Two theorems on Galois representations and Shimura varieties

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TWO THEOREMS ON GALOIS REPRESENTATIONS AND
SHIMURA VARIETIES

by

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ABSTRACT

One of the central themes of modern Number Theory is to study properties of Galois
and automorphic representations and connections between them. In our dissertation, we
describe two different projects that study properties of these objects. In our first project,
which is analytic in nature, we consider Artin representations of $\mathbb{Q}$ of dimension 3 that
are self-dual. We show that these occur with density 0 when counted using the conductor.
This provides evidence that self-dual representations should be rare in all dimensions. Our
second project, which is more algebraic in nature, is related to automorphic representations.
We show the existence of canonical models for certain unitary Shimura varieties. This
should help us in computing certain cohomology groups of these varieties, in which regular
algebraic automorphic representations having useful properties should be found.
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List of Symbols

\(\rho\) Artin representation

\(q(\rho)\) Conductor ideal of Artin representation \(\rho\)

\(\Delta(\rho)\) Absolute norm of the conductor \(q(\rho)\)

\(\vartheta_{F,n}(x)\) Number of isomorphism classes of Artin representations over \(F\) of dimension \(n\) with \(q(\rho) \leq x\)

\(\vartheta_{F,n}^{sd}(x)\) Number of isomorphism classes of self-dual Artin representations over \(F\) of dimension \(n\) with \(q(\rho) \leq x\)

\(\mathfrak{d}_K\) Relative discriminant ideal of a field extension \(K/\mathbb{Q}\)

\(d_K\) Absolute norm of the relative discriminant ideal \(\mathfrak{d}_K\)

\(\eta_{\mathbb{Q},m}(x)\) Number of extensions \(K/\mathbb{Q}\) inside a fixed algebraic closure \(\bar{\mathbb{Q}}\) such that \([K : \mathbb{Q}] = m\) and \(d_K \leq x\)

\(\eta_{\mathbb{Q},m}^T(x)\) Number of extensions \(K/\mathbb{Q}\) inside a fixed algebraic closure \(\bar{\mathbb{Q}}\) such that \([K : \mathbb{Q}] = m\) and \(d_K \leq x\) with Galois group of normal closure of \(K\) isomorphic to \(T\)

\(h^*_{M,c}(q)\) Number of idele class characters \(\chi\) of \(M\) of conductor \(q\) such that \(\chi^c = 1\)

\(S(G,U)(\mathbb{C})\) Complex Shimura varieties associated to group \(G\) and level subgroup \(U\)

\(S(G',U')(\mathbb{C})\) Locally symmetric spaces associated to group \(G'\) and level subgroup \(U'\)

\(\Omega_{\mathcal{O}_K}\) Drinfel’d \(p\)-adic upper half plane over \(\mathcal{O}_K\)
Chapter 1

Introduction

The theory of automorphic forms is a central area of mathematics with a rich history and deep connections to number theory, harmonic analysis and algebraic geometry. Automorphic forms can be construed as analytic objects by virtue of being functions on locally symmetric spaces. They also carry algebraic structure as functions defined on adelic points of a reductive group $G$ satisfying certain transformation laws and growth properties. This gives them a representation-theoretic context by work of Gelfand et al. who related them to the classical problem of decomposing the Hilbert space $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ into irreducible representations. Work of Weil on converse theorems for Hasse-Weil zeta functions showed that automorphic representations were related to algebraic geometry as well. Shimura worked with moduli of abelian varieties and cemented this connection by noting that many geometrical objects such as elliptic curves seemed to arise from automorphic forms. Automorphic representations can be found in the cohomology of these moduli spaces which are objects defined over number fields.

This tripartite view of automorphic forms provides us with insight into the interplay of algebraic number theory and automorphic forms. The understanding that these were related by what is now known as ‘modularity’ evolved gradually. Langlands made the celebrated conjecture on ‘functoriality’ of automorphic representations and perceived its relationship with the Artin conjecture for Artin $L$-functions coming from Galois representations. More generally, he saw that his automorphic $L$-functions were the right candidates to be motivic $L$-functions. The progress made in the theory of representations of real Lie groups by Harish Chandra, and in the theory of representations of $p$-adic groups by Bern-
stein and Zelevinsky supported his broad theory. His conjectures relating the ‘automorphic side’ which roughly represents the analytic information to the ‘Galois side’ which roughly represents the arithmetic information are a source of several outstanding open problems and a lot of the progress in modern number theory has been driven by his program.

This thesis is concerned with both sides of Langlands program, involving analysis, representation theory, geometry. It deploys techniques from various areas of mathematics such as analytic number theory, complex analysis, algebraic geometry. On the Galois side we have representations of a Galois group arising from number theory or geometry. Artin representations are such representations of the absolute Galois group of a number field $F$ over a finite dimensional complex vector space. In particular, there exists a notion of duality for Artin representations. The question of how frequently self-dual representations occur can be made precise once the dimension is fixed and was considered by Rohrlich in [Roh13].

In the first half of the thesis, we show that the density of self-dual Artin representations is 0 in dimension 3. We relate this problem to the classical problem of counting number fields by their discriminants, studied by Bhargava and his collaborators ([Bha05], [Bha10], [BCT]), and outline a possible strategy for higher dimensional representations. The result takes into account all Artin representations including those arising from Maass forms. This question of ‘letting the level vary’ is analogous to the work of DeGeorge and Wallach [DW78] on the automorphic side, but doesn’t directly relate to it.

In the second half of this thesis, we study spaces on the automorphic side similar to Shimura varieties. These are adelic quotients arising from unitary groups. My aim is to find models of these spaces over number fields. This has applications to constructing congruences of automorphic forms by level-raising. Many authors such as Kottwitz [Kot92a], [Kot92b], Harris-Taylor [HT01], et al. have studied automorphic representations arising from cohomology of certain simple unitary similitude Shimura varieties associated to a CM imaginary field $E$ with totally real subfield $F$. It was shown by Rapoport and Zink [RZ96] that these varieties admit $p$-adic uniformizations by the Drinfeld $p$-adic upper half plane. Thorne [Tho14] uses these uniformizations to prove new instances of level-raising for au-
tomorphic forms on $GL_n(\mathbb{A}_E)$ and to establish analogues of Ihara’s lemma. Our goal is to write down models over a number field for similar Shimura variety-like quotients that arise from unitary subgroups of the similitude groups. For the applications to level-raising, we show that this number field is unramified at a chosen prime $p$. These models should then in principle be amenable to $p$-adic uniformizations and thus help in generalizing previous results.

We briefly introduce these results below. A larger introduction for each individual result will follow in the later chapters.

1.1 Artin Representations

Before describing our work, we set some notations. Let $F$ be a number field, and if $\rho$ is an Artin representation of $F$, let $q(\rho)$ be the absolute norm of the conductor ideal $q(\rho)$. We denote by $\vartheta_{F,n}(x)$ the number of isomorphism classes of Artin representations over $F$ of dimension $n$ with $q(\rho) \leq x$ and by $\vartheta_{F,n}^{sd}(x)$ the number of isomorphism classes of self-dual Artin representations over $F$ of dimension $n$ with $q(\rho) \leq x$.

In our dissertation, we prove that self-dual Artin representations of dimension 3 have density 0, viz.

$$\lim_{x \to \infty} \frac{\vartheta_{\mathbb{Q},3}^{sd}(x)}{\vartheta_{\mathbb{Q},3}(x)} = 0$$

Our general strategy is to relate our problem with a classical problem, viz. that of counting field extensions of $\mathbb{Q}$ by discriminants. The Artin representations $\rho$ that we consider are faithful representations of certain finite Galois groups $\text{Gal}(K/\mathbb{Q})$. We translate the problem of counting $\rho$ to a problem of counting such number fields $K$ by carefully understanding the relation between conductor and discriminant using our understanding of ramification in these number fields. Finally, we appeal to results of Bhargava [Bha05, Bha10] and Bhargava, Cojocaru, Thorne [BCT] to obtain the density result. This provides evidence for the intuition that self-dual objects should be rare and outlines a possible
strategy for higher dimensional representations. In particular, we see that representations that are *primitive* correspond to the dominant term in previous as well as our analysis.

1.2 Shimura Varieties

Let $E$ be a CM imaginary field with totally real subfield $F$. There is a group of unitary similitudes $G$ over $\mathbb{Q}$ attached to such an extension under certain hypotheses. It is equipped with a similitude character $\theta : G \to \mathbb{G}_m$. Let $G'$ denote the algebraic group over $\mathbb{Q}$ that is the kernel of $\theta$. There exists $X$, a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \text{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$, such that $(G, X)$ is a Shimura datum. When certain hypotheses are satisfied by $E, F$, and $G$, there exist PEL data associated to $G$. For $U \subset G(\mathbb{A}_c)$ a neat open compact subgroup, there are Shimura varieties $S(G, U)$ with $S(G, U)(\mathbb{C}) := G(\mathbb{Q}) \backslash G(\mathbb{A}_c) \times X/U$ and admit canonical models over the reflex field $F$.

In our dissertation, we aim to extend such a result to the group $G'$. We define $S(G', U')(\mathbb{C}) := G'(\mathbb{Q}) \backslash G'(\mathbb{A}_c) \times X'/U'$ where $U' = G \cap U$. We show that $S(G', U')(\mathbb{C})$ can be identified with a union of connected components of $S(G, U)(\mathbb{C})$ and then descend using explicit Galois action on the set of connected components. It is not reasonable to expect that these models will be defined over $F$, but at the cost of enlarging the reflex field by a finite extension, we can achieve what we want.

This work is inspired by work of Thorne [Tho14]. He uses the canonical models for $G$ to prove instances of level-raising for automorphic forms for $GL_n(\mathbb{A}_E)$. Working with the groups $G$ results in certain restrictions on fields $E$ and ramification in them. It is our hope that working with the groups $G'$ will help lift such conditions and prove his results in greater generality.
Chapter 2

First Result

It is expected that essentially self-dual motives (i.e. motives that are dual to a Tate twist of themselves) should occur with density 0. This statement is not amenable to a precise formulation for motives with weights higher than 0, but for Artin motives of fixed dimension, this is a precise question as shown in [ABCZ94]. Let $F$ be a number field, and if $\rho$ is an Artin representation of $F$, let $q(\rho)$ be the absolute norm of the conductor ideal $q(\rho)$. We denote by $\vartheta_{F,n}(x)$ the number of isomorphism classes of Artin representations over $F$ of dimension $n$ with $q(\rho) \leq x$ and by $\vartheta_{sd_{F,n}}(x)$ the number of isomorphism classes of self-dual Artin representations over $F$ of dimension $n$ with $q(\rho) \leq x$. Rohrlich proved in [Roh13] that the quotient $\vartheta_{sd_{Q,2}}(x)/\vartheta_{Q,2}(x)$ goes to 0 as $x$ goes to $\infty$. Thus, our density 0 expectations are true for dimension 2. (The 1-dimensional case is elementary.) He proved the same result for $\mathbb{Q}$ and dimension 3 under a weak form of Malle’s Conjecture. In our dissertation, we remove this condition, viz. we confirm unconditionally,

**Theorem 1.**

$$\lim_{x \to \infty} \frac{\vartheta_{sd_{Q,3}}(x)}{\vartheta_{Q,3}(x)} = 0$$

We can replace $\mathbb{Q}$ by any number field and ask a similar question. But that case seems considerably harder. In dimension 1, the density result for a general number field follows from work of Taylor [Tay84].

