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Perturbed polyhedra and the construction of local Euler-Maclaurin formulas

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Dissertation

PERTURBED POLYHEDRA AND THE CONSTRUCTION
OF LOCAL EULER-MACLAURIN FORMULAS

by

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ABSTRACT

A polyhedron $P$ is a subset of a rational vector space $V$ bounded by hyperplanes. If we fix a lattice in $V$, then we may consider the exponential integral and sum, two meromorphic functions on the dual vector space which serve to generalize the notion of volume of and number of lattice points contained in $P$, respectively. In 2007, Berline and Vergne constructed an Euler-Maclaurin formula that relates the exponential sum of a given polyhedron to the exponential integral of each face. This formula was "local", meaning that the coefficients in this formula had certain properties independent of the given polyhedron. In this dissertation, the author finds a new construction for this formula which is very different from that of Berline and Vergne.

We may ‘perturb’ any polyhedron by translating its bounding hyperplanes. The author defines a ring of differential operators $R(P)$ on the exponential volume of the perturbed polyhedron. This definition is inspired by methods in the theory of toric varieties, although no knowledge of toric varieties is necessary to understand the construction or the resulting Euler-Maclaurin formula. Each polyhedron corresponds to a toric variety, and there is a dictionary between combinatorial properties of the polyhedron and algebro-geometric properties of this variety. In particular, the equivariant
cohomology ring and the group of equivariant algebraic cycles on the corresponding toric variety are equal to a quotient ring and subgroup of $R(P)$, respectively. Given an inner product (or, more generally, a complement map) on $V$, there is a canonical section of the equivariant cohomology ring into the group of algebraic cycles. One can use the image under this section of a particular differential operator called the Todd class to define the Euler-Maclaurin formula. The author shows that this formula satisfies the same properties which characterize the Berline-Vergne formula.
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List of Notation

$L \cong \mathbb{Z}^n$ - a free, finitely generated abelian group

$V \cong L \otimes \mathbb{Q}$ - a rational vector space with lattice $L$

$P$ - a given (full-dimensional) polyhedron in $V$

$\text{Face}(P)$ - the set of faces of $P$

$F_1, \ldots, F_k$ – the facets (faces of codimension one) of $P$

$H_1, \ldots, H_k$ – the (minimal) set of bounding hyperplanes of $P$

$S_1, \ldots, S_k$ – the (minimal) set of half-spaces whose intersection is $P$

$\mathcal{F}_P$ – the set of face indices of $P$ (isomorphic to the set of subsets of $\{1, \ldots, k\}$)

$f$ – a given face index of $P$

$f := (\cap_{i \in f} H_i) \cap P$ – the face corresponding to $f$

$\text{Span}(f)$ – the affine span of $f$

$K_f := \cap_{i \in f} S_i$ – the tangent cone of $f$ in $P$

$W_f := \cap_{i \in f} H_i$ – the affine space corresponding to $f$

$\Lambda \cong \mathbb{Q}[v_1, \ldots, v_n]$ - the completed symmetric tensor algebra of $V$

$i(K)$ - the index of the cone $K$

$I(P)$ - the exponential volume of $P$

$S(P)$ - the exponential sum of $P$

$P_{\vec{h}}$ - the polyhedron $P$ perturbed by $\vec{h} \in \mathbb{Q}^k$

$R(P) \cong \Lambda[[D_1, \ldots, D_k]]$ - the ring of differential operators

$\mathcal{E} : R(P) \to \Lambda$ - the evaluation map

$I_P, J_P$ - two ideals in the kernel of $\mathcal{E}$

$A(P) \cong R(P)/I_P + J_P$ - the Chow ring of $P$
\[ Z(P) \cong \Lambda\{D_f; f \neq \emptyset\} \] - the cycle group of \( P \)

\[ \Psi \] - a chosen complement map on \( V \)

\[ J^\Psi(P) \] - a sub-ideal of \( J_P \) dependent on \( \Psi \)

\[ Z^\Psi(P) \cong R(P)/I_P + J^\Psi_P \] - a ring structure on the cycle group

\[ s^\Psi : A(P) \to Z(P) \] - a section of the natural (surjective) quotient map

\[ \tilde{\pi}_* : \text{Face}(P) \to \text{Face}(P') \] - pushforward of faces

\[ \pi_* : Z(P) \to Z(P') \] - pushforward of cycles

\[ S^c(P) \] - the smoothed exponential sum of \( P \)

\[ \text{Td}(P) \] - the Todd class of \( P \)

\[ \mu^\Psi(F, P) \] - the SI-interpolator function on the face \( F \) of \( P \)
Chapter 1

Introduction

Let $V$ be a rational vector space of dimension $n$, with a lattice $L$. A polytope is defined to be the convex hull of finitely many points in $V$. One can consider two distinct, but related measures of the size of any polytope: the volume, $\text{Vol}(P)$, and the number of lattice points contained in $P$, or $|P \cap L|$. The study of the relationship between these two measures dates back at least as far as Pick’s Theorem of 1899. One version of the theorem states that if $\dim(V) = 2$, and a polytope in $V$ has vertices in $L$ (i.e. is integral), then its volume is equal to the sum over each lattice point of the solid angle formed by the lattice point in $P$. Taking a small enough circle $C$ centered at the point, the solid angle is equal to $\text{Vol}(C \cap P)/\text{Vol}(C)$. The solid angle of a point in $P$ depends only on what face the point is contained in: here, by convention, faces include vertices, edges, and the polytope itself. Then, through an inductive calculation, we may assign a number $\nu(F,P)$ to each face $F$ of a polytope $P$, such that

$$\text{Vol}(P) = \sum_{F \subset P} \nu(F,P)|F \cap L|.$$ 

Furthermore, the coefficients $\nu(F,P)$ depend only on the solid angle at any point in the interior of $F$.

Pick’s Theorem itself does not generalize to higher-dimensional polytopes $P$. However, it is possible to assign coefficients $\nu(F,P)$ to each $F$ in $P$ that satisfy the same formula. These coefficients do not depend only on the solid angle of $F$ in $P$, but will instead depend on the “cone of feasible directions” (see Theorem 5.5.1), which
is still quite remarkable. These coefficients, which are examples of what we will call interpolator functions, have been constructed independently by a number of mathematicians.

Perhaps most notably, in 2005, Berline and Vergne [BV1] constructed infinitely many interpolator functions, which they called local Euler-Maclaurin formulas. (The standard Euler-Maclaurin formula was a special case in dimension one. The ‘localness’ refers to the fact that the coefficients depend only the cone of feasible directions.) To do so, they first broadened their consideration from polytopes to polyhedra. A polyhedron is the intersection of finitely many closed ‘half-spaces’, each bounded by a hyperplane in $V$. Every polytope is a polyhedron, but the class of polyhedra includes unbounded sets, such as cones. Since $\text{Vol}(P)$ and $|P \cap L|$ are undefined if $P$ is unbounded, they instead used the exponential integral and exponential sum functions, called $I(P)$ and $S(P)$ (see Section 3.1 or [La]). $I(P)$ and $S(P)$ are meromorphic functions on the dual space of $V$. When $P$ is a polytope, these functions evaluate at 0 to $\text{Vol}(P)$ and $|P \cap L|$, respectively. Thus a function that interpolates between $S(P)$ and $I(P)$ is indeed a generalization of the previous interpolator functions. They construct $\tilde{\mu}(F, P)$, itself a function on the dual space, such that

$$S(P) = \sum_{F \subset P} \tilde{\mu}(F, P) I(F),$$

for all polyhedra $P$. (Note the change in order is somewhat irrelevant, since it is possible to ‘invert’ this interpolator function so that the exponential integral $I(P)$ is equal to a combination of the exponential sums $S(F)$. See Appendix B.) They construct this by inducting on dimension: to define the interpolator function in dimension $n$, they make use of the function defined in all smaller dimensions. Their construction required the choice of an inner product on $V$; since there are in general infinitely many possible inner products on a vector space, there are infinitely many
local Euler-Maclaurin formulas. Garoufalidis and Pommersheim \[\text{GP}\] showed that the same construction extends to a vector space with a choice of complement map (see 4.2.3).

However, this was not the first example of an interpolator function: there were earlier examples originating in the study of toric varieties. Corresponding to each polytope in \(V\) is a toric variety, a special type of complex algebraic variety. Although the correspondence is not bijective, there is a well-known ‘dictionary’ via which properties of the toric variety correspond to properties of the polytope, and vice-versa. In particular, one can construct a vector space homomorphism between the rational cohomology (or Chow) ring of the toric variety and \(\mathbb{Q}\), that sends cohomology classes coming from certain canonical algebraic cycles to volumes of faces of the polytope \(P\).

As an application of the Riemann-Roch theorem, the map also takes a well-known canonical element of the Chow ring called the Todd class to \(|P \cap L|\), as long as \(P\) is a special type of polytope called unimodular. In 2003, Pommersheim and Thomas \[\text{cite}\], building on the work of Morelli \[\text{cite}\], found a cycle-level expression of the Todd class for a more general class of polytopes, and used it to construct an interpolator function (between standard volume and number of lattice points). Remarkably, this expression also required a complement map. With computational evidence, Pommersheim and Garoufalidis conjectured in 2012 that the Pommersheim-Thomas function was equal to the Berline-Vergne function evaluated at 0.

In 2014, the author, working jointly with Pommersheim, proved that conjecture by generalizing on the algebro-geometric methods of Pommersheim and Thomas. A polyhedron also corresponds to a toric variety, although for unbounded polyhedra, it is not enough to consider rational cohomology. Indeed, in some cases an unbounded polyhedron corresponds to an affine variety, where rational cohomology is trivial. Instead, one may use of the action of the torus on a toric variety to define an equivariant
cohomology ring, which is well-behaved even in the affine case. The coefficients in this cohomology correspond naturally to functions on the dual space of $V$, and there is a homomorphism as before between this ring and a field of such functions. This homomorphism takes certain equivariant cycles to the exponential integral on faces of $P$, and the (equivariant) Todd class to $S(P)$. Thus to construct an interpolator function between $S$ and $I$, it is enough to find a cycle-level expression of the Todd class. More generally, one can put an intersection product on the group of equivariant cycles, given the choice of an inner product. The resulting interpolator function is equal to that of Berline and Vergne’s, since it satisfies the same inductive properties that define their construction. It also specializes to Pommersheim and Thomas’s interpolator function, via the natural map from equivariant cohomology to rational cohomology.

In this dissertation, we present the details of this latest construction without assuming any knowledge of toric varieties; this is possible because of the remarkable properties of the exponential sum and integral functions. As has long been known, rational cohomology classes of toric varieties are closely related to partial derivatives of the volume of $P$ as one translates, or ‘perturbs’ the hyperplanes that bound the corresponding polytope. In particular, Khovanskii and Pukhlikov [KP] used this connection to prove that a differential operator closely related to the Todd class, known as the Todd operator, evaluated to the number of lattice points in the polytope. Analogously, equivariant cohomology classes correspond to differential operators on the exponential integral. Certain algebraic invariants such as the equivariant Chow ring (resp. the group of equivariant cycles) may be constructed as a quotient ring (resp. subgroup) of a completed ring of partial derivatives. There is naturally a surjective map from the Chow ring to the cycle group. The interpolator function is constructed via a section of this map, which in turn is induced by a complement map.
Although the terminology (e.g. Chow ring, cycle group, pushforward, etc.) comes from algebraic geometry, neither the construction itself nor the proofs in this dissertation make any reference to geometric facts. (See Appendix A.)
Chapter 2

Polyhedra, Faces, and Cones

Most of the elementary statements about polyhedra are left unproven. A careful development can be found in [Bv2].

2.1 Polyhedra and Faces

Let $L$ be a free abelian group of rank $n$, and let $V := L \otimes \mathbb{Z} \mathbb{Q} \cong \mathbb{Q}^n$. A (rational) half-space is the closed subset bounded on one side by a hyperplane in $V$: after choosing an isomorphism $V \cong \mathbb{Q}^n$, a half-space is the set of vectors $\{(v^1, \ldots, v^n) : \sum_{i=1}^n a_i v^i \leq b\}$ for some $a_1, \ldots, a_n, b \in \mathbb{Q}$.

A (rational) polyhedron is the intersection of finitely many (rational) half-spaces. The dimension of a polyhedron is the dimension of the smallest affine space containing it. Except where we say otherwise, we will always assume that a given polyhedron $P$ has dimension $n$. (This is not as restrictive as it sounds, since one can without loss of generality restrict $V$ to the affine span of $P$ so that $P$ becomes full-dimensional.)

If $S$ is any half-space bounded by hyperplane $H$, and $P$ is a polyhedron contained in $S$, then we say that $H \cap P$ is a face of $P$. By convention, we always include the empty set and $P$ itself as faces of $P$. Non-empty faces $F$ also have a well-defined dimension ranging from 0 to $n$: this is the dimension of the smallest affine space containing $F$. Faces of dimension 0 are called vertices, faces of dimension 1 are called edges and faces of dimension $n - 1$ are called facets. We will occasionally use Face$(P)$ and Vert$(P)$ to refer to the sets of faces and vertices of $P$, respectively.
There are a few natural and intuitive facts about faces which we will need for what follows. These facts usually have simple proofs which we will omit. For example:

- A polyhedron will have finitely many faces.
- A face of a polyhedron is itself a polyhedron.
- The intersection of faces is always a face.
- Excepting $P$ and the empty set, every face of $P$ is an intersection of facets.
- If $F$ is a face of $P$, then any face of $F$ is also a face of $P$.
- If $P$ contains an affine space of dimension $k$, then all non-empty faces of $P$ will also contain affine spaces of dimension $k$, and thus $P$ will have no faces of dimension $< k$. Conversely, if any face $F$ of $P$, with $\dim(F) = k$, contains no proper sub-faces, then $P$ contains an affine space of dimension $k$.

We refer to faces of $P$ using the following notation. If $P$ is a polyhedron with $k$ facets, let $\mathcal{F}_P$ be the set of all subsets of facets of $P$. If we choose an ordering of the facets, and label them $F_1, \ldots, F_k$, then $\mathcal{F}_P$ is in natural bijection to the set of subsets of $\{1, \ldots, k\}$. An element $f \in \mathcal{F}_P$ is called a face index. Each face index $f$ corresponds to a face $f := \cap_{i \in f} F_i$ of $P$. Note that $f$ may be empty. Also, the map $f \mapsto f$ is not necessarily injective. However, since every non-empty face of $P$ is the intersection of facets of $P$, it must have at least one corresponding face index.

By the definition of a face, any facet $F_i$ is the intersection of some bounding hyperplane $H$ with $P$. But since a facet has dimension $n-1$, there is only only one hyperplane in $V$ that contains it, which we call $H_i$. One can show that $\{H_1, \ldots, H_k\}$ is a minimal set of bounding hyperplanes, in the sense that any other set of bounding hyperplanes must contain it. For $f$ any face index of $P$, let

$$W_f = \cap_{i \in f} H_i,$$
an affine space in $V$. Note that $f \subset W_f$, although $W_f$ might not be the affine span of $f$.

A (rational) polytope is a bounded (rational) polyhedron. A polytope is the convex hull of its vertices; conversely, the convex hull of finitely many points is a polytope. This fact, though intuitive, is difficult to prove and not necessary for what follows.

There is a natural volume for polytopes $P$ of dimension $n$: any choice of isomorphism $L \cong \mathbb{Z}^n$ will induce a Euclidean measure $dv$ on $V = L \otimes \mathbb{Q} \cong \mathbb{Z}^n \otimes \mathbb{Q} = \mathbb{Q}^n$. This measure is independent of that initial choice. Then we may define $\text{vol}(P) = \int_P 1 \, dv$.

Given any polyhedron $P$ that has dimension smaller than $n$, we let $\text{Span}(P)$ be the affine space spanned by $P$, and $P\parallel$ be the vector space parallel to $\text{Span}(P)$ (i.e. $\text{Span}(P) = P\parallel + v$ for some $v \in V$).

Given any $\mathbb{Q}$-linear subspace $W$ of $V$, the set $L \cap W$ will be a free abelian group of rank equal to the dimension of $W$. Thus by the same procedure, there is a canonical, nontrivial Euclidean measure on $W$. By translation, any affine subspace of $V$ has a canonical measure as well. If $\text{dim}(P) < n$, we define the volume of $P$ to be its measure when considered as a subset of $\text{Span}(P)$. Thus the volume of any non-empty polyhedron will be a positive rational number. The volume of a point is conventionally 1, and the volume of the empty set is 0.

(Note that, if $\text{dim}(V) > 1$, the volume form on one-dimensional polytopes will not induce a metric on $V$, since the triangle inequality will not hold. $V$ has no canonical metric, unless we fix a basis of $L$.)

**Example 2.1.1.** Let $P$ be the triangle in Figure 2-1. There are three facets $F_1$, $F_2$, and $F_3$, each with a corresponding hyperplane $H_i$ and half-space $S_i$. Notice that $\text{Vol}(P) = 3$, while $\text{Vol}(F_1) = 1$, $\text{Vol}(F_2) = 2$, and $\text{Vol}(F_3) = 3$. All vertices have volume 1.

