1948

Trans-sonic flow of a compressible fluid

Engelbrecht, Arthur Frederick

Boston University

http://hdl.handle.net/2144/18535

Boston University
Thesis

Trans-sonic Flow of a Compressible Fluid

Arthur F. Engelbrecht
BOSTON UNIVERSITY
GRADUATE SCHOOL

Thesis

TRANS-SONIC FLOW OF A COMPRESSIBLE FLUID

by

Arthur Frederick Engelbrecht
(A.B., Boston University, 1944)

submitted in partial fulfilment of
the requirements for the degree of

Master of Arts

1948
**TABLE OF CONTENTS**

**INTRODUCTION**
- Statement of problem: 2

**DEFINITIONS OF TERMS**
- Definitions of terms used: 5

**EXPERIMENTAL METHODS**
- Physical plane treatment: 14

**MATHEMATICAL FORMULATION OF THE PROBLEM**
- Non-linear differential equation of flow: 17
- Attempts to linearize the equation: 18
- Stationary, irrotational representation: 22

**FLOW PATTERNS**
- Flow pattern analysis: 25
- Series expansion of velocity potential: 26

**PERTURBATION THEORY**
- Perturbation theory applied: 29

**HODOGRAPH REPRESENTATION**
- Hodograph representation: 30

**LOGARITHMIC REPRESENTATION**
- Logarithmic representation: 35

**CONCLUSION**
- Conclusion: 36

**ABSTRACT OF THE THESIS**
- Abstract of the thesis: 39

**BIBLIOGRAPHY**
- Bibliography: 42
<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
</tr>
<tr>
<td>2. Formulation of problem</td>
</tr>
<tr>
<td>3. Definitions of frame need</td>
</tr>
<tr>
<td>10. EXPERIMENTAL METHODS</td>
</tr>
<tr>
<td>11. MATHEMATICAL FORMULATION OF THE PROBLEM</td>
</tr>
<tr>
<td>12. Approximate plane treatment</td>
</tr>
<tr>
<td>14. Non-linear differential equation of flow</td>
</tr>
<tr>
<td>15. Attempts to linearize the equation</td>
</tr>
<tr>
<td>28. Elementary, higher order representation</td>
</tr>
<tr>
<td>38. Flow pattern analysis</td>
</tr>
<tr>
<td>42. Series expansion of velocity potential</td>
</tr>
<tr>
<td>47. Representation formula applying</td>
</tr>
<tr>
<td>50. Logarithmic representation</td>
</tr>
<tr>
<td>52. Potential representation</td>
</tr>
<tr>
<td>56. CONCLUSION</td>
</tr>
<tr>
<td>58. ABSTRACT OF THE THESIS</td>
</tr>
<tr>
<td>63. BIBLIOGRAPHY</td>
</tr>
</tbody>
</table>
TRANS-SONIC FLOW OF A COMPRESSIBLE FLUID

INTRODUCTION

On stating the problem undertaken, it is well to note that in the theory of fluid flow (hydrodynamic theory) as in general applied mathematical theory, it is customary first to form a mathematical model of the fluid by stating certain axioms or laws to be obeyed and then to draw mathematical deductions of the conduct of the model. The next logical step in the applied theory is to compare deductions from the mathematical model with results of experiments performed on the physical object. If there is a discrepancy between the observations and deductions and the experiment is regarded correct, the deductions are carefully examined and possibly the mathematical model, the postulates and axioms are revised. The purpose of this paper is to draw conclusions from the application of certain axioms and laws to a model fluid, keeping in mind the experimental results.

From photographs of bodies travelling at trans-sonic speed\(^1\), shock waves are actually observed, which waves are not deduced from the mathematically formulated model. The difficulty must be found in the axioms and laws governing the mathematical model.

In the following it is intended to assume the expressions for the conservation of matter, irrotational flow of the fluid,

---

TRANSIENT FLOW OF A COMRESSIBLE FLUID

INTRODUCTION

On studying the properties of fluid mechanics in the context of fluid flow (hydrodynamics), it is customary first to form a semi-physical model of the fluid. Theoretical, numerical, and experimental investigations of the model's behavior are conducted. The next logical step is to form the mathematical model and conduct theoretical and numerical investigations of the model. The purpose of this paper is to consider aspects of these investigations and draw conclusions from the application of certain theorems.

Idea of a model theory relating in mind the experimental results from the basic principles of fluid mechanics. The study of classical fluid mechanics and the use of the mathematical formulation model from the mathematical viewpoint. The difficulty arises in the sense and understanding of the mathematical model. In the following sections, the mathematical model of the fluid mechanics is considered.
the change of state relation, and the existence of a velocity
and stream potential. Using these relations it is possible to
obtain differential equations for the velocity or stream
potential of the flow. The purpose of the hydrodynamic theory
as formulated in this paper is to obtain solutions to these
differential equations. It is to be shown that the different-
tial equations obtained for the stream and velocity potentials
are of an elliptic character when the velocity of flow is
subsonic and of an hyperbolic character when the velocity is
supersonic. No actual method for solving such mixed subsonic
and supersonic problems has been presented to date and it is
not intended to present such a method in this thesis.

The problem is to investigate the various existing
solutions for the velocity or stream potential in the case of
trans-sonic flow of a compressible fluid, to decide which
deductions better suit the model fluid considered, and to
indicate the problems which still lie before the compressible
flow theory.
The purpose of this research is to investigate the nature and role of the faculty in the process of science. The focus is on the faculty as an essential component of the academic community. The role of the faculty in the dissemination of knowledge and the development of new ideas is explored. The impact of faculty decisions on the academic environment is also discussed. The faculty's responsibilities include not only teaching but also research and service. The faculty's role in the academic community is essential for the maintenance of high standards in education and research.
As for the importance of the subject matter, flow at high speeds, in the neighborhood of the velocity of sound has been of lay interest for quite some time. That an object arrive at its destination prior to the arrival of the sound wave produced by the object motivated considerable thought before the subject had been thoroughly analyzed.

It is known that projectiles in flight, high speed bullets, attain supersonic velocities, discussion of these having been presented by Lamb in 1923 \(^2\).

G. I. Taylor noted in 1928 \(^3\) that the propeller tips of a plane travelling at four tenths the speed of sound move at 1.3 times the speed of sound.

Also in ballistics, the problem of trans-sonic flow has presented itself in that the terminal velocity of bombs dropped from great heights is in the trans-sonic range.

More recently, chiefly as a result of wartime interest in faster aircraft, rockets, and guided missiles, trans-sonsics has become of great importance in physical and mathematical research.

From the point of view of the speed of the fluid, below the speed of sound the solutions to the differential equations of flow are well known and experiment has checked the theory.


For speeds above the speed of sound, solutions for the velocity and stream potentials may be obtained and it is assumed that in the supersonic range of speeds experiment again checks theory. But, on increasing speed through the speed of sound, very high pressures are experienced with formation of the shock wave on the exposed surfaces of the object in flight. The effect of this shock wave on different surface contours leads to many possible experimental studies. The mathematical formulation of the problem has not yet been completed in a manner which yields satisfactory solutions.

In classical hydrodynamics the fluids are usually assumed incompressible since solutions to the differential equations are more readily obtained with this assumption and these solutions do not differ excessively from the results of experiment when the flow is of subsonic speed. On increasing the speed of flow, the thermodynamic nature of the fluid enters the problem and the compressibility of the fluid must not be ignored.

Study of the flow of a compressible fluid is also not entirely recent, a paper having been presented in the year 1876 by Riemann ⁴. The analysis of motion of a compressible fluid through jets or ducts was well presented by Chaplygin in 1902 ⁵.

---

⁵ Chaplygin, A. "Gas jets" Sci. Memoirs of the Univ. of Moscow, 1902.
...
It is to be defined at the outset that only macroscopic phenomena are dealt with. In terms of the particle theory of matter, the unit volume element considered shall contain a large number of particles and be assumed continuous.

The fluid under study is formed of continuous matter which may move about freely and thus be readily subject to the action of pressures. More definitely, any continuous matter which offers no resistance to a change in shape with no corresponding change in volume is to be considered a fluid. Thus, liquids and gases qualify for treatment under the definition given above.

That the fluid be compressible, capable of being compressed, refers to the change in volume accompanying a change in pressure on the fluid. In terms of the internal energy of the fluid, isothermal and adiabatic compressibilities are usually distinguished. The isothermal compressibility is the fractional decrease of volume per unit increase of pressure at constant temperature and the adiabatic quantity is the fractional decrease of volume per unit increase of pressure when there is no heat flow. We assert that the sound wave in air travels so rapidly that there is not time enough in the process for a heat transfer, and thus the expression for the adiabatic compressibility of the fluid will be required. For elaboration of the chemical definitions reference is made to Slater.  

---

If it is to be achieved it is essential that only the necessary changes be made in the remainder of the document.

In terms of the particular function of a system, the only volume element contributed by the movement of particles can be measured continuously.

The long answer is "I am not an Continue to page 8."
The state of the fluid is defined by assigning to each point in the fluid a velocity vector indicating the velocity of flow at that point. The fluid is said to be in equilibrium when it is in a state of rest. When the fluid is not in equilibrium it is in a state of flow. The state of flow is given by the velocity vectors assigned to the points throughout the fluid. If the flow velocity vector at a given point does not change with respect to time, the state of flow is called the stationary state; if the flow velocity vector does change with time, the flow is in a non-stationary state. The flow is also described by curves called the streamlines of flow, drawn by connecting the flow velocity vectors from point to point, thus forming the tangent curve or streamline of the flow. The streamlines of flow are equal to the lines of motion of the fluid only in the stationary state. The state of flow is also described by a set of curves, related to the streamlines, called the velocity potential lines and defined as the lines of the flow along which the magnitude of the velocity is constant.

The fluid, which is compressible, has elastic properties. It is known that a disturbance in an elastic body will be propagated through the body with a speed dependent on the coefficient of elasticity and the density of the body. In the case of the fluid air, the disturbance is called a sound wave and its magnitude is defined in standard air, that is, air at sea level, (15° centigrade, 760 mm. of mercury) as 763 miles per hour.  

The rules of the third law involve the determination of forces by the factor of the moment. The third law is said to be in the problem of the state of the body. When the third law is not to apply, the forces of the body determine the forces of the body. The third law is said to be in the state of the body. When the third law is given, the forces of the body determine the forces of the body. If the forces of the body determine the forces of the body, then the forces of the body determine the forces of the body. The third law is said to be in a non-Newtonian state. The third law is also considered a set of various situations to the question. The forces of the body determine the forces of the body. The question of the state of the body is also considered a set of various situations to the question. The forces of the body determine the forces of the body. If it is known that a phenomenon in an existing body will be included, when a person goes through the body. If the body is considered, then the question is called a certain way and the forces of the body determine the forces of the body.
Pending further definition, the ratio of the speed of an object to the speed of sound in air will be called the Mach number of the flow hereafter. The range of velocities has been arbitrarily divided in terms of the speed of sound in air as follows:

Subsonic range...... up to 600 miles per hour;
Trans-sonic range.... 600 to 900 mph;
Supersonic range..... over 900 mph.

