically sufficient.
4.3 Dimensionless Form of the Burgers Equation

Before we present the computer algorithm to solve the extended Burgers equation, it shall be written in dimensionless form. The classical Burgers equation, Eq. 2.1, is

\[ \frac{\partial p'}{\partial x} - \frac{\beta}{2\rho_0 c_0^2} \frac{\partial p'^2}{\partial t'} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 p'}{\partial t'^2} \].

A characteristic acoustic pressure \( p_0 \) and characteristic frequency \( \omega_0 \) are used to produce dimensionless pressure \( P = p'/p_0 \) and time \( \tau = \omega_0 t' \). The Burgers equation can be written in the following dimensionless form:

\[ \frac{\partial P}{\partial x} - \frac{\beta}{\rho_0 c_0^2} \frac{\omega_0}{2} \frac{\partial P^2}{\partial \tau} = \frac{b\omega_0^2}{2\rho_0 c_0^3} \frac{\partial^2 P}{\partial \tau^2} \].

Two length scales are apparent in this equation, the shock formation distance of the characteristic frequency, \( x = 1/\beta \rho_0 c_0^2 \), and the absorption length of the characteristic frequency, \( b\omega_0^2/2\rho_0 c_0^3 \). Recall from Chapter Two that the length scale related to the nonlinear distortion \( \varepsilon \) was used, on the grounds that, for sonic boom propagation at least, it is the shortest length scale. We use the shock formation distance of the characteristic frequency to define the dimensionless range variable \( \sigma = x/x \). The dimensionless Burgers equation is

\[ \frac{\partial P}{\partial \sigma} - \frac{1}{2} \frac{\partial P^2}{\partial \tau} = \frac{1}{\Gamma} \frac{\partial^2 P}{\partial \tau^2} \]. \hspace{1cm} (4.2)

The term \( \Gamma \) is the Gol'berg number (see, for example, Blackstock 1972). It is the ratio of the absorption length to the shock formation distance of the characteristic frequency. For large \( \Gamma \) the shock formation distance is much shorter than the absorption length and so nonlinear effects are more important than absorption effects. For small \( \Gamma \) absorption dominates.

The dimensionless version of the extended Burgers equation (the effects of relaxation processes, geometrical spreading and propagation through a stratified medium are included) is

\[ \frac{\partial P}{\partial \sigma} = P \frac{\partial P}{\partial \tau} + \frac{1}{\Gamma} \frac{\partial^2 P}{\partial \tau^2} + \sum_{\nu} C_{\nu} \frac{\partial^2 P}{\partial x^2} - \frac{\partial P}{2S} + \frac{\partial}{\partial \sigma} \left( \rho_0 c_0 \right) P \]. \hspace{1cm} (4.3)
The same nondimensionless variables are used. Recall, that the retarded time frame is defined as 

\[ t' = t - \int \frac{dx}{c_0} \]

and the range variable is now \( \sigma = s/\bar{x} \), where \( s \) is the path length. The variable \( C_{\nu} = \frac{m_{\nu} \tau_{\nu} \omega_0^2 \bar{x}}{2c_0} = \frac{m_{\nu} \tau_{\nu} \omega_0 \rho_0 c_0^2}{\bar{\rho}_{\rho_0}} \) is a dimensionless dispersion parameter and \( \theta_{\nu} = \omega_0 \tau_{\nu} \) is the dimensionless time of the \( \nu^{th} \) relaxation process.

### 4.4 Code Details

The extended Burgers equation is solved numerically by decoupling Eq. 4.3 into each of its separate effects:

\[
\frac{\partial P}{\partial \sigma} = P \frac{\partial P}{\partial \tau}, \\
\frac{\partial P}{\partial \sigma} = \frac{1}{1} \frac{\partial^2 P}{\partial \tau^2}, \\
\frac{\partial P}{\partial \sigma} = \sum \frac{C_{\nu} \frac{\partial^2}{\partial \tau^2}}{\nu} \frac{\partial}{\partial \sigma} P, \\
\frac{\partial P}{\partial \sigma} = -\frac{1}{2S} \frac{\partial S}{\partial \sigma} P, \\
\frac{\partial P}{\partial \sigma} = \frac{1}{2\rho_0 c_0} \frac{\partial (\rho_0 c_0)}{\partial \sigma} P.
\]

The propagation problem is approximated by successively solving Eqs. 4.4–4.8 over some small step \( \Delta \sigma \). The size of the step \( \Delta \sigma \) is chosen so that the error associated with solving each effect independently is insignificant. Lee and Hamilton (1995, see also, Lee 1993) examined the convergence of the splitting method.

To represent the waveform in the computer it is uniformly sampled at frequency \( f_s \). The dimensionless sampling period is \( \Delta \tau = \omega_0 / f_s \). The number of samples in the whole waveform is denoted by \( M \). The notation used to describe the pressure in the \( i^{th} \) sample at the \( k^{th} \) step is

\[ P(\sigma_k, \tau_i) = P_{i}^{k}. \]

The integer \( i \) runs from 1 to \( M \), i.e., over the number of points in the waveform. The integer \( k \) ranges from 0 upwards, where \( k = 0 \) is the initial condition.

In the code, each algorithm monitors how much it changes the waveform at each step. The step size is dynamically altered to ensure that it is as large as possible to
keep run time down but small enough to keep errors associated with decoupling effects to a minimum.

4.4.1 Implementation of Nonlinearity

Equation 4.4 is solved using the Poisson solution presented in Chapter Three (Eq. 3.2). The analytic solution of Eq. 4.4 at \( \sigma + \Delta \sigma \), for a known waveform at \( \sigma \), is

\[
P(\sigma + \Delta \sigma, \tau) = P(\sigma, \tau + P \Delta \sigma).
\] (4.9)

Multivalued waveforms are avoided if the restriction

\[
\Delta \sigma < \frac{1}{\max(\frac{\partial P}{\partial \tau})}
\] (4.10)

is satisfied. Equation 4.9 is implemented by transforming the time base onto a distorted grid:

\[
\tau^d_i = \tau_i - P^k_i \Delta \sigma.
\] (4.11)

The waveform is returned to the uniform time grid by using linear interpolation. The linear interpolation scheme is accurate to \( O[(\Delta \tau)^2] \).

The interpolation must be done with care. Consider a positive point in the waveform, for which the distortion is such that \( \tau^d_i < \tau_i \). It is possible for the distorted waveform to move such that \( \tau^d_i < \tau_{i-1} \) without a shock forming, in which case \( P^{k+1}_i \) should be calculated by interpolating between \( \tau^d_{i+1} \) and \( \tau^d_{i+2} \). A similar effect can happen if \( P \) is negative; to calculate \( P^{k+1}_i \) the code may need to interpolate between \( \tau^d_{i-2} \) and \( \tau^d_{i-1} \).

The nonlinear algorithm presented by Lee and Hamilton (1995), where distortion and interpolation are combined into one equation, does not account for this gross transport of the time grid. The restriction on their space step is \( \Delta \sigma_k < \Delta \tau / \max|P^k| \), which can be unnecessarily strict if the time step \( \Delta \tau \) is small. In THOR the interpolation algorithm is written to cope with any movement of the time grid.

4.4.2 Implementation of Classical Attenuation

The equation for classical, or thermoviscous, attenuation is Eq. 4.5:

\[
\frac{\partial P}{\partial \sigma} = \frac{1}{\Gamma} \frac{\partial^2 P}{\partial \tau^2},
\]
The equation is equivalent to a standard heat diffusion equation, with the space and time coordinates switched. We will solve the equation by a finite difference method.

The derivative on the left-hand side is approximated by the forward-space finite difference
\[
\frac{\partial P}{\partial \sigma} = \frac{P(\sigma + \Delta \sigma) - P(\sigma)}{\Delta \sigma} + O[(\Delta \sigma)^2].
\]
The right hand side is approximated by the centered-time finite difference
\[
\frac{\partial^2 P}{\partial \tau^2} = \frac{P(\tau + \Delta \tau) - 2P(\tau) + P(\tau - \Delta \tau)}{(\Delta \tau)^2} + O[(\Delta \tau)^2].
\]

When these equations are expressed in explicit form (the right-hand side is evaluated in terms of \(P(\sigma)\)) the resulting equation is
\[
P_j^{k+1} = 2\lambda P_{j-1}^k + (1 - \lambda)P_j^k + 2\lambda P_{j+1}^k,
\]
where \(\lambda = \Delta \sigma/(2\Gamma(\Delta \tau)^2)\). This expression is referred to as explicit because it calculates the new \(P(\sigma + \Delta \sigma, \tau)\) in terms of pressures at \(\sigma\), i.e., values that are already known. This method is very accurate (Ames 1977, Chap. 2-1) but becomes unstable for larger step sizes (Ames 1977, Chap. 2-2).

The fully implicit, or O'Brien, method evaluates the right hand side using the unknown values of \(P\) at \(\sigma + \Delta \sigma\). The resulting equation is
\[
-2\lambda P_{j-1}^{k+1} + (1 + \lambda)P_j^{k+1} - 2\lambda P_{j+1}^{k+1} = -2\lambda P_j^k,
\]
which is a set of \(M\) coupled linear equations. The implicit method is unconditionally stable but is not as accurate and is susceptible to numerical error in the inversion process (Ames 1977, Chap. 2-3).

A popular numerical technique for solving the diffusion equation is the Crank-Nicolson method, which combines the implicit and explicit methods. The finite difference approximation of Eq. 4.5 is
\[
-\lambda P_{j-1}^{k+1} + (1 + 2\lambda)P_j^{k+1} - \lambda P_{j+1}^{k+1} = \lambda P_j^k + (1 - 2\lambda)P_j^k + \lambda P_j^{k+1}. \quad (4.12)
\]
It is convenient to write the absorption equations in matrix form. The vector $P^k$ contains the entire waveform at step $k$, that is,

$$P^k = \begin{pmatrix} p_1^k \\ p_2^k \\ \vdots \\ p_{M-1}^k \\ p_M^k \end{pmatrix}. \quad (4.13)$$

It follows that the vector $P^{k+1}$ contains the waveform at step $k + 1$. We define the following matrices

$$A_{tv} = \begin{pmatrix} 1 & 0 & \cdots & -\lambda & (1 + 2\lambda) & -\lambda \\ -\lambda & (1 + 2\lambda) & -\lambda & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ -\lambda & (1 + 2\lambda) & -\lambda & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots & \cdots \\ \end{pmatrix}, \quad (4.14)$$

$$B_{tv} = \begin{pmatrix} 1 & 0 & \lambda & (1 - 2\lambda) & \lambda \\ \lambda & (1 - 2\lambda) & \lambda & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \lambda & (1 - 2\lambda) & \lambda & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots & \cdots \\ \end{pmatrix}, \quad (4.15)$$

Equation 4.12 can be described by the matrix equation

$$A_{tv}P^{k+1} = B_{tv}P^k.$$

The pressure at step $k + 1$ is obtained by premultiplying both sides with the inverse of $A_{tv}$,

$$P^{k+1} = A_{tv}^{-1}B_{tv}P^k.$$

The matrix $A_{tv}$ is tridiagonal and can be inverted in order $M$ operations by the Thomas algorithm (Ames 1977, Chap. 2.3).

The matrices are constructed so that the end points of the waveform remain constant, that is,

$$P_1^{k+1} = P_1^k,$$
This makes the system ideal for propagating step shocks. For pulses the restriction should be \( P_{M+1}^{k+1} = 0 \) and \( P_{M}^{k+1} = 0 \); however, if the waveform starts off with zeroes at the end points the present scheme will ensure that the end points remain zero. For periodic signals \( P_1^k = P_{M+1}^k \) and the \( \mathbf{A}_{tv} \) matrix becomes

\[
\mathbf{A}_{tv} = \begin{pmatrix}
(1 + 2\lambda) & -\lambda & \cdots & -\lambda \\
-\lambda & (1 + 2\lambda) & \cdots & -\lambda \\
& \ddots & \ddots & \ddots \\
-\lambda & \cdots & \cdots & (1 + 2\lambda)
\end{pmatrix}.
\]

The triadiagonal nature of the matrix is lost. Note that it is very close to tridiagonal and there may exist efficient algorithms for inverting this system. In this research we are only interested in pulses and this avenue is not explored. When results for sinusoidal waveforms are required, tone bursts of five cycles are used as the input to THOR. The behavior of the middle cycle appears to be close to that of a continuous wave.

**4.4.3 Implementation of Relaxation Processes**

The equation for a single relaxation process, Eq. 4.6, is

\[
\left(1 + \theta \frac{\partial}{\partial \tau}\right) \frac{\partial P}{\partial \sigma} = C \frac{\partial^2 P}{\partial \tau^2}.
\]

This is also solved by a finite-difference technique. The finite difference operator for \( \frac{\partial P}{\partial \sigma} \) remains the same. The term \( \frac{\partial^2 P}{\partial \tau \partial \sigma} \) is approximated by the following forward-space, centered-time finite difference:

\[
\frac{\partial^2 P}{\partial \tau \partial \sigma} = \frac{1}{2\Delta\tau} \left( \frac{\partial P}{\partial \sigma} \bigg|_{\tau+\Delta\tau} - \frac{\partial P}{\partial \sigma} \bigg|_{\tau-\Delta\tau} \right) + O[(\Delta \tau)^2],
\]

\[
= \frac{P_{i+1}^{k+1} - P_{i-1}^{k+1} - (P_{i+1}^k - P_{i-1}^k)}{2\Delta\tau \Delta\sigma} + O[(\Delta \sigma)^2, (\Delta \tau)^2].
\]

The effect of using forward time differencing is discussed in Sec. 4.5.5.
The implicit and explicit finite differences for the right hand side of Eq. 4.6 are the same as for classical absorption. Some stability problems occurred with the relaxation algorithm for cases with very small $\Delta \sigma$. To improve the stability a control parameter, $\alpha$, is introduced which allows the weighting of the explicit and implicit differences to be varied (Wilson 1994). When $\alpha = 0$ the method is fully explicit, when $\alpha = 1$ the method is fully implicit, and when $\alpha = 0.5$ the Crank-Nicolson method is recovered (Ames 1977, Chap. 2-3). For cases where instability occurs increasing $\alpha$ to 0.55 or 0.6 stabilized the code, that is, more weight is given to the stable implicit method. For all the results presented in this work no stability problems occurred and the Crank-Nicolson method with $\alpha = 0.5$ is used.

The finite difference used for a single relaxation process is

$$-(\alpha \lambda + \mu) P_{j-1}^{k+1} + (1 + 2\alpha \lambda) P_j^{k+1} - (\alpha \lambda - \mu) P_{j+1}^{k+1} =$$

$$(\alpha' \lambda - \mu) P_{j-1}^k + (1 - 2\alpha' \lambda) P_j^k + (\alpha' \lambda + \mu) P_{j+1}^k,$$

where $\alpha' = 1 - \alpha$, $\lambda = C_\nu \Delta \sigma/(\Delta \tau)^2$, and $\mu = 0_\nu/2\Delta \tau$.

We define the matrices

$$A_\nu = \begin{pmatrix}
1 & 0 & & & \\
-(\alpha \lambda + \mu) & (1 + 2\alpha \lambda) & -(\alpha \lambda - \mu) & & \\
& \ddots & \ddots & \ddots & \\
& & -(\alpha \lambda + \mu) & (1 + 2\alpha \lambda) & -(\alpha \lambda - \mu) & \\
& & & 0 & 1
\end{pmatrix}, \quad (4.16)$$

$$B_\nu = \begin{pmatrix}
1 & 0 & & & \\
(\alpha' \lambda - \mu) & (1 - 2\alpha' \lambda) & (\alpha' \lambda + \mu) & & \\
& \ddots & \ddots & \ddots & \\
& & (\alpha' \lambda - \mu) & (1 - 2\alpha' \lambda) & (\alpha' \lambda + \mu) & \\
& & & 0 & 1
\end{pmatrix}. \quad (4.17)$$

Again the matrices have been constructed so the boundary conditions applied at the ends of the time window are $P_1^{k+1} = P_1^k$ and $P_M^{k+1} = P_M^k$. The system to be solved for each relaxation process is

$$A_\nu P^{k+1} = B_\nu P^k.$$
It is a tridiagonal system and can be solved in order $M$ operations.

Note it is not possible to combine thermoviscous absorption and relaxation processes into a single matrix equation without losing the tridiagonal nature of the system. For every added absorption or relaxation process two more diagonal components are added to the $A$ and $B$ matrices. For example, for thermoviscous and one relaxation process one must solve a pentadiagonal matrix system. Therefore at each step $A_{tv}$ must be inverted and for each relaxation process the corresponding $A_{v}$ must be inverted.

4.4.4 Implementation of Spreading

Spherical or cylindrical spreading is taken into account by the equation

$$\frac{\partial P}{\partial \sigma} = -\frac{m}{\sigma} P,$$

where $m = 1/2$ for cylindrical waves and $m = 1$ for spherical waves. The exact solution is

$$P(\sigma + \Delta\sigma, \tau) = \frac{P(\sigma, \tau)}{(1 + \Delta\sigma/\sigma)^{m}}. \quad (4.18)$$

For the more general case of arbitrary ray tube area the equation describing the evolution of the wave is

$$\frac{\partial P}{\partial \sigma} = -\frac{1}{2S} \frac{\partial S}{\partial \sigma} P,$$

for which the exact solution is

$$P(\sigma + \Delta\sigma, \tau) = \sqrt{\frac{S(\sigma)}{S(\sigma + \Delta\sigma)}} P(\sigma, \tau). \quad (4.19)$$

It is assumed that the step size $\Delta\sigma$ is small enough that the curvature of the ray can be neglected.

Both of these solutions, Eqs. 4.18 and 4.19, are implemented in the code.

4.4.5 Implementation of Stratification

The effect of stratification is taken into account by Eq. 4.8,

$$\frac{\partial P}{\partial \sigma} = \frac{1}{2\rho_{0} c_{0}} \frac{\partial (\rho_{0} c_{0})}{\partial \sigma} P.$$
The exact solution for this is

\[ P(\sigma + \Delta \sigma, \tau) = \sqrt{\frac{(\rho_0 c_0)_{\sigma + \Delta \sigma}}{(\rho_0 c_0)_{\sigma}}} P(\sigma, \tau). \] (4.20)

Once again it is assumed that the curvature of the ray path is negligible over the step size \( \Delta \sigma \).

The variation in density and sound speed adds a small complication. Because \( \rho_0 \) and \( c_0 \) can vary along the path, the shock formation distance can vary with path length. This means the relationship between the nondimensional range \( \sigma \) and the path length \( s \) is no longer a simple scaling. It is usual for input and output data to be expressed in terms of the path length \( s \), not the dimensionless range \( \sigma \). Therefore THOR calculates the true path length \( s \) at each step by the formula

\[ s^{k+1} = s^k + \bar{s}^k \Delta s^k. \]

The spatial derivative at a given range is approximated by

\[ \frac{\partial}{\partial \sigma} = \bar{s}^k \frac{\partial}{\partial s}. \]

The path dependent nature of all the loss terms, e.g., \( b_i \), is incorporated by simply changing the coefficients as a function of distance. The code must be supplied with the coefficients at a number of points along the ray path. Linear interpolation is used to obtain their values at any other point on the path. The data must be provided regularly so that linear interpolation does not introduce a significant error. It is recommended that no quantity change by more than 5% between samples along the ray path.

4.5 Analysis of Numerical Algorithms

The various algorithms described above can be examined by considering their effect on harmonic waves. This is similar to von Neumann/Fourier stability analysis which is commonly used in numerical analysis (Ames 1977, Chap. 2.2, Strang 1986, p. 579). Each of the equations is spectrally decomposed by expressing the waveform as the \( n^{th} \) harmonic of the characteristic frequency \( \omega_0 \). The pressure at a given location \( \sigma \) can
be written as

\[ P(\sigma, \tau) = \tilde{P}_n(\sigma)e^{in\tau}, \]

\[ = \tilde{P}_ne^{in\Delta\tau}, \]

where \( \tilde{P}_n \) is the amplitude of the \( n \)th harmonic, the integer \( i \) denotes the time point in the waveform, and \( j^2 = -1 \).

4.5.1 Fourier Analysis of Thermoviscous Attenuation

The behavior of the Crank-Nicolson algorithm is analyzed in most textbooks on numerical methods (see, for example, Ames 1977, Chap. 2). Lee (1993, Chap. 4.1) shows excellent agreement between results from the finite-difference thermoviscous absorption algorithm and analytical predictions from linear theory.

In their work on nonlinear Rayleigh waves, Hamilton et al. (1995) analyzed the finite difference approximation of thermoviscous absorption. Recall that the absorption equation is solved using a centred-time finite difference equation:

\[ \frac{\partial P}{\partial \sigma} = A \frac{\partial^2 P}{\partial \tau^2}, \]

\[ \sim \frac{A}{(\Delta \tau)^2} [P(\tau - \Delta \tau) - 2P(\tau) + P(\tau + \Delta \tau)]. \quad (4.21) \]

When the harmonic form of \( P \) is substituted into Eq. 4.21 one obtains

\[ \frac{\partial P}{\partial \sigma} = \frac{A}{\Delta \tau^2} (\tilde{P}_ne^{jn(i+1)\Delta\tau} - 2\tilde{P}_ne^{jn(i+1)\Delta\tau} + \tilde{P}_ne^{jn(i-1)\Delta\tau}), \]

\[ \frac{\partial \tilde{P}_ne^{jn\Delta\tau}}{\partial \sigma} = \frac{A}{\Delta \tau^2} \tilde{P}_ne^{jn\Delta\tau}(e^{jn\Delta\tau} - 2 + e^{-jn\Delta\tau}), \]

\[ \frac{\partial \tilde{P}_n}{\partial \sigma} = \frac{A}{\Delta \tau^2} \tilde{P}_n(2 \cos(n\Delta\tau) - 2), \]

\[ = -\frac{A}{\Delta \tau^2} 4\tilde{P}_n \sin^2(n\Delta\tau/2), \]

\[ = -An^2 \sin^2(n\Delta\tau/2) \tilde{P}_n. \]

The exact relation for a sinusoidal waveform propagating in a thermoviscous medium is

\[ \frac{\partial \tilde{P}_n}{\partial \sigma} = -An^2 \tilde{P}_n. \]
The effective absorption applied by the finite difference algorithm is not the exact absorption $\alpha_{\text{exact}} = -An^2$, but

$$\alpha_{\text{fd}} = -An^2 \sin^2(\pi f/f_s) / \left(\pi f/f_s\right)^2.$$  \hspace{1cm} (4.22)

The argument has been rewritten using $\eta = f/f_0$ and $\Delta \tau = 2\pi f_0/f_s$, where $f_0$ is the characteristic frequency of the waveform and $f_s$ is the frequency used to sample the waveform. We can rewrite the effective absorption in terms of the sinc function $\alpha_{\text{fd}} = -An^2 \text{sinc}^2(f/f_s)$, where $\text{sinc} x = \sin \pi x / \pi x$.