Before describing our work, we set some notations. For a finite extension $K/F$ of number fields, we denote by $\mathfrak{d}_{K/F}$ the relative discriminant ideal and by $d_{K/F}$ its absolute norm. For $F = \mathbb{Q}$, we simply write $\mathfrak{d}_K$ and $d_K$. We denote by $\eta_{F,m}(x)$ the number of
extensions $K/F$ inside a fixed algebraic closure $\bar{F}$ such that $[K : F] = m$ and $d_{K/F} \leq x$. Also, if $T$ is a transitive subgroup of the symmetric group $S_m$, we denote by $\eta^T_{F,m}(x)$ the number of extensions $K/F$ for which $\text{Gal}(L/F) \cong T$ as permutation groups, where $L/F$ is the normal closure of $K$ over $F$ and $\text{Gal}(L/F)$ is viewed as a permutation group via its action on conjugates of a primitive element of $K$ over $F$.

We now describe the structure of the proof. In section 1 of the next chapter, we shall use results from [Roh13] to see that the problem reduces to bounding the number of irreducible self-dual Artin representations of $\mathbb{Q}$. Thus, we wish to count irreducible self-dual Artin representations. Such a representation has to be either orthogonal or symplectic. Since the dimension is odd, we see that we are reduced to the orthogonal case. We are thus reduced to analyzing irreducible orthogonal finite subgroups $G$ of $\text{GL}_3(\mathbb{C})$, where $G = \text{Gal}(K/\mathbb{Q})$ and $K$ is the fixed field of $\ker \rho$.

Our general strategy is to replace $\vartheta$, which count conductors of representations, by $\eta$, which count discriminants of number fields. We then appeal to results of Bhargava ([Bha05], [Bha10]) and Bhargava, Cojocaru, and Thorne [BCT].

We’ll divide our analysis into two cases: (1) Those $G$ which are contained in $SO_3$ and, (2) those which are not. In section 3, we analyze the subgroups occurring in case (1). Using bounds on ramification of primes, we obtain bounds in this case in terms of $\eta^A_4(x)$, $\eta^S_4(x)$, and $\eta^{A_5}(x)$.

Having obtained these, we turn our attention to case (2). We further divide this case into two parts. Part 1 is the case where $-1 \not\in G$. We analyze this case in section 4. If $-1 \not\in G$, then we show that $G \cong S_4$ and $\rho$ is monomial, induced from a quadratic character of a cubic subextension. Therefore, we reduce our problem to counting such extensions and characters, using the interplay between conductors and discriminants, and obtain bounds in terms of $\eta_{\mathbb{Q},3}(x)$. This requires a result of Rohrlich, which is proven in Appendix A.

In section 5, we deal with the case where $-1 \in G$. This implies that $G$ can be written as $H \times \{\pm 1\}$. In this case, in addition to monomial representations with $H \cong A_4$ or $S_4$, we must also contend with representations coming from a primitive $H \cong A_5$. It is worth noting
that the irreducible primitive case corresponds to the dominant term in all our analysis and it is only the power-saving result of Bhargava, Cojocaru and Thorne that helps us establish our result.

Finally, in section 6, we combine results from section 4, 5, 6 to get the main theorem.
Chapter 3

Proof of First Result

3.1 Reduction to the Irreducible Case

As in [Roh13], we have

\[
\vartheta_{Q,3}(x) = \vartheta_{Q,3}^{ab,sd}(x) + \vartheta_{Q,3}^{1+2,sd}(x) + \vartheta_{Q,3}^{irr,sd}(x)
\]  

(3.1)

where

(a) \(\vartheta_{Q,3}^{ab,sd}(x)\) is the number of abelian self-dual Artin representations of \(Q\) of dimension 3 with \(q(\rho) \leq x\)

(b) \(\vartheta_{Q,3}^{1+2,sd}(x)\) is the number of isomorphism classes of self-dual Artin representations of \(Q\) of dimension 3 of the form \(\rho \cong \rho' \oplus \rho''\) with \(\rho'\) one-dimensional, \(\rho''\) irreducible and two-dimensional and \(q(\rho')q(\rho'') \leq x\), and

(c) \(\vartheta_{Q,3}^{irr,sd}(x)\) is the number of irreducible self-dual Artin representations of \(Q\) of dimension 3 with \(q(\rho) \leq x\).

From Theorem 2 of [Roh13], we see that

\[
\vartheta_{Q,3}^{ab,sd}(x) = O(x(\log x)²)
\]

while from equation (80) of the same paper, we see that

\[
\vartheta_{Q,3}^{1+2,sd}(x) \ll x^{2-\epsilon}
\]
Since, by Theorem 1 of [Roh13],

\[ \vartheta_{Q,3}^{\text{sh}}(x) \sim O(x^2 (\log x)^2) \]

we see that, if we can prove

\[ \vartheta_{Q,3}^{\text{irr, sd}}(x) \ll x^{2-\epsilon}, \tag{3.2} \]

then we conclude that the self-dual representations have density zero.

### 3.2 Finite Irreducible Orthogonal Subgroups of $\text{GL}_3(\mathbb{C})$

We are interested in irreducible self-dual Artin representations. These have to be either orthogonal or symplectic. But since the dimension is odd, these have to be in $O_3$, where $O_3$ is the orthogonal group of real matrices. We’ll first concentrate on finite subgroups of $SO_3$. These are well-studied. Referring to a standard reference like [Art10], chapter 5, we see that every finite subgroup $G$ of $SO_3$ is one of the following:

1. $C_k$ : The cyclic group of rotations by multiples of $2\pi/k$ about a line

2. $D_k$ : The dihedral group of symmetries of a regular $k$-gon

3. $A_4$ : The alternating group on 4 variables (One embedding up to isomorphism in $O_3$)

4. $S_4$ : The symmetric group on 4 variables (Two embeddings up to isomorphism in $O_3$)

5. $A_5$ : The alternating group on 5 variables (Two embeddings up to isomorphism in $O_3$)

The cyclic groups and dihedral groups do not possess irreducible 3-dimensional representations. The last three groups do have irreducible 3-dimensional representations. Note that $S_4$ has two irreducible 3-dimensional representations, but the image of only one of
them is contained in $SO_3$. We call a subgroup $G$ of $GL_n(\mathbb{C})$ irreducible if the inclusion $i: G \to GL_n(\mathbb{C})$ is an irreducible representation of $G$.

### 3.3 A Bound on Discriminants for Finite Subgroups of $SO_3$

In this section, $\rho$ is an irreducible self-dual Artin representation and $K$ denotes the fixed field of $\ker \rho$. We know that $Gal(K/\mathbb{Q})$ is a finite irreducible subgroup of $O_3$. We divide our analysis into two cases, depending upon whether $Gal(K/\mathbb{Q})$ is a subgroup of $SO_3$ or not. Thus, we write:

$$\vartheta^{irr,sd}_{Q,3}(x) = \vartheta_1(x) + \vartheta_2(x)$$  \hspace{1cm} (3.3)

where

$$\vartheta_1(x) = \sum_{Gal(K/\mathbb{Q}) \subset SO(3), q(\rho) \leq x} 1$$

and

$$\vartheta_2(x) = \sum_{Gal(K/\mathbb{Q}) \not\subset SO(3), q(\rho) \leq x} 1$$

For the rest of the section, we focus on bounding $\vartheta_1(x)$. Thus, we assume $Gal(K/\mathbb{Q}) \subset SO_3$. We have seen in the previous section that this implies that $Gal(K/\mathbb{Q})$ is isomorphic to $A_4$, $S_4$, or $A_5$, and we write $m$ for the degree of the permutation group in question. Thus, $m = 4$ in the first two cases and $m = 5$ for the third. In all that follows, $M$ is any subfield of $K$ with $[M : \mathbb{Q}] = m$. The choice of $M$ is arbitrary, but the normal closure of $M$ is $K$ for every one of them.

**Proposition 1.** If $Gal(K/\mathbb{Q}) \cong A_4$,

$$d_M \leq cq(\rho)^{3/2}$$

with an absolute constant $c > 1$.  

Proof. We quote a standard bound (cf. [Ser81], p. 127, Proposition 2) which is

\[ d_M \leq c \prod_{p \mid d_M \atop p > m} p^3 \tag{3.4} \]

with \( c = 2^{11} \cdot 3^7 \).

Now, if \( p > 4 \) and \( p \mid d_M \), then \( \rho \) restricted to an inertia group \( I \) at \( p \) factors through its tame quotient (since 2 or 3 are the only wildly ramified primes for \( \text{im} \rho \cong A_4 \)) and hence, by the standard formula for the (local) Artin conductor,

\[ \text{ord}_p(q(\rho)) = \dim(V/V^I) \tag{3.5} \]

where \( V \) is the space of \( \rho \) and \( V^I \) is the subspace of inertial invariants.

**Case 1: I is a cyclic subgroup of order 2.** Since all elements of order 2 are conjugate to each other in \( A_4 \), only one computation will suffice for all the three subgroups of order 2. We see from a character table (see e.g. [FH91]) that

\[
\text{Multiplicity of trivial character} = \frac{3(1) + (-1)(1)}{2} = 1
\]

hence, \( \dim(V^I) = 1 \). So that \( \dim(V/V^I) = 2 \).

Alternatively, we can also argue without referring to a character table as follows: We see that since the determinant of \( \rho \) is 1, the image under \( \rho \) of a non-trivial element is conjugate to

\[
\begin{pmatrix}
1 \\
-1 \\
-1
\end{pmatrix}
\]

from which it is immediate that \( \dim(V^I) = 1 \).

**Case 2: I is a cyclic subgroup of order 3.** There are two conjugacy classes, each containing 4 elements of order 3, which cover all the elements of order 3 in \( A_4 \). The
character of our 3-dimensional representation is valued 0 on both of these classes. Hence, for restriction to any subgroup of order 3, we see that

\[
\text{Multiplicity of trivial character} = \frac{3(1) + 0(1) + 0(1)}{3} = 1
\]

hence, \(dim(V^I) = 1\). So that \(dim(V/V^I) = 2\).

Therefore, by (3.5), we see that

\[
q(\rho) \geq \prod_{\substack{p | q(\rho) \\ p > m}} p^2
\]

and combined with (3.4), it completes the proof.

Again, we can argue without using a character table: Since the determinant of \(\rho\) is 1, the image under \(\rho\) of a non-trivial element is conjugate to

\[
\begin{pmatrix}
1 \\
\zeta \\
\zeta^2
\end{pmatrix}
\]

where \(\zeta\) is a primary cube root of unity. Thus, it is immediate that \(dim(V^I) = 1\).

**Proposition 2.** If \(Gal(K/Q) \cong S_4\),

\[
d_M \leq cq(\rho)^{3/2}
\]

with an absolute constant \(c > 1\).

**Proof.** We use a strategy similar to the one we used earlier. Note that our representation is the twist of the standard representation by the alternating character.

**Case 1: I is a cyclic subgroup of order 2.** There are two conjugacy classes, one containing 6 elements of order 2 and one containing 3 elements of order 2, which cover all
the elements of order 2 in $S_4$. The character of our 3-dimensional representation is valued $-1$ on both these classes. Hence, for restriction to any subgroup of order 2, we see that

\[ \text{Multiplicity of trivial character} = \frac{3(1) + (-1)(1)}{2} = 1 \]

hence, $\dim(V^I) = 1$. So that $\dim(V/V^I) = 2$.

**Case 2: I is a cyclic subgroup of order 3.** The character is valued 0 on the unique conjugacy class of elements of order 3. Hence, for restriction to any subgroup of order 3, we see that

\[ \text{Multiplicity of trivial character} = \frac{3(1) + 0(1) + 0(1)}{3} = 1 \]

hence, $\dim(V^I) = 1$. So that $\dim(V/V^I) = 2$.

**Case 3: I is a cyclic subgroup of order 4.** Any subgroup of order 4 contains, apart from the identity, two elements of order 4 and one element of order 2. Our character is valued 1 on elements of order 4 and $-1$ on elements of order 2. (Elements of order 4 all belong to the same conjugacy class and the class does not matter for elements of order 2 as our character is valued the same on both of them as mentioned above.) Hence, for restriction to any subgroup of order 4, we see that

\[ \text{Multiplicity of trivial character} = \frac{3(1) + 1(1) + (-1)(1) + 1(1)}{4} = 1 \]

hence, $\dim(V^I) = 1$. So that $\dim(V/V^I) = 2$.