It is interesting to note that the mathematical models prescribed for certain different physical phenomena are alike in form. For example, the mathematical fluid equations which apply to electrodynamics, mechanics, and hydrodynamics are alike in form. Due to the similarity of the mathematical expressions it is possible to devise analogous experiments from the three physical realms. Whereas it is difficult to perform experiments involving the transonic flow of a compressible fluid, it is much easier to perform experiments analogous to the former by means of an electrolytic bath, or a devised mechanical system. A brief exposition of the experimental analogies used to approximate the trans-sonic flow of a compressible fluid is next given.

The mathematical treatment of the problem follows, starting with the five equations assumed on pages one and two. The non-linear differential equations of the flow are formed and attempts to linearize this equation are presented. The representation of the flow by a stationary, two-dimensional flow in
Completing the introduction, the plan of procedure is next presented. In the statement of the aims of applied mathematical theory, it was noted that the deductions on a mathematical model must be confirmed by experiment on the physical object. The first sections of the thesis concern the experimental results, including added descriptions of the flow patterns, means of generating supersonic speeds for observation, and the methods used in measuring the phenomena.

It is interesting to note that the mathematical models prescribed for certain different physical phenomena are alike in form. For example, the mathematical field equations which apply to electrodynamics, mechanics, and hydrodynamics are alike in form. Due to the similarity of the mathematical expressions it is possible to devise analogous experiments from the three physical realms. Whereas it is difficult to perform experiments involving the transonic flow of a compressible fluid, it is much easier to perform experiments analogous to the former by means of an electrolytic bath, or a devised mechanical system. A brief exposition of the experimental analogies used to approximate the trans-sonic flow of a compressible fluid is next given.

The mathematical treatment of the problem follows, starting with the five equations assumed on pages one and two. The non-linear differential equations of the flow are formed and attempts to linearize this equation are presented. The representation of the flow by a stationary, two-dimensional flow in
Complementary to the introduction, the plan of procedure in
next previewed. In the introduction of the main point of
metamaterial research, it was noted that the predominance of a
metamaterial cannot merely be considered as experimental on the
important aspect. The theoretical analysis of the microwave
experiments revealed interesting and geometric properties of the
flows bettering means of generating superconductive devices for applications,
flow, and the means to achieve metamaterial phenomena.

It is important to note that the metamaterials model
emerging in certain different, intrinsically emerging and
intrinsically lying superconductive flows, which are macroscopic, metamagnetics,
and microscopic, the fields to which one refers to the similarity of the metamagnetic
expression in terms of view of the metamagnetic expression.

It is of interest to realize that it is difficult to obtain
experimental evidence of a metamagnetic phase transition involving
measuring of an electrostatic field, to a non-linear conducting
scheme. A partial explanation of the experimental anomalies needs
next view.

The metamagnetic transition to the principal transistor
were

with the five experimental scenario to observe one and two.

non-linear differential equation of the form we following the
acceleration to investigate this anomaly and comparing the
temperature of the form, an experimental two-dimensional view in
specificity of the form of a transistor, the two-dimensional view in

the physical, hodograph, and logarithmic coordinate systems is presented with criticism. Various series expansions are indicated with the results of the expansions noted.

Sources of data are recorded in the bibliography. A most complete list of printed matter concerning compressible flow theory and non-linear physical theory in general is given by von Karman 8.

EXPERIMENTAL METHODS

The usual method of examining flow phenomena in aerodynamics is by use of a wind tunnel. The power unit driving the propeller is here of prime importance. A large power unit is required to approach supersonic speeds in a wind tunnel. Usually the pressure in the observation chamber is varied so that supersonic speeds may be more easily attained.

In this way, Dr. Rudolph Hermann achieved speeds of more than four times the speed of sound. His method involved the use of a tunnel which was intermittently connected to a large sphere. The sphere was evacuated and observations on models were made during the time required for the evacuated sphere to be filled with air flowing through the wind tunnel. According to periodicals the German government in the closing days of the second world war was building a wind tunnel of this nature designed to produce speeds near 7,000 miles per hour.

Methods of recording the flow patterns include the surface, pitot, and static pressure tubes familiar to subsonic wind tunnel experiments. The photographic methods used are the Schlieren interferometer, Toepler striation, and the direct shadow methods. The photographic methods rely on the fact

9 Newsweek Dec. 6, 1946: "German Scientists working for the United States Army" p.64.
EXPERIMENTAL圖片

The present method of examining the properties of salmon
is to use a wire mesh. The lower part of the fish
breaks in a manner to indicate its
deficiency in a certain respect. An attempt will be
made to extend the observations to a wider
pertaining to the presence of the operation competes at a rate so

In this way, Dr. William Watson's sensitive device of more
than four times the speed of sound. Theoretical
investigations of the
are all based upon which are introduced into the
field
through the use of an electron microscope at
a rate of more than 100 per second. The device was
exacerbated and experimental equipment was
made during the time reserved for the experimental phase of
an attempt to work upon the machine from the
second. According
to calculations, the German government in the autumn of
the second world war was participating with certain of this
feature

In the presence of limitations the low latitude indicates the
surface, point, and during the observation, up to the point of

The present experimental device was
the
reduction in the intensity of light on the
topical reaction of the.

According to calculations, the low latitude indicates the
surface, point, and during the observation, up to the point of

The present experimental device was
the
reduction in the intensity of light on the
topical reaction of the.

According to calculations, the low latitude indicates the
surface, point, and during the observation, up to the point of

The present experimental device was
the
reduction in the intensity of light on the
topical reaction of the.

According to calculations, the low latitude indicates the
surface, point, and during the observation, up to the point of

The present experimental device was
the
reduction in the intensity of light on the
topical reaction of the.
that the flow pattern consists of layers of air of varying density and light will be distorted on being transmitted through the fluid. The Toepler striation method of photography yields a very detailed picture of the flow especially in the boundary layers between regions of different density. The direct shadow method yields a well defined image of the shock wave proper with less detail of the flow.

The study of flow patterns for projectiles which travel at trans-sonic speeds indicates the physical problem at hand. At the nose of the projectile, the air undergoes a condensation, a pressure wave is formed. Since air is an elastic fluid, the pressure wave travels from the point of formation in all directions at approximately the speed of sound. For speeds of the projectile below the speed of the pressure wave, the latter travels away from the projectile in all directions. For speeds of the projectile greater than the speed of propagation of sound in air, the pressure wave will not be transmitted in all directions, since only lateral transmissions will be possible. At the nose of the projectile a cushion of air is present. The air in the cushion is compressed causing the projectile to experience much greater pressures when travelling with velocities greater than the velocity of sound.

Referring to flow over the surface of an airfoil, the above named pressure wave is accompanied by the so-called shock wave. This shock wave appears where the direction of the
The study of how particles form protective layers, which act as barriers against energy. The interaction between layers of different materials, the direct exchange between particles of the same material, and the effect of light on the protective layer. This process involves a well-defined layer of the material which forms a protective layer against light. The protective layer is formed at the interface of the particle and the material. The effectiveness of this layer depends on the properties of the materials involved. The protective layer provides a barrier against the penetration of light into the material, preventing it from being absorbed or reflected. This process is essential for the formation of protective layers on the surface of materials in various applications.
flow changes. That is, the air in the cushion between the pressure wave and the foil rushes over the surface of the foil's upward curvature at supersonic speeds. On changing the directions to the foil's downward curvature, the speed of the flow becomes subsonic as it slows down. The transition from the super- to the subsonic flow is accompanied by a shock wave at the point of transition.

A report on the experimental analogies used to approximate conditions observed at the trans-sonic flow of the compressible fluid now follows.

An electrical analogue is used to solve flow problems for incompressible fluids when the flow is limited to two dimensions. The method consists in passing an alternating current through a layer of liquid of the proper electrolytic conductivity and mapping the amplitude of the alternating voltage. If the height of the layer of electrolyte is variable (the base of the tray may be made of material which may be carved to any desired shape) the density of the fluid analogue (air) may be said to be proportional to the height of the electrolyte. By means of modifying the shape of the bottom of the tray it is possible to solve problems of compressible flow theory by means of an iteration method. The process is applicable only under the initial assumption that the flow of electricity be two dimensional. This assumption is no longer true if the depth of the electrolyte is too great. In problems
In the context of the connection between the presence of any kind of brown and the surface of the material, the presence of particles at a microscopic or submicroscopic scale may affect the process of the formation of the material. The presence of these particles may be due to the interaction of the material with its environment. The formation of these particles may be a result of the action of a process that may be described as a reaction to the presence of particles at a microscopic scale.

A report on the experimental evidence suggests that the formation of these particles is limited to two conditions: the presence of a layer of a material on the surface of the material, and the compatibility and structure of the material itself. Additional evidence suggests that the formation of these particles may be part of the process of the formation of the material itself.

The presence of these particles may affect the formation of the material itself. As a result, the formation of these particles may be part of the process of the formation of the material. The presence of these particles may also affect the formation of the material itself, and the formation of these particles may be part of the process of the formation of the material.

In the context of the formation of the material, the presence of these particles may affect the formation of the material itself. As a result, the presence of these particles may be part of the process of the formation of the material.
of compressible flow the method is not accurate as the velocity of flow approaches the velocity of sound. The use of the electrical analogue thus becomes increasingly erroneous as the velocity approaches the velocity of sound.

A mechanical analogue to two dimensional flow of a compressible fluid of constant depth presents itself in the flow of an incompressible fluid in a channel of gradually varying depth. A consideration of the flow through a duct of variable cross-sectional area may be used to show the essential difference between the mechanics of the flow of a compressible and that of an incompressible fluid.

The stepping stones of classical hydrodynamic theory of a compressible fluid are the mathematical expressions for the conservation of matter, the condition of irrotational flow of the fluid, the adiabatic change of state relation, the velocity and the stream potentials of the flow. These five expressions form a set of non-linear differential equations. A transformation of coordinates may be effected whereby the non-linear partial differential equations are linearized and solutions may be obtained to satisfy the mathematical model fluid flow.
MATHEMATICAL FORMULATION OF THE PROBLEM

The limitations imposed on the classical model for hydrodynamic phenomena must be stated and kept in mind in the following pages. The classical theory formulation assumes a continuous space and the presence of force fields derivable from potential functions. From these assumptions solutions to a given flow pattern may be obtained. However, the shock wave, on either side of which different densities of fluid are observed, cannot be completely understood with the classical model. A solution may be approximated if the shock wave be considered a discontinuity in the compressible fluid. Other treatments of the subject consider the flow of a perfect incompressible fluid which is allowed to have a small viscosity and heat conductivity rather than introduce the more advanced thermodynamic relations.

The stepping stones of classical hydrodynamic theory of a compressible fluid are the mathematical expressions for the conservation of matter, the condition of irrotational flow of the fluid, the adiabatic change of state relation, the velocity and the stream potentials of the flow. These five expressions form a set of non-linear differential equations. A transformation of coordinates may be effected whereby the non-linear partial differential equations are linearized and solutions may be obtained to satisfy the mathematical model fluid flow.
The limitations imposed on the theoretical model so far

Dynamic behaviors must be studied and kept in mind in the
following phases. The theoretical lecture on technical measures
contributes greatly to the understanding of causes leading to
them. However, the scoops may vary from one to another.

