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Cohen, Michael A.
Boston University Center for Adaptive Systems and Department of Cognitive and Neural Systems

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Michael A. Cohen

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Boston University Center for Adaptive Systems and
Department of Cognitive and Neural Systems
111 Cummington Street
Boston, MA 02215
THE CONSTRUCTION OF ARBITRARY STABLE DYNAMICS IN NON-LINEAR NEURAL NETWORKS

M. A. Cohen†
Center for Adaptive Systems
and
Department of Computer Science
Boston University
111 Cummington Street
Boston Mass 02215

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Abstract

In this paper, two methods for constructing systems of ordinary differential equations realizing any fixed finite set of equilibria in any fixed finite dimension are introduced; no spurious equilibria are possible for either method. By using the first method, one can construct a system with the fewest number of equilibria, given a fixed set of attractors.

Using a strict Lyapunov function for each of these differential equations, a large class of systems with the same set of equilibria is constructed. A method of fitting these nonlinear systems to trajectories is proposed. In addition, a general method which will produce an arbitrary number of periodic orbits of shapes of arbitrary complexity is also discussed.

A more general second method is given to construct a differential equation which converges to a fixed given finite set of equilibria. This technique is much more general in that it allows this set of equilibria to have any of a large class of indices which are consistent with the Morse Inequalities. It is clear that this class is not universal, because there is a large class of additional vector fields with convergent dynamics which cannot be constructed by the above method.

The easiest way to see this is to enumerate the set of Morse indices which can be obtained by the above method and compare this class with the class of Morse indices of arbitrary differential equations with convergent dynamics. The former set of indices are a proper subclass of the latter, therefore, the above construction cannot be universal. In general, it is a difficult open problem to construct a specific example of a differential equation with a given fixed set of equilibria, permissible Morse indices, and permissible connections between stable and unstable manifolds.

A strict Lyapunov function is given for this second case as well. This strict Lyapunov function as above enables construction of a large class of examples consistent with these more complicated dynamics and indices. The determination of all the basins of attraction in the general case for these systems is also difficult and open.
1. Introduction. In the majority of applications of ordinary differential equations to neural networks of which this author is aware, there is a concern in synthesizing stable dynamical systems, often called neural architectures, which predict or model certain phenomena. The dynamics of the system are often motivated through analogy to well understood physical processes like the shunting equations of resistor–capacitor networks or of neurophysiology. The model is judged a success when the output of the system fits behavioral or neurophysiological data in a coherent and systematic fashion, especially when both neurophysiological and behavioral responses are accurately predicted by the same model.

However, these fits are done using either non-linear, or time varying linear systems of differential equations, usually because the richness of the data do not allow for adequate fits using a linear model. For such systems, there is no guarantee that a shift of parameters or a change in initial data or inputs will produce a model which will fit the modified data in a new parameter range. This is because explicit construction of the exact transduction of the model is virtually impossible, and numerical simulation of one form or another is essential, given this uncertainty.

One minimal constraint for such a model is that it transduces a pattern. Precisely, given any arbitrary fixed input pattern a neural network transduces a pattern if its activations converge to a fixed pattern. Such a convergent Neural Network is called a Content Addressable Memory or CAM. Much effort has been devoted in the recent and early literature to showing that architectural candidates for CAMs were stable (see Grossberg (1982b) for early examples).

More recently, Cohen & Grossberg (1983), announced in Grossberg (1982a p. 322), proved that a large class of neural networks can function as stable content addressable memories or CAMs. these Cohen–Grossberg networks were designed to include additive neural networks, later studied by Hopfield (Hopfield, 1984), and shunting neural networks. The shunting networks included cooperative–competitive networks, also called on–center, off–surround feedback networks. The coefficients of the negative feedback signals between populations were drawn from a symmetric matrix. The question was next raised concerning how much the Cohen–Grossberg form for a content addressable memory could be generalized. Cohen (Cohen, 1988; 1990) showed that if the excitatory on–center was broadened to admit positive feedback from neighboring populations, then persistent oscillations were possible.

Further research has broadened these results in a number of ways. For example, Hirsch (Hirsch, 1982; 1985) has shown that all bounded trajectories in a strictly cooperative network converge with probability one. A strictly cooperative network is a neural network whose Jacobian Matrix has off–diagonal positive elements. Moreover, no strictly cooperative system has a stable periodic orbit. It is convenient to discuss cooperative systems of Cohen–Grossberg form,

\[
\dot{x}_i = a_i(x_i)(b_i(x_i) + \sum_{j=1}^{n} c_{ij}d_j(x_j)),
\]

where \(a_i(x_i)\) is positive except at possibly the origin, \(b_i(x_i)\) is unbounded and of the
opposite sign as $x_i$ for $x_i$ sufficiently large and the derivative of $d_i(x_i), d_i'(x_i) > 0$ is positive and bounded. If the matrix $c_{ij}, i \neq j$ is positive, the system is cooperative. Since the above constraints imply that every trajectory is bounded, it follows that all trajectories converge with probability one. Examples of neural networks which are cooperative include the Wilson–Cowan (Wilson & Cowan, 1972) equations.

