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Consideration of certain vibrational modes of polarized multi-resonant barium titanate transducers

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Boston University
CONSIDERATION OF CERTAIN VIBRATIONAL MODES OF \textsc{Polarized Multi-Resonant} Barium Titanate Transducers

by

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CONSIDERATION OF CERTAIN VIBRATIONAL MODES OF MULTI-RESONANT BARIUM TITANATE TRANSDUCERS

CHAPTER I
INTRODUCTION

1.1 HISTORICAL BACKGROUND

In 1880 the brothers Pierre and Jacques Curie of France, who had been investigating the relation between pyroelectric phenomena and quartz symmetry discovered the piezoelectric property of quartz. The term pyroelectricity is the name given to the phenomenon whereby two opposite electric charges are induced at the extremities of the so-called electric axes of certain crystals when they are subjected to a temperature gradient. The Curie brothers found a new method for the development of polar electricity in these same crystals, which consisted of subjecting them to variations in mechanical pressure along their hemihedral axes. A hemihedral crystal is one with oblique faces having one or more axes whose ends are unlike. Such crystals have the special physical property of giving rise to electric charges of opposite signs at the extremities of these axes when they undergo a change in temperature. The Curies experimented on a large number of crystals, among which were zinc blende, sodium chlorate, boracite, tourmaline, quartz, calamine, topaz, Rochelle salt, sugar, and d-tartaric acid. They came to the conclusion that all pyroelectric crystals were also piezoelectric. When such crystals are compressed in particular directions, they show positive and negative charges on
certain portions of their surfaces, and these charges are directly pro-
portional to the pressure exerted on the crystal and they disappear
when the pressure is withdrawn.

In 1881 Lippmann discussed the application of thermodynamic
principles to reversible processes involving electric quantities.\textsuperscript{2,3}
He treated the special cases of electrostriction, pyroelectricity, and
the Curies' discovery of the piezoelectric effect. He asserted that there
should exist a converse phenomenon\textsuperscript{4} corresponding to each of these
effects, and all of these predictions have been verified. He predicted
that the electric polarization of the ends of the hemihedral axes would
produce contraction or expansion along the axis. The Curie brothers
verified Lippmann's prediction before the end of 1881 of this converse
piezoelectric effect and they also showed that the piezoelectric coeffi-
cient of quartz had the same value for the converse as for the direct
effect. Piezoelectric deformations are directly proportional to the
electric field and reverse their sign upon reversal of the field. This
effect is distinct from electrostriction (which is a deformation produced
by electric stress) whose deformations are proportional to the square
of the applied electric field and, therefore, are independent of the direc-
tion of the field. It may be noted further that electrostriction is a
universal property of dielectrics, whereas the piezoelectric effect is
characteristic of only those crystals which possess the required degree
of asymmetry.
Lord Kelvin had predicted the converse of pyroelectricity (the electrocaloric effect) in 1887 based on thermodynamic grounds. In 1893 Lord Kelvin, extending thermodynamic principles, presented a molecular theory of piezoelectricity, which marked a great advance in the study of crystal physics.  

The electro-optic effect in crystals was investigated by R. Pockels and he made many experimental determinations from 1890-1897. P. Duhem carried out more completely the piezoelectric formulation in this same period. W. Voigt systematized the work of his predecessors more fully and rigorously. In 1910 his monumental "Lehrbuch der Kristallphysik" contained most of what was known about piezoelectricity up to World War I. The first application or utilization of piezoelectric devices was developed in 1917 by Professor Paul Langevin of France, who conceived the idea of exciting quartz plates electrically to serve as emitters, and later as receivers, of high-frequency sound waves under water. His device has been used extensively as a sonic depth finder. A. M. Nicholson, then in the research department of the Western Electric Company, U.S.A., independently initiated work along these same lines, working with quartz and Rochelle salt.  

In 1918 Professor Walter G. Cady, in his studies of underwater signaling, examined certain peculiarities in the electrical behavior of Rochelle salt crystal plates in the neighborhood of mechanical re-
sonance. From this, he developed the piezoelectric resonator and its various uses as stabilizer, oscillator, and filter, for which quartz was soon found to be a most suitable material. By means of observations with piezo-resonators, knowledge has been gained of the nature of vibrations in crystalline media and the dynamic values of the elastic and piezo-electric constants.

Professor George W. Pierce of Harvard University devised a number of circuits and showed that it was possible to do with quartz what Nicolson had been able to do with Rochelle salt. He constructed oscillators with a piezoelectric element and a single electron tube but without the electrical resonant circuits which were used by Cady.

The work of Cady and Pierce, making possible the use of the crystal resonator to determine elastic properties of crystals, stimulated activity and interest in the science of crystal physics and its applications. In 1921 Cady published a paper on longitudinal vibrations in damped isotropic rods. Soon after this several papers on crystal vibrations appeared, dealing with rods, plates, and rings cut from various piezoelectric crystals. The longitudinal vibrations of piezoelectric bars was considered in detail. This type of vibration refers to modes in which the particles of the specimen move to and fro in the direction of the applied field.

It is also possible to produce transverse vibrations in which the particle motion takes place in a direction perpendicular to the applied
field. This was pointed out by Professor J. R. Harrison who made a study of flexural vibrations in 1927. E. Giebe and A. Scheibe, in 1928, studied transverse and torsional vibrations. In the torsional vibration motion takes place about an axis, preferably the electric axis, and the material is subjected to shear strain. Torsional vibrations about the optic axis were first studied by Tawil, and Hund and Wright.

The increasing use of electronics, especially stimulated by World War II, created a need for reliable frequency control and selective filter circuits. The unique ability of piezoelectric crystals (when properly cut and mounted) to satisfy these requirements caused increasing interest and attention to be focused on their use. One of the leading research and development teams in the study of piezoelectricity is the Technical Staff of the Bell Telephone Laboratories. This group introduced several synthetic crystals with interesting piezoelectric properties which made them most suitable for selective filter applications.

During this period the ferroelectric type crystal was investigated also and considerable attention has been directed to the physical characteristics and properties of such crystals during the past few years. A ferroelectric type of crystal is one which exhibits spontaneous polarization in one or more directions of the crystal within one or several definite temperature ranges. To date, three separate types
of ferroelectric crystals have been found: the Rochelle salt type, the potassium dihydrogen phosphate, and the barium titanate type. The Rochelle salt type ferroelectric crystal has ferroelectric and non-ferroelectric regions over the temperature range with upper and lower Curie points marking the separation temperatures. The potassium dihydrogen phosphate crystal has a single Curie temperature at -152°C is ferroelectric below and non-ferroelectric above this point. The barium titanate type ferroelectric crystal is capable of becoming ferroelectric in any one of three directions. It has two transitions between absolute zero and an upper Curie temperature at 120°C. The transitions are due to the crystal becoming ferroelectric in two and three directions, simultaneously.

In ferroelectric crystals a large distortion occurs when the crystal is subjected to an electric displacement. This results in piezoelectric constants much greater than those of non-ferroelectric crystals such as quartz. From 1935 to 1941 H. Mueller published several papers\textsuperscript{21-27} discussing the elastic properties of Rochelle salt.

In 1942 E. Wainer and A. N. Salomon while studying the properties of numerous titania ceramics as dielectrics first noted the anomalous polarization characteristics.\textsuperscript{28} The Laboratory for Insulation Research at the Massachusetts Institute of Technology investigated the dielectric properties of barium titanate and established that it was a new type of ferroelectric material.\textsuperscript{29,30,31} In 1945 the British and
Russians were introduced to the new material, and they likewise became interested in its possibilities. In Russia B. Wul reported on the dielectric constant of barium titanate and announced the discovery as a new class of materials.\textsuperscript{32,33} At the same time a collaborator, V. Ginsburg\textsuperscript{34} expounded on the theory of ferroelectricity in barium titanate, also mentioning its piezoelectric nature. Almost simultaneously the British announced the dielectric constants of the material, but they appeared to concentrate more on the crystallographic analysis of the material.\textsuperscript{35,36}

In 1947, it was discovered that barium titanate in polycrystalline form could be polarized to saturation by applying a direct voltage field. The remanent polarization, after removing the applied field, may be ten to fifteen percent of the saturation value. If now a small alternating electric field is applied so as not to reduce the remanent polarization the barium titanate plate will act as a thickness vibrating piezoelectric crystal.\textsuperscript{37,38,29}

The application of alternating voltage can excite resonances in a polarized polycrystalline barium titanate ceramic. The effects that have been measured are the radial vibrations of a disk of the material\textsuperscript{37}; a thickness vibration in the direction of the applied field\textsuperscript{38}, and a thickness-shear mode.\textsuperscript{39} In his work Roberts\textsuperscript{37} also demonstrated a resonant frequency shift due to dimensioning the sample. In 1956 E. A. G. Shaw discussed resonant vibrations of thick polarized
barium titanate ceramic disks in conjunction with his experimental studies using an optical interference technique employing stroboscopically illuminated multiple Fizeau fringes to display surface motion. 41

1.2 THESIS OBJECTIVE

In this investigation the frequencies and relative amplitudes of normal vibrational modes of polarized barium titanate ceramic plates excited by sinusoidal electrical stimuli are studied. An attempt is then made to correlate some of these experimental findings with an analytic consideration of pertinent extensional-compressional vibrations in such plates. Special attention is directed toward the principal two-dimensional mode of extensional-compressional electromechanic vibrations in thin, essentially flat rectangular, polarized, barium titanate ceramic plates. The large effective piezoelectric coefficient for such plates, the favorable electric characteristics of the admittance-frequency spectra, and the great mechanical stability of this solid makes it desirable for use in selective electrical filter and electromechanical transducer applications. This investigation extends the knowledge of these characteristics.

1.3 BARIUM TITANATE STRUCTURE

Investigators as early as 1945 began to study the structure of barium titanate. It has since been determined 47, 48, 49 to have a face centered body centered cubic structure of the perovskite type or the oxygen octahedra family. The faces of the cube are oxygen atoms with the corners occupied by barium atoms, while the central atom is titanium. This material in the nonpolar phase is cubic above the 120°C Curie point, see Figure 1a. Below this temperature, down to 5°C, the material is tetragonal in structure, see Figure 1b. At 5°C, the symmetry again changes, this time to orthorhombic, which is main-
FIGURE 1
tained down to -80°C. Below -80°C the symmetry is rhombohedral. These structural changes produce corresponding changes in the polar axis. In general the crystal is spontaneously ferroelectric below its Curie point of 120°C. In traversing other temperature transitions the ferroelectric axis is changed as noted previously. The cause has been investigated at length and although an exact answer cannot be given as yet, the main source of ferroelectricity appears due to dipole interactions, although a change in chemical bonding has been postulated. In single crystal barium titanate a strong piezoelectric effect is observed below the 120°C curie point. The polycrystalline ceramic is piezoelectric only after electric polarization.
2.1 PREPARATION OF SAMPLES

In this experimental investigation it was necessary to fabricate thin, essentially flat rectangular plates from raw barium titanate ceramic stock, to apply full-area conducting electrodes to the two major surfaces, and then to subject the finished plate to a polarizing process in order to establish in it a permanent and strong piezoelectric effect.

The stock ceramic was the unmodified type obtained from the Gulton Industries, Metuchen, New Jersey. It was supplied in disk form, approximately four inches in diameter by three-fourths of an inch in thickness. In the preparation of the thin rectangular plates the first step was to cut bar stock from the original disk. This was done by the method used in processing quartz. The ceramic disk was first attached to a sheet of plate glass approximately 6 x 6 x 1/2 inch by thermoplastic cement, flat sides together. The glass carrier was then fastened to the adjustable table of a Felker high speed sawing machine. This machine was equipped with a circular disk copper saw blade with industrial diamonds imbedded in the edge. The blade was mounted on a spindle and belt driven at speeds up to 3600 rpm. The spindle mounting is pivoted and counterbalanced to provide cutting pressures at the blade contact area from zero to several pounds per square inch.
The saw table is capable of linear translational motion in two directions perpendicular to each other via the action of feed screws; 360° rotation about a vertical axis; and up to 20° tilt about the horizontal axis. This table flexibility provides a means for orientation of the stock for cuts at various angles. In cutting the rectangular bar pieces the table with plate and ceramic attached was locked in a horizontal position and two parallel cuts through the entire thickness of the ceramic and across the disk were made with about three-fourths of an inch spacing between cuts. To minimize local heating during the sawing action a coolant oil was sprayed at the cutting point. A satisfactory linear cutting rate was found to be a feed of three feet per hour in sawing the three-fourths of an inch stock, with the counterbalance adjusted to yield a six ounce bearing force.