The following ideas can be applied:

1. A scientific can be simplified if the scoops are
not critical to any significant part of the
original concept. Certain phenomena may be
combined with the classical

2. Conceptual simplicity in the commercial field.

3. Concentration of the subject concept, i.e., the
topic of a particular focus of

4. A subject worth of attention to have a smooth transition

5. Keeping in mind the factors from which the model

The subject of the study is the commercial field. The

The following ideas can be applied in the

The above statements are a result of non-negative scientific

A conclusion of the above can be selected:

Based on the above, a model is proposed.
It is to be noted that the subject of hydrodynamics, as here considered, exists and is to be developed in plane (and solid) Euclidean space.

Considering now the flow of a compressible fluid, the following notation will be used. Let the orthogonal axes be represented by $x_i$; $u_i$ be the components if the velocity of flow $U$. Two dimensional flow is assumed in the following and therefore the letter subscripts are allowed values 1, 2.

The summation notation employed: wherever a subscript appears twice in the same term of the equation, summation is indicated.

As stated before, the following expressions are assumed;

1. Conservation of matter,
   $$(\rho u_i)_i = -\frac{\partial p}{\partial t} = \frac{\partial (\rho u_i)}{\partial x_i} + \frac{\partial (\rho u_j)}{\partial x_j}$$

2. Irrotational flow of the fluid,
   $$(u_i)_j = (u_j)_i \quad i \neq j$$

3. Thermodynamic change of state,
   $$p = \alpha \rho^\gamma + \beta$$

Notation used in (3): $p$ is the pressure of the fluid,
\(\rho\) variable density of the fluid,
\(\gamma\) adiabatic index, constant for a fluid,
\(\alpha, \beta\) constants.

From (1) and vector analysis, it is seen that, in the stationary case, $\rho U$ must be the curl of some function which is
The equation for the flow of a compressible fluid is the following potential, with a mean. The core form of the potential due to the velocity of the source is expressed as

\[ \text{Flow} = \text{The stream function of the flow vector in the following and}
\]

\[ \text{Source term of the stream function, stream function in the source term of the}
\]

\[ \frac{1}{2} \left( \nabla \times \mathbf{v} \right) \cdot \nabla \phi = 0
\]

\[ v = \phi
\]

\[ q = \nabla \phi = q
\]

\[ q \cdot \nabla \phi = q
\]

\[ \boldsymbol{c} \cdot \nabla \phi = q
\]

From (1), we see that to find the

\[ \text{Potential energy of the stream function, stream function in the energy equation}
\]

\[ \text{of the stream function, stream function in the energy equation}
\]

\[ \text{of the stream function, stream function in the energy equation}
\]
The partial derivative of a quantity with respect to the coordinates is denoted by a comma subscript following the quantity, the subscript indicating the coordinate. See the expression (1) for the Conservation of matter on page 15, as an example.

\[ \partial_q = \partial_q, \]

In order to set up the equation connecting the density with the velocity of flow, we make use of Bernoulli's relation,

\[ \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{p}{\rho} \right) + \int_{x_0}^{x} \frac{d\rho}{\rho \sqrt{2g}} = 0, \]

wherein the expression in the denominator of the integral is relation (3). The evaluation of (6) follows, the integral equation is solved by use of relation number 155 in the Pedroc Integral tables.

From (5),

\[ \rho = \frac{(p-\alpha)}{\alpha \gamma}, \]

\[ \int_{x_0}^{x} \frac{d\rho}{\rho} = \alpha \int_{x_0}^{x} \frac{d\rho}{(\rho-\alpha) \gamma} = \left[ -\frac{\rho}{\sqrt{2g} (p-\alpha)} \right]_{x_0}^{x}. \]

To evaluate the lower limit, we define a stagnation point with values of the quantities, \( \rho_0, \) \( \gamma, \) \( \beta = 0. \) Let \( \frac{\rho_0}{\gamma} = \beta \) be the velocity of sound at the defined point.

Since \( \alpha = \frac{\gamma \rho_0}{\beta} \) and \( \beta = \frac{\rho_0}{\gamma} \rho_0 + \beta \) thus, \( \rho_0 = \frac{\beta}{\gamma}. \)

\[ \frac{\partial}{\partial x} \left[ \frac{\gamma \rho_0}{(\rho_0-\alpha)^{\gamma/2}} \right]_{x_0}^{x} = \frac{\partial}{\partial x} \left[ \frac{\gamma \rho_0}{(\rho_0-\alpha)^{\gamma/2}} - \frac{\gamma \rho_0}{(\rho_0-\alpha)^{\gamma/2}} \right] = \frac{\partial}{\partial x} \left[ \frac{\gamma \rho_0}{(\rho_0-\alpha)^{\gamma/2}} \right]_{x_0}^{x}. \]
The present activities of a community with regard to the cooperator program, which is the subject of the previous paragraph, are of particular interest to the cooperative de-
called the stream function, stream potential, $\Psi$, (4)  
$$u_i = \frac{1}{\rho} \psi_j, \quad i \neq j.$$  
From vector analysis and (2) it is seen that $U$ must be the gradient of some scalar function which we define the velocity potential, $\phi$. (5)  
$$u_i = \phi, i.$$  

In order to set up the equation connecting the density with the velocity of flow, we make use of Bernoulli's relation, (6)  
$$\frac{(u_i)^2}{2} + \int_{\rho_0}^\rho \frac{dp}{\rho(p)} = 0,$$  
wherein the expression in the denominator of the integral is relation (3). The evaluation of (6) follows, the integral equation is solved by use of relation number 108 in the Peirce integral tables.  

From (3)  
$$\rho = \frac{(p-\beta)^{\frac{1}{\gamma}}}{\alpha^\frac{1}{\gamma}},$$  
$$\int_{\rho_0}^\rho \frac{dp}{\rho} = \alpha^\frac{1}{\gamma} \int_{\rho_0}^\rho \frac{dp}{(p-\beta)^{\frac{1}{\gamma}}} = \left[ \frac{\gamma \alpha^{\frac{1}{\gamma}}}{\gamma - 1} (p-\beta)^{\frac{1}{\gamma}} \right]_{\rho_0}^\rho.$$  
To evaluate the lower limit, we define a stagnation point with values of the quantities, $\rho_0$, $p_0$, $\beta = 0$. Let $\frac{dp_0}{d\rho_0} = c^2$ be the velocity of sound at the defined point.  

Since $\alpha = \frac{\rho_p}{\rho_0}$ and $p = \frac{\rho_p}{\rho_0} \rho^\gamma + \beta \quad \text{thus,} \quad c^2 = \frac{\rho_p}{\rho_0} \gamma$.  

$$\frac{\rho_0^{\frac{1}{\gamma}}}{\rho_0} \left[ \frac{\gamma}{\gamma - 1} (p-\beta)^{\frac{1}{\gamma}} \right]_{\rho_0}^\rho = \frac{\rho_0^{\frac{1}{\gamma}}}{\rho_0} \left[ \frac{\gamma}{\gamma - 1} (p-\beta)^{\frac{1}{\gamma}} - \frac{\gamma}{\gamma - 1} (p_0)^{\frac{1}{\gamma}} \right]_{\rho_0}^\rho.$$  

$$= \frac{c^2 \rho_0^{\frac{1}{\gamma}}}{\gamma - 1} \left( \frac{\rho_0^\gamma}{\rho_0^\gamma} \right)^{\frac{1}{\gamma}} - \frac{c^2}{\gamma - 1},$$
\( \psi \)

Calling the above function, assume particular

\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle \tag{1} \]

From vector subtraction and \( H \) it is seen that \( H \) must be the

eigenvalue of some scalar function which we call the

potential.

\[ i\hbar \frac{\partial}{\partial t} |\phi\rangle = \hat{\mu} \tag{2} \]

In order to set up the equation connecting the general
with the velocity of the wave, we make use of Poynting's relation

\[ S = \frac{\partial E}{\partial t} = \frac{\partial H}{\partial t} \tag{3} \]

This particular expression in the denominator of the integral is

\[ \frac{d}{dt}(\phi_{\alpha} - \phi_{\beta}) = \eta \tag{4} \]

\[ \left[ \frac{\partial}{\partial t} \phi_{\alpha} - \phi_{\beta} \right] \left[ \phi_{\alpha} - \phi_{\beta} \right] = \frac{d}{dt} \phi_{\alpha} - \phi_{\beta} \]

\[ \phi_{\alpha} = \frac{d}{dt} \phi_{\alpha} - \phi_{\beta} \]

To evaluate the lower limit, we call in a stationary point, with

\[ \frac{d}{dt} \phi_{\alpha} = \phi_{\beta} \]

\[ \frac{d}{dt} \phi_{\alpha} \left[ \frac{d}{dt} \phi_{\alpha} - \phi_{\beta} \right] = \frac{d}{dt} \phi_{\alpha} - \phi_{\beta} \]

\[ \phi_{\alpha} = \frac{d}{dt} \phi_{\alpha} - \phi_{\beta} \]

\[ \phi_{\alpha} = \phi_{\beta} \]

\[ \phi_{\alpha} = \frac{d}{dt} \phi_{\alpha} - \phi_{\beta} \]

\[ \phi_{\alpha} = \phi_{\beta} \]

\[ \phi_{\alpha} = \phi_{\beta} \]
Substituting into (6) the value of the integral and equating the resultant expression in terms of the density, with the given boundary conditions, (7) is obtained,

\[
\frac{u_1 u_1}{2} + \frac{c^2}{\gamma - 1} \left[ \left( \frac{\rho}{\rho_0} \right)^{\gamma - 1} - 1 \right] = 0
\]

(7)

\[
\rho = \rho_0 \left[ 1 - \frac{\gamma - 1}{2} \frac{u_1 u_1}{c^2} \right]^{1/(\gamma - 1)}
\]

The three equations (4), (5), and (7), comprising the equations of motion of the compressible fluid, are seen to form a non-linear differential equation.

In order to attempt a linearization of the non-linear partial differential equations obtained for the flow of a compressible fluid, the following approach to the theory is employed. The effects of volume forces on the system are to be neglected. Volume forces are those forces exerted on the infinitesimal volume elements, such as might be caused by the field of gravity, electricity, or magnetism on the system of fluid. Consideration falls, then, on the area forces, defined as the forces existing between the infinitesimal volume elements and proportional to the area of the face shared by the two volume elements. The area forces in hydrodynamic theory are usually taken to be proportional to the pressure of the fluid.

The equations of motion of the fluid under the above definitions of existing forces are,

\[
\frac{\partial \left( \rho u_1 \right)}{\partial t} + \frac{\partial}{\partial x_i \left( \rho u_1 u_i + p \right) = 0 ,
\]

(8)
Substituting into (6) the value of the integrals and substituting the result of the expression for terms of the general form, we obtain

\[ 0 = \left( 1 - \frac{1}{\gamma} \left( \frac{c_1}{c_2} \right) \right) \frac{c_1}{1 - \gamma} + \frac{c_2}{c_2} \]

\[ \left( \frac{1}{\gamma} \left( \frac{c_1}{c_2} \right) \right) \left( -c_1 \right) = c_1 \]

(7)

The phase equations (7) and (8), (9) imply the absence of motion of the composites (the line is seen to form a non-linear differential equation).