Therefore, by (3.5), we see that

\[ q(\rho) \geq \prod_{\substack{p|q(\rho) \\ p > m}} p^2 \]

and combined with (3.4), it completes the proof.

We remark that we can follow the alternate method mentioned in the previous proposition here as well.
Proposition 3. If $\text{Gal}(K/Q) \cong A_5$,

$$d_M \leq c q(\rho)^2$$

with an absolute constant $c > 1$.

Proof. Since $m$ is now 5 rather than 4, we have a different bound

$$d_M \leq c \prod_{\substack{p|d_K \\ p > 5}} p^4$$

with $c = 2^{14} 3^9 5^9$.

Now, we again consider cases of cyclic subgroups. Note that, in this case, we have two 3-dimensional representations.

Case 1: I is a cyclic subgroup of order 2. All the elements of order 2 in $A_5$ are conjugate to each other. Both our characters are valued $-1$ on this class. Hence, for restriction of either of the representations to any cyclic subgroup of order 2, we see that

$$\text{Multiplicity of trivial character} = \frac{3(1) + (-1)(1)}{2} = 1$$

hence, $\dim(V^I) = 1$. So that $\dim(V/V^I) = 2$.

Case 2: I is a cyclic subgroup of order 3. All the elements of order 3 in $A_5$ are conjugate to each other. Both our characters are valued 0 on this class. Hence, for restriction of either of the representations to any cyclic subgroup of order 3, we see that

$$\text{Multiplicity of trivial character} = \frac{3(1) + 0(1) + 0(1)}{3} = 1$$

hence, $\dim(V^I) = 1$. So that $\dim(V/V^I) = 2$.

Case 2: I is a cyclic subgroup of order 5. There are two conjugacy classes in $A_5$,
each containing 12 elements of order 5, which cover all the elements of order 5. As before, we can compute \( \text{codim} \ V^I \) using a character table. But, in this case, it is more efficient to use the alternate method. We just note that since the determinant of \( \rho \) is 1, the image under \( \rho \) of a non-trivial element of \( I \) is conjugate to

\[
\begin{pmatrix}
1 \\
\omega \\
\omega^2
\end{pmatrix}
\]

where \( \omega \) is a primary 5th root of unity. Thus, it is immediate that \( \dim(V^I) = 1 \). So that \( \dim(V/V^I) = 2 \).

Therefore, by (3.5), we see that

\[
q(\rho) \geq \prod_{p \mid q(\rho), p > m} p^2
\]

and combined with (3.6), it completes the proof.

\[\square\]

Finally, we have

\[
\vartheta_1(x) = \sum_{\text{Gal}(K/Q) \subseteq SO(3), \text{Gal}(K/Q) \cong A_4, q(\rho) \leq x} 1 + \sum_{\text{Gal}(K/Q) \subseteq SO(3), \text{Gal}(K/Q) \cong S_4, q(\rho) \leq x} 1 + \sum_{\text{Gal}(K/Q) \subseteq SO(3), \text{Gal}(K/Q) \cong A_5, q(\rho) \leq x} 1
\]

which translates using propositions 1, 2, 3 to

\[
\vartheta_1(x) \leq \sum_{\text{Gal}(K/Q) \subseteq A_4, d_M \leq cx^{\frac{3}{2}}} 1 + \sum_{\text{Gal}(K/Q) \subseteq S_4, d_M \leq cx^{\frac{3}{2}}} 1 + \sum_{\text{Gal}(K/Q) \subseteq A_5, d_M \leq cx^2} 1
\]
which, after appealing to [Bha05], [Bha10] and [BCT], imply that

\[ \vartheta_1(x) = O(x^{2-2\beta+\epsilon}) \] (3.7)

3.4 Finite Subgroups of $O(3)$ Not Contained in $SO(3)$ - Part 1

In this section and the next, we focus on bounding $\vartheta_2(x)$.

We deal with this case in two parts, depending upon whether $-1$, the negative of the identity matrix in 3 dimensions, is in the image of $Gal(K/\mathbb{Q})$. Part 1 is devoted to the case:

\[ Gal(K/\mathbb{Q}) \not\subset SO(3), -1 \notin Gal(K/\mathbb{Q}) \]

In this case, we prove a lemma which straightaway tells us what $Gal(K/\mathbb{Q})$ is.

Lemma 1. Let $G$ be a finite irreducible subgroup of $O(3)$ which is not contained in $SO(3)$. Assume further that $-1 \notin G$. Then $G \cong S_4$.

Proof. Let

\[ H = G \cap SO(3) \]

Then we can write

\[ G = H \cup \kappa H \]

where $det(\kappa) = -1$.

Then we can define

\[ H^* = H \cup (-\kappa)H \]

so that $H^* \subset SO(3)$. Note that $G$ and $H^*$ are “isoclinic”. Therefore, $H^*$ is an irreducible subgroup of $SO_3$. As a result, we see that $H^* \cong A_4, S_4$ or $A_5$. But, $A_4$ or $A_5$ do not possess index 2 subgroups. Hence,

\[ H^* \cong S_4 \]
Then, we can give an explicit isomorphism from $G$ to $H^*$ using the index 2 subgroup $H$, viz. send $h \to h$ and $\kappa h \to (-\kappa)h$. It is easily seen that this is an isomorphism, and thus $G \cong S_4$.

\[ \begin{pmatrix} -1 & \quad & \quad \\ \quad & 1 & \quad \\ \quad & \quad & 1 \end{pmatrix} \]

which yields a weaker lower bound for $q(\rho)$ and thus allows for a larger number of $\rho$. This does not happen, but we need to take a different path to prove this, which we describe below.

Indeed we see that $S_4$ has a faithful 3-dimensional representation which satisfies our hypotheses. Our earlier method of comparing conductors and discriminants yields weaker bounds than what are needed for our purpose, because we cannot rule out the possibility that $\det \rho$ is nontrivial on $I$. If $I$ is of order 2, then the matrix corresponding to the non-trivial element of inertia would then be conjugate to

The standard representation (i.e. the 3-dimensional representation with non-trivial determinant) of $S_4$ is monomial. This means that there exists a subfield $M$ of $K$ such that $[M : \mathbb{Q}] = 3$, $Gal(K/M) = D_8$, and a quadratic 1-dimensional character $\chi$ of $Gal(K/M)$ such that $\rho = Ind_{M/\mathbb{Q}} \chi$. Thus, we can try to count pairs $(M, \chi)$. We have classical results by Davenport and Heilbronn ([DH71]) on counting cubic number fields by discriminants. Since we are only concerned about the main term here, we do not need the error improvement by Bhargava, Shankar and Tsimerman ([BST13]). Note that for each such pair $(M, \chi)$, we have two more pairs $(M', \chi')$ and $(M'', \chi'')$, corresponding to conjugate copies of $D_8 \subset S_4$ that will give us the same representation, but that will only affect the constant term in our bounds.
We begin by denoting our counting function:

$$\Theta(x) := \sum_{\substack{Gal(K/Q) \not\subset SO_3 \\ -1 \notin Gal(K/Q) \\ q(\rho) \leq x}} 1$$

The Conductor-Discriminant formula gives

$$q(\rho) = d_M q(\chi) \quad (3.8)$$

where $d_M$ is the discriminant of the field $M$ and $q(\chi)$ is the absolute norm of the conductor of the character $\chi$.

Thus, we see that

$$\Theta(x) = \sum_{\substack{Gal(K/Q) \not\subset SO_3 \\ Gal(K/Q) \cong S_4 \\ q(\rho) \leq x}} 1 \leq \sum_{\substack{[M:Q]=3 \\ \chi^2=1 \\ q(\chi) d_M \leq x}} 1 \quad (3.9)$$

where the first equality is due to the lemma.

We need to count the extensions $M$ as well as characters $\chi$. We write $\theta_{M,2}(x)$ for the number of characters $\chi$ of $M$ with $\chi^2 = 1$ and $q(\chi) \leq x$. We’ll use upper bounds for $\theta_{M,2}(x)$ from the appendix of this paper. These bounds are slightly weaker than the actual asymptotic if we work with a field $M$ that is fixed, but since we are working with varying fields $M$ at the same time, these bounds, which are uniform as long as the degree $[M : Q]$ is fixed (which is true in our case), will work better, and the only expense incurred is a power of logarithm. It can be seen from the final proposition of this section below, that this increased power does not affect our result. We note that the asymptotic is an interesting result in itself, which follows from computing the residue of an appropriate Zeta Function and knowledge of bounds on the class number and the regulator of a number field.

From the corollary to Proposition 2 in the appendix to this paper, we see that

$$\theta_{M,2}(x) \ll \sqrt{d_M} (\log d_M)^2 x (\log x)^2 \quad (3.10)$$
where the implied constant is independent of $M$, since $c$ and $m$ are now fixed.

We now prove our main result of this section.

**Proposition 4.** Let $\rho, K$ be as before. Then

\[ \Theta(x) = O(x^{\frac{3}{2} + \epsilon}) \]

**Proof.** By (3.9), it is sufficient to prove

\[ \sum_{[M:Q]=3}^{Q \leq x} \sum_{q_{\chi} \leq x/d_M} 1 = O(x^{\frac{3}{2} + \epsilon}) \]  \hspace{1cm} (3.11)

where $\chi$ is a quadratic character of $M$. That is, we wish to prove

\[ \sum_{[M:Q]=3}^{d_M \leq x} \sum_{q_{\chi} \leq x/d_M} 1 = O(x^{\frac{3}{2} + \epsilon}) \]  \hspace{1cm} (3.12)

From (3.10), we see that

\[ \sum_{[M:Q]=3}^{d_M \leq x} \sum_{q_{\chi} \leq x/d_M} 1 \ll \sum_{[M:Q]=3}^{d_M \leq x} \sqrt{d_M} (\log d_M)^2 (\log \frac{x}{d_M})^2 \frac{x}{d_M} \]  \hspace{1cm} (3.13)

The implied constant is uniform. Hence, we get

\[ \sum_{[M:Q]=3}^{d_M \leq x} \sum_{q_{\chi} \leq x/d_M} 1 = x(\log x)^2 \cdot O \left( \sum_{[M:Q]=3}^{d_M \leq x} \frac{(\log d_M)^2}{\sqrt{d_M}} \right) \]  \hspace{1cm} (3.14)

Using the fact from [DH71] that

\[ \sum_{[M:Q]=3}^{d_M \leq x} \sum_{q_{\chi} \leq x/d_M} 1 \sim cx \]  \hspace{1cm} (3.15)
where $c$ is an absolute constant, we see that the above sum can be estimated as

$$
\left( \sum_{\substack{[M:Q]=3 \\ d_M \leq x}} \frac{(\log d_M)^2}{\sqrt{d_M}} \right) = O(x^{1/2+\epsilon}) \tag{3.16}
$$

which proves (3.12). Thus, the proposition follows.

\[ \Box \]

### 3.5 Finite Subgroups of $O(3)$ Not Contained in $SO(3)$ - Part 2

We deal with the remaining cases in this section. These cases are characterized by:

$$Gal(K/Q) \not\subset SO(3), -1 \in Gal(K/Q)$$

Put $H = Gal(K/Q) \cap SO(3)$. Then we see that $Gal(K/Q) \cong H \times \{ \pm 1 \}$. The Artin representation $\rho$ we are considering can be written as $\rho \cong \sigma \otimes \epsilon$, where $\sigma$ is an irreducible three-dimensional representation of $H \subset SO(3)$ and $\epsilon$ is a quadratic character of $Q$. Thus, we can hope to estimate the number of pairs $(\sigma, \epsilon)$ and obtain the bounds that we need. We’ll first do this in the case where $H \cong A_5$. Let’s denote the corresponding $A_5$-subextension of $K$ by $L$. In this case, the representation $\sigma$ of $A_5$ is a primitive representation. In fact, there are two representations of $A_5$ possible, and both of them are primitive. As before, let $M$ be a subfield of $L$ such that $[M:Q] = 5$. The normal closure of $M$ is $L$ for any choice of $M$.