Under this numbering, the face index $\{1,2\}$ corresponds to the top vertex $F_1 \cap F_2$, while $\{1,3\}$ and $\{2,3\}$ correspond to the other two vertices. The face index $\{1,2,3\}$ corresponds to the empty face, since $F_1 \cap F_2 \cap F_3 = \emptyset$. 
2.2 Simple Polyhedra and Tangent Cones

Suppose that \( \{S_i\} \) is a finite set of half-spaces with bounding hyperplanes \( \{H_i\} \). Suppose there exists a point \( v \in V \) such that \( v \in \bigcap \{H_i\} \). Then the polyhedron \( \bigcap \{S_i\} \) is called a cone. By convention, \( V \) itself is also a cone. The following intuitive facts are true of cones:

- The face of a cone is itself a cone.

- A cone cannot have more than one vertex. A cone has a vertex at \( v \) if and only if \( \bigcap \{H_i\} = \{v\} \). A cone with a vertex is called a pointed cone.

- A set \( K \) in \( V \) is a cone if and only if \( K = \{v + \sum a_i v_i : a_i \geq 0\} \) for some \( v \), and finite set \( \{v_i\} \subset V \). We call \( \{v_i\} \) a generating set.
If $K$ is a pointed cone, then there is a canonical generating set. For each edge $F$ of $K$, the set $L \cap F$ takes the form $\{v + a_i w : a_i \in \mathbb{Z}_{\geq 0}\}$ for a unique vector $w \in L$, called a primitive generator. $K$ is generated by the set consisting of primitive generators, one from each edge of $K$.

A pointed cone $K$ is simple if a generating set forms a basis of $\text{Span}(K)$ as a rational vector space. If a pointed cone is simple with primitive generators $v_1, \ldots, v_l$, then the set $\{\sum_{i=1}^l a_i v_i : 0 \leq a_i < 1\}$ is a parallelipiped in $V$, which we call the fundamental parallelipiped of $K$. We let the index of $K$, or $i(K)$, be the volume of this parallelipiped (as a subset of $\text{Span}(K)$). There is another characterization of the index (see [Bv2], Theorem 10.9): it is also the index $[L_K : L \cap \text{Span}(K)]$, where $L_K$ is the sublattice of $L \cap \text{Span}(K)$ generated by the primitive generators of $K$. In particular, $i(K)$ is always a positive integer.

We wish to generalize these definitions to all cones. First note that any cone $K$ has an expression $W + K_p$ where $W$ is a subspace of $V$, and $K_p$ is a pointed cone (see [Bv2], Theorem 4.13). The subspace $W$ in this expression is unique; we will consider the $\mathbb{Q}$-vector space $\bar{V} := V/W$ with lattice $\bar{L} = L/(L \cap W)$. Let $\bar{K}$ be the image of $K$ in $\bar{V}$. Then $\bar{K}$ is a pointed cone. We say that $K$ is simple if $\bar{K}$ is simple, and define the index $i(K) := i(\bar{K})$.

We say that a set of hyperplanes $\{H_i\}$ in $V$ are in general position if:

- $|\{H_i\}| \leq \dim(V)$.

- $\cap\{H_i\}$ has dimension $\dim(V) - |\{H_i\}|$.

It is easy to show that a polyhedron is a simple cone if and only if it is bounded by a set of hyperplanes that are in general position.

Let $P$ be any full-dimensional polyhedron with minimal set of bounding hyperplanes $\{H_i\}$, each corresponding to a half-space $S_i$. For any non-empty face $F \subset P$, define the tangent cone of $F$ in $P$, or $K_F$, equal to $\cap_{i:F \subset H_i} S_i$. (We will also notate
this $K_f$, where $f$ is a corresponding face index.) $K_F$ is the minimal cone (under inclusion) containing $P$ such that all non-empty faces contain $F$. We will be particularly interested in the case where $F$ is a vertex $v$, in which case we call $K_v$ a *vertex cone*.

We say that $P$ is *simple* if the tangent cone of every face of $P$ is simple. The terminology is consistent in the case when $P$ is itself a cone. The following are equivalent:

- $P$ is simple.
- Given any $f \in \mathcal{F}_P$, $f = \emptyset$ or $\{H_i : i \in f\}$ are in general position.
- Any non-empty face of $P$ has exactly one corresponding face index.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1a}
\caption{$K_1$}
\end{subfigure}
\hspace{0.5cm}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1b}
\caption{$K_2$}
\end{subfigure}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1c}
\caption{$K_3$ (dimension 3)}
\end{figure}

*Figure 2.2. Three Pointed Cones and Their Primitive Generators*
Example 2.2.1. The two pointed cones $K_1$ and $K_2$ in Figure 2.2 are both simple. $K_1$ has index two: this is the volume of the parallelogram formed by the primitive generators $v_1$, and $v_2$. It is also the index of the sublattice generated by $v_1$ and $v_2$. Similarly, $K_2$ has index one.

The three-dimensional cone $K_3$ is a non-simple cone, since the generating set $v_1, \ldots, v_4$ does not form a basis of $V = \text{Span}(K_3)$. (There are no non-simple cones of dimension less than three.)
Chapter 3

The Perturbed Polyhedron

The exponential integral and sum were introduced by Brion [Br1] and Lawrence [La]. The perturbed polyhedron was used by Khovanskii and Pukhlikov [KP] to show that cohomology classes could be constructed as differential operators.

3.1 Exponential Volume and Sum

Let $V^*$ be the dual space of $V$. We always use $\langle \cdot, \cdot \rangle$ to refer to the natural evaluation map of $V^*$ on $V$. We also put the natural (archimedean) topology on $V$ and $V^*$ as finite-dimensional rational vector spaces. We say that a function from an open subset of $V$ to $\mathbb{C}$ is meromorphic if it extends (uniquely) to a function from $\mathbb{C} \otimes \mathbb{Q} V$ to $\mathbb{C}$ which is meromorphic in complex-geometric terms. We similarly define meromorphic functions on $V^*$.

Let $\Lambda$ be the completed symmetric tensor algebra of $V$. Given any basis $\{v_1, \ldots, v_n\}$ on $V$, $\Lambda$ is isomorphic to $\mathbb{Q}[[v_1, \ldots, v_n]]$. Since $V$ is the dual space of $V^*$, an element of $\text{Frac}(\Lambda)$ may, with the right convergence conditions, be a meromorphic function on $V^*$.

Let $\Lambda$ be the completed symmetric tensor algebra of $V$. Given any basis $\{v_1, \ldots, v_n\}$ on $V$, $\Lambda$ is isomorphic to $\mathbb{Q}[[v_1, \ldots, v_n]]$. Since $V$ is the dual space of $V^*$, an element of $\text{Frac}(\Lambda)$ may, with the right convergence conditions, be a meromorphic function on $V^*$.

Let $K$ be a pointed cone. The function $\int_K e^{-\langle w, v \rangle} dv : V^* \to \mathbb{R}$ given by $w \mapsto \int_K e^{-\langle w, v \rangle} dv$ converges uniformly on some open subset of $V^*$. (For this and all future integrals, we use the volume measure on the affine span of $K$, from the previous section.) The function has a meromorphic extension: if $K$ is simple with vertex $u$ and primitive generators $v_1, \ldots, v_l$, then $\int_K e^{-u} = i(K)e^{-u}(\prod_{i=1}^{l} v_i^{-1}) \in \text{Frac}(\Lambda)$, by a straightforward-
ward application of Fubini’s Theorem. (Here $e^{-u}$ is - by its Taylor series expansion - an element of $\Lambda$.)

It is possible to extend this definition to non-simple $K$ as follows:

Let $K, K_1, \ldots, K_l$ be cones in $V$, all of the same dimension. We say that $\{K_i\}$ form a decomposition of $K$ if:

- $K = \cup\{K_i\}$
- $K_i \cap K_j$ is a proper face of both $K_i$ and $K_j$, for each $i, j = 1, \ldots, l$.

We use the following strong result about decompositions:

**Lemma 3.1.1.** Every cone decomposes into simple cones.

(See [Bv2], Chapter 16 for a proof in the form of an algorithm.)

Then given any pointed cone $K$, we have that $\int_K e^{-v} = \sum_i \int_{K_i} e^{-v}$, where $\{K_i\}$ is some simple decomposition of $K$. This shows that $\int_K e^{-v}$ has a meromorphic extension, which must be independent of the chosen decomposition (by uniqueness of meromorphic extensions).

For any pointed $K$, we define $I(K) \in \text{Frac}(\Lambda)$ to be the meromorphic extension of $\int_K e^{-v}$. $I(K)$ is called the exponential volume of $K$.

We may similarly define the exponential sum $S(P)$ on a pointed cone to be the meromorphic extension of $\sum_{P \cap L} e^{-v}$ in $\text{Frac}(\Lambda)$. For a simple cone $K$ with vertex $u \in L$ and primitive generators $v_1, \ldots, v_l$, there is a geometric sum formula: $S(K) = i(K)e^{-u} \prod_{i=1}^l (1 - e^{-v_i})^{-1}$. (If the vertex is not in $L$, there is still a formula, which must consider whether $K \cap L$ is the empty set; if so, $S(K) = 0$.) As above, we use the simple decomposition to find a meromorphic extension for all pointed cones.

If $K$ is a non-pointed cone, then neither $\int_K e^{-v}$ nor $\sum_{P \cap L} e^{-v}$ converges for any $w \in V^*$. In this case, we set $I(K) = S(K) = 0$. 

Let \( P \) be any polyhedron. We define \( I(P) \) to be \( \sum_u I(\text{Tan}(u, P)) \), a sum over all vertices in \( P \). Similarly, we define \( S(P) = \sum_u S(\text{Tan}(u, P)) \) Certainly these definitions agree with the previous ones if \( P \) is itself a cone. Note that if \( P \) is a polyhedron without any vertices - or equivalently, if it contains an affine space of positive dimension - then \( I(P) = S(P) = 0 \). It deserves the name exponential volume, due to the following remarkable theorem of Brion and Lawrence (see [Br1], [La]):

**Theorem 3.1.2.** Let \( P \) be any polyhedron in \( V \). Then the sum over all vertices of \( P \), \( \sum_u I(\text{Tan}(u, P)) \), is equal to \( \int_P e^{-v} \) wherever the latter converges. Similarly, \( \sum_u S(\text{Tan}(u, P)) \), is equal to \( \sum_{P \cap L} e^{-v} \) wherever the latter converges.

For example, if \( P \) is a polytope, then \( \sum_u I(\text{Tan}(u, P)) \) is equal to the volume of \( P \) at \( w = 0 \), even though the exponential volume of each vertex cone has a pole at 0. This is often a very efficient way to compute the volume of \( P \).

The following theorem is another way in which the exponential volume and sum generalize on \( \text{Vol}(P) \) and \( |P \cap L| \). In preparation, we define, for any polyhedron, \( \chi(P) \) to be the characteristic function of \( P \) in \( V \).

**Theorem 3.1.3.** Let \( P, P_1, \ldots, P_k \) be polyhedra, and \( a_1, \ldots, a_k \in \mathbb{Q} \), such that \( \chi(P) = \sum_{i=1}^k a_i \chi(P_i) \).

Then

\[
I(P) = \sum_{i: \dim(P_i) = \dim(P)} a_i I(P_i)
\]

and

\[
S(P) = \sum_{i=1}^k a_i S(P_i)
\]

If \( P, P_1, \ldots, P_k \) are all polyhedra that do not contain straight lines, then the theorem is clear, since both sides of the equation are meromorphic functions that are equal, by definition, on some open subset of \( V^* \). But this also holds true when one
or more of the polyhedra does contain a straight line. Thus it was not arbitrary to set $I(P)$ and $S(P)$ equal to 0 in this case.

### 3.2 The Perturbed Polyhedron and the Perturbation Domain

Let $P$ be a polyhedron in $V$ of dimension $n$. (In the future, we will often assume without loss of generality that $\dim(P) = n$. This is possible because, for polyhedra of smaller dimension, we may restrict $L$ and $V$ to the affine span of $P$, where $P$ is full-dimensional. The exponential sum and volume will be invariant under this restriction.)

Let $V^*$ be the dual space of $V$. We reserve the notation $\langle \cdot, \cdot \rangle$ for the natural evaluation of $V^*$ on $V$. Let $L^* = \{ \bar{v} \in V^* : \langle \bar{v}, v \rangle \in \mathbb{Z} \text{ for all } v \in L \}$. Like $L \subset V$, $L^*$ is a lattice in $V^*$: it is a free abelian group of rank $n$, and $V = L^* \otimes \mathbb{Q}$. $L^*$ is the dual lattice of $L$: the natural isomorphism $V^{**} \cong V$ takes $L^{**}$ to $L$. We use the dual lattice when considering properties of polyhedra, in particular cones, that are contained in $V^*$.

Now let $K \subset V$ be any cone. Let $K^* = \{ \bar{v} \in V^* : \langle \bar{v}, v \rangle \geq 0 \text{ for all } v \in K \}$. $K^*$ is a cone in $V^*$: it is called the dual cone of $K$. As we expect, $K^{**} = K$.

Now let $P$ be any polyhedron in $V$ of full dimension $n$. Let $F_1, \ldots, F_k$ be the facets of $P$ (with a chosen numbering). Given that $P$ has dimension $n$, each facet $F_i$ corresponds naturally to a unique half-space $S_i$, such that $P = \cap \{S_i\}$. The dual cone to $S_i$ has a single primitive generator: we call this generator, $\bar{u}_i$, the primitive (inward) normal vector to $F_i$.

Next, choose $v_i \in V$ such that $\langle \bar{u}_i, v_i \rangle = -1$. The possible choices of $v_i$ form a hyperplane parallel to $H_i$.

Let $\vec{h} = (h_1, \ldots, h_k) \in \mathbb{Q}^k$. Let $P_{\vec{h}}$ be the polyhedron equal to $\cap_i (S_i + h_i v_i)$. It
is easy to see that this is independent of the choice of $v_i$. Certainly, $P_0 = P$, and for small $\vec{h}$, $P_{\vec{h}}$ will have $k$ facets, corresponding to each half-space $S_i + h_i v_i$. The number of faces of smaller dimension might change even for small $\vec{h}$, however, if $P$ is not simple. (In this case, the bounding hyperplanes of $P$ are not in general position.)

Let $P$ be a simple polyhedron, with minimal bounding hyperplanes $H_1, \ldots, H_k$. Given $\vec{h} \in \mathbb{Q}^k$, we use $H_{i,\vec{h}}$ to refer to the hyperplane $H_i + h_i v_i$ that bounds $P_{\vec{h}}$. Given any $f \in \mathcal{F}_P$, we will also refer to $W_{f,\vec{h}}$ to be the affine space $\bigcap_{i \in f} H_{i,\vec{h}}$.

Let $f \in \mathcal{F}_P$ be such that $f \neq \emptyset$, and let $i \in \{1, \ldots, k\}$ be such that $f \not\subset H_i$. Define

$$\mathcal{H}(f, i) := \{\vec{h} \in \mathbb{Q}^k : W_{f,\vec{h}} \subset H_{i,\vec{h}}\}.$$ 

The condition $W_{f,\vec{h}} \subset H_{i,\vec{h}}$ says that the set of $|f| + 1$ hyperplanes is not in general position. Since $\{H_{j,\vec{h}} : j \in f\}$ is in general position by the simplicity of $P$, the set $\mathcal{H}(f, i) \in \mathbb{Q}^k$ is either empty or it is a hyperplane itself in $\mathbb{Q}^k$, given by an equation of the form $h_i = a + l(h_j : j \in f)$ for some $a \in \mathbb{Q}$ and $\mathbb{Q}$-linear function $l$.

Let

$$\mathcal{O}_P = \cap\{\mathcal{H}(f, i)^c : f \neq \emptyset, i \not\in f\}.$$ 

The set $\mathcal{O}_P$, as the complement of the union of finitely many hyperplanes, is open and dense in $\mathbb{Q}^k$. Since $P$ itself was assumed to be simple, $0 \in \mathcal{O}_P$. Let $\mathcal{D}_P$ be the set of $\vec{h}$ such that:

- $\vec{h}$ is in the closure of the connected component of $\mathcal{O}_P$ containing the origin.
- $P_{\vec{h}}$ is full-dimensional.

$\mathcal{D}_P$ is then equal to a polyhedron in $\mathbb{Q}^k$ (bounded by certain of the $\mathcal{H}(f, i)$) with some of the boundary points removed.
Example 3.2.1. Let $P$ be the polyhedron in Figure 3.1. Then $\vec{h} = (0, 1, 0, 0)$ is in the interior of $D_P$. Note that the faces of $P_{\vec{h}}$ are in bijective correspondence to the faces of $P$. Furthermore, this correspondence preserves the tangent cones, up to translation.

If $\vec{h} = (1, 0, 0, 0)$, then $\vec{h} \in \mathcal{H}(\{1, 2\}, 4)$, so $\vec{h}$ is on the boundary of $D_P$. Note that $P_{\vec{h}}$ has fewer facets than $P$.

If $\vec{h} = (-2, 0, 0, 0)$, then $\vec{h}$ is in the closure of $D_P$, but is not itself in $D_P$ since $P_{\vec{h}}$ is not full-dimensional.

The following lemma shows that $P_{\vec{h}}$ is very similar to $P$ for $\vec{h}$ in the interior of $D_P$.

Lemma 3.2.2. Let $P$ be simple with $k$ facets, $\vec{h}$ be in the interior of $D_P$, and $P' := P_{\vec{h}}$. Then $P'$ is simple with $k$ facets, and there is a natural bijection $\mathcal{F}_{P'} \rightarrow \mathcal{F}_P$, which we notate $f' \mapsto f$. The following is true:

- $f \neq \emptyset \iff f' \neq \emptyset$. 