Figure 4.1 plots $\alpha_{\text{fd}} / \alpha_{\text{exact}}$ as a function of frequency. Also plotted are both attenuation laws on a log scale. It is apparent that attenuation at high frequencies is underestimated by the finite difference approach. At the Nyquist frequency $f = 0.5f_s$, $\alpha_{\text{fd}} = 0.5\alpha$. Recall that when the waveform is reconstructed, frequencies above the Nyquist frequency do not contribute to the waveform.

Hamilton \textit{et al.} (1995) found that the shock amplitudes predicted by a time domain computer code, for the propagation of nonlinear Rayleigh waves, were slightly overestimated. Lee and Hamilton (1995) suggest that the time domain algorithm for acoustic waves should exhibit similar behavior. However, this assumes that the other algorithms do not add absorption artificially. A likely source of numerical absorption is the nonlinear distortion algorithm, as it uses linear interpolation to resample the distorted waveform back onto a uniform time grid. Interpolation is a smoothing process, equivalent to applying a low pass filter to the data, and so high frequencies are attenuated. This could compensate, in some form, for the lack of attenuation in the thermoviscous step. The code of Hamilton \textit{et al.} (1995) did not distort the time base and so interpolation was not applied.

It is not possible to fully analyze the nonlinear distortion routine using Fourier analysis because the distortion is nonlinear. In the next section the presence of numerical absorption in the nonlinear algorithm is demonstrated by analysis of the linear wave equation.

A crude approach is to neglect the distortion of the nonlinear routine altogether and consider just the effect of linear interpolation. The process of linear interpolation is equivalent to convolution in the time domain by a triangular waveform of duration $2\Delta \tau$.
Figure 4.1: The finite difference approximation of thermoviscous absorption. (a) The finite difference operator. (b) The attenuation curves of the analytic and finite difference approximation of thermoviscous attenuation.

(Oppenheim and Schafer 1989, Chap. 3.6). In the frequency domain this is equivalent to multiplying by the function \( \text{sinc}^2(f/f_s) \). In practice the distortion of the time base means that the length of the triangular convolution kernal will be different along the waveform. We neglect the distortion here and use the low-pass filter for a uniform \( \Delta \tau \). Absorption is inversly proportional to the transfer function (or gain) and so the effective absorption applied by linear interpolation is \( \alpha_{\text{li}} = 1/\text{sinc}^2(f/f_s) \). Therefore, \( \alpha_{\text{li}} \alpha_{\text{li}} = \alpha_{\text{exact}} \), and the linear interpolation exactly compensates for the lack of attenuation in the finite difference absorption.

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*This operation turns a sampled waveform into a piecewise continuous waveform. To return to a sampled waveform we should convolve with a series of \( \delta \) functions (Oppenheim and Schafer 1989, Chap. 3.2).
4.5.2 Fourier Analysis of Nonlinear Distortion

The nonlinear nature of the distortion does not lend itself to study by Fourier analysis (which is best suited to linear systems). Recall that the equation used to model nonlinear distortion is the nonlinear first-order hyperbolic equation

$$\frac{\partial P}{\partial \sigma} = P \frac{\partial P}{\partial \tau}.$$  

To use Fourier analysis the equation must be linearized. In this case we reduce it to a linear first order wave equation,

$$\frac{\partial P}{\partial \sigma} = \frac{\partial P}{\partial \tau}.$$  

The hyperbolic nature of the nonlinear equation is retained but the effective sound speed is a constant, $\frac{d\sigma}{d\tau} = 1$ for all points, rather than varying with the amplitude of the wave. However, the linear equation can demonstrate the behavior of the algorithm used.

The nonlinear algorithm uses the Poisson solution, which shifts the time base at the local sound speed, and then interpolates back onto the uniform grid. For the linear wave equation above the time shift to the distorted time grid is

$$\tau_i^d = \tau_i - \Delta \sigma.$$  

Note that $\tau^d$ does not depend on $P$ and so the grid is not really distorted. Linear interpolation yields

$$P_i^{k+1} = P_i^k + \frac{P_i^{k+1} - P_i^k}{\Delta \tau} (\tau_i - \tau_i^d).$$  

To use this formula we assume that $\Delta \sigma \leq \Delta \tau$, that is, the points do not move further than $\Delta \tau$ at any given step. In practice the points are not restricted by this but we use it in this analysis to simplify the algebra.

If we assume a time harmonic waveform the distortion algorithm becomes

$$\tilde{P}_n^{k+1} = \tilde{P}_n^k e^{j n \Delta \tau} + (e^{j n \Delta \tau} - 1) \tilde{P}_n^k e^{j n \Delta \tau} \frac{\Delta \sigma}{\Delta \tau},$$

$$\frac{\tilde{P}_n^{k+1}}{\tilde{P}_n^k} = 1 - \frac{\Delta \sigma}{\Delta \tau} (1 - e^{j n \Delta \tau}).$$
The analytic solution is $\tilde{P}_{n}^{k+1} = e^{jn\Delta\sigma} \tilde{P}_{n}^{k}$, a travelling (hence the phase shift) time harmonic wave of constant amplitude. The Courant number (Strang 1986, p. 579, see also, Ames 1977, Chap. 4.6-4.7) is defined as $C = \Delta\sigma/\Delta\tau$. The numerical solution is exact when $C = 1$, that is, if the grid is set up so that the grid points lie exactly on the characteristics. For finite-amplitude waves it is impossible to generate such a grid because the sound speed, and hence the Courant number, is different at various points on the waveform. Some error in the algorithm has to be expected. A new algorithm is presented in the final section of this chapter to circumvent this problem.

To examine the behavior of the finite difference approximation of the first order wave equation we introduced the amplification or growth factor $G$ (Strang 1986, Chap. 6.5). The amplification factor is the gain of a given frequency component from one step to the next, $G = \tilde{P}_{n}^{k+1}/\tilde{P}_{n}^{k}$. The finite difference and exact solutions have the following amplification factors:

$$G_{fd} = 1 - C(1 - e^{j2\pi f/\tau}),$$
$$G_{exact} = e^{jn\Delta\tau(\Delta\sigma/\Delta\tau)}.$$

The implementation of the nonlinear algorithm is such that the Courant number varies from 0 to 1 depending on how close a distorted sample is to the nearest uniform sample. Figure 4.2 shows numerical absorption and phase error (numerical dispersion) as a function of Courant number for various frequencies. Because the nonlinear algorithm has Courant numbers varying from 0 to 1, we should expect the absorption and dispersion to be some sort of average value.

The strongest numerical attenuation occurs when $C = 0.5$. The uniform point lies right in between two distorted points and the error due to linear interpolation is a maximum. Attenuation also gets worse with higher frequency, that is, less samples in the waveform. The worst dispersion occurs for high frequencies (low sampling rates).

Recall that the finite difference algorithm is exact if $C = 1$. However, this is impossible to ensure for the nonlinear algorithm. For high sampling rates ($f_S \gg f$)

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*Actually Strang defines the Courant number as $\Delta\tau/\Delta\sigma$, because he marches in time rather than space.*
Figure 4.2: The numerical gain and phase error of the finite difference growth factor $G_{fd}$.

The finite difference amplification factor can be written as

$$G_{fd} = 1 - C(1 - (1 + j2\pi f/f_s)),$$

$$= 1 - jC^2\pi f/f_s.$$  

We have made use of the fact that the Courant number is less than 1. The analytical amplification factor for low frequencies (high sampling rates) is

$$G_{exact} = 1 - jC^2\pi f/f_s.$$  

Therefore as long as the frequencies of interest are well sampled the finite difference method should yield good results. We demonstrate this in the next section.

4.5.3 Comparison to Steady State

To check the algorithm for thermoviscous attenuation and nonlinear distortion, output from THOR is compared to the steady-state solution of the classical Burgers equation.
for a thermoviscous fluid. The known analytical steady state solution is the hyperbolic tangent function, see Appendix E. Figure 4.3 shows a symmetric step shock propagated by THOR. The first figure $\sigma = 0$ is the initial profile, selected because it looked interesting. The other figures show how the profile develops. The final figure, at distance $\sigma = 2$, shows that the numerical result agrees very well with the analytical steady-state solution.

Figure 4.3: Propagation of a shock in a thermoviscous medium.

In the Fourier analysis above it is apparent that the thermoviscous algorithm underestimates absorption, whereas the nonlinear algorithm adds some numerical absorption and dispersion. To examine how nonlinearity and absorption are modeled, close attention is paid to the steady-state solution. The profile of a unipolar shock is given by
(see Appendix E)

\[ P(\tau) = 0.5 \left[ 1 + \tanh\left(\frac{\tau - \sigma/2}{\Gamma/2}\right) \right] . \]  

Note that the shock moves with respect to the small signal sound speed at a speed \( \frac{\partial \sigma}{\partial \tau} = 1/2 \).

The steady state profile is used as the initial waveform for the code. The profile and the propagation of the shock are then monitored. Changes in the profile describe how well thermoviscous attenuation is being modeled. If the attenuation is overestimated, the rise time of the shock increases. If attenuation is being underestimated, the rise time decreases.

The arrival time of the shock at various locations determines how well the shock speed is being modeled by the nonlinear algorithm. If the shock arrives later than expected, then it is travelling a little slow. If it arrives earlier than expected it is travelling too fast. In what follows \( \Gamma = 8 \), which means that the 10% to 90% rise time of the exact solution is 0.5 ln 9 \( \sim 1.10 \).

Figure 4.4 compares the analytical step shock profile to the numerical profiles for various numbers of points in the 10% to 90% rise phase of the shock. For less than 20 points in the shock the numerical rise time is longer than the analytical result. This implies that absorption is being overestimated. The numerical absorption introduced by the nonlinear algorithm overcompensates for the lack of attenuation in the thermoviscous algorithm. When there are more points in the shock, the linear interpolation introduces less error and the profile is very close to the exact result. An average Courant number of 0.5 was used to obtain these results (\( C = 0 \) where \( P = 0 \), and \( C = 1 \) where \( P = 1 \)).

Figure 4.5(a) shows a plot of computed rise time versus step size for either 5, 10 or 20 points in the shock. The step size, \( \Delta \sigma \), is the fraction of the distance to the occurrence of a shock. The shock rise time is almost always overestimated, implying that high frequency attenuation is overestimated in this scheme—not underestimated. The error is less for large step size and many points in the shock. This is apparently because the low pass filtering effect of the nonlinear algorithm is least for these conditions and so does not compensate for the lack of attenuation in the thermoviscous algorithm.
In Fig. 4.5(b) the error in arrival time as a function of step size is plotted for 5, 10 and 20 sample points in the shock. It is seen that the step size should be approximately $1/(2 \times \text{number of points in shock})$. A step size this large is when the Courant number for the shock first covers the whole range 0 to 1. From the analysis in Fig. 4.2 we expect the average error in dispersion to be zero. It is clear that ten to twenty points are required to adequately describe a shock front and that the step size should be no smaller than $1/10^{th}$ of a shock formation distance.

4.5.4 Comparison to Fay Solution

A second test of the nonlinear and theromviscous algorithms is to compare THOR to the Fay solution. Lee (1993, Chap. 4.3) made a similar comparison but did not focus on the discrepancies. The Fay solution (see, for example, Blackstock 1972) is an asymptotic solution to the Burgers equation for an initially sinusoidal waveform for large $\Gamma$ and $\sigma > 3$. It can be written as follows:

$$P(\sigma, \tau) = \sum_{n=1}^{\infty} \frac{\sin(n\tau)}{\sinh(n(1 + \sigma)/\Gamma)}.$$
Figure 4.5: (a) Shock rise time as a function of points in shock and step size. (b) The error in arrival time of a step shock as a function of step size. A positive time indicates the numerical shock is late in arriving. Results are shown for shocks with a different number of points describing the shock.

Figure 4.6 compares THOR to the Fay solution for $\Gamma = 20$ and $\Gamma = 200$; sine waves with either 100 or 400 points per cycle are shown. The shock amplitudes predicted by THOR appear to be a little less than predicted by the Fay solution. This implies that attenuation is slightly overestimated. We also see that more points per cycle, that is, a higher sampling rate, provides a better description of the shock. With the higher sampling rate there is less numerical absorption.
The finite-difference algorithm for relaxation processes can also be examined by Fourier analysis. The differential equation used to model a relaxation process is, Eq. 4.6,

\[
\left(1 + \theta \frac{\partial}{\partial \tau}\right) \frac{\partial P}{\partial \sigma} = C \frac{\partial^2 P}{\partial \tau^2},
\]

\[
\frac{\partial P}{\partial \sigma} + \theta \frac{\partial^2 P}{\partial \tau \partial \sigma} = C \frac{\partial^2 P}{\partial \tau^2}.
\]

The finite difference operator for the right hand side is, Eq. 4.22,

\[
\frac{\partial^2}{\partial \tau^2} = -n^2 \text{sinc}^2(f/f_s),
\]

where \(f/f_s = n\Delta \tau/2\). The centered-time finite difference for the first order derivative \(\partial P/\partial \tau\) is

\[
\frac{\partial P}{\partial \tau} = \frac{1}{2\Delta \tau} (\tilde{P}_n e^{jn(i+1)\Delta \tau} - \tilde{P}_n e^{jn(i-1)\Delta \tau}),
\]

\[
\tilde{P}_n e^{jn i \Delta \tau} = \frac{1}{2\Delta \tau} (e^{jn \Delta \tau} - e^{-jn \Delta \tau}),
\]

\[
\frac{\partial \tilde{P}_n}{\partial \tau} = \frac{\tilde{P}_n}{\Delta \tau} j \sin(n \Delta \tau),
\]
The exact derivative is \( jn \hat{P}_n \). Figure 4.7 shows how the finite difference approximation compares to the actual derivative. Note that the finite difference approximation is zero at \( f/f_s = 0.5 \) and 1.0. It also has a negative sign in the region \( 0.5 < f/f_s < 1.0 \).

When the finite difference operators are substituted into Eq. 4.6 the result is

\[
[1 + jn \theta \text{sinc}(2f/f_s)] \frac{\partial \hat{P}_n}{\partial \sigma} = -Cn^2 \text{sinc}^2(f/f_s) \hat{P}_n,
\]

\[
\frac{\partial \hat{P}_n}{\partial \sigma} = \frac{-Cn^2 \text{sinc}^2(f/f_s)}{1 + jn \theta \text{sinc}(2f/f_s)} \hat{P}_n.
\]

The quantity \( n\theta = f/f_r \), where \( f_r \) is the relaxation frequency, \( f_r = 1/2\pi \nu_p \). The attenuation and dispersion for the finite difference approximation can be written as

\[
\alpha_{\text{fd}} = \frac{-(f/f_0)^2 C \text{sinc}^2(f/f_s)}{1 + (f/f_r)^2 \text{sinc}^2(2f/f_s)} [1 - j(f/f_r) \text{sinc}(2f/f_s)]. \quad (4.24)
\]

The exact attenuation coefficient is

\[
\alpha_{\text{exact}} = \frac{-(f/f_0)^2 C}{1 + (f/f_r)^2} (1 - jf/f_r). \quad (4.25)
\]

Plots of the attenuation and the phase speed for a single relaxation process, with a relaxation frequency \( f_r = 0.05f_s \), are shown in Fig. 4.8. Notice that the numerical
absorption is much larger than the exact absorption for $f > 0.2f_s$. The numerical absorption peaks at $f/f_s = 0.5$, where the first order operator goes through a zero. Similarly, the numerical algorithm significantly overestimates the phase speed for $f > 0.2f_s$. The numerical phase speed then drops below $c_0$ for $f/f_s > 0.5$ due to the change in sign of the first order derivative operator.

Figure 4.8: Fourier analysis of the finite difference approximation of the relaxation differential operator.

It is apparent from these plots that the relaxation frequency must be much less than $f_s$ for the relaxation to be modeled properly. A value of $f = 0.05f_s$ is probably the upper limit for the relaxation frequency. For $f > 0.2f_s$ neither the absorption nor the dispersion behavior is very well modeled. In practice the large attenuation at higher frequencies is good for numerical stability as it inhibits the development of multivalued waveforms.

Relaxation introduces dispersion as well as absorption. The high frequency components of the wave move a little faster than the retarded time frame and almost behave
like a progressive wave,* see Sec. 5.2. A forward finite difference is more appropriate for progressive wave motion than a centered finite difference (Ames 1977, Chap. 4-14). The first order derivative should be approximated by the following expression:

$$\frac{\partial P}{\partial \tau} = \frac{P_{j+1} - P_j}{\Delta \tau}.$$  

In Fig. 4.9 the frequency behavior of the centered differencing and forward differencing methods are compared. It was found that a pure forward time difference for both the first order and second order derivatives does not give good results. The best behavior was found using a pure forward difference for the first order time derivative and a mix of 52% forward and 48% centered differencing for the second order derivative. Still there is not a significant improvement over the pure centered time differencing scheme, and the centered differencing scheme was therefore implemented in THOR.

*The high frequency components tend to move to the left in the time window.
4.5.6 Comparison to Steady State Solution for Relaxation

The modeling of relaxation can be verified by comparing the code with a steady-state solution by Polyakova et al. (1962) for a finite amplitude wave in a medium with one relaxation process but no thermoviscous losses. Their result is (see Appendix E)

\[ \frac{t - t_0}{\tau} = \ln \frac{(1 + \rho/\rho_0)^{D-1}}{(1 + \rho/\rho_0)^{D+1}} , \]

where \( t_0 \) is an integration constant and

\[ D = \frac{\Delta \epsilon \rho_0 c_0}{\rho_0 \beta} , \]

\[ = C_\nu \theta_\nu . \]

Figure 4.10 (Cleveland et al. 1994b, 1995) compares the analytic result (denoted PSK) with the result from THOR. For the values chosen, relaxation was insufficient to stop the waveform from becoming multivalued. In the analytical result, Fig. 4.10(a), weak shock theory was used to ensure a single valued function. Multivaluedness was prevented in the numerical algorithm, Fig. 4.10(b), by including a small amount of thermoviscous attenuation. The comparison in Fig. 4.10(c) shows excellent agreement between the analytical and numerical predictions.

4.5.7 Comparison to Sinusoidal Propagation with One Relaxation Process

Hamilton and Zabolotskaya (1995) show the result for a finite-amplitude sine wave propagating in a thermoviscous medium with one relaxation process. They obtained their result from a computer code which solves the spectral form of the augmented Burgers equation entirely in the frequency domain (Fenlon 1971). Figure 4.11 shows excellent agreement between the output from THOR and Hamilton and Zabolotskaya’s result.

4.6 Determining the Step Size

The choice a step size is a balancing act between shorter run time and increased accuracy. We desire the step size to be small enough that the decoupling of the
Figure 4.10: (a) The analytical result for the steady-state solution in a relaxing medium with no thermoviscous effects; $D = 0.5$. (b) The initial and steady-state profiles obtained by THOR ($f_s = 1000f_r$). (c) Comparison of the analytical and numerical steady-state profiles.
Figure 4.11: A finite-amplitude sine wave propagating in a thermoviscous medium with one relaxation process; $D = 0.5$, $\tau_1 \omega_0 = 1$, and $\Gamma = 400$. The solid line is output from THOR ($f_s = 1600f_r$). The dashed line is Hamilton and Zabolotskaya's result (1995, Fig. 5.3). Range is scaled by the classical shock formation distance of the sinusoidal waveform.

various processes is realized. On the other hand, large step size makes the code run faster. THOR achieves the balance by getting each part of the code to calculate the largest step size possible that does not introduce unnecessary error. THOR then uses the smallest of the recommended step sizes for the next step. The algorithms for choice of step size is now discussed.

4.6.1 Nonlinearity

As discussed in Sec. 4.5.2 the nonlinear algorithm behaves best if the step size is close to a shock formation length. In particular if the step size is such that Courant number
within the shock varies from 0 to 1, the error in both the shock rise time and shock speed is minimized. It is important, however, not to take a step that is so large a multivalued waveform is predicted. The code stops execution if this occurs.

The nonlinear algorithm in THOR ensures it does not calculate a multivalued waveform by explicitly calculating the "local" shock formation distance. The "local" shock formation distance can be determined from the steepest slope in the waveform. The nonlinear algorithm sets the largest step size as some fraction of the "local" shock formation distance (typically 20%). The code adapts the step size to suit the waveform. THOR's approach is different from some other codes which take fixed steps depending on the shock formation distance of the initial waveform. In their purely spectral code, Trivett and Van Buren (1981) recommend a fixed step size of about 5% of a shock formation distance of the fundamental frequency component.

We demonstrate THOR's algorithm with a sinusoidal source waveform. If THOR steps 20% of a local shock formation distance, it does not take five steps to get to the classical shock formation distance of the sine wave. As the wave distorts, and steepens, the "local" shock formation distance gets smaller. THOR adjusts by taking shorter steps. For example, for a 100 point/cycle sinusoid with $\Gamma = 20$ it takes THOR ten steps to get to the classical shock formation distance. If $\Gamma = 50$ (the wave gets steeper) the code requires twelve steps. THOR has an advantage over codes with a fixed step size as THOR adjusts its step size to suit the waveform and the absorption in the medium. In fixed step size codes, the user needs to estimate the best step size before the code runs.

A step size that is 20% to 25% of a local shock formation distance is recommended. Recall from Sec. 4.5.2 that this should still give reasonable results even if there are only a couple of points in the shock. The Anderson type code maintained at NCPA, University of Mississippi also calculates the local shock formation distance as it propagates and has been successfully run with step sizes up to 95% of a local shock formation distance (Chambers 1994).

To recapitulate, THOR explicitly calculates the local shock formation distance at each step. When applying nonlinear distortion, THOR finds the steepest positive slope in the waveform, from which it calculates the shock formation distance of the
The nonlinear algorithm returns a maximum step size that is some fraction of the local shock formation distance. Unless otherwise noted, the results presented in this work are for, \( \Delta \sigma_{\text{max}} = 0.2 \varepsilon \).