If the sign $c_{ij}$ is arbitrary, the methods used by Hirsch are generally inapplicable. However, if $c$ is symmetric, the Lyapunov methods used by Cohen and Grossberg (Cohen & Grossberg, 1983) and later by Michel et al (Michel et al., 1988; 1989) can be applied to show stability. Such systems include most of the convergent neural networks discussed in the literature, for example the shunting neural networks mentioned above, masking fields (Cohen & Grossberg, 1986; 1987), additive networks (Grossberg, 1968; 1969; 1971; 1972), Volterra–Lotka models (Lotka, 1956), the Brain State in a Box (Anderson et al., 1977) and many others, as shown in (Grossberg, 1988).

Many questions concerning the stability of these and similar systems remain. The results of Cohen (Cohen, 1988; 1990) suggest that further progress will require sophisticated mathematical analysis to ensure convergence. Cohen (Cohen, 1988; 1990) showed that some systems of the form

\begin{equation}
\dot{x}_i = -Ax_i + (B - x_i) \sum_k C_{ik} f_k - x_i \sum_k D_{ik} g_k
\end{equation}

where all constants are positive, $f_k$ and $g_k$ are strictly monotone increasing, and $C$ and $D$ are symmetric, can engage in persistent oscillations. These oscillations can occur even if the positive feedback through the the interaction strength matrix $C$ is nearest neighbor only. Stability could only be obtained if complicated numerical constraints were satisfied. Unfortunately, verifying constraints of this sort in neural networks of large dimension can become intractable, limiting the practical usefulness of this sort of result. Pearlmutter (Pearlmutter, 1989) and Barhen et al (Barhen & Gulati, 1989) found related Cohen–Grossberg networks (1) which engaged in persistent oscillations and had chaotic dynamics, respectively.

Recently, there has been much interest in the literature, in synthesizing systems with specified point attractors. For example, Pineda (Pineda, 1987; 1989) has discussed conditions under which the weights $c_{ij}$ of an additive neural network

\begin{equation}
\dot{x}_i = -ax_i + \sum_{j=1}^n c_{ij} d_j(x_j) + I_i
\end{equation}

can be trained to produce a specific point attractor. Li et al (Li et al., 1989) and Michel et al (Michel et al., 1991), given a fixed signal functions $d_j(x_i)$ and equilibria, have derived constraints using linear algebra which are necessary to insure that the given point is an attractor for (3). Pearlmutter (Pearlmutter, 1989) and Sato (Sato, 1990b; 1990a) have introduced algorithms derived from optimal control theory to train a network to approximate given trajectories. General fits to given trajectories have been constructed
by Pearlmutter (Pearlmutter, 1989). Guez et al (Guez et al., 1988) have constructed a scheme to adjust the signal functions of a neural network to extend its capacity. These results are based on an observation in Grossberg (1973) concerning how to construct a CAM such that every pattern can be stored.

However, for these networks there is no guarantee of global stability. For some of these algorithms, i.e. those of Sato (Sato, 1990b) there is no guarantee of convergence of the algorithms. Even when the proscribed patterns are stored as point attractors there is often no guarantee that other point attractors called spurious memories do not occur. In a network with spurious memories, there is no guarantee that the system of differential or difference equations will store an appropriate pattern given disadvantageous input data. In many applications such as the design of oscillators (see below) or the design of analogue circuitry noise in the input could lead therefore to incorrect and potentially disastrous behavior. At least in part for these reasons, the use of analogue CAMs in the design of memories has been quite limited.

This work begins a study of the theory of synthesis of neural networks for which the stored states are known ahead of time. Every shift of parameters within this theory preserves stable equilibria, unstable equilibria, and saddle points. It appears that since the free parameters are functions and there are \( n \) functional degrees of freedom in an \( n \) dimensional system that this theory has the maximal degree of flexibility possible in any system using autonomous differential equations or arbitrary neural networks for modeling. Such a theory is a precondition for the explicit design of decision regions and relative convergence rates for distinct regions within CAMs. Because the construction on which this theory is based is done using polynomial functions, these vector fields can be explicitly evaluated quickly. Furthermore synthesis in VLSI should be possible and could be the basis for sophisticated neural controllers.

Furthermore as a corollary arbitrarily complicated stable periodic orbits and invariant tori can be constructed by these techniques. The Lyapunov techniques used in this work makes possible detailed control of the rates of convergence and the shape of the decision regions for different attracting periodic orbits.