After completing the two parallel cuts the glass plate and barium titanate ceramic were reheated to loosen the thermoplastic cement, then separated. The rectangular bar of ceramic was then remounted on the glass plate. The mounted specimen was attached to the saw table with the long axis of the bar at right angles with the plane of the saw blade. Wafer slices were then cut at thicknesses about ten percent greater than the expected finished plate thickness.

These rough rectangular plates were finished to the correct thickness by a two step lapping process. First a rough lap was done in a drill press lapping machine with the spindle driving a plastic
carrier rotating between two heavy cast iron lapping plates. The plastic carrier is provided with several openings so that several sample plates may be processed and lapped to the same thickness simultaneously. Medium fine Number 200 Aloxite powdered lapping abrasive, mixed with Number 10 SAE lapping oil was dripped into the input funnels of the top cast iron plate to activate the lapping plates during the rotation of the carrier. Finish lapping was done on a Sipp-Eastwood single plate lapping machine. Number 600 fine Aloxite powder mixed with lapping oil was the abrasive used and smooth, flat, parallel major surfaces resulted. A sequence of tests was carried out to determine the rate at which the thickness of the ceramic plates were reduced per revolution of the lapping carrier. With this knowledge it was possible to lap the ceramic plates to within \( 0.0001 \) inch of the desired thickness. Variation in thickness over a three-fourths of an inch square test plate was always less than this tolerance and a flatness to within a few wavelengths of sodium light was obtained for the major surfaces as tested on an electro-limit gauge. Sample plates were made ranging in thickness from 0.0200 to 0.1500 inch.

The edges of the flat thin wafers were next squared-up on a modified heavy duty Landis surface grinder with a Dumore high speed grinding head attached. The Dumore head was equipped with a fine grit diamond loaded side-faced grinding wheel. Precision edge finishing as well as accurate plate length-width dimensioning could
be accomplished on this machine. Mr. MacLeod (see acknowledgement) had designed and constructed a special jig-holder for the surface grinder to be used in holding the barium titanate plates during the edge finishing process. The jig unit has two precision flat ground reference surfaces which are at exactly right angles to each other; and one is parallel to the ways of the surface grinder; while the other is perpendicular to the ways. These surfaces may in turn be accurately adjusted to be respectively, perpendicular and parallel to the cutting plane of the rotating face of the grinding wheel.

A micrometer positioning screw and dial indicator gauge were included to allow the machine operator to preset the depth of cut or the amount of material to be removed from the sample during each cut. Auxiliary instruments such as a Wilder Shadowgraph, a precision Ames Comparator Dial Gauge Indicator unit, a Pratt-Whitney Electrolimit Gauge, and a Hilger Interferometer were at hand for checking angles, parallelism of edges, and making linear measurements. In the fabrication process it was possible to remove as little as 0.00005 inch depth in an edge grind, while maintaining parallelism and perpendicularity to the expected tolerances. Careful attention was given in all steps of the preparation of the sample plates to assure them to approach the thin rectangular parallelepiped geometric shape.

It was next necessary to provide the finished ceramic plates with metallic electrodes. In order to avoid the effects of any air-gap
between the electrode and the barium titanate surface a thin coating of DuPont "non-firing" silver paint was applied to the full-area of the two major surfaces of each sample plate. When dry these silver electrodes are of nearly negligible mass, eliminate air-gaps, and are excellent electrical conductors.

2.2 POLARIZATION TECHNIQUE

In order to produce a large degree of permanent polarization in a barium titanate ceramic plate and thus cause it to exhibit a large piezoelectric effect it was necessary to subject the sample plates to a polarization process. There are two basic methods for inducing permanent polarization: first the cold method and second the hot method. The cold method is to apply a high voltage polarizing potential to the metallic electrodes of the sample and to leave in this state for several hours. On removal the sample should retain a large degree of remanent polarization.

The second, or hot method, developed by Mr. MacLeod (see acknowledgment) proved to be most applicable, resulting in fewer plate fractures, and was adopted. In this method the sample plate is immersed in a temperature environment so that its ambient temperature is at the upper Curie point of 120°C, the high voltage polarizing potential is applied for about thirty minutes, then the plate rapidly cooled with the electric field still applied. Mr. MacLeod
had designed and constructed a special double walled water jacket calorimeter. An inner metallic cup of about one liter capacity contained special high voltage breakdown transformer oil, an electric stirrer, a thermostatically controlled electric heater, a copper tubing heat exchanger coil, thermometer mount, and a holder to support the barium titanate sample during the polarizing process. This holder was made of linen bakelite and supported two electrodes, one fixed and the other movable and spring loaded. These electrodes were constructed of brass in the form of a piston with faces about one centimeter in diameter and machine finished to smooth parallel bearing surfaces. Each electrode was connected by conducting leads to external high voltage terminals located on the cover of the calorimeter. Insulation to allow for as much as 25,000 volts potential difference to be applied to the terminals without flashover was provided.

A sample plate was mounted between the electrodes of the holder with good contact between them and the silver surfaces of the plate. The holder unit was then submerged into position in the oil of the inner calorimeter cup. The thermostat control regulator was set at 117°C and the heater turned on. During the warm up period, which usually took about thirty minutes for the temperature of the inner cup system to stabilize at the preset temperature, the d-c polarizing potential was applied and very gradually and uniformly increased from
zero up to the required value which depended upon the thickness of
the sample plate. Sudden sizable increase in polarizing potential
often initiated voltage breakdown and this usually resulted in
mechanical fracture of the plate. In order to avoid this, a Variac
transformer was rigged with an electric motor drive through a large
ratio step-down gear train and connected in series with the high
toltage transformer primary. This provided a smooth and slow
control for increasing the polarizing potential. A secondary precau-
tion was to monitor the presence of any polarizing current to the
sample while it was undergoing polarization. This was accomplished
by connecting a 0 - 10 d-c milliampere meter in the high voltage lead.
The object was to try to prevent any surges of polarizing current and
it was found by experimentation that a safe upper limit was about
three or four milliamperes. If the surge current exceeded this value
breakdown usually occurred. However, it was possible to prevent
this many times by shutting off the polarizing potential when the
current surge started; although in many cases the breakdown took
place so rapidly that the shut-down could not be initiated in time to
save the test plate. In general the desirable upper polarizing poten-
tial was sixty volts per 0.001 inch plate thickness. This voltage
gradient is close to the breakdown potential of barium titanate un-
modified ceramic and due care must be exercised in the polarizing
process, especially at the hot oil temperatures used, to minimize
flash-over at the edges of the plate. A flash-over may occur if
impurities such as carbon particles or water are present in the oil.

At the end of the polarizing period and with the electric field
still on, the heat source was shut off and cold water circulated through
the heat exchanger coil. This would drop the temperature of the
inner calorimeter cup and its contents to room temperature in a few
minutes. After this was accomplished the high voltage was shut off and
the sample removed from the unit. This method resulted in the
barium titanate ceramic test plates being polarized to approximately
the saturation point and they would retain a large percentage of the
polarization for long periods of time. Most of the samples showed
little or no appreciable decrease in the degree of polarization for
periods up to two years. The polarized sample now exhibited a
large piezoelectric effect and was ready for use in the experimental
device for investigating the relative amplitudes and resonance
frequencies of its normal vibrational modes.

2.3 INSTRUMENTATION

The electronic system developed by the author to stimulate some
of the various vibrational modes in polarized barium titanate ceramic
plates and to observe and record frequency and relative amplitude of
such modes is as follows. The polarized ceramic plate (with its full-
area silver electrodes on the major surfaces) acts as a one-port
network whose driving point admittance is a function of frequency. This functional admittance-frequency relationship, in turn depends upon the elastic constants of the test sample, the boundary, and the mounting conditions. The one-port test sample is a branch of the general network, a block diagram of which is given in Figure 2.

The basic circuit consists of a pi-network assembly in which the test sample forms the series branch. The shunt branches are two equal-valued fixed non-inductive type resistors. Auxiliary equipment consists of a variable frequency sinusoidal wide-range radio frequency drive oscillator, General Radio Type 1211, followed by a stabilized buffer wide-band amplifier, General Radio Type 1233, forming the exciter unit. A detector and Leeds and Northrup Micro-max pen recording potentiometer form the output unit. A General Radio Type 1100AP primary frequency standard is available to determine the absolute frequency of the exciter oscillator at convenient frequency intervals. The ten and fifty kilocycle per second harmonics of the reference standard make excellent and accurately known marker frequencies for the frequency reference bench-marks. A frequency comparison method was employed; a direct reading frequency counter was also available.

![Figure 2α](image-url)
In operation, the system provides an essentially constant amplitude sinusoidal excitation voltage to the input terminals of the two-port pi-network redrawn in Figure 2b. The frequency of the essentially constant amplitude a-c input signal voltage may be varied smoothly and continuously from a few cycles per second to several megacycles per second. A variable speed double worm gear reduction drive system with synchronous motor drive is employed to control the rate of change in frequency of the excitation oscillator. This allows the desired flexibility of adjustment so that fine structure in the frequency-amplitude response spectrum of the sample may be resolved. The pi-network serves as a frequency-amplitude dependent voltage divider. When a steady state constant amplitude sinusoidal signal is applied to the input terminals of the pi, the magnitude of the output voltage is a function of its transfer impedance. Since two of the three branches are essentially pure resistances, then the third branch (which is the test plate) is the major parameter to cause deviations in
the magnitude of the output signal with frequency variations.

Van Dyke\textsuperscript{44} developed the equivalent electrical circuit of the one-port electromechanical transducer in the neighborhood of one of its simpler vibrational resonances, such as a one-dimensional extensional mode, free from the coupling effect of other modes. This equivalent circuit is illustrated in Figure 3 and consists of a simple RLC resonant series circuit shunted by a capacitance $C_0$ all in series with a third capacitance $C_A$. $C_0$ represents the effective static capacitance formed by the barium titanate as the dielectric between the parallel metallic electrodes. $C_A$ represents the capacitance formed by any air gap existing between electrodes and the dielectric. However, the $C_A$ does not exist in this investigation since the silver paint is applied directly to the barium titanate to form the electrodes. $L_x$ is the effective mechanical inductance due to the mass of the test plate, $R_x$ is the effective resistance due to its internal loss, and $C_x$ is the effective mechanical capacitance due to the stiffness of the test plate. This circuit will have a resonant and antiresonant frequency. $C_0$ is usually very much larger than $C_x$.

When more complex modes are excited, such as a two-dimensional extensional mode with intercoupling between the individual extensional (length and width) modes, the equivalent electrical circuit becomes more complex. To a first approximation, for small strain amplitudes, the principal length-width extensional mode of a thin rectangular plate may be represented by an equivalent electrical
circuit in which the intermechanical coupling has an analogous common variational magnetic flux between the two individual inductors of the series chains as indicated in Figure 4. The frequency-amplitude response characteristic of the mode has the behavior of coupled electrical circuits. It follows that the more complex the mode of operation, the more complex is the equivalent electrical circuit and its analysis.

In order to observe and record the pertinent response modes of a test plate, the a-c output signal from the pi-network is applied to a detector. Here the signal is converted to a unidirectional signal whose amplitude is proportional to the magnitude of the a-c signal applied to the input of the detector. This variational d-c signal from the detector is scaled down by a linear voltage divider and then used to actuate the Micromax self-balancing recording potentiometer. The adjustable voltage divider provides a means for controlling the magnitude of the d-c signal variation applied to the recorder so as to prevent it from deflecting the recorder off scale.

![Figure 3](image3.png)

![Figure 4](image4.png)
The recorder provides a permanent chart record of the amplitude-frequency response spectrum of the sample over a selected frequency range of interest. Typical examples of such response spectra are given in Figure 5. Actually the independent scale of these graphs is on a linear time basis, since the recorder chart is driven by a synchronous motor. However, frequency markers are inserted on the time scale as the variable frequency of the exciter drive oscillator passes through a known harmonic of the primary frequency standard producing an audible zero beat signal on the radio receiver which has been tuned to the channel of the oscillator.

Figure 5a is a photograph showing the response near the principal length-width extensional mode of a test plate approximately one-half inch square. Figure 5b is a photograph of the response spectrum of a plate in the vicinity of a high frequency overtone of the principal mode. The eigenvalues are \( m = 11, n = 9 \), of this particular overtone mode. This will be discussed in a later section of this investigation.