### 4.6.2 Absorption and Dispersion

For thermoviscous absorption small steps, with respect to an absorption length, are best. The absorption length \( l_{tv} \) is frequency dependent:

\[
l_{tv} \propto \Gamma / n^2,
\]

where \( n = \omega / \omega_0 \) is the harmonic number. In many waveforms a large number of frequency components are present. Strictly, the absorption length of the highest harmonic should be used to control the step size, as this has the shortest absorption length. However, in the Fourier analysis of the finite difference algorithm it was seen that high frequency attenuation is not accurately modeled. It is not worthwhile choosing a step size to control the error in these terms. A good choice for \( n \) was found to be the square root of the number of points in the waveform, i.e., \( n^2 = M \). The results in this work were run with a thermoviscous step size controlled by

\[
\Delta \sigma_{\text{max}} = 0.1 \Gamma / M. \tag{4.26}
\]

For relaxation two effects need to be monitored. First, that the attenuation is small over the step. Second, that dispersion is small over the step. The first requirement is similar to that for thermoviscous absorption:

\[
\Delta \sigma \ll \frac{1 + (n \theta_v)^2}{C_v n^2}.
\]

The second requires that

\[
\Delta \sigma \ll 2 \pi \frac{1 + (n \theta_v)^2}{C_\nu \theta_v n^3}.
\]

It was found that 10% is suitably small with \( n^2 = M \). The step size restriction is

\[
\Delta \sigma_{\text{max}} = 0.1 \frac{1 + M \theta_v^2}{C_\nu M} \min \left( 1, \frac{2 \pi}{\sqrt{M \theta_v}} \right). \tag{4.27}
\]
4.6.3 Spreading and Stratification

For spreading it was found that consistent results were achieved if the amplitude changes by less than 5% at each step. For cylindrical or spherical spreading the requirement is

$$\Delta \sigma_{\text{max}} = 0.05 \frac{\sigma}{m}.$$  (4.28)

For general ray tube area spreading the requirement is

$$\Delta \sigma_{\text{max}} = 0.05 \frac{S}{\partial S/\partial \sigma}.$$  (4.29)

Finally, for the effect of stratification, good results were obtained if the amplitude changed by less than 5% at each step. The maximum step size is

$$\Delta \sigma_{\text{max}} = 0.05 \rho_0 c_0 \frac{\partial (\rho_0 c_0)}{\partial \sigma}.$$  (4.30)

It is also necessary to ensure that the curvature of the ray is small over a given step size. That is, the propagation over $\Delta \sigma$ can be approximated as propagation along a straight line. Recall that the radius of curvature for a ray is

$$R_c = \frac{1}{g \cos \theta}.$$  

The smallest possible curvature, when $\theta = 0$, is $R_{\text{min}} = c/\partial g/\partial \sigma$. A step size of 5% of the radius of curvature requires

$$\Delta \sigma_{\text{max}} = 0.05 \frac{c_0}{\partial c_0/\partial \sigma}.$$  (4.31)

At each step, each process returns the maximum $\Delta \sigma$ it recommends for the next step. The smallest recomendated step is used as the next $\Delta \sigma$. It is possible for the user to override these suggestions with an even smaller step—but not to force a larger step. If larger steps are desired the control parameters must be adjusted.

4.7 Nonuniform Sampling Algorithm

The numerical algorithm described above uses a uniform time grid to describe the waveform. After applying nonlinear distortion, the waveform is linearly interpolated to return it back onto the uniform grid. However, it is possible for all parts of the
algorithm to cope with a waveform that is not uniformly sampled. This removes the need to resample the waveform after distortion. The two major advantages are:

1. No numerical attenuation and dispersion should be introduced in the nonlinear algorithm, as these effects are due to the linear interpolation.  
2. The number of points necessary to describe a waveform should be greatly reduced. For example, in sonic boom propagation a high sampling rate is required to properly determine the shocks. However, this rate is not required for the long sloping sections that form the majority of the waveform (typically more than 90%). With uniform sampling it is necessary to sample the smooth sections of the waveform as finely as the shocks are sampled. With nonuniform sampling it should be possible to finely sample only the shocks and sample the smooth sections with far fewer points. This would drastically reduce the number of points used to describe the waveform and hence cut down the computation time.

Even with initially smooth waveforms a nonuniform time grid should be better. In general, all finite amplitude waveforms that start off smooth, a sine wave for example, form shocks where the slope is positive. It is necessary to sample the whole waveform finely enough so that the profile of any shocks that are formed can be properly resolved. In the scheme under consideration we can use the time grid distortion of the nonlinear algorithm to automatically move samples into the shock regions. This alleviates the necessity to heavily sample the waveform, as samples are moved into the shock regions as required and, on the face of it, continual resampling of the waveform is not required.

Unfortunately, everything is not so rosy. The nonlinear algorithm does push points into the shock region, but there is no mechanism to pull points back out of the shock to the rest of the waveform. This can be demonstrated by considering a finite amplitude sine wave. Initially it starts to distort and turn into a sawtooth, which decays away. Eventually, in the old age region, all the high frequencies are stripped out and the resultant waveform returns to a sine wave, albeit with a much reduced amplitude. The ideal algorithm moves points into the shocks in the sawtooth region and then redistributes them over the entire waveform as the waveform goes into the old age region. In Table 4.1 is listed the percentage of samples of a uniformly sampled sine wave that are moved into the shock region as a function of shock formation distance.
At a distance of 31 shock formation distances nearly 97% of the points in the original

<table>
<thead>
<tr>
<th>Range</th>
<th>Points in shock.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0257</td>
<td>12.39%</td>
</tr>
<tr>
<td>1.1057</td>
<td>24.47%</td>
</tr>
<tr>
<td>1.2589</td>
<td>36.55%</td>
</tr>
<tr>
<td>1.5290</td>
<td>48.62%</td>
</tr>
<tr>
<td>2.0200</td>
<td>60.70%</td>
</tr>
<tr>
<td>3.0294</td>
<td>72.78%</td>
</tr>
<tr>
<td>5.8194</td>
<td>84.85%</td>
</tr>
<tr>
<td>31.6242</td>
<td>96.93%</td>
</tr>
</tbody>
</table>

Table 4.1: Percentage of an initially sinusoid at the shock as a function of distance.

waveform have moved into the shock region. At a distance of 200 shock formation
distances — 99.5% of the waveform is at the shock. It is not possible to recreate a
sinusoid if all samples are located where the shocks used to be.

It turns out that before the waveform can get into the sawtooth region a numerical
problem occurs. The nonlinear distortion moves time points so close together that
the matrices required to calculate the absorption become badly conditioned. The
numerical errors that result produce unstable results.

In what follows, the modeling of nonlinear, thermoviscous and relaxation equations
for nonuniform sampling is described. Then the problem with the nonlinear distortion
and time grid generation is addressed.

4.8 Finite Difference Equations on a Nonuniform Time Grid

Recall that the expressions for thermoviscous attenuation and relaxation processes
involve time derivatives of the pressure waveform. Previously the time derivatives
were approximated with a finite difference algorithm assuming that $\Delta \tau$ is a constant,
which is no longer the case. Finite difference approximations of the time derivatives
are now derived for a nonuniform time grid.
The Taylor series expansions for two points either side of \( \tau \), \( \tau + \Delta \tau_+ \) and \( \tau - \Delta \tau_- \), are

\[
P(\tau + \Delta \tau_+) = \frac{P(\tau) + \Delta \tau_+ \frac{\partial P}{\partial \tau}|_\tau + \frac{(\Delta \tau_+)^2 \frac{\partial^2 P}{\partial \tau^2}|_\tau}{2} + O((\Delta \tau_+)^3), \tag{4.32}
\]

\[
P(\tau - \Delta \tau_-) = \frac{P(\tau) - \Delta \tau_- \frac{\partial P}{\partial \tau}|_\tau + \frac{(\Delta \tau_-)^2 \frac{\partial^2 P}{\partial \tau^2}|_\tau}{2} + O((\Delta \tau_-)^3). \tag{4.33}
\]

The first order derivative can be obtained by multiplying Eq. 4.32 by \((\Delta \tau_-)^2\) and Eq. 4.33 by \((\Delta \tau_+)^2\). Subtraction yields

\[
(\Delta \tau_-)^2 P(\tau + \Delta \tau_+) - (\Delta \tau_+)^2 P(\tau - \Delta \tau_-) = [(\Delta \tau_-)^2 - (\Delta \tau_+)^2]P(\tau) + [(\Delta \tau_-)^2 \Delta \tau_+ + (\Delta \tau_+)^2 \Delta \tau_-] \frac{\partial P}{\partial \tau} + O((\Delta \tau)^3). \tag{4.34}
\]

It follows that

\[
\frac{\partial P}{\partial \tau} = \frac{(\Delta \tau_-)^2 [P(\tau + \Delta \tau_+) - P(\tau)] + (\Delta \tau_+)^2 [P(\tau) - P(\tau - \Delta \tau_-)]}{\Delta \tau_+ \Delta \tau_- (\Delta \tau_+ + \Delta \tau_-)} + O((\Delta \tau)^2),
\]

where for the error term it is assumed \(O(\Delta \tau_+) = O(\Delta \tau_-) = O(\Delta \tau)\). Note that the time derivative is a weighted average of the slope on either side of \( \tau \), with the heaviest weighting applied to the point that is closest. For \( \Delta \tau_+ \sim \Delta \tau_- \) this reduces to the intuitive finite difference equation

\[
\frac{\partial P}{\partial \tau} = \frac{P(\tau + \Delta \tau_+) - P(\tau - \Delta \tau_-)}{\Delta \tau_+ + \Delta \tau_-} + O(\Delta \tau).
\]

The second order derivative is obtained by multiplying Eq. 4.32 by \( \Delta \tau_- \) and Eq. 4.33 by \( \Delta \tau_+ \). Addition and simple manipulation yields

\[
\frac{\partial^2 P}{\partial \tau^2} = \frac{\Delta \tau_- P(\tau + \Delta \tau_+) - (\Delta \tau_+ + \Delta \tau_-) P(\tau) + \Delta \tau_+ P(\tau - \Delta \tau_-)}{\frac{1}{2} (\Delta \tau_+ + \Delta \tau_-) \Delta \tau_+ \Delta \tau_-} + O(\Delta \tau). \tag{4.35}
\]

With these two expressions it is now possible to obtain finite difference equations for the thermoviscous absorption and relaxation equations.

### 4.8.1 Implementation of Thermoviscous Absorption and Relaxation

The classical absorption equation is

\[
\frac{\partial P}{\partial \sigma} = \hat{A} \frac{\partial^2 P}{\partial \tau^2}.
\]
The Crank-Nicolson finite-difference can now be written as

\[
\frac{P_i^{k+1} - P_i^k}{\Delta \sigma} = \frac{\alpha}{2} \left( \frac{\Delta \tau_+ P_{i+1}^k - (\Delta \tau_+ + \Delta \tau_-) P_i^k + \Delta \tau_- P_{i-1}^k}{\frac{1}{2}(\Delta \tau_+ + \Delta \tau_-) \Delta \tau_+ \Delta \tau_-} \right) \\
+ \frac{\alpha}{2} \left( \frac{\Delta \tau_- P_{i-1}^{k+1} - (\Delta \tau_+ + \Delta \tau_-) P_i^{k+1} + \Delta \tau_+ P_{i+1}^{k+1}}{\frac{1}{2}(\Delta \tau_+ + \Delta \tau_-) \Delta \tau_+ \Delta \tau_-} \right),
\]

\[- \lambda \Delta \tau_+ P_{i-1}^{k+1} + [1 + \lambda(\Delta \tau_+ + \Delta \tau_-)] P_i^{k+1} - \lambda \Delta \tau_- P_{i+1}^{k+1} = \lambda \Delta \tau_+ P_{i-1}^k \\
+ [1 - \lambda(\Delta \tau_+ + \Delta \tau_-)] P_i^k + \lambda \Delta \tau_- P_{i+1}^k,
\]

(4.36)

where \( \lambda = A \Delta \sigma / (\Delta \tau_+ + \Delta \tau_-) \Delta \tau_+ \Delta \tau_- \). This yields a tridiagonal matrix system, except that now the values of the coefficients are no longer constant but vary with position.

The relaxation equation is

\[
\frac{\partial P}{\partial \sigma} + \theta \frac{\partial^2 P}{\partial \sigma \partial \tau} = C \frac{\partial^2 P}{\partial \tau^2}.
\]

The derivative \( \theta \frac{\partial^2 P}{\partial \sigma \partial \tau} \) follows from evaluating Eq. 4.34 at \( \sigma \) and \( \sigma + \Delta \sigma \):

\[
\frac{\partial^2 P}{\partial \tau \partial \sigma} = \frac{1}{\Delta \sigma} \left( \frac{(\Delta \tau_-)^2 [P_{i+1}^{k+1} - P_i^{k+1}] + (\Delta \tau_+)^2 [P_i^{k+1} - P_{i-1}^{k+1}]}{\Delta \tau_+ \Delta \tau_- (\Delta \tau_+ + \Delta \tau_-)} \\
- \frac{(\Delta \tau_-)^2 [P_{i+1}^k - P_i^k] + (\Delta \tau_+)^2 [P_i^k - P_{i-1}^k]}{\Delta \tau_+ \Delta \tau_- (\Delta \tau_+ + \Delta \tau_-)} \right).
\]

The finite difference equation for one relaxation process is therefore

\[
[-\mu(\Delta \tau_-)^2 - \lambda \Delta \tau_+] P_{i-1}^{k+1} + \{1 + \lambda(\Delta \tau_+ + \Delta \tau_-) + \mu((\Delta \tau_+)^2 - (\Delta \tau_-)^2)\} P_i^{k+1} \\
+ \{\mu(\Delta \tau_-)^2 - \lambda \Delta \tau_-\} P_{i+1}^{k+1} = [-\mu(\Delta \tau_+)^2 + \lambda \Delta \tau_+] P_{i-1}^k \\
+ \{1 - \lambda(\Delta \tau_+ + \Delta \tau_-) + \mu((\Delta \tau_+)^2 - (\Delta \tau_-)^2)\} P_i^k + \{\mu(\Delta \tau_-)^2 + \lambda \Delta \tau_-\} P_{i+1}^k,
\]

where \( \mu = \theta_\nu / \Delta \tau_+ \Delta \tau_- (\Delta \tau_+ + \Delta \tau_-) \) and \( \lambda = C \Delta \sigma / \Delta \tau_+ \Delta \tau_- (\Delta \tau_+ + \Delta \tau_-) \). This can be rewritten in the somewhat more compact form

\[
(-\mu_+ - \lambda_+) P_{i-1}^{k+1} + (1 + \lambda_+ + \lambda_- + \mu_+ - \mu_-) P_i^{k+1} + (\mu_- - \lambda_-) P_{i+1}^{k+1} \\
= (-\mu_+ + \lambda_+) P_{i-1}^k + (1 - \lambda_+ - \lambda_- + \mu_+ - \mu_-) P_i^k + (\mu_+ + \lambda_-) P_{i+1}^k,
\]

(4.37)

where \( \mu_+ = \mu(\Delta \tau_+)^2, \mu_- = \mu(\Delta \tau_-)^2, \lambda_+ = \lambda \Delta \tau_+ \) and \( \lambda_- = \lambda \Delta \tau_- \). This equation is also in the form of a tridiagonal matrix system, where the values on the diagonal vary with position.
4.8.2 Implementation of Nonlinearity

The Poisson solution is used to implement the nonlinear distortion. The algorithm simply consists of distorting the time grid according to

$$\tau_j^{k+1} = \tau_j^k - P_j^k \Delta \sigma.$$  (4.38)

It is not necessary to interpolate the waveform back onto a uniform grid. That is, the algorithm returns a new $\tau$ array for step $k + 1$, but leaves the pressure array $P$ alone. As discussed above, this yields a physical problem of moving all points into the shock regions, and a numerical problem occurs with the absorption algorithm.

4.8.3 Implementation of Other Effects

All other effects can be implemented using the algorithms developed for the uniform code.

- The absorption matrices are again chosen to ensure that the end points are constant at each step.
- The effects of spreading and stratification of the impedance are simply scaling problems and use exactly the same algorithm as the uniform code.
- The step-size is chosen dynamically using the same scheme used in the uniform code.

4.9 Conditioning Problem with the Nonuniform Algorithm

When the nonuniform algorithm is applied to the case of a sinusoid, a typical result is shown in Fig 4.12. The code behaves well up to the shock formation distance, and shortly afterwards the code fails because a multivalued waveform occurs. At the steepest part of the shock the points get very close together. This ends up making the tridiagonal matrices used to calculate the absorption badly conditioned. It appears that the inversion process introduces numerical errors, as shown by the expanded view of the zero crossing of the shock. These small fluctuations make the waveform multivalued at the next nonlinear step, and so the code halts.
Figure 4.12: Propagation of a sine wave with $\Gamma = 100$; (a) shows a sine wave with 100 samples/cycle, (b) shows a sine wave with 30 samples/cycle. The zero crossing of the shock at the point the code fails is shown on an expanded scale.

Figure 4.13 shows the conditioning number of the absorption algorithm as a function of distance. The conditioning number becomes very large as the waveform passes through the shock formation distance and the time points get very close together. Therefore errors in the inversion process should be expected. The worst case is for large $\Gamma$, where the shock is very sharp.

A number of attempts have been made to alleviate this problem. It appears that some form of resampling is necessary to stop time points in the grid from getting too close together. To take full advantage of the nonuniform grid the resampling algorithm should dynamically alter the grid as the waveform distorts. There is a trade-off of the computational price paid to constantly generate a grid against requiring far fewer points to describe a waveform.

4.9.1 Slope Based Grid

The best algorithm developed so far dynamically picks a grid based on the slope of the waveform, that is, places of high slope are given more samples than places of
lower slope. This algorithm has some basis in sampling theory. One expects the high frequency components to be important at places of high slope. By the Nyquist sampling theory, the sampling rate should be at least twice the highest frequency of interest. So the sampling density should be related to the slope of the waveform.

An attempt was made to weight positive slopes more than negative slopes. No improvement in the results was observed, principally because the nonlinear effects quickly turn large negative slopes into small negative slopes. Also, if sampling theory is the basis for the slope based grid, then positive and negative slopes need the same sampling rate.

The weight of a given interval $\Delta \tau_i$ should be equal to the slope multiplied by the size of the interval. The resampling algorithm calculates the slope along the whole waveform and assigns a sampling weight $w_i$ to each interval using the following formula:

$$w_i = \Delta \tau_i \left( 1 + \text{scale} \times \frac{|\Delta P / \Delta \tau|}{\Delta P / \Delta \tau} \right).$$

The slope is divided by the average slope $|\Delta P / \Delta \tau|$ to produce a dimensionless slope which is multiplied by a scaling constant. The addition of unity to the slope is to ensure that regions of low or zero slope have some weighting. The new time grid is formed by giving each interval a fraction of the samples proportional to its fractional weight.
$w_i/\sum w_i$. The resampling of the waveform onto the new time grid is done using linear interpolation. An alternative is to consider using a higher order interpolation scheme, although the computation time is increased. The resampling is calculated following the absorption routine, rather than after the nonlinear algorithm. This is done to reduce the error created by interpolation. After the absorption routine the waveform should be smoother and thus better approximated by straight line segments. The resampling need not be done at every step but can be done at any integral number of steps.

4.9.2 Other Grid Generation Algorithms

Two other solutions to the resampling problem were considered. They are briefly outlined here.

The first solution ensures that the time points do not get too close together. The distance between all the points in the $\tau$ array is monitored. When two points get too close together they are agglomerated into one point. This means the number of points in the waveform is reduced by one. To compensate an extra point is added into the waveform elsewhere to maintain the same number of points in the waveform. The positioning of this extra point is somewhat arbitrary. The two algorithms used are:

1. The largest $\Delta\tau$ is found and the extra point is used to make this smaller. This stops the variation in $\Delta\tau$ along the waveform from becoming very large. It has the disadvantage of requiring the largest $\Delta\tau$ to be found and the shifting of many points every time too small a $\Delta\tau$ is found.

2. The point is shifted to the largest $\Delta\tau$ either side of the critical $\Delta\tau$. Numerically this has the advantage of not requiring a large $\Delta\tau$ to be found and involves the shifting of a minimal number of points. It also means that the points are kept in the shock region where they are probably of highest value.

Neither of these methods proved to be very satisfactory and a substantial improvement over uniform step sizing was never obtained without careful tweaking of the parameters.

The second solution considered, is to let both the nonlinear distortion routine and the absorption routines move the time grid. As mentioned before, the problem is that
the distortion algorithm moves points into the shock region and there is no mechanism to pull them out. In this algorithm the nonlinear distortion is calculated by distorting the time base with no resampling. The input waveform $p_{in}$ is not affected. Absorption is then applied to the $p_{in}$ to produce $p_{abs}$. However, rather than returning $p_{abs}$ the code distorts the time base a second time so that the $p_{in}$ is interpolated onto the absorbed waveform $p_{abs}$. Because this provides a mechanism by which absorption can pull points out of the shock it balances the nonlinear routine which pushes points into the shock. This algorithm was abandoned because the waveform is attenuated by the absorption routine. Therefore $p_{in}$ has a higher amplitude than $p_{abs}$ and it would not be possible to interpolate $p_{in}$ onto $p_{abs}$ unless some scaling exercise is done first.

4.10 Comparison between the Uniform and the Nonuniform Codes

A few representative results are presented in this section to compare the uniform and nonuniform algorithms and in particular to show the strengths of the nonuniform time grid.

Figure 4.14 shows comparison between the nonuniform and uniform time grid and the Fay solution for a sinusoid with $\Gamma = 500$ at two ranges. Each code uses 102 points in one cycle. Notice that the nonuniform algorithm does a much better job of modeling the shock. The disagreement on the shock amplitude between the Fay and the numerical solutions is because the Fay solution is only approximate.

Figure 4.15 compares the results for a step shock in a medium with one relaxation process. We chose $D = C/\theta = 0.5$ and include a small amount of thermoviscous absorption, that is, the same conditions used to produce Fig. 4.10. However in this case just 512 points are used to determine the shock. We see that both model the rise due to the relaxation process well but that the nonuniform grid also does a good job of modeling the shock, whereas the uniform grid cannot properly capture the shock due to a lack of resolution.

The final example is the propagation of N waves. The duration predicted by weak shock theory (Eq. 3.12) is compared to that predicted by the uniform and nonuniform codes. Both codes require a small amount of thermoviscous attenuation to keep the
waveforms single valued. As the attenuation is reduced the output from the numerical codes should behave more like weak shock theory because the shocks become thinner. However there is a limit. Recall from Sec. 4.5.2 that if too few points are used to describe a shock it suffers from numerical dispersion. When attenuation is reduced and the shock becomes thinner there are also less samples in the shock (assuming a fixed number of points in the waveform). Eventually numerical disperison becomes significant. The nonuniform algorithm will tend to move points into the shock regions because of the high slope. The effective sampling rate at the shock should be quite high. Therefore, the nonuniform code should not suffer anywhere near as badly from numerical dispersion as the uniform code.