Lyapunov functions are used as the major synthesis tool in this work. Normally, Lyapunov functions for systems are used to guarantee the stability of a system. Here, the procedure is reversed. The local behavior of a system is characterized independently of the Lyapunov function. Then a Lyapunov function is constructed. If a function \( L \) is a strict Lyapunov function for a convergent system then it is shown that any system which has \( L \) as a strict Lyapunov function is also convergent, and has the same equilibria. Furthermore the dimension of the set of points which diverge from corresponding equilibria is the same for both the original and the constructed gradient systems. This dimension is called the Morse Index of an equilibrium point.

The role of these indices have been clarified by work of Smale (Smale, 1960; Smale, 1967) and Morse (Morse, 1925; Morse, 1934). They have shown that any two systems which have the same set of Morse indices and same set of equilibrium have the same qualitative dynamics in any sufficiently small region. However, globally the qualitative dynamics of two such systems may be distinct. Since it is relatively easy to write
down most systems which have a specific function as a strict Lyapunov function, it is relatively easy to synthesize a large class of systems which have the same local dynamics. The theory presented in this paper in this sense is the start of a Universal Theory of Synthesis using Lyapunov functions.

This theory of synthesis, even when complete, is not meant to supplant standard methods of modeling, or the analysis of particular well motivated models. However, it could be used as a method to collapse large sets of data and reveal underlying unsuspected regularities contained therein.

The bulk of the argument is an application of techniques discussed by Hirsch (1989) in his discussion of feedforward systems. The key idea of the proof is to build up stable equilibria inductively in each higher dimension using interpolation polynomials. For each of the two constructions, a strict Lyapunov function is given. Unfortunately, however, there is an infinite class of convergent vector fields whose Morse indices differ from the ones which can be constructed using the methods discussed in this paper. It appears feedback may be necessary to obtain examples with Morse indices which are not constructible using the techniques discussed here. Lyapunov functions have not been constructed for these cases.

The methods of proof include classical techniques of commutative algebra: the Hilbert Nullstellensatz is used here, to show that the second construction has a Lyapunov function of a given form. Using these classical techniques we generalize Lagrange interpolation to interpolate arbitrary polynomial functions which vanish on a fixed set of zeros in $\mathbb{R}^n$. This construction is key to showing the Nullstellensatz can be applied. It may be possible to construct differential equations which converge to one of a countable set of isolated zeros using the above methods and real analytic functions. However, it appears that technical difficulties such as dealing with essential singularities at infinity and the appropriate replacement for the Nullstellensatz make such an extension very difficult.

To do the synthesis described above two distinct but related constructive techniques are used to produce networks with arbitrary numbers of attractors. Both are conceptually very simple.

Consider first the construction which produces an arbitrary set of stable memories with arbitrary coordinates. Generically it is the case (see the proof of Lemma 1) that all sets of points have one coordinate value which is distinct for each of the points, which we assume for definiteness is the first coordinate. If not one can always find a linear transformation of coordinates such that this is the case. Now choose Lagrange Interpolation $p_i(x_1)$ polynomials which interpolate the remaining coordinate values as a function of the first coordinate. We let the remainder of the coordinates exponentially decay to this interpolated value. The construction is schematized in figure 1 below. This construction is minimal in the sense that the fewest possible equilibria exist for the given specified set of attractors.
CONSTRUCTION ONE
AN ARBITRARY SET OF ATTRACTIONS

Fig. 1. This is a pictorial representation of the primary construction. The bottom line represents the initial one dimensional system which converges to a given set of distinct values. The remaining lines are values interpolated by constructed Lagrange Interpolation Polynomials.
The second construction is a generalized variant of the first. Using generalized Lagrange Interpolation Polynomials, one inductively constructs a feedforward subsystem of differential equations which vanish on the projection of the set of given equilibria on that subsystem. This is done for each dimension. The details of the construction of these interpolation polynomials (see section 3) ensure that a Lyapunov function exists for these differential equations. One can construct many additional types of qualitative dynamics by using this method than by using the method above. The second construction is schematized in the following figure.
### CONSTRUCTION TWO

**GENERALIZED STABLE ATTRACTOR**

Converges To | Equation
--- | ---
$Z$ | $\dot{x}$
$\Pi_1(Z)$ | $\dot{y}$
$\Pi_2(Z)$ | $\dot{z}$
$\Pi_3(Z)$ | $\dot{w}$

**Fig. 2.** Using a generalization of Lagrange Interpolation, a nested set of feedforward differential equations are constructed. The first $i$ of these equations converge to the projection of a given finite set of zeros $Z$, onto their first $i$ coordinates. The figure shows a schematic representation of such a four dimensional system. Such a system is an example of a system of equations precisely defined in (43) below.