2.4 EXPERIMENTAL CONSIDERATION OF MODE CHARACTERISTICS

The question naturally arises as to what actually is the character of the modes excited. Is there mechanical displacement and vibration associated with a resonance? In order to resolve the question, several tests were conducted. One experimental technique used was
to finish the major outer surfaces of the silver-plated electrodes extremely flat, polish to good luster, and then apply a special powder-pattern technique on the plate while it was being electrically stimulated near a resonance frequency. The results of these tests was most gratifying and photographs of the resulting powder patterns are illustrated in Figure 6. Figure 6a shows the powder pattern of a high overtone mode of the length-width extensional mode of a half-inch size rectangular plate. In this case, the frequency of the excitation source is set to the resonant frequency of the particular mode and very fine powder dusted above the surface of the plate. This surface was oriented in a horizontal position, and the cloud of fine powder slowly settling under the influence of gravity was intercepted by the projected area of the plate normal to the direction of motion of the powder cloud. The resulting quadrille powder pattern shows where the powder accumulates under the dynamic action of the excitation. The positions of minimum disturbances represent nodal points similar to the Chladni plate and sand experiment used in college physics laboratories or lecture halls to demonstrate various modes of vibration in metal plates. For the case at hand, the powder nodes have eigen values $m = 11$ and $n = 9$ for this particular length-width extensional overtone mode. Figure 6b is the photograph of a thin circular disk approximately one-half inch in diameter when excited near a resonant frequency. The powder pattern shown has a concentric circular ring
configuration. At different resonant frequencies powder patterns have been obtained representing the nodal configuration for radial, twister, and flexural overtone modes. It is usually rather difficult to stimulate high overtone modes unless they are in the neighborhood of a fundamental high frequency response such as the fundamental thickness extensional mode, or special techniques are used. The special techniques which have been used to stimulate particular responses are preferential electrode configuration and oblique polarization.

Figure 6c and 6d are photographs of the same rectangular plate under excitation with an oil film used on the major surface instead of fine powder. Here again the quadrille pattern is observed, although a different mode of operation is being excited in this case. These experiments provide ample verification that periodic mechanical displacements or resonance vibrations may be set up in thin flat polarized barium titanate test plates by the electronic system described in this investigation.

The recorder charts yield information as to the position in the frequency spectrum of the various resonances of a test plate, together with the relative amplitude and effective sharpness of such responses. The relative amplitude of a response peak for a test plate depends upon many factors. These include the magnitude of the excitation signal, the degree of polarization of the ceramic sample, the particular mode being excited, the comparative magnitude of the
pi-network parameters and electromechanical parameters of the barium titanate, the type and orientation of the electrodes, and the size and dimension ratios of the plate.
CHAPTER 3
EXTENSIONAL MODES IN BARIUM TITANATE MULTI-RESONANT TRANSUDERS.

3.1 INTRODUCTION

In this chapter the development of a simplified expression for the extensional normal modes of vibration of a homogeneous, free, thin, polarized, rectangular plate of barium titanate ceramic is presented. The modes of particular interest are the two-dimensional vibrations in the plane of the plate with a uniform electric field normal to the plane of the plate. The development starts with the equations of state which are then specialized to apply to transducer systems whose vibrations are purely extensional. The resulting equations are then linearized to approximate the case for small signal analysis; that is to say, where the variations in the system parameters are small compared with the initial values of the parameters. It is assumed that adiabatic conditions exist in this treatment. The electromechanical coupling is determined by the electric displacement and the independent variables are the stress and the electric displacement. Symbols used together with a statement of their meaning are given in Table I.

3.2 EQUATIONS OF STATE

Following the work of Mason\textsuperscript{45}, section 12.3, page 296 the internal energy per unit volume of a body can be expressed in tensor form as

$$
\text{d}U = T_{ij} \text{d}S_{ij} + E_m \frac{\text{d}D_m}{\text{d}m} + \Theta \text{d}S
$$

(3.1)

where $T_{ij}$ is the stress tensor, $S_{ij}$ the strain tensor, $E_m$ the electric field, $D_m$ the electric displacement, $\Theta$ the absolute temperature, and $S$ the entropy. $U$ is the free energy function. (Note: hereafter we
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition in MKS Units</th>
<th>Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>Free energy function</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>Newton/meter$^2$</td>
<td>Stress</td>
</tr>
<tr>
<td>$S$</td>
<td>Meter/meter</td>
<td>Strain</td>
</tr>
<tr>
<td>$E$</td>
<td>Volt/meter</td>
<td>Electric field</td>
</tr>
<tr>
<td>$D$</td>
<td>Coulomb/meter$^2$</td>
<td>Electric displacement</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>$0^\circ$K</td>
<td>Absolute temperature</td>
</tr>
<tr>
<td>$\sigma_f$</td>
<td>Joules/Meter$^3$ $0^\circ$K</td>
<td>Entropy</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Transverse strain</td>
<td>Poisson ratio</td>
</tr>
<tr>
<td></td>
<td>Longitudinal extension</td>
<td></td>
</tr>
<tr>
<td>$u$</td>
<td>Meter</td>
<td>Elastic displacement</td>
</tr>
<tr>
<td>$H^{(1)}$</td>
<td></td>
<td>Elastic enthalpy</td>
</tr>
<tr>
<td>$H^{(2)}$</td>
<td></td>
<td>Electric enthalpy</td>
</tr>
<tr>
<td>$Y^{(E)}$</td>
<td>Newton/meter$^2$</td>
<td>Young's modulus</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(constant electric field)</td>
</tr>
<tr>
<td>$\beta T$</td>
<td>Meter/farad</td>
<td>Dielectric impermeability</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(constant stress)</td>
</tr>
<tr>
<td>$\beta S_{1lz}$</td>
<td>Meter/farad</td>
<td>Dielectric impermeability</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(constant strain)</td>
</tr>
<tr>
<td>$D$</td>
<td>Kilograms/meter$^3$</td>
<td>Density</td>
</tr>
<tr>
<td>$d$</td>
<td>Meter$^2$/newton</td>
<td>Elastic compliance</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(constant electric disp.)</td>
</tr>
<tr>
<td>$\varepsilon^{E}_{11}$</td>
<td>$1/Y^{E}$ Meter$^2$/newton</td>
<td>Elastic compliance</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(constant electric field)</td>
</tr>
<tr>
<td>$\varepsilon^{E}_{12}$</td>
<td>$-\sigma/Y^{E}$ Meter$^2$/newton</td>
<td>Transverse elastic compliance</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(constant electric field)</td>
</tr>
<tr>
<td>$\varepsilon^{E}<em>{12}/d</em>{11}$</td>
<td>Transverse strain Longitudinal strain</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(constant electric field)</td>
</tr>
<tr>
<td>$\varepsilon^{E}_{12z}$</td>
<td>Meter$^2$/coulomb</td>
<td>Effective piezoelectric constant</td>
</tr>
<tr>
<td>$d_{12z}$</td>
<td>Coulomb/newton</td>
<td></td>
</tr>
<tr>
<td>$k^2$</td>
<td>Volt·coulomb/Meter·newton</td>
<td>Coefficient of electromechanical coupling (simple longitudinal vibrator)</td>
</tr>
</tbody>
</table>
will consider the 4π to be included in the symbol $D_m$.

The strain components are defined

$$S_{ij} = \frac{1}{a} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

(3.2)

where $u_i$ are the displacements along the $x_i$-axes. $T_{ij}$, $M/4\pi$, and $\sigma_s$ are the independent variables. Let us now introduce a potential $H(1)$, called the elastic enthalpy, defined by the equation

$$H(0) = U - S_{ij} T_{ij}$$

(3.3)

The total differential is then

$$dH(0) = -S_{ij} dT_{ij} + E_{m} dM + \Theta d\sigma_s$$

(3.4)

Since this is a perfect differential one has under adiabatic conditions ($d\sigma_s = 0$)

$$S_{ij} = -\frac{\partial H(0)}{\partial T_{ij}}; \quad E_{m} = \frac{\partial H(0)}{\partial M}$$

(3.5)

Noting that the strain $S$ and field $E$ may each be expressed as functions of $T$ and $D$ only, they may be expanded in a MacLaurin series about the position of zero strain and zero electric field. We have, up to second-order terms,

$$dS_i = \frac{\partial S_i}{\partial T_{kl}} dT_{kl} + \frac{\partial S_i}{\partial D_{n}} dD_{n} + \frac{1}{2!} \left[ \frac{\partial^2 S_i}{\partial T_{kl} \partial T_{qr}} dT_{kl} dT_{qr} + \frac{\partial^2 S_i}{\partial T_{kl} \partial D_{n}} dT_{kl} dD_{n} \right]$$

(3.6)

To follow a from equation (3.5) that
Furthermore from equation (3.5)
\[ \frac{\partial E_{n}}{\partial D_{n}} = - \frac{\partial^{2} H^{(0)}}{\partial D_{n} \partial T_{ij}} = - \frac{\partial^{2} H^{(0)}}{\partial T_{ij} \partial D_{n}} \]

Since the unpolarized ceramic is composed of a uniform distribution of the crystals in all directions there is no direct piezoelectric effect; therefore,
\[ \frac{\partial E_{n}}{\partial T_{ij}} = 0 = - \frac{\partial S_{ij}}{\partial D_{n}} \] (3.7)

Barium titanate ceramic can be described as soft electrically. That is, it may be polarized readily by the application of an external electric field; however, the polarization vanishes on the removal of the field (analogous to the magnetization of soft iron). On the other hand, the material is mechanically hard and when subjected to mechanical stress the departure from linearity of Hooke's law will be small over a considerable range. Consequently, not much change in the elastic constants with stress will occur and we may set the second derivatives
\[ \frac{\partial^{2} S_{ij}}{\partial T_{kl} \partial T_{r}} = 0 \]

In addition the slight change of elastic constants with electric displacement will be neglected. Then
\[ \frac{\partial^{2} S_{ij}}{\partial T_{kl} \partial D_{n}} = - \frac{\partial^{2} H^{(0)}}{\partial T_{kl} \partial D_{n} \partial T_{r}} = - \frac{\partial^{2} H^{(0)}}{\partial T_{kl} \partial T_{r} \partial D_{n}} = - \frac{\partial^{2} E_{m}}{\partial T_{kl} \partial T_{r} \partial D_{n}} = 0. \] (3.8)

Hence under the above conditions equation (3.6) reduces to
\[ dS_{ij} = \frac{\partial S_{ij}}{\partial T_{kl}} dT_{kl} + \frac{1}{2} \frac{\partial^{2} S_{ij}}{\partial D_{n} \partial D_{p}} dD_{n} dD_{p} \]
\[ dE_{m} = \sum_{n=1}^{3} \frac{\partial E_{m}}{\partial D_{n}} dD_{n} + \sum_{k,l} \frac{\partial^{2} E_{m}}{\partial T_{kl} \partial D_{n}} dT_{kl} dD_{n} + \frac{1}{2} \frac{\partial^{2} E_{m}}{\partial D_{n} \partial D_{p}} dD_{n} dD_{p} \] (3.9)
Considering the three second-order partial derivatives in equation (3.9) we note that two of them are related via equation (3.5); i.e.,

$$\frac{\partial^2 S_{ij}}{\partial D_n \partial D_p} = \frac{\partial^2}{\partial D_n \partial D_p} \left( \frac{\partial H^{(i)}}{\partial u_i} \right) = - \frac{\partial^2 H^{(i)}}{\partial D_n \partial D_p}$$

and

$$\frac{\partial^2 E_p}{\partial T_{kl} \partial D_n} = \frac{\partial^2}{\partial T_{kl} \partial D_n} \left( \frac{\partial H^{(i)}}{\partial D_P} \right) = - \frac{\partial^2 H^{(i)}}{\partial T_{kl} \partial D_n \partial D_P}.$$

Let us designate the second-order partial derivatives as follows:

$$\frac{\partial^2 S_{ij}}{\partial D_n \partial D_p} = - \frac{\partial^2 E_p}{\partial T_{ij} \partial D_n} = \frac{2}{3} \Phi_{ijnp} \quad (3.10)$$

where $\Phi_{ijnp}$ represents the electrostrictive tensor and $O_{mnp}$ the correction to the dielectric constant. The first-order partial derivatives in equation (3.9) we designate as

$$\frac{\partial S_{ij}}{\partial T_{kl}} = \kappa_{ijkl} \quad j \quad \frac{\partial E_m}{\partial D_n} = \zeta_{mn}$$

where $\kappa_{ijkl}$ represents the elastic constants measured at constant electric displacement and $\zeta_{mn}$ represents the dielectric impermeability constants measured at constant stress. Note: see appendix for a more detailed explanation of these terms.