Figure 4.16 shows the duration and peak pressure of an N wave as a function of distance for various parameters. The N wave is sampled with 2048 samples and is zero padded with 1024 samples at either end (total length of the time window is 4096 samples). The N wave is multiplied by an envelope function to ensure it is smooth (Lee and Hamilton 1995, Eq. 12) and each shock initially has 10 samples. As the N wave propagates it slowly fills up the window. The attenuation $TV = 1/\Gamma$ is based
Figure 4.15: Comparison with step shock in a relaxing medium with $D = 0.5$.

on a frequency that has a period twice as long as the full initial duration of the N wave, that is, it has the period of the time window. For low attenuation both codes do a good job of estimating the rise time but for very low attenuation the uniform code suffers from numerical dispersion and vastly underestimates both the duration and peak pressure of the N wave. To maintain accuracy in this situation one must increase the number of samples in the waveform. The nonuniform code, however, does a much better job of modeling the N wave over a large range of absorptions.

Although the slope based weighting algorithm works reasonably well, it is not yet robust enough to handle a large parameter space. The selection of the scale variable and number of steps between each resampling was found to be different for various situations, for example,

- Changing $\Gamma$ from 20 to 500.
- Propagating N waves instead of sine waves.
- Including relaxation processes instead of just thermoviscous absorption.

It may be necessary to come up with a different implementation of the weighting algorithm to make it practical.
Figure 4.16: N wave duration and peak pressure as a function of distance for various amounts of attenuation. The range is expressed in terms of the dimensionless quantity $ax$, where $a$ is related to the shock formation distance (see Sec. 3.4).

Because of the variability of the nonuniform code the results presented in the next chapter are calculated using the uniform algorithm. This algorithm is much better behaved but has the down side of requiring a large number of samples.
Chapter 5

Comparison of Sonic Boom Propagation Codes

5.1 Introduction

In this chapter we compare output from THOR to output from other computer programs that model the propagation of sonic booms through the atmosphere. NASA orchestrated the exercise in an attempt to validate the large number of codes in the community. A number of codes were invited and in the end three were used. THOR was one of them. SHOCKN, maintained at the University of Mississippi, and ZEPHYRUS, from the University of Texas at Austin, were the other two. A fourth "code" was also used; the analytical solutions to the lossless equation described in Chapter Three. These analytical results are equivalent to the Thomas and Hayes computer codes developed by NASA in the late 1960s and early 1970s.

So far only two atmospheres have been considered, uniform and isothermal. First, propagation through each atmosphere is done with thermoviscous attenuation only. The purpose behind these runs was to allow fair comparison between the THOR and the analytical solutions. The analytical solutions can not account for relaxation as they only use weak shock theory. THOR with only thermoviscous behaves in a similar manner to weak shock theory.

In the later comparisons atmospheric absorption was calculated using the ISO 9613-1 (1993) standard (described in the next section). The standard includes classical-rotational absorption and absorption and dispersion due to oxygen and nitrogen relaxation. Sonic boom shocks have been shown to be strongly dependent on the relaxation effects in the air. We discuss this in detail in the next chapter.

Results from this test are considered good verification of THOR as a model for sonic boom propagation. THOR is also compared against the predictions of waveform chilling of sonic booms mentioned in Chapter Three.
5.2 Absorption of Sound in the Atmosphere

It has long been recognized (see, for example, Lighthill 1956) that the absorption of sound in the air, in the audio range at least, is controlled by the relaxation processes of N\textsubscript{2} and O\textsubscript{2}, not thermal and viscous losses. In this section the effect of a relaxation process on acoustic waves is analyzed. The equations for the calculation of the absorption of sound in air are then presented.

5.2.1 Analysis of a Single Relaxation Process

The progressive wave equation for a small signal, plane, acoustic wave in a medium with a single relaxation process (Eq. 2.62 with \(\beta = 0, b = 0,\) and \(\nu = 1\)) is

\[
\frac{\partial p'}{\partial x} = \frac{m\tau}{2c_0} \frac{\partial^2 p'}{\partial \tau^2} ,
\]

where the subscript on \(m\) and \(\tau\) has been dropped. The analytical form of the impulse response of this system includes an infinite sum of incomplete Bessel functions (Hamilton 1994) and is rather unwieldy.*

In the frequency domain Eq. 5.1 becomes

\[
\frac{\partial P'}{\partial x} = \frac{m\tau}{2c_0} \frac{-\omega^2 P'}{1 + j\omega\tau} ,
\]

where \(P'\) is the Fourier transform of \(p'\). The solution is \(P' = \hat{P} e^{-\alpha x + j\Phi x}\), where \(\alpha\) is the absorption coefficient

\[
\alpha = \frac{m\tau}{2c_0} \frac{\omega^2}{1 + (\omega\tau)^2} ,
\]

*Hamilton finds the response at range \(x\) from an impulse at \(x = 0\) is

\[
h(t') = e^{-m\omega/\alpha\tau} \sum_n I_n \left( \frac{m\omega}{\alpha\tau} \right) \tau^n e^{-t'/\tau} \delta^{(n)}(t') ,
\]

where \(I_n\) is the incomplete Bessel function of order \(n\) and \(\delta^{(n)} = \frac{d^n}{dt^n} \delta(t)\). The pressure field due to a source condition \(p'(0,t')\) is therefore \(p'(x,t') = p'(0,t') \ast h(t')\), or

\[
p'(x,t') = e^{-m\omega/\alpha\tau} e^{-t'/\tau} \sum_n I_n \left( \frac{m\omega}{\alpha\tau} \right) \left( \frac{d}{dt'} \right)^n [p'(0,t') e^{t'/\tau}] .
\]
and \( \Phi \) the dispersion coefficient

\[
\Phi = \frac{m\tau^2 \omega^3}{2c_0 1 + (\omega \tau)^2}.
\]

We identify the relaxation frequency \( f_r = 1/2\pi \tau \). The absorption and dispersion coefficients can now be expressed as

\[
\alpha = \frac{\pi mf_r}{c_0} \left( \frac{f^2}{1 + f^2/f_r^2} \right),
\]

\[
\Phi = \frac{\pi m f^3/f_r^2}{c_0} \left( 1 + f^2/f_r^2 \right).
\]

For a time harmonic waveform the solution is

\[
p = p e^{-(\alpha\tau + j\omega t - \frac{\omega^2}{c_0} - \Phi).}
\]

If the relaxation coefficient is defined as \( A_r = \pi m/c_0 \), the following absorption law is obtained:

\[
\alpha = A_r f_r f^2 / f_r^2 + f^2. \tag{5.2}
\]

The dispersion law can be rewritten in terms of the phase speed:

\[
c_{ph} = \frac{\omega}{-\Phi + \omega/c_0},
\]

\[
= c_0 \left( 1 - \frac{A_r c_0}{2\pi} \frac{f^2}{f_r^2 + f^2} \right)^{-1}. \tag{5.3}
\]

The change in sound speed is \( \Delta c \approx A_r c_0^2/2\pi = mc_0/2 \) for \( A_r c_0 \ll 2\pi \). This is consistent with the definition of \( m \) used in Sec. 2.5.

Figure 5.1 shows the attenuation behavior and phase speed as a function of frequency. At low frequencies the absorption is proportional to \( \omega^2 \). Above the relaxation frequency the absorption reaches a plateau. The phase speed undergoes a smooth transition from \( c_0 \) for \( f \ll f_r \), to \( c_\infty = c_0 + mc_0/2 \) for \( f \gg f_r \). At very low frequencies (time scales much greater than \( \tau \)) the system has ample time to react to the acoustic wave and remains in equilibrium at all times; \( c_0 \) is therefore called the equilibrium sound speed. At high frequencies (time scales much less than \( \tau \)) there is virtually no time for the relaxation processes to act. Since the state of the fluid is frozen, the sound speed \( c_\infty \) is referred to as the frozen sound speed.

The response of relaxing fluid to a Heaviside step function can be approximated numerically by using a square wave train and the frequency law of absorption. Figure 5.2
Figure 5.1: The absorption and phase speed of sound in a medium with one relaxation process, where $\alpha_{\text{max}} = A_r f_r$.

shows the response at various propagation distances. As the step travels into the relaxing fluid, dispersion makes the leading edge advance in the retarded time frame. However, absorption also diffuses the shock and eventually a very smooth waveform develops. The waveform is always asymmetric.

5.2.2 Formulae for the Absorption of Sound in Air

The rotational energy states of $O_2$ and $N_2$ are usually very close together (less than the average thermal energy) and can be described by a single relaxation frequency (Bass 1994). This frequency is usually greater than 10 MHz. Therefore, for frequencies in the audio range the rotational processes have the same frequency dependence as thermoviscous attenuation, they are normally included with the thermoviscous absorption coefficient.

However, both oxygen and nitrogen have a significant vibrational relaxation frequency that lies in the audio range. Other relaxation processes of these and other gases are present but may be neglected. At high altitude the relaxation of $CO_2$ can
become important (Raspet 1994).

The ISO standard (ISO 9613-1 1993) and the ANSI standard (ANSI S1.26-1978) describe the absorption processes in air as a superposition of three components: classical-rotational absorption (thermoviscous attenuation and rotational relaxation), nitrogen vibrational relaxation, and oxygen vibrational relaxation, that is,

\[ \alpha = \alpha_{cr} + \alpha_{v,N} + \alpha_{v,O}. \]

The terms are dependent on the temperature, pressure, and water content of the air.

The classical-rotational absorption is given as \( \alpha_{cr} = A_{cr} f_{cr}^2 \), where

\[ A_{cr} = 1.84 \times 10^{-11} \frac{P_r}{P_b} \sqrt{\frac{T_0}{T_r}} \frac{N_p}{(Hz^2 \text{ m})}. \]  \hspace{1cm} (5.4)

The vibrational relaxation absorption has the form \( \alpha_{v,\nu} = A_{v,\nu} f_{v,\nu} f_0^2 / (f_{v,\nu}^2 + f_0^2) \), see Eq. 5.2. The oxygen relaxation is defined by

\[ A_{v,O} = 0.01275 \left( \frac{T_0}{T_r} \right)^{-5/2} e^{-2339.1/T_0} \frac{N_p}{(Hz \text{ m})}, \] \hspace{1cm} (5.5)

\[ f_{v,O} = \frac{P_0}{P_r} \sqrt{\frac{T_0}{T_r}} \left( 9 + 280he^{-4.170(7/T_0)^{1/3}-1} \right) \text{ Hz}. \] \hspace{1cm} (5.6)
For the nitrogen relaxation the expressions are

\[ A_{v,N} = 0.1068 \left( \frac{T_0}{T_r} \right)^{-5/2} e^{-3352/T_b} \text{ Np}/(\text{Hz m}), \quad (5.7) \]

\[ f_{v,N} = \frac{P_0}{P_r} \left( 24 + 4.04 \times 10^4 h \frac{0.02 + h}{0.391 + h} \right) \text{ Hz}. \quad (5.8) \]

The ISO standard defines the frozen (high frequency) small-signal sound speed to be

\[ c = 343.2 \sqrt{\frac{T_0}{T_r}} \text{ m/s}. \quad (5.9) \]

The reference pressure is \( P_r = 101.325 \text{ kPa} \) (1 atmosphere) and the reference temperature is \( T_r = 293.15 \text{ K} \) (20°C).*

The quantity \( h \) is the percent mole fraction of water vapor in the air. It is the ratio of the vapor pressure \( P_w \) to the ambient atmospheric pressure,

\[ h = 100 \frac{P_w}{P_0}. \quad (5.10) \]

Humidity is commonly given in terms of the relative humidity \( RH \), which is defined as the ratio of vapor pressure to the saturation vapor pressure \( P_{\text{sat}} \), \( RH = 100 \frac{P_w}{P_{\text{sat}}} \).

The following empirical law gives the dependence of saturation pressure on the temperature:

\[ \log_{10}(P_{\text{sat}}/P_r) = -6.8346 \left( \frac{T_r}{T_0} \right)^{1.261} + 4.6151, \quad (5.11) \]

*The ANSI 1978 standard has the same classical absorption as the ISO 9613-1 standard, but has slightly different definitions for the relaxation terms.

\[ A_{v,O} = 0.01278 \left( \frac{T_b}{T_r} \right)^{-5/2} e^{-2339.1/T_b} \text{ Np/Hz/m}, \]

\[ f_{v,O} = \frac{P_b}{P_r} \sqrt{\frac{T_r}{T_b}} \left( 9 + 350 h e^{-6.142(T_r/T_b)^{1/3} - 1} \right) \text{ Hz}, \]

\[ A_{v,N} = 0.1068 \left( \frac{T_b}{T_r} \right)^{-5/2} e^{-3352/T_b} \text{ Np/Hz/m}, \]

\[ f_{v,N} = \frac{P_b}{P_r} \left( 24 + 4.41 \times 10^4 h \frac{0.05 + h}{0.391 + h} \right) \text{ Hz}. \]

The ANSI standard is presently under revision and the new draft, as it stood on 23 August 1994, uses the same constants as the ISO standard. The new ANSI draft also allows for a humidity variation in the small-signal sound speed:

\[ c = 343.2(1 + 0.0016h) \sqrt{T_0/T_r} \text{ m/s}. \]
where $T_z = 273.16$ K, the triple-point of water. It is therefore possible to calculate $h$ from the relative humidity.

Figure 5.3 shows the absorption curves for air at the ground for various relative humidities. Also shown is the small-signal sound speed as a function of frequency. The small-signal sound speed is calculated using Eq. 5.3.

Figure 5.3: The absorption and dispersion of sound in air at the ground, using the ISO standard at various relative humidities.

Because of stratification absorption and dispersion vary with altitude. Figure 5.4 shows the temperature, pressure, and relative humidity profiles for the ISO 9613-1 atmosphere (1993, the data originally came from the ISO 2533:1975 standard atmosphere). In Fig. 5.5 the absorption for three frequency components is shown as a function of altitude. Because the rise time of a sonic boom is typically 1 ms these absorption curves are representative of the variation of absorption at the shock. The ANSI standard gives a more accurate, though much more involved, formula for the saturation vapor pressure.
large peak in the absorption curves at an altitude around 5 km is due to the increase of the relaxation frequency of $O_2$ from 10 Hz at 7 km to nearly 30 kHz at the ground.

5.3 Operation Just ‘Cause

In 1994 “Operation Just ‘Cause”* was initiated by K. E. Needleman at NASA Langley Research Center to compare computational codes for sonic boom propagation. Of several groups who indicated an interest in the exercise, in the end only two participated, Nonlinear Acoustics Division of Applied Research Laboratories (ARL:UT), University of Texas at Austin, and National Center for Physical Acoustics (NCPA), University of Mississippi. The author (ARL:UT) used THOR and Dr. James P. Chambers (NCPA) used SHOCKN (Bass et al. 1987), which is based on an Anderson (1974) type algorithm. With the blessing of Dr. Leick D. Robinson, the author also ran ZEPHYRUS (Robinson 1991), a modified Pestorius (1973) type code in the exercise. The author also calculated the waveforms predicted by weak shock theory, as a simple version of the Thomas code (1972). The codes were evaluated by comparing the waveforms predicted

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*Needleman’s choice of title for the exercise was influenced by the code name for the US invasion of Panama in December 1989. The NASA exercise was conducted just (be)cause NASA wanted to know how well the codes compare.
Figure 5.5: Absorption in the ISO atmosphere as a function of altitude.

on the ground. The following results are reported elsewhere (Chambers et al. 1995, Cleveland et al. 1995c).

The codes use the same mathematical equation to model sonic boom propagation but solve it using different techniques. All three codes use the same type of algorithm to calculate nonlinear distortion: they distort the time base with the Poisson solution. Before applying absorption all use linear interpolation to resample the waveform and restore a uniform time grid. One presumes that all should therefore suffer from similar numerical dispersion at the shock fronts. Their principal difference, as mentioned in the previous chapter, is in their implementation of absorption and dispersion.

THOR accounts for absorption and dispersion at each step by a calculation in the time domain. A finite difference approach is used, and a tridiagonal matrix system is solved at each step.

SHOCKN accounts for absorption and dispersion by a frequency domain calculation. At each step the FFT is used to transform the signal to the frequency
domain, where each frequency component is multiplied by a complex coefficient which accounts for absorption and dispersion at that frequency. The inverse FFT transforms the signal back to the time domain.

ZEPHYRUS uses a mix of frequency domain attenuation and weak shock theory. Absorption and dispersion for the bulk of the wave are calculated by transferring to the frequency domain. Any shocks that are too thin to be modeled with the given sampling rate are treated with weak shock theory. Because of this ZEPHYRUS can use fewer samples to describe a waveform. A consequence is that the step size can be larger. The number of transformations to the frequency domain is therefore reduced and the waveform is resampled less often, both of which reduce numerical error.

Finally, the effects of spreading and inhomogeneity are included by using two techniques. THOR and SHOCKN both analytically scale the amplitude of the waveform at each step. ZEPHYRUS, on the other hand, applies the transformations that were used in Chapter Three for solving the lossless Burgers equation. Rather than having to scale the whole waveform at each step, ZEPHYRUS scales the absorption instead. The form of the scaling is described in Sec. 6.4.2. The two techniques for accounting for spreading and inhomogeneity are equivalent.

In this exercise THOR and SHOCKN are run using a step size that is 20% of the local shock formation distance; see Sec. 4.6. ZEPHYRUS has two step sizes, which were optimized by Robinson (1991). The steps for nonlinear distortion are 10% of the local shock formation distance. The application of absorption occurs no more regularly than two local shock formation distances. However, ZEPHYRUS usually applies absorption less regularly than this. It accumulates absorption at each nonlinear step and only transforms into the frequency domain when a significant amount of absorption has accrued.

The sampling rate, and hence number of samples, vary for each code and each atmosphere. Typically THOR requires a time window just a little longer than the duration of the waveform on the ground. ZEPHYRUS and SHOCKN need a time
window approximately twice as long as the waveform duration to eliminate aliasing effects produced by applying absorption in the frequency domain.

It was hoped that the Thomas code (1972) would also take part in the exercise. This code, developed by NASA, uses pure weak shock theory over the entire propagation path to predict sonic boom waveforms on the ground. The code uses ray theory to account for stratification. It does not explicitly include absorption and dispersion except to the extent that weak shock theory includes absorption. However, Needleman was not able to get the code to run with the desired parameters. As a simple alternative, the author applied weak shock theory in the form developed in Chapter Three. The equal area rule was used to make predicted waveforms single valued. In principle the waveforms calculated in this manner should be identical to those predicted by the Thomas code.

5.3.1 Operating Parameters

The idealized atmospheres used in the exercise are as follows:

- A uniform (i.e., homogeneous) atmosphere with only thermoviscous absorption. By including thermoviscous absorption only, we achieve a fair comparison with weak shock theory, which cannot account for relaxation. It is not possible to run THOR and SHOCKN without absorption, as multivalued waveforms are predicted.

- An isothermal atmosphere with only thermoviscous absorption.

- A uniform atmosphere with absorption and dispersion as given by the ISO 9613 standard, that is, both classical absorption and two relaxation processes.

- An isothermal atmosphere with atmospheric absorption and dispersion given by the ISO 9613 standard.

In all four atmospheres no refraction takes place because the small-signal sound speed is constant; the spreading is cylindrical. The effect of the ground impedance is ignored. Because ZEPHYRUS automatically includes relaxation, it could not be run for the first two cases.
The aircraft is at an altitude of 14 630 m (48 000 ft) and flying at Mach 1.8. Three fabricated test waveforms, 183 m (600 ft) directly below the aircraft, were provided by Needleman and Brenda Sullivan of NASA Langley. The waveforms are not associated with any specific aircraft design that NASA is considering. Although they bear some resemblance to a waveform near a supersonic aircraft, they are not representative of any real nearfield waveform. The three test waveforms—given the names flat top, ramp, and "N wave"—are shown in Fig. 5.6. Because the zero crossing in the middle of the waveform moves at the small-signal (equilibrium) sound speed it is used as an anchor point for the alignment of all subsequent waveforms.

Figure 5.6: The initial test waveforms used in operation Just "Cause.

5.3.2 Uniform Atmosphere

The uniform atmosphere has the following properties: ambient temperature $T_0 = 273.15$ K ($0^\circ$C), ambient pressure $P_0 = 101.3$ kPa, universal gas constant $R = 287$ J/(kg·K), and ratio of specific heats $\gamma = 1.4$. The ambient density and small-signal sound speed are calculated assuming an ideal gas, that is, $\rho_0 = P_0/(RT_0)$ and
The uniform atmosphere is quite contrived because it neglects the presence of gravity which is substantial over the ranges we are considering. However, it is a useful first comparison to be sure the codes are solving the same problem.

We neglect relaxation in this atmosphere and the coefficient of classical-rotational attenuation is chosen to be $A_{cr} = 2 \times 10^{-9} \text{Np/(Hz}^2 \text{m)}$—the ISO 9613 standard yields a smaller value $A_{cr} = 1.78 \times 10^{-11} \text{Np/(Hz}^2 \text{m)}$ for the same conditions. We used the larger value to increase the shock thickness and hence reduce the number of samples required to describe the waveform. For this absorption a sampling rate of 27.6 kHz can be used to model the shocks (THOR needed approximately 9000 samples to describe the test waves and SHOCKN 16 384 samples). Although the absorption is stronger than in the standards, it is still weak enough that weak shock theory is valid. For the case of an N wave, Blackstock (1972, Sec. 3n-11) states that weak shock theory is valid if the rise time $t_{rt}$ of the shock is much less than the duration of the wave.

The risetime for a steady-state shock (Eq. 6.5) is

$$t_{rt} = \ln(9) A_{cr} \frac{4 \rho_0 c_0^3}{\beta p_0}.$$ 

If we use this as a measure of the rise time of the shock in N wave, it follows that weak shock theory is valid if the following inequality is true

$$\rho_0 \gg \frac{A_{cr} \rho_0 c_0^3}{T_h \beta}.$$  \hspace{1cm} (5.12)

If we choose the characteristic frequency of a sonic boom as $1/2T_h$,* a similar expression to that found in Pierce (1981, Eq. 11-3.10), for a general absorbing fluid, $\frac{\dot{\rho} \rho_0}{\rho_0 c_0^2} > \alpha(\omega)$ is obtained except his $>$ is replaced with $\gg$. In this case the test waves at the ground have a duration of roughly $2T_h = 0.25 \text{ s}$ and $\omega = 25.1 \text{ rad/s}$. The inequality Eq. 5.12 therefore requires that the peak overpressure be

$$p_0 \gg 0.28 \text{ Pa}.$$  

This is true for the test waves used in this exercise.

The ground signatures predicted by THOR and weak shock theory are shown in Fig. 5.7 for all three test waveforms. The agreement is excellent. Figure 5.8 shows the

*Note that the peak in the spectrum of an N wave actually occurs at $f = 0.66/2T_h$. 