2.1. Notation and Definitions. Let \( X, Y, Z \) be points in \( \mathbb{R}^n \), for some \( n \). \( \alpha_1 \ldots \alpha_n \), be arbitrary points in \( \mathbb{R}^n \) which are all distinct, and which will represent the stable equilibrium points of our equation. For any point \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \), let \( \pi_i(\beta) = (\beta_1, \ldots, \beta_i) \) be the projection on the first \( i \) coordinates.

We let \( F, G, H \) denote arbitrary vector fields in \( \mathbb{R}^n \). \( U, V, W \), denote Lyapunov functions, that is functions whose derivative along trajectories are non-positive. A Lyapunov function is said to be strict if its derivative along trajectories vanishes only at the equilibria. An integral of a vector field is a function whose derivative along trajectories defined by the vector field is zero. We let \( \| X \| \) denote the \( L_2 \) norm of \( X \). For any point \( \beta = (\beta_1, \ldots, \beta_n) \), we let \( x_i(\beta) = \beta_i \) be the \( i \)th coordinate function in \( \mathbb{R}^n \). Let \( \nu, \lambda, \mu \) be vectors in \( \mathbb{R}^n \) which are used to represent hyperplanes through the origin, and let \( \langle X, Y \rangle \) denote the standard dot product in \( \mathbb{R}^n \). \( \nabla F \) denotes the gradient, and the Jacobian of \( F \), is \( D(F) = \partial F_i/\partial x_j, \ i, j = 1 \ldots n \). The Hessian of \( W, H(W) = \partial^2 W/\partial x_i \partial x_j, \ i, j = 1 \ldots n \). A Riemannian Metric is a smooth choice of positive definite matrices.

A critical point of \( g : \mathbb{R}^n \rightarrow \mathbb{R}^1 \), is a point where the gradient of \( g \) vanishes. \( g \) is a Morse function if the Hessian of \( g \) is non-singular at each of its critical points. A differential equation is said to be gradient-like with respect to \( g \) if \( g \) is a strict Lyapunov function for the differential equation.

The index of a vector field \( F \) at an equilibrium point is the dimension of the subspace spanned by the eigenvectors of \( D(F) \) whose eigenvalues have positive real part. The index of a function \( W \), at a critical point, is the dimension of the subspace spanned by the eigenvectors with positive real eigenvalues for the Hessian of \( W \). The index of a real matrix \( A \) is the dimension of the largest subspace spanned by the generalized eigenvectors whose eigenvalues have positive real part. A vector field \( F \) is said to be hyperbolic at an equilibrium point \( E \) if \( DF|_E \) has no eigenvalues with zero real part. A matrix is sometimes also called hyperbolic if it has no eigenvectors whose eigenvalues have zero real part. A system of differential equations is said to have the indices \( (i_0, i_1, \ldots, i_n) \) if the equation has \( i_0 \) equilibrium points of index 0, \( i_1 \) equilibrium points of index 1, \ldots \( i_n \) equilibrium points of index \( n \).

To avoid technicalities, we assume that our vector fields are always \( C^\infty \) although much less smoothness is necessary to derive the majority of the results below. However occasionally, (Theorem X) the full generality of the Picard–Lindelöf existence theorem for ordinary differential equations is necessary.

We define a function \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) to be locally Lipschitzian if for every bounded open set \( U \subset \mathbb{R}^n \) there is a positive constant \( K_U \) such that \( \| F(X) - F(Y) \| \leq K_U \| X - Y \| \). The Picard–Lindelöf theorem states that if \( F \) is locally lipshtizzean, the differential equation \( \dot{X} = F(X) \) has a unique solution, which is locally lipshitzean but not necessarily differentiable, as a function of the initial data.

It is stated that a quantity \( f(x) \) is \( o(x) \) if \( \lim_{x \to 0} f(x)/x = 0 \).
It is necessary to state restrictions on a function of two variables $H$, which will guarantee that limit sets contained in the level sets of $H$ are periodic orbits for our discussion. Accordingly, a smooth function of two variables is weakly coercive if and only if

i. $H(0,0) = 0$, $H > 0$ except at 0.

ii. The only critical point of $H$ is the point $(0,0)$.

iii. $\lim_{\|X\| \to +\infty} H(X) = \infty$.

A smooth function of two variables is strongly coercive if $x\partial H/\partial x + y\partial H/\partial y > 0$, except at $(0,0)$. It can be shown that $H$ is strongly coercive if $H$ is convex, i.e. the Hessian of $H$ is everywhere positive definite.

Examples of such functions are $H(x,y) = a^2x^2 + cxy + b^2y^2$, when $a^2b^2 - c^2/4 > 0$, $H(x,y) = e^{(ax)^2} + e^{(by)^2} - 2$, $a, b \neq 0$ or even $H(x,y) = x^2(1 + 1/2 \sin y) + y^6$.