![Figure 3.1. A Simple Polyhedron $P$ and Three of Its Perturbations](image-url)
When \( f \neq \emptyset \), the tangent cones \( K_f \) and \( K_f' \) are translates.

Now suppose that \( \vec{h} \) is on the boundary of \( D_P \), and \( P' := P_{\vec{h}} \). Then \( P' \) has at most \( k \) facets. For any \( f \in F_P \),

\[
    f \neq \emptyset \implies (\cap_{i \in f} H_{i,\vec{h}}) \cap P' \neq \emptyset.
\]

**Proof.** Given any \( \vec{h} \in Q^k \), \( P_{\vec{h}} \) is, by definition, bounded by \( H_{1,\vec{h}}, \ldots, H_{k,\vec{h}} \), where \( H_{i,\vec{h}} \) is a translate of \( H_i \). Thus \( P_{\vec{h}} \) has at most \( k \) facets. (It does not necessarily have exactly \( k \) facets, since \( H_{1,\vec{h}}, \ldots, H_{k,\vec{h}} \) may not be a minimal bounding set.) Also, every facet of \( P' \) is parallel to a unique facet of \( P \); this induces a natural injective map \( F_{P'} \to F_P \).

Now suppose that \( \vec{h} \in D_P \). Suppose there exists some face index \( f \in F_P \), such that \( f \neq \emptyset \), but \( W_{f,\vec{h}} \cap P' = \emptyset \). We suppose without loss of generality that there are no proper subfaces of \( f \), since if there were, we could substitute one for \( f \).

Since \( W_{f,\vec{h}} \) contains \( H_{i,\vec{h}} \) for each \( i \in f \), it must be the case that \( W_{f,\vec{h}} \cap (\cap_{j \notin f} S_j) = \emptyset \).

Note that the map \( q \mapsto P_{q\vec{h}} \) for \( 0 \leq q \leq 1 \) gives us a continuous ‘deformation’ of the closed set \( P \) (and thus of \( W_f \) and \( \cap_{j \notin f} S_j \)), so there must exist a real number \( 0 < r < 1 \) such that:

- \( W_{f,q\vec{h}} \cap (\cap_{j \notin f} S_{j,q\vec{h}}) \neq \emptyset \) for all \( 0 \leq q \leq r \).
- \( W_{f,q\vec{h}} \cap (\cap_{j \notin f} S_{j,q\vec{h}}) = \emptyset \) for all \( r < q \leq 1 \).

Furthermore, since all polyhedra involved are rational, \( r \in Q \). Then \( W_{f,r\vec{h}} \) must intersect nontrivially with the boundary of \( \cap_{j \notin f} S_{j,r\vec{h}} \), which is itself contained in \( \cup_{j \notin f} H_{j,r\vec{h}} \). The fact that \( f \) contains no proper faces means that all hyperplanes \( H_i \) are parallel to the affine space \( W_f \). The same is true with the translates \( H_{i,\vec{h}} \) and \( W_{f,\vec{h}} \), so in fact \( W_{f,\vec{h}} \subset \cup_{j \notin f} H_{j,\vec{h}} \). This means that there must be some \( j \notin f \) such that \( f_{r\vec{h}} \in H_{j,\vec{h}} \).

However, \( D_P \) is a polyhedron; in particular, it is convex. Thus \( r\vec{h} \) is in the interior of \( D_P \), so it cannot be contained in \( \mathcal{H}(f,j) \) as we have concluded. This proves the second part of the theorem.

Next we further suppose that \( \vec{h} \) is in the interior of \( D_P \). We first show that \( P' \) has \( k \) facets. If not, then there exists some hyperplane \( H_{i,\vec{h}} \) that is not part of a minimal bounding set, meaning that \( P' = \cap_{j \notin i} S_{j,\vec{h}} \). Then \( H_{i,\vec{h}} \cap P' \) is a face (possibly empty) of dimension smaller than \( n \). We may use an argument identical to our previous one
to show that there must exist some \( r \in \mathbb{Q} \), \( 0 < r \leq 1 \), such that \( H_{i,r\vec{h}} \cap P_{r\vec{h}} \) is not empty and has dimension smaller than \( n \).

Then there exists some face index \( f^r \in \mathcal{F}_{P_{r\vec{h}}} \), \( f^r \neq \emptyset \) and \( i \notin f^r \), such that \( f^r \in H_{i,r\vec{h}} \). Next, let \( \tilde{f} \in \mathcal{F}_P \) be such that \( i \in f \) and \( |\tilde{f}| = |f^r| \). (There must exist a proper face of \( F_i \) of this dimension - otherwise, \( f_{r\vec{h}} \) must be empty.) In particular, \( \tilde{f} \neq \emptyset \). Then by the first half of the proof, \((\cap_{j \in f} H_{j,r\vec{h}}) \cap P_{r\vec{h}} \neq \emptyset \). But also,

\[
(\cap_{j \in f} H_{j,r\vec{h}}) \cap P_{r\vec{h}} \subset H_{i,\vec{h}} \cap P_{r\vec{h}} = f_{r\vec{h}}.
\]

Since \( i \notin f^r \) and \( |\tilde{f}| = |f^r| \), there must exist some \( j \in f^r \), \( j \notin \tilde{f} \). Then we have shown that \( r\vec{h} \in \mathcal{H}(f,j) \); this is a contradiction, since \( \vec{h} \), and thus \( r\vec{h} \), is in the interior of \( D_P \).

Then \( P' \) has \( k \) facets, and there is a natural bijection \( \mathcal{F}_{P'} \to \mathcal{F}_P \). Furthermore, \( P = P'_{-\vec{h}} \), and it is easy to show that \( D_P \) and \( D_{P'} \) are in natural bijection under the map \( j \mapsto (j - \vec{h}) \). Thus, \( -\vec{h} \in D_{P'} \), and so we may use the first half of the proof in reverse to show that \( f' \neq \emptyset \implies f \neq \emptyset \).

To see that \( K_f \) and \( K_{f'} \) are translates, we note that they are simple cones defined by the same hyperplanes up to translation.

Given a simple polyhedron \( P \), and \( \vec{h} \in \mathcal{D}_P \), we call \( P_{\vec{h}} \) a *perturbation* of \( P \).
Chapter 4

The Chow Ring and Cycle Group

The algebraic objects constructed in this chapter have (canonically isomorphic) counterparts in the study of equivariant cohomology of toric varieties. This explains the terminology and notation that we use. For a quick primer on toric geometry, see Appendix A. For a thorough development, see [Fu]. For a discussion of the equivariant cohomology, including the equivariant Chow ring, cycle group, and pushforward maps, see [Fu2] and [Br2].

In particular, the ring structure $Z^0(P)$ that we put on the cycle group was first discovered by Pommersheim and Thomas ([PT] and [Th]), before the author and Pommersheim [FP] found it in the equivariant case.

4.1 The Ring of Differential Operators

We wish to study how $I(P) \in \Lambda$ changes as we perturb $P$. Fortunately, we can describe this behavior explicitly using the known formulas.

Theorem 4.1.1. Given any simple polyhedron $P$ with $k$ facets, there exists unique $\mathcal{F} \in \text{Frac}(\Lambda)[[h_1, \ldots, h_k]]$ such that:

- $\mathcal{F}(\vec{h}) = I(P_{\vec{h}})$ for all $\vec{h} \in \mathcal{D}_P$.

- $\mathcal{F}(\vec{h})$ is uniformly convergent on compact sets of $\mathbb{Q}^k$. In particular, $\mathcal{F}$ is smooth for all $\vec{h}$.

The most difficult part of this theorem is proving that $I(P_{\vec{h}})$ itself is continuous on the boundary of $\mathcal{D}_P$. 

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Proof. First recall that $I(P) = \sum_v I(K_v)$, where $v$ varies over vertices of $P$, and $K_v$ is the tangent cone of $v$. By Lemma 3.2.2, if $\vec{h}$ is in the interior of $D_P$, then $P$ and $P_{\vec{h}}$ have the same vertex cones, up to translation. These translations are, themselves, perturbations of each respective tangent cone. In other words,

$$I(P_{\vec{h}}) = \sum_v I(K_v, \vec{h}_v),$$

where, for each vertex $v$, there is a natural function $\vec{h} \mapsto \vec{h}_v$ that simply removes the components of $\vec{h}$ corresponding to facets that do not contain $v$.

Suppose $K_v$ has vertex $v$ and primitive generators $v_1, \ldots, v_l$. Recall from Section 3.1 that $I(K_v) = i(K_v) e^{-v} \prod_{i=1}^l v_i^{-1}$. It is easy to see that for $K$ a simple cone, $D_K = Q^k$, and for all $\vec{h} \in Q^k$, $K_{\vec{h}}$ is a translation of $K$. The vertex of $K_{\vec{h}}$ is the unique vector $v(\vec{h})$ such that

$$\langle \tilde{u}_i, v(\vec{h}) - v \rangle = -h_i.$$

One is led to the following formula:

$$v(\vec{h}) = v - \sum_{i=1}^l \langle \tilde{u}_i, v_i \rangle^{-1} h_i v_i.$$

In particular, $v(\vec{h}_v)$ is a polynomial in the components of $\vec{h}_v$, which are themselves components of $\vec{h}$, so it is clearly an analytic function in $\vec{h}$. Then

$$I(K_v, \vec{h}_v) = i(K_v) e^{-v(\vec{h}_v)} \prod_{i=1}^l v_i^{-1}.$$

This expression is a power series in $\vec{h}_v$ (and thus in $\vec{h}$) with coefficients in Frac($\Lambda$) that clearly satisfies the required convergence properties. The function $F$ is the sum of these analytic functions over all vertices of $P$.

It remains to show that $I(P_{\vec{h}}) = F(\vec{h})$ when $\vec{h}$ is on the boundary of $D_P$. Fix $\vec{h}$ on the boundary of $D_P$, and let $\{\vec{h}_i : i \in \mathbb{Z}_+\}$ be a sequence in the interior of $D_P$ approaching $\vec{h}$. Then since $F$ is continuous everywhere, we have $\lim_{i \to \infty} I(P_{\vec{h}_i}) = F(\vec{h})$, so it suffices to show that $\lim_{i \to \infty} I(P_{\vec{h}_i}) = I(P_{\vec{h}})$.

First we define a map $p : \text{Vert}(P) \to \text{Vert}(P_{\vec{h}})$. (This is a special case of the pushforward map $\tilde{\pi}_*$ defined in Section 4.3.) If $u$ is a vertex of $P$, there is a unique
corresponding face index \( f_u \in \mathcal{F}_P \). Define

\[
p(u) := \cap_{i \in f_u} (H_i + h_i v_i) \cap \bar{P}_\bar{h},
\]

By Lemma 3.2.2, this intersection is non-empty. It is also a face of \( P_\bar{h} \) with dimension less than or equal to \( \dim(\{ u \}) \), so it must be a vertex.

Given \( v \in \text{Vert}(P_\bar{h}) \), let \( U_v \) be the set of all vertices \( u \) of \( P \) such that \( p(u) = v \). Then let \( P_v = \cap_{u \in U_v} K_u \), where \( K_u \) is the tangent cone of \( u \) in \( P \). Since each such \( K_u \) is bounded by a subset of the hyperplanes \( H_1, \ldots, H_k \), so is \( P_v \). Then without stretching notation too far, it is reasonable to write \( P_{v,\bar{h}} \) to refer to the perturbation of \( P_v \) by the components of \( \bar{h} \) corresponding to the subset of hyperplanes bounding \( P_v \).

We claim that \( P_{v,\bar{h}} = K_v \), the tangent cone of \( v \) in \( P_\bar{h} \). To see this, note that the set of hyperplanes \( H_i \) bounding \( P_v \) is exactly the set of hyperplanes \( H_i \) such that \( H_i + h_i \bar{v} \) contains \( v \). Then \( P_{v,\bar{h}} \) and \( K_v \) are equal, since they are bounded by the same hyperplanes.

Now note that

\[
I(P) = \sum_{u \in \text{Vert}(P)} I(K_u)
= \sum_{\bar{v} \in \text{Vert}(P_\bar{h})} \sum_{u \in U_{\bar{v}}} (I(K_u))
= \sum_{\bar{v} \in \text{Vert}(P_\bar{h})} I(P_\bar{v})
\]

A similar argument shows that \( I(P_{\bar{h}_i}) = \sum_{u \in U_{\bar{v}}} I(P_{v,\bar{h}_i}) \). Then since \( I(P_{\bar{h}_i}) = \sum_{v \in \text{Vert}(P_v)} I(K_v) \), we have reduced our problem to showing that \( \lim_{i \to \infty} I(P_{v,\bar{h}_i}) = I(K_v) \) for any \( v \in \text{Vert}(P_{\bar{h}}) \).

Since this equation is invariant to translation by any \( w \in V \) (both sides are multiplied by \( e^{-w} \)), we may suppose that \( v = \bar{0} \). Now recall that the vertex cones of \( I(P_{v,\bar{h}_i}) \) are translates of the vertex cones of \( P_v \). Then \( I(P_{v,\bar{h}_i}) = \sum_{u \in \text{Vert}(P_v)} e^{u - u(\bar{h}_i)} I(K_u) \). Then
\[
\lim_{i \to \infty} I(P,v_i) = \lim_{i \to \infty} \sum_{u \in \text{Vert}(P_v)} e^{u-v(h_i)} I(K_u) \\
= \sum_{u \in \text{Vert}(P_v)} e^{u-\tilde{0}} I(K_u) \\
= \sum_{u \in \text{Vert}(P_v)} I(K_u - u),
\]

where \( K_u - u \) is the vertex cone of \( u \) in \( P \) translated so that the vertex is \( 0 \). This has a special name: it is called the cone of feasible directions of \( u \) in \( P \), or \( \text{feas}(u, P) \). To finish the proof, we need to rely on a result from [Bv2], Chapter 6:

**Lemma 4.1.2.** Let \( P \) be any polyhedron. Let

\[
\text{rec}(P) = \{w \in V : \text{there exists } v \in V \text{ such that } v + aw \in P \text{ for all } a \in \mathbb{Q}_{\geq 0}\}
\]

be the recession cone of \( P \). (See [Bv2], Theorem 4.12 for a proof that \( \text{rec}(P) \) is a cone.) Then

\[
\sum_{u \in \text{Vert}(P)} I(\text{feas}(u, P)) = I(\text{rec}(P)).
\]

Then the following is true:

\[
\lim_{i \to \infty} I(P,v_i) = \sum_{u \in \text{Vert}(P_v)} I(\text{feas}(u, P)) = \text{rec}(P_v)
\]

Also, \( K_v = \text{feas}(v, K_v) = \text{rec}(K_v) \), since the vertex \( v = \tilde{0} \), so we must show that \( \text{rec}(P_v) = \text{rec}(K_v) = \text{rec}(P_{v,i}) \).

But we claim that \( \text{rec}(P) = \text{rec}(P') \) whenever \( P' \) is a finite perturbation of \( P \), and both \( P \) and \( P' \) are non-empty. We use the definiton of \( \text{rec}(P) \): for any \( w \in P \), \( w' \in P' \),

\[
w + aw_1 \in P \text{ for all } a \in \mathbb{Q}_{\geq 0} \iff w + aw_1 \in S_i \text{ for all } a \in \mathbb{Q}_{\geq 0} \text{ and } i \in \{1, \ldots, k\} \\
\iff w' + aw_1 \in S'_i \text{ for all } a \in \mathbb{Q}_{\geq 0} \text{ and } i \in \{1, \ldots, k\} \\
\iff w' + aw_1 \in P' \text{ for all } a \in \mathbb{Q}_{\geq 0}
\]
It follows that $w_1 \in \text{rec}(P) \iff w_1 \in \text{rec}(P')$.

We will be considering partial derivatives of $F$. To help keep track of these, we will construct an abstract ring of differential operators. Let $P$ be any polyhedron with $k$ facets. Define $R(P) := \Lambda[[D_1, \ldots, D_k]]$. Then if $P$ is simple, there exists an evaluation map $\mathcal{E} : R(P) \to \text{Frac}(\Lambda)[[h_1, \ldots, h_k]]$. The evaluation map is a $\Lambda$-module homomorphism, determined by the following formula for monomials in $R(P)$:

$$\mathcal{E}(D_1^{a_1} \cdots D_k^{a_k}) = \left[ \frac{\partial^{a_1}}{\partial h_1^{a_1}} \cdots \frac{\partial^{a_k}}{\partial h_k^{a_k}} F \right]$$

Also, $\mathcal{E}(1)$ is the ‘trivial’ differential operator $I(P_\vec{h})$. Note that the evaluation map is generally not a ring homomorphism. The following theorem illustrates one of the main useful properties of the evaluation map.

**Theorem 4.1.3.** Let $P$ be a simple polyhedron with $k$ facets, and let $f \subset \mathcal{F}_P$. Then for $\vec{h}$ in the interior of $D_P$,

$$\mathcal{E}\left( \prod_{i \in f} D_i \right)(\vec{h}) = \begin{cases} 
  i(K_f)^{-1}I(f_{\vec{h}}) & \text{if } f \neq \emptyset \\
  0 & \text{otherwise}
\end{cases}$$

where $K_f$ is the tangent cone of $f$ in $P$, and $f_{\vec{h}} = W_{f,\vec{h}} \cap P_{\vec{h}}$.