$c_0 = \sqrt{\gamma R T_0}$. The uniform atmosphere is quite contrived because it neglects the presence of gravity which is substantial over the ranges we are considering. However, it is a useful first comparison to be sure the codes are solving the same problem.
front and rear shocks of each calculated ground waveform on an expanded scale. The slight differences between THOR and SHOCKN are attributed to numerical error. Shock location in both codes is slightly dependent on the step size and number of points in the waveform. Note that the near identical shapes of the shock profile in each case is an indication that absorption and nonlinear effects are properly accounted for. However, the slight discrepancy in the shock location predicted by weak shock theory is a systematic problem with weak shock theory. Weak shock theory, by neglecting the rounding of the shocks, slightly overestimates the head shock speed and slightly underestimates the tail shock speed. This effect is addressed in Sec. 5.4. As a further test, THOR was run with a higher sampling rate, 100 kHz. The same results were observed.

5.3.3 Isothermal Atmosphere

The isothermal atmosphere has the same ambient temperature as the uniform atmosphere, $T_0 = 273.15$ K. The ambient pressure at the ground is 101.3 kPa and decreases exponentially with altitude (see Chapter Three), $P_0 = P_b(0) e^{-z/H}$, where $H = R T_b / g = 7991.2$ m is the scale height of the atmosphere. The gas constant and ratio of specific heats remain the same: $R = 287 J/(kg \cdot K)$ and $\gamma = 1.4$. The absorption varies with altitude in an isothermal atmosphere because $A_{\sigma r} = b/(2 \rho_0 c_0^2)$ and the density decreases with altitude. For the isothermal atmosphere then we have $A_{\sigma r} = 2 \times 10^{-9} e^{z/2H} \text{Np}/(\text{m} \cdot \text{Hz}^2)$. The same sampling rate, 27.6 kHz, is used by both THOR and SHOCKN.

The ground waveforms predicted by THOR, SHOCKN, and weak shock theory are shown in Fig. 5.9. There is excellent agreement between THOR and SHOCKN. Again THOR was also run with a higher sampling rate of 100 kHz and the same results were observed. The arrival time discrepancy for weak shock theory is similar to that seen for the uniform atmosphere (see Sec. 5.4).

5.3.4 Uniform Atmosphere with Relaxation

Relaxation effects were included for both the uniform and isothermal atmospheres. Results in the uniform atmosphere are presented here. Absorption is calculated using
Figure 5.7: Uniform atmosphere: Ground signatures computed by THOR, SHOCKN, and weak shock theory for the three test waves.
Figure 5.8: Uniform Atmosphere: Close up of the shocks in the ground signatures computed by THOR, SHOCKN and weak shock theory for the three test waves.
Figure 5.9: Isothermal atmosphere: Ground signatures computed by THOR, SHOCKN, and weak shock theory for the three test waves.
the formulae in the ISO standard. A uniform relative humidity of 20% is used. Only the flat top and ramp waveforms are presented for this atmosphere.

Figure 5.10 shows the predicted waveforms at the ground. There is excellent agreement between THOR, SHOCKN, weak shock theory, and ZEPHYRUS. Figure 5.11 shows the front and rear shocks of each calculated ground waveform on an expanded scale.* The overprediction of shock amplitude by weak shock theory is now very apparent.

5.3.5 Isothermal Atmosphere with Relaxation

Relaxation effects were included for the isothermal atmosphere with uniform relative humidities of 20% and 80%. A significant problem is encountered when trying to model the shocks in the isothermal atmosphere. Near the aircraft the shock amplitudes are so large, and the relaxation frequencies so low, that relaxation has a negligible effect on the shock rise time. The absorption at the shock is dominated by thermoviscous effects. Thermoviscous absorption is relatively weak and the shock rise times are on the order of 1 μs. To successfully model a shock this thin, we need a sampling period of 0.1 μs, i.e., a sampling rate of 10 MHz. Such a rate applied to the entire waveform requires more than 3 million samples. Although as the wave propagates it would be possible to reduce the number of samples in the waveform,† the computational cost would be extreme. ZEPHYRUS does not have this problem. It samples the waveform at a rate sufficient for analysis purposes on the ground. If this rate is too low to properly capture any shocks, ZEPHYRUS automatically uses weak shock theory to calculate them. ZEPHYRUS continues to use 16 384 samples (a sampling rate of approximately 27.6 kHz) to describe the waveforms in the following calculations.

In order for THOR to avoid the very high sampling rate we used two schemes. In the first scheme, weak shock theory is used initially. When the wave is far enough away from the aircraft that the required sampling rate is more practical, the program

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*The waveform predicted by ZEPHYRUS is not smooth because ZEPHYRUS only uses 1 000 samples when it outputs waveforms.
†If the scaling of the waveform (due to spreading and impedance variation) were monitored, the code could automatically reduce the number of samples required to describe the waveform.
Figure 5.10: Uniform atmosphere with ISO absorption (20% relative humidity): Ground signatures computed by THOR, SHOCKN, weak shock theory, and ZEPHYRUS for the flat top and ramp test waves.
Figure 5.11: Uniform atmosphere with ISO absorption (20% relative humidity): Close up of the shocks in the ground signatures computed by THOR, SHOCKN, weak shock theory, and ZEPHYRUS for the flat top and ramp test waves.
shifts to the normal computation. Weak shock theory is a good model near the aircraft because the shocks are very thin, less than 1 part in 100,000 of the duration. It was found that if weak shock theory is used to propagate the wave to an altitude 2 km below the source point (a path length of about 2.4 km, leaving about 15 km of propagation to the ground), the waveform could be adequately modeled with a sampling rate of 100 kHz. This rate requires about 35,000 points to describe the waveform, a number that is considered reasonable. Although, as discussed above, the code could gradually reduce the number of samples as the wave propagates, the results presented here are for a constant number of samples all the way to the ground.

SHOCKN has a routine built in to simulate weak shock theory. When the shocks are too thin to be properly captured, it switches into weak shock mode. It then applies only classical $f^2$ absorption but chooses $A_{cr}$ high enough that the shocks are represented by at least ten points. In this way SHOCKN simulates weak shock theory whilst avoiding numerical dispersion problems associated with too few points in the shock. For this exercise SHOCKN was programmed to be in weak shock theory mode until the wave was 4 km below the source, at which stage it shifts into regular calculation. A sampling rate of 27.6 kHz is used, requiring 16,384 samples.

The price paid for using weak shock theory near the aircraft is that the shocks may not be correctly modeled in the first 2.4 km of propagation. However, it is expected the error should be very small because the shocks near the aircraft are so thin.

Figure 5.12 shows the predicted ground signatures for the flat top and ramp waveforms. THOR, SHOCKN, weak shock theory, and ZEPHYRUS are in excellent agreement, except for the overprediction of shock amplitude by weak shock theory.

The validity of beginning the calculation with weak shock theory and then switching over to THOR (or SHOCKN) was tested by using the procedure with THOR for two different switchover distances. If the same ground waveforms are predicted, the conclusion is the use of weak shock theory, for these switchover distances, introduces no errors. The predicted ground signatures of the ramp test wave for switchover distances of 2 km and 4 km are shown in Fig. 5.13. The waveforms are in excellent agreement.

\[2.183 \text{ km below the aircraft.}\]
Figure 5.12: Isothermal atmosphere with ISO absorption (20% relative humidity). Ground signatures computed by THOR, SHOCKN, weak shock theory, and ZEPHYRUS for the flat top and ramp test waves.
The use of weak shock theory near the aircraft appears to have a negligible effect on the ground signature.

Figure 5.13: Ground signatures for the ramp waveform predicted by THOR in the isothermal atmosphere with ISO absorption (20% relative humidity). Weak shock theory is used to propagate the test wave for either the first 2 km of 4 km.

The second scheme used by THOR to cope with very thin shocks is a mixture of the methods used by SHOCKN and ZEPHYRUS. An adaptive absorption routine was written for THOR. It applies absorption in the same way as described in Chapter Four. However, when it calculates the $A_{tv}$ and $B_{tv}$ matrices (Eqs. 4.14 and 4.15) it monitors the jump in pressure between samples. If any jump is too large for the prescribed sampling rate, then the absorption routine artificially increases the local absorption coefficient to compensate. The artificial absorption coefficient is chosen so that at least ten points fall within each shock in the waveform. The shock is therefore expected to behave in a manner similar to that predicted by weak shock theory, without having to explicitly implement weak shock theory. When the second scheme for THOR is used a 27.6 kHz sampling rate can be used to model all the waveforms. This dramatically reduces the number of points required (9000 as opposed to 35000) and hence reduces
the computational time. It also avoids the need to precalculate waveforms using weak shock theory.

Figure 5.14 compares the ground waveforms calculated by THOR with those of weak shock theory and ZEPHYRUS for the isothermal atmosphere with a relative humidity of 20%. There is excellent agreement between the results. This implies that the second THOR scheme produces results nearly identical to those of the first scheme at a much lower computational cost.

Finally we compare results for the same atmosphere but with 80% relative humidity. The absorption due to relaxation is less in this atmosphere. The results for THOR are obtained using the second scheme. SHOCKN was not run for this atmosphere. Figure 5.15 shows the ground signatures for all three test waves. THOR, ZEPHYRUS, and weak shock theory all predict the same duration of the ground signature. As for shock overpressure, we find again that weak shock theory overestimates the peaks.

In order to compare run times, we ran all three codes on a single dedicated IBM RISC 6000 computer. The IMSL routine dtime was used to calculate the CPU run time for THOR (second scheme) and SHOCKN. For the ramp waveform in the uniform atmosphere with relaxation the run times were 37 seconds for THOR and 75 seconds for SHOCKN. For the isothermal atmosphere with relaxation the run times were 6.1 minutes for THOR and 12 minutes for SHOCKN. Because ZEPHYRUS is written in C it was not possible to use the FORTRAN library command to calculate CPU run time for ZEPHYRUS. From the wall clock measurements it appears that ZEPHYRUS has a run time about 50% longer than that of THOR.

The results in this section imply excellent agreement between THOR and other sonic booms codes. It has been demonstrated that weak shock theory overestimates the amplitude of sonic boom shocks. Therefore one expects that use of weak shock theory should lead to an overestimate of loudness.

5.4 Comparison to Weak Shock Theory and Waveform Freezing

In the course of Operation Just ’Cause a slight discrepancy in shock locations predicted by THOR and weak shock theory was observed. In this section we investigate the effect
Figure 5.14: Isothermal atmosphere with ISO absorption (20% relative humidity). Ground signatures computed by THOR (scheme 2), weak shock theory, and ZEPHYRUS for the ramp test wave.
Figure 5.15: Isothermal atmosphere with ISO absorption (80% relative humidity). Ground signatures computed by THOR (scheme 2), weak shock theory, and ZEPHYRUS for the ramp test wave.
in more detail using N waves. We also compare output from THOR to the predictions of waveform freezing presented in Chapter Three.

We now investigate the difference in shock speed between N waves propagated by THOR and weak shock theory. The discrepancy is a real effect and is not due to the numerical dispersion described in Chapter Four, see Fig. 4.16. It appears because weak shock theory cannot model the rounding of shocks due to absorption. In the case of N wave like shocks this lack of rounding means that weak shock theory overpredicts the shock amplitude and hence overpredicts the shock speed. We examine this effect by considering the propagation of N waves.

The half duration and peak overpressure of an N wave predicted by using weak shock theory are given by Eqs. 3.12 and 3.13, repeated here for convenience,

\[ T_h = T_{ho}\sqrt{1 + ax} \]
\[ \hat{p} = \frac{\hat{p}_0}{\sqrt{1 + ax}} \]

where \( a = \beta\rho_0/\rho_0c_0^3 T_{ho} \).

Figure 5.16 compares the half duration and peak pressure of a plane N wave predicted by weak shock theory to that predicted by THOR for varying amounts of absorption. It is clear that the peak pressure predicted by weak shock theory is greater than that predicted by THOR. We contend that weak shock theory is in error because it neglects the rounding of the shocks. The effect on half duration is hard to see from this plot.

Figure 5.17 shows N wave signatures at two ranges for the case \( \Gamma = 100 \). At the range \( ax = 1.27 \) the peak predicted by THOR is clearly less than predicted by weak shock theory; THOR’s predicted duration is also slightly less. At \( ax = 2.55 \) the duration predicted by THOR is noticeably less than that predicted by weak shock theory. The reduction in duration is attributed to the rounding of the shocks by THOR. Because of rounding, the head shock calculated by THOR has a smaller amplitude and therefore a lower propagation speed than the “ideal” head shock of weak shock theory. In the same way, the THOR-calculated tail shock is weaker and thus not quite so slow as the ideal tail shock. The effect of rounding on shock speed is not, however, as pronounced as the effect on amplitude. The reason is that the diffusion of the shock
pushes the foot of the shock ahead. This partially compensates for the “fallback” effect of the diffusion on the peak of the shock. These results corroborate the explanation given in the previous section about the discrepancies between the THOR and weak shock theory waveforms calculated for Operation Just ’Cause.

The THOR-weak shock theory comparison also has application to the calibration of microphones in our laboratory. Calibration is achieved by exploiting the fact that the product of peak overpressure and half duration $pT_h$ is constant for a plane N wave (see, for example, Hester 1992). That is, waveforms measured in the laboratory are similar to those shown in Fig. 5.17. The peak pressure and half duration are estimated

---

*For a step shock in a thermoviscous fluid the shock amplitude predicted by THOR remains constant. The diffusion gives the shock some thickness but does not decrease the amplitude. In this case the shock speed predicted by THOR and weak shock theory agree.

*For a spherical N wave $r^2T_h$ is a constant, where $r$ is the propagation distance.
Figure 5.17: N wave profiles predicted by weak shock theory and THOR at two ranges; \( \Gamma = 100 \).

from the waveform. Figure 5.18 shows the product of half duration and overpressure for the N wave shown in Fig. 5.17. It is clear that \( \tilde{p}T_h \) is not constant. The major reason is that the peak pressure is less than weak shock theory predicts. By inspection of Fig. 5.17, one gets much better agreement with weak shock theory if the linear section of the measured waveform is extrapolated the pressure to the midpoint of the shock. Even then there will be a slight error because the midpoint of the measured waveform does not quite correspond to the shock location predicted by weak shock theory.
5.4.1 Effects of Spreading and Stratification

Sonic booms propagated through a stratified medium by THOR should experience waveform freezing, or at least chilling, as described in Chapter Three. Recall that the analysis in Chapter Three is done without taking explicit account of absorption; weak shock theory is incorporated to remove multivaluedness. THOR does not use weak shock theory, so a small amount of thermoviscous absorption is used to keep the waveforms well behaved. To compare results we shall use the half duration of an N wave.

For downward wave propagation of plane waves in an isothermal atmosphere, where the ambient density varies as $\rho_0 = \bar{\rho}_0 e^{x/H}$, the distortion distance is (Eq. 3.27)

$$\bar{x} = 2H(1 - e^{-x/2H}),$$

where $H$ is the scale height of the atmosphere. Figure 5.19 shows the normalized half duration for a plane N wave propagating straight downwards through an isothermal atmosphere (1) predicted by weak shock theory (Eq. 3.12), and (2) predicted by THOR. The agreement is excellent.

In the case where the waveform is also suffering cylindrical spreading the distortion distance (Eq. 3.37 with $\theta_0 = -\pi/2$ and $s = r - r_0$) is

$$\bar{x} = \sqrt{2r_0H}e^{r_0/2H} \{\text{erf}(r/2H) - \text{erf}(r_0/2H)\}.$$
Recall that $r_0$ is the initial source radius of the waveform. In Fig. 5.20 the half duration predicted by weak shock theory and that predicted by THOR is shown. Again the agreement is excellent. THOR’s slight underestimation of the half duration is due to rounding of the shocks. This is more noticeable than in the plane wave case because the spreading reduces the amplitude of the shocks. Therefore the shocks predicted by THOR become more rounded.

Figure 5.19: Comparison between THOR and weak shock theory predictions of the half duration of an N wave propagating through an isothermal atmosphere, $H = 8.5$ km.
Figure 5.20: Comparison between THOR and weak shock theory predictions of the half duration of a cylindrically spreading N wave propagating in an isothermal atmosphere, $H = 8.5$ km and $r_0 = 100$ m.
Chapter 6

Sonic Boom Rise Time

6.1 Introduction

This chapter focuses on the effect that stratification of the atmosphere has on sonic boom shocks. The shocks play a major role in determining the loudness of the sonic boom, which is of great interest to NASA. The rise time of shocks has been shown to be dependent on the relaxation effects in the air (Lighthill 1956, Polyakova it et al. 1962, Ockendon and Spence 1969). Stratification means that absorption mechanisms will vary with altitude. In particular, the relaxation processes of oxygen and nitrogen, which dominate sound absorption, are strongly dependent on the water content of the atmosphere and vary significantly with altitude.

In the next few sections the question of whether the shocks in a sonic boom are in steady state at the ground is addressed. First, we give a little background to the prediction of shock rise time. THOR is then used to check a method proposed by Kang and Pierce (1990) to estimate the rise time of sonic boom shocks on the ground. In an attempt to reduce the high computational cost of including absorption in a propagation model Kang and Pierce came up with an attractive scheme where they balance nonlinear steepening and absorption effects at the ground to construct a shock profile. Because they assume the forces on the shock are in balance they can neglect the variation of absorption with altitude, that is, the path history. Their assumptions are analyzed using THOR. A parametric study of the effect of variation in relative humidity on the rise time of step shocks is undertaken. The effect of spreading and waveform shape on rise time is also considered. The rise time of sonic boom waveforms predicted by THOR are compared to the rise time predicted by the Kang-Pierce model. It is seen that path history is important in the prediction of the rise time of sonic boom shocks.
Finally, THOR is used to examine the effect of measurement error of atmospheric data on predicted ground waveforms. As mentioned in Chapter Three the stratification of the density and sound speed has a major effect on the ray paths and nonlinear distortion of a finite-amplitude wave. Of some concern in the NASA exercise was the sensitivity of shock location to the ambient values used. The effect of slight fluctuations in the ambient quantities on the ground waveform is demonstrated. It is shown that fluctuations can lead to a large variation in the prediction of sonic boom rise time.

6.2 Sonic Boom Rise Time

The prediction of sonic boom rise time* on the ground is important in determining the loudness of the sonic boom (von Gierke and Nixon 1972). Sonic booms are usually hundreds of milliseconds in duration and the rise time of the shocks is normally of order 1 ms. The shocks are the major factor in determining the loudness of the sonic boom, at least for observers outdoors.† It is important therefore to properly characterize the shock profile on the ground to determine the sonic boom’s loudness and hence its acceptability.

An early code developed by NASA (Hayes et al. 1969, see also Thomas 1972) for the propagation of sonic booms through the atmosphere used a lossless Burgers equation, ray theory, and weak shock theory, much like the method described in Chapter Three. Because the shocks were treated as discontinuities no information about the profile, in particular the rise time was available.

In the 1960s and 1970s comparisons between the rise time given by the classical Burgers equation (plane waves in a thermoviscous fluid) and field data showed that measured sonic booms have rise times two orders of magnitude longer than the predictions (see, for example, Pierce and Maglieri 1972). It was thought that atmospheric turbulence was the mechanism responsible for the long rise times (see, for example, Crow 1969, Pierce 1971). Plotkin and George (1972) derived a Burgers equation for the propagation through turbulence, where the absorption term is replaced by an ef-

---

*Rise time is defined to be the time it takes a shock to go from 10% to 90% of its peak value.
†For people indoors the rise time is not so important (Slutsky and Arnold 1971).
fective absorption which depends on the strength of the turbulence. They were able to fit their results to the measured data.

To allow codes which use weak shock theory to estimate rise time Plotkin (see, for example, Plotkin 1989) developed the “3/p” rule. The 3/p rule is an empirical fit of the rise time to the overpressure for a number of measured sonic booms and blast waves from the 1960s. Given the peak overpressure \( p \) of the lead shock in lbs/ft\(^2\) the rise time, in milliseconds, is 3/p. The rule is a nominal curve fit to the measured data and as such yields an average value for different aircraft and atmospheric conditions. The ease of use of the 3/p rule has lead to its common use in the literature. However, it provides no information about the fine structure of the shock. More important in the original data (Plotkin and George 1972) the scatter in the measured rise time was on the order of 10. This much scatter should correspond to significant variation in the loudness, but is not represented in the 3/p rule.

Investigation in the 1970s (Hodgson and Johannesen 1971, Hodgson 1973, Reed 1977, Bass and Raspet 1978) recognized the importance of relaxation in determining sonic boom rise time. Kang and Pierce (1990) developed a scheme to account easily for nonlinearity, absorption, and dispersion, using atmospheric conditions at the ground. In the absence of turbulence the rise time of a shock is controlled by the opposing forces of waveform steepening and absorption and dispersion (including relaxation). Kang and Pierce assumed that the lead shock of a sonic boom can be modeled as a step shock\(^*\) where these forces are in balance. Furthermore, they assumed that near the ground the cylindrically spreading sonic boom can be approximated as a plane wave. The consequence of the assumptions is a great simplification of the problem: a step shock in a relaxing fluid can be modeled by a set of coupled ordinary differential equations. The equations were integrated numerically and the result provided the complete shock profile, from which the rise time was calculated. Their scheme is particularly attractive because it predicts the shock profile using only the atmospheric conditions at the ground. The path history of the sonic boom can be neglected.

The U.S. Air Force measured a large number of sonic booms in an exercise in

\(^*\)For a sonic boom the pressure behind the shock falls off so slowly that it seems reasonable that the field behind the shock has little influence on the rise time.
1982 at White Sands Missile Range in the Mojave Desert (Lee and Downing 1991). Kang and Pierce (Pierce 1993a, 1992, Kang 1991), using atmospheric ground data only, compared their prediction of the rise time and overpressure to the measured values. Figure 6.1 (Pierce 1992, 1993a)\textsuperscript{1} shows good agreement between their curve and the measured data.

![Figure 6.1: Kang and Pierce prediction of rise time compared to measured sonic boom rise time.](image)

Figure 6.1: Kang and Pierce prediction of rise time compared to measured sonic boom rise time.

Their work demonstrates that including relaxation processes greatly improves the prediction of sonic boom rise times. Kang and Pierce argue that turbulence is no longer required to describe a two orders of magnitude increase in rise time. Turbulence simply introduces fluctuations about the no-turbulence prediction. However, model experiments carried out at Applied Research Laboratories (Lipkens and Blackstock 1991, 1992, 1995, Lipkens 1993) indicate that turbulence almost always increases the rise time of N waves.\textsuperscript{*} If turbulence is indeed the mechanism that produces the spread in the measured data then, except for an odd outlier, one would expect the rise time of

\textsuperscript{1}Earlier published results (Pierce and Kang 1990, Pierce and Sparrow 1991) were erroneous in that account had not been taken of pressure doubling due to the reflection at the ground.