For any trajectory $\gamma$ of a differential equation the $\omega$ limit set $\omega(\gamma)$ is the set of limit points of a forward trajectory. If we denote $x.t$, the value at time $t$ of the trajectory starting at $x$, $w(\gamma, t) = n(x, t, \infty)$, where $x$ is some point on $\gamma$, and $\text{cl}(S)$ denotes the closure under limits of the set $S$. If $\gamma$ is bounded, $\omega(\gamma)$ is a closed, bounded set which is invariant. That is, it contains all the trajectories of all the points in the set.

Also some concepts from commutative algebra are used in this paper. We restrict our attention to the set of real valued polynomials in $n$ real variables. A set of polynomial functions closed under addition, subtraction, multiplication, and scalar multiplication is called a ring of polynomial functions. A ring of polynomial functions $I$ is an ideal, if for any polynomials $f, g$, if $f \in I$ then $fg \in I$.

2.2. A constructed Example, which has a fixed Arbitrary Set of Points as Stable Equilibria.

Lemma 1. Let $\alpha_1 \ldots \alpha_m$ be an arbitrary set of $m$ distinct points in $R^n$. Then:

i. There are at most a measure zero set of vectors $\lambda$ in $R^n$ (in fact a finite union of hyperplanes) such that the values $<\lambda, \alpha_i>$, $i = 1, \ldots m$ are not all distinct.

ii. There is an orthogonal change of coordinates (in fact all but a measure zero set of orthogonal changes of coordinates) such that $x_1(\alpha_i), i = 1, \ldots n$, are all distinct.

Proof: (with help from John Merrill)

i. Form the difference vectors, $\beta_{ij} = \alpha_i - \alpha_j$, $i, j = 1 \ldots m$ which are non-zero. $<\lambda, \alpha_i> = <\lambda, \alpha_j>$ for some $i, j$ if and only if $<\lambda, \beta_{ij}> = 0$. In this case, $\lambda$ must lie in one of the $m(m-1)/2$ hyperplanes orthogonal to one of the $\beta_{ij}$. It can be shown that the complement of this set is non-empty in $R^n$ by a simple diagonal argument, and, in fact, the same argument shows that this set of vectors has measure zero.

ii. The set of orthogonal linear transformations can be represented as a set of orthonormal frames in $R^n$ (see Sternberg (1964) for details). The subspace of first elements in this frame which agree on two $\beta_{ij}$ consists of a finite union of $n-2$ dimensional spheres. The result follows from the fact that this subspace has measure zero in the set of permissible first components (a $n - 1$ dimensional sphere).

The following result proved by (Hirsch, 1989) is also used throughout:
THEOREM II. Let
\begin{align*}
\dot{X} &= F(X) \\
\dot{Y} &= G(X,Y)
\end{align*}

Be a feedforward Network (called a cascade by (Hirsch, 1989)) and suppose the dynamics of $F$ are convergent. Suppose for each fixed equilibrium $p$ of $F$ there is a Lyapunov function for the system $\dot{y} = G(p,y)$ and that $G(p,y)$ has only a finite number of equilibria. Then the system (4) is convergent.

The following standard result (LaSalle, 1976; Hale, 1980) is also used.

THEOREM III (LASALLE INVARIANCE PRINCIPLE). Let the system $\dot{X} = F(X)$ have a Lyapunov function $U$ and let $\gamma$ be a bounded trajectory of this system. Then the $\omega$ limit set of $\gamma$, $\omega(\gamma) \subset E$, where $E$ is the largest invariant set contained in the set where $U = 0$.

Theorem II combined with the next produces a minimal system with a fixed set of attractors.

THEOREM IV. Let $\alpha_1, \ldots, \alpha_m$, be a given set of points in $\mathbb{R}^n$. and suppose that $x_1(\alpha_i) \neq x_1(\alpha_j)$ for all $i, j$. Let $\alpha_1 = x_1(\alpha_1), \ldots, \alpha_m = x_1(\alpha_m)$. Choose an arbitrary set of numbers $b_i$, $i = 1, \ldots, m - 1$ such that $a_i < b_i < a_{i+1}$. For each $i > 1$, let $p_i(x)$ be the Lagrange Interpolation Polynomial assuming the values $x_i(\alpha_j)$ at the point $\alpha_j$ and let $p_i(x) = \prod_{j=1}^{i-1} (x - a_j) \prod_{j=i+1}^{m} (x - b_j)$ and let $\beta_i = (b_i, p_1(b_i), \ldots, p_n(b_i))^T$, for each $i$, $i = 1, \ldots, m - 1$.

\begin{align*}
\dot{x}_1 &= -p_1(x_1) \\
\dot{x}_i &= -(x_i - p_i(x_1)) \quad \text{for } i = 2, \ldots, n
\end{align*}

converge to one of the equilibrium points $\alpha_i, \beta_i$, $i = 1, \ldots, m$. Each of the $\alpha_i$ are attractors and the $\beta_i$ are saddles.