**Proof.** We first suppose that $P = K$ is a simple, pointed cone. Without loss of generality, we may assume that $K$ has vertex at $\vec{0}$, since translating $K$ by $v \in V$ has the effect of multiplying both sides of the equation in the theorem by $e^{-v}$.

Let $v_1, \ldots, v_k$ be the primitive generators of $K$, $\vec{u}_1, \ldots, \vec{u}_k$ be the primitive normal vectors in $V^*$. Recall from the proof of Theorem 4.1.1 that

$$I(K_{\vec{h}}) = i(K) \prod_{i=1}^k v_i^{-1} e^{-v(\vec{h})},$$

where

$$v(\vec{h}) = -\sum_{i=1}^k \langle \vec{u}_i, v_i \rangle^{-1} h_i v_i.$$
We are going to prove the theorem by induction on $|f|$, the size of the face index. For $f = \emptyset$, the theorem follows trivially from the fact that $i(K_\emptyset) = i(V) = 1$. Now suppose that the theorem is true for $|f| = j$. Fix a face index $f$ of size $j$, suppose that $i \notin f$, and let $g = f \cup \{i\}$. The face $f$ is itself a simple cone with primitive generators $\{v_j : j \notin f\}$. Any perturbation of $f$ is also a translation. Then

$$E(D_g) = \frac{\partial}{\partial h_i} E(D_f) = \frac{\partial}{\partial h_i} i(K_f)^{-1} I(f_h) = \frac{\partial}{\partial h_i} i(K_f)^{-1} i(f) \prod_{j \notin f} v_j^{-1} e^{-v(h)} = \langle \bar{u}_i, v_i \rangle^{-1} i(K_f)^{-1} i(f) \prod_{j \notin f} v_j^{-1} e^{-v(h)} = \langle \bar{u}_i, v_i \rangle^{-1} i(K_f)^{-1} i(f) \prod_{j \notin g} v_j^{-1} e^{-v(h)}$$

To show this equals $i(K_g)^{-1} I(g_h) = i(K_g)^{-1} i(g) \prod_{j \notin g} v_j^{-1} e^{-v(h)}$, it suffices to show that

$$\frac{i(K_f)}{i(K_g)} \langle \bar{u}_i, v_i \rangle = \frac{i(f)}{i(g)}.$$

Let $p : V \rightarrow V/\parallel g\parallel$ be the natural projection map, where $g\parallel$ is the unique subspace of $V$ parallel to the face $g$. Then $p(K_g)$ is a pointed cone, and $i(K_g)$ is equal to the lattice index $[L_g : p(L)]$, where $L_g$ is the lattice generated by the primitive generators of $p(K_g)$, which we notate $\{w_j : j \in g\}$. The generator $w_i$ is a multiple of $p(v_i)$: the key property of $w_i$ is that $\langle w_i, w_i \rangle = 1$, where $\bar{w}_i \in (f\parallel)^\ast$ is the primitive normal vector of $g$ as a facet of $f$.

The cone $p(K_f)$ is the (non-pointed) cone generated by $\{w_j : j \in g\}$ and $-w_i$. $i(K_f)$ is the index of the lattice $[L_f : p(L)]$, where $L_f$ is the lattice generated by $p(L)$ intersected with the two-dimensional faces of $p(K_f)$. Close inspection reveals this lattice is generated by $\{w_j : j \in f\}$ and some vector $p(u_i)$, where $\langle \bar{u}_i, u_i \rangle = 1$.

Then since $L_g$ is contained in $L_f$, $i(K_f)/i(K_g) = [L_g : L_f]$. This index is $\langle \bar{u}_i, w_i \rangle$, or $\langle \bar{u}_i, v_i \rangle/\langle \bar{w}_i, v_i \rangle$. (The latter expression is sensible, since $v_i \in f\parallel$.) Then

$$\frac{i(K_f)}{i(K_g)} \langle \bar{u}_i, v_i \rangle = \langle \bar{w}_i, v_i \rangle.$$
Then recalling that $i(g)$ is also the volume of the fundamental paralleliped of $g$, a simple volume computation shows that $i(f) = \langle \bar{w}, v_i \rangle i(g)$. This finishes the argument for $P = K$.

Now suppose $P$ is a general simple polyhedron. Then given $f \in \mathcal{F}_P$, and $\vec{h}$ in the interior of $\mathcal{D}_P$, we have that

\[
E(D_f) = \sum_u E_{K_u}(D_{f_u})
= \sum_{u \in f} i(K_f)^{-1} I(W_{f,\vec{h}} \cap K_{u,\vec{h}})
= i(K_f)^{-1} I(f_{\vec{h}})
\]

The last line is true because the vertex cones of the face $f = W_f \cap P$ are precisely $\{W_f \cap K_u\}$, where $u$ ranges over all vertices of $P$ that are contained in $f$. The same is true for all perturbations $f_{\vec{h}}$ with $\vec{h}$ in the interior of $\mathcal{D}_P$. Since $I(f_{\vec{h}})$ is equal to the sum of $I$ over all vertex cones, we are done.

\[\square\]

**Corollary 4.1.4.** Let $P$ be a simple polyhedron with $k$ facets, and let $f \subset \mathcal{F}_P$. Then for $\vec{h}$ on the boundary of $\mathcal{D}_P$,

\[
E(\prod_{i \in f} D_i)(\vec{h}) = \begin{cases} 
  i(K_f)^{-1} \ast I(f_{\vec{h}}) & \text{if } f \neq \emptyset \text{ and } \dim(f_{\vec{h}}) = \dim(f), \\
  0 & \text{otherwise}
\end{cases}
\]

where $K_f$ is the tangent cone of $f$ in $P$, and $f_{\vec{h}} = W_{f,\vec{h}} \cap P_{\vec{h}}$.

**Proof.** Let $f \in \mathcal{F}_P$ such that $f \neq \emptyset$ and $\dim(f_{\vec{h}}) = \dim(f)$. Let $\vec{h}$ be on the boundary of $\mathcal{D}_P$. For any $\vec{g}$ in the interior of $\mathcal{D}_P$, $E(D_f)(\vec{g}) = i(K_f)^{-1} \ast I(f_{\vec{g}})$. Then it is enough to show that, as $\vec{g}$ approaches $\vec{h}$ from the interior of $\mathcal{D}_P$, $I(f_{\vec{g}})$ approaches $I(f_{\vec{h}})$. We would like to use Theorem 4.1.1 to show this - specifically, the continuity of $I$ at the boundary of the perturbation domain. However, this is problematic, since the polyhedron $f$ is not (necessarily) full-dimensional.

To get around this, we may choose a linear projection map $p : V \to f^\parallel$. (This is equivalent to choosing a complementary vector space to $f^\perp$ in $V$. Then $p(f)$ is full-dimensional in $f^\parallel$. Furthermore, $p(W_{f,\vec{h}} \cap P')$ is a perturbation of $p(f)$: if one is careful
about ordering the facets of $f$, it is reasonable to say that $p(W_{f,\vec{h}} \cap P') = p(f)_{p(\vec{h})}$, with $p(\vec{h}) \in D_p(f)$ (since $W_{f,\vec{h}} \cap P'$ is also full-dimensional in $f^\parallel$). Then Theorem 4.1.1 does imply the continuity that we need.

If dim$(W_{f,\vec{h}} \cap P') <$ dim$(f)$, then the set $f_{\vec{g}}$ is deforming to a set of smaller dimension, as $\vec{g} \rightarrow \vec{h}$. Recall that $I(f_{\vec{g}}) = \int_{f_{\vec{g}}} e^{-v} dv$, where $dv$ is the canonical measure on $f^\parallel$. Any polyhedron of smaller dimension will have measure zero under $dv$; then $I(f_{\vec{g}})$ must approach 0 as $\vec{g}$ approaches $\vec{h}$.

If $f = \emptyset$, then $\mathcal{E}(D_f)(\vec{h}) = 0$ for all $\vec{g}$ in the interior of $D_P$, so by continuity it is 0 on the boundary as well.

4.2 Chow Ring and Cycle Group for Simple Polyhedra

Recall that if $P$ is a simple polyhedron with $k$ facets, we define the $\Lambda$-algebra $R(P) := \Lambda[[D_1, \ldots, D_k]]$. If $f \in \mathcal{F}_P$, then $f$ corresponds to a monomial

$$D_f := \prod_{i \in f} D_i \in R(P).$$

Also, recall that $W_f = \cap_{i \in f} H_i$.

Let

$$I_P := < D_f : f = \emptyset >$$

be the smallest ideal of $R(P)$ containing the given elements.

Recall that given $i = 1, \ldots, k$, $\bar{u}_i$ is the (unique) primitive inward normal vector in $V^*$ to the hyperplane $H_i$. Given $v \in V$, we define

$$g_v := \left( \sum_{i=1}^{k} \langle \bar{u}_i, v \rangle D_i \right) - v,$$

an element of $R(P)$. Now we define

$$J_P := < g_v : v \in V >$$

to be the ideal of $R(P)$ generated by $g_v$ with $v$ ranging over all of $V$. Since the map
Theorem 4.2.1. Let $P$ be a simple polyhedron with vertices. Then $\ker(\mathcal{E}) = I_P + J_P$.

Proof. We first show that $I_P + J_P \subset \ker(\mathcal{E})$. Note that $\ker(\mathcal{E})$ is closed under addition, since $\mathcal{E}$ is a $\Lambda$-module homomorphism. It is also closed under scalar multiplication: if $\mathcal{E}(G) = 0$, then all partial derivatives of $\mathcal{E}(G)$ equal 0 as well. Then to show $I_P + J_P \subset \ker(\mathcal{E})$, it is enough to show that the generators of each ideal are in $\ker(\mathcal{E})$.

If $D_f$ is a generator of $I_P$, then $f(\vec{h}) = 0$. By Theorem 4.1.3, $\mathcal{E}(D_f)(\vec{h}) = 0$ for $\vec{h}$ in the interior of $D_P$. But the interior is an open set, so by the analyticity of $\mathcal{E}(D_f)$, $\mathcal{E}(D_f) = 0$.

Now suppose $g_v$ is a generator of $J_P$. Let $u$ be a vertex of $P$, and let $K_u$ be the tangent cone of $u$ in $V$. Then there is a natural ring homomorphism $R(P) = \Lambda[1, \ldots, D_k] \to R(K_u) \cong \Lambda[[D_j : u \in F_j]]$ that maps $D_i \mapsto 0$ if $u \notin F_i$. It is easy to see that $g_v, P \mapsto g_v, K_u$ under this map. We notate this homomorphism $G \mapsto G^u$ for $G \in R(P)$.

Then since $I(P_{\vec{h}}) = \sum_u I(K_u, \vec{h})$ (see the proof of Theorem 4.1.1),

$$\mathcal{E}_P(G) = \sum_u \mathcal{E}_{K_u}(G^u).$$

In particular, $\mathcal{E}_P(g_v, P) = \sum_u \mathcal{E}_{K_u}(g_v, K_u)$. Thus we have reduced our problem to the case where $P$ is itself a simple cone $K$.

Then using notation from the proof of Theorem 4.1.1, we have that $I(P_{\vec{h}}) = e^{-u(\vec{h})}I(P)$, where $u(\vec{h})$ is the vertex of $K_{\vec{h}}$. Recall that $u(\vec{h}) = u(\vec{0}) - \sum_{i=1}^k h_i u_i$, where $\{u_i\}$ is the dual basis to the basis of primitive normal vectors $\{\tilde{u}_i\}$. Then $\mathcal{E}(D_i) = u_i I(P_{\vec{h}})$, or more generally,

$$\mathcal{E}(\sum_{i=1}^k \langle \tilde{u}_i, v \rangle D_i) = \left(\sum_{i=1}^k \langle \tilde{u}_i, v \rangle u_i\right) I(P_{\vec{h}}) = v I(P_{\vec{h}}) = \mathcal{E}(v)$$

so $\mathcal{E}(g_v) = 0$.

In order to prove that, for simple $P$, $I_P + J_P = \ker(\mathcal{E})$, it is enough to show that $\mathcal{E}$ is injective when factored through $R(P)/I_P + J_P$. We make use of the following lemma:
Lemma 4.2.2. Let \( P \) be a simple polyhedron. Then \( R(P)/(I_P + J_P) \) is naturally isomorphic as a \( \mathbb{Q} \)-algebra to

\[
\mathbb{Q}[[D_1, \ldots, D_k]]/(I_P \cap \mathbb{Q}[[D_1, \ldots, D_k]]).
\]

Proof of Lemma: There is a natural \( \mathbb{Q} \)-algebra homomorphism \( \mathbb{Q}[[D_1, \ldots, D_k]]/(I_P \cap \mathbb{Q}[[D_1, \ldots, D_k]]) \to R(P)/(I_P + J_P) \). To construct the inverse, note that \( v \equiv \sum_i \langle \bar{u}_i, v \rangle D_i \) modulo \( J_P \), inducing a \( \mathbb{Q} \)-algebra homomorphism \( \Lambda \to \mathbb{Q}[[D_1, \ldots, D_k]] \). Extending that map to \( R(P)/(I_P + J_P) \) is clear.

We wish to show that given \( G \in R(P)/(I_P + J_P) \), \( \mathcal{E}(G) = 0 \) implies \( G = 0 \). By the lemma, it is enough to assume that \( G = g(D_1, \ldots, D_k) \) for some power series \( g \in \mathbb{Q}[[D_1, \ldots, D_k]] \).

We first assume that \( P \) is a cone \( K \). Then using our earlier notation,

\[
\mathcal{E}(G) = g(u_1, \ldots, u_k) i(K_h^{-1} I(K_h)).
\]

Then since we assumed at the beginning that \( I(K) \neq 0 \), \( \mathcal{E}(G) = 0 \implies g = 0 \).

Next, suppose \( P \) is a simple polyhedron. We again use the fact that \( \mathcal{E}_P(G) = \sum_u \mathcal{E}_{K_u}(G^u) \). If \( G \notin I(P) \), then there exists some vertex \( u \) such that \( G^u \neq 0 \). If we assume that \( \mathcal{E}_P(G) = 0 \), then

\[
\mathcal{E}_P((\prod_{u \in F_i} D_i) G) = \mathcal{E}_{K_u}((\prod_{u \in F_i} D_i) G^u) = 0,
\]

which contradicts the fact that \( (\prod_{u \in F_i} D_i) G^u \neq 0 \).

Now for any polyhedron \( P \), define the Chow ring

\[
A(P) := R(P)/(I_P + J_P).
\]

The previous result shows that the evaluation map \( \mathcal{E} \) factors through the Chow ring, and is injective there.

Next, let

\[
Z(P) := \Lambda \{ D_f : f \in \mathcal{F}_P, D_f \notin I_P \}
\]
be the sub-$\Lambda$-module of $R(P)$ generated by all squarefree monomials $D_f$ that are not in $I_P$. Since $Z(P) \subseteq R(P)$, there is a natural $\Lambda$-module homomorphism $Z(P) \to A(P)$.

We wish to first show two things:

- The map $Z(P) \to A(P)$ is surjective.
- With a little extra structure on $V$, there will exist a natural section $A(P) \to Z(P)$ of this epimorphism.

The 'extra structure' we are referring to is a complement map:

**Definition 4.2.3.** A complement map is a function $\Psi$ from the set of subspaces of $V$ to itself, such that:

- For every subspace $W$, $W \oplus \Psi(W) = V$.
- If $W_1 \subseteq W_2$, then $\Psi(W_2) \subseteq \Psi(W_1)$.

A complement map $\Psi$ (for example, an inner product on $V$) allows us to define, for every linear or affine subspace $W \subseteq V$, a linear subspace $W^\perp = \Psi(W')$, where $W'$ is the linear space parallel to $W$.

Let

$$J_P^\Psi := \langle D_f g_v : D_f \not\in I_P, v \in W_f^\perp \rangle$$

be a sub-ideal of $J_P$. (Like $J_P$, it is finitely generated.) We will define

$$Z^\Psi(P) := R(P) / (I_P + J_P^\Psi).$$

**Lemma 4.2.4.** $Z^\Psi(P)$ is spanned over $\Lambda$ by the images of squarefree monomials from $R(P)$. In other words, the natural map $Z(P) \to Z^\Psi(P)$ is surjective.

**Proof.** We will exhibit an algorithm to express any element $z \in R(P)$, modulo $I_P + J_P^\Psi$, as a $\Lambda$-combination of squarefree monomials from $\{D_f : f \in \mathcal{F}_P\}$.

It suffices to consider $z$ to be a monomial that is not in $I_P$. Any such $z$ has an expression of the form

$$z = D_{i_1} \ldots D_{i_m} D_f.$$
where \( \mathbf{f} \neq \emptyset \), \( \dim(W_f) = n - |\mathbf{f}| \), and \( i_1, \ldots, i_m \in \mathbf{f} \). If \( m = 0 \), the expression is already squarefree. We proceed by induction on \( m \).

Since \( \dim(W_f) = n - |\mathbf{f}| \), \( \{ \tilde{u}_i : i \in \mathbf{f} \} \) restricts to a basis of \( (W_f^\perp)^* \). Then fix \( v \in W_f^\perp \) so that \( \langle \tilde{u}_{i_1}, v \rangle = 1 \), and \( \langle \tilde{u}_j, v \rangle = 0 \) for \( j \in \mathbf{f}, j \neq i_1 \). Then \( z - D_{i_2} \ldots D_{i_m} D_f g_v \equiv z \mod J_P^\psi \). We then expand the left hand side and consider each term in the expansion.