We wish to estimate

$$\Psi(x) := \sum_{\substack{q(\rho) \leq x \\ \rho \cong \sigma \otimes \epsilon}} 1$$

where $\sigma$ is a faithful irreducible representation of $A_5 \cong Gal(L/Q)$, considered as an Artin representation of $Q$, and $\epsilon$ is a quadratic character of $Q$. 
For a fixed $\sigma$ we look at $q(\sigma \otimes \epsilon)$ and $q(\sigma)$. Let

$$q_{tame}(\sigma) = \prod_{p \in X} p^{e_p}$$

be the tame conductor of $\sigma$, where $X$ is the set of tamely ramified primes in $L$. By a computation done previously, each $e_p = 2$. Thus,

$$q_{tame}(\sigma) = \prod_{p \in X} p^2 \quad (3.17)$$

Let $X = A \cup B \cup C$, where $A$, $B$, and $C$ are the sets of tamely ramified primes with inertia subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/5\mathbb{Z}$ respectively.

Let’s write the conductor of $\epsilon$ as

$$q(\epsilon) = 2^\alpha \prod_{j=1}^s q_j \quad (3.18)$$

where $\alpha = 0, 2, \text{ or } 3$, and $q_j$ are distinct odd primes. We can write $\epsilon = \chi \chi'$ where $\chi$ and $\chi'$ are quadratic characters of $\mathbb{Q}$ such that

$$p | q(\chi) \iff p = 2, 3, 5 \text{ or } p \in X$$

Thus, primes that divide $q(\chi')$ are primes that are unramified in $L$ which divide the conductor of $\epsilon$.

**Proposition 5.** Let $\sigma, \epsilon, \chi, \chi'$ be as above. Then

$$q(\sigma \otimes \epsilon) = q(\sigma \otimes \chi \chi') = q(\sigma \otimes \chi)q(\chi')^3$$

**Proof.** This follows from a well-known local computation for each prime dividing $q(\sigma \otimes \chi)$ and $q(\chi')$. \qed
Thus, we see that the condition

\[ q(\rho) = q(\sigma \otimes \epsilon) \leq x \]

translates to the condition

\[ q(\sigma \otimes \chi) \leq x \text{ and } q(\chi') \leq \left( \frac{x}{q(\sigma \otimes \chi)} \right)^{\frac{1}{3}} \quad (3.19) \]

Hence, we get

\[ \Psi(x) \leq \sum_{q(\sigma \otimes \chi) \leq x} \sum_{q(\chi') \leq \left( \frac{x}{q(\sigma \otimes \chi)} \right)^{\frac{1}{3}}} 1 \quad (3.20) \]

The number of quadratic characters of \( \mathbb{Q} \) with conductor \( \leq x \) is \( O(x) \). So (3.20) yields

\[ \Psi(x) \ll \sum_{q(\sigma \otimes \chi) \leq x} \left( \frac{x^{\frac{1}{3}}}{q(\sigma \otimes \chi)^{\frac{1}{3}}} \right) \]

which gives

**Proposition 6.**

\[ \Psi(x) \ll x^{\frac{1}{3}} \sum_{q(\sigma \otimes \chi) \leq x} \frac{1}{q(\sigma \otimes \chi)^{\frac{1}{3}}} \quad (3.21) \]

Let

\[ \Theta(x) = \sum_{q(\sigma \otimes \chi) \leq x} 1 \]

where \( \sigma \) and \( \chi \) are as above.

**Proposition 7.** We have

\[ \Theta(x) = O(x^{2-2\beta}) \]

where \( \beta \) is any positive constant less than \( \frac{1}{120} \).

**Proof.** We’ll convert the problem of estimating \( \Theta(x) \) into a problem of counting \( A_5 \)-extensions of \( \mathbb{Q} \) and quadratic characters of \( \mathbb{Q} \). For this, we look at the conductors and
discriminants.

Since we only let 2, 3, 5 or primes that are tamely ramified in $L$ remain in the conductor of $\chi$, we see that

$$q(\chi) = 2^\alpha 3^\beta 5^\gamma \prod_{p \in Y} p$$

for some $Y \subset X$ and $\beta, \gamma \in \{0, 1\}$.

Let $A' = A \cap Y, B' = B \cap Y, C' = C \cap Y$ and $A'' = A \setminus A', B'' = B \setminus B', C'' = C \setminus C'$.

We can compute the effect of twisting by $\chi$ at each prime locally by looking at image under $\rho$ of $I$, cf. proof of Proposition 1. We see using equations (3.17) and (3.18) that the tame conductor of $\sigma \otimes \chi$ is given by

$$q_{tame}(\sigma \otimes \chi) = \left( \prod_{p \in A'} p \prod_{p \in A''} p^2 \right) \left( \prod_{p \in B'} p^3 \prod_{p \in B''} p^2 \right) \left( \prod_{p \in C'} p^3 \prod_{p \in C''} p^2 \right) \tag{3.22}$$

On the other hand, by a computation involving ramification degrees, we obtain a bound on the tame discriminant of $M$:

$$d_{M}^{tame} \leq \prod_{p \in A} p^2 \prod_{p \in B} p^2 \prod_{p \in C} p^4 \tag{3.23}$$

Comparing the above expressions, we see that

$$d_{M}^{tame} \leq q_{tame}(\sigma \otimes \chi)^2 \tag{3.24}$$

This is the inequality which helps us translate the bound on the conductor to a bound on the discriminants. We have not yet dealt with the primes 2, 3, 5 which might be wild primes. But we have a uniform bound for them as we have seen before, cf. [Ser81]. Letting $c = 2^{14}3^95^9$, we see from loc. cit. and (3.19) that

$$d_{M} \leq cd_{M}^{tame} \leq cq_{tame}(\sigma \otimes \chi)^2 \leq cq(\sigma \otimes \chi)^2 \leq cx^2 \tag{3.25}$$
Equipped with these results, we now obtain our result. We have

\[ \Theta(x) \leq \sum_{\sigma} \sum_{\chi} 1 \]

where \( \sigma \) and \( \chi \) are as above and the sum runs only over pairs such that \( q(\sigma \otimes \chi) \leq x \).

Thus, we get

\[ \Theta(x) \ll \sum_{d_M \leq cx^2} \sum_{\chi \mid d_M} 1 \]

(3.26)

The inner sum is over quadratic characters \( \chi \) and is thus \( O(x^\epsilon) \) once \( \sigma \) is fixed, because the number of divisors of \( d_M \) is \( d_M^\epsilon \). Hence, we get

\[ \Theta(x) \ll x^\epsilon \sum_{d_M \leq cx^2} 1 \]

(3.27)

We then appeal to [BCT] to obtain

\[ \Theta(x) \ll x^\epsilon (x^2)^{1-\beta} \]

(3.28)

where \( \beta \) is a positive constant less than 1/120, which finishes our proof.

Proof. This follows from combining the previous propositions 6 and 7 using Abel partial summation.

Using this proposition, we obtain our main result:

**Proposition 8.**

\[ \Psi(x) = O(x^{2-2\beta}) \]

(3.29)

where \( \beta \) is as before.

Proof. This follows from combining the previous propositions 6 and 7 using Abel partial summation.

We move on to the remaining cases. Recall that \( \text{Gal}(K/\mathbb{Q}) \cong H \times \{\pm 1\} \), where \( H \) is an
irreducible finite subgroup of $SO(3)$. We have dealt with the case $H \cong A_5$. We deal with the cases $H \cong A_4$, $H \cong S_4$ below. Let $L/\mathbb{Q}$ be a subextension such that $Gal(L/\mathbb{Q}) \cong H$.

In these cases, the representations $\rho$ can be again written as $\sigma \otimes \epsilon$, where now $\sigma$ is an irreducible 3-dimensional representation of $A_4$ or $S_4$ with trivial determinant. Such representations are necessarily monomial, say induced from a cubic subextension $M/\mathbb{Q}$ of $L$. $\epsilon$ is a quadratic character of $\mathbb{Q}$. Note that $\epsilon = \det \rho$.

**Proposition 9.** Let $K, M, H$ be as before where $H \cong A_4$ or $S_4$. Define

$$\Phi(x) := \sum_{\substack{q(\rho) \leq x \\ \rho \cong \sigma \otimes \epsilon}} 1$$

where $\rho, \sigma, \epsilon$ are also as before.

Then,

$$\Phi(x) = O(x^{\frac{3}{2} + \epsilon'})$$

where $\epsilon'$ is arbitrarily small.

**Proof.** We prove the result for $H \cong S_4$. The case $H \cong A_4$ is completely analogous. If $\sigma = Ind_M^L(\chi)$ for some $\chi$, we see that

$$\rho \cong Ind_M^L(\chi \otimes \text{res } \epsilon)$$

Here, $M$ determines $\sigma$ which in turn fixes $\epsilon$. Moreover, both $\chi$ and $\epsilon$ are quadratic characters. We denote the quadratic character $\chi \otimes \text{res } \epsilon$ by $\chi'$.

This is analogous to our methods in section 5, where we counted pairs $(M, \chi')$ where $M$ is a cubic extension of $\mathbb{Q}$ and $\chi'$ is a quadratic character of $M$. We again write:

$$q(\rho) = d_Mq(\chi')$$
Then, using the same method, from corollary to Proposition 2 in the appendix, we have:

\[ \theta_{M,2}(x) \ll \sqrt{d_M}(\log d_M)^2 x (\log x)^2 \]

and thus, we get

\[ \Phi(x) \ll \sum_{\substack{[M:Q]=3 \\ d_M \leq x}} \sqrt{d_M}(\log d_M)^2 (\log \frac{x}{d_M})^2 \frac{x}{d_M} \]

from which we get the required result.

\[ \square \]

3.6 Proof of Theorem 1

By Propositions 4, 8, 9, we see that

\[ \vartheta_2(x) = O(x^{2-2\beta}) \quad (3.31) \]

which coupled with equations (3.3) and (3.7) finishes the proof.
Chapter 4

Second Result

Many mathematicians have studied automorphic representations arising from cohomology of certain unitary similitude Shimura varieties associated to a CM imaginary field $E$ with totally real subfield $F$. Information about these can be transferred under right circumstances to information about representations of $GL_n$. For example, Clozel [Clo91], using Kottwitz’s results ([Kot92a, Kot92b]) on the Shimura varieties associated to some of these groups, attached $n$-dimensional Galois representations to self-dual automorphic representations of $GL_n$ over CM fields satisfying local-global compatibility at the unramified primes. Harris and Taylor [HT01] studied bad reduction of such varieties to obtain proof of local-global compatibility at ramified primes and the proof of local Langlands conjecture for $GL_n$ over $p$-adic fields.

In another such instance, Thorne [Tho14] uses torsion-vanishing results about the cohomology of these varieties proven by Lan and Suh [LS12] to prove new instances of level-raising for automorphic forms on $GL_n(\mathbb{A}_E)$ and to establish analogues of Ihara’s lemma. He uses $p$-adic uniformizations of these varieties to relate certain spaces of automorphic forms to the cohomology. Inspired by his work, our aim is to write down models for certain Shimura variety-like quotients that arise from unitary subgroups of the similitude groups, which should in principle be amenable to $p$-adic uniformizations. These would in turn generalize the level-raising results to a larger class of fields $E$.