On substituting equations (3.10) and (3.11) back into equation (3.9) we have in tensor form:

$$S_{ij} = \kappa_{ijkl} T_{kl} + \Phi_{ijnp} D_n D_p$$

$$E_m = \left( \zeta_{mn} + O_{mnp} D_p \right) D_n - 2 \Phi_{ijklmn} T_{kl}$$

(3.12)

where $S_{ij}$ are the strain tensor components, $T_{kl}$ the stress tensor.
components, $E_m$ the electric field components, and $D_m$ the electric displacement components. Equations (3.12) are the electrostrictive equations of state in tensor form.

Making use of the following relations (see appendix for explanation)

\[
\Delta_{\xi \eta \kappa \lambda} = \left[(A_{n} - A_{12}) \delta_{\epsilon \kappa} \delta_{\ell \lambda} + A_{12} \delta_{\xi \rho} \delta_{\rho \lambda}\right]
\]

where $A_{n} = \frac{1}{\gamma}$ and $-A_{12} = \frac{\sigma}{\gamma}$

\[
\Phi_{i \rho m} = \left[(Q_{ii} - Q_{i2}) \delta_{mn} \delta_{\rho \rho} + Q_{i2} \delta_{\xi \xi} \delta_{\rho \rho}\right]
\]

\[
\Theta_{mn} = \frac{1}{\gamma_{m}} \epsilon \delta_{mn} = \theta_{ii}
\]

(3.13)

The term $O_{11}$ represents the decrease in dielectric constant with
applied fields, which may be considerable.

Let us now consider the special case where only purely extensional vibrations exist. Thus the shear stress components $T_{12}$, $T_{31}$, $T_{23}$ vanish and equations (3.14) reduce to six.

In the most general case, the electric displacement $D_n$ may be due to either an applied d-c field or remanent polarization plus the effect of an applied a-c voltage. In terms of the various components of $D_n$, namely: $D_1$, $D_2$, and $D_3$, let the terms $D_1^{(0)}$, $D_2^{(0)}$, and $D_3^{(0)}$ represent respectively the electric displacement due to either the applied d-c field or the remanent polarization along the 1, 2, and 3-axis, and $D_1^{(1)}$, $D_2^{(1)}$, and $D_3^{(1)}$ the a-c components. The a-c component may be included by rewriting equations (3.14)

\[
\begin{align*}
S_{11} &= A_{11}^{(0)}T_{11} + A_{12}^{(0)}T_{12} + A_{13}^{(0)}T_{13} + \mathcal{G}_{111}D_1^{(0)} + \mathcal{G}_{122}D_2^{(0)} + \mathcal{G}_{133}D_3^{(0)} \\
S_{22} &= A_{21}^{(0)}T_{11} + A_{22}^{(0)}T_{22} + A_{23}^{(0)}T_{23} + \mathcal{G}_{112}D_1^{(0)} + \mathcal{G}_{222}D_2^{(0)} + \mathcal{G}_{233}D_3^{(0)} \\
S_{33} &= A_{31}^{(0)}T_{11} + A_{32}^{(0)}T_{22} + A_{33}^{(0)}T_{33} + \mathcal{G}_{113}D_1^{(0)} + \mathcal{G}_{223}D_2^{(0)} + \mathcal{G}_{333}D_3^{(0)} \\
E_1 &= -\mathcal{G}_{111}T_{11} - \mathcal{G}_{121}T_{12} - \mathcal{G}_{131}T_{13} + \mathcal{G}_{112}D_1^{(0)} \\
E_2 &= -\mathcal{G}_{122}T_{11} - \mathcal{G}_{132}T_{12} - \mathcal{G}_{132}T_{13} + \mathcal{G}_{113}D_1^{(0)} \\
E_3 &= -\mathcal{G}_{133}T_{11} - \mathcal{G}_{133}T_{12} - \mathcal{G}_{133}T_{13} + \mathcal{G}_{113}D_1^{(0)} \\
\end{align*}
\]

(3.15)

where we have dropped the pure d-c and higher order terms, and

\[
\begin{align*}
\mathcal{G}_{111} &= 2Q_{11}D_1^{(0)} & \mathcal{G}_{121} &= 2Q_{12}D_2^{(0)} & \beta_{111}^T &= \beta_{11}^T + Q_{11}D_1^{(0)} \\
\mathcal{G}_{112} &= 2Q_{12}D_2^{(0)} & \mathcal{G}_{122} &= 2Q_{13}D_2^{(0)} & \beta_{112}^T &= \beta_{12}^T + Q_{11}D_2^{(0)} \\
\mathcal{G}_{113} &= 2Q_{13}D_3^{(0)} & \mathcal{G}_{123} &= 2Q_{12}D_3^{(0)} & \beta_{113}^T &= \beta_{13}^T + Q_{11}D_3^{(0)}
\end{align*}
\]

(3.16)
We assumed that the vibrations in the system parameters would be small compared with the initial values of the parameters; therefore, the second-order electrostrictive equations may be represented as effectively first-order. The resulting equations are thus linearized under the approximation. In this case, to present a first-order simplification, shear and flexural effects have been neglected. If these were included a more comprehensive and exacting relationship would evolve. Nevertheless the above equations give enough relations to solve the case of the barium titanate plate undergoing extensional vibrations in the plane of the plate.

3.3 EXTENSIONAL MODES IN BARIUM TITANATE THIN RECTANGULAR PLATES

Let us now extend the preceding introductory treatment and focus attention on the development of an expression for the normal modes of vibration of a homogeneous free thin rectangular plate of barium titanate. The orientation of the plate with respect to the rectangular coordinate system is illustrated in Figure 7.

![Figure 7](image_url)

Here the principal sides of the plate are parallel to the coordinate axes. \( f(x,y) \) and \( g(x,y) \) denote the vibration displacements.
along the x- and y-axes respectively. A linear uniform electric field is directed along the positive z-axis. The term thin plate implies that the thickness of the plate be so small that the change of stress is negligible in the z-direction, and the stresses required to vanish everywhere on the two major surfaces. Then the above implies that $T_{zz} = 0$ throughout the plate. Furthermore, since the electric field is applied along the z-axis only, $D_x^{(1)} = D_y^{(1)} = 0$. Thus the equations (3.15) reduce to

\[
\begin{align*}
S_{xx} &= \Delta_D^{\varepsilon} F_{xx} + \Delta_{12}^{\varepsilon} F_{yy} + G_{rz} D_z^{\varepsilon} \\
S_{yy} &= \Delta_{12}^{\varepsilon} F_{xx} + \Delta_D^{\varepsilon} F_{yy} + G_{rz} D_z^{\varepsilon} \\
E_z &= -G_{rz} F_{xx} - G_{rz} F_{yy} + \beta_{niz} D_z^{\varepsilon}
\end{align*}
\]

(3.17)

Since the strains and the electric field are more convenient independent variables, we will express the stresses and electric displacement as linear functions of these quantities. Thus

\[
\begin{align*}
\tau_{xx} &= \left( \frac{\Delta_D^{\varepsilon}}{\Delta_{11}^{\varepsilon} + \Delta_{12}^{\varepsilon} T_{xx}} \right) S_{xx} - \left( \frac{\Delta_{12}^{\varepsilon}}{\Delta_{11}^{\varepsilon} + \Delta_{12}^{\varepsilon} T_{xx}} \right) S_{yy} - \frac{G_{rz}}{\beta_{niz} (A_n^e + A_{12}^e)} E_z \\
\tau_{yy} &= -\left( \frac{\Delta_{12}^{\varepsilon}}{\Delta_{11}^{\varepsilon} + \Delta_{12}^{\varepsilon} T_{xx}} \right) S_{xx} + \left( \frac{\Delta_D^{\varepsilon}}{\Delta_{11}^{\varepsilon} + \Delta_{12}^{\varepsilon} T_{xx}} \right) S_{yy} - \frac{G_{rz}}{\beta_{niz} (A_n^e + A_{12}^e)} E_z \\
D_z &= \frac{G_{rz}}{\beta_{niz} (A_n^e + A_{12}^e)} S_{xx} + \frac{G_{rz}}{\beta_{niz} (A_n^e + A_{12}^e)} S_{yy} + \frac{1}{\beta_S} E_z
\end{align*}
\]

(3.18)

where

\[
\begin{align*}
\Delta_D^{\varepsilon} &= \frac{\Delta_D^{\varepsilon}}{1 - \kappa_x^2} + \frac{\Delta_{12}^{\varepsilon}}{1 - \kappa_z^{2}} \\
\Delta_{12}^{\varepsilon} &= \Delta_{12}^{\varepsilon} \left[ 1 + \frac{\Delta_D^{\varepsilon}}{\Delta_{12}^{\varepsilon}} \left( \frac{\kappa_x^2}{1 - \kappa_x^2} \right) \right] \\
\kappa_x^2 &= \frac{G_{rz}^2}{\beta_{niz} (A_n^e + A_{12}^e)} \\
\kappa_z^2 &= \frac{1}{\beta_S} = \left( 1 - \kappa_x^2 \right) \frac{1}{\beta_{niz}} \\
\kappa_\varphi^2 &= \frac{G_{rz}^2}{\beta_{niz} (A_n^e + A_{12}^e)} = \left( \frac{2 A_{12}^e}{A_n^e + A_{12}^e} \right) \kappa_x^2
\end{align*}
\]

(3.19)
In the above equations $\dot{\epsilon}_{11}$ and $\dot{\epsilon}_{12}$ are the isagric (constant field) elastic compliance constants; $\Theta_{112}$ the constant strain impermittivity constant as distinguished from $\Theta_{112}^T$; $k_\ell$ the coefficient of electromechanical coupling for the simple longitudinal vibrator, and $k_p$ the coefficient of electromechanical coupling for the plate. We note that $\frac{d}{d x} \gamma^E$ the Young's modulus, $\frac{d^2}{d x^2} \sigma^E$ the Poisson's ratio, and $\frac{d^2}{d x^2} \sigma^E$ the effective piezoelectric constant, so that the above equations can be simplified to

$$
T_{xx} = \left( \frac{Y^E}{1 - \sigma^E} \right) S_{xx} + \left( \frac{\sigma^E \gamma^E}{1 - \sigma^E} \right) S_{xy} - \left( \frac{d_{12} \gamma^E}{1 - \sigma^E} \right) E_z
$$

$$
T_{yy} = \left( \frac{\sigma^E \gamma^E}{1 - \sigma^E} \right) S_{xx} + \left( \frac{Y^E}{1 - \sigma^E} \right) S_{xy} - \left( \frac{d_{12} \gamma^E}{1 - \sigma^E} \right) E_z
$$

$$
D_z = \left( \frac{d_{32} \gamma^E}{1 - \sigma^E} \right) S_{xx} + \left( \frac{d_{12} \gamma^E}{1 - \sigma^E} \right) S_{xy} + \frac{1}{\Theta_{112}^S} E_z
$$

(3.20)

From the last of equations (3.19), the coefficient of electromechanical coupling now reads

$$
k_p = \left( \frac{d_{12}}{1 - \sigma^E} \right)^2 \frac{d}{d x} \gamma^E
$$

(3.21)

and is slightly greater than the coupling coefficient for the simple longitudinal vibrator. In equations (3.20) the independent variables read

$$
S_{xx} = \frac{\partial \xi}{\partial x}, \quad S_{xy} = \frac{\partial \eta}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}
$$

(3.22)

where $\xi$ and $\eta$ denote the vibration displacements along the $x$- and $y$-axes respectively and $V$ is the potential distribution in the plate.

For inspection at a single frequency $\omega$ the time dependent factor is $e^{j\omega t}$ and will not be written explicitly.