One of these terms is \( (1 - \langle \tilde{u}_{i_1}, v \rangle) D_{i_1} \ldots D_{i_m} D_f = 0 \). The last term is \( (-v) D_{i_2} \ldots D_{i_m} D_f \). All other terms are of the form

\[ -D_j D_{i_2} \ldots D_{i_k} D_f = -D_{i_2} \ldots D_{i_k} D_{f \cup \{j\}}, \]

since for each of these terms, \( j \not\in \mathbf{f} \). We are done by our induction hypothesis. \( \square \)

We say that an element of \( R(P) \) is squarefree if it has the form \( \sum_{f \in \mathcal{F}_P} a_F D_F \), where \( a_F \in \Lambda \).

**Lemma 4.2.5.** \( J_P^\psi \) contains no nontrivial squarefree elements that are not in \( I_P \).

**Proof.** It is clear from Theorem 4.2.1 that \( J_P^\psi \) and \( I_P \) are both invariant under small perturbations of the polyhedron \( P \). If \( P \) is non-simple, then there exists a small perturbation that is simple, so we may assume that \( P \) is a simple polyhedron.

Next, we will reduce the problem to simple cones. If \( K_v \) is any vertex cone of \( P \), then there is a natural \( \Lambda \)-homomorphism \( R(P) \to R(K_v) \) that sends \( D_i \mapsto 0 \) if the vertex \( v \not\in F_i \). It is easy to see that \( J_P^\psi \to J_{K_v}^\psi \) under this map, and that squarefree elements map to squarefree elements. Furthermore, given any squarefree element \( z \not\in \mathcal{I}_P \), there is some vertex cone \( K_v \) such that \( z \not\in 0 \in R(K_v) \).

Therefore, we may assume \( P = K \) is a simple cone. We will prove this by induction on \( n \), the dimension of \( K \). If \( n = 1 \), \( J_P^\psi \) has one generator that is not squarefree, and thus clearly has no squarefree elements.

Recall that if \( P \) is not a full-dimensional polyhedron, we restrict \( V \) to the affine span of \( P \), and thus \( \Lambda \) is restricted as well. Since we are about to consider cones of different dimensions, we will briefly introduce the notation \( \Lambda_K \) to specify the cone.

Let \( v_1, \ldots, v_n \) be the primitive generators of \( K \), and let \( F_1, \ldots, F_n \) be the facets of \( K \). Now fix \( i \in \{1, \ldots, n\} \). It is easy to check that \( F_i \) is itself a simple cone, and that there are natural maps \( \Lambda_K \to \Lambda_{F_i} \) and \( J_P^\psi \to J_{F_i}^\psi \). (The first is given by killing
Span($v_i$), the second by killing both Span($v_i$) and $D_i$.) The latter factors as

$$J^\Psi_K \rightarrow \tilde{J}^\Psi_{F_i} \rightarrow J^\Psi_{F_i},$$

where \( \tilde{J}^\Psi_{F_i} = J^\Psi_{F_i} \otimes_{\Lambda_{F_i}} \Lambda_K \). Note that \( J^\Psi_{F_i} \) has no squarefree elements by the induction hypothesis, and thus neither does \( \tilde{J}^\Psi_{F_i} \). Thus any squarefree element of \( J^\Psi_K \) must be in the kernel of the first map in the factorization, which is the \( \Lambda_K \)-module map given by sending any term containing \( D_i \) to 0.

Since \( i \) is arbitrary, any squarefree element of \( J^\Psi_K \) must be of the form \( z := aD_1 \ldots D_n \), where \( a \in \Lambda \). We apply the evaluation map \( \mathcal{E} : \mathcal{E}(z) = a(i(K)I(F_1 \cap \ldots \cap F_n)) = a(i(K)e^{-v}) \), where \( v \) is the vertex of \( K \). But also, since \( z \in J^\Psi_K \subset J_K \), we have \( \mathcal{E}(z) = 0 \). Thus \( z = 0 \), and we are done.

**Theorem 4.2.6.** As \( \Lambda \)-modules, \( Z^\Psi(P) \cong Z(P) \).

**Proof.** Since \( Z(P) \subset R(P) \), there is a natural map \( Z(P) \rightarrow Z^\Psi(P) \). By Lemma 4.2.4, it is surjective, and by Lemma 4.2.5, it is injective.

**Corollary 4.2.7.** The natural \( \Lambda \)-module homomorphism \( Z(P) \rightarrow A(P) \) is surjective. In other words,

$$A(P) \cong Z(P)/Z(P) \cap (I_P + J_P).$$

**Proof.** By Lemma 4.2.4, any element of \( R(P) \) can be expressed modulo \( I_P + J^\Psi_P \) as a \( \Lambda \)-combination of squarefree monomials. The corollary follows, since \( J^\Psi_P \subset J_P \).

**Corollary 4.2.8.** Given a complement map \( \Psi \), there is a natural \( \Lambda \)-module section \( s^\Psi : A(P) \rightarrow Z(P) \) of the epimorphism \( Z(P) \rightarrow A(P) \).

**Proof.** The section \( s^\Psi \) is equal to the composition

$$A(P) \cong Q[[D_1, \ldots, D_k]]/I_P \rightarrow \Lambda[[D_1, \ldots, D_k]]/(I_P + J^\Psi_P) \cong Z(P).$$

The section \( s^\Psi \) will be used to construct the interpolator function in Section 5.3.

We next compute some low-dimensional examples:

**Example 4.2.9.** Suppose \( V \) is one-dimensional. For such \( V \), there is only one complement map \( \Psi \). Let \( K \) be a cone with vertex \( v \) and primitive generator \( v_1 \). Then

$$J^\Psi_K := \langle D_1 g_{v_1} \rangle = \langle D_1(D_1 - v_1) \rangle.$$
In particular, for each \( n > 1 \), \( D_1^n - v_1 D_1^{n-1} \in J^\Psi \). Then by induction, it can be shown that \( s^\Psi(D_1^n) = v_1^{n-1}D_1 \). More generally, given any power series \( g \) with rational coefficients,

\[
s^\Psi(g(D_1)D_1) = g(v_1)D_1.
\]

**Example 4.2.10.** Now suppose \( V \) is two-dimensional. Let \( K \) be a pointed cone in \( V \) with vertex \( v \) and primitive generators \( v_1 \) and \( v_2 \). We number the facets of \( V \) so that \( v_1 \) is not contained in \( F_2 \) (and vice-versa). As before, let \( \tilde{u}_i \) be the primitive (inward) normal vector to \( F_i \).

Now we have a choice of complement map \( \Psi \). Given such a \( \Psi \), let \( w_i \in F_i^\perp \) for \( i = 1, 2 \), normalized so that \( \langle \tilde{u}_i, w_i \rangle = 1 \). Finally, let \( u_1, u_2 \) be the dual basis to \( \tilde{u}_1, \tilde{u}_2 \).

The following are all elements of \( J^\Psi_K \):

\[
D_1 g_{w_1} = D_1(D_1 + \langle \tilde{u}_2, w_1 \rangle D_2 - w_1),
\]

\[
D_2 g_{w_2} = D_2(D_2 + \langle \tilde{u}_1, w_2 \rangle D_1 - w_2),
\]

\[
D_1 D_2 g_{u_1} = D_1 D_2(D_1 - u_1 D_1),
\]

\[
D_1 D_2 g_{u_2} = D_1 D_2(D_2 - u_2 D_2).
\]

The last two elements imply, via induction as in the one-dimensional case, that for \( m, n > 0 \),

\[
s^\Psi(D_1^m D_2^n) = u_1^{m-1} u_2^{n-1} D_1 D_2,
\]

implying that

\[
s^\Psi(g(D_1, D_2)D_1 D_2) = g(u_1, u_2)D_1 D_2
\]

for any power series \( g \) in two variables (with rational coefficients). Also, for any \( n > 1 \),

\[
D_1^n g_{w_1} = D_1^n - w_1 D_1^{n-1} + \langle \tilde{u}_2, w_1 \rangle D_1^{n-1} D_2 \in J^\Psi_K. \]

Then one can prove by induction that

\[
s^\Psi(D_1^n) = w_1^{n-1} D_1 - \langle \tilde{u}_2, w_1 \rangle (w_1^{n-2} + w_1^{n-3} u_1 + \cdots + w_1^{n-2}) D_1 D_2
\]

\[
= w_1^{n-1} D_1 - \langle \tilde{u}_2, w_1 \rangle \frac{w_1^{n-1} - w_1^{n-1}}{u_2 - u_1} D_1 D_2
\]

\[
= w_1^{n-1} D_1 - \frac{w_1^{n-1} - u_1^{n-1}}{u_2} D_1 D_2
\]

where the last line follows from the fact that \( w_1 = \langle \tilde{u}_2, w_1 \rangle u_1 + \langle \tilde{u}_2, w_2 \rangle u_2 \) and
\(\langle \tilde{u}_1, w_1 \rangle = 1\). Then we have that

\[
s^\Psi(g(D_1)D_1) = g(w_1)D_1 - \frac{g(w_1) - g(u_1)}{u_2} D_1 D_2
\]

for any rational power series \(g\). (The coefficient \((g(w_1) - g(u_1))/u_2\) is necessarily an element of \(\Lambda\).) A symmetrical formula holds for \(s^\Psi(g(D_2)D_2)\).

### 4.3 Pushforward on Cycle Groups and Chow Rings

Let \(P\) be a simple polyhedron in \(V\), and let \(P' = P_\tilde{h}\) for some \(\tilde{h} \in D_P\). (Then \(P'\) is full-dimensional, but not necessarily simple.) We will use the notation \(H'_i\) and \(S'_i\) for \(H_i(h_i)\) and \(S_i(h_i)\), respectively. Then \(P' = \cap_{i=1}^k S'_i\).

Recall that \(F_P\) is the set of face indices of \(P\). Let \(\text{Face}(P')\) be the set of faces of \(P'\). We will first define a map \(\tilde{\pi}^* : \{ f \in F_P : D_f \notin I_P \} \to \text{Face}(P')\) as follows:

\[
\tilde{\pi}^*(f) = \cap_{i \in f} H'_i \cap P'
\]

Note that since each \(H'_i\) is a bounding hyperplane of \(P'\), \(H'_i \cap P'\) is a face of \(P'\), so the intersection over \(i \in f\) will be the intersection of faces, which is also a face.

The following lemma clearly follows from Lemma 3.2.2.

**Lemma 4.3.1.** Let \(P\) be a simple polyhedron. Then given \(f \in F_P\) such that \(D_f \notin I_P\), the face \(\tilde{\pi}^*(f)\) is non-empty.

Recall that for simple polyhedra \(P'\), every non-empty face \(F\) of \(P'\) corresponds naturally to a unique face index \(f \in F_P\). Then if \(P, P'\) are both simple, Lemma 3.2.2 allows us to define a map \(\pi^* : \{ f \in f_P : D_f \notin I_P \} \to \{ f \in f_{P'} : D_f \notin I_{P'} \}\) so that \(\pi^*(f)\) is the unique face index corresponding to \(\tilde{\pi}^*(f)\).

**Lemma 4.3.2.** Let \(P, P', P''\) be polyhedra in \(V\) such that \(P, P'\) are simple. Suppose also that \(P'\) is a finite perturbation of \(P\) and \(P''\) is a finite perturbation of \(P'\). Then given any \(f \in F_P\) such that \(D_f \notin I_P\), and let \(f' = \pi^*_{P', P''}(f)\), and \(F'' = \tilde{\pi}^*_{P'', P''}(f')\). We
will notate \( \tilde{\pi}_*(f) := \tilde{\pi}_*^{P,P''}(f) \in \text{Face}(P'') \). Then

\[ F'' = \tilde{\pi}_*(f). \]

In other words, the pushforward map is transitive.

Proof. We require yet more notation: \( P = \cap_{i=1}^k S_i \), each \( S_i \) a closed half-space bounded by \( H_i \). Also, \( P' = \cap_{i=1}^k S'_i \), \( P'' = \cap_{i=1}^k S''_i \).

Then by the definition of the pushforward, the face \( f' \) of \( P' \) is equal to \( \cap_{i \in f} H'_i \cap P' \). Furthermore, \( P' \) is contained in \( \cap_{i \in f} S'_i \). Then since \( \cap_{i \in f} S'_i \) is the tangent cone of \( f' \) in \( P' \) (i.e. the minimal cone that satisfies the same properties), we must have \( \cap_{i \in f} S'_i \subset \cap_{i \in f} S'_i \) and \( \cap_{i \in f} H'_i \subset \cap_{i \in f} H'_i \).

Now we focus our consideration on \( P'' \). Once again, we must have \( P'' \) contained in both \( \cap_{i \in f} S''_i \) and \( \cap_{i \in f} S''_i \).

First, note that, since \( P \) and \( P' \) are simple, \( \cap_{i \in f} H''_i \) and \( \cap_{i \in f} H''_i \) are just translations of the above affine spaces, so either \( \cap_{i \in f} H''_i \subset \cap_{i \in f} H''_i \) or their intersection is empty. Assume for contradiction that the intersection is empty. Then since \( \cap_{i \in f} S''_i \) and \( \cap_{i \in f} S''_i \) are translations of the previous two simple cones, we must have either \( \cap_{i \in f} H''_i \not\subset \cap_{i \in f} S''_i \) or \( \cap_{i \in f} H''_i \not\subset \cap_{i \in f} S''_i \). But either case would contradict the fact that \( P'' \) is contained in both cones, and that by Lemma 3.2.2, the intersection of \( P'' \) with both \( \cap_{i \in f} H''_i \) and \( \cap_{i \in f} H''_i \) is non-empty.

Then we must have \( \cap_{i \in f} H''_i \subset \cap_{i \in f} H''_i \). Then \( F'' = \cap_{i \in f} H''_i \cap P'' \subset \pi_*(f) = \cap_{i \in f} H''_i \cap P'' \). The inclusion must be an equality, since \( P'' \subset \cap_{i \in f} S''_i \) and \( \cap_{i \in f} S''_i \cap \cap_{i \in f} H''_i = \cap_{i \in f} H''_i \).

Now let \( P, P' \) be simple polyhedra, with \( P' \) a perturbation (finite or infinite) of \( P \). We define a \( \Lambda \)-module homomorphism, also called \( \pi_* \), from \( Z(P) \) to \( Z(P') \). (Context will determine which map \( \pi_* \) we are referring to.)

As a \( \Lambda \)-module homomorphism, the map is determined uniquely by where it sends \( D_f \) for every \( f \in F_P \). Given \( f \in F_P \), let \( K_{f,P} \) be the tangent cone of the face \( f \) in \( P \), and let \( i(K_{f,P}) \) be the index of that cone.

- If the bounding hyperplane \( H_i \) is perturbed to infinity in \( P' \) (i.e. \( P' = P'' \) where \( \infty \) is the smallest possible value of \( h_i \)), then \( \pi_*(D_f) = 0 \) for every \( f \) containing \( i \).
(For all other $f$, the face index $\pi_*(f) \in \mathcal{F}_{P'}$ is sensible under the earlier definition.)

- If $|\pi_*(f)| < |f|$, then $\pi_*(D_f) = 0$.

- If $|\pi_*(f)| = |f|$, then
  \[ \pi_*(D_f) = \frac{i(K_{\tilde{\pi}_*(f), P'})}{i(K_f, P')} D_{\pi_*(f)}. \]

Note that the definition covers all possible $f$; it is not possible for $|\pi_*(f)| > |f|$. It is also easy to see, by Lemma 4.3.2, that this pushforward map is transitive as well (in the case where $P, P', P''$ are all simple). The unusual constants in the pushforward formula are included so that the following lemma holds:

**Lemma 4.3.3.** Let $P, P'$ be simple polyhedra, such that $P' = P_{\tilde{h}_0}$ for some $\tilde{h}_0 \in \mathcal{D}_P$. Suppose that $P$ has $k$ facets, and $P'$ has $l$ facets. By Lemma 3.2.2, $l \leq k$. Then there exists a linear map $s : Q^l \to Q^k$ such that, for any $G \in \mathcal{Z}(P)$, $\mathcal{E}_{P'}(\pi_*(G))(\tilde{g}) = \mathcal{E}_P(G)(\tilde{h}_0 + s(\tilde{g}))$. In other words, the following diagram commutes:

\[
\begin{array}{ccc}
Z(P) & \xrightarrow{\mathcal{E}_P} & \Lambda[[h_1, \ldots, h_k]] \\
\downarrow{\pi_*} & & \downarrow{f \mapsto f(\tilde{h}_0 + s(\tilde{g}))} \\
Z(P') & \xrightarrow{\mathcal{E}_{P'}} & \Lambda[[h_1, \ldots, h_l]]
\end{array}
\]

**Proof.** Let $H_1, \ldots, H_k$ be the minimal bounding hyperplanes of $P$. Then there exists some set $S \subset \{1, \ldots, k\}$ of size $l$ such that $H'_i$ is a bounding hyperplane of $P'$ if and only if $i \in S$. Fix $i \notin S$. Then the face index $\pi_*({\{i}\})$ in $\mathcal{F}_{P'}$ corresponds to some face index $f \subset S$. If we write $\tilde{h}_0 = (a_1, \ldots, a_k)$, then there exists a linear map $h_i : Q^l \to Q$ such that
\[
[H_i + (a_i + h_i(\tilde{g}))v_i] \cap P'_i = \cap_{j \in f} [H_j + (a_j + g_j)v_j]
\]
for small enough $\tilde{g}$. Informally, $h_i(\tilde{g})$ is how much we must perturb the hyperplane $H'_i$ so that it continues to intersect the boundary of $P'_i$.