In this chapter, we define the unitary similitude group $G$ over $\mathbb{Q}$ and its unitary subgroup $G'$ associated to a CM imaginary field $E/F$ in section 1. We write down the corresponding Shimura varieties and state our result in section 2. In the next chapter, we
first describe the theory of canonical models of Shimura varieties and briefly illustrate the example of a modular curve in sections 1 and 2. In section 3, we describe how canonical models can be defined for locally symmetric spaces arising from groups $G'$ over number fields that are suited for future applications.

### 4.1 Defining the Groups

The material in this section is largely taken from [RZ96]. In this section we give some general definitions. In the subsequent sections, these might be restricted under some hypotheses in order to accommodate certain results.

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $n \in \mathbb{Z}_{>1}$. Let $E$ be a CM imaginary field with totally real subfield $F$, $[F : \mathbb{Q}] = d$. Let $D$ be a central division algebra of dimension $n^2$ over $E$. Let $*$ be a positive involution of the second kind on $D$ so that the ring of invariants of $*$ on $E$ is the subfield $F$. Let $V$ be a left $D$-module and $\psi$ an alternating pairing $\psi : V \times V \to \mathbb{Q}$ satisfying

$$\psi(dv, w) = \psi(v, d^* w)$$

for all $d \in D, v, w \in V$.

We define an algebraic group $G$ over $\mathbb{Q}$ by its functor of points:

$$G(R) = \{ g \in GL_D(V \otimes R) \mid \psi(gv, gw) = c(g) \psi(v, w), c(g) \in R^\times \}$$

where $R$ is a $\mathbb{Q}$-algebra. In particular, $c = c(g)$ is the similitude character of $G$.

$G_{\mathbb{R}}$ can be embedded into a product of unitary similitude groups by making a choice of a CM-type $\Phi \subset Hom(E, \mathbb{C})$ as follows. We can choose an isomorphism $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau \in \Phi} D \otimes_{E, \tau} \mathbb{C} \cong \prod_{\tau \in \Phi} M_n(\mathbb{C})$, such that $*$ corresponds to the operation $X \to X^\tau$.

This decomposition induces an orthogonal decomposition with respect to $\psi$ on $V \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau \in \Phi} V \otimes_{E, \tau} \mathbb{C}$. There are isomorphisms $V \otimes_{E, \tau} \mathbb{C} \cong \mathbb{C}^n \otimes_{\mathbb{C}} W_\tau$ where $M_n(\mathbb{C})$ acts on the
first factor. Then we can find a skew-hermitian form \( h_\tau \) on \( W_\tau \) so that

\[
\psi_\tau(z_1 \otimes w_1, z_2 \otimes w_2) = tr_{\mathbb{C}/\mathbb{R}}(\overline{z}_1^t z_2 h_\tau(w_1, w_2)) 
\]

(4.3)

We can find a basis of \( W_\tau \) such that \( h_\tau \) is represented by the matrix

\[
diag(-i, -i, \cdots, -i, i, i, \cdots, i)
\]

(4.4)

We denote the number of times \(-i\) appears in this matrix by \( r_\tau \) and the number of times \( i \) appears by \( r_{\tau c} \). It can be seen that \( r_\tau + r_{\tau c} = (1/n) \dim_E V \). Thus, the choices we have made so far imply that

\[
G_{\mathbb{R}} \leftarrow \prod_{\tau \in \Phi} GU(r_\tau, r_{\tau c})
\]

(4.5)

such that \( G_{\mathbb{R}} \) is a normal subgroup with torus cokernel.

Under the above identifications, we write \( h : \text{Res}_{\mathbb{C}/\mathbb{R}} G_m \rightarrow G_{\mathbb{R}} \) for the homomorphism

\[
h : z \in \mathbb{C}^\times \rightarrow (\text{diag}(z, z, \cdots, z, \bar{z}, \bar{z}, \cdots, \bar{z}))_{\tau \in \Phi}
\]

(4.6)

where again the number of times \( z \) appears is \( r_\tau \) and the number of times \( \bar{z} \) appears is \( r_{\tau c} \). Let \( X \) denote the \( G(\mathbb{R}) \)-conjugacy class of \( h \) inside the set of homomorphisms \( \text{Res}_{\mathbb{C}/\mathbb{R}} G_m \rightarrow G_{\mathbb{R}} \).

We define another algebraic group \( G' \) over \( \mathbb{Q} \) by its functor of points :

\[
G'(R) = \{ g \in GL_D(V \otimes R) \mid \psi(gv, gw) = \psi(v, w) \}
\]

(4.7)

where \( R \) is a \( \mathbb{Q} \)-algebra. There is an exact sequence of algebraic groups

\[
1 \rightarrow G' \rightarrow G \xrightarrow{\xi} G_m \rightarrow 1
\]

(4.8)

We let \( G_0 \) denote the derived subgroup of \( G \). \( G \) and \( G' \) have the same derived sub-
group.) It is a subgroup of $G'$ consisting of matrices of determinant 1. It is a form of $SL_n$ and in particular, simply connected as an algebraic group. There is an exact sequence of algebraic groups

$$1 \to G_0 \to G' \to (E^\times)^{N=1} \to 1$$

(4.9)

where $(E^\times)^{N=1}$ is the torus of elements of $E^\times$ with reduced norm 1 in $F$. We let $T := G/G_0$ and $T' := G'/G_0$. Note that $T$ and $T'$ are the largest commutative quotients of the respective groups. We denote the natural map $G \to T$ by $\nu$. Let $Z, Z'$ denote the centre of $G, G'$ respectively.

### 4.2 Shimura Varieties, PEL Data and $p$-adic Uniformizations

All the definitions have been global so far. Hereon they will acquire a local flavour for the purpose of application to automorphic forms. We fix a rational prime $p$ and suppose that $p$ is inert in $F$. Let $\nu$ be the unique prime of $F$ over $p$. We assume that $\nu$ splits in $E$ as $\omega \omega^c$.

We fix embeddings $\phi_\infty, \phi_p$ of $\mathbb{Q}$ into $\mathbb{C}, \mathbb{Q}_p$ respectively. This choice induces a bijection of sets $\text{Hom}(E, \mathbb{C}) \leftrightarrow \text{Hom}(E, \mathbb{Q}_p)$.

We assume that invariants of $D$ at the places $\omega$ and $\omega^c$ are given respectively by $1/n$ and $-1/n$. At every other place of $F$, $D$ is split. We take $V$ to be a free module over $D$ of rank 1.

We suppose that $\Phi$ corresponds to the set of embeddings inducing the $p$-adic place $\omega$ of $E$. Let $\tau_1, \cdots, \tau_d$ be the elements of $\Phi$. We assume that $r_{\tau_1} = 1$ and $r_{\tau_i} = 0$ for $i = 2, \cdots, d$. Note that $r_{\tau_j} + r_{\tau_j^c} = n$ for all $j$ by our hypothesis on $V$. We also assume that PEL data $(D, E, \ast, F, V, \psi)$ is such that $G$ is quasi-split at every finite place not dividing $p$. PEL data satisfying these assumptions exist, in particular if $d$ is even, which we assume, cf. [HT01].

We pause to note that under these assumptions

$$G'_R \cong \prod_{i=1}^d U(r_{\tau_i}, r_{\tau_i^c}) \cong U(1, n-1) \times U(0, n)^{d-1}$$

(4.10)
as shown in [HT01]. It follows from the same reference that \( G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times D_\omega^\times \) and thus, \( G'(\mathbb{Q}_p) \cong (D_\omega)^\times \).

With these assumptions, we have the following result, primarily due to Kottwitz [Kot92b]:

**Proposition 10.** The pair \((G, X)\) is a Shimura datum. For \( U \subset G(\mathbb{A}_\infty) \) a neat open compact subgroup, the Shimura varieties \( S(G, U) \) with \( S(G, U)(\mathbb{C}) := G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times X/U \) are smooth projective algebraic varieties over \( \mathbb{C} \) and admit canonical models over \( \tau_1(F) \subset \mathbb{C} \).

We’ll briefly recall the theory of canonical models in the next chapter.

The varieties \( S(G, U) \) admit \( p \)-adic uniformizations. We describe them briefly. We refer the reader to [RZ96] and [Tho14] for more details.

Let \( K \) be a finite extension of \( \mathbb{Q}_p \). We write \( \Omega_{O_K} \) for the Drinfeld \( p \)-adic upper half plane over \( O_K \). It is a \( p \)-adic formal scheme, formally locally of finite type over \( \text{Spf} \ O_K \).

There is a faithful action of \( \text{PGL}_n(K) \) on \( \Omega_{O_K} \). Following Thorne [Tho14], we define a \( p \)-adic formal scheme \( \mathbb{M}^{\text{split}} \) by the formula

\[
\mathbb{M}^{\text{split}} := \Omega_{O_K} \times \mathbb{Q}_p^\times / \mathbb{Z}_p^\times \times \text{GL}_n(K)/\text{GL}_n(K)^0 \tag{4.11}
\]

where \( \text{GL}_n(K)^0 \subset \text{GL}_n(K) \) is the open subgroup of matrices with determinant a \( p \)-adic unit and the sets on the right hand side are identified with the corresponding constant formal schemes over \( O_K \). We define \( \mathbb{M} := \mathbb{M}^{\text{split}} \hat{\otimes}_{O_K} K \) where \( K \) denotes the completion of a maximal unramified extension of \( O_K \).

Let \( \nu = \phi_p \phi_{\infty}^{-1} \) be the embedding of \( \tau_1(F) \) into \( \mathbb{Q}_p \). There exists an inner form \( I \) of \( G \) over \( \mathbb{Q} \) such that \( G(\mathbb{A}^{p, \infty}) \cong I(\mathbb{A}^{p, \infty}) \), \( I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \text{GL}_n(F_\nu) \), cf. [RZ96]. Let \( F \) be the completion of the maximal unramified extension of \( F_\nu \). Then the group \( I(\mathbb{Q}) \) acts on \( \Omega_{O_{F_\nu}} \hat{\otimes} O_F \) through the map \( I(\mathbb{Q}) \subset I(\mathbb{Q}_p) \rightarrow \text{PGL}_n(F_\nu) \) where the latter group acts on the Drinfeld \( p \)-adic upper half plane. It also acts on \( G(\mathbb{A}^{\infty})/U_p \) where \( U_p \subset G(\mathbb{Q}_p) \) is the unique maximal compact subgroup. We describe this action below. We have \( G(\mathbb{A}^{\infty})/U_p \cong \)
\[ G(\mathbb{A}^{p,\infty}) \times G(\mathbb{Q}_p)/U_p \cong I(\mathbb{A}^{\infty}) \times G(\mathbb{Q}_p)/U_p. \]

\[ I(\mathbb{Q}) \text{ acts diagonally on this product. It acts naturally on } I(\mathbb{A}^{p,\infty}) \text{ and it acts on } G(\mathbb{Q}_p)/U_p \text{ through } I(\mathbb{Q}_p) \text{ as follows, cf. [RZ96].} \]

Let \((c,a) \in I(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times GL_n(F_\nu)\) and \((c',a') \in G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times D_\omega^\times\).

\[ \text{Let } \Pi \text{ denote a uniformizer in } D_\omega^\times. \text{ Then } \]

\[ (c,a) \cdot (c',a') \mod U_p = (cc',\Pi^{val_{F\nu}}\det a a') \mod U_p \]

where \(\text{val}_{F\nu}\) is normalized so that \(\text{val}_{F\nu}(F_\nu) = \mathbb{Z}\).

**Definition 1.** We say that an open compact subgroup \(U_p\) of \(G(\mathbb{A}^{p,\infty}) \cong I(\mathbb{A}^{p,\infty})\) is sufficiently small if there exists a prime \(q \neq p\) such that the projection of \(U_p\) to \(G(\mathbb{Q}_q)\) contains no non-trivial elements of finite order.

With this notation, the following is Corollary 6.51 from [RZ96].