3.4 ENERGY FUNCTIONS

The potential energy density of the vibrating plate, using for the independent variables the strains and electric field (as developed
above), is given by the thermodynamic potential $H^{(2)}$. Where $H^{(2)}$ is the electric enthalpy in differential form

$$\Delta H^{(2)} = T_{xx} \Delta S_{xx} + T_{yy} \Delta S_{yy} - D_z \Delta E_z$$

(3.23)

Substitute equations (3.20) into equation (3.23) and since the expression is a perfect differential the potential energy density is

$$H^{(2)} = \left(\frac{\nu \varepsilon}{1 - \sigma \varepsilon^2}\right) \frac{S_{xx}^2}{2} + \left(\frac{\nu \varepsilon}{1 - \sigma \varepsilon^2}\right) \frac{S_{yy}^2}{2} + \left(-\frac{\rho \sigma \varepsilon \varepsilon^2}{1 - \sigma \varepsilon^2}\right) S_{xx} S_{yy}
- \left(-\frac{\rho \sigma \varepsilon \varepsilon^2}{1 - \sigma \varepsilon^2}\right) S_{xx} E_z - \left(-\frac{\rho \sigma \varepsilon \varepsilon^2}{1 - \sigma \varepsilon^2}\right) S_{yy} E_z - \frac{1}{\rho \sigma \varepsilon^2} \frac{E_z^2}{2}$$

(3.24)

Now integrate the above expression over the volume of the plate to obtain the potential energy of the system

$$W = \int_{x_1}^{x_2} \int_{y_1}^{y_2} H^{(2)} \, dx \, dy$$

(3.25)

on substituting for $H^{(2)}$ the expression (3.24), the total potential energy becomes

$$W = \frac{\rho \varepsilon}{2 (1 - \sigma \varepsilon^2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} S_{xx}^2 \, dx \, dy + \frac{\rho \varepsilon}{2 (1 - \sigma \varepsilon^2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} S_{yy}^2 \, dx \, dy + \frac{\rho \sigma \varepsilon \varepsilon^2}{1 - \sigma \varepsilon^2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} S_{xx} S_{yy} \, dx \, dy
- \frac{\rho \sigma \varepsilon \varepsilon^2}{1 - \sigma \varepsilon^2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} E_z S_{xx} \, dx \, dy - \frac{\rho \sigma \varepsilon \varepsilon^2}{1 - \sigma \varepsilon^2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} E_z S_{yy} \, dx \, dy - \frac{\rho \sigma \varepsilon \varepsilon^2}{1 - \sigma \varepsilon^2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} E_z^2 \, dx \, dy$$

(3.26)

The total kinetic energy of the vibrating plate is given as

$$T' = \frac{\rho}{2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left(\dddot{x}^2 + \dddot{y}^2\right) \, dx \, dy$$

(3.27)

where $\rho$ is the density of the material.

In order to evaluate the energy functions $T$ and $W$ we must specify the independent variables $S_{xx}, S_{yy}$, and $E_z$ or equivalent $F(x, y), F(y, x)$, and $V(z)$. First, let us consider $V(z)$ and make use of $\nabla D = 0$ inside a dielectric. In the case at hand, $D$ has only a $z$-component so that the equation to be satisfied is

$$\frac{\partial D_z}{\partial z} = 0$$

(3.28)
This may be applied to the third equation of equations (3.20).

Remember that $S_{xx}$ and $S_{yy}$ are both independent of $z$.

\[
\frac{\partial D_z}{\partial z} = -\frac{1}{\varepsilon_{nz}} \frac{\partial E_z}{\partial z} = -\frac{1}{\varepsilon_{nz}} \frac{\partial^2 V}{\partial z^2} = 0
\]

or
\[V(z) = c_1 z + c_2\]

The two arbitrary constants $c_1$ and $c_2$ are determined by the two electrical boundary conditions

\[V(0) = V_0\]
\[V(l_z) = 0\]

Thus
\[V(z) = -V_0 \left( \frac{z}{l_z} - 1 \right)\]  \hspace{1cm} (3.22)

from which the electric field follows from the third of equations (3.22)

\[E_z = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z} \left[-V_0 \left( \frac{z}{l_z} - 1 \right)\right] = \frac{V_0}{l_z}\]  \hspace{1cm} (3.30)

and is uniform throughout the plate.

In specifying the vibration displacements of the plate the most general displacement can be considered to arise as a result of a superposition or summation over all the possible characteristic or normal functions of the plate.

3.5 NORMAL MODES OF VIBRATION OF A FREE PLATE

When an elastic solid body is excited into a state of free vibration its motion, if of small amplitude, may be analyzed into a large number of "normal" vibrational modes. To each normal mode corresponds a normal frequency. For any normal frequency the necessary characteristics
are that all particles move in phase with simple harmonic motion and with amplitudes in constant ratios to one another.

To obtain the generalized electromechanical equations it is necessary to express the displacements of the plate in terms of the characteristic functions. These functions satisfy both the dynamical differential equations of motion and the boundary conditions. From dynamical considerations the differential equations of motion to be satisfied are given by

\[ \rho \frac{\partial^2 \delta \varepsilon}{\partial t^2} = \frac{\partial T_{xx}}{\partial x} \]
\[ \rho \frac{\partial^2 \delta \varepsilon}{\partial t^2} = \frac{\partial T_{yy}}{\partial y} \]

The vibration displacements must also satisfy the boundary condition for vanishing stress at each of the four edges of the free plate, i.e.,

\[ T_{xx}(0,y) = T_{xx}(L_x,y) = 0 \]
\[ T_{yy}(x,0) = T_{yy}(x,L_y) = 0 \]  

For the free plate, devoid of electric fields and external forces, the stress distributions from equation (3.20) are

\[ T_{xx} = \left( \frac{\varepsilon - \varepsilon^2}{1 - \varepsilon^2} \right) \frac{\partial \varepsilon}{\partial x} + \left( \frac{\varepsilon^2}{1 - \varepsilon^2} \right) \frac{\partial^2 \varepsilon}{\partial y^2} \]
\[ T_{yy} = \left( \frac{\varepsilon^2}{1 - \varepsilon^2} \right) \frac{\partial \varepsilon}{\partial y} + \left( \frac{\varepsilon - \varepsilon^2}{1 - \varepsilon^2} \right) \frac{\partial^2 \varepsilon}{\partial x^2} \]
Substitution of equations (3.33) into equations (3.31) yields

\[
\rho \frac{\partial^2 \phi}{\partial t^2} = \left( \frac{\gamma^2}{1 - \epsilon^2} \right) \frac{\partial^2 \phi}{\partial x^2} + \left( \frac{\sigma^2 \gamma^2}{1 - \epsilon^2} \right) \frac{\partial^2 \phi}{\partial x \partial y}
\]

\[
\rho \frac{\partial^2 \psi}{\partial t^2} = \left( \frac{\gamma^2}{1 - \epsilon^2} \right) \frac{\partial^2 \psi}{\partial y^2} + \left( \frac{\sigma^2 \gamma^2}{1 - \epsilon^2} \right) \frac{\partial^2 \psi}{\partial x \partial y}
\]

(3.34)

For periodic motion at angular frequency \( \omega \) the equations reduce to

\[
\frac{\partial^2 \phi}{\partial x^2} + \sigma^2 \frac{\partial^2 \phi}{\partial x \partial y} + \beta^2 \phi = 0
\]

\[
\frac{\partial^2 \psi}{\partial y^2} + \sigma^2 \frac{\partial^2 \psi}{\partial x \partial y} + \beta^2 \psi = 0
\]

(3.35)

where

\[
\Omega = \frac{\omega}{C_\infty}, \quad C_\infty = \left( \frac{\gamma^2}{\rho (1 - \epsilon^2)} \right)^{\frac{1}{2}}
\]

The following functions satisfy the boundary conditions, equation (3.32)

\[
\phi_n(x,y) = A_{mn} \sin \frac{m \pi x}{l_x} \cos \frac{n \pi y}{l_y}
\]

\[
\psi_n(x,y) = B_{mn} \cos \frac{m \pi y}{l_y} \sin \frac{n \pi x}{l_x}
\]

(3.36)

Substituting these expressions in the differential equations (3.35) yields

\[
(- \omega^2 + \frac{mn^2 C_\infty^2 \gamma^2}{l_x^2}) A_{mn} + \frac{mn^2 C_\infty^2 \gamma^2}{l_x l_y} B_{mn} = 0
\]

\[
\frac{mn^2 C_\infty^2 \gamma^2}{l_x l_y} A_{mn} + \left(- \omega^2 + \frac{mn^2 C_\infty^2 \gamma^2}{l_y^2}\right) B_{mn} = 0
\]

(3.37)
Setting the determinant of equations (3.37) equal to zero and solving for the roots of the determinantal equation then leads to the frequency equation

\[
\omega_{mn}^2 = \frac{n^2 \nu^2 E}{\alpha_0 (1-\ell^2)} \left\{ \left( \frac{n^2}{l_x^2} - \frac{m^2}{l_y^2} \right)^2 + \frac{4mn^2 \ell^2}{l_x^2 l_y^2} \right\}^{1/2}
\]  

(3.38)

and serves to define the natural frequencies of the system.

The most general displacement of the plate is obtained by summing over all the normal functions (3.36)

\[
\phi(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn} \sin \frac{m \pi y}{l_y} \cos \frac{n \pi x}{l_x}
\]  

(3.39)

\[
\gamma(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{mn} \cos \frac{m \pi y}{l_y} \sin \frac{n \pi x}{l_x}
\]

This represents two Fourier series in which the Fourier coefficients \(\alpha_{mn}\) and \(\beta_{mn}\) are functions of time only and are taken as the generalized coordinates for the plate. From equation (3.22) the strains read

\[
\sigma_{xx} = \frac{\partial \phi}{\partial x} = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n \pi}{l_x} \alpha_{mn} \sin \frac{m \pi y}{l_y} \sin \frac{n \pi x}{l_x}
\]  

(3.40)

\[
\sigma_{yy} = \frac{\partial \phi}{\partial y} = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m \pi}{l_y} \beta_{mn} \cos \frac{m \pi y}{l_y} \cos \frac{n \pi x}{l_x}
\]

Finally, the energy expressions may be obtained by substituting equations (3.30), (3.39), and (3.40) into (3.25), (3.26), and (3.27)
The integrals appearing in equations (3.26) and (3.27) may be evaluated as:

\[
\begin{align*}
\int_{0}^{b} \int_{0}^{a} S_{xx} \, dx \, dy &= \frac{\pi b y}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 a^2 \quad ; \quad \int_{0}^{b} \int_{0}^{a} S_{yy} \, dx \, dy = \frac{\pi b y}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 b^2 \\
\int_{0}^{b} \int_{0}^{a} S_{yx} \, dx \, dy &= \frac{\pi b y}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m n b_{mn} \quad ; \quad \int_{0}^{b} \int_{0}^{a} E_x \, dx \, dy = -\frac{b y}{l_z^2} V_0^2 \\
\int_{0}^{b} \int_{0}^{a} E_y \, dx \, dy &= -\frac{b y}{l_z^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m n}{m} (\cos m\pi - 1) \left( \cos n\pi - 1 \right) \\
\int_{0}^{b} \int_{0}^{a} \phi^2 \, dx \, dy &= \frac{b y}{\beta_l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn}^2 \\
\int_{0}^{b} \int_{0}^{a} \phi^2 \, dx \, dy &= \frac{b y}{\beta_l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{mn}^2 
\end{align*}
\]

Using these expressions the energy equations may be rewritten to yield

\[
I = \frac{\phi b y}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn}^2 + \frac{\phi b y}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{mn}^2 
\quad (3.41)
\]

\[
W = \frac{b x Y_0^2}{2(1 - \xi^2)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^2 a_{mn}^2 + \frac{b x Y_0^2}{2(1 - \xi^2)} \frac{1}{b y} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 b_{mn}^2
\]

\[
+ \frac{b x Y_0^2}{(1 - \xi^2)^2} \frac{1}{\beta_l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m a_{mn} b_{mn} \quad + \frac{b x Y_0^2 V_0}{(1 - \xi^2)^2 \beta_l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \cos m\pi - 1 \right) \left( \cos n\pi - 1 \right) \\
+ \frac{b x Y_0^2}{(1 - \xi^2)^2 \beta_l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{n} \left( \cos m\pi - 1 \right) \left( \cos n\pi - 1 \right) - \frac{\phi b y}{\alpha l_z l_z} V_0^2 
\quad (3.42)
\]
Equation (3.42) indicates the condition under which the various modes of motion may be excited by the electric field. The terms with \((\cos ma \pi - i)(\cos nb \pi - i)\) must not vanish. Since both terms include \(\cos ma\pi\) and \(\cos nb\pi\), if either \(m\) or \(n\) is even the product vanishes. Therefore, the only modes which can be excited with fully plated electrodes are those for which \(m\) and \(n\) take on odd integral values. For example, the \((11), (31), (13), (33), (35), (53), (55), (15), \ldots\) are possible modes of vibration. Special electrode configurations have been used to enhance various mode responses. For example, it is possible to excite the even-order modes if divided electrodes are used.