Now for $i \in S$, we define $h_i : Q^l \to Q$ so that $h_i(\tilde{g}) = g_i$. (For this to be sensible, we must number the facets of $P'$ so that $F'_i$ of $P'$ and $F_i$ of $P$ are parallel.) Then collecting these maps together, and defining $s = (h_1, \ldots, h_k)$, we have that $P'_\tilde{g} = P_{\tilde{h}_0 + s(\tilde{g})}$. Furthermore, it is not hard to see that for small enough $\tilde{g}$, $\tilde{h}_0 + s(\tilde{g}) \in \mathcal{D}_P$. 

To simplify the notation a bit, we write $\vec{h} := h_0 + s(\vec{g})$. Then given $D_f \in \mathcal{Z}(P)$ such that $f \neq 0$ and $\pi_*(D_f) \neq 0$,

$$E_{P'}(\pi_*(D_f))(\vec{g}) = E_{P'}(\frac{i(K_{\pi_*(f),P'})}{i(K_{f,P})}D_{\pi_*(f)})(\vec{g})$$

$$= \frac{i(K_{\pi_*(f),P'})}{i(K_{f,P})}[i(K_{\pi_*(f),P'})^{-1}I(\pi_*(f)\vec{g})] \text{ by Theorem 4.1.3}$$

$$= i(K_{f,P})^{-1}I(\pi_*(f)\vec{g})$$

$$= E_P(D_f)(\vec{h}) \text{ by Corollary 4.1.4}$$

Since the above is true for small enough $\vec{g}$, analyticity guarantees that it is true for all $\vec{g}$. Also, $\Lambda$-linearity implies that the result is true for arbitrary $G \in \mathcal{Z}(P)$.

**Corollary 4.3.4.** Let $P, P'$ be as in the theorem. Then $\pi_*(Z(P) \cap (I_P + J_P)) \subset (Z(P') \cap (I_{P'} + J_{P'}))$. In other words, $\pi_*$ induces a map $A(P) \to A(P')$. (We call this map $\pi_*$ as well.)

**Proof.** Let $G \in Z(P) \cap (I_P + J_P)$. Then $E_P(G)(h_0 + s(\vec{g})) = 0$, by Lemma 4.3.3. Then $E'_P(\pi_*(G)) = 0$ for all $\vec{g}$. Using Theorem 4.3.3 again, it must be that $\pi_*(G) \in Z(P) \cap (I_{P'} + J_{P'})$.

**Theorem 4.3.5.** Given any complement map $\Psi$, and $G \in A(P)$, $\pi_*(s^\Psi(G)) = s^\Psi(\pi_*(G))$. In other words, the following diagram commutes:

$$\begin{array}{ccc}
Z(P) & \xrightarrow{\pi_*} & Z(P') \\
\downarrow{s^\Psi} & & \downarrow{s^\Psi} \\
A(P) & \xrightarrow{\pi_*} & A(P')
\end{array}$$

**Proof.** In this proof, we notate $q : Z(P) \to A(P)$ to be the canonical quotient map (we use the same letter for $P'$).

Let $G \in A(P)$. Since $\pi_* : A(P) \to A(P')$ is induced from $\pi_* : Z(P) \to Z(P')$, $\pi_*(G) = (q \circ \pi_*)(H)$, where $H$ is any element of $Z(P)$ such that $q(H) = G$. We may choose $H = \sigma^\Psi(G)$, since $s^\Psi$ is a section of $q$. It suffices to show that

$$(s^\Psi \circ q \circ \pi_* \circ s^\Psi)(G) = (\pi_* \circ s^\Psi)(G).$$
Since $q \circ s^\Psi = \text{id}$, $s^\Psi \circ q \circ s^\Psi = s^\Psi$. Then it suffices to show that $(\pi_* \circ s^\Psi)(G) \in s^\Psi(A(P'))$, or more generally, that

$$(\pi_* \circ s^\Psi)(A(P)) \subset s^\Psi(A(P')).$$

To see this, we make use of the following characterization of $s^\Psi(A(P))$:

**Lemma 4.3.6.** An element $G = \sum_{f \in F} \lambda_f D_f \in Z(P)$ is in $s^\Psi(A(P))$ if and only if, for each $f$,

$$\lambda_f \in \Lambda_{f^\perp},$$

where $\Lambda_{f^\perp}$ is the symmetric tensor algebra of the vector space $f^\perp \subset V$ (which itself depends on $\Psi$).

Before we prove the lemma, we show how this implies our theorem. If $G = \sum_{f \in F, f \neq \emptyset} \lambda_f D_f$ as in the lemma, then we may prove that $\pi_*(G) \in \sigma^\Psi(A(P'))$ by examining the pushforward term-by-term. Fixing $f$, we have that $\pi_*(\lambda_f D_f)$, when non-zero, equals $a \lambda_f D_{\tilde{\pi}_*(f)}$, with $a \in \mathbb{Q}$, and $\dim(\tilde{\pi}_*(f)) = \dim(f)$. Since $f$ and $\tilde{\pi}_*(f)$ span the same affine space up to translation, $f^\perp = \tilde{\pi}_*(f)^\perp$. Then $\pi_*(\lambda_f D_f) \in s^\Psi(A(P'))$.

**Proof of Lemma:** Recall that $s^\Psi : A(P) \to Z(P)$ is a $\mathbb{Q}$-vector space homomorphism $A(P) \cong \mathbb{Q}[\![D_1, \ldots, D_k]\!]/(I_P \cap \mathbb{Q}[\![D_1, \ldots, D_k]\!]) \to R(P)/(I_P + J^\Psi_P) \cong Z(P)$. Then, if we take $G \in \mathbb{Q}[\![D_1, \ldots, D_k]\!]$, one may show that $G \mapsto H \in Z(P)$ of the proposed form, by considering the algorithm in the proof of Lemma 4.2.4, for resolving $G$ into a squarefree element of $R(P)$ modulo $J^\Psi$. Specifically, one can show, inductively, that at each stage of the algorithm, the coefficient in front of each monomial $M$ is in $\Lambda_{f^\perp}$, where $f = \{i : D_i \text{ divides } M\}$.

Conversely, given $H = \lambda_f D_f \in R(P)$, where $\lambda_f \in \Lambda_{f^\perp}$, it is possible to construct an element $G \in A(P)$ such that $s^\Psi(G) = H$. Note that given any $v \in \Lambda_{f^\perp}$,

$$v D_f \equiv (\sum_i \langle \tilde{u}_i, v \rangle D_i) D_f \text{ modulo } J^\Psi_P.$$ 

Generalizing from this, it is possible to find an element $G \in \mathbb{Q}[\![D_1, \ldots, D_k]\!]$ that is equivalent to $H$ modulo $J^\Psi_P$. \qed
Note: Both directions of the above argument make subtle use of the fact that

$$f \circ f' \implies f'^\perp \subset f^\perp.$$
Chapter 5

The SI-interpolator Function

The interpolator function constructed here was first discovered by Berline and Vergne [BV2] in the case where $\Psi$ is an inner product, and by Garoufalidis and Pommersheim [GP] in general, who first called it an SI-interpolator function. We use the notation from [GP].

The connection between the Todd class and interpolator functions was first developed by Morelli [Mo]. The equivariant version of this was proved by Brion and Vergne [BV3]. Khovanskii and Pukhlikov showed that the Todd class was related to a differential operator on the perturbed polyhedron [KP].

5.1 Smoothing Out the Exponential Sum

One advantage that the exponential integral has over the exponential sum is that $I(P_{\vec{h}})$ is continuous in $\vec{h}$ (and analytic outside of certain hyperplanes). The exponential sum, since it generalizes a sum over discrete lattice points, is not continuous. However, there is another function, which we call the continuous exponential sum, $S_c(P)$, which is continuous in $\vec{h}$, and can be thought of as a continuous interpolation of the exponential sum on integral polyhedra.

We first define it on pointed cones. If $K$ is a pointed cone, then there exists some $u$ in the lattice $L$ such that $K - u$ is integral. We define $S_c(K) = e^{-u}(S(K - u))$. This definition is independent of the choice of $u$: it is easy to see from the fact that $S(K) =$
$\sum_{v \in (L \cap K)} e^{-v}$ that if $K_1$ and $K_2$ are two integral cones such that $K_1 - u = K_2$, then $S(K_2) = e^{-u}S(K_1)$. Also, it is clear that if $K$ itself is integral, then $S^c(K) = S(K)$.

For general polyhedra $P$, then $S^c(P) := \sum_v S^c(K_v)$, where $v$ ranges over vertices of $P$, and $K_v$ is the tangent cone of $v$ in $P$. Again, it is clear that $S^c(P) = S(P)$ if $P$ is integral. The following lemma shows that $S^c(P_{\vec{h}})$ behaves analogously to $I(P_{\vec{h}})$:

**Lemma 5.1.1.** Given any simple polyhedron $P$ with $k$ facets, there exists $G \in \text{Frac}(\Lambda)[[h_1, \ldots, h_k]]$ such that

- $G(\vec{h})$ is absolutely convergent for all $\vec{h} \in \mathbb{Q}^k$. In particular, $G$ is smooth for all $\vec{h}$.
- $G(\vec{h}) = S^c(P_{\vec{h}})$ for all $\vec{h} \in \mathcal{D}_P$.

**Proof.** There are no substantial differences between the proof of this result and Theorem 4.1.1. The important properties of $S^c(P)$ that allow the same proof to hold are as follows:

- $S^c(P) = \sum_u S^c(K_u)$, a sum over vertex cones of $P$.
- $S^c(P + v) = e^{-v}S^c(P)$.
- For $K$ a simple cone, and $\vec{h}$ in the interior of $\mathcal{D}_P$. $S^c(P_{\vec{h}}) = e^{-v(\vec{h})}S^c(P)$

An analogous version of Lemma 4.1.2 is also true: that lemma held because

$$\sum_u \chi(\text{feas}(u, P)) = \chi(\text{rec}(P)),$$

with all cones full-dimensional. Thus

$$\sum_u S(\chi(\text{feas}(u, P))) = S(\text{rec}(P)).$$

Since all cones have vertex at the origin,

$$\sum_u S^c(\chi(\text{feas}(u, P))) = S^c(\text{rec}(P)).$$
5.2 Unimodular Polyhedra and the Todd Operator

Our goal over the next section is to show that given a simple polyhedron $P$ without straight lines, then the power series $G$ from Lemma 5.1.1 is in the image of the evaluation map $E : A(P) \to \Lambda[[h_1, \ldots, h_k]]$. Since the evaluation map is injective on such $P$, there is at most one element that maps to $G$. It is that element that we will call the Todd class of $P$.

There is one class of polyhedra for which the Todd class is easy to construct. We say that a simple cone $K$ is unimodular if the index $i(K) = 1$. A simple polyhedron is unimodular if the tangent cone to each face is unimodular.

Lemma 5.2.1. Let $P$ be unimodular. Then if we define

$$\text{Td}(P) := \prod_{i=1}^{k} \frac{D_i}{1 - e^{-D_i}} \in R(P),$$

we have that

$$\mathcal{E}(\text{Td}(P))(\vec{h}) = S^c(P_{\vec{h}})$$

for all $\vec{h} \in D_P$.

Proof. Let us first consider the case when $P = K$ is a unimodular cone with vertex $u$ and primitive generators $v_1, \ldots, v_k$. Then $D_K = Q^k$.

If $K = K_1 + v$, where $K_1$ is integral, then $S^c(K_{\vec{h}}) = e^{-v}S^c(K_{1,\vec{h}})$ and $I(K_{\vec{h}}) = e^{-v}I(K_{1,\vec{h}})$. Then we may further suppose that $K$ is integral.

Next recall that for all simple cones, $\mathcal{E}(G) = g(u_1, \ldots, u_k)I(K_{\vec{h}})$, where $G = g(D_1, \ldots, D_k) \in R(K)$, and $u_1, \ldots, u_k$ is the dual basis to the basis of primitive normal vectors $\tilde{u}_1, \ldots, \tilde{u}_k$. When $i(K) = 1$, then $u_i = v_i$ for each $i$. 
Then for all $\vec{h} \in \mathbb{Q}^k$,

$$
\mathcal{E}(\text{Td}(K))(\vec{h}) = \left( \prod_{i=1}^{k} \frac{v_i}{1 - e^{-v_i}} \right) I(K_\vec{h})
$$

$$
= \left( \prod_{i=1}^{k} \frac{v_i}{1 - e^{-v_i}} \right) e^{-u(\vec{h})} \prod_{i=1}^{k} v_i^{-1}
$$

$$
= e^{-u(\vec{h})} \prod_{i=1}^{k} \frac{1}{1 - e^{-v_i}}
$$

$$
= \mathcal{S}^c(K_\vec{h})
$$

The last equality is true since $\mathcal{S}^c(K) = S(K) = \sum_{v \in (L \cap K)} e^{-v}$ for integral $K$.

Next suppose $P$ is any unimodular polyhedron. Recall that given any $G \in \mathcal{R}(P)$, $\mathcal{E}(G) = \sum_u \mathcal{E}_{K_u}(G_u)$, where $u$ ranges over vertices of $P$, $K_u$ is the tangent cone of $u$ in $P$, and $G_u$ is the natural ring homomorphism from $\mathcal{R}(P)$ to $\mathcal{R}(K_u)$ that maps any $D_i$ such that $u \notin F_i$ to 0. It is easy to see from the Todd class formula that $\text{Td}(P)_u = \text{Td}(K_u)$ for any vertex $u$.

Then given any $\vec{h}$ in the interior of $\mathcal{D}_P$,

$$
\mathcal{E}(\text{Td}(P))(\vec{h}) = \sum_u \mathcal{E}_{K_u}(\text{Td}(P)_u)(\vec{h})
$$

$$
= \sum_u \mathcal{E}_{K_u}(\text{Td}(K_u))(\vec{h})
$$

$$
= \sum_u \mathcal{S}^c(K_u, \vec{h})
$$

$$
= \mathcal{S}^c(P_\vec{h})
$$

The last line is true, since vertex cones of $P_\vec{h}$ are translates of the vertex cones of $P$, for $\vec{h}$ in the interior of $\mathcal{D}_P$. By the continuity of both $\mathcal{E}(\text{Td}(P))$ and $\mathcal{S}^c(P_\vec{h})$, the two must be equal for $\vec{h}$ on the boundary of $\mathcal{D}_P$ as well.

For unimodular polyhedra $P$, we define the Todd class $\text{Td}(P) \in A(P)$ equal to (the equivalence class of)

$$
\prod_{i=1}^{k} \frac{D_i}{1 - e^{-D_i}}
$$
To extend the construction to simple polyhedra, we must use the fact that any simple polyhedron is a perturbation of some unimodular polyhedron.

Lemma 5.2.2. Let $P'$ be a simple polyhedron with vertices. Then there exists a unimodular polyhedron $P$ such that $P' = P + \vec{h}$ for some $\vec{h} \in \mathcal{D}_P$.

Proof. We will provide an inductive algorithm for constructing such a unimodular $P$:

If $P'$ is not unimodular, then there exists some face index $f$ such that the tangent cone $K_f$ has index $> 1$. We may choose such $f$ such that:

- For all $g \in \mathcal{F}_P$ such that $|g| < |f|$, $K_g$ is unimodular.
- $i(K_f) = \max\{i(K_g) : |g| = |f|\}$.

Recall that $K_f$ is the intersection of half-spaces $\{S_i : i \in f\}$. Then the dual cone $K_f^*$ is the pointed, simple cone generated by the primitive normal vectors $\{\tilde{u}_i : i \in f\}$. It is not hard to show (see [Bv2]) that $i(K_f) = i(K_f^*)$. Recall that the index of the pointed cone $K_f^*$ is the index of the lattice $L_f^* \cap \text{Span}(K_f^*)$ formed by its primitive generators inside $L_f^* \cap \text{Span}(K_f^*)$. Since $i(K_f^*) > 1$, there must be some $\tilde{u} \in L_f^* \cap \text{Span}(K_f^*)$ such that $u \notin L_f^*$. After translating by an element of $L_f^*$, we may assume that $\tilde{u}$ is in the closed parallelepiped formed by the vectors $\tilde{u}_i$. Note that $\tilde{u}$ cannot be on the boundary of this parallelepiped: otherwise there would be some proper subset $g \subset f$ such that $i(K_g) = i(K_g^*) > 1$.

Since $\tilde{u}$ is a positive linear combination of the primitive inward normal vectors $\tilde{u}_i : i \in f$, there exists a hyperplane $H \subset V$ with normal vector $\tilde{u}$ such that $H$ contains $f$ and bounds $P'$. Let $S$ be the half-space bounded by $H$ which contains $P'$, and let $v \in V$ be chosen so that $\langle \tilde{u}, v \rangle = -1$. Then for small enough $h \in \mathbb{Q}_{< 0}$, the interior of $S - hv$ contains all vertices of $P'$ that are not contained in $f$. Let $P_1 = P' \cap (S - hv)$. Then the following is true about $P_1$:

- $H - hv \cap \tilde{P}$ is a facet of $\tilde{P}$. All bounding hyperplanes of $P'$ are also bounding hyperplanes of $\tilde{P}$. Then $\mathcal{F}_P$ is naturally a subset of $\mathcal{F}_{P_1}$. We will notate $H_{k+1} = H - hv$.