**Proposition 11.** \((p\text{-adic uniformization})\) For each sufficiently small open compact subgroup \(U_p \subset G(\mathbb{A}^{p,\infty})\) there is an integral model of \(S(G, U_p U_p) \otimes_{\mathcal{O}(F_\nu)} F_\nu\) over \(\mathcal{O}_{F_\nu}\) and a canonical isomorphism of formal schemes over \(\text{Spf } \mathcal{O}_F:\)

\[ I(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}^{p,\infty})/U_p \cong (S(G, U_p U_p) \otimes_{\mathcal{O}(F_\nu)} \mathcal{O}_F)^\wedge \]

This isomorphism is equivariant for the action of the prime-to-\(p\) Hecke algebra which acts on both sides.

For \(U = U_p U_p\) as above, we set \(U' := G'(\mathbb{A}) \cap U\). Note that \(U'_p = O_{D_\omega}\). We define

\[ S(G',U')(\mathbb{C}) := G'(\mathbb{Q}) \backslash G'(\mathbb{A}) \times X/U'. \tag{4.12} \]

This is a complex variety. There is no moduli problem associated with it, so methods of Rapoport and Zink do not apply here directly. Our main theorem in this chapter is a first step towards \(p\)-adic uniformization for the groups \(G'\):
Theorem 2. Assume $U$ is as above with the additional hypothesis that

$$\mathbb{G}_m(\mathbb{Q}) \cap c(U) = \{1\}.$$ 

Then for every $U$, there exists an integral model $S(G', U')$ of the complex variety $S(G', U')(\mathbb{C})$ over $O_L$ where $L$ is a finite extension of $F_\nu$ unramified at $p$. 
Chapter 5

Proof of second result

5.1 Canonical Models

We briefly recall the definition of a canonical model of a Shimura variety. Briefly, the canonical model of a Shimura variety is characterized by reciprocity laws at certain special points. Most of the material in this section is taken from section 12 of Milne [Mil05] and is well-known classically.

For a reductive group $G$ over $\mathbb{Q}$ and a subfield $k$ of $\mathbb{C}$, we write $\mathcal{E}(k)$ for the set of $G(k)$-conjugacy class of cocharacters of $G_k$ defined over $k$. Let $(G, X)$ be a Shimura datum. Let $S$ denote the Deligne torus $Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. For each $x \in X$, we have a cocharacter $\mu_x$ of $G_{\mathbb{C}}$:

$$\mu_x(z) = h_x,_{\mathbb{C}}(z, 1)$$

where $h_x : S \to G_{\mathbb{R}}$ is the homomorphism corresponding to $x$. A different choice of $x$ will give a conjugate $\mu_x$. Thus $X$ defines an element $c(X)$ of $\mathcal{E}(\mathbb{C})$. It can be shown that $c(X)$ contains a $\mu$ defined over $\overline{\mathbb{Q}}$ and that the $G(\overline{\mathbb{Q}})$-conjugacy class of $\mu$ is independent of the choice of $\mu$. This shows that we can consider $c(X)$ as element of $\mathcal{E}(\overline{\mathbb{Q}})$.

**Definition 2.** The reflex field $E(G, X)$ is the field of definition of $c(X)$, i.e. it is the fixed field of the subgroup of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing $c(X)$ as an element of $\mathcal{E}(\overline{\mathbb{Q}})$.

**Remark.** It can be shown that any subfield $k$ of $\overline{\mathbb{Q}}$ splitting $G$ contains $E(G, X)$. It follows that $E(G, X)$ is finite over $\mathbb{Q}$.

We now define canonical models of Shimura varieties over reflex fields. As mentioned
earlier, these will be defined by explicit reciprocity laws at certain points. We define these points first:

**Definition 3.** A point \( x \in X \) is said to be special if there exists a torus \( \mathcal{T} \subset G \), defined over \( \mathbb{Q} \), such that \( h_x(C^\times) \subset \mathcal{T}(\mathbb{R}) \). The pair \( (\mathcal{T}, x) \) or \( (\mathcal{T}, h_x) \) is called a special pair in \( (G, X) \).

**Example.** Let \( G = \text{GL}_2 \) and let \( \mathcal{X}^\pm = \mathbb{C} \setminus \mathbb{R} \). \( G(\mathbb{R}) \) acts on \( \mathcal{X}^\pm \) by fractional linear transformations. Suppose that \( z \in \mathbb{C} \setminus \mathbb{R} \) generates an imaginary quadratic extension \( E \) of \( \mathbb{Q} \). This gives an embedding \( E \hookrightarrow M_2(\mathbb{Q}) \) using the \( \mathbb{Q} \)-basis \( \{1, z\} \), and thus a maximal subtorus \( \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m \subset G \). Since \( \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m(\mathbb{R}) \) fixes \( z \), this shows that \( z \) is a special point.

We now describe the reciprocity law. Examples of such laws were first obtained by Shimura in the context of Shimura curves. Deligne [Del71] generalized them for an abstract group \( G \) in an adelic setting.

Let \( \mathcal{T} \) be a torus over \( \mathbb{Q} \) and let \( \mu \) be a cocharacter of \( \mathcal{T} \) defined over a finite extension \( E \) of \( \mathbb{Q} \). For \( Q \in \mathcal{T}(E) \), the element \( \sum_{\sigma:E \rightarrow \overline{\mathbb{Q}}} \sigma(Q) \) of \( \mathcal{T}(\overline{\mathbb{Q}}) \) is stable under the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and thus lies in \( \mathcal{T}(\mathbb{Q}) \). Define \( r(\mathcal{T}, \mu) \) to be the homomorphism \( r(\mathcal{T}, \mu) : \text{Res}_{E/\mathbb{Q}} \mathbb{G}_m \rightarrow \mathcal{T} \) such that:

\[
r(\mathcal{T}, \mu)(P) = \sum_{\sigma:E \rightarrow \overline{\mathbb{Q}}} \sigma(\mu(P)), \text{ all } P \in E^\times
\]

(5.1)

Let \( (\mathcal{T}, x) \) be a special pair, and let \( E(x) \) be the field of definition of \( \mu_x \). We define a homomorphism \( r_x : \mathbb{A}_E^\times \rightarrow \mathcal{T}(\mathbb{A}_\mathbb{Q}^f) \) by

\[
\mathbb{A}_E^\times \xrightarrow{r(\mathcal{T}, \mu)} \mathcal{T}(\mathbb{A}_\mathbb{Q}) \rightarrow \mathcal{T}(\mathbb{A}_\mathbb{Q}^f)
\]

(5.2)

where the first map is the natural extension of \( r(\mathcal{T}, \mu) \) to the ideles, and the second map
is the natural projection to finite adeles. If $a \in \mathbb{A}^\times_{E(x)}$ is denoted as $(a_\infty, a_f)$, then

$$r_x(a) = \sum_{\sigma : E \hookrightarrow \mathbb{Q}} \sigma(\mu_x(a_f)).$$

Finally, we define a canonical model. Recall from Class Field Theory that for a special pair $(T, x)$, we have the Artin homomorphism (defined by the geometric Frobenius in this case, as opposed to the usual arithmetic one)

$$\text{art}_{E(x)} : \mathbb{A}_{E(x)} \to \text{Gal}(E(x)^{ab}/E(x)).$$

**Definition 4.** Let $(G, X)$ be a Shimura datum, and let $K$ be a compact open subgroup of $G(\mathbb{A}^f)$. We define

$$\text{Sh}_K(G, X) := G(\mathbb{Q}) \setminus X \times G(\mathbb{A}^f)/K.$$ 

This has the structure of a complex variety. Then a model $M_K(G, X)$ of $\text{Sh}_K(G, X)$ is said to be canonical if, for every special pair $(T, x) \subset (G, X)$ and $a \in G(\mathbb{A}^f)$, the image $[x, a]_K \in \text{Sh}_K(G, X)$ of $(x, a)$ has coordinates in $E(x)^{ab}$ (thus has an action of $\text{Gal}(E(x)^{ab}/E(x))$, and

$$\delta.[x, a]_K = [x, r_x(s).a]_K, \quad (5.3)$$

for all $\delta \in \text{Gal}(E(x)^{ab}/E(x))$ and $s \in \mathbb{A}^\times_{E(x)}$ with $\text{art}_{E(x)}(s) = \delta$.

In other words, $M_K(G, X)$ is canonical if every automorphism $\delta$ of $\mathbb{C}$ fixing $E(x)$ acts on the point $[x, a]_K$ according to the reciprocity law defined above, where $s$ is any idele such that $\text{art}_{E(x)}(s) = \delta|_{E(x)^{ab}}$.

While it can be shown that a canonical model has to be unique if it exists, it indeed is surprising that any canonical models exist at all. While we will not delve into the beautiful and deep theory underlying canonical models here, we make the comment that for certain Shimura varieties of PEL-type which arise from moduli problems, these models can be shown to exist due to their underlying geometry and theory of complex multiplication.
A canonical model for $Sh_K(G, X)$ will define an action of $Aut(\mathbb{C}/E(G, X))$ on the set of connected components $\pi_0(Sh_K(G, X))$. In the case that $G^{der}$ is simply connected, it can be shown that

$$\pi_0(Sh_K(G, X)) \cong T(\mathbb{Q}) \times T(\mathbb{A}^f) / \nu(K)$$

where $\nu : G \to T$ is the quotient of $G$ by $G^{der}$ (so that the notation for $T$ is consistent with the notation from earlier chapter) and $Y$ is the quotient of $T(\mathbb{R})$ by the image $T(\mathbb{R})^\perp$ of $Z(\mathbb{R})$ in $T(\mathbb{R})$. Let $h = \nu \circ h_x : S \to T_\mathbb{R}$ for any $x \in X$. Then the cocharacter $\mu_h$ of $T$ is defined over $E(G, x)$ and thus it defines as before a homomorphism

$$r := r(T, \mu_h) : \mathbb{A}_{E(x)}^\times \to T(\mathbb{A}_\mathbb{Q}).$$

The action of $\delta \in Aut(\mathbb{C}/E(G, X))$ on $\pi_0(Sh_K(G, X))$ is given as follows: let $s \in \mathbb{A}_{E(x)}^\times$ such that $art_{E(x)}(s) = \delta|_{E(x)^{ab}}^{\perp}$, and let $r(s) = (r(s)_\infty, r(s)_f) \in T(\mathbb{R}) \times T(\mathbb{A}_\mathbb{Q}^f)$. Then,

$$\delta.[y, a]_K = [r(s)_\infty y, r(s)_f a]_K, \text{ for all } y \in Y, a \in T(\mathbb{A}_{\mathbb{Q}}^f). \quad (5.4)$$

Note that the set of connected components $\pi_0(Sh_K(G, X))$ is a finite set, so that the action of $Aut(\mathbb{C}/E(G, X))$ has to factor through a finite quotient. In fact, since the reciprocity law is only given by restriction to $Gal(E(G, X)^{ab}/E(G, X))$, it has to factor through an abelian extension of $E(G, X)$.

### 5.2 An Example: Modular Curve

We recall that $\mathcal{M}^\pm = \mathbb{C} \setminus \mathbb{R}$. We call it $X = X^+ \sqcup X^-$ for notational ease. Let $G = GL_2$ for this section. Then for a compact open subgroup $K$ of $G(\mathbb{A}^f)$, we define

$$Sh_K(G, X) := G(\mathbb{Q}) \setminus X \times G(\mathbb{A}^f)/K.$$
Then unraveling the definitions we have made before, it can be proven that

\[
\pi_0(Sh_K(G, X)) \cong \mathbb{Q}^\times \setminus \{\pm 1\} \times (\mathbb{A}^f)^\times / \text{det}(K).
\]

(For any \(a_0 \in GL_2(\mathbb{A}^f)\), the fiber of the map

\[
G(\mathbb{Q}) \setminus X \times G(\mathbb{A}^f)/K \to \mathbb{Q}^\times \setminus \{\pm 1\} \times (\mathbb{A}^f)^\times / \text{det}(K)
\]

given by

\[
[x, a]_K \to [\text{sgn}(x), \text{det}(a)]_K
\]

containing \([X^+, a_0]\) is \(SL_2(\mathbb{Q}) \setminus X^+ \times SL_2(\mathbb{A}^f).a_0/\text{det}(K)\).