In order to express a differential equation for the motion of a composites, the following approach is taken:

The absence of volume forces on the axes are to be expected. The absence of volume forces on the axes are expressed on the assumption of infinitesimal volume elements, such as might be common of the field of brane, specifically to represent the field of infinitesimal volume elements between two infinitesi volume elements. Consequently, these forces on the axes are neglected, and the forces acting between the infinitesimal volume elements are the forces acting between the infinitesimal volume elements.

In the expression of volume of the line formed by the two and the plane normal to the line of the axes parallel to the plane and perpendicular to the line of the line are taken to be proportional to the pressure of the line.

The absence of motion of the line under the space get

\[ 0 = (a + b + c + d) \frac{c_1}{a + b + c + d} + \frac{c_2}{c_2} \]

(8)
which may be reduced to the following by use of the expression for the conservation of matter, (1),

\[ \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial \rho}{\partial x_i} = 0 \]

The velocity of sound, \( c \), is defined by the relation

\[ c^2 = \frac{\partial p}{\partial \rho} \]

From the thermodynamic change of state relation (3), and, noting that \( \frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x_i} = c^2 \frac{\partial \rho}{\partial x_i} \), (9) becomes,

\[ \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + c^2 \frac{\partial \rho}{\partial x_i} = 0 \]

Equations (11) and (1) together form a non-linear set of equations of motion of the fluid which will be called (11a),

\[ \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -c^2 \frac{\partial \rho}{\partial x_i} \]

\[ \frac{\partial (\rho \partial u_i)}{\partial x_i} + \frac{\partial \rho}{\partial t} = 0 \]

The first method of linearizing the set of equations (11a) assumes that \( u_i \) is small and the deviation of \( \rho \) from average value \( \rho_0 \) is small. These assumptions reduce (11a) to,

\[ \frac{\partial u_i}{\partial t} + \frac{c^2}{\rho_0} \frac{\partial \rho}{\partial x_i} = 0 \]

\[ \frac{1}{\rho_0} \frac{\partial \rho}{\partial t} + \frac{\partial u_i}{\partial x_i} = 0 \]

Eliminating \( u_i \) from (12), we obtain the wave equation for \( \rho \),

\[ \Delta \rho = \frac{\partial^2 \rho}{\partial x_1^2} = \frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2} \]
\[ 0 = \frac{\phi}{x^2 c} + \frac{\mu}{x c^3 \sigma} \phi + \frac{\mu}{J c} \phi \]  

Note the similarity of equation (1) and the following by this relation

\[ \frac{\phi}{c^2} = \frac{\sigma}{c} \]  

The similarity of equation (1) and (2) was described by the transformation

\[ 0 = \frac{\phi}{x^2 c} \phi + \frac{\mu}{x c^3 \sigma} \phi + \frac{\mu}{J c} \phi = \frac{\phi}{J c} + \frac{\mu}{x c^3 \sigma} \phi \]

To eliminate the second of these (11) and apply (11) to the first, we have

\[ 0 = \frac{\phi}{J c} + \frac{\mu}{x c^3 \sigma} \phi \]

And, taking (II) into consideration, we have

\[ \frac{\phi}{x^2 c} \phi = \frac{\mu}{J c} \phi + \frac{\mu}{x c^3 \sigma} \phi \]  

Eliminating \( \phi \) from (11) and applying the same substitution for \( \phi \) to (II), we have

\[ \frac{\phi}{x^2 c} \frac{\mu}{x c^3 \sigma} \phi = \frac{\phi}{J c \times \sigma} \phi = \phi \Delta \]
Under the assumptions made, the equations of motion are
linearized, solutions to the wave equation (13) being known.
A second method of linearizing equations (11a) makes use of the
assumption that $u_2$ be small, $u_1$ differ only slightly from $U$
where $U$ is the mean velocity of flow and may be large. Also,
the deviation of $\rho$ from $\rho_0$ is still to be small. Let

$$u_1 = U + u_1' \quad u_2 = u_2'$$
in (11a), obtaining,

$$\frac{\partial u_1'}{\partial t} + u_2 \frac{\partial u_1'}{\partial x_1} + u_1 \frac{\partial u_1'}{\partial x_1} = -\frac{c^2}{\rho} \frac{\partial \rho}{\partial x_1}$$
$$+ \frac{\partial (\rho u_1')}{\partial x_1} + \rho \frac{\partial \rho}{\partial x_1} = 0$$

$$\frac{\partial u_2'}{\partial t} + u_2 \frac{\partial u_2'}{\partial x_2} = -\frac{c^2}{\rho} \frac{\partial \rho}{\partial x_2}$$
$$+ \frac{\partial (\rho u_2')}{\partial x_2} = 0$$

From the above and using the assumptions made, obtain (14),

$$\frac{\partial u_1'}{\partial t} + U \frac{\partial u_1'}{\partial x_1} = -\frac{c^2}{\rho} \frac{\partial \rho}{\partial x_1}$$
$$+ \rho \frac{\partial \rho}{\partial x_1} = 0$$

It is now desired to eliminate $u_1$ from both equations (14).
This is done by differentiating both equations with respect to
$x_1$ and again with respect to $t$. Differentiate the first
equation in (14) with respect to $x_1$, obtaining,

$$\frac{\partial^2 u_1'}{\partial x_1 \partial t} + U \frac{\partial^2 u_1'}{\partial x_1 \partial x_1} + \frac{c^2}{\rho} \frac{\partial^2 \rho}{\partial x_1^2} = 0$$

differentiate the second equation in (14) with respect to $t$,

$$\frac{\partial^2 \rho}{\partial t^2} + U \frac{\partial^2 \rho}{\partial t \partial x_1} + \rho \frac{\partial^2 u_1'}{\partial t \partial x_1} = 0$$
The assumption that $U$ is small in a differential region is based on the fact that $U$ is the mean atmospheric or flow, and may be large, the flow itself is to be split into small parts, with the assumption of the atmosphere, to obtain:

$$0 = \frac{\partial U}{\partial x} + \frac{1}{x} \left( \frac{\partial}{\partial y} \right) U + \frac{1}{x^2} \frac{\partial^2 U}{\partial y^2} = \frac{1}{x} \frac{\partial U}{\partial y} + \frac{1}{x^2} \frac{\partial^2 U}{\partial y^2}$$

From this, we obtain:

$$0 = \frac{\partial U}{\partial x} + \frac{1}{x} \frac{\partial U}{\partial y} + \frac{1}{x^2} \frac{\partial^2 U}{\partial y^2} = \frac{1}{x} \frac{\partial U}{\partial y} + \frac{1}{x^2} \frac{\partial^2 U}{\partial y^2}$$

This is the linear differential equation for $U(x,y)$. The solution to this equation is:

$$0 = \frac{\partial U}{\partial x} + \frac{1}{x} \frac{\partial U}{\partial y} + \frac{1}{x^2} \frac{\partial^2 U}{\partial y^2}$$

Solving (14) with respect to $x$, we obtain:

$$0 = \frac{\partial U}{\partial x} = \frac{1}{x} \frac{\partial U}{\partial y} + \frac{1}{x^2} \frac{\partial^2 U}{\partial y^2}$$

This is a first-order differential equation with respect to $x$. The solution to this equation is:

$$0 = \frac{\partial U}{\partial x}$$

Solving (14) with respect to $y$, we obtain:

$$0 = \frac{\partial U}{\partial y}$$

This is a first-order differential equation with respect to $y$. The solution to this equation is:

$$0 = \frac{\partial U}{\partial y}$$
now differentiating the second equation in (14) with respect to 
\( x_1 \) obtain,

\[
\frac{\partial^2 \rho}{\partial x_1^2} + U \frac{\partial^2 \rho}{\partial x_1 \partial x_2} + \rho \frac{\partial^2 u_1}{\partial x_1^2} = 0
\]

From the last two equations on page nineteen,

\[
- \frac{\partial^2 u_1}{\partial x_1 \partial t} = U \frac{\partial^2 u_1}{\partial x_1 \partial x_1} + \frac{c^2}{\rho} \frac{\partial^2 \rho}{\partial x_1^2} = \frac{1}{\rho} \frac{\partial^2 \rho}{\partial t^2} + \frac{U}{\rho} \frac{\partial^2 \rho}{\partial t \partial x_1}.
\]

Rearranging the two right-hand members,

\[
U \frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{\rho} \frac{\partial^2 \rho}{\partial t^2} + \frac{U}{\rho} \frac{\partial^2 \rho}{\partial t \partial x_1} = \frac{c^2}{\rho} \frac{\partial^2 \rho}{\partial x_1^2}.
\]

Using the expression for \( \frac{\partial^2 u_1}{\partial u_l^2} \) from the relation at the top of page twenty, obtain

\[
\frac{\partial^2 \rho}{\partial x_1^2} = \frac{1}{c^2} \left[ \frac{\partial^2 \rho}{\partial \tau^2} + 2U \frac{\partial \rho}{\partial \tau \partial x_1} + U^2 \frac{\partial^2 \rho}{\partial x_1^2} \right]
\]

The quantity in brackets has the form of a quadratic and the expression may be written

\[
\Delta \rho = \frac{1}{c^2} \left( \frac{\partial}{\partial \tau} + U \frac{\partial}{\partial x_1} \right)^2 \rho.
\]

The relation (15) is reducible by an affine transformation to the Laplace equation having solutions for the case \( U < c \).

Considering the stationary state of the fluid flow (that state of flow for which time derivatives vanish), equation (15) is changed as follows,

\[
\left(1 - \frac{U^2}{c^2}\right) \frac{\partial^2 \rho}{\partial x_1^2} + \frac{\partial^2 \rho}{\partial x_2^2} = 0
\]
\[ 0 = \frac{a_{12}}{g^2 G^2} \frac{S}{f^2} u + \frac{a_{12}}{g^2 G^2} \frac{S}{f^2} \theta + \frac{a_{12}}{g^2 G^2} \frac{S}{f^2} \theta \]

From the first two equations we have,

\[ \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} u + \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta = \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta + \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta = \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta \]

Representing the two right-hand members,

\[ \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} u + \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta = \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta \]

Use the expression for \( \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta \) from the relation of the form

\[ \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} u + \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta + \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta \]

The equation in question then has the form of a differential and the expression may be written

\[ \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} u + \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta = \frac{\partial^2 \theta}{\partial x^2} \frac{S}{g^2 G^2} \theta \]

The relation \( 1 \) leads to the expression for the vanishing of the reaction of the case \( \theta > 0 \).

Knowing the solution of the equation of the type (1),

\[ 0 = \frac{a_{12}}{g^2 G^2} \frac{S}{f^2} u + \frac{a_{12}}{g^2 G^2} \frac{S}{f^2} \theta \]

(15)
The Mach number of the flow was defined previously as the ratio of the speed of flow to the speed of sound. By the definition of \( U \) and of \( c \), the Mach number may be defined in terms of these as \( M = \frac{U}{c} \). With this value, (16) is written,

\[
(1 - M^2) \frac{\partial^2 \rho}{\partial x_1^2} - \frac{\partial^2 \rho}{\partial x_2^2} = 0
\]

This linear differential equation of the second order contains acceptable solutions for values of \( M \) below,

(a) \( U < c \) \quad 0 < M < 1

(b) \( U > c \) \quad M > 1

In case (a) the differential equation is of the elliptic type and may be reduced to Laplace's equation by the transformation,

\[
x_1' = \frac{x_1}{\sqrt{1-M^2}}, \quad x_2' = x_2
\]

In case (b), the differential equation is of the hyperbolic type which may be solved by various methods.