\textsuperscript{*}The coherence in arrival time required to make sharp shocks is disturbed by the turbulence. The notable exception being where the sonic boom is focussed.
the measured sonic booms always to be greater than the Kang-Pierce prediction. The Kang-Pierce prediction should be a lower bound. So, why does their prediction go through the middle of the data?

We suspected that the answer is associated with steady-state assumption. The question then is whether path history may be neglected for sonic booms. If the steady-state assumption is correct, nonlinear steepening and absorption are always in balance. However, because of stratification of the atmosphere, absorption varies markedly with altitude, as shown in Fig. 5.5. For example, molecular relaxation, which is a major factor controlling sonic boom rise time, is strongly dependent on relative humidity. Because humidity varies with altitude, so does the absorption. Sonic boom rise time therefore varies a great deal as the boom propagates downward. In addition the amplitude of the shock, and hence the steepening strength, changes because (1) the impedance of the atmosphere is increasing, (2) the wave is spreading, and (3) the sonic boom is really an N wave, not a step shock. In summary, nonlinear steepening and absorption processes continuously vary along the propagation path. Kang and Pierce assume the shocks adjust immediately to the change in these opposing forces, and thus the shock retains no memory of conditions previously encountered along the ray path.

Kang (1991, Chap. 7.2) argues that sonic boom shocks respond so quickly that the memory of the shock is less than 30 meters. In other words the forces at the shock can be considered in balance, and only local conditions are important. In his work with ZEPHYRUS, Robinson (1991, Chap. 5.2) observed memory effects on the order of a kilometer or so and thus disagreed with Kang's argument. Using SHOCKN, Raspet et al. (1992) found that perturbed 100 Pa shocks (step waveform) require a propagation distance of order 1 km for the rise time to return to within 10% of its steady shock value. There is some indication then that the lead shock of a sonic boom does not respond quickly enough to variation in atmospheric conditions (and to other changes that affect the profile, such as geometrical spreading and wave shape) to justify the steady-state assumption.

If the shocks are not in balance, past history along the propagation path must be significant. Figure 6.2 shows profiles of relative humidity and temperature measured during the sonic boom exercise over the Mojave desert (Lee and Downing 1991,
Kang 1991) and the calculated absorption at three frequencies, 500 Hz, 1 kHz, and 2 kHz. The measured sonic booms had rise times of order 1 ms so these curves give an indication of the behavior of the absorption at the shock front. It is seen that absorption changes rapidly, particularly during the lower part of the propagation path.

Figure 6.2: Relative humidity and temperature and calculated absorption (three pure tones) for the atmosphere over the Mojave desert during the sonic boom exercise. Altitude is kilometers above sea level. The Mojave desert is at an altitude of 723 m.

Note that in the last 5 km the attenuation increases steadily. The shock is therefore trying to increase its rise time as it approaches the ground. If it is not in steady state, then the rise time at the ground should contain some memory of the shorter rise time that existed somewhere above the ground. That is, the steady-state assumption based on the ground conditions should overestimate the rise time of a shock that has propagated through this atmosphere.

In addition, geometrical spreading reduces the amplitude of the shock as it propagates. Nonlinear effects are therefore weakened as the sonic boom propagates and the rise time should increase. Finally, since the sonic boom is an N wave, which suffers amplitude decrease as it propagates, nonlinear effects at the shock are weakened because of this effect as well.

To answer the question of whether sonic boom shocks are in balance, we use THOR as a propagation model. The processes that are examined are (1) stratifi-
cation, which includes variation in density, temperature, and relative humidity, (2) geometrical spreading, and (3) the shape of the sonic boom. First, we review the Kang-Pierce model.

6.3 The Kang-Pierce Model

The extended version of the Burgers equation (Eq. 2.84) is

$$\frac{\partial p'}{\partial z} - \frac{1}{2} \left( \frac{1}{S} \frac{\partial S}{\partial z} + \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} + \frac{1}{c_0} \frac{\partial c_0}{\partial z} \right) p' = \frac{\beta}{2 \rho_0 c_0^3} \frac{\partial p'^2}{\partial \nu} + \frac{(b + \mathcal{R})}{2 \rho_0 c_0^3} \frac{\partial^2 p'}{\partial \nu^2}.$$ 

For plane waves propagating through a homogeneous medium the equation reduces to a form of, what Pierce calls, the augmented Burgers equation (Eq. 2.3):

$$\frac{\partial p'}{\partial x} - \frac{\beta}{2 \rho_0 c_0^3} \frac{\partial p'^2}{\partial \tau} = \frac{b}{2 \rho_0 c_0^3} \frac{\partial^2 p'}{\partial \nu^2} + \sum_{\nu} \frac{m_{\nu} \tau_{\nu}}{2 c_0} \frac{\partial^2}{\partial \tilde{\nu}^2} p'. $$

The relaxation operator \( \mathcal{R} \) has been expanded into a sum over all relaxational processes. Note the operator \( \tau_{\nu}/(1 + \tau_{\nu} \frac{\partial}{\partial \nu}) \) may be expressed as the following integral:

$$\frac{\tau_{\nu}}{1 + \tau_{\nu} \frac{\partial}{\partial \nu}} f(t') = e^{-t'/\tau_{\nu}} \int_{-\infty}^{t'} e^{\tau'/\tau_{\nu}} f(\tau) d\tau.$$ 

In Pierce’s formulation (Kang 1991, Chap 2.2.4) the augmented Burgers equation is a set of coupled equations which removes the need for the awkward relaxation operator used in the equation shown here, see Sec. 2.8. Also, the time and spatial derivatives are swapped and there is no transformation to a retarded time frame.

Kang and Pierce used the augmented Burgers equation to calculate the shock profile of sonic booms (Pierce and Kang 1990, Kang 1991). The assumption that the shocks are always in balance (i.e., steady state) implies there is no evolution of the waveform, \( \frac{\partial p'}{\partial x} = 0 \). The resulting ordinary differential equation is

$$-\frac{\beta}{2 \rho_0 c_0^3} \frac{d p'^2}{d \tilde{\nu}} = \frac{b}{2 \rho_0 c_0^3} \frac{d^2 p'}{d \tilde{\nu}^2} + \sum_{\nu} \frac{m_{\nu} \tau_{\nu}}{2 c_0} \frac{\partial^2}{\partial \tilde{\nu}^2} p'.$$

(6.1)

Kang and Pierce included the relaxation processes of oxygen and nitrogen to obtain a set of coupled differential equations which they solved numerically. An equivalent problem can be posed within the framework of Eq. 6.1.
The relaxation dispersion and relaxation time due to nitrogen are denoted by \( m_N \) and \( \tau_N \), and for oxygen by \( m_O \) and \( \tau_O \). By multiplying Eq. 6.1 by \((1 + \tau_N \frac{d}{dt'}) (1 + \tau_O \frac{d}{dt'})\), an ordinary differential equation is obtained:

\[
- \frac{\beta}{2 \rho_0 c_0^2} \left( 1 + (\tau_N + \tau_O) \frac{d}{dt'} + \tau_N \tau_O \frac{d^2}{dt'^2} \right) \frac{dp'^2}{dt'} = \frac{b}{2 \rho_0 c_0^2} \left( 1 + (\tau_N + \tau_O) \frac{d}{dt'} \right)
\]

\[
+ \tau_N \tau_O \frac{d}{dt'^2} \frac{d^2p'}{dt'^2} + \frac{m_N \tau_N}{2 c_0} \left( 1 + \tau_O \frac{d}{dt'} \right) \frac{d^2p'}{dt'^2} + \frac{m_O \tau_O}{2 c_0} \left( 1 + \tau_N \frac{d}{dt'} \right) \frac{d^2p'}{dt'^2}.
\]

This equation can be integrated once with respect to \( t' \). It is assumed at \( t' = \pm \infty \), \( P'^2 = P_0^2 \) and all the derivatives are zero, i.e., the profile is a step shock from \(-P_0\) to \(P_0\). The resulting equation is

\[
- \left( p'^2 - P_0^2 + (\tau_N + \tau_O) \frac{dp'^2}{dt'} + \tau_N \tau_O \frac{d^2p'^2}{dt'^2} \right) = \frac{b}{\beta} \left( \frac{dp'}{dt'} + (\tau_N + \tau_O) \frac{d^2p'}{dt'^2} + \tau_N \tau_O \frac{d^3p'}{dt'^3} \right)
\]

\[
+ \frac{m_N \tau_N P_0 c_0^2}{\beta} \left( \frac{dp'}{dt'} + \tau_O \frac{d^2p'}{dt'^2} \right) + \frac{m_O \tau_O P_0 c_0^2}{\beta} \left( \frac{dp'}{dt'} + \tau_N \frac{d^2p'}{dt'^2} \right). \quad (6.2)
\]

This equation is equivalent to Kang and Pierce's coupled differential equations.

It is more convenient to express Eq. 6.2 in dimensionless form. The reference pressure is taken to be \( P_0 \), i.e., half the shock amplitude. Use of the dimensionless variables introduced in Chapter Four and simple manipulation yields

\[
\theta_N \theta_0 \frac{d^3P}{\Gamma d\tau^3} = 1 - P^2 - (\theta_N + \theta_O) P \frac{dP}{d\tau} - \theta_N \theta_0 \left( \frac{dP}{d\tau} \right)^2 - \theta_N \theta_0 \theta_0 \left( \frac{d^2P}{d\tau^2} \right) - (\alpha + C_N \theta_N + C_O \theta_O) \frac{dP}{d\tau} - (\theta_N + \theta_O + \theta_N \theta_0 (C_N + C_O)) \frac{d^2P}{d\tau^2}. \quad (6.3)
\]

Equation 6.3 is written in the form of a third order derivative on the left-hand side, and only lower order derivatives on the right-hand side. This makes the equation amenable to numerical integration using a Runge-Kutta algorithm. We define the variables \( P_1 = \frac{dP}{d\tau} \) and \( P_2 = \frac{d^2P}{d\tau^2} = \frac{dP_1}{d\tau} \) and Eq. 6.3 can be written as the following set of coupled equations:

\[
\frac{dP}{d\tau} = P_1,
\]

\[
\frac{dP_1}{d\tau} = P_2,
\]

\[
\frac{dP_2}{d\tau} = \frac{\Gamma}{\theta_N \theta_0} \left( 1 - P^2 - (\theta_N + \theta_O) PP_1 - \theta_N \theta_0 (P_1^2 + PP_2) \right)
\]

\[
- (\alpha + C_N \theta_N + C_O \theta_O) P_1 - (\theta_N + \theta_O + \theta_N \theta_0 (C_N + C_O)) P_2.
\]
In Fig. 6.3 the predictions of Kang (1991, Fig. 5.2), the Runge-Kutta solution of Eq. 6.3, and the steady-state solution from THOR are shown. The conditions are: a 100 Pa shock, in a medium at 20 °C, 1 atmosphere pressure, and 10% relative humidity. The agreement is excellent. The results from the Runge-Kutta solution are used in subsequent sections to compare the results from THOR to the Kang-Pierce model.

### 6.4 Use of THOR to Predict Rise Time

The notion that sonic boom shocks remember their path history is now examined. The term transition distance was introduced (Cleveland et al. 1994a) to describe how far a step-shock must propagate to go from one steady state profile to another. A somewhat similar term, “healing distance,” is commonly used in literature related to turbulence for the distance a perturbed shock needs to return to its original state.
(Raspet et al. 1992). In their prediction of sonic boom rise time Kang and Pierce assume that the lead shock in a sonic boom immediately adjusts to changes in the absorption, i.e., the transition distance is very small. Kang (1991, Chap. 7.2) claims transition distances for sonic booms in the atmosphere are on the order of 30 m. THOR is used to examine this claim by conducting a parametric study on the reaction of a step-shock to changes in relative humidity. Relative humidity is chosen as the variable parameter as both the ISO 9613-1 (1993) atmosphere and the Mojave desert atmosphere, measured during the sonic boom exercise, show significant changes in relative humidity with altitude. The parametric study considered the effect of a linear change of 10% in relative humidity over a 2 km path length.

6.4.1 Transition Distances

In this section THOR is used to examine the claim that sonic boom shocks require 30 m.

A parametric study of the reaction of a step-shock to changes in relative humidity is carried out. Relative humidity is chosen as the variable parameter, as both the ISO 9613-1 (1993) atmosphere and the Mojave desert atmosphere, measured during the sonic boom exercise, show significant changes in relative humidity with altitude. In particular, for the Mojave desert data the relative humidity decreases from 87% (4.46 km above the ground) to 23% at the ground. A rate of 13.9%/km. If we assume an aircraft flying at Mach 1.44, the change in relative humidity along the ray path is 10%/km. The parametric study considered the effect of a linear change of 5%/km over a 2 km path length. An aircraft would need to fly at Mach 1.07 for the ray to be traveling at a shallow enough angle for the relative humidity to be changing at this rate.

The transition distance is calculated in the following manner. An initial steady-state waveform is obtained by using THOR to propagate a step-shock in a uniform atmosphere for a long distance. This steady-state waveform is compared to that predicted by the Kang-Pierce steady-state profile (described by Eq. 6.3) to ensure THOR's profile is accurate. The steady-state waveform is then used as the input to an atmosphere where the relative humidity changes linearly with distance for 2 km and then
remains constant. The rate of change is either 5%/km, 0%/km, or -5%/km. The rise
time of the shock is used as a measure of the state of the profile. When the rise time
reaches a stable value, it is assumed that the shock has reached steady state. This
technique is similar to that used to determine healing distance (Raspet et al. 1992).
Transition distances for three base relative humidities 20%, 50%, and 80% are calcu-
lated.

The technique used to carry out the parametric study is now demonstrated. A step
shock of amplitude 100 Pa (-50 Pa to 50 Pa) and a profile corresponding to a steady-
state shock in a medium of 20% relative humidity is used as an initial waveform. The
shock is propagated into three different atmospheres with final relative humidities of
either 10%, 20%, or 30%. The upper plot in Fig. 6.4 shows the relative humidity
as a function of propagation distance for the three cases. The lower plot in Fig. 6.4
shows the rise time of a plane step shock as a function of propagation distance. The
initial fluctuations in rise time (e.g., the dip in the 10% curve in Fig. 6.4) are due
to rather gross changes in the profile which are not very well characterized by the
standard definition of rise time. The results are however sufficient to determine the
distance at which steady state occurs. Similar fluctuations were observed by Raspet et
al. (1992). The results show the transition distance to be at least 5 km. This distance
is significantly longer than the 30 m claimed by Kang.

The parametric study of the effect of changing relative humidity on shock rise time
is now carried out. Results of the parametric study are shown in Fig. 6.5. Shock
amplitudes of 25 Pa, 50 Pa and 75 Pa are used which correspond to shock amplitudes
at the ground of 50 Pa, 100 Pa and 150 Pa.* Initial relative humidities of 20%, 50% and
80% are used. These humidities and pressures cover the parameter space appropriate
for sonic booms. It is clear from Fig. 6.5 that the transition distance for step-shock to
react to a change in relative humidity varies from as few as 2 km to over 10 km. The
Kang-Pierce steady state assumption does not appear to be valid.

*If the ground is modeled as an infinite impedance boundary then a sonic boom suffers pressure
doubling at the ground.
Figure 6.4: Change in rise time for a 100 Pa step shock leaving a medium of 20% relative humidity at 15°C and 1 atmosphere.

6.4.2 Effect of Spreading on Rise Time

In this section the balance of nonlinear effects and absorption at a geometrically spreading step-shock is examined. In an isothermal atmosphere the sonic boom generated by an aircraft in steady supersonic flight spreads cylindrically. Therefore the amplitude of the boom decreases as it propagates away from the aircraft. This in turn leads to a decrease in the nonlinear steepening force, as was seen with the reduction of $\beta_{\text{eff}}$ in Chapter Three. An alternative viewpoint is that the wave propagates, without loss of nonlinear steepening, into a medium with increasing absorption (Naugol’nykh et al. 1963). Either viewpoint leads to the conclusion that the rise time of a spreading shock tends to increase. The effects of both cylindrical and spherical spreading on the rise time of a step shock are considered here.

The extended Burgers equation, Eq. 2.84, is

$$
\frac{\partial \rho'}{\partial s} + \frac{1}{2S} \frac{\partial S'}{\partial s} \rho' - \frac{1}{2\rho_0} \frac{\partial \rho_0}{\partial s} \rho' - \frac{1}{2c_0} \frac{\partial c_0}{\partial s} \rho' = \frac{\beta}{2\rho_0 c_0^3} \frac{\partial \rho'}{\partial t} + \frac{(b + R)}{2\rho_0 c_0^3} \frac{\partial \rho'}{\partial t^2}.
$$
Figure 6.5: Change in rise time for various shock amplitudes and relative humidities. The title gives the shock pressure and initial relative humidity. The shock pressure observed on the ground would be double what is indicated. All atmospheres are at 20 °C and 1 atmosphere.
When the transformations introduced in Chapter Three are used (Eq. 3.15 removes the effect of spreading and impedance variation through a new "pressure" variable $q$, and Eq. 3.17 removes the effective coefficient of nonlinearity through a new range variable $\tilde{x}$), one obtains
\begin{equation}
\frac{\partial q}{\partial \tilde{x}} + \frac{\beta}{2\rho_0 c_0^3} \frac{\partial q^2}{\partial \tilde{t}} = \sqrt{\frac{S \rho_0 c_0^5}{S \rho_0 c_0^3}} \frac{(b + \mathcal{R})}{2\rho_0 c_0^3} \frac{\partial^2 q}{\partial \tilde{t}^2}. \tag{6.4}
\end{equation}

In the rest of this section relaxation is ignored for analytical simplicity although it could have been included in the analysis. The problem now appears to be that of plane wave propagation in a medium where the effective coefficient of absorption is

$$b_{\text{eff}} = b \sqrt{\frac{S \rho_0 c_0^3}{S \rho_0 c_0^5}},$$

which increases with range. For spherical spreading the effective coefficient of absorption is $b_{\text{eff}} = b r / r_0$, that is, it increases as $r$. For cylindrical spreading $b_{\text{eff}} \approx b \sqrt{r / r_0}$.

In this section we consider only the effect of spreading on the shock rise time. Although the variation in sound speed and density is neglected, the effect of their variation on shock rise time can be inferred from the results for spreading.

The classical Burgers equation has a steady-state solution where the opposing effects of steepening due to nonlinearity and diffusion due to thermoviscous absorption exactly balance. The solution is (see Appendix E)

$$p' = p_0 \tanh(t' p_0 \beta / b),$$

where the shock amplitude is $2p_0$. The 10% to 90% rise time $t_{\text{rt}}$ for the shock is

$$t_{\text{rt}} = \ln(9) \frac{2b}{\beta p_0}. \tag{6.5}$$

The question is what happens to the rise time when the effective absorption varies with range? Does a step shock immediately adjust to the variation so that nonlinearity and absorption are always balance? An increase of absorption should diffuse the shock and thus increase the rise time.* For steady state to be maintained, the diffusion of

*Another viewpoint is that the amplitude of the shock (nonlinear steepening) is reduced due to spreading. Because nonlinear steepening is weakened, absorption can diffuse the shock and the rise time increases.
the shock will have to occur immediately in response to the spreading. Naugol'nykh (1973) argued that absorption cannot act fast enough and a spreading shock in a thermoviscous medium should have a rise time that is shorter than the steady-state value.

If a spreading shock were indeed to remain in steady state then, from Eq. 6.5, the rise time would vary with the effective absorption. Since for spherically spreading waves the absorption increases with distance, one would expect

$$t_{rt} \propto r,$$

and for a cylindrically spreading wave

$$t_{rt} \propto \sqrt{r}.$$

The same results are apparent if one assumes \( b \) is constant but the shock amplitude \( p_0 \) decreases with range.

THOR is used to investigate the validity of these relations, between range and rise time, for spreading step shocks. The initial waveform is the steady-state hyperbolic tangent profile appropriate at the source condition. The shock is then propagated as a spreading wave, Figure 6.6(a) shows the initial waveform and the profile at three subsequent ranges. Note, the pressure has been scaled to remove the effect of the spreading and ease comparison. As the absorption increases with range (or the nonlinear steepening decreases) the shock does indeed diffuse. However, we see in Fig 6.6(b), at a range of \( r = 20r_0 \), that the absorption has not been able to diffuse the shock sufficiently for steady state to be maintained. At this range the steady-state shock has a rise time that is about 50% longer than the actual shock.

Figure 6.7 compares the steady-state prediction of the rise time to the numerically calculated rise time as a function of propagation distance. In the upper plots observe that for cylindrical spreading, absorption can almost keep up with the spreading but quickly falls behind for spherical spreading. In the lower plots the initial amplitude is increased by four. In this case absorption is four times weaker and cannot even keep up when the spreading is cylindrical. Note that the steady-state prediction always overestimates the rise time. Absorption cannot act quickly enough to diffuse the
profile before more amplitude decrease, due to spreading, occurs. These tests confirm Naugol'nykh's hypothesis.

An interesting result is the farfield behavior of the step shocks. At very large ranges the spreading wave becomes more planar in nature. One might expect that this would give absorption the opportunity to catch up. However, the results indicate that absorption keeps falling further and further behind. An explanation for this is that the length scale on which absorption acts increases as the shock rise time increases. If the shock rise time is characterized by a frequency component with a period $2\tau_t$, i.e., $f_t = 1/(2\tau_t)$, then the characteristic absorption length for the shock is $1/(A_\sigma 4\pi^2 f_t^2 \tau_t^3)$. The absorption length increases as rise time squared. As the shock disperses it takes absorption longer and longer to cause further changes in the waveform. Therefore, absorption never catches up with the effect of spreading.

We choose parameters typical for a sonic boom to demonstrate the significance of the effect of spreading on the rise time of sonic booms in the atmosphere. The propagation of a cylindrically spreading shock wave in a uniform atmosphere with only thermoviscous losses is examined.* Note from Eq. 6.4 that the increasing density

---

*This is a little simplistic because in the atmosphere there are relaxation processes present. Fur-
Figure 6.7: The rise time of a step shock in a thermoviscous medium. The wave starts off in steady state and is propagated as either a spherically or cylindrically spreading wave. Two different initial amplitudes are used. Rise time is normalized to the initial rise time and distance is normalized to the source radius.
and sound speed in the atmosphere should exacerbate the effect due to spreading. A source radius of $r_0 = 100$ m is used and a shock with an initial pressure jump of 200 Pa. The rise time is chosen to be slightly longer than the steady-state value; absorption is given a head start. Figure 6.8 shows a few representative profiles, in terms of $p'$ rather than $q$, and the rise time predicted by THOR compared to the expected steady-state rise time. Quickly the steady-state assumption slips behind and overestimates the rise time.