Consider again the frequency equation (3.38). For each pair of integers \((mn)\) there exist two sets of natural frequencies. Call the set corresponding to the negative sign \(\omega_{mn}'\) and the set corresponding to the positive sign \(\omega_{mn}''\). Then

\[
\begin{align*}
\omega_{mn}'^2 + \omega_{mn}''^2 &= \frac{\rho^2 \gamma^E}{\rho'(1-\sigma^2)} \left( \frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right) \\
\omega_{mn}'^2 - \omega_{mn}''^2 &= \frac{\rho^2 \gamma^E}{\rho'(1-\sigma^2)} \left[ \left( \frac{n^2}{l_x^2} - \frac{m^2}{l_y^2} \right)^2 + \left( \frac{2\sigma^5 mn}{l_x l_y} \right)^2 \right]^{1/2}
\end{align*}
\]

(3.43)

With this set of equations it is possible to discuss several cases of interest. For instance if \(l_x, l_y \to \infty\), then the two sets of natural frequencies approach the values

\[
\begin{align*}
\omega_{mn}' &= 0 \\
\omega_{mn}'' &= \frac{m \sigma}{l_y} \left( \frac{\gamma^E}{\rho'(1-\sigma^2)} \right)^{1/2}
\end{align*}
\]

(3.44)

and the vibrations take place along the \(y\)-axis. Of particular interest are the \((mn)\) modes, \(m = n\). The two sets of frequencies for this case are

(*Since only these terms contain \(d_{12z}\) the effective piezoelectric constant)
\[ \omega^2_{mn} = \frac{n^2\pi^2 Y^E}{2\rho x_y (1 - \sigma^2)} \left\{ \left( \frac{dy}{dx} + \frac{dx}{dy} \right) \right\} \left[ \left( \frac{dy}{dx} - \frac{dx}{dy} \right)^2 + 4 \sigma^2 \right] \] (3.45)

If the plate is perfectly square, that is if \( l_x = l_y \), the equations yield the following two sets of natural frequencies.

\[ \omega^1_{mn} = \frac{n\pi}{l_x} \left[ \frac{Y^E}{\rho (1 - \sigma^2)} \right]^{1/2} \] \[ \omega^2_{mn} = \frac{n\pi}{l_x} \left[ \frac{Y^E}{\rho (1 - \sigma^2)} \right]^{1/2} \] (3.46)

If the plate is perfectly square and the limit \( \sigma \to 0 \) the general expression (3.38) reduces to those from the theory of simple longitudinal vibrations. The two sets of resonant frequencies are then independent and the plate executes independent vibrations along the x- and y-axis. Thus

\[ \omega_{nx} = \frac{n\pi}{l_x} \left[ \frac{Y^E}{\rho} \right]^{1/2} \] \[ \omega_{my} = \frac{m\pi}{l_y} \left[ \frac{Y^E}{\rho} \right]^{1/2} \] (3.47)

If the frequency equation (3.38) is expressed in terms of \( \omega_{nx} \) and \( \omega_{my} \) one obtains the simpler equation

\[ \omega^2_{mn} = \frac{1}{2(1 - \sigma^2)} \left[ (\omega_{nx}^2 + \omega_{my}^2) + \left( \omega_{nx}^2 - \omega_{my}^2 \right) + (2\sigma^2 \omega_{nx} \omega_{my})^2 \right]^{1/2} \] (3.48)
CHAPTER 4

EXPERIMENTAL RESULTS

4.1 SQUARE PLATE: PRINCIPAL MODE FREQUENCY VS. PLATE THICKNESS

In order to investigate the behavior of the frequency of the principal mode of a square plate as a function of plate thickness, a set of flat plates was fabricated from the same piece of raw stock with as nearly identical side dimensions as possible, but with various plate thicknesses. The principal mode \((m=n=1)\) refers to the fundamental length-width two-dimensional extensional-compressional mode of vibration. Thicknesses used were 0.020, 0.030, 0.040, \(\ldots\) 0.150 = 0.0002 inch. The edges were carefully ground so that each plate was square with final edge lengths \(l_x = l_y = 0.5447 \pm 0.0005\) inch. Silver electrodes were formed on the two opposite major surfaces with DuPont silver paint and each plate polarized to saturation with the applied electric field normal to the electrodes.

Experimental runs were taken on each plate to obtain a response spectrum recording near the principal length-width mode. The frequencies of this mode together with pertinent dimensions are listed in Table II and displayed graphically in Figure 8. The frequencies listed here are accurate to \(\pm 2\) kcps. A major point of interest in the results of these tests is that within the limitations of small size tolerance and experimental error all plates have the same fundamental
# TABLE II

## SQUARE PLATES

<table>
<thead>
<tr>
<th>PLATE NUMBER</th>
<th>THICKNESS IN INCHES</th>
<th>LENGTH IN INCHES</th>
<th>WIDTH IN INCHES</th>
<th>PRINCIPAL RESPONSE IN KILOCYCLES/SEC.</th>
</tr>
</thead>
<tbody>
<tr>
<td>GL-10</td>
<td>0.030 ± 0.0002</td>
<td>0.5446 ± 0.0003</td>
<td>0.5449 ± 0.0001</td>
<td>172</td>
</tr>
<tr>
<td>GL-9</td>
<td>0.040 ± 0.0002</td>
<td>0.5451 ± 0.0005</td>
<td>0.5451 ± 0.0005</td>
<td>170</td>
</tr>
<tr>
<td>GL-8</td>
<td>0.050 ± 0.0002</td>
<td>0.5450 ± 0.0001</td>
<td>0.5450 ± 0.0001</td>
<td>173</td>
</tr>
<tr>
<td>GL-7</td>
<td>0.060 ± 0.0002</td>
<td>0.5452 ± 0.0005</td>
<td>0.5452 ± 0.0001</td>
<td>166</td>
</tr>
<tr>
<td>GL-7R</td>
<td>0.060 ± 0.0002</td>
<td>0.5449 ± 0.0003</td>
<td>0.5451 ± 0.0001</td>
<td>169</td>
</tr>
<tr>
<td>GL-6</td>
<td>0.070 ± 0.0002</td>
<td>0.5449 ± 0.0001</td>
<td>0.5451 ± 0.0001</td>
<td>170</td>
</tr>
<tr>
<td>GL-5R</td>
<td>0.080 ± 0.0002</td>
<td>0.5448 ± 0.0001</td>
<td>0.5447 ± 0.0001</td>
<td>168</td>
</tr>
<tr>
<td>GL-4R</td>
<td>0.090 ± 0.0002</td>
<td>0.5447 ± 0.0005</td>
<td>0.5447 ± 0.0005</td>
<td>169</td>
</tr>
<tr>
<td>GL-3</td>
<td>0.100</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>GL-2</td>
<td>0.110 ± 0.0002</td>
<td>0.5453 ± 0.0002</td>
<td>0.5450 ± 0.0005</td>
<td>166</td>
</tr>
<tr>
<td>GL-1R</td>
<td>0.130 ± 0.0002</td>
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<td>0.5451 ± 0.0005</td>
<td>168</td>
</tr>
<tr>
<td>GL-20</td>
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<td>0.5442 ± 0.0002</td>
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</tr>
<tr>
<td>GL-21</td>
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<td>0.5451 ± 0.0005</td>
<td>0.5448 ± 0.0005</td>
<td>171</td>
</tr>
</tbody>
</table>

**Average**: 169

**NOTE**: The frequency readings are accurate to ±2Kc/sec.
or principal response frequency. The thinnest plate \((l_z = 0.030\) inch) satisfies approximately the "thin-plate" classification. However, the thickest plate of the set \((l_z = 0.150\) inch) certainly does not, as its length-thickness ratio is only about 4 to 1. Nevertheless, the results verify that for these plates and associated dimension range, the principal length-width extensional response mode is independent of the plate thickness. This is as it should be, for equation (3.46) which applies to square thin flat plates, does not call for the plate thickness parameter.

4.2 RECTANGULAR PLATE

a. Principal Extensional Mode

The experimental development was extended to include a study of the dependence of the frequency response of the principal mode of a rectangular plate as a function of plate width and plate length. The test runs to obtain the principal length-width response frequency spectrum gave results as expected; namely, the response frequencies were different, except in the cases of identical dimensions. Furthermore, the principal frequencies of various plates with identical thickness appeared to depend on both length and width dimensions simultaneously. It then became the major experimental objective to determine how the principal mode response frequency depends upon the length and width dimensions of the plate.
This task was accomplished by making a set of test plates with identical thicknesses of 0.020 inch, of plate length 0.560 inch and plate width 0.500 inch. In studying the shift in frequency of the principal mode with decreasing plate width, 0.010 inch of material was removed so as to decrease the plate width by this amount for each test run. For rectangular plates in this size range the frequency of the principal mode is less than 200 kcps. A reduction in plate width by 0.010 inch will cause a shift upward in the response frequency by approximately 5 kcps. However, when studying overtone modes of the same plate, which fall in the low megacycle range, it was necessary to grind off but 0.0020 inch from the plate width to give a 5 kcps shift in frequency of the overtone mode. Hence, in order to follow the frequency shift of a particular overtone mode, whose response may be in the vicinity of many other modes, it was necessary to remove as little as 0.002 inch from the width dimension for each experimental test run. This is illustrated in Figure 5b where several high frequency modes fall close to the fundamental thickness extensional mode. Those which are within 100 kcps of it are enhanced in amplitude. This particular spectrum recording is one of a series in which the plate width was reduced in steps of approximately 0.002 inch and the resulting resonance peaks were displaced upward in frequency by 4.8 kcps with each width reduction.

An example of a detailed study used to determine the frequency
response of the principal mode \((m = n = 1)\) as the plate width was reduced is given in Table III. This table gives the calculated and experimental principal response frequencies \(f'_{11}\) and \(f''_{11}\) of a rectangular plate as a function of plate width. The calculated results were determined by equation (3.45) in which the dimensions \(l_x\) and \(l_y\) were measured with a micrometer comparator. The density of the barium titanate used was determined experimentally via Archimedes' principal (with chainomatic balance and distilled water). The value used for the elastic modulus \(Y_E = 1.12 \times 10^{12}\) dynes/cm was obtained from Mason\textsuperscript{45} (page 293), and he points out that the value of Young's modulus is increased slightly with bias, being \(1.18 \times 10^{12}\) at 30,000 volts per centimeter. In searching the literature for a value of the Poisson ratio for barium titanate a range of values \(0 = 0.15\) to \(0.30\) was obtained. In order to establish an optimum value for this parameter several comparison runs were made between experimental and calculated values and a value of \(0 = 0.20\) resulted. The effective results when the upper and lower values for this parameter are used in place of the optimum value is illustrated and discussed below. The experimental response frequencies \(f'_{11}\) and \(f''_{11}\) were read directly from the spectrum response chart taken at each plate width and recorded in Table III.

For convenience the data of Table III are shown in a graph in Figure 9. This is a plot of the theoretical frequencies of the princi-
<table>
<thead>
<tr>
<th>Run Number</th>
<th>( y ) (inches)</th>
<th>( f'_{11} ) Cal</th>
<th>( f'_{11} ) Exp</th>
<th>( f''_{11} ) Cal</th>
<th>( f''_{11} ) Exp</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.467</td>
<td>157</td>
<td>143</td>
<td>192</td>
<td>192</td>
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<tr>
<td>2</td>
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<td>3</td>
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<td>5</td>
<td>0.461</td>
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<td>144</td>
<td>195</td>
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**Note:**
\( l_x = 0.556'',\) \( l_z = 0.020'',\) \( \varphi = 5.24 \text{ gm/cm}^3,\) \( \sigma = 0.20,\) \( \gamma^E = 1.12 \times 10^{12} \text{ dynes/cm}^2\)
Plate P-10
Length = 0.556 inch
Thickness = 0.020 inch

Solid curves are theoretical curves
Calculated from Eqn 288
With \( f = 0.24 \sqrt{d/n} \text{ rpm} \)

Points are experimental values

Principal Length-Width Extension Mode

Frequency vs. Plate Width

Frequency in Kilohertz Per Second

FIGURE 9
pal mode as calculated from equation (3.45) using the dimensions listed in Table III. In this case the plate length and thickness remain constant as the plate width is changed. The calculated results are shown as a smooth solid line curve and the experimental results appear as points. The agreement is within ±4% over the entire range tested.