- Given any face index $g \in \mathcal{F}_P$ such $f \not\subset g$, the corresponding face of $P_1$ is non-empty if and only if $g \subset P'$ is non-empty. Furthermore, if they are non-empty, then they have the same tangent cones. (This is because we were careful to make sure $S - hv$ contained all vertices outside $f$).
• The face index \( f \) corresponds to the empty face in \( P_1 \).

• \( P' \) is a perturbation of \( P_1 \). Specifically, \( P' = P_1,\vec{h} \), where \( \vec{h} = (0, \ldots, 0, h) \).

• For any \( j \in f \), define \( f_j = (f \cup \{ k + 1 \}) \setminus \{ j \} \). Then \( i(K_{f_j}) < i(K_f) \).

Only the last point deserves explanation. The dual cone to \( K_{f_j} \) has primitive generators

\[
\{ \tilde{u}_i : i \in f, i \neq j \} \cup \{ \tilde{u} \}.
\]

Since the index is the volume of the parallelipiped formed by these vectors, and \( \tilde{u} \) is in the interior of the parallelipiped formed by \( \{ \tilde{u}_i : i \in f \} \), \( i(K_{f_j}) < i(K_f) \).

Then we repeat the same procedure if necessary, creating a \( P_2 \). and so forth. The algorithm must end, since at each stage we are either reducing the maximum index for tangent cones corresponding to faces of a certain dimension, or reducing the number of cones that reach that maximum.

Note: It is not necessary in the above algorithm that \( P' \) contain vertices, as long as one is careful not to perturb the new facet in each step too far.

**Corollary 5.2.3.** Let \( P \) be a simple polyhedron without straight lines. Then there exists some element \( \text{Td}(P) \in A(P) \) such that \( \mathcal{E}(\text{Td}(P)) (\vec{h}) = S^c(P_{\vec{h}}) \) for \( \vec{h} \in D_P \).

**Proof.** By the previous lemma, there exists some unimodular polyhedron \( \tilde{P} \) such that \( P = \tilde{P}_{\vec{h}_0} \) for some \( \vec{h}_0 \in D_{\tilde{P}} \). By Lemma 5.2.1, \( \mathcal{E}(\text{Td}(\tilde{P})) (\vec{h}) = S^c(P_{\vec{h}}) \) for all \( \vec{h} \in D_{\tilde{P}} \).

Suppose \( \tilde{P} \) has \( k \) facets, and \( P \) has \( l \) facets. By Lemma 4.3.3, there exists a linear map \( s : \mathbb{Q}^l \rightarrow \mathbb{Q}^k \), such that \( \mathcal{E}_{P'}(\pi_*(\text{Td}(\tilde{P})))(\vec{g}) = \mathcal{E}_P(\text{Td}(\tilde{P}))((\vec{h}_0 + s(\vec{g}))) \). Furthermore, for small enough \( \vec{g} \in Q^l, h_0 + s(\vec{g}) \in D_{\tilde{P}} \), and \( \tilde{P}_{\vec{h}_0 + s(\vec{g})} = P_{\vec{g}} \). For such \( \vec{g} \),

\[
\mathcal{E}_{P'}(\pi_*(\text{Td}(\tilde{P})))(\vec{g}) = \mathcal{E}_P(\text{Td}(\tilde{P}))((\vec{h}_0 + s(\vec{g})))
\]

\[
= S^c(\tilde{P}_{\vec{h}_0 + s(\vec{g})})
\]

\[
= S^c(P_{\vec{g}})
\]

Then since \( \mathcal{E}_{P'}(\pi_*(\text{Td}(\tilde{P})))(\vec{g}) \) and \( S^c(P_{\vec{g}}) \) are both analytic on the interior of \( D_P \) and continuous on the boundary, they must be equal for any \( \vec{g} \in D_P \).
Since the evaluation map $A(P) \to \Lambda[[h_1, \ldots, h_k]]$ is injective, the element $Td(P)$ from Corollary 5.2.3 is unique. We define it to be the Todd class of $P$.

## 5.3 The SI-interpolator Function on Simple Polyhedra

For all future sections, we fix a complement map $\Psi$. We are finally ready to define our interpolator function on simple polyhedra.

**Definition 5.3.1.** Let $P$ be a simple polyhedron without straight lines. Let $s^\Psi : A(P) \to Z(P)$ be the section defined in Corollary 4.2.8. Then since $Z(P)$ is generated
over \( \Lambda \) by \( \{ D_f : D_f \in I_P \} \), we have a unique expression

\[
s^\Psi(Td(P)) = \sum_{f \in F_P} a^\Psi_f D_f,
\]

where each \( a^\Psi_f \in \Lambda \). (To ease the notation, we just set \( a^\Psi_f = 0 \) for \( f \) such that \( D_f \in I_P \).) Then for any non-empty face \( F \in \text{Face}(P) \), define

\[
\mu^\Psi(F, P) := \frac{a^\Psi_f}{i(K_f)},
\]

where \( f \) is the unique face index corresponding to \( F \), and \( K_F \) is the tangent cone of \( F \) in \( P \). We also set \( \mu(\emptyset, P) = 0 \).

The most important property of our interpolator function is that it relates \( S(P) \) to \( I \) on each face of \( P \).

**Theorem 5.3.2.** Let \( P \) be a simple polyhedron without straight lines. Then

\[
S^c(P) = \sum_{F \in \text{Face}(P)} \mu^\Psi(F, P) I(F).
\]

**Proof.** Using notation from the definition above, we have that

\[
\sum_{F \in \text{Face}(P)} \mu^\Psi(F, P) I(F) = \sum_{F \in \text{Face}(P)} (a^\Psi_f/i(K_f))I(F)
\]

\[
= \sum_{F \in \text{Face}(P)} (a^\Psi_f/i(K_f))[i(K_f)(\mathcal{E}(D_f)(\vec{0}))]
\]

\[
= \mathcal{E}\left( \sum_{F \in \text{Face}(P)} a^\Psi_f D_f \right)(\vec{0})
\]

\[
= \mathcal{E}(Td(P))(\vec{0})
\]

\[
= S^c(P)
\]

\[\square\]

**Example 5.3.3.** Let \( K \) be as in Example 4.2.9. Since \( K \) is one-dimensional, \( K \) is necessarily unimodular, so

\[
Td(K) = \frac{D_1}{1 - e^{-D_1}} = 1 + B(D_1)D_1,
\]
where

\[ B(D_1) := \frac{1}{1 - e^{-D_1}} - \frac{1}{D_1} \]

is a power series in \( D_1 \) whose \( n \)th coefficient is \((-1)^{n+1}B_{n+1}/(n+1)!\), where \( B_n \) is the \( n \)th Bernoulli number. Then by Example 4.2.9,

\[ s^\Psi(Td(K)) = 1 + B(v_1)D_1, \]

implying that\[ \mu^\Psi(K, K) = 1 \]

and\[ \mu^\Psi(v, K) = B(v_1). \]

**Example 5.3.4.** Let \( K \) be as in Example 4.2.10. Furthermore, suppose that \( K \) is unimodular. Then we have that

\[ Td(K) = \frac{D_1}{1 - e^{-D_1}} \frac{D_2}{1 - e^{-D_2}} \]

\[ = 1 + B(D_1)D_1 + B(D_2)D_2 + B(D_1)B(D_2)D_1D_2 \]

so that, by Example 4.2.10,

\[ s^\Psi(Td(K)) = 1 + B(w_1)D_1 + B(w_2)D_2 \]

\[ + [B(w_1)B(w_2) - \frac{B(w_1) - B(u_1)}{u_2} - \frac{B(w_2) - B(u_2)}{u_1}]D_1D_2. \]

This implies that\[ \mu^\Psi(K, K) = 1, \]

\[ \mu^\Psi(F_1, K) = B(w_1), \]

\[ \mu^\Psi(F_2, K) = B(w_2), \]

\[ \mu^\Psi(v, K) = B(w_1)B(w_2) - \frac{B(w_1) - B(u_1)}{u_2} - \frac{B(w_2) - B(u_2)}{u_1}. \]
5.4 The SI-Interpolator Function on Non-Simple Polyhedra

We must take a more ad-hoc approach to define the interpolator function on non-simple polyhedra. There are two key ideas that will guarantee the success of this approach. Firstly, just as simple polyhedra are always perturbations of unimodular polyhedra, non-simple polyhedra are always perturbations of simple polyhedra. Second, even though the functions $I(P_h)$ and $S^c(P_h)$ are not smooth at $\vec{0}$ for non-simple $P$, they are both still continuous there.

By definition, the facets of non-simple polyhedra are not in general position. If $P$ is non-simple, then $\vec{0}$ is contained in at least one of the hyperplanes $\mathcal{H}(f, i)$. However, there exist vectors $\vec{h}$ that are not contained in any $\mathcal{H}(f, i)$, meaning that $P_{\vec{h}}$ will be simple. Furthermore, we may assume $\vec{h}$ is small enough so that:

- $P_{\vec{h}}$ has as many facets as $P$.
- $-\vec{h} \in \mathcal{D}_{P_{\vec{h}}}$, meaning that $P$ is a finite perturbation of $P_{\vec{h}}$.

**Lemma 5.4.1.** Let $P' = P_{\vec{h}}$ where $\vec{h}$ is chosen to satisfy the above conditions. Let $\vec{\pi}: \mathcal{F}_{P'} \to \text{Face}(P')$ be the pushforward map defined in [Section 7]. Let $F \in \text{Face}(P)$ be given, and let $A_F$ be the set of $f \in \mathcal{F}_{P'}$ such that

- $\vec{\pi}(f) = F$.
- $\dim(f) = \dim(F)$.

Set $\mu_{\vec{h}}^\Psi(F, P) := \sum_{f \in A_F} \mu^\Psi(f, P')$. Then $\mu_{\vec{h}}^\Psi(F, P)$ is independent of our choice of $\vec{h}$, so we may retitle it $\mu^\Psi(F, P)$. Furthermore, $S^c(P) = \sum_{F \in \text{Face}(P)} \mu^\Psi(F, P)I(F)$.

**Proof.** If $P$ is non-simple, there exists some face $F$ such that $K_F$ is non-simple. We may choose such an $F$ such that given any face $G$ containing $F$, $K_G$ is simple. The dual cone $K_F^*$ is generated by $\vec{u}_i$ for all $i$ such that $F \subset F_i$. This is a pointed, non-simple cone. Choose any $\vec{u}$ in the interior of this cone. Then since $\vec{u}$ is a linear combination of the generators $\vec{u}_i$ for all $F_i$ containing $F$, $\vec{u}$ is normal to the affine space $F^\|$. 
What follows is very similar to the proof of Lemma 5.2.2. There exists a hyper-
plane $H$ normal to $\tilde{u}$ that contains $F$ and bounds $P$. Since $\tilde{u}$ is in the interior of the
dual cone, $H$ doesn’t contain any larger faces. Letting $S$ be the half-space bounded
by $H$ and containing $P$, and choosing an outward normal vector $u \in V$ to $H$, we
choose $g_{k+1} \in \mathbb{Q}_{\leq 0}$ to be small enough so that $S - g_{k+1}u$ contains all vertices of $P$
not contained in $F$. Then let $P_1 = P \cap S - g_{k+1}u$. $P_1$ has one more bounding facet
$F_{k+1}$, bounded by $H_{k+1} := H - g_{k+1}u$.

$K_F$ is not a tangent cone of $P_1$. Furthermore, for any face $G \subset F$ with $\dim(G) = 
\dim(F) + 1$, the face $G \cap F_{k+1}$ has dimension equal to $\dim(F)$, since the hyperplane
$H_{k+1}$ does not contain $G$. This means that all of the new faces of dimension equal
to $\dim(F)$ are simple, and all faces of larger dimension remain simple. Repeating the
algorithm with any remaining non-simple tangent cones, we eventually construct a
simple polyhedron $\tilde{P}$. Suppose that $\tilde{P}$ has $l$ facets. Then $P = \tilde{P}_{\tilde{h}}$, where

$$g_i = \begin{cases} 
0 & \text{if } 0 \leq i \leq k \\
g_i & \text{if } k + 1 \leq i \leq l
\end{cases}$$

Now let $\tilde{h} = (h_1, \ldots, h_k) \in \mathbb{Q}^k$ be given from the theorem. Then

$$P_{\tilde{h}} = \tilde{P}_{\tilde{h}},$$

where

$$h_i = \begin{cases} 
h_i & \text{if } 0 \leq i \leq k \\
g_i & \text{if } k + 1 \leq i \leq l
\end{cases}$$

Furthermore, for small enough $\tilde{h}$, $\tilde{h} \in \mathcal{D}_{\tilde{P}}$. Since $\tilde{h}$ can always be scaled down
without changing $\mu_{\tilde{h}}^\Psi(F, P)$ for any faces $F$, we may assume that $\tilde{h}$ is small enough
so this is true.

Now we fix a face $F$ in $P$. Similar to our definition of $A_F \in \mathcal{F}_{P'}$, we define
$B_F \in \mathcal{D}_{\tilde{P}}$ as the set of $f \in \mathcal{F}_{\tilde{P}}$ such that

- $\tilde{\pi}_{\ast,\tilde{P}}^\tilde{P}(f) = F$.
- $\dim(f) = \dim(F)$.

Let $\pi_{\ast,\tilde{P}}^{\tilde{P},P'} : \mathcal{F}_{\tilde{P}} \to \mathcal{F}_{P'}$ is the pushforward map between $\tilde{P}$ and $P'$. We claim that
$\pi_{\ast}(B_F) = A_F$. This follows from the transitivity of the pushforward map, given by
Lemma 4.3.2.
The commutativity of the diagram in Theorem 4.3.5 means that

\[ \pi_*(s^\Psi(Td(\tilde{P}))) = s^\Psi(Td(P')) \]

\[ \pi_*\left( \sum_{f \in \mathcal{F}_P} a^\Psi_{f,\tilde{P}} D_f \right) = \sum_{g \in \mathcal{F}_{P'}} a^\Psi_{g,P'} D_g, \]

Then in particular,

\[ \pi_*\left( \sum_{f \in B_F} a^\Psi_{f,\tilde{P}} D_f \right) = \sum_{g \in A_F} a^\Psi_{g,P'} D_g. \]

Adding up the coefficients in front of all terms on both sides,

\[ \sum_{f \in B_F} a^\Psi_{f,\tilde{P}} [i(K_{\pi_*(f)})/i(K_f)] = \sum_{g \in A_F} a^\Psi_{g,P'} \]

\[ \sum_{f \in B_F} a^\Psi_{f,\tilde{P}}/i(K_f) = \sum_{g \in A_F} a^\Psi_{g,P'}/i(K_g) \]

\[ \sum_{f \in B_F} \mu^\Psi(f, \tilde{P}) = \sum_{g \in A_F} \mu^\Psi(g, P') = \mu_{\tilde{h}}^\Psi(F, P) \]

Note that the left hand sum is independent of \( \tilde{h} \), since the construction \( \tilde{P} \) is independent of \( \tilde{h} \), and the set \( B_F \) uses only the pushforward from \( \tilde{P} \) to \( P' \). This proves that \( \mu_{\tilde{h}}^\Psi(F, P) \) is independent of \( \tilde{h} \), and can be called \( \mu^\Psi(F, P) \).

By Theorem 5.3.2,

\[ S^e(P) = \sum_{f \in \mathcal{F}_{P'}} \mu^\Psi(f, P') I(f). \]

But since the Todd class, and thus \( \mu(F, P) \), do not change under perturbations \( P_{\tilde{h}} \).
where \( \vec{g} \) is in the interior of \( \mathcal{D}_{P'} \),

\[
S^c(P'_\vec{g}) = \sum_{f \in F_{P'}} \mu^\Psi(f, P') I(f_{\vec{g}}).
\]

Next, since \(-\vec{h}\) is on the boundary of \( \mathcal{D}_{P'} \), we have that

\[
\lim_{\vec{g} \to \vec{h}} S^c(P'_\vec{g}) = \sum_{f \in F_{P'}} \mu^\Psi(f, P') \lim_{\vec{g} \to \vec{h}} I(f_{\vec{g}}).
\]

But the continuity of \( S^c(P'_\vec{g}) \) (see Lemma 5.1.1) and of \( I(f_{\vec{g}}) \) when \( \dim(\tilde{\pi}_*f) = \dim(f) \) (see the proof of Corollary 4.1.4 implies that

\[
S^c(P) = \sum_{f \in F_{P'}} \mu^\Psi(f, P') I(\tilde{\pi}_*f) = \sum_{F \subset P} \left( \sum_{f \in A_F} \mu^\Psi(f, P') I(F) \right) = \sum_{F \subset P} \mu^\Psi(F, P) (I(F))
\]

\[\square\]

### 5.5 Properties of the Interpolator Function

**Theorem 5.5.1.** The interpolator coefficient \( \mu^\Psi(F, P) \in \Lambda \) depends only on \( \Psi \) and the tangent cone of \( F \) in \( P \). (In other words, it is “local”.)