The two methods of approximation (to linearize equations) herein described do not provide a close analogy to the physical problem in most cases, and other solutions to the non-linear equations must be obtained to better this analogy.

The most criticized assumption is that of a mean velocity of flow, \( U \). Imposing a mean velocity on the fluid amounts,

\[\text{For a general treatment of both hyperbolic and elliptic differential equations, reference is made to, Courant, R. and Hilbert, D., "Methoden der Mathematischen Physik" vol. II, 1937 Julius Springer Verlag, Berlin.}\]
The mean weight of the iron and yellow, developed as the result of the growth of the iron to the power of an unknown. By the help of these results, we obtain a value of the mean weight, which is given by the following equation:}

\[ \frac{C}{\sqrt{C}} - \frac{C}{\sqrt{C}} (C - 1) \]

The mean weight is determined by the following equations:

\[ I > 0 \quad c > 0 \quad (a) \]

\[ I < 0 \quad c < 0 \quad (b) \]

In case (a) the alternative equation is of the elliptical type and may be reduced to Lagrange's equation by the transformation:

\[ \frac{x}{x} = \frac{1}{x} \quad \frac{y}{y} = \frac{1}{y} \]

In case (b), the alternative equation is of the hyperbolic type.

The two methods of superposition (to transform the problem into a problem of the wave equation) may not be used in our case, and other methods to the problem must be applied to partial differential equations. The work of partial differential equations is based on a mean, not of iron, but of the iron's properties.
usually, to saying that the velocity of flow nowhere differs greatly from the undisturbed velocity, at a distance from the point under observation. This assumption is known to be quite erroneous when the velocity near an object approaches supersonic velocities. In the following section no mean velocity will be specified so that this criticism at least, will be removed.

Using the same notation as before, it is intended to represent the trans-sonic flow by a stationary, irrotational, two dimensional flow. To better distinguish the two sections, the notation for velocities will be changed so that in the following $v_i$ indicates the components of the velocity. Thus, the flow considered must be:

(a) Stationary
\[ \frac{\partial \rho}{\partial t} = \frac{\partial v_i}{\partial t} = 0 \]

(b) Irrotational
\[ \frac{\partial v_i}{\partial x_i} = \frac{\partial v_j}{\partial x_j} \]

(c) Two-dimensional $i = 1, 2 \quad j = 1, 2 \quad i \neq j$.

Applying condition (a) to the conservation of matter relation (1), we obtain for the continuity equation in our flow,

(17)
\[ \frac{\partial v_i}{\partial x_i} = - \frac{v_i \partial \rho}{\rho \partial x_i} \]

evaluate $\frac{\partial \rho}{\partial x_i}$ by use of relation (7) for $\rho$ and (10) for $c^2$ as follows; from (7)
\[ \frac{\partial v_i}{\partial x_i} = \rho \left( - \frac{1}{2} \frac{l}{\alpha \gamma \rho^{\gamma-1}} \right) \frac{\partial v_i^2}{\partial x_i} \]
and from (10), $\frac{d \rho}{d \rho} = \alpha \gamma \rho^{\gamma-1} = c^2$. 
\[ \frac{\partial u}{\partial x} = \frac{\partial \phi}{\partial x} \]

\[ \frac{\partial v}{\partial x} = \frac{\partial \phi}{\partial x} \]

\[ \int_{-1}^{1} \phi_1 = 0 \]

\[ \int_{-1}^{1} \phi_2 = 1 \]

\[ \int_{-1}^{1} \phi_3 = 0 \]

\[ \int_{-1}^{1} \phi_4 = 0 \]

\[ \int_{-1}^{1} \phi_5 = 0 \]

\[ \int_{-1}^{1} \phi_6 = 1 \]

\[ \int_{-1}^{1} \phi_7 = 0 \]

\[ \int_{-1}^{1} \phi_8 = 0 \]

\[ \int_{-1}^{1} \phi_9 = 0 \]

\[ \int_{-1}^{1} \phi_{10} = 0 \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]

\[ \frac{\partial v}{\partial x} \]

\[ \frac{\partial u}{\partial x} \]
Combining the above relations, obtain,

\[ \frac{\partial p}{\partial x_1} = - \frac{\rho}{2c^2} \frac{\partial v_i^2}{\partial x_1} \quad \text{Equation (18)} \]

Equations (17) and (18) are our modified continuity and Bernoulli equations, respectively. Substituting (18) into (17) yields,

\[ \frac{\partial v_i}{\partial x_1} = \frac{v_i v_k}{c^2} \frac{\partial v_i}{\partial x_k} \quad \text{Equation (19)} \]

Using the velocity potential function, \( \varphi \), (19) becomes,

\[ \frac{\partial^2 \varphi}{\partial x_1^2} = \frac{1}{c^4} \frac{\partial \varphi}{\partial x_1} \frac{\partial \varphi}{\partial x_k} \frac{\partial \varphi}{\partial x_k} \frac{\partial \varphi}{\partial x_1} \quad \text{Equation (20)} \]

and, on expanding subscripts \( i \) and \( k \), the following is obtained,

\[ \frac{1}{c} \frac{\partial \varphi}{\partial x_1} \frac{\partial \varphi}{\partial x_1} \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \frac{\partial \varphi}{\partial x_2} \frac{\partial \varphi}{\partial x_2} - \frac{\partial \varphi}{\partial x_1} \frac{\partial \varphi}{\partial x_2} \frac{\partial \varphi}{\partial x_1} \frac{\partial \varphi}{\partial x_2} 2 \]

The cross derivatives are summed by use of the condition that the flow be irrotational. The following is obtained by arranging the above in terms of the second derivatives present,

\[ \left[ 1 - \left( \frac{v_i}{c} \right)^2 \right] \frac{\partial \varphi}{\partial x_1} - \frac{2v_i v_k}{c^2} \frac{\partial \varphi}{\partial x_k} \frac{\partial \varphi}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \left[ 1 - \left( \frac{v_k}{c} \right)^2 \right] \frac{\partial^2 \varphi}{\partial x_1^2} = 0 \quad \text{Equation (21)} \]

In order to find the character of possible solutions of (21), the primary assumption will be made that (21) is a second ordered differential equation linear in its second derivatives. The assumption on (21) requires that the velocity components \( v_i \) be dependent on the coordinates, that is,

\[ v_i = f_i(x_1, x_2) \quad i = 1, 2 \]

Then we may say, along with the theory of differential equations,
\[ \frac{\partial u}{\partial x} \frac{u}{x} = \frac{\partial v}{\partial y} \frac{v}{y} \]

(19)

Equation (19) states that the ratio of the partial derivatives of the functions in question is constant. This is a fundamental property of certain differential equations.

Let the velocity perturbation function

\[ \frac{\partial u}{\partial x} \frac{u}{x} + \frac{\partial v}{\partial y} \frac{v}{y} = \frac{\partial w}{\partial z} \frac{w}{z} \]

(20)

be rewritten as

\[ \frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} \frac{u}{x} + \frac{\partial v}{\partial y} \frac{v}{y} \]

The above expression is known as the continuity equation, which expresses the conservation of mass.

The above equation is obtained by integrating the partial derivatives of the functions along the path of the fluid elements.

\[ \theta = \frac{3}{2} \int \left( \frac{1}{x} \sqrt{x} \right) \left( \frac{1}{x} \sqrt{x} \right) \left( \frac{1}{x} \right) + \frac{3}{2} \int \left( \frac{1}{x} \sqrt{x} \right) \left( \frac{1}{x} \sqrt{x} \right) \left( \frac{1}{x} \right) \]

(21)

In order to find the characteristic of possible solutions of (21).

The continuity assumption will be made that (21) is a second order differential equation. Therefore, the differential equation is obtained as

\[ \frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} \frac{u}{x} + \frac{\partial v}{\partial y} \frac{v}{y} \]

and the equation for the characteristic of the solution is

\[ s, t = 1 \quad (x, t) \sqrt{t} = 1 \]

The above equation is known as the characteristic of the differential equation.
that the type of the differential equation (21) at a point \((x_1,x_2)\) is determined by the value of the determinant of the coefficients of (21). If \(G\) is the value of the determinant consisting of elements \(g_{ij}\) which are the coefficients of the \(i\) and \(j\)th partial derivative of \(\phi\), then the following conditions determine the character of (21):

- if \(G > 0\), the differential equation is of elliptic type,
- if \(G = 0\), the differential equation is of parabolic type,
- if \(G < 0\), the differential equation is of hyperbolic type.

In the case at hand,

\[
G = \left| \begin{array}{cc}
1 - \left(\frac{v_1}{c}\right)^2 & - \frac{v_1v_2}{c^2} \\
- \frac{v_1v_2}{c^2} & 1 - \left(\frac{v_2}{c}\right)^2
\end{array} \right| = 1 - \frac{v_1^2 + v_2^2}{c^2} = 1 - \frac{v^2}{c^2}
\]

According to the statements on the type of the differential equation given above, (21) is seen to be of:

- elliptic type if \(v < c\),
- hyperbolic type if \(v > c\),
- parabolic type if \(v = c\).

From (22) the difficulties involved in obtaining solutions for the velocity potential in the physical plane are realized. Three different types of equations must be solved as the velocity of flow in the channel passes through the velocity of sound. It is to be remembered that these difficulties have occurred when the flow is described in the physical plane, that is, the frame of reference with coordinates \(x_1, x_2\).
More will be said about other frames of reference. Again, it is proper to investigate the initial assumptions leading to the conclusions (22). It was assumed that the coefficients of the second derivatives in (21) were linear. The degree of error of this assumption should be pointed out.

The value \( c \) was defined by (10) to be, \( c = \frac{dp}{d\rho} \). By this definition, the speed of sound in an incompressible fluid would be infinite. This infinite value of \( c \) reduces (20) to the Laplace equation before mentioned as having been thoroughly treated in potential theory. For compressible fluids, \( c^2 \) has a finite value and (20) has coefficients dependent of the velocity of flow and the velocity of sound in the fluid. Solutions to (21) in the compressible case, with \( c \) finite, are obtained by methods used to solve non-linear differential equations.

Dealing exclusively with similarities in flow patterns, it is possible to set up the pattern for the flow of a compressible fluid past a given object by using the flow pattern obtained for an incompressible fluid past the same body. The empirical methods, using alternating current in an electrolytic solution, or incompressible fluid (water to a close approximation) in a channel of gradually increasing depth, have been described (on pages 12 and 13). As results of experiment, G.I. Taylor and C.F. Sharman have stated that the approximation to the compressible flow potential in the electrical case is not very
von Karman states that the variable depth channel method yields qualitative information of the possible flow patterns in the case of supersonic flow of a gas corresponding to "supercritical flow" in the channel.