![Figure 6.8: The rise time of a cylindrically spreading, sonic boom like, step shock in a thermoviscous medium. The top plot shows the profiles at various ranges and the lower plot the rise time predicted by THOR and the steady-state model.](image)

One can expect then for sonic booms a reasonable distance away from the aircraft that the rise time is shorter than the steady-state prediction. It follows that the Kang-Pierce model, by neglecting spreading and impedance variation, thermore, the absorption varies with altitude. However, we wish only to demonstrate the effect of spreading here. A realistic atmosphere is used in Sec. 6.4.4.
overestimates rise time.

6.4.3 Effect of Signature on Sonic Boom Rise Time

Kang and Pierce assumed that a sonic boom shock is a step shock. In this section the effect of modeling a sonic boom shock as a step shock is examined. We model the sonic boom as an N wave. THOR is used to propagate an N wave through a number of different atmospheres, and the rise time of the N wave is compared to the rise time from the Kang-Pierce model for the same overpressure. In Fig. 6.9 the profile of the lead shock of an N wave and the equivalent step shock is shown in a medium with 50% relative humidity. Note that the step shock overestimates rise time. This illustrates the fact that N wave shock retains some memory of the fact that it was a higher amplitude shock (with a shorter rise time) in its past history.

In Fig. 6.10 the development of the rise time of a plane N wave as it propagates is compared to the rise time of what would be the local steady-state shock. This was done for a number of different atmospheres as indicated on the plot. In each case the N wave started with a duration of 81.9 ms and peak pressure of 400 Pa. In an attempt to give absorption a head start the initial rise time was chosen to be very large. Depending on the humidity the Kang-Pierce model overestimates the rise time by 10% to 100%.

6.4.4 Sonic Boom Propagation in Real Atmospheres

In this section an attempt is made to show the combined effect of variation in absorption, spreading, and waveform shape on sonic boom rise time. That is, we demonstrate the importance of path history for sonic boom propagation in realistic atmospheres. The ISO 9613 atmosphere and the Mojave desert atmosphere are used as model atmospheres. The ISO atmosphere is an average atmosphere over a year for mid-latitudes. The Mojave desert data is the atmosphere measured at one location at 10:30 AM on August 5, 1987. The initial waveform for all of the sonic booms is an N wave. The initial amplitude and duration were varied.

In Fig. 6.11 shows a ground waveform predicted by THOR for the Mojave desert. The upper plot shows the complete ground signature. The lower plot compares the
Figure 6.9: The profile of a plane N wave propagating in a medium of 50% relative humidity compared to the steady state step shock.

Figure 6.10: The rise time of a plane N wave (\(p_0 = 400\) Pa, \(2T_{h0} = 81.9\) ms) as it propagates compared to the rise time of a steady state step shock.
Figure 6.11: Sonic boom propagation in the Mojave desert atmosphere. The upper plot shows the complete ground signature predicted by THOR (aircraft at Mach 2 and altitude 17 km; pressure doubling due to the ground reflection has been applied). The lower plot compares the profile predicted by THOR to the profile predicted by the Kang-Pierce model.
shock profile predicted by THOR to the profile predicted by the Kang-Pierce model. The initial waveform was an N wave of duration 81.9 ms and peak overpressure 300 Pa. The steady-state rise time is about three times longer than that predicted by THOR, a value that is consistent with the results from the previous sections. In particular recall from Fig. 6.2 that in the lower atmosphere the absorption increases as the sonic boom approaches the ground. The shocks seem to remember the effect of the lower absorption they encountered in their immediate path history. The memory of the lower absorption, combined with the effect of spreading and waveform shape, leads to a rise time shorter than the steady-state rise time.

Figure 6.12 shows a similar situation except the sonic boom is propagated through ISO 9613 atmosphere. In this case the Kang-Pierce model overestimates the rise time by only 20%. Recall from Fig. 5.5 that in the ISO 9613 atmosphere the absorption near the ground is nearly constant. One would expect that it is spreading, wave shape and change in acoustic impedance that makes the Kang-Pierce model slightly overestimate rise time in this case.

In an attempt to allow the waveform to get close to steady state, THOR propagated a sonic boom from an aircraft flying at Mach 1.3. Because of the lower Mach number and refraction, the rays travel a much longer distance to the ground. Therefore they have more time to reach steady state. Figure 6.13 compares the profiles for the Mojave desert and Fig. 6.14 for the ISO atmosphere.

The agreement for the ISO atmosphere is very good. However, for the Mojave desert the Kang-Pierce model still overestimates the rise time. There are two apparent reasons why the Kang-Pierce model does not do so well in the Mojave desert. First, absorption increases as the boom approaches the ground. Second, the relative humidity is much lower and Fig. 6.5 shows that shocks take much longer to react to changes in low humidities. However, in the ISO atmosphere absorption is nearly constant as the sonic boom approaches the ground so the shock does not need to change very much as it propagates.* Also, the humidity is much higher and the shock responds quickly

---

*Ideally absorption should be slightly decreasing to offset the effects of waveform shape and spreading.
Figure 6.12: Sonic boom propagation in the ISO 9613-1 atmosphere. The upper plot shows the complete ground signature predicted by THOR (aircraft at Mach 2 and altitude 14 km; pressure doubling due to the ground reflection has been applied). The lower plot compares the profile predicted by THOR to the profile predicted by the Kang-Pierce model.
Figure 6.13: Sonic boom profiles at the ground, predicted by THOR and the Kang-Pierce model for the Mojave desert atmosphere (pressure doubling due to the ground reflection has been applied). THOR assumed a sonic boom with initial duration 82 ms and initial overpressure 300 Pa generated by an aircraft flying at Mach 1.3 and 14 km altitude.

Figure 6.14: Sonic boom profiles at the ground, predicted by THOR and the Kang-Pierce model for the ISO 9613-1 atmosphere (pressure doubling due to the ground reflection has been applied). THOR assumed a sonic boom with initial duration 82 ms and initial overpressure 400 Pa generated by an aircraft flying at Mach 1.3 and 17 km altitude.
to the small variations that do occur. It is for these two reasons that the Kang-Pierce model does a better job of predicting the rise time in the ISO9613-1 atmosphere.

If the factor of 2–3 overprediction of the rise time is representative of the Kang-Pierce prediction for the Mojave desert data, this would explain why the Kang-Pierce model goes through the middle of the data in Fig. 6.1. If the curve of Kang and Pierce is shifted down by a factor of two or three it would become a lower bound on the data, except for the odd outlier. In other words, almost all of the measured points would have a rise time greater than predicted by the no-turbulence model. This would support our present understanding of the effect of turbulence on sonic booms.

6.5 Effect of Fluctuations in Atmospheric Data

In this section an attempt is made to show the effect of measurement errors of atmospheric data on the predictions of the ground waveform. This was initiated partly because of the sensitivity of the shock locations in Operation Just 'Cause. In addition, we did this work because a field exercise to measure sonic booms was scheduled at Dryden A.F.B. for sometime in 1995. We wished to indicate to NASA how accurately atmospheric data should be acquired.

The ISO standard gives the temperature, pressure, and humidity at the ground, 500 m, 1000 m, and every kilometer after that. To simulate measurement errors, we perturb each of these quantities at each altitude by a small amount. The normal distribution random number generator of MATLAB is used to generate the perturbations. A number of realizations are generated and then identical source N waves are propagated to the ground through each atmosphere. Unfortunately the effect of the fluctuations on ray paths and ray tube areas cannot be evaluated with the tools developed in this research. The ray paths and ray tube areas are calculated using the undisturbed bilinear profile. The source condition is for an aircraft flying at 600 m/s, approximately Mach 2.0, at an altitude of 14 km. The source signal is an N wave of duration 245.8 ms and peak pressure 400 Pa.

Figure 6.15 shows the shock profiles for the various realizations of the atmosphere with a 5% variation in the ambient pressure only. The percent variation in shock arrival
time is surprisingly small, about 0.2%. The percent variation in the shock overpressures and rise times is somewhat larger, about 0.7%.

Figure 6.15: The predicted profiles of sonic boom shocks propagated through the ISO atmosphere with 5% fluctuations in the ambient pressure only. The solid line is the profile for the undisturbed atmosphere. Pressure doubling has been applied to simulate the ground reflection.

Figure 6.16 shows variation in shock profile caused by a 5% and a 10% fluctuation in temperature. Again the arrival times are very similar. The variation in shock overpressure is similar to that due to pressure fluctuation. However, compared with the pressure test, the rise time varies more for the 5% temperature fluctuation (about 28%) and even more for the 10% fluctuation.

Figure 6.17 shows the variation in shock profile due to 5% variations in temperature, pressure and relative humidity. Once again the arrival times are very similar. The variation in shock overpressure is similar to the previous two cases. Variation in the rise times is larger, about 33%. The principal reason for this is the sensitivity of
absorption to the humidity.

The results in this section are qualitative only. They show that the prediction of shock overpressure and in particular rise time varies due to the fluctuations in the atmospheric data. It is not possible to draw quantitative conclusions without including the effect of the fluctuations on the ray paths and ray tube areas. The inclusion of such effects is beyond the scope of this work. These results indicate that the measurement error must be less than 5% for accurate predictions to be possible.
Figure 6.17: The predicted profiles of sonic boom shocks propagated through the ISO atmosphere with 5% fluctuations in the temperature, pressure and relative humidity. The solid line is the profile for the undisturbed atmosphere. Pressure doubling has been applied to simulate the ground reflection.
Chapter 7

Conclusions

7.1 Summary

The two questions posed at the beginning of this work have been answered. (1) In general, waveform freezing does not occur to finite-amplitude waves in the atmosphere (source altitudes less than 20 km). However, for sonic booms the distortion is such that the ground waveform is very close to its frozen state. Sonic booms can be considered to be frozen at the ground. (2) Sonic boom shocks are not in steady state at the ground. The variation of both the nonlinear steepening and absorption with altitude means that they do not balance.

In the course of answering the questions, a very general form of the Burgers equation has been derived from the basic conservation equations. This equation is appropriate for the propagation of sonic booms through a still, stratified atmosphere. The effects of nonlinear distortion, thermoviscous absorption, multiple relaxation processes, geometrical spreading, and stratification of the medium are all accounted for in a systematic manner.

Waveform freezing was investigated by dropping the absorption terms in the Burgers equation and using an analytical solution. We demonstrated how spreading and stratification can slow down nonlinear distortion. In extreme cases the cumulative amount of nonlinear distortion is finite—the waveform freezes. The physical interpretation is that finite-amplitude waves appear to travel in a medium with an effective coefficient of nonlinearity $\beta_{\text{eff}}$ that is range dependent. Waveform freezing occurs when $\beta_{\text{eff}}$ vanishes with propagation distance in such a way that $\int_0^\infty \beta_{\text{eff}} \, ds < \infty$. For the aircraft altitudes under consideration, the analysis using $\beta_{\text{eff}}$ implies that sonic booms at the ground are “chilled” rather than frozen. However, when the duration of an N wave is used to determine if freezing occurs we find for aircraft at 18 km sonic booms are indeed frozen at the ground.
A new time domain code, THOR, is developed to solve the Burgers equation (including the absorption and dispersion). Output from the code agrees very well with known analytical solutions. In NASA exercise Just ’Cause, set up to compare sonic boom codes, THOR was found to agree well with the two other participating codes. Both of the other codes, SHOCKN and ZEPHYRUS, apply absorption in the frequency domain. THOR was also compared to weak shock theory. Results from indicate that weak shock theory overpredicts shock amplitudes of sonic boom waveforms in the atmosphere.

THOR was used to investigate the Kang-Pierce claim that at the ground sonic boom shocks are in steady state. A parametric study was done on the effect of a change in relative humidity on the rise time of sonic boom shocks. The results indicate that a shock remembers the relative humidity from the last 5 km or so. It was also demonstrated that geometrical spreading prevents steady state conditions from being attained by the shock. A third effect that prevents the attainment of steady state conditions is the loss of amplitude because a sonic boom is an N wave rather than a step shock. Simulations of sonic boom propagation through realistic atmospheres demonstrated that the steady-state assumption tends to overestimate sonic boom rise time. The path history of a sonic boom must be taken into account to make an accurate prediction of rise time. For the case of the Mojave desert atmosphere the steady-state assumption overestimated the rise time by a factor of 2–3. It appears that the rise time predicted by a no-turbulence model, which includes path history, is a lower bound to the measured data.

A variant on the algorithm used by THOR was presented. The new algorithm implements all the effects included in THOR on a nonuniform time grid. The time grid can be dynamically altered to match waveform distortion. The preliminary results appear promising. The concept has the potential for significant computational savings. A robust scheme for generating the time grid has been elusive thus far.

7.2 Future Work

An important effect in sonic boom propagation that is not accounted for in THOR is the presence of wind. Robinson (1991) showed that the jet stream, in particular, can
have a significant effect on sonic boom propagation. The jet stream can have velocities of 100 m/s or more. To include wind in the conservation equations the particle velocity becomes $u = w + u'$, where $w$ is the steady state wind and $u'$ the acoustic particle velocity. The continuity equation, Eq. 2.4, becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 w) + \nabla \cdot (\rho' w) + \nabla \cdot (\rho_0 u') + \nabla \cdot (\rho'u') = 0.$$

In the absence of an acoustic wave $\nabla \cdot (\rho_0 w) = 0$. The new term introduced in the continuity equation is $\nabla \cdot (\rho' w)$. Because the wind is not necessarily small compared to the small-signal sound speed, this term can be first order. Similar terms enter the other conservation equations. Previous workers (see, for example, Robinson 1991) have derived a transport equation for finite-amplitude propagation in the atmosphere. An added complication is that ray tracing problem becomes much more involved. It appears possible to include winds in THOR.

The algorithm THOR uses comes from a code that solves the KZK equation. The relaxation algorithm has been successfully incorporated into the KZK code (Cleveland et al. 1995). The KZK code can now model nonlinearity, thermoviscous absorption, absorption and dispersion due to multiple relaxation processes, and diffraction in the parabolic approximation. It is plausible that the KZK code could be used to simulate the propagation of sonic booms through the turbulent boundary layer. A lot of recent work on the propagation of small-signal acoustic waves through turbulence is based on a parabolic type wave equation with a fluctuating small signal sound speed to model the turbulence (Gilbert et al. 1990). The KZK code could be used to solve similar problem but including the extra essential ingredient of nonlinear distortion. A single realization of a turbulent field can be easily constructed with a suitable generation scheme (Karweit et al. 1991).

THOR is not restricted to the problem of sonic boom propagation. There are a number of other problems where the Burgers equation is a good propagation model, for example, underwater blast waves in the ocean. There are some problems in medical acoustics where the Burgers equation is valid. However, for many problem in this fields diffraction is important and the KZK equation is a better model. In addition,
tissue has a very complicated absorption and dispersion behavior. Tissue appears to have a continuum of relaxation processes. We can only approximate the frequency dependence by modeling it with a finite number of relaxation processes.

Finally, the development of a successful grid generation scheme for the nonuniform time code will offer great reward. The number of points required to describe a given waveform should be significantly less than required by the present uniform sampling codes. The computational time should be reduced dramatically because of this.
Appendix A

Turbulence Experiment

Previous work with a model experiment for the propagation of sonic booms through turbulence has been quite successful (Lipkens 1993, Lipkens and Blackstock 1995). However, the experiments could not follow the development of a single N wave as it propagated through turbulence. As part of this research an attempt was made to measure the propagation of N waves within a turbulent field in the laboratory. A turbulent field was to be generated over a flat plate as shown in Fig A.1. A line of microphones were to be flush-fitted into the plate, which would allow measurement of an N wave as it progressed through the turbulent field.

![Diagram of turbulence experiment](image)

Figure A.1: Proposed set up for turbulence experiment.

A spark generated N wave has a rise time of about 1 μs. To measure the N waves we build condensor microphones with a bandwidth greater than 1 MHz (Wright 1971, Anderson 1974). Previous work has always been for N waves that impinge the microphone at normal incidence, and the microphones have a large active area—about 2 mm in diameter. In the proposed experiment, the N waves would be incident at a grazing angle; therefore, the microphones need very small active areas—less than the
shortest length scale of the N wave. The shortest scale in an N wave is the rise time, which corresponds to a length scale of about 0.3 mm. Therefore it is necessary for the diameter of the microphone to be smaller than 0.3 mm for the rise time of the N wave to be properly characterized. We attempted to build microphones with the same technique used previously, but with a much smaller active area. This enterprise was quite unsuccessful.

Figure A.2 shows the response of a microphone with a diameter of 0.64 mm to an N wave for both normal and grazing incidence. The microphone is unacceptable because the N wave profile is smeared out at grazing incidence.

A number of microphones with diameters of about 0.35 mm were built. Figure A.3 shows the response of the best microphone, with a diameter of 0.38 mm. The output from the microphone is shown for grazing N waves incident from various directions. The response is very directional. The directionality is described by marking the points of the compass on each microphone. The best response, from the north, had a rise time of 1.3 $\mu$s. The response from the south is N wave like but has pronounced ringing. N waves incident from the east and west produced a response with little resemblance to an N wave.
Figure A.3: Response of a microphone with a diameter of 0.38 mm. Response is shown for oblique incident N waves from four different directions.

The smallest microphone which worked had a diameter of 0.30 mm but it was very insensitive and did not produce a good waveform. In general the response of the microphones with small active areas contained a lot of ringing. It appears that this is because the microphone construction requires a uniform distribution of pits in the surface of the active area. For the small areas required here it does not appear possible to generate such a distribution. Signal processing was applied to filter the response of the received signals and remove the ringing. Unfortunately, the ringing frequency of a given microphone would vary from day to day. Each of the microphones in the array would need to be calibrated frequently to ensure the filtering was accurate. The experiment was abandoned due to the problem with microphone construction.
Appendix B

Ray Theory

It is common to model the atmosphere as a moving, stratified medium. Because the stratification of the atmosphere occurs on a large length scale with respect to the characteristic length of a sonic boom, geometrical acoustics can be used. Much work has been done to predict ray paths and ray tube areas in a moving, stratified medium (see, for example, Robinson 1991, Foreman 1983, Uginčiūs 1970, 1965, Blokhintzev 1946a, 1946b).

In this section a number of simplifying assumptions are made. First, wind is ignored, because it is stratification of the ambient properties, sound speed, density, and dissipation coefficients, that is of interest. Although THOR is presently not capable of dealing with winds, they could be incorporated into the fluid dynamics equations. Second, it is assumed that the aircraft is in steady, level flight. For a supersonic aircraft that is doing manoeuvres, e.g., turning, accelerating, or climbing, there is the added complication of multiple paths to certain receivers. The multiple paths can create super booms (Maglieri 1966) which need to be taken into account to properly predict sonic boom signatures on the ground. Finally, it is assumed that the sound speed profile can be constructed from linear sections. THOR is capable of handling arbitrary ray tube areas and if necessary other codes could be used to predict ray paths and ray tube areas and their results used by THOR.

The ray theory presented here follows Blackstock (1996, Chap. 8-C), a similar approach is presented by Kinser, Frey et al. (1982, Chap. 15.4). The derivation of an expression for the ray tube area is given in Appendix C.

B.1 Ray Paths in the Atmosphere

Rays in an isothermal medium travel on straight lines because the sound speed does not vary. A ray is characterized by its initial location and grazing angle $\theta_0$ with respect to a
horizontal plane, see Fig. B.1. If a ray travels a
distance $s$, then the vertical change in altitude is

$$\Delta z = s \sin \theta_0. \quad (B.1)$$

The convention used here is for a downward
propagating ray $\theta_0 < 0$, and for an upward prop­
agating ray $\theta_0 > 0$. The horizontal range is
given by

$$\Delta r = s \cos \theta_0. \quad (B.2)$$

Rays in a medium with a linear sound speed profile travel circular paths, see
Fig. B.2. The small-signal sound speed is expressed as

$$c_0(z) = c_0(0) + g z, \quad (B.3)$$

where $g = \frac{dc_0}{dz}$ is the gradient of the sound speed (not the acceleration due to gravity)
and $z$ is the altitude. A ray having an initial grazing angle $\theta_0$ travels on a circle of
radius

$$R_c = -\frac{\overline{c_0}}{g \cos \theta_0}, \quad (B.4)$$

where $\overline{c_0}$ is the sound speed at the launch altitude. Recall for a downward propagating
ray $\theta_0 < 0$ and for an upward propagating ray $\theta_0 > 0$. For a downward propa­
gating in the atmosphere $g < 0$ one finds $R_c > 0$. As a ray propagates, the grazing
angle $\theta$ varies according to Snell’s law

$$\frac{\overline{c_0}}{\cos \theta_0} = \frac{c_0}{\cos \theta}. \quad (B.5)$$

The horizontal range traversed by a ray having
an initial grazing angle $\theta_0$ and a final grazing
angle $\theta$ is

$$\Delta r = R_c(\sin \theta - \sin \theta_0), \quad (B.6)$$

the change in altitude is

$$\Delta z = R_c(\cos \theta_0 - \cos \theta). \quad (B.7)$$
The travel time along the path is

\[ t = \int_0^s \frac{1}{c_0} \, ds. \]

We identify \( ds = R_0 \, d\theta' = -d\theta' \frac{\cos \theta_0}{(g \cos \theta_0)} \) and \( c_0 = \frac{\cos \theta}{\cos \theta_0} \). The travel time becomes

\[ t = -\int_{\theta_0}^{\theta} \frac{1}{g \cos \theta'} \, d\theta', \]

\[ = \frac{1}{g} \left[ \ln \tan \left( \frac{\theta_0}{2} + \frac{\pi}{4} \right) - \ln \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) \right]. \quad (B.8) \]

### B.2 Ray Paths from a Sonic Cone

The hydrodynamic field close to a supersonic aircraft is very involved (see, for example, Whitham 1952, 1956, Hayes et al. 1969). However, at distances further than a few body lengths from the aircraft, the acoustical wavefront can be well modelled by a cone (the sonic cone). A simple geometric argument can be used to obtain the angle of the cone. Assume that disturbances, created by a supersonic aircraft, travel as spherically spreading wavefronts at speed \( c_0 \). Figure B.3 shows a number of equally spaced wavefronts created by an aircraft in supersonic flight. The sonic cone generated by the aircraft is apparent. The angle of the cone is given by \( \psi = \arcsin(1/M) \), where \( M \), the Mach number, is the speed of the aircraft relative to the local sound speed (not the sound speed on the ground).