**Proof.** First suppose that \( P \) is simple, and let \( f \) be the unique face index corresponding to \( F \). Recall that \( F^\parallel \) is the linear space parallel to the affine space \( \text{Span}(F) \), and \( F^\perp \) is its complement under \( \Psi \). Then there is a surjective map \( V \to F^\perp \), given by quotienting along the subspace \( F^\parallel \). Let \( \tilde{K}_f \) be the image of \( K_f \) under this map. Then \( \tilde{K}_f \) is a pointed, simple cone that is full-dimensional in \( F^\perp \), and \( K_f = \tilde{K}_f + F^\parallel \). Let \( v \) be the vertex of \( \tilde{K}_f \)

To prove the theorem, it suffices to show that \( \mu^\Psi(F, P) = \mu^\Psi(v, \tilde{K}_f) \). In order to
do this, we first note that there is a natural ring homomorphism

\[ p_* : R(P) = \Lambda[[D_i : H_i \text{ bounds } P]] \to R(\bar{K}_f) = \Lambda_{F^\perp}[[D_i : \bar{H}_i \text{ bounds } \bar{K}_f]]. \]

\( (p_* \text{ is similar to a pushforward.}) \) Note that any facet of \( \bar{K}_f \) corresponds to a facet of \( K_f \), which in turn corresponds to a facet of \( P \) contained in \( f \). Then it is sensible to say that \( p_*(D_i) = D_i \) if \( i \in f \). If \( i \notin f \), then we set \( p_*(D_i) = 0 \). Next, the quotient map \( V \to F^\perp \) induces a map \( \Lambda \to \Lambda_{F^\perp}. \)

This restricts to a map \( Z(P) \to Z(\bar{K}_f). \) We claim this further induces a map

\[ A(P) \to A(\bar{K}_f). \]

Indeed, it is trivial to see that \( I_P \leftrightarrow I_{\bar{K}_f}. \) Also, if we let \( g_v \) be a generator of \( J_P \), for \( v \in V \). It is easy to show that \( p_*(g_v) = g_{\bar{v}}, \) the generator of \( J_{\bar{K}_f} \) corresponding to \( \bar{v} \), the image of \( v \) under the quotient map.

We note that the commutative diagram

\[
\begin{array}{ccc}
Z(P) & \xrightarrow{p_*} & Z(\bar{K}_f) \\
\downarrow{s^\Psi} & & \downarrow{s^\Psi} \\
A(P) & \xrightarrow{p_*} & A(\bar{K}_f)
\end{array}
\]

still holds true.

To show this, one can use the characterization of the image \( s^\Psi \) under Lemma 4.3.6, which is easily preserved by \( p_* \). Alternatively, it is easy enough to show that \( p_* \) sends the ideal \( J^\Psi_P \) to \( J^\Psi_{\bar{K}_f}. \) (Either way, it is crucial that we chose the subspace \( F^\perp \) coming from the complement map.)

Next, we must show that \( p_*(\text{Td}(P)) = \text{Td}(\bar{K}_f). \) Given the definition of \( \text{Td}(\bar{K}_f) \), it is enough to show that \( E(p_*(\text{Td}(P))) = S^c(\bar{K}_f). \) The basic idea is that there is a map \( \Lambda[[h_1, \ldots, h_k]] \to \Lambda[[\bar{h}_i : i \in f]] \) (very similar to \( p_* \) in definition) such that \( E(G) \mapsto E(p_*(G)). \) Using the formulas for \( S^c \) on simple polyhedra, it is easy to see that \( S^c(P) \mapsto S^c(\bar{K}_f) \) under this map.
Writing $s^\Psi(\text{Td}(P)) = \sum_g a_g^\Psi D_g$, we have that

$$s^\Psi(\text{Td}(\bar{K}_f)) = (s^\Psi \circ p_*)(\text{Td}(P))$$

by the commutative diagram above

$$= (p_\ast \circ s^\Psi)(\text{Td}(P))$$

$$= p_\ast(\sum_g a_g^\Psi D_g)$$

$$= \sum_{g \subset f} p_\ast(a_g^\Psi)D_g$$

Then in particular, $p_\ast(a_f^\Psi)$ is the coefficient in front of $D_f$ in $s^\Psi(\text{Td}(\bar{K}_f))$, so

$$\mu(v, \bar{K}_f) = p_\ast(a_f^\Psi)/i(\bar{K}_f).$$

By the characterization of the image of $s^\Psi$ in [Lemma 7.5], $s^\Psi \in \Lambda_{F_\perp}$, so $p_\ast(a_f^\Psi) = a_f^\Psi$. Also, $i(\bar{K}_f) = i(K_f)$. Finally, we have

$$\mu(v, \bar{K}_f) = a_f^\Psi/i(\bar{K}_f) = \mu(F, P).$$

Lastly, suppose $P$ is non-simple, and $F \subset P$. Recall from Lemma 5.4.1 that $\mu^\Psi(F, P) = \sum_{f \in A_F} \mu^\Psi(f, P')$, where $P' = P_{\vec{h}}$ is a small, simple perturbation of $P$ and $A_F$ is the set of face indices of $P'$ that pushforward to $F$ and have the same dimension as $F$.

Let $K_F$ be the tangent cone of $F$ in $P$. Note that if $P_{\vec{h}}$ is a small, simple perturbation of $P$, then $K_{F,\vec{h}_F}$ is a small, simple perturbation of $K_F$. (Here $\vec{h}_F$ takes only the components of $\vec{h}$ corresponding to facets containing $F$. Furthermore, $A_F \in \mathcal{F}_{P'}$ is naturally identified with the corresponding set in $\mathcal{F}_{K_{F,\vec{h}_F}}$, and for each $f \in A_F$, $\mu^\Psi(f, P') = \mu^\Psi(f, K_{F,\vec{h}_F}')$, since both polyhedra are simple, and the tangent cone of $f$ in $P'$ is equal to the tangent cone of $f$ in $K_{F,\vec{h}_F}$.)

The following corollary comes directly from the previous proof.

**Corollary 5.5.2.** Let $F$ be a face of a polyhedron $P$. Suppose that $F$ contains an affine space $W + v$, such that $W$ is a sub-vector space of $V$. Let $\bar{P}$ and $\bar{F}$ be the images of $P$ and $F$, respectively, under the projection map $V \to V/W$.

Furthermore, let

$$i^\Psi : V/W \to W_\perp \subset V$$
be the section of the projection map induced by $\Psi$. Then

$$\mu^\Psi(F, P) = i^\Psi(\mu^\Psi(\bar{F}, \bar{P})).$$

The interpolator function found by Berline and Vergne [BV1] also satisfies the property in this corollary. In fact, they showed that, given $\Psi$, there is a unique interpolator function satisfying Corollary 5.5.2. For this reason, $\mu^\Psi$ is equal to the local Euler-Maclaurin formula in [BV1].
Appendix A

A Toric Geometry Primer

The purpose of this appendix is to briefly describe how a toric variety corresponds to a polyhedron in a vector space with a lattice, and how the objects $Z(P)$ and $A(P)$ [cite] are equal to the group of equivariant cycles and the ring of equivariant cohomology classes, respectively. There are no proofs here: see [cite] and [cite].

As before, let $L \cong \mathbb{Z}^n$ be a free group with $n$ generators, and let $V \cong L \otimes \mathbb{Q}$. Let $P$ be an $n$-dimensional polyhedron in $V$. For any face $F$ in $V$, let $\sigma_F$ be the cone of feasible directions of $F$ in $P$: recall that this is the tangent cone translated so that its face of smallest dimension contains the origin.

Then $\sigma_F \cap L$ is a monoid, so we may consider the monoid algebra $\mathbb{C}[\sigma_F \cap L]$. Let

$$U_F := \text{Spec}(\mathbb{C}[\sigma_F \cap L]).$$

This is an $n$-dimensional complex affine variety. If $F \subset G$, then $\sigma_F \cap L \subset \sigma_G \cap L$, so there is a natural injection $U_G \subset U_F$. We define $X_P$ to be the union $\cup\{U_F : F \subset P\}$, modulo the identification of $U_G$ with its image in $U_F$ for each $F \subset G$. Note that $P$ is always a face of itself. Since $\sigma_P = V$,

$$U_P \cong \text{Spec}(\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]) \cong (\mathbb{C}^*)^n.$$

$U_P$ is a complex torus which is open and dense in $X_P$. Furthermore, the natural action of $U_P$ on itself (by multiplication) extends to an action on $X_P$. [cite] This designates $X_P$ as a toric variety, with embedded torus $T := U_P$. $X_P$ is always
projective: conversely, if a toric variety \( X \) is projective, it equals \( X_P \) for some \( P \).

*Example* A.0.1. Let \( P \) be a pointed unimodular cone of dimension \( n \), with vertex \( v \) and primitive generators \( v_1, \ldots, v_n \). Then \( \sigma_v = P - v \), and \( \sigma_v \cap L \) is the monoid generated by \( v_1, \ldots, v_n \). Then

\[
X_P = U_v \cong \text{Spec}(\mathbb{C}[X_1, \ldots, X_n]) \cong \mathbb{C}^n.
\]

The action of \( T \) on \( X_P \) is multiplication, once the appropriate isomorphism \( T \cong (\mathbb{C}^*)^n \) is chosen.

*Example* A.0.2. Let \( P \) be the simple cone \( K \) with vertex \( v \) in Figure A·1. Then \( \sigma_v \) has three generators \( v_1, v_2, v_3 \), with \( v_1 + v_2 = 2v_3 \). Then \( X_P = U_v \) is the locus of \( z^2 - xy \) in \( \mathbb{C}^3 \). Note that \( X_P \) is singular. If we identify \( X_P \) as two copies of \( \mathbb{C}^2 \) glued together along \( xy = 0 \), and choose the correct isomorphism \( T \cong \mathbb{C}^2 \), then the action of \( T \) on \( X_P \) is multiplication.

Many properties of \( P \) translate to properties of \( X_P \). For example, \( P \) is unimodular if and only if \( X_P \) is nonsingular. Also, \( P \) is a polytope if and only if \( X_P \) is complete.

From now on, we suppose that \( P \) is simple. Let \( F \) be any face of \( P \) of dimension \( m \). Recall that \( F \) is a full-dimensional polyhedron in the affine space \( \text{Span}(F) \). Then there is a toric variety \( X_F \) of dimension \( m \). (The construction requires choosing an origin of \( W_F \), although the resulting variety is independent of that choice.)
We claim that $X_F$ is naturally a closed subvariety of $X_P$. Given any face $G \subset F$, let $\sigma_{G,P}$ be the cone of directions of $G$ in $P$. Similarly, let $\sigma_{G,F}$ be the cone of directions of $F$ in $P$. Then $\sigma_{G,F}$ is a face of $\sigma_{G,P}$, so there is a natural ring homomorphism

$$\mathbb{C}[\sigma_{G,P} \cap L] \to \mathbb{C}[\sigma_{G,F} \cap L]$$

that maps $v \mapsto 0$ if $v \notin \sigma_{G,F}$. This induces an injective map $U_{G,F} \to U_{G,P}$. As $G$ ranges over faces of $F$, we induce a map $X_F \to X_P$.

For all details on equivariant cohomology, see [cite] and [cite]. We restrict to the case when $P$ is simple. Then $\{X_F : F \subset P\}$ is exactly the set of $T$-invariant closed subvarieties of $P$, and they generate the equivariant cycle group

$$Z(P) \cong \Lambda[X_F : F \subset P]$$

over $\Lambda$.

Also, each subvariety $[X_F]$ corresponds to an equivariant cohomology class $[X_F]$. If $F_1, \ldots, F_k$ are the facets of $P$, then we notate $D_i = [X_{F_i}]$. The set $D_1, \ldots, D_k$ are the $T$-invariant divisors of $X_P$. Let $F$ be a face with corresponding face index $f$. Then $X_F = \cap_{i \in f} X_{F_i}$, with all intersections transversal. Because of the singularities that occur when $P$ is not unimodular,

$$\prod_{i \in f} D_i = i(K_f)^{-1}[X_F].$$

Then the equivariant cohomology ring is generated over $\Lambda$ by $D_1, \ldots, D_k$. The ideal $I_P$ corresponds to empty intersections: if $f$ is a face index corresponding to the empty face, then $\prod_{i \in f} D_i = 0$ because $\cap_{i \in f} X_{F_i} = \emptyset$. The ideal $J_P$ is constructed as follows: an element $v \in L$ corresponds naturally to a rational function on $X_P$, with divisor is $\sum_{i=1}^k \langle \bar{u}_i, v \rangle D_i$. Since this divisor is equivariant, the character $v$ is equal to $\sum_{i=1}^k \langle \bar{u}_i, v \rangle D_i$ in the equivariant cohomology ring.
See [Fu2] or [Br3] for a full argument that $\Lambda[[D_1, \ldots, D_k]]/I + J_P$ is the equivariant cohomology ring.

The equivariant cohomology ring, like the rational cohomology ring, has a canonical Todd class. (See [BV2]. For a discussion of the relationship with the equivariant Todd class and the exponential sum function, see [BV3]. For a construction of the cycle-level expression of the Todd class, see [FP].

Suppose that $P, P'$ are full-dimensional simple polyhedra, with $P' = P_{\vec{h}}$ for $\vec{h} \in D_P$. Then we claim there is a canonical morphism of varieties $X_P \to X_{P'}$ that respects the action of the torus. If $\vec{h}$ is in the interior of $D_P$, then all tangent cones of $P$ are translates of tangent cones of $P'$. In this case, $X_P \cong X_{P'}$.

Suppose then that $\vec{h}$ is on the boundary. Let $f \in \mathcal{F}_P$, and let $f' := \tilde{\pi}^* (f) \in \mathcal{F}_{P'}$. Then one can prove that $\sigma_{f'} \subset \sigma_f$, and that the induced map $C[\sigma_{f'} \cap L] \to C[\sigma_f \cap L]$ is a ring homomorphism. This induces a morphism $U_f \to U_{f'}$. Letting $f$ vary all over elements of $\mathcal{F}_P$, it can be shown that the maps respect the various glueings to form a map $X_P \to X_{P'}$.

This map induces a pushforward map on both equivariant cycles and cohomology classes. To see that these agree with the definitions in Section 4.3, see [Br2].
Appendix B

Inverting the Interpolator Function

This appendix is to show an interpolator function can easily be ‘inverted’. For each face $F$ of any polyhedron $P$ in $V$, let $\mu(F, P) \in \Lambda$ be given such that:

1. $\sum_{F \subset P} \mu(F, P)I(F) = S^c(P)$
2. $\mu(P, P) \in \Lambda^\times$ for any polyhedron $P$

In particular, $\mu^\Psi(F, P)$ satisfies these properties for any complement map $\Psi$. The first property is exactly [Theorem 8 - Lemma 5.4.1 in the case of non-simple $P$], while the second may be argued as follows. By [Theorem 9], $\mu^\Psi(P, P) = \mu^\Psi(\text{Span}(P), \text{Span}(P))$, since $\text{Span}(P)$ is the tangent cone of $P$ in itself. By [cite], $\mu^\Psi(\text{Span}(P), \text{Span}(P)) = \mu^\Psi(\{v\}, \{v\})$, where $v$ is any element of $\text{Span}(P)$. But this is easily seen to be 1 by the first property and the fact that $I(\{v\}) = S^c(\{v\}) = e^{-v}$.

Given such a $\mu$, we will construct an interpolator function $\nu$ in the other direction: namely

$$\sum_{F \subset P} \nu(F, P)S^c(F) = I(P).$$

Let a polyhedron $P$ in $V$ be given. Suppose that $P$ has $l$ non-empty faces: we choose a numbering $F_1, \ldots, F_l$ of decreasing dimension. (This departs from our usual practice of numbering only the facets.) Note that since $P$ is the face of highest dimension,
$F_1 = P$. We define an $l$-by-$l$ matrix $M$ with coefficients in $\Lambda$ such that

$$
M_{i,j} = \begin{cases} 
\mu(F_j, F_i) & \text{if } F_j \text{ is a face of } F_i, \\
0 & \text{otherwise}.
\end{cases}
$$

Then $M$ induces a linear map from $\Lambda^l$ to itself. We define two vectors on $\Lambda^l$:

$$
\tilde{I} = (I(F_1), \ldots, I(F_l))
$$

and

$$
\tilde{S}^c = (S^c(F_1), \ldots, S^c(F_l)).
$$

The first property of $\mu$ says precisely that

$$
M \cdot \tilde{I} = \tilde{S}^c.
$$

Since the faces are numbered in decreasing dimension, $M$ is an upper-triangular matrix. The second property of $\mu$ implies that $M$ has units along the diagonal, so $M$ is invertible over $\Lambda$. Let $N := M^{-1}$. Then $N \cdot \tilde{S}^c = \tilde{I}$. Restricting this expression to the first row, we have that

$$
\sum_{j=1}^{l} N_{1,j} S^c(F_j) = I(F_1) = I(P).
$$

We may then define $\nu(F_j, P) := N_{1,j}$.

In the case where $\mu = \mu^\Psi$, the resulting function $\nu(F, P)$ is easily seen to be local - it satisfies [Theorem 9?] - and to satisfy an inductive property analogous to [Corollary 9.1.1 -not there yet]. This implies that it is the same function constructed in [cite Pommersheim-Garoufalidis], there called an IS-interpolator function.
References


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