Proof

The first equation of the set (5) is a one dimensional differential equation with stable limit points $a_i$, $i = 1, \ldots, m$, and unstable limit points $b_i$, $i = 1, \ldots, m - 1$. All trajectories must converge to these points; the equilibrium points $b_i, i = 1, \ldots, m - 1$ attract themselves only.

A direct way to see this is to note that

\begin{align*}
V(x) &= \int p_1(x) dx \\
\dot{V}(x) &= -p_1(x)^2.
\end{align*}

Since this function is defined up to a constant and is of even order with positive top order term, we can assume it is positive and unbounded as $x \to \infty$. It follows that all trajectories are bounded, and converge to the set of equilibria. Because the limit
set of any trajectory is connected and each equilibrium point is isolated, it follows that each trajectory converges to an equilibrium. Calculating the derivative \( p'_1(a_i), p'_1(b_i) \) immediately shows that the \( a_i \) are stable and the \( b_i \) are unstable.

For each fixed \( i, j \) the functions

\[
V_{ij}(x_i) = \frac{1}{2}(x_i - p_i(a_j))^2 \\
V'_{ij}(x_i) = \frac{1}{2}(x_i - p_i(b_j))^2
\]

are Lyapunov whose derivatives are respectively,

\[
\dot{V}_{ij}(x_i) = -(x_i - p_i(a_j))^2 \\
\dot{V}'_{ij}(x_i) = -(x_i - p_i(b_j))^2
\]

when \( x_1 = a_i, b_i \) respectively, for the remainder of the equations of system (5). The result now follows immediately from Theorem II.

**Corollary V.** Let \( \alpha_i, i = 1 \ldots m \) be distinct. Then there is a linear transformation \( A \) such that each trajectory of the system

\[
\dot{X} = AF(A^{-1}x)
\]

converges to an equilibrium point \( \alpha_i \) or \( \beta_j, j = 1 \ldots m - 1 \). The \( \alpha_i, i = 1 \ldots m \) are attractors and the \( \beta_j, i = 1 \ldots m - 1 \) are saddles and \( F \) takes on the form (5).

**Proof**

Apply Lemma I to show there is a isometric change of coordinates \( X' = AX \) such that after the change of coordinates, \( x_1(\alpha_i), i = 1 \ldots m \) are distinct. Apply Theorem IV in this coordinate system and transform back to the original coordinates to complete the proof.

As an example of such a minimal system, suppose we want to construct a two dimensional convergent system of differential equations where the set \( A = \{(0, 0), (1/2, 1/2), (1, 1)\} \) are attracting equilibria, the set \( S = \{(0, 1), (1, 0)\} \) are saddles and there are no other equilibria. The system of differential equations satisfying these criteria is given by:

\[
\frac{dx}{dt} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix}
\]

where

\[
f(u, v) = -u(u - 1/2)(u - \sqrt{3}/4 - 1/4)(u - \sqrt{3}/2)(u - \sqrt{3}/2 - 1/2) \\
g(u, v) = -v + p(u)
\]

and \( p(u) \) is the unique polynomial of lowest degree such that

\[
p(0) = 0, \quad p(\sqrt{3}/2) = -1/2, \\
p(\sqrt{3}/4 + 1/4) = \sqrt{3}/4 - 1/4, \quad p(1/2) = \sqrt{3}/2, \\
p(1/2 + \sqrt{3}/2) = -1/2 + \sqrt{3}/2.
\]

The matrix \( A \) rotates the coordinates by 30 degrees. After the rotation the first coordinate is distinct for each of the points. In these coordinates the methods of Theorem IV can be directly applied. A representation of the phase portrait of this example system is contained in figure 3 below.
FIG. 3. This figure is a schematic of the flow of (11) above. The basins of attraction of (0, 0) and (1, 1) are crosshatched in the vertical direction; the basin of attraction of (1/2, 1/2) is crosshatched in the horizontal direction. The abscissa is distinct for each of the marked equilibria for the rotated axes.
The argument of Lemma I is not constructive in that it does not generate an appropriate change of coordinates should there not be a coordinate $x_i(\alpha)$ which is distinct for each equilibrium point. The fact that all coordinates have this property with probability one suggests one can simply use a random number generator and with probability one any normalized initial vector will have this property. If not, a small perturbation of the initial vector will again with probability one produce a vector $\lambda$ such that $<\lambda, \alpha> \neq <\lambda, \alpha'>$ for $\alpha \neq \alpha'$. For the given set of equilibrium points. If one wants to choose a vector whose dot product maximally separates each of the given elements of the given set of equilibrium points, then a constrained optimization can be set up to accomplish this. The details of this construction are beyond the scope of this paper.