A study of flow patterns indicate that if the element of an incompressible flow field be taken to be a square shaped element (the distances between two stream potential lines and two velocity potential lines be approximately equal), then the corresponding field assuming a compressible fluid will be made up of rectangular elements. Noting this, Bryan attempted to generalize that, due to the adiabatic compressibility of a compressible fluid, the square shaped elements in the flow pattern of an incompressible fluid are distorted from squares to rectangles. This planned distortion method failed.

A second method used in solving the non-linear partial differential equations consists in representing the potential functions of the trans-sonic flow with the assumption of axial symmetry on the part of the body around which the potential functions are to be determined. The velocity potential about a cylinder and a sphere have been approximated by Lord Rayleigh with certain symmetry assumptions, and treatments of other surfaces usually start with these approximations. The assumption that the velocity vector nowhere differ much from the

---

The process of determining the potential of a system involves:

1. Identifying the relevant variables and constants.
2. Formulating the governing equations.
3. Applying boundary conditions and initial conditions.
4. Solving the equations numerically or analytically.
5. Interpreting the results and validating the solution.

The potential energy of a system is given by:

\[ U = \sum \frac{1}{2} k_i x_i^2 + \sum \frac{1}{2} k_{ij} x_i x_j \]

where \( k_i \) and \( k_{ij} \) are the spring constants, and \( x_i \) and \( x_j \) are the displacements of the system.
undisturbed velocity, or mean velocity (as in the development starting on page 19) is equal to the assumption that the body around which the potential functions are being determined be axially symmetric. If the body under investigation presents a small area to the flow, such as a thin airfoil, then the method of small perturbations on the velocity potential is used.

Otherwise, in the symmetrical body treatment, the differential equations (21) may be solved by a series expansion of the velocity potential in terms of powers of the Mach number of the flow. As in the previous work, $U$ shall be the velocity of the undisturbed flow, and the value $M = \frac{U}{c}$ with $c$ the sonic speed in the undisturbed region, is the Mach number of undisturbed flow. The values $\phi$ used in the successive approximations are all assumed functions of the coordinates $x_1, x_2$. Given the series expansion of $\phi$,

$$
\phi = U \left[ \phi_0 + \phi_1 M^2 + \phi_2 M^4 + \ldots \right]
$$

On substitution into (20) note that (23) provides as a first approximation,

$$
\Delta \phi_0 = 0
$$

an expression with the form of the Laplace equation which is the differential equation obtained for the potential of the incompressible fluid. Obtaining the next approximation in terms of $\phi_0$,

$$
\Delta \phi_1 = \frac{\partial^2 \phi_0}{\partial x_1^2} + \frac{\partial^2 \phi_0}{\partial x_2^2} + \frac{\partial^2 \phi_0}{\partial x_3^2}
$$
in the development

of small perturbations on the velocity potential to near

smallness in the streamfunction form treatment of the

steady assumption (ii) may be solved by a series expansion of the

velocity potential in terms of powers of the Mach number of the

flow. As in the previous work, we shall use the analogy of the

method described above and the virial theorem in the mathematical

treatment of the expansion of the source strength of the co-ordinates

Given

the series expansion of

\[
\left[ \ldots + \frac{1}{M_1^2} + \frac{1}{M_2^2} + \ldots \right] U = \omega
\]

\( \theta \)

or that

\[ \frac{1}{M_1^2} + \frac{1}{M_2^2} + \ldots = \omega \Delta \]

\( \theta \)

as an approximation to (88) with the law of leading expansion

which is the differential equation for the potential of

Optimizing the next approximation

\[ \frac{1}{M_1^2} \frac{1}{M_2^2} \frac{1}{M_3^2} \ldots = \omega \Delta \]
The partial derivatives in (25) may be computed from the solutions to the analogous problem for the incompressible fluid, (24), and then substituted in (25). Thus, the first approximation to the velocity potential is expressed as a function of the coordinates \( x_1, x_2 \). Similarly, having the function \( \Phi \) as dependent on the coordinates, it is possible to obtain the second approximation, \( \Phi_2 \), and express it as a function of position, and so on, to any desired degree of approximation. Thus, the solution to (25) for the given boundary conditions at the surface of the body is made equivalent to obtaining expressions of the velocity potential around a body in an incompressible fluid, in which the potential is produced by a field of continuously distributed sources. The sources are assumed in the problem when the statement is made that the first approximation to the velocity potential shall be a function of coordinates, i.e., \( \Delta \Phi = F(x_1, x_2) \).

The method outlined provides reasonable solutions if the flow is around bodies of symmetrical form, cylinders and spheres for example, because the solutions of (25) may be obtained by superposition for these symmetrical bodies. The method is laborious for approximations higher than the first and the series expansion converges very slowly if the Mach number of the flow is greater than 0.4. Clearly, the series expansion here indicated does not find place in problems of trans-sonic flow where the Mach number is greater than 0.6.
The partial derivatives in (5) may be computed from the 

\( \partial^2 \phi / \partial x^2 \) and \( \partial^2 \phi / \partial y^2 \) with respect to the variables \( x \) and \( y \). These are the first derivatives of \( \phi \) with respect to \( x \) and \( y \), respectively. The second derivatives are \( \partial^2 \phi / \partial x^2 \) and \( \partial^2 \phi / \partial y^2 \) and so on. To find higher derivatives of \( \phi \), we proceed as described above.

The solution to (5) for the given boundary conditions is the superposition of the parts of the solution that correspond to each of these boundary conditions. The superposition principle states that a solution is the sum of the solutions corresponding to each individual boundary condition.

The function \( \phi(x, y) \) is a solution of the partial differential equation if it satisfies the equation

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \Delta \phi \]

where \( \Delta \) is the Laplacian operator.

The method of characteristics may be applied in this case. The method of characteristics is a powerful tool for solving partial differential equations. The method relies on the superposition of the solutions corresponding to each of the boundary conditions and the imposition of the boundary conditions on the solution.
The method of small perturbations, as indicated above, rises from the assumption that the velocity of flow nowhere differ greatly from the undisturbed velocity. On page 19 and following it was assumed that the density nowhere differ greatly from a mean density. Expressions analogous to those derived on page 21 are obtained for the velocity potential. In the development below it is assumed that the velocity potential must contain an added perturbing potential function which does not differ much from the given velocity potential. Thus, it is assumed that \( \phi = Ux + \phi' \), and, analogous to (16),

\[
(26) \quad \left(1 - \frac{U^2}{c^2}\right) \frac{\partial^2 \phi'}{\partial x_1^2} + \frac{\partial^2 \phi'}{\partial x_2^2} = 0
\]

In an analysis similar to that proposed on page 21, it is seen possible to reduce (26) to the Laplace equation when \( U < c \), in the case for subsonic flow by the transformation,

\[
(27) \quad x_1' = x_1 \quad x_2' = x_2 \sqrt{1 - M^2}
\]

If \( U > c \), that is, the undisturbed flow is supersonic, (26) again has the form of a wave equation and solutions are known.

Due to the transformation (27) it has been stated that the flow of a compressible fluid around a thin body with Mach number of undisturbed flow \( M \), may be found approximately by expanding the dimensions of the body normal to the main flow such that the dimensions in the incompressible flow are multiplied by \( \sqrt{1 - M^2} \) to obtain the above stated dimensions in the compressible case. It is to be noted that this method is
The method of small perturbations as incorporated above

tries to capture the essential part of the velocity of flow, however

While the strategy from the multistationed velocity on page 12 may

follow Section II. Therefore, extend the general conclusion which

rests on a mean velocity. Expressions and methods for those yielding in

place of the approximating for the velocity potential. In the

development below, it is assumed that the velocity potential

were contrary to a single potential function which uses

the velocity known from the given velocity potential. Thus, if

and substitute to \( \dot{q} = 0 \),

\[ 0 = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \left( \frac{x}{x_0} - 1 \right) \]  

(53)

As can be seen, in an entirely similar to that proposed on page 55, it is seen

that in the multistationed flow is unsatisfactory. (53)

If \( U > 0 \), then the form of a wave motion is satisfied and is known

due to the transformation (57). In the case of a flow with mean

flow of a compressible fluid some a finite body with mean

velocity of multistationed flow \( M \) may be taken of a compressible

shock front. The generalization of the body norlity to the major

Lore the shock front. The generalization in the incompressible flow is

multiplied by \( \sqrt{\frac{M - 1}{1}} \) to obtain the proper velocity dimension. In

the compressible case, it is to be noted that this method to
still one of obtaining solutions to a differential equation which is non-linear. Also that no discussion is made of the physical case where the Mach number of the flow is 1.0.

The next representation of flow patterns to be discussed is the transformation from the physical to the hodograph plane, first effected by Molenbroek\(^{14}\) and Chaplygin\(^{15}\). The velocity and stream potentials are here considered functions of the magnitude, \(v\), and the direction, \(\theta\), of the flow velocity vector. It is possible to obtain two linear partial differential equations where the potential functions satisfy the systems,

\[
\frac{\partial \phi}{\partial \theta} = \frac{v}{c} \frac{\partial \psi}{\partial v}
\]

(28)

\[
\frac{\partial \phi}{\partial v} = \frac{1 - (\frac{v}{c})^2}{c \nu} \frac{\partial \psi}{\partial \theta}
\]

The new coordinates in terms of the former are, \(v = \sqrt{v_1^2 + v_2^2}\) and \(\theta = \tan^{-1} \frac{v_2}{v_1}\).

The density and speed of sound are expressible as functions of the flow velocity and the thermodynamic constants so that it is possible to eliminate them using relations (3), (7), and (10) and the definition of the stagnation point. Lighthill\(^{16}\) using

---

\[
\begin{align*}
\frac{\nabla \phi}{\nabla \phi} &= \frac{\phi}{\phi} \\
\frac{\nabla \left( \frac{\nabla \phi}{\phi} \right) - I}{\frac{\nabla \phi}{\phi}} &= \frac{\phi}{\phi}
\end{align*}
\]

The new coordinates in terms of the original are:

\[\frac{\nabla \phi}{\nabla \phi} \quad \text{and} \quad \frac{\nabla \left( \frac{\nabla \phi}{\phi} \right) - I}{\frac{\nabla \phi}{\phi}}\]
the above method, obtains,

\[
\frac{\partial \varphi}{\partial \theta} = \frac{\partial v}{\partial \nu} \frac{v}{(1 - v^2)^{1/2}}
\]

(29)

\[
\frac{\partial \varphi}{\partial \nu} = \frac{\partial v}{\partial \theta} \frac{(\gamma + 1)v^2 - (\gamma - 1)}{(\gamma - 1)v(1 - v^2)^{1/2}}
\]

\( \varphi \) is eliminated from the above equations by differentiating the first with respect to \( \nu \) and the second with respect to \( \theta \), then equating the two expressions to obtain,

\[
(30) \quad \frac{(\gamma - 1)(1 - v^2)}{(\gamma + 1)v^2 - (\gamma - 1)} \frac{\partial^2 \psi}{\partial \nu^2} + \frac{2v(\gamma - 1)}{(\gamma + 1)v^2 - (\gamma - 1)} \frac{\partial^2 \psi}{\partial \nu \partial \theta} - \frac{\partial^2 \psi}{\partial \theta^2} = 0
\]

Rearranging the terms in the above,

\[
(31) \quad \frac{\partial^2 \psi}{\partial \theta^2} = \frac{(\gamma - 1)}{(\gamma + 1)v^2 - (\gamma - 1)} \left[ v^2(1 - v^2) \frac{\partial^2 \psi}{\partial \nu^2} + 2v \left( 1 + \frac{2 - \gamma}{\gamma - 1} \right) \frac{\partial \psi}{\partial \nu} \right]
\]

is obtained. It is noted that (31) is linear, since the coefficients of the right hand side are functions of the velocity \((\gamma \) is assumed to be the adiabatic constant).