The theoretical curve shows a transition point or effective cross-over region. In the region to the left of this break point the frequency \( f'_{11} \) levels off approaching a constant value with reduction in plate width; whereas to the right of this point \( f'_{11} \) begins to slope down sharply, goes through a point of flexure and then ultimately approaches zero value asymptotically. The converse effect of frequency \( f''_{11} \) is observed. That is to say, to the right of the break point the frequency \( f''_{11} \) levels off and approaches a constant value; while to the left it slopes upward rapidly increasing without limit as the plate width goes to zero. This behavior was predicted by equation (3.88). The transition point is located at the point at which the plate becomes square. Data taken on another rectangular plate CS-4 agree qualitatively with this in that a greater slope was observed experimentally for \( f'_{11} \) than for \( f''_{11} \) as the greater edge dimension was reduced causing the plate to approach the square configuration.

b. Coupled Modes -- Frequency Splitting
The experimental points show that in practice the response characteristic exhibits a frequency splitting in one of the principal modes over a certain segment of the dimensioning range. The experimental curve for $f''_{11}$ splits into a double valued function because neighboring resonances are excited simultaneously. This occurs when the length-width ratio of the plate enhances the stimulation of even order flexures along with the principal length-width extensional mode of the plate.

A similar effect is characteristic in stimulated rectangular quartz plates and was pointed out by Mason$^{45}$ (page 92). In this work Mason states that the frequency splitting effect in the principal face-shear spectrum of flat rectangular quartz plates as the width is reduced is due to the coupling to even-order flexures. When the plate is dimensioned so that the main shear mode lies half way between two coupled flexure modes a good agreement is obtained between the experimental frequency and the frequency equation which predicts the face-shear mode (derived by Mason$^{46}$) if the effect of other modes is neglected. When the shear mode is coupled to even-order flexures the coupling probably occurs through the stress conditions at the boundary. However, the analytic treatment in Chapter 3 of this investigation does not include the effect of face shear, flexural, torsional, etc, type modes in the development of the normal frequency equation. Hence, the graphical display of equation $^{(3.38)}$
is a continuous curve. The experimental frequency splitting would occur in \( f'_{11} \) to the right of the transition point and in \( f''_{11} \) to the left.

It is of interest to note that the orientation of principal mode frequency \( f''_{11} \) vs. edge size for the nearly square plates with various thicknesses listed in Table II fall almost exactly on the theoretical curve of Figure 9. The agreement for all plates listed falls within \( \pm 2\% \). A second comparison between the theoretical and experimental principal mode frequency for two rectangular plates (one of which is two and one half times the thickness of the other) is given for plate CS-4 with \( l_y = 0.556, l_x = 0.480, l_z = 0.050 \) inch which yields an experimental frequency \( f''_{11} = 182 \) kcps. On checking these figures on the curve of Figure 9 for plate P-10; the theoretical frequency for these dimensions is 186 kcps which gives a difference of approximately 2%.

The \( f''_{11} \) splitting into two responses is illustrated by the photographic reproduction shown in Figure 10. This shows chart photos for successive runs 24 through 29 of plate P-10 (whose data are listed in Table III) for six different plate widths when the length-width ratio is in the critical region to enhance coupling between two neighboring modes; the principal length-width extensional and probably a flexural. The result is a splitting in the \( f''_{11} \) mode. At the beginning of the reduction in plate width the amplitude of the lower
frequency response is greater than the higher. However, the amplitude of the lower response decreases while that of the higher increases and eventually exceeds that of the lower frequency response as the plate width reduction is continued. Both modes increase in frequency with width reduction but one increases at a greater rate than the other. Perhaps a better way of looking at this phenomenon is to think of the principal mode as passing through the flexural response which is coupled in in an increasing then decreasing degree as the plate length-width ratio goes through a critical value. When the width is well beyond the critical value the coupling between the two neighboring modes becomes negligible and $f'_{11}$ then returns to a single valued function.

The effect of response $f'_{11}$ over the same width range is also shown in these six views and appears in the lower right hand corner of each of the chart reproductions. Little or no frequency shift is observed for the six dimension changes and is as expected for at this plate length-width size the region of operation with width reduction is to the left of the transition point and the $f'_{11}$ response should remain essentially constant in this region as shown in Figure 9. In addition it should be a single valued function in this region as the region of frequency splitting with respect to $f'_{11}$ is to the right of the transition point.

It is natural to expect that if more complex response modes
were taken into consideration the theoretical results would yield a more exacting picture when compared with the experimental findings. However, the mathematical expression required to predict the actual behavior of the plate would become extremely complex. The prime objective of this investigation is to report on the development of a simplified expression which would fit, rather well, the experimental observations. This end has been achieved for the principal length-width (two-dimensional) extensional modes of small rectangular polarized barium titanate plates.

4.3 RELATED EXPERIMENTS

a. Overtone Modes of Length-Width Extensional Family

The experimental method for obtaining the frequency spectrum of polarized barium titanate plates yielded results which gave many responses besides the principal length-width extensional modes. These included higher overtone modes whose eigen values \( m \) and \( n \) take on values 1, 2, 3, \( \ldots \) etc. In addition, responses for shear, flexural, and thickness families were recorded. However, little attempt was made to identify the responses except for the length-width extensional modes with low order eigen numbers. In general the agreement between calculated and experimental results for this family was good.

An example, in the case of a flat square plate for \( m = 3, n = 1 \)
was worked out for plate CS-4, Run Number 22. In this test the plate dimensions were \( l_x = 0.479 \), \( l_y = 0.479 \), \( l_z = 0.050 \) inch and for \( m = 3 \), \( n = 1 \) the experimental frequencies were \( f'_{31} = 192 \) kc, \( f''_{31} = 595 \) kc. The calculated values applying equation 3.38 for these dimensions gives \( f'_{31} = 194 \) kc and \( f''_{31} = 580 \) kc. The experimental values were read directly from the spectrum response recorded chart. The maximum discrepancy between the experimental and calculated values is less than 4%.

A second example is for a rectangular plate P-10, Run Number 9 in which \( l_x = 0.556 \), \( l_y = 0.431 \), \( l_z = 0.020 \) inch, \( m = 3 \), \( n = 1 \). The experimental frequencies for these modes read from the spectrum chart were \( f'_{31} = 205 \) kc, \( f''_{31} = 475 \) kc while the calculated values were \( f'_{31} = 213 \) kc, \( f''_{31} = 503 \) kc. The maximum discrepancy between these results is less than 6%. It is usually difficult to excite overtone modes whose eigen numbers are greater than one. However, when such overtone modes fall close to other major modes, such as the fundamental thickness extensional mode, they may be stimulated rather easily. It is also possible to stimulate special responses by preferential plating.

b. Thickness Extensional Mode

A third example is the comparison between experimental and calculated frequency of the fundamental thickness longitudinal-
extensional mode for a flat rectangular plate. Again using plate P-10, Run Number 9 for which the dimensions are the same as in the previous example, the experimental value was $f_{1z} = 4.9 \text{mc}$ and the calculated value using equation 3 gives $5.05 \text{mc}$. The difference between these values is 3%. The length and width dimensions of such plates are large in comparison with the thickness and the overtone modes of the principal length-width extensional mode which fall near the fundamental extensional mode are of high eigen numbers. Furthermore, the response frequencies of such overtone modes are relatively close together. Hence, if the fundamental thickness mode is being stimulated and the coupling reasonable, it is quite possible to stimulate a neighboring high overtone mode of the length-width extensional family. Many examples of this were encountered in test runs and a typical response chart for such a case is shown in Figure 5b. It was possible to obtain powder patterns in such cases and an example is illustrated in Figure 6a. In this case the overtone mode has eigen values $m = 11$, $n = 9$. The vibrational motion pattern is rather complex in such cases as it represents the interaction between two or more classes of vibration. These classes are the overtone length-width extensional and the thickness extensional modes. It is also possible to have higher order flexural modes excited. Hence, it becomes difficult to identify which mode is being excited or if a composite coupled response is being stimulated in the high frequency...
region of the spectrum.

c. **Transverse Polarization**

Several other areas of interest in observing response modes of polarized barium titanate plates were investigated. These included the effect of transverse polarization, preferential electrode configuration, and fabrication and testing modes of flat circular plates. In the case of transverse polarization a cube of barium titanate stock was surface finished and silver electrodes painted on a pair of opposite faces. The polarizing field was applied to these electrodes and after saturation the electrodes were dissolved. The block was then wafered into rectangular plates with the normal to the major parallel surfaces at various angles with respect to the direction of polarization. The edges were finished to give a rectangular parallelepiped shape to the plate. Silver paint was then applied to the major surfaces to form electrodes. The plates were tested in the response spectrometer and exhibited modes associated with transverse vibration which included thickness shear and flexural modes in addition to the extensional thickness and length-width families. It was possible to excite coupled as well as uncoupled modes in these obliquely polarized plates as the excitation source swept over a portion of the frequency spectrum. The relative amplitude of the fundamental thickness extensional resonance was found to depend upon the cosine of the angle between
the plate thickness direction $l_z$ and the direction of polarization. There was no apparent difference in response frequencies of the fundamental thickness extensional modes for plates with different angles of polarization, other parameters being the same.

d. Preferential Electrode Configuration

A second method used to stimulate flexural modes was to apply multiple electrodes. Many possible configurations may be used to achieve special results. It was found possible to stimulate flexural modes of vibration in polarized barium titanate plates and bars using the technique introduced by Harrison in his pioneer work in flexural vibration for quartz plates and bars. The photoetch method for applying precision electrode configurations was tried and found to work satisfactorily on barium titanate.

e. Circular Plates

A preliminary study of the response spectrum of flat circular polarized barium titanate plates was carried out and the usual modes of longitudinal thickness, shear, flexural, radial, twister, etc. were recorded. Figure 6b is the photograph of a circular plate which when under excitation at one of its resonance frequencies was subjected to the powder pattern technique. The resulting center symmetrical or concentric circular powder rings show the location of the modes for
one of the many possible types of vibration for circular plates.
CHAPTER 5

CONCLUSIONS

The principal objective of this work was three-fold. First, to develop a method for stimulating various vibrational response modes in polarized barium titanate ceramic plates. Second, to develop an experimental technique for the continuous and automatic recording of such responses. Third, to develop from physical considerations a simplified expression for the normal modes of the two-dimensional length-width extensional-compressional family for essentially thin flat rectangular plates.

Using the system outlined in Chapter 2 many of the common types of response modes have been excited and measured in polarized barium titanate ceramic plates. Of particular interest, in this study, are the two-dimensional length-width extensional-compressional modes in the plane of the plate. It was found that the frequency of this principal mode is essentially independent of the plate thickness. This holds for considerable variation in the plate thickness. This condition was found to hold for plates in the "thin-plate" classification, that is to say, for a plate whose thickness is less than 5% its width up to plates in which the thickness is as much as 30% the plate width. Plates with thickness greater than this figure were not available for measurement at the time of this study. Further study should be made in this area.

The observation of the node-antinode configuration for higher order modes of the family was achieved by using a special powder-pattern technique. In addition the oil film and electrical admittance
recording techniques were used to investigate these resonances. The characteristic behavior of the electrical driving point admittance of the one-port transducer network as a function of excitation frequency was possible by this system. The pen recordings of the frequency response spectrum for a plate provide a means for locating accurately the response modes. In addition the relative amplitude and selectivity characteristic of a response is also readily available. It was found that the principal and higher order overtone modes for the length-width family are not harmonically related.

In the section on the analytic development, Chapter 3, the second-order electrostrictive equations of state are specialized to apply to transducer systems whose vibrations are purely extensional. The resulting equations are then linearized to approximate the case where vibrations in the system parameters are small compared with the initial values of the parameters. For thin rectangular plates with a linear uniform electric field directed parallel to the thickness of the plate the normal modes are derived as

$$\omega_{mn}^2 = \frac{p^2 Y^5}{2\rho (1 - \sigma^2)} \left\{ \frac{(m^2 \alpha_n^2 + m^2 \beta_n^2)}{\alpha_n^2 + \beta_n^2} + \left[ \frac{(m^2 \alpha_n^2 - m^2 \beta_n^2)^2}{\alpha_n^2 \beta_n^2} + \frac{\mu m^2 Y^5 \sigma^2}{\alpha_n^2 \beta_n^2} \right] \right\}$$

and this serves to define the natural frequencies of the system.

A sequence of tests on rectangular plates with constant width (approximately one-half inch), fixed thickness, and with lengths varying over the range from 0.125 to 0.545 inch was made. The experimental and analytical results for the principal mode \((m = n = 1)\) agree to within 4% in the frequency range in which shear and flexural modes are inhibited. The error for higher
order modes was 10% and increased as the eigen values m and n increase. This is probably due to the perturbation of neighboring shear and flexural modes.