2.3. This Reconstruction has The Fewest Possible Equilibria. In this subsection we show that the constructed system (5) is minimal in the sense that it has the fewest possible number of equilibrium points consistent with having $m$ distinct attractors in an open set in $\mathbb{R}^n$.

We will assume that the given system (5), has a strict Lyapunov function, whose derivative vanishes only at the set of equilibrium points. Such a function will be constructed in the next section. However, the existence of such a function follows from a result cited in Shub (1986 p 19) as long as convergence is established via an iterative use of Theorem II using Lyapunov functions. Shub's result states that any $C^0$ vector field with unique solutions has a $C^\infty$ Lyapunov function whose derivative is negative except at points of the Chain Recurrent set of the system. The definition of Chain Recurrence is not necessary here but only the fact that any Lyapunov function must have zero derivative on a Chain Recurrent Set.

The following result of Franks (Franks, 1982) will be used:

**Theorem VI.** Suppose the flow $f_t$ is gradient-like with respect to a Morse function $g$ on a compact Manifold $M$ (in our case a closed smoothed box or ball in $\mathbb{R}^n$) and $V$ is the union of those components of $\partial M$ on which the flow is exiting ($V$ may be empty). If there are $c_k$ critical points of index $k$ and $\beta_k = \dim \mathcal{H}_k(M, V, F)$ for a field $F$, then

$$c_k - c_{k-1} + \cdots \pm c_0 \geq \beta_k - \beta_{k-1} + \cdots \pm \beta_0$$

for all $k \geq 0$

As a consequence, $\sum (-1)^j c_j = \chi(M, V)$, the Euler characteristic.

In this theorem, $\mathcal{H}_k(M, V, F)$ is the $k$ dimensional homology vector space of the pair $(M, V)$ with coefficients in a field $F$. The definition or the detailed properties of these vector spaces in general are not relevant to our discussion here. But in all cases treated in our paper $V = \emptyset$, in which case $\mathcal{H}_k(M, V, F)$ reduces to $\mathcal{H}_k(M, F)$, the homology of the manifold in the field $F$. In all cases treated in this paper $M$ is a closed ball or box. Since this set is contractible:

$$\mathcal{H}_k(M, F) = \begin{cases} F & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

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Note, in all cases this theory requires the equilibrium points to be non-degenerate
(in fact, hyperbolic). There is no satisfactory general theory for vector fields which allow
degenerate equilibria. Indeed, no such theory may be possible! (see Arnold (1988) for
such an argument).

**Corollary VII.** System (5), of section 2.2 is a minimal system which has a given
fixed set \( \alpha_i, i = 1 \ldots k \) of attractors in the sense that no other hyperbolic \( C^\infty \) system
which has \( k \) attractors has fewer than \( 2k - 1 \) equilibria.

**Proof:**

Let \( m = \min \{ x_1(\alpha_i) \mid i = 1 \ldots k \} \), and let \( M = \max \{ x_1(\alpha_i) \mid i = 1 \ldots k \} \), choose
a positive constant \( K > 1 \) and let \( m_1 = m - K \), and \( M_1 = M + K \). Let \( m_i = \min_{x \in [m_i, M_i]} p_i(x) - K, M_i = \max_{x \in [m_i, M_i]} p_i(x) + K \) for \( i = 2 \ldots m \), where \( \{ p_i(x) \mid i = 2, \ldots n \} \) are the Lagrange Interpolation Polynomials defined in Theorem IV. Let \( B^n = [m_1, M_1] \times \ldots \times [m_n, M_n] \). Using the form of equation (5), and checking each variable it
can be seen that the flow defined by the vector field (5), is entering \( B^n \) on \( \partial B^n \), so the
exit set is empty. If we smooth the corners of the boundary of \( B^n \) by a sufficiently small
perturbation, we obtain a compact manifold with boundary \( f \) to which Theorem VI applies.

System (5), has \( k \) rest points of index zero and \( 2k - 1 \) rest points of index 1,
and no other equilibria. One simply checks for our system that all the other Morse
Inequalities are satisfied. Any system which satisfies the Morse inequalities with \( k \)
attracting equilibria has to satisfy \( c_1 - k \geq -1 \), so \( c_1 \), the number of equilibria with a
one dimensional unstable manifold, has to be at least \( k - 1 \). Therefore there must be
at least \( 2k - 1 \) equilibria.