In particular, if \(K = K(N)\) for some level \(N\) (take \(N > 2\) for the sake of this example), the above discussion shows that the Shimura variety \(Sh_K(G, X)\) can be written as a disjoint union of connected components. (Since the determinant map is not surjective in this case.) This is the disconnected modular curve, which has a canonical model defined over \(\mathbb{Q}\) as a Galois orbit, but individual components of it are not necessarily defined over \(\mathbb{Q}\). In fact, the reciprocity law shows that this action factors through the ray class field of modulus \(N\infty\) which is \(\mathbb{Q}(\zeta_N)\) in this case, i.e. there are \(\phi(N)\) components with \(\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times\) acting on them. It follows that each connected component gets a canonical model over \(\mathbb{Q}(\zeta_N)\), since it trivializes this action. Since the modular curve \(\Gamma(N) \setminus X^+\) is a connected component, we see that it has a canonical model over \(\mathbb{Q}(\zeta_N)\).

The reciprocity law thus shows that the field of definition of the connected components is unramified outside \(N\infty\). This is so because the reciprocity law factors through a ray class group. In particular, if a prime \(\nu\) is such that \(O_\nu^\times \subset K\) (all primes not dividing \(N\) in our example), then \(\text{det}(a) = 1\), which leads to the field of definition being unramified at \(\nu\). This is the key observation in the proof of the main theorem.
5.3 Proof of Theorem

There is a natural map \( \phi \) from \( S(G', U')(\mathbb{C}) \) to \( S(G, U)(\mathbb{C}) \) induced from the inclusion \( i : G' \hookrightarrow G \). We show in the following lemma that it is injective under a mild hypothesis. This hypothesis is inspired by a similar calculation in Carayol’s paper [Car86]. Roughly, it says that the global image of \( c \) on \( U \) is trivial. Note that \( U' \) is by definition the subgroup of \( U \) where image of \( c \) is trivial on adelic points.

**Lemma 2.** Let \( c \) be the similitude character as before. Assume that \( \mathbb{G}_m(\mathbb{Q}) \cap c(U) = \{1\} \). Then

\[ \phi : S(G', U')(\mathbb{C}) \to S(G, U)(\mathbb{C}) \]

sending a double coset representative \((g, x)\) to \((g, x)\) is injective.

**Proof.** Suppose \((g, x), (h, y) \in G'(\mathbb{A}) \times X\) are such that their image in \( S(G, U)(\mathbb{C}) \) coincides. By definition, this implies that there exist \( a \in G(\mathbb{Q}), u \in U \) such that \( a.(g, x).u = (h, y) \). Hence, it follows that

\[ agu = h \text{ and } ax = y. \]

If we prove that, in fact \( a \in G'(\mathbb{Q}) \) and \( u \in U' \), then injectivity of \( \phi \) follows. Recalling the exact sequence (4.8), we see that

\[ c(agu) = c(au) = c(h) = 1. \]

Thus,

\[ c(a) = c(u^{-1}). \]

By our assumption on the image of \( U' \) under \( c \), this forces

\[ c(a) = c(u) = 1 \]

which completes the proof. \( \square \)
Since $\mathbb{G}_m(\mathbb{Q}) \cap c(U)$ is a congruence subgroup of $\mathbb{Z}^\times$ by definition, we see that this hypothesis is not very strong. Indeed, a congruence subgroup of $\mathbb{Z}^\times$ can be trivial or the group $\{1, -1\}$. So that if $U$ is sufficiently small, then this intersection should be trivial.

We wish to identify $S(G', U')(\mathbb{C})$ with its image under $\phi$. This is the content of our next proposition.

**Proposition 12.** $\phi(S(G', U')(\mathbb{C}))$ is a union of connected components of $S(G, U)(\mathbb{C})$ and $\phi$ is an isomorphism onto its image.

**Proof.** This can be seen by writing each space of double cosets as a finite disjoint union of quotients of $X$ by arithmetic subgroups indexed by the set of double coset representatives. (cf. Milne [Mil05].)

Alternatively, from the definition of $\phi$, it can be seen that it is a local homeomorphism and any bijective local homeomorphism is in fact a homeomorphism.

This identification of connected components can be further explored. There is a zero-dimensional Shimura variety associated with the data of $T \subset G$ which can be identified with the set of connected components of $S(G, U)$. We describe it briefly. Recall that $Z$ is the center of $G$. There is a surjective homomorphism $Z \to T$. We define

\begin{equation}
T(\mathbb{R})^\dagger := \text{Im}(Z(\mathbb{R}) \to T(\mathbb{R})),
\end{equation}

\begin{equation}
T(\mathbb{Q})^\dagger := T(\mathbb{Q}) \cap T(\mathbb{R})^\dagger.
\end{equation}

Then, let $Y := T(\mathbb{R})/T(\mathbb{R})^\dagger$. $Y$ is a finite set. For any compact open set $K \subset T(\mathbb{A}^f)$, we can define a zero-dimensional Shimura variety

\begin{equation}
S(T, K) := T(\mathbb{Q}) \backslash T(\mathbb{A}^f) \times Y / K.
\end{equation}

It is known that

\begin{equation}
\pi_0(S(G, U)) \cong S(T, \nu(U)).
\end{equation}
Similarly, we define

\[ S(T',K') := T'(\mathbb{Q}) \backslash T'(A^f) \times Y'/K'. \tag{5.9} \]

where \( T'(\mathbb{Q})^\dagger \) and \( Y' \) are defined in a similar fashion. We show that \( S(T',\nu(U')) \), like its \( G' \)-counterpart, can be understood as a subgroup of \( S(T,\nu(V)) \).

**Lemma 3.** Under the assumption \( \mathbb{G}_m(\mathbb{Q}) \cap c(U) = \{1\} \), the natural map

\[ \theta : S(T',\nu(U')) \to S(T,\nu(U)) \]

is injective.

**Proof.** Note that since \( G_0 \) is contained in the kernel of the similitude character \( c \), it is well-defined at the level of \( T \). The proof follows the same path we took for Lemma 1. \( \square \)

Since \( T(\mathbb{Q}) \) is dense in \( T(\mathbb{R}) \), \( Y \cong T(\mathbb{Q})/T(\mathbb{Q})^\dagger \). Hence, the space \( T(\mathbb{Q}) \backslash T(A^f) \times Y/K \) can be identified with the set \( T(\mathbb{Q})^\dagger \backslash T(A^f)/K \). There is a natural map

\[ \eta : G(\mathbb{Q}) \backslash G(A^f) \times X/U \to T(\mathbb{Q})^\dagger \backslash T(A^f)/\nu(U) \tag{5.10} \]

given by \( (g,x) \to \nu(g) \). There is a similar map \( \eta' \) induced by \( \nu \) from \( S(G',U')(\mathbb{C}) \) to \( S(T',\nu(U')) \). Since the maps \( \phi \) and \( \theta \) are injective, we can identify \( S(G',U')(\mathbb{C}) \) with a union of the connected components of \( S(G,U)(\mathbb{C}) \) that map to \( S(T',\nu(U')) \) under \( \eta \). We collect this information in the proposition below.

**Proposition 13.** The diagram

\[
\begin{array}{ccc}
S(G',U')(\mathbb{C}) & \hookrightarrow & S(G,U)(\mathbb{C}) \\
\downarrow & & \downarrow \\
S(T',\nu(U')) & \hookrightarrow & S(T,\nu(U))
\end{array}
\tag{5.11}
\]

is cartesian.
Theorem. (Main Theorem)

For each $U'$ satisfying our hypotheses, there exists an integral model $S(G', U')$ of the complex variety $S(G', U')(\mathbb{C})$ over $O_L$ where $L$ is a finite extension of $F_\nu$ unramified at $p$.

Proof. We use the fact that $S(G, U)$ already has an integral canonical model over $O_{F_\nu}$, cf. proposition 11. We have the reciprocity law that describes Galois action on the connected components. (In this case, $E(G, X) = F$.) We see that it factors through a ray class field of $F$. In particular, since the level subgroup $U'$ at $p$ is $O_{D_\omega}^\times$, we see that $p$ is unramified in this ray class field over which the complex variety $S(G', U')(\mathbb{C})$ has a canonical model. (In particular, this field should be the field fixing the subgroup $S(T', \nu(U'))(\mathbb{C}) \subset S(T, \nu(U))(\mathbb{C})$.) This yields the required result and completes the proof.
Appendix A

Ray class characters of bounded order and bounded conductor - By David E. Rohrlich

We introduce a partial order on the set of formal Dirichlet series with nonnegative real coefficients. Given two such series \( A(s) = \sum_{q \geq 1} a(q)q^{-s} \) and \( B(s) = \sum_{q \geq 1} b(q)q^{-s} \), write \( A(s) \preceq B(s) \) to mean that \( a(q) \leq b(q) \) for all \( q \geq 1 \). It is readily verified that if \( A(s) \preceq B(s) \) and \( C(s) \preceq D(s) \) then \( A(s)C(s) \preceq B(s)D(s) \). Furthermore the implication holds at the level of Euler products: if \( A(s) = \prod_p A_p(s) \) and \( B(s) = \prod_p B_p(s) \) with \( A_p(s) \preceq B_p(s) \) for all \( p \) then \( A(s) \preceq B(s) \).

By way of illustration, if \( M \) is a number field and \( \zeta_M(s) \) is the Dedekind zeta function of \( M \) then it is a standard remark that

\[
\zeta_M(s) \preceq \zeta(s)^m, \tag{A.1}
\]

where \( m = [M : \mathbb{Q}] \). Indeed the Euler factor of \( \zeta_M(s) \) at a prime ideal \( p \) of \( M \) of residue class degree \( f \) satisfies

\[
(1 - (Np)^{-s})^{-1} = \sum_{\nu \geq 0} p^{-\nu fs} \preceq \sum_{\nu \geq 0} p^{-\nu s} = (1 - p^{-s})^{-1},
\]

and thus if there are exactly \( r \) prime ideals \( p \) of \( M \) lying over a given rational prime \( p \) then

\[
\prod_{p | p}(1 - (Np)^{-s})^{-1} \preceq (1 - p^{-s})^{-r} \preceq (1 - p^{-s})^{-m}.
\]
Passing to Euler products we obtain (A.1).

It is immediate from the definitions that if $A(s) \ll B(s)$ then the associated summatory functions $\vartheta_A(x) = \sum_{n \leq x} a(n)$ and $\vartheta_B(x) = \sum_{n \leq x} b(n)$ satisfy $\vartheta_A(x) \leq \vartheta_B(x)$ for all $x$. For example, let $A(s)$ and $B(s)$ be the two sides of (A.1): using Theorem 7.7 on p. 154 of [BD04] to estimate the summatory function of $\zeta(s)m$, we obtain

$$\sum_{Na \leq x} 1 \ll x(\log x)^{m-1}, \quad (x \geq 2), \quad (A.2)$$

where the implied constant depends only on $m$.