The general solution of (31), regular when \( v = 0 \) and odd in \( \theta \) is

\[
(32) \quad \psi(v^2, \theta) = A_0 \theta + \sum_{n=1}^{\infty} A_n \psi_n(v^2) \sin n\theta
\]

where the \( \psi \) under the summation sign indicates the hypergeometric functions,
\[
\frac{v}{r_0 N_0 (\gamma - 1)} \frac{\gamma}{v_0} = \frac{\gamma \nu}{v_0} = \frac{\gamma \nu}{v_0} \\
\frac{(\gamma - 1) - \frac{v_0}{v}(1 + \gamma)}{r_0 N_0 (\gamma - 1)v(1 + \gamma)} \frac{\gamma}{v_0} = \frac{\gamma \nu}{v_0} = \frac{\gamma \nu}{v_0}
\]

From these equations, we can eliminate \( v \) from the following equation to find the value of the second term to an excellent approximation. Then, substituting these two expressions into equation (1) gives:

\[
0 = \frac{v_0^2}{r_0 N_0} - \frac{\gamma \nu}{v} \left[ \frac{\gamma - 1}{\gamma + (\gamma - 1)v_0} \frac{(1 - \gamma)(1 - \gamma)}{(1 - \gamma) - \frac{v_0}{v}(1 + \gamma)} + \frac{\gamma \nu}{v_0} \left( \gamma - 1 \right) \right] - \frac{\gamma \nu}{v} \frac{(1 - \gamma)}{(1 - \gamma) - \frac{v_0}{v}(1 + \gamma)} - \frac{\gamma \nu}{v_0} = \frac{\gamma \nu}{v_0}
\]

Thus, the velocity curve is given by:

\[
\left[ \frac{\gamma \nu}{v_0} \left( \frac{\gamma - 1}{\gamma + (\gamma - 1)v_0} + \left( \gamma - 1 \right) \right) - \frac{\gamma \nu}{v} \left( \gamma - 1 \right) \right] - \frac{(1 - \gamma)}{(1 - \gamma) - \frac{v_0}{v}(1 + \gamma)} = \frac{\gamma \nu}{v_0}
\]

The general solution of equation (15) is:

\[
\Psi = \left( \frac{\nu}{v_0} \right) \int_{0}^{\infty} \left[ \Psi_{\infty} + \frac{\gamma \nu}{v_0} \right] \psi_0 (x) + \left( 0, \frac{\nu}{v_0} \right) \Psi
\]

where the integral is the connection with the initial function.
\[ \Psi_n(v^2) = v^n F(a_n, b_n; n+1; v^2) \]

where

\[ a_n + b_n = n - \frac{1}{\gamma - 1} \]

\[ a_n b_n = -\frac{n(n+1)}{2(\gamma - 1)} \]

\[ a_n^2 b_n = \frac{1}{2} \left[ n - \frac{1}{\gamma - 1} \pm \sqrt{\frac{(\gamma+1)n^2}{(\gamma-1)^2} + \frac{1}{(\gamma-1)^2}} \right] \]

The expansion of (33) follows from the definition of the hypergeometric series,

\[ \Psi_n(v^2) = v^n \left[ 1 + \frac{a_n b_n}{1.(n+1)} v^2 + \frac{a_n(a_n+1)b_n(b_n+1)}{1.2.(n+1)(n+2)} v^4 + \ldots \right] \]

The analysis of the hypergeometric function is given in Whittaker and Watson \(^{17}\) including convergence of the series and the residues about the ennumerated poles, important to the theory of flow in the hodograph plane. Lighthill evaluates the subsonic and supersonic expressions for (32) and gives a connected account of the properties of the expansion as a function of \( n \) in the complex plane.

At the sonic speed, \( v = \sqrt{\frac{\gamma-1}{\gamma+1}} \) and at the velocity \( v = \infty \) the hypergeometric series expansion fails so that at these points singularities are observed in the hodograph plane.

The following conditions are imposed on the \( \Psi_n \) in order to obtain a steadily increasing velocity on the axis of the

\[
\frac{1}{I - \gamma} - \beta = n^d + n^a
\]

\[
\frac{(I + \beta)\gamma}{(I - \gamma)} = n^d n^a
\]

\[
\left[ \frac{\beta (I + \beta)}{(I - \gamma)} \right]^{1/2} \frac{1}{I - \gamma} - \beta = n^d n^a
\]

The expression of the dehuyequity function is given in

\[
\sum_{k} \frac{1}{I - \gamma} \left[ \frac{1}{I + \gamma} \right]^{1/2} - \beta = n^d n^a
\]

The expression of the dehuyequity function is given in

\[
\sum_{k} \frac{1}{I - \gamma} \left[ \frac{1}{I + \gamma} \right]^{1/2} - \beta = n^d n^a
\]

The expression of the dehuyequity function is given in

\[
\sum_{k} \frac{1}{I - \gamma} \left[ \frac{1}{I + \gamma} \right]^{1/2} - \beta = n^d n^a
\]

The expression of the dehuyequity function is given in

\[
\sum_{k} \frac{1}{I - \gamma} \left[ \frac{1}{I + \gamma} \right]^{1/2} - \beta = n^d n^a
\]

The expression of the dehuyequity function is given in

\[
\sum_{k} \frac{1}{I - \gamma} \left[ \frac{1}{I + \gamma} \right]^{1/2} - \beta = n^d n^a
\]

The expression of the dehuyequity function is given in

\[
\sum_{k} \frac{1}{I - \gamma} \left[ \frac{1}{I + \gamma} \right]^{1/2} - \beta = n^d n^a
\]

The expression of the dehuyequity function is given in

\[
\sum_{k} \frac{1}{I - \gamma} \left[ \frac{1}{I + \gamma} \right]^{1/2} - \beta = n^d n^a
\]
channel from the value 0 at \( x = -\infty \) to the value \( v_1 \), a
supersonic value, at \( x = +\infty \):
\[
\begin{align*}
\psi_0 &< 0 \quad \text{for} \quad 0 < v < \sqrt{\frac{\gamma - 1}{\gamma + 1}} \\
\psi_0 &= \infty \quad \text{at} \quad v = -\sqrt{\frac{\gamma - 1}{\gamma + 1}} \\
\psi_0 &> 0 \quad \text{for} \quad \sqrt{\frac{\gamma - 1}{\gamma + 1}} < v < v_1 \\
\psi_0 &= \infty \quad \text{at} \quad v = v_1
\end{align*}
\]
(35)

With these conditions imposed on the flow, a linear
combination of values (32) represents the flow in a channel
with \( v \) rising steadily on the axis from 0 to \( v_1 \). The linear
combination may be written in terms of \( \psi \), since the sine terms
may be expanded in series, thus,
\[
\psi = \theta Q_0 + \theta^3 Q_1 + \theta^5 Q_2 + \theta^7 Q_3 + \ldots
\]
(36)

If (36) diverges, an expression connecting \( \psi \), \( v \), and \( \theta \) is
obtained by inverting (36) obtaining the following series,
\[
\theta = \psi R_0 + \psi^3 R_1 + \psi^5 R_2 + \psi^7 R_3 + \ldots
\]
(37)

The chief advantage of this representation is that (37) con-
verges uniformly in \( v \) in any range \( v \geq v_0 > 0 \) and in partic-
ular near the speed of sound, \( v = \sqrt{\frac{\gamma - 1}{\gamma + 1}} \).

The above discussion has been for flow in a channel, with
no body present. Using the hypergeometric expansion, a very
general solution for subsonic flow around a body which reduces
to the incompressible flow as the Mach number approaches zero,
is possible. The characteristic fault of the analysis in the
\[ I^V = v \Rightarrow \int_0^\infty \frac{\partial \psi}{\partial t} \, dt \]

\[ \int_0^\infty \frac{\partial \psi}{\partial t} \, dt = 0 \]

With these conditions imposed on the flow, it is seen that the flow is continuous with a positive speed on the axis from 0 to \( I^V \). Hence, the terms on the right side of the equation may be written in terms of \( v \), since the sign remains.

\[ \ldots + \partial \psi_0^0 + \partial \psi_0^0 + \partial \psi_0^0 + \partial \psi_0^0 = v \]

\[ \ldots + \partial \psi_0^0 + \partial \psi_0^0 + \partial \psi_0^0 + \partial \psi_0^0 = \psi \]

Using the potential representation of the complex variable solution, the following equation is obtained:

\[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + \frac{\partial}{\partial \psi} = \psi \]

The space available for flow is a bounded domain.

Using the hypergeometric expression, we have the general solution for the axisymmetric flow and in particular,

\[ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial \psi} = \psi \]

The complex potential is the complex function of the variables in the bounded domain.
subsonic region of flow is that the solution in one part of the
flow does not connect with the solution for another region.
The solutions in the hodograph representation are valid every-
where, although there is a singularity at the sonic speed.
The representation of formula (33) is quite complicated
when the velocity of flow is greater than the speed of sound,
that is, when \( v > \sqrt{\frac{y-1}{y+1}} \). When the supersonic flow is being
calculated around a body, series expansions, of the type of the
generalized Laurent expansion, must be used around such symmet-
rical bodies as circles, ellipses, polygons, or the Joukowski
airfoils. The expansions are very difficult to obtain in the
cases of more general airfoils.

It is expected that inherent difficulties with the supersonic representation in the hodograph plane will result in
attempts to express the flow by other means. Mathematical work
on the essentially mathematical problem of obtaining contin-
uous solutions in the different regions of investigation noted
above must yet be perfected. As a step in this direction,
Bergman\(^\text{18}\) notes that it still seems preferable to use the
hodograph method, linking with it the method of operators
which transform solutions of one partial differential equation
into solutions of another one. For, example, it is possible
to find operators which will transform the solutions of an
equation of elliptic type into solutions of an hyperbolic type.

\(^{18}\) Bergman, S. "Two-dimensional subsonic flow of a compressible
Supposing the region of flow is part of the solution, if the solution is not another region.

The solutions in the hypothesis that the solution is another region are very easy.

Where solution exists is a singularity of the same kind.

The intersection of a cone with a sphere.

the solution at a point is a singularity if it is higher than the other.

When the solution of flow is studied by the method of some

When the superscript than to the power.

As a result in this situation,

I hope that it is still as a perspective to use the

Because of the intersection, mixing with the method of operators

which contains solutions of the partial differential equation

If it is possible.