For acoustic propagation purposes, the structure of the near field can be ignored. Initial sonic boom waveforms are normally supplied a few body lengths away from the aircraft, that is, beyond the near field. It is assumed that the rays travel in straight lines from the aircraft to the source point.
Figure B.4: The description of rays in terms of the heading $\nu$ and grazing angle $\theta_0$.

Each ray leaving the sonic cone can be uniquely characterized by the Mach number of the aircraft and the azimuthal angle $\phi$. It is more convenient, however, to identify each ray by its initial grazing angle $\theta_0$ and its heading relative to the aircraft heading $\nu$. In a quiet medium, the heading $\nu$ does not change along the ray path. The grazing angle changes according to Snell’s Law. Figure B.4 shows the orientation of the angles. The relationship of the Mach cone angle and azimuthal angle, to the heading and initial grazing angle is given by:

$$
\sin \theta_0 = \cos \psi \cos \phi,
$$

$$
\sin \nu = \frac{\cos \psi \sin \phi}{\cos \theta_0}.
$$
The trajectory of the rays is calculated assuming that the initial ray location is $(x, y, z) = (0, 0, z)$. In an isothermal atmosphere, after travelling a path length $s$, the ray is at a location

\[ \begin{align*}
    x &= s \sin \nu \cos \theta_0, \\
    y &= s \cos \nu \cos \theta_0, \\
    z &= z + s \sin \theta_0.
\end{align*} \]

In a linear sound speed profile the grazing angle at path length $s$ is

\[ \theta = \theta_0 - s/R_0. \]

The location of the ray is given by

\[ \begin{align*}
    x &= \sin \nu R_0 (\sin \theta - \sin \theta_0), \\
    y &= \cos \nu R_0 (\sin \theta - \sin \theta_0), \\
    z &= z + R_0 (\cos \theta - \cos \theta_0).
\end{align*} \]

The model for the atmosphere used in this dissertation is taken from ISO 9613-1 (1993). This ISO atmosphere is very close to the U. S. Standard Atmosphere (1962). The variation of the small-signal sound speed with altitude is shown in Fig. B.5. The sound speed at the ground is 340.26 m/s ($T = 288.15$ K). The sound speed decreases linearly* with altitude to 295.04 m/s ($T = 216.65$ K) at an altitude of 11 km. The sound speed gradient is $g = -4.1109$ m/s/km. For altitudes in the range 11 km to 20 km the atmosphere is close to isothermal and $c_0 = 295.04$ m/s.

\[ \begin{align*}
    \frac{\partial c}{\partial z} &= -4.1109, \\
    c_0 &= 295.04.
\end{align*} \]

*Actually the temperature decreases linearly and the sound speed goes as $\sqrt{T}$. However, the variation in temperature is so small that the sound speed decrease is very nearly linear too.
Because the small-signal sound speed is highest at the ground, the rays from a supersonic aircraft tend to bend away from the ground and back up into the atmosphere. For typical cruising altitudes, the direct ray $\phi = 0$ always reaches the ground, except for very low Mach numbers $M < 1.15$. However, rays at higher azimuthal angles $\phi$ (shallower launch angles $\theta_0$) may not reach the ground. Figure B.6 shows the rays from an aircraft flying at Mach 2 and altitude 17 km for various azimuthal angles. For angles $\phi > 55^\circ$, the rays never reach the ground. The sonic boom is heard only in an 85 km wide strip under the aircraft—the sonic boom carpet. The primary sonic boom carpet becomes very complicated when the aircraft is undergoing manoeuvres (Maglieri 1966). A secondary sonic boom carpet exists hundreds of kilometers from the aircraft (Pierce 1993b) because initially upward travelling rays are refracted downwards by the upper atmosphere.

Figure B.6: Rays from an aircraft flying straight out of the page at speed Mach 2 and altitude 17 km. Rays are drawn for various azimuthal angles: at $\phi > 55^\circ$ the ray just grazes the ground at $\Delta r = 42$ km. For angles $\phi > 55^\circ$ the rays never reach the ground.
Appendix C

Ray Tube Area in a Linear Sound Speed Profile

In this Appendix the expression for the ray tube area for sonic booms in a linear sound speed profile is derived. We generalize the equations for the ray paths presented in Appendix B to include the effect of the source radius:

\[ s = R_c(\theta - \theta_0), \]
\[ x = M c_0 t_a + \cos \nu R_c(\sin \theta - \sin \theta_0), \]
\[ y = r_s \sin \phi + \cos \nu R_c(\sin \theta - \sin \theta_0), \]
\[ z = z_a - r_s \cos \phi + R_c(\cos \theta_0 - \cos \theta). \]  

To calculate the ray tube area, rays are characterized by the time \( t_a \) they leave the aircraft and \( \phi \) the azimuthal angle. Figure C.1 shows the ray tube area enclosed by the rays launched at \( t_a, t_a + \Delta t_a, \phi, t_a + \Delta t_a, \phi + \Delta \phi \) and \( t_a + \Delta t_a, \phi + \Delta \phi \). Rays that are launched at times \( t_a \) and \( t_a + \Delta t_a \), follow exactly the same trajectory but are separated by a distance \( \Delta t_a M c_0 \). Rays launched at angles \( \phi \) and \( \phi + \Delta \phi \) travel on different trajectories. The ray tube area is commonly defined as the area, normal to the wavefront, enclosed by the four rays (Hayes et al. 1969). The ray tube area in the horizontal plane is given by

\[ r_{ta_h} = \frac{\partial x}{\partial t_a} \Delta t_a \frac{\partial y}{\partial \phi} \Delta \phi. \]

The ray tube area normal to the ray is

\[ r_{ta} = r_{ta_h} \sin \theta. \]  

The time derivative is simply

\[ \frac{\partial x}{\partial t_a} = M c_0. \]
Figure C.1: The ray tube area is the area enclosed by the four rays shown.

The derivatives with respect to $\phi$ are somewhat more involved. 

\[
\frac{\partial x}{\partial \phi} = \frac{\partial \cos \nu}{\partial \phi} R_c (\sin \theta - \sin \theta_0) + \cos \nu \frac{\partial R_c}{\partial \phi} (\sin \theta - \sin \theta_0) \\
\quad + \cos \nu R_c \left( \frac{\partial \sin \theta}{\partial \phi} - \frac{\partial \sin \theta_0}{\partial \phi} \right) ,
\]

\[
\frac{\partial y}{\partial \phi} = r_s \cos \phi + \frac{\partial \cos \nu}{\partial \phi} R_c (\sin \theta - \sin \theta_0) + \cos \nu \frac{\partial R_c}{\partial \phi} (\sin \theta - \sin \theta_0) \\
\quad + \cos \nu R_c \left( \frac{\partial \sin \theta}{\partial \phi} - \frac{\partial \sin \theta_0}{\partial \phi} \right) ,
\]

\[
\frac{\partial z}{\partial \phi} = r_s \sin \phi + \frac{\partial R_c}{\partial \phi} (\cos \theta_0 - \cos \theta) + R_c \left( \frac{\partial \cos \theta_0}{\partial \phi} - \frac{\partial \cos \theta}{\partial \phi} \right) .
\]

These equations need to be manipulated so the derivatives are removed and the equations are in terms of $\theta$ only so that they can be evaluated at a given altitude.

We make use of the identity $\sin \theta_0 = -\cos \psi \cos \phi$, to remove derivatives involving $\theta_0$,

\[
\frac{\partial \sin \theta_0}{\partial \phi} = \cos \psi \sin \phi = \cos \theta_0 \sin \nu .
\]

It follows then that

\[
\frac{\partial \theta_0}{\partial \phi} = \sin \nu .
\]

The expression for $\partial \theta / \partial \phi$ can be obtained from the expression for $\partial \theta_0 / \partial \phi$ using Snell's law. Snell's law relates $\theta$ to $\theta_0$, 

\[
\frac{\partial \cos \theta}{\partial \phi} = \frac{c_0}{c_0} \sin \theta_0 \frac{\partial \theta_0}{\partial \phi} ,
\]
\[ \frac{\partial \theta}{\partial \phi} = -\frac{\cos \theta \sin \theta_0}{\cos \theta \sin \theta} \sin \nu. \]

It follows then that
\[ \frac{\partial \theta}{\partial \phi} = -\frac{\cos \theta \sin \theta_0}{\cos \theta_0 \sin \theta} \sin \nu. \]

The radius of curvature is dependent on \( \theta_0 \), and therefore the derivative of the radius of curvature is
\[ \frac{\partial R_c}{\partial \phi} = \frac{-\cos \theta_0}{g \cos^2 \theta_0} (-\sin \theta_0) \sin \nu, \]
\[ = R_c \tan \theta_0 \sin \nu. \]

The identity \( \cos \nu = \cos \psi / \cos \theta_0 \) allows us to evaluate the derivative of the heading:
\[ \frac{\partial \cos \nu}{\partial \phi} = \cos \psi \frac{-\sin \theta_0}{\cos^2 \theta_0} \sin \nu, \]
\[ = -\tan \theta_0 \cos \nu \sin \nu. \]

We use the chain rule on the left-hand side to obtain
\[ \frac{\partial \nu}{\partial \phi} = \tan \theta_0 \sin \nu. \]

All derivatives have now been calculated and yield the following expressions
\[ \frac{\partial x}{\partial \phi} = R_c \cos \nu \sin \nu \left( 2 \tan \theta_0 \left( \sin \theta - \sin \theta_0 \right) + \cos \theta_0 \left( -1 + \frac{\cos^2 \theta \sin \theta_0}{\cos^2 \theta_0 \sin \theta} \right) \right), \tag{C.6} \]
\[ \frac{\partial y}{\partial \phi} = r_s \cos \phi + R_c \left( \tan \theta_0 \left( \sin \theta - \sin \theta_0 \right) \left( \sin^2 \nu - \cos^2 \nu \right) \right. \]
\[ + \cos \theta_0 \left( -1 + \frac{\cos^2 \theta \sin \theta_0}{\cos^2 \theta_0 \sin \theta} \right) \sin^2 \nu \right), \tag{C.7} \]
\[ \frac{\partial z}{\partial \phi} = -r_s \sin \phi. \tag{C.9} \]

The ray path and ray tube area of a ray are calculated in the following manner.

The initial data for the aircraft allows one to determine \( z_a, r_s, \) and \( \mu \). The initial data for the ray allows one to calculate \( \phi, \nu, \theta_0, \) and \( R_c \). For a given path length \( s \) the grazing angle \( \theta \) can be calculated from Eq. C.1. From this information the coordinate of the ray can be calculated from Eqs. C.2–C.4 and the ray tube area can be calculated from Eq. C.5.
Patching Isothermal and Linear Sound Speed Gradients

In the bilinear atmosphere rays travel on a straight line in the isothermal section until they reach the knee and then they travel on a circular path. In the isothermal section the spreading is simply

\[ \text{rta} = \frac{s + s_0}{s_0}, \]

where \( s_0 \) is the apparent path length from the flight path to the source location. Note that \( \text{rta}|_{s=0} = 1 \). Beyond the knee the spreading is described by the analysis above. To patch the results together the ray tube area at the knee must be continuous. From the isothermal section the ray tube area at the knee is

\[ \text{rta}_k = \frac{s_k + s_0}{s_0}, \]

where \( s_k \) is the path length to the knee.

In the linear region of the atmosphere the ray tube area is proportional to \( \frac{\partial y}{\partial \phi} \). If the source radius in Eq. C.7 is chosen to be \( r_s = s_k \) (the path length to the knee) then the expression for the ray tube area in the linear section of the sound speed profile is

\[ \text{rta} = \frac{\frac{\partial y}{\partial \phi}}{s_0 \cos \phi}. \]
Appendix D

Waveform Freezing in an Exponential Horn

An experiment that was considered as part of the attempt to observe waveform freezing was the propagation of finite amplitude waves down a horn (Blackstock 1973). The lossless form of the Burgers equation for propagation down a horn can be obtained from Eq. 2.69,

$$\frac{\partial \eta'}{\partial x} + \frac{1}{2S} \frac{\partial S}{\partial x} \eta' = \frac{\beta}{2\rho_0 c_0^2} \frac{\partial \eta'^2}{\partial x},$$

where $S$ is the cross-sectional area and $x$ is the distance along the horn.

For an exponential horn the surface area of the horn is $S(x) = S_0 e^{\alpha x}$ and the radius is $y(x) = y_0 e^{\frac{\alpha x}{2}}$. The distortion distance is

$$\tilde{x} = \int_0^s \sqrt{\frac{S}{S_0}} \, ds,$$

$$= \int_0^x e^{-\alpha x/2} \, dx,$$

$$= -\frac{2}{\alpha} \left[ e^{-\alpha x/2} \right]_0^x,$$

$$= \frac{2}{\alpha} \left[ 1 - e^{-\alpha x/2} \right].$$

Note this is similar in form to the distortion distance for an isothermal atmosphere, Eq. 3.27, where $\alpha = 1/H$. As $x \to \infty$, the distortion distance $\tilde{x} \to \frac{2}{\alpha}$ and waveform freezing occurs. The distortion distance will be within 5% of its final value when $\frac{\alpha}{2} x = 3$, i.e., $x = 6/\alpha$.

In the derivation of the Burgers equation we require that the area of the horn change slowly over a characteristic length $l_c$. This demands that

$$\left| \frac{S(x + l_c) - S(x)}{S(x)} \right| \ll 1,$$

$$\left| \frac{S_0 e^{\alpha(x+l_c)} - S_0 e^{\alpha x}}{S_0 e^{\alpha x}} \right| \ll 1,$$

$$\left| e^{\alpha l_c} - 1 \right| \ll 1.$$
If the surface area is allowed to change by 1% then this means $e^{\alpha l_e} = 1.01$, whence
\[
\alpha = \frac{\ln(1.01)}{l_e}.
\]

An N wave generated by a spark in air has a typical length of 14 mm. Based on a characteristic length scale $l_e = 14$ mm the flare constant is
\[
\alpha = 0.725 \text{ m}^{-1}.
\] (D.1)

If waveform freezing is going to be observed then $\alpha L$ must be at least 6 at the far end of the horn. For this to be true the length of the horn must be at least $L = 8.3$ m.

The fluid dynamics equations used to derive the Burgers equation also require that the slope or flare of the horn be small. The flare is given by
\[
\frac{dy}{dx} = \frac{\alpha}{2y_0} e^{\frac{a_f}{2}}.
\] (D.2)

The greatest flare occurs at the mouth. For the values above we find
\[
\left. \frac{dy}{dx} \right|_{x=L} = 7.28 y_0.
\]

If we choose a maximum flare of 0.01, the throat radius has to be
\[
y_0 = 1.4 \text{ mm}.
\]

The throat diameter of the horn is 3 mm. At the mouth ($x = 8.3$ m) the diameter will be about 60 mm.

These requirements are quite challenging but by easing the restrictions a somewhat more practical horn can be realised. For example, using a 10% criterion for the area change and slope yields: $\alpha = 6.81 \text{ m}^{-1}$, $L = 0.88$ m, and $y_0 = 1.47$ mm.

One could enhance waveform freezing by heating the horn in the same way as the proposed box in Chapter Three. The horn must point upwards to maintain a temperature gradient. We assume the ambient pressure within the horn is constant and air is an ideal gas. The distortion distance for a heated horn is
\[
\tilde{x} = \int_0^s \sqrt{\frac{S \rho_0 c_0^5}{S \rho_0 c_0^5}} ds,
\]
There does not appear to be a closed form solution of this integral.

The results of evaluating the distortion distance numerically are shown in Fig. D.1. It seems plausible that waveform freezing could be observed in a horn. It also appears that a large temperature gradient will be needed to make an noticeable change to the onset of waveform freezing.

\[
\int_0^x e^{-\frac{3}{2}x} \sqrt{\frac{c_0^3}{c_0^3}} \, dx,
\]

\[
= \int_0^x e^{-\frac{3}{2}x} \sqrt{\frac{\sqrt{\gamma R T_0^3}}{\sqrt{\gamma R (T_0 + m x)^3}}} \, dx,
\]

\[
= \int_0^x e^{-\frac{3}{2}x} (1 + m x/T_0)^{-3/4} \, dx.
\]

Preliminary investigations were carried out into an exponential wedge. This has the qualitative appeal that it would yield not just the horn effect but also cylindrical spreading. The propagation of N waves in the wedge is somewhat equivalent to the
propagation of sonic booms in an isothermal atmosphere. The design of the wedge however was impractical.

In the end the horn experiment was not followed up. Partly because we did not have easy access to a horn of the right dimensions. Also, because the link between propagation down a horn and propagation in the atmosphere is tenuous to say the least. Finally, there is a question of whether acoustic propagation down a horn remains one dimensional over the distances we would require (Post 1994).
Appendix E

Stationary Solutions

Recall from Sec. 6.3 that the equation for a stationary waveform in a medium with thermoviscous attenuation and multiple relaxation processes is, Eq. 6.1,

\[- \frac{\beta}{2\rho_0 c_0^2} \frac{dp'^2}{dt'} = \frac{b}{2\rho_0 c_0^2} \frac{d^2 p'}{dt'^2} + \sum \frac{m_\nu \tau_\nu}{2c_0} \frac{d^2}{d^2 v^2} p'.\]

Analytical solutions exist for a thermoviscous medium and for a medium with one relaxation process.

E.1 Steady State Solution in a Thermoviscous Fluid

The classical Burgers equation reduces to the following form for a stationary wave

\[ \frac{d p'^2}{dt'} = -\frac{b}{\beta} \frac{d^2 p'}{dt'^2}. \]  

(E.1)

Equation E.1 can be integrated once with respect to time. A symmetric step shock of amplitude \(2p_0\) is assumed, that is, at \(t' = \pm \infty\), \(p'^2 = p_0^2\) and the time derivatives are zero. Equation E.1 becomes

\[ p'^2 = -\frac{b}{\beta} \frac{dp'}{dt'} + p_0^2. \]

This equation is separable

\[ \int_0^{p'} \frac{dp'}{p_0^2 - p'^2} = \int_{t_0}^{t'} \frac{\beta}{b} dt'. \]

The integration constant \(t_0\) is defined such that \(p'(t_0) = 0\); without loss of generality we choose \(t_0 = 0\). The left-hand side is expressed in terms of partial fractions:

\[ \frac{1}{2p_0} \int_0^{p'} \frac{1}{p_0 + p'} + \frac{1}{p_0 - p'} dp' = \frac{\beta}{b} \int_0^{t'} dt', \]

\[ \{\ln(p_0 + p') - \ln p_0 - (\ln(p_0 - p') - \ln p_0)\} = \frac{2p_0 \beta}{b} t', \]

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\[ \frac{1 + p'/p_0}{1 - p'/p_0} = e^{t'2p_0\beta/b}. \]

Simple algebraic manipulation allows us to obtain an expression for the acoustic pressure

\[ \frac{p'}{p_0} = \frac{e^{t'2p_0\beta/b} - 1}{e^{t'2p_0\beta/b} + 1}, \]
\[ p' = p_0 \tanh(t'p_0\beta/b). \quad (E.2) \]

For a unipolar shock, that is, a shock that rises from \( p' = 0 \) to \( p' = 2p_0 \), the solution is (Pierce 1981, Chap. 11-6)

\[ p' = p_0 \left[ 1 + \tanh \left( (t' - \frac{\beta p_0}{\rho_0 c_0^2}) \frac{p_0\beta}{b} \right) \right]. \]

The shock moves, in the retarded frame, with speed \( \beta p_0/(\rho_0 c_0^2) \).

**E.2 Steady State Solution in a Fluid with a Single Relaxation Process**

If the distortion term and only one relaxation process are retained, the resulting steady state equation is

\[ -\frac{dp'^2}{dt'^2} = \frac{m\nu\tau\rho_0 c_0^2}{\beta} \frac{d^2}{dt'^2} \frac{1}{1 + \tau'p'^2}, \]
\[ \left( 1 + \tau' \frac{d}{dt'} \right) \frac{dp'^2}{dt'^2} = -\frac{m\nu\tau\rho_0 c_0^2}{\beta} \frac{d^2p'}{dt'^2}. \quad (E.3) \]

Equation E.3 can be integrated once, assuming a symmetric step shock from \(-p_0\) to \( p_0\),

\[ \left( 1 + \tau' \frac{d}{dt'} \right) p'^2 = -\frac{m\nu\tau\rho_0 c_0^2}{\beta} \frac{dp'}{dt'} + p_0^2. \]

Polyakova et al. (1962) define \( D = \frac{m\nu\rho_0 c_0^2}{2\beta p_0} \) (\( D = C/\theta \) in terms of the dimensionless variables defined in Chapter Four) and one obtains

\[ p_0^2 - p'^2 = (\tau'2p' + 2Dp_0\tau) \frac{dp'}{dt'}. \]

This equation is separable

\[ \int_0^{p'} \frac{2(p' + Dp_0)dp'}{p_0^2 - p'^2} = \int_{t_0}^{t'} \frac{dt'}{\tau'}. \]
The integration constant $t_0$ is chosen so $p'(t_0) = 0$ and again $t_0$ is set to zero. If the left hand-side is expressed in terms of partial fractions one obtains:

$$
\int_0^{t'} \left( \frac{D - 1}{p_0 + p'} + \frac{D + 1}{p_0 - p'} \right) \, dp' = \frac{1}{\tau} \int_0^{t'} \, dt',
$$

$$(D - 1) \ln(1 + p'/p_0) - (D + 1) \ln(1 - p'/p_0) = t'/\tau.$$

The solution is

$$\frac{t'}{\tau} = \ln \left( \frac{1 - p'/p_0}{1 + p'/p_0} \right)^{1+D}. \quad (E.4)$$

This is the result obtained by Polyakova et al. (1963). Equation E.4 cannot be inverted to make $p'$ a function of $t'$. 
Bibliography


VITA

Robin Cleveland was born in London, England, on 18th February, 1967, the son of Roger Cleveland and Hæge Hestnes. His family emigrated to New Zealand in 1973. After completing high school at Western Heights High School in Rotorua, he entered the University of Auckland (New Zealand) in March 1986. In May 1989 he received the degree of Bachelor of Science in Physics. He remained in Auckland to earn his Master's degree under Dr. Murray Johns and Dr. Sze Tan. His research project was to design an acoustical system for estimating fish biomass in the ocean. After receiving his Master of Science degree with First Class Honours in Physics in May 1991, Robin worked with the Ministry of Agriculture and Fisheries for six months on the implementation of the fish sonar system. He entered The University of Texas at Austin in August 1991 to start his doctoral work. He is a student member of the Acoustical Society of America, and will be the F. V. Hunt Fellow of the Acoustical Society of America for 1995/96. Robin will marry Christine Anne Cotton on 20th May 1995.

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