To illustrate the use of (A.2), let us deduce a standard bound for the class number $h_M$ of $M$. Let $r_1$ and $r_2$ be the number of real embeddings and half the number of complex embeddings of $M$, so that $r_1 + 2r_2 = m$. Thus the Minkowski constant, namely $(4/\pi)^{r_2}m!/m^m$, is bounded above by

$$\mu = (4/\pi)^{m/2} \frac{m!}{m^m},$$

and therefore Minkowski’s theorem gives

$$h_M \leq \sum_{Na \leq \mu \sqrt{d_M}} 1, \quad (A.3)$$

where $d_M$ is the absolute value of the discriminant of $M$ (cf. [Lan94], pp. 119-120). Hence by inserting (A.2) we recover the well-known bound

$$h_M \ll \sqrt{d_M}(\log d_M)^{m-1}, \quad (\mu \sqrt{d_M} \geq 2), \quad (A.4)$$

where the implicit constant depends only on $m$. We shall regard $m$ as a fixed integer $\geq 2$, and thus the condition $\mu \sqrt{d_M} \geq 2$ is satisfied for all but finitely many $M$ with $[M : \mathbb{Q}] = m$. Furthermore, since $m \geq 2$, we have $d_M \geq 2$. Therefore we can remove the condition $\mu \sqrt{d_K} \geq 2$ from (A.4) and still assert that the implicit constant in (A.4) depends
only on \( m \). Actually it is more useful to state (A.4) for \( h_{M}^{\text{nar}} \), the narrow ray class number of \( M \). Since \( h_{M}^{\text{nar}} < 2^{r_{1}}h_{M} \), we have

\[
h_{M}^{\text{nar}} \ll \sqrt{d_{M}(\log d_{M})^{m-1}}, \tag{A.5}
\]

where the implicit constant depends only on \( m \).

It is convenient to refine the relation \( \preceq \) slightly. Suppose that \( A(s) \) and \( B(s) \) are Dirichlet series over \( M \) in the sense that they are presented to us in the form \( A(s) = \sum_{q} a(q)(Nq)^{-s} \) and \( B(s) = \sum_{q} b(q)(Nq)^{-s} \), where \( q \) runs over the nonzero integral ideals of \( M \). We write \( A(s) \preceq_{M} B(s) \) to mean that \( a(q) \preceq b(q) \) for all \( q \). Thus \( \preceq \) coincides with \( \preceq_{\mathbb{Q}} \). Of course every Dirichlet series is a Dirichlet series over \( \mathbb{Q} \), and one readily verifies that if \( A(s) \preceq_{M} B(s) \) then \( A(s) \preceq B(s) \).

Given a nonzero integral ideal \( q \) of \( M \) and a rational integer \( c \geq 2 \), let

\[
R_{M,c}(s) = \sum_{q} h_{M,c}^{*}(q)(Nq)^{-s}
\]

where \( h_{M,c}^{*}(q) \) is the number of idele class characters \( \chi \) of \( M \) of conductor \( q \) such that \( \chi^{c} = 1 \). Also put

\[
E_{M,c}(s) = \prod_{p|c} \prod_{p|p} \left( \sum_{\nu=0}^{e(p)(v_{p}(c)+1)} (Np)^{\nu(1-s)} \right),
\]

where \( e(p) \) is the absolute ramification index of \( p \) and \( v_{p}(c) \) is the \( p \)-adic valuation of \( c \).

**Proposition 1.** \( R_{M,c}(s) \preceq_{M} h_{M}^{\text{nar}} \cdot (\zeta_{M}(s)/\zeta_{M}(2s))^{c-1} \cdot E_{M,c}(s) \).

Define

\[
E_{m,c} = \prod_{p|c} \prod_{e=1}^{m} \prod_{f=1}^{m} \left( \sum_{\nu=0}^{e(v_{p}(c)+1)} p^{f(1-\nu)s} \right)^{m},
\]

The following variant of Proposition 1 is cruder but actually more useful for the intended application, because the dependence on \( M \) (as opposed to \( m \)) is confined to \( h_{M}^{\text{nar}} \):

**Proposition 2.** \( R_{M,c}(s) \preceq h_{M}^{\text{nar}} \cdot (\zeta(s)/\zeta(2s))^{m(c-1)} \cdot E_{m,c}(s) \).
Proof. We have $\zeta_M(s)/\zeta_M(2s) = \prod_p (1 + (Np)^{-s})$, whence $\zeta_M(s)/\zeta_M(2s) \propto \zeta(s)^m$. Also $E_{M,c}(s) \approx E_{m,c}(s)$.

Let $\vartheta_{M,c}(x)$ and $\vartheta_{M,c}(x)$ denote the summatory function associated to $R_{M,c}(s)$ and $(\zeta(s)/\zeta(2s))^{m(c-1)} E_{m,c}(s)$ respectively. Then Proposition 2 gives $\vartheta_{M,c}(x) \leq h_{M}^{\text{har}} \vartheta_{m,c}(x)$. Combining this with (A.5), we find

$$\vartheta_{M,c}(x) \ll \sqrt{d_M} (\log d_M)^{m-1} \vartheta_{m,c}(x),$$

(A.6)

where the implicit constant depends only on $c$ and $m$. Since $E_{m,c}(s)$ is an entire function while $\zeta(s)/\zeta(2s)$ is meromorphic for $\Re(s) > 1/2$ and in fact holomorphic in this region except for a simple pole at $s = 1$, we obtain (cf. [BD04], p. 154, Theorem 7.7):

**Corollary.** $\vartheta_{M,c}(x) \ll \sqrt{d_M} (\log d_M)^{m-1} x (\log x)^{m(c-1)-1}$, where the implied constant depends only on $c$ and $m = [M : \mathbb{Q}]$.

We turn to the proof of Proposition 1. Put

$$\varphi_M(q) = |(\mathcal{O}_K/q)^\times|$$

Let $\mathbb{A}_M^\times$ be the group of ideles of $K$. As usual, we think of $\mathbb{A}_M^\times$ as the restricted direct product $\prod_v M_v^\times$, where $v$ runs over the places of $M$ and $M_v$ is the completion of $M$ at $v$, and we identify $M^\times$ with its image in $\mathbb{A}_M^\times$ under the diagonal embedding. We also put

$$\widehat{\mathcal{O}_M}^\times = \prod_{v \mid \infty} \mathcal{O}_v^\times,$$

(A.7)

where $v$ runs over the finite places of $M$ and $\mathcal{O}_v$ is the ring of integers of $M_v$. By appending the coordinate 1 at the infinite places, we may view $\widehat{\mathcal{O}_M}^\times$ as a subgroup of $\mathbb{A}_M^\times$. Similarly, the product $M_\infty^\times = \prod_{v \mid \infty} M_v^\times$ and its identity component $(M_\infty^\times)^0$ are subgroups of $\mathbb{A}_M^\times$.
with coordinate 1 at the finite places. With these conventions,

\[ h_{M}^{\text{nar}} = |\mathbb{A}_M^{\times} / (M^{\times} \cdot \mathcal{O}_M^{\times} \cdot (\mathcal{M}_\infty^{\times})^0)| \]  

(A.8)

(cf. [Lan94], pp. 146-147). As idele class characters are trivial on the principal ideles and idele class characters of finite order are trivial on the identity component at infinity, we deduce that there are at most \( h_{M}^{\text{nar}} \) extensions of a given character of \( \mathcal{O}_M^{\times} \) to a finite-order idele class character of \( M \). It follows that if we write \( \varphi_{M,c}^*(q) \) for the number of characters \( \chi \) of \( \mathcal{O}_K^{\times} \) with \( \chi^c = 1 \) and \( q(\chi) = q \) (the conductor of a character of \( \mathcal{O}_K^{\times} \) being defined in the same way as for idele class characters) then

\[ h_{M,c}^*(q) \leq h_{M}^{\text{nar}} \varphi_{M,c}^*(q). \]

Now \( \varphi_{M,c}^* \) is multiplicative by virtue of (A.7). It follows that

\[ \sum q h_{M,c}^*(q)(Nq)^{-s} \leq_M h_{M}^{\text{nar}} \prod_{p} (\sum_{\nu \geq 0} \varphi_{K,c}^*(p^\nu)(Np)^{-\nu s}), \]  

(A.9)

where \( p \) runs over the nonzero prime ideals of \( \mathcal{O}_K \).

We now focus on the Euler factor in (A.9) corresponding to a particular prime ideal \( p \). Let \( v \) be the corresponding place of \( M \) and \( p \) the residue characteristic of \( p \). We consider cases according as \( p | c \) or \( p \nmid c \). In both cases we use the fact that if \( \nu \geq 2 \) then \( \varphi_{K,c}^*(p^\nu) \) is the number of characters of \( \mathcal{O}_v^{\times} \) of order dividing \( c \) which factor through \( \mathcal{O}_v^{\times} / (1 + p^\nu \mathcal{O}_v) \) but not through \( \mathcal{O}_v^{\times} / (1 + p^{\nu-1} \mathcal{O}_v) \).

Suppose first that \( p \nmid c \). Then any character of \( \mathcal{O}_v^{\times} \) of order dividing \( c \) is trivial on the pro-\( p \)-group \( 1 + p \mathcal{O}_v \). Hence if \( \nu \geq 2 \) then \( \varphi_{M,c}^*(p^\nu) = 0 \). Furthermore

\[ \varphi_{M,c}^*(p) = \gcd(c, Np - 1) - 1 \]

because \( \mathcal{O}_v^{\times} / (1 + p \mathcal{O}_v) \) is cyclic and the trivial character of \( \mathcal{O}_v^{\times} \) does not have conductor \( p \).
In particular we have $\varphi^*_{M,c}(p) \leq c - 1$, whence
\[
\sum_{\nu \geq 0} \varphi^*_{M,c}(p^\nu)(Np)^{-\nu s} \leq 1 + (c - 1)(Np)^{-s}.
\]

Therefore
\[
\sum_{\nu \geq 0} \varphi^*_{M,c}(p^\nu)(Np)^{-\nu s} \leq M(1 + (Np)^{-s})^{c-1} \tag{A.10}
\]
by the binomial theorem.

Next suppose that $p | c$. If $k \geq e(p)/(p - 1) + 1$ then every element of $1 + cp^kO_v$ is a $c$th power (cf. [Lan94], p. 186). It follows in particular that every element of $1 + cp^{e(p)+1}O_v$ is a $c$th power. It follows that $\varphi^*_{M,c}(p^\nu O_v) = 0$ for $\nu \geq e(p)(v_p(c) + 1) + 1$. Now for $1 \leq \nu \leq e(p)(v_p(c) + 1)$ we apply the trivial estimate
\[
\varphi^*_{M,c}(p^\nu) \leq |O_v^\times/(1 + p^\nu O_v)|.
\]

Since $|O_v^\times/(1 + p^\nu O_v)| = (Np)^{\nu-1}(Np - 1) \leq (Np)^\nu$, we obtain
\[
\sum_{\nu \geq 0} \varphi^*_{M,c}(p^\nu)(Np)^{-\nu s} \leq M \sum_{\nu = 0}^{e(p)(v_p(c)+1)} (Np)^{\nu(1-s)}. \tag{A.11}
\]

This completes our discussion of the individual Euler factors in (A.9).

Now combine (A.9), (A.10), and (A.11). We obtain
\[
\sum_q h^*_{M,c}(q)(Nq)^{-s} \leq M h^\text{nar}_M \prod_{p | c} \prod_{p | p^\nu} (1 + (Np)^{-s})^{c-1} \cdot E_{M,c}(s) \tag{A.12}
\]

We weaken the estimate in (A.12) by extending the product over $p \nmid c$ to a product over all $p$: In other words, we replace the product over $p \nmid c$ by
\[
\zeta_M(s)/\zeta_M(2s) = \prod_p (1 + (Np)^{-s})
\]
Making this substitution in (A.12), we obtain Proposition 1.
List of Journal Abbreviations

Compositio Math.  .............. Compositio Mathematica
Forum Math.  .................. Forum Mathematicum
Forum Math. Σ  ............... Forum of Mathematics Σ
Invent. Math.  ................ Inventiones Mathematicae
Jour. of the AMS ............. Journal of the American Mathematical Society
Publ. Math. IHES ............ Publications Mathématiques de l’IHÉS
Bibliography


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