It is impossible to link operators with a connection of the solution of an

emergence of difficult cases into solution or in preparation cases.
However, since the solutions of both equations have quite different character, it is stated that the operators cannot preserve various properties of the functions upon which they act. The same article by Bergman listed above, contains a description of the flow in a logarithmic and a pseudo-logarithmic plane. This representation allows the introduction of imaginary and real components of the variables by a transformation of coordinates from $v$ and $\theta$ to $\bar{Z}$ and $\bar{\lambda}$,

$$\bar{Z} = i \log \bar{v} \quad \bar{v} = v_1 + iv_2 = ve^{-i\theta}$$ (38)

and $v_1, v_2, \theta$, are the components and direction respectively of the flow velocity vector. However desirable the separation of flow into real and imaginary parts may be, it is concluded by Bergman that further work should be done in the hodograph method for better representation of the flow in the different regions. This is to be expected, (b) that the velocity nowhere differ much from an undisturbed velocity, both cripple the attempted solutions of the problem in the case of trans-sonic flow.

It is found (p.84) that the velocity potential in the physical plane may, from the differential equations be, solution of an elliptic, parabolic, or hyperbolic differential equation dependent on the velocity of flow being less than, equal to, or greater than the velocity of sound in air.

Flow pattern analysis points out the insufficiency of the
However, since the existence of both boxplots has little or no correlation, it is interesting that the distribution of the boxplots among the different datasets and species is the same. Although the boxplot of the data in the forest is a lognormal distribution of the boxplot among the different datasets and species, the boxplot is a lognormal distribution of the boxplot among the different datasets and species. However, the boxplot of the data in the forest is a lognormal distribution of the boxplot among the different datasets and species. However, the boxplot of the data in the forest is a lognormal distribution of the boxplot among the different datasets and species. However, the boxplot of the data in the forest is a lognormal distribution of the boxplot among the different datasets and species. However, the boxplot of the data in the forest is a lognormal distribution of the boxplot among the different datasets and species. However, the boxplot of the data in the forest is a lognormal distribution of the boxplot among the different datasets and species.
CONCLUSION

Flow patterns in various representations have been investigated in the case of the trans-sonic flow of a compressible fluid. The fluid, in addition to being compressible, is assumed to obey the law of conservation of matter, be irrotational in flow and obey the adiabatic change of state relation. The discussion further limits the kind of fluid to air, since we speak throughout of trans-sonic speed, implying the region of the speed of sound in air.

The non-linear partial differential equations of flow of the fluid expressed in the Cartesian (physical) plane have been developed starting on page 17.

On page 18 two attempts at linearization of the non-linear partial differential equations are presented. The assumptions made in the attempts, (a) that the velocity be small, (b) that the velocity nowhere differ much from an undisturbed velocity, both cripple the attempted solutions of the problem in the case of trans-sonic flow.

It is found (p. 24) that the velocity potential in the physical plane may, from the differential equations be a solution of an elliptic, parabolic, or hyperbolic differential equation dependent on the velocity of flow being less than, equal to, or greater than the velocity of sound in air.

Flow pattern analysis points out the insufficiency of the
CONCLUSION

Even differences in various representations have been found in the case of the visual-spatial type of a compensable injury. The injury is applicable to both compensation of material or moral injury. The association between the kind of injury to the sense and the area of the brain that has sufficient representation of the sense in the cortex is at issue. The question is whether or not the sense representation area of the cortex is at issue. The question is whether or not the sense representation area of the cortex is at issue. The question is whether or not the sense representation area of the cortex is at issue.
physical plane representation. The attempt to form a flow pattern in the case of a compressible fluid by mathematically altering the flow pattern of a corresponding incompressible fluid was declared unsuccessful (p.26).

On the assumption of axial symmetry, the velocity potential is expanded in terms of the Mach number in order to approach a continuous value for the velocity potential in the trans-sonic region. It is seen that the convergence of the series slows down considerably before the trans-sonic region is neared (p.26).

Perturbation theory is also shown to be inadequate on considering trans-sonic flow of the fluid. The differential equations obtained are still non-linear and the basic assumption that there be no great velocity difference from an undisturbed velocity is not a correct assumption for trans-sonic flow (p.29).

The hodograph transformation, however, as indicated on page 30, yields linear partial differential equations whose solutions are the desired potentials of flow. The stream potential is here evaluated, since it is more easily formulated. Solutions are given in the form of the hypergeometric series. The speed of sound and the speed at infinity are still singularities of the flow. A method is indicated (p.33) by which the velocity potential may be approximated by a series expansion even closer to the speed of sound.
The attempt to solve a flow physical plane representation. The attempt to solve a flow physical plane representation. The attempt to solve a flow physical plane representation.

First we express the flow in terms of the stream function to the approach a continuous value to the original problem in the stream-sound region. To see how the condition of the stream-sound region to

```latex
\text{see eq. (36).}
```

Concerning the properties of the shock, the differential equation is still not linear and the proof requires

```latex
\text{see eq. (36).}
```

Concerning the representation, however, as indicated on

```latex
\text{see eq. (36).}
```

The stream function, however, as indicated on
A brief indication of the coordinates used in a logarithmic representation closes the thesis.

Hovering over the problem still is the fact pointed out on page two that no mathematical solutions to non-linear partial differential equations of the type here discussed have yet been presented.

A model fluid has been created which: (a) is continuous; (b) obeys the law of conservation of matter; (c) is irrotational in flow; (d) obeys the adiabatic change of state relation; (e) may be described by a system of curves, called the velocity and stream potentials of flow. The above statements, not supposed independent of one another, are taken to be the most prominent laws or axioms imposed on the model fluid. The problem at hand is to investigate the behaviour of the model fluid in transonic flow as far as law (e) is concerned. The question to be proposed is how is the model fluid, traveling at transonic speeds, described in terms of the velocity and stream potentials of the flow?

The general term fluid is limited throughout the thesis to air, but the same mathematical discussion follows for any fluid satisfying laws (a) to (e). Also, in the analysis performed, it is assumed that if solutions for both the velocity and stream potential could be obtained, it would be sufficient for the problem to attempt solution for that potential which could be more easily formulated. Thus, in the physical plane, the velocity
A partial explanation of the contradictions seen in a formalism in representation closes the thesis. 

Having made the argument early in the first section of the second chapter, two facts in the first section of new literature have been presented here: that evidence for the perception of the phenomenon has been seen.
ABSTRACT OF THE THESIS

At the outset, it is stated that a model fluid is defined as obeying certain laws or axioms from which deductions are to be drawn. The deductions should agree with experiments performed on the fluid.

A model fluid has been created which: (a) is continuous; (b) obeys the law of conservation of matter; (c) is irrotational in flow; (d) obeys the adiabatic change of state relation; (e) may be described by a system of curves, called the velocity and stream potentials of flow. The above statements, not supposed independent of one another, are taken to be the most prominent laws or axioms imposed on the model fluid. The problem at hand is to investigate the behaviour of the model fluid in trans-sonic flow as far as law (e) is concerned. The question to be proposed is how is the model fluid, traveling at trans-sonic speeds, described in terms of the velocity and stream potentials of the flow?

The general term fluid is limited throughout the thesis to air, but the same mathematical discussion follows for any fluid satisfying laws (a) to (e). Also, in the analysis performed, it is assumed that if solutions for both the velocity and stream potential could be obtained, it would be sufficient for the problem to attempt solution for that potential which could be more easily formulated. Thus, in the physical plane, the velocity
ABSTRACT OF THE TREATISE

At the outset, it is essential that a model (m) is to be defined as the

The abstract contains the following sections:

1. The abstract starts with a definition of the model (m) and its significance.
2. It then explains the purpose and objectives of the study.
3. The methodology used in the research is outlined.
4. The results obtained from the experiments are presented.
5. The conclusions drawn from the analysis are discussed.
6. The implications of the findings are highlighted.
7. Finally, the limitations of the study are acknowledged.

The abstract concludes with a brief summary of the research and its contributions.
potential and in the hodograph plane the stream potentials are investigated.

In the section entitled Experimental Methods, the photographic means of recording flow patterns are given. It is there noted that on passing through the speed of sound, a shock wave is formed, on either side of which different densities of the fluid are observed. This shock wave might be taken as an indication that the fluid at the speed of sound, is undergoing an un-adiabatic change of state and thereby invalidate the law (d) above. Whatever the condition of flow at this speed might be, the mathematical formulation yields singularities which are the subject of the investigation.

The physical plane representation is especially singular at the speed of sound. A set of non-linear partial differential equations is obtained which would yield for the velocity potential solutions of elliptic, parabolic, and hyperbolic character depending on the speed of flow being just below, equal to, or just above the speed of sound.

Attempts to obtain a solution for the velocity potential, which is continuous in the trans-sonic region are developed and criticised. The non-linear differential equation obtained for the velocity potential is linearized by different assumptions which limit the validity of the velocity potential in the trans-sonic region.

The perturbation theory application allows speculation
In the section entitled 'Experimental Work', the phrase 'the photon' was incorrectly presented as 'the photo'.

The correct presentation should read: 'In the section entitled 'Experimental Work', the photon was incorrectly presented as 'the photo'.

This error might be taken as an indication of which different conclusions were to be drawn. However, the correctness of the approach of these conclusions remains to be evaluated.

Indications that the photon is not a thing to be sought or measured in our experimental apparatus or theory may render invalid the assumption of the existence of the photon.

The prevalent phrase 'the photon is a-matter' is not true in all cases and theoretical inferences may be misleading.

A clear statement of the correctness of the experiment is needed.

A correct statement of the experiment is needed. The experiment is not to be taken as completed, nor is the theory to be taken as complete. The experimental evidence is not to be taken as conclusive.

Analysis of the data of the experiment is necessary to determine the significance of the experimental results.

The experimental data must be interpreted with care. The experimental results must be carefully analyzed to determine the significance of the experimental findings.

The present portion of the analysis suggests the possible limitations of the experimental results.

The presentation of the experimental findings should be clear and concise.
on the nature of the supersonic flow pattern, but the treatment fails in the trans-sonic region. The series expansion method of approximation of the velocity potential fails here also.

The hodograph representation of the flow yields linear differential equations which are solved in terms of the hypergeometric series. The study of the convergence, singularities and residues of the series has been made (p.32). Approximations for the stream potential are obtained. The solutions are valid close to the points of singularity of the flow, for, if the approximating series does not converge, the inverse series is taken as the solution.

Thus, in the hodograph plane it is possible to approximate the stream and velocity potentials by using a series expansion based on the hypergeometric series, and these approximations are valid close to the speed of sound.

It is to be noted that in the hodograph plane the singularities at the speed of sound and at infinite speed are still present. The presence of this singularity in the mathematical treatment of the flow indicates that other mathematical methods for solving such mixed problems (sub- to trans- to supersonic) are to be sought.
The essence of the supercritical flow process.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.

The supercritical flow process. The series expansion method.
BIBLIOGRAPHY


Bers, L and Gelbert, A. *Quarterly of Appl. Math.* 1, 168-88 (1943)

Chaplygin, S.A. "Gas Jets" *Sci. Mem. of Univ. of Moscow*, 1902


Lamb, H. et al. *Mechanical properties of fluids.* Blackie and Sons Ltd. (1923)


Newsweek; "German Scientists working for the U.S. Army" Dec. 6, 1946 p. 64


Riemann, B. "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite" Ges. der Wiss. Göttingen, p144 (1876)