The findings of these investigations indicate promising results. However, it represents but a comparatively small phase of the very large field for further investigations. To date, very little has been published on the normal modes of polarized barium titanate ceramic symmetrical bodies. An extensive study could be made on rectangular plates having a greater range of dimensions. The use of preferential electrode configuration and oblique polarization may be used to enhance additional and special vibrational modes of interest. An extensive study should be made on circular flat plates. The effect of controlled surface contour on the plates should be investigated. The study of polarized barium titanate ceramic transducers for use in selective electrical filters should be extended. Parallel to the experimental investigations an extension of the analytical study should be made. This should include the effect of shear, flexural, and torsional modes; together with their inter-coupling action.
APPENDIX

STRESS-STRAIN RELATIONS IN AN ELASTIC SOLID: $\mathcal{A}_{ijkl}^{D}$, $Q_{ijnp}$

The fourth rank tensors $\mathcal{A}_{ijkl}^{D}$ and $Q_{ijnp}$ may be expressed as 81 term transformation matrices, Mason, page 298. Because of the symmetry of the stress and strain tensors each reduce to 21 independent components. For barium titanate ceramic which is elastically and electrically isotropic; equivalences reduce the 21 terms to only two independent constants. In the case of the elastic stress-strain relations these two constants are simply Young's modulus $Y$, and Poisson's ratio $\sigma$ as is shown in the following discussion, (ref: Page 50, pp. 173-183).

A simple displacement or rotation of an elastic solid to a new position of equilibrium requires no stress. Therefore, the stress components are independent of rotation. Now the fundamental relationship between stress and strain is expressed by Hooke's law. It states that as long as the strain components are small the stress is directly (linearly) related to them. This is true for the most anisotropic crystals.

In the case of polycrystalline barium titanate ceramic we have a more restricted case. That is, this material is assumed homogeneous and isotropic. This means that the stress tensor and the strain tensor always have the same principal axes. Hence, when a tension is applied along a uniform bar, the bar extends longitudinally and contracts laterally. Furthermore, the change of length of equal elements in all lateral directions are equal.

Let us take the x-axis along the bar and assume the stress is
simply $T_{11}$. Then $S_{12} = S_{23} = S_{31} = 0$; since, in an isotropic material there is no shear in the absence of a shearing stress. The non-vanishing components of strain are

$$S_\mu = \frac{1}{\gamma} T_\mu \quad j \quad S_{22} = S_{33} = -\frac{\sigma}{\gamma} T_\mu$$  \hspace{1cm} (A-1)

Since the barium titanate ceramic stress-strain relation is assumed linear we can consider also stresses $T_{22}$ and $T_{33}$. In order to produce a strain in only one dimension, we require three stress components.

$$S_\mu = \frac{1}{\gamma} T_\mu - \frac{\sigma}{\gamma} (T_{22} + T_{33})$$ \hspace{1cm} (A-2)

Equation (A-2) may be written

$$S_\mu = \frac{1}{\gamma} [T_\mu (1+\sigma') - \sigma' (T_{22} + T_{33})] = (\frac{1+\sigma'}{\gamma}) T_\mu - \frac{\sigma}{\gamma} \sum_m T_{mm}$$ \hspace{1cm} (A-3)

which is a tensor component relation between parallel components.

Similar relations hold for the other diagonal components, and since the stress and strain tensors may be diagonalized, this applies to the off diagonal terms as well as the diagonal terms.

Thus

$$S_{ij} = (\frac{1+\sigma'}{\gamma}) T_{ij} - \frac{\sigma}{\gamma} \sum_m T_{mm} \delta_{ij}$$ \hspace{1cm} (A-4)

where $S_{ij}$ is the unit tensor. From the first part of equation (3.12), in which the electric displacement is held constant,

$$S_{ij} = \sum_{k,l} d_{ijkl} T_{kl}$$

$$S_{ij} = (\frac{1+\sigma'}{\gamma}) T_{ij} - \frac{\sigma}{\gamma} \sum_m T_{mm} \delta_{ij}$$ \hspace{1cm} (A-5)

$$S_{ij} = \sum_{k,l} \left[ (\frac{1+\sigma'}{\gamma}) \delta_{ik} \delta_{jl} - \frac{\sigma}{\gamma} \delta_{ij} \delta_{kl} \right]$$
Thus

\[ \theta^D_{ijkl} = \left[ \frac{(1+\varepsilon)}{Y} \right] \delta_{ik} \delta_{jl} - \frac{\sigma}{Y} \delta_{ij} \delta_{kl} \]  

\[ \theta^D_{ijkl} = \left( e_{ii} - e_{ij}^D \right) \delta_{ik} \delta_{jl} + A_{ij}^D \delta_{ij} \delta_{kl} \]  

(A-6)

\( Q_{ijnp} \) represents the electrostrictive effect in the body and it relates the strains existing in the body which are proportional to the square of the electric displacement. Equation (3.1) may be interpreted as follows: twice the value of the electrostrictive tensor \( Q_{ijnp} \) is the change in the inverse dielectric constant or impermeability constant per unit stress. In general, electrostriction in dielectrics applies to any interaction between an electric field and the deformation of a dielectric in the field. The two components to which \( Q_{ijnp} \) reduces are isomorphic with the coefficients above, \( (\theta^D_{11} \text{ and } \theta^D_{12}) \), but are not simply related to physical constants which may be measured directly.

\[ Q_{ijnp} = \left[ (Q_{ii} - Q_{ij}) \delta_{in} \delta_{jp} + Q_{ij} \delta_{ij} \delta_{np} \right] \]  

(A-7)

The dielectric impermeability constants \( \mathbf{G}^T_{mn} \) measured at constant stress for the isotropic media are

\[ \mathbf{G}^T_{mn} = \frac{1}{\varepsilon} \frac{\partial E_m}{\partial D_n} = \frac{1}{\varepsilon} \frac{\partial E_m}{\partial D_n} \frac{\partial D_n}{\partial P} = \mathbf{G}^T_{in} \]  

(A-8)

where \( \varepsilon \) is the scalar dielectric constant.

The dielectric constants may change appreciably with applied electric fields and this effect is recognized by including the term

\[ O_{mnp} = \frac{1}{2} \frac{\partial^2 E_m}{\partial D_n \partial P} = \frac{1}{2} \frac{\partial^2 E_m}{\partial P \partial D_n} = \frac{1}{2} \frac{\partial}{\partial P} \left( \frac{\partial E_m}{\partial D_n} \right) = \frac{1}{2} \frac{\partial}{\partial P} \left( \mathbf{G}^T_{mn} \right) \]
which has for a scalar, three components all of which are equal in the isotropic case. Hence, we will call

\[ C_{mn} = C_{n} \]  

(A-9)

In an isotropic elastic solid the elastic constants, Young's modulus \( Y \), rigidity or shear modulus \( n \), and the bulk or volume elasticity \( K \) are not independent. Any one of these can be expressed in terms of the others. In addition the Poisson ratio \( \sigma \) (transverse contraction: longitudinal extension) is included. In the case of an elastic solid which is piezoelectric or ferroelectric attention must be given to the physical meaning of the elastic constants with respect to ambient conditions. That is to say, the specification as to the physical constraint or constraints under which the elastic constant applies must be given. The relationship between the elastic constants used in this work are given as follows:

\[
Y = \frac{1}{\varepsilon_{E}^D} = -\frac{\sigma}{\varepsilon_{E}^D} \quad \frac{1}{\varepsilon_{E}^D} = -\frac{\sigma}{\varepsilon_{E}^D}
\]

\[
\varepsilon_{E}^D = \varepsilon_{E} (1 - \kappa_e^2) \quad \varepsilon_{E}^D = \varepsilon_{E}^D - \varepsilon_{E}^D \kappa_e^2
\]

\[
Y = Y(1 - \kappa_e^2) \quad \sigma = -\frac{\varepsilon_{E}^D}{\varepsilon_{E}^D} \quad \sigma = -\frac{\varepsilon_{E}^D}{\varepsilon_{E}^D}
\]

\[
\frac{\sigma}{\sigma} = \frac{\varepsilon_{E}^D}{\varepsilon_{E}^D} \kappa_e^2 + (1 - \kappa_e^2)
\]
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CONSIDERATION OF CERTAIN VIBRATIONAL MODES OF POLARIZED
MULTI-RESONANT BARIUM TITANATE TRANSDUCERS

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A report is made on experimental findings for stimulating various vibrational response modes in polarized barium titanate ceramic plates. An experimental technique was developed for the continuous and automatic recording of such responses. An analytic consideration evolved and a simplified expression for the normal modes of the two-dimensional length-width extensional-compressional family for essentially thin flat plates resulted.

The polarized ceramic plate, with its full-area silver plated major surfaces as electrodes, acts as a one-port network whose driving point admittance is a function of frequency. This functional admittance-frequency relationship, in turn, depends upon the elastic and electric constants of the test sample, the boundary, and the mounting conditions. The thin rectangular transducer can be classified conveniently as an essentially double-stress system since the dynamical behavior is completely described by two components of stress. The change of stress along the thin dimension is considered negligible. The vibrational patterns or modes of motion investigated are the two-dimensional modes for in each of the transducers considered Poisson coupling effects between the various modes of vibration studied are exhibited.

The second-order electrostrictive equations of state are specialized to apply to transducer systems whose vibrations are purely extensional. The resulting equations are then linearized to approximate the case where vibrations in the system parameters are small compared with the initial values of the parameters. For thin rectangular plates with a linear uniform electric field directed along the thickness; the normal modes are derived as

\[ \omega_{mn}^2 = \frac{m^2 \gamma E}{l_x (-\sigma^2)} \left\{ \left( \frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} \right) + \left[ \left( \frac{n^2}{l_x^2} - \frac{m^2}{l_y^2} \right)^2 + \frac{2 \mu n^2 \gamma E}{l_x l_y} \right] \right\}^{1/2} \]

and this serves to define the natural frequencies of the system.

Many of the common types of response modes have been excited in rectangular plates approximately one-half inch along a major side. For square plates, it was found that the frequency of the principal mode \((m=n=1)\) was essentially independent of the plate thickness. This held for plate thicknesses ranging from less than 5% to greater than 30% the width of the plate.

The observation of node-antinode configurations for thin plates stimulated at high overtone modes was achieved by using a special powder-pattern technique. The experimental and analytical results for the principal mode \((m=n=1)\) agree to within 4% in the frequency range in which shear and flexural modes are inhibited. The error increases with an increase in eigen values \(m\) and \(n\). This is probably due to the perturbation of neighboring shear and flexural modes.

If the plate is perfectly square, that is if \(l_x = l_y\), the above equation yields the following two sets of natural frequencies.

\[ \omega_{mn} = \frac{m \pi}{l_x} \left( \frac{\gamma E}{\rho (1+\sigma^2)} \right)^{1/2} \quad \text{and} \quad \omega_{mn} = \frac{n \pi}{l_y} \left( \frac{\gamma E}{\rho (1+\sigma^2)} \right)^{1/2} \]

The experimental curve of principal mode frequency vs. plate length for rectangular plates exhibit frequency splitting over certain segments of the
ABSTRACT: CONSIDERATION OF CERTAIN VIBRATIONAL MODES OF POLARIZED
MULTI-RESONANT BARIUM TITANATE TRANSDUCERS.

dimensioning range. This occurs when the length-width ratio of the plate is
favorable to enhance the stimulation of even order flexures along with the
principal length-width extensional-compressional mode in the plane of the
plate.

Related experiments were conducted in several areas of interest. These
included the effect of transverse polarization, preferential electrode con-
figuration, and the testing of vibrational modes for thin flat circular
barium titanate ceramic polarized plates. A preliminary study of the response
spectrum for thin flat circular plates was carried out and the usual modes
of longitudinal thickness, shear, flexural, radial, and twister were recorded.
Carl A. Stevens was born in Malden, Mass. on November 21, 1911. He received the BSEE degree from Tufts University in 1935, the MS in physics from Tufts University in 1937, and the Sc.M. in physics from Brown University, Providence, Rhode Island in 1950. From 1953 to 1960 he continued graduate study, on a part time basis, working for the doctorate in physics at Boston University, Boston, Mass. From 1935 to 1955 he was a member of the teaching and research staff of the Department of Physics at Tufts University. From 1955 to date he has been a member of the teaching and research staff at Lowell Technological Institute, Lowell, Mass. He has been engaged in research and development programs in the fields of frequency control and ionospheric sounding.