**2.4. Lyapunov Functions: Increasing the Class of Examples with the
Same Stable Dynamics.** The constructed systems except for the linear change of
coordinates, are really "lifts of one dimensional systems". In fact, these systems are
really independent two dimensional systems. Aside from the existence proof that one
can explicitly construct differential equations with a given finite set of attracting
equilibria, there is question as to why these systems might be useful. One would like
to explicitly generate a large family of systems with the same equilibria and the same
attractors as the original system. The tool that is used to do this is a strict Lyapunov
function. First we construct the function:

**Proposition VIII.** For any positive constant \( K \), the function

\[
W(X) = \sum_{i=2}^{n} \left[ \frac{1}{2} \left( (x_i - p_i(x_1))^2 + \int p_1(x_1)p_i'(x_1)dx_1 \right) \right] + K \int p_1(x_1)dx_1
\]

is a strict Lyapunov function for the system (5) with derivative

\[
\dot{W}(X) = -Kp_i^2 - \sum_{i=2}^{n} \left[ (x_i - p_i - p_1p_i'/2)^2 + 1/4p_i'^2p_i^2 \right]
\]

**Proof**
The form of $\dot{W}(X)$ shows the Lyapunov function is strict. Since $p_1$ is a polynomial of odd degree, whose highest order coefficient is positive, its integral is an even function of $x_1$ and is therefore bounded below, as is the integral of $p_1^2 p_1$. By convention, we choose $\int p_1, \int p_1^2 p_1$ so that they are always non-negative.

One can now construct a large class of examples which have the same attractors and saddles as the original equation. We can use these to produce fits to a given set of dynamics with a fixed set of attracting equilibria. But first a technical proposition is proved relating the indices of an equilibrium point to the indices of a strict Lyapunov function in a neighborhood of the equilibrium point.

**Proposition IX.** Let $\dot{X} = AX$, be a linear differential equation with constant coefficients and suppose that $AX$ is hyperbolic — the eigenvalues of $A$ have non-zero real parts. Then

i. There is a quadratic form $V(X) = X^T S X$ which is a strict Lyapunov function for the system $\dot{X} = AX$.

ii. Let $X^T S X$ be a strict Lyapunov function for the system $\dot{X} = AX$. The dimension of the subspace of $\mathbb{R}^n$, $A_-$, $(A_+)$ spanned by the eigenvectors of $A$ whose eigenvalues have negative (positive) real parts is the same as the dimension of the subspace of $\mathbb{R}^n$, spanned by the eigenvectors of $S$ whose eigenvalues have positive (negative) real parts. By definition therefore the index of $-S$ is the same as the index of $A$.

iii. Let $X^T S X$ be a strict Lyapunov function for the system $\dot{X} = BX$ and let $-S$ have index $m$. Then $B$ is hyperbolic and also has index $m$.

The proof uses standard techniques of Linear Algebra such as found in (Taussky, 1961) and (Wilkinson, 1961) and is included in the Appendix.

**Theorem X.** Choose a Locally Lipshitz Frame of Vector Fields $F_2, \ldots F_n$ such that for each $i$, $\langle F_i, \nabla W \rangle = 0$ where $W$ is defined as in equation (5) and $F_i$ vanishes at the critical points of $W$. Choose arbitrary locally Lipshitz scalar functions, $f_i \quad i = 2, \ldots n$ and choose $f_1$ to be a strictly positive function defined everywhere except possibly at the critical points of $\nabla W$, such that $f_1 \nabla W$ is locally lipshitz. Then

$$\dot{X} = -f_1 \nabla W(X) + \sum_{i=2}^{n} f_i(X) F_i(X) \tag{16}$$

converges to one of the same set of equilibria as (5) of Theorem IV. Moreover, the set of attractors for (5), $\alpha_i, \quad i = 2, \ldots n$ are the same as the set of attractors for (16). If equation (16) can be rewritten as

$$\dot{X} = -A(X) \nabla W(X) + \sum_{i=2}^{n} f_i(X) F_i(X) \tag{17}$$

where $A(X)$ is a $C^\infty$ Riemannian Metric and the $f_i, F_i$ are all $C^\infty$ then the index of each of the critical points of (17) is the same as the index the corresponding critical points of (5).
Remarks

Although this result can be proved by direct calculation, the proof presented here is via Proposition IX because this method of proof is used later in the paper. The result concerning the index is a result which depends upon the fact that $W$ is a strict Lyapunov function for the systems (17) and (5), not on the specific form of $W$ and the proof reflects this fact. It is probable that a similar proof which holds for these $W$, which are degenerate at the critical points of (5), can be constructed.

The theorem was stated in this level of generality so as to be able to treat the equation

$$\dot{X} = -A(X)\nabla W$$

where $A(X)$ is an arbitrary Riemannian Metric on $\mathbb{R}^n$ (a $C^\infty$ matrix function which is positive definite at each point in $\mathbb{R}^n$). We can rewrite this system as

$$(18) \quad \dot{X} = -\frac{\nabla W^T A(X) \nabla W}{\|\nabla W\|^2} \nabla W - \left( \nabla W^T A(X) - \frac{\nabla W^T A(X) \nabla W}{\|\nabla W\|^2} \nabla W \right)$$

In this case, the function $f_1 = \frac{\nabla W^T A(X) \nabla W}{\|\nabla W\|^2}$ which multiplies $\nabla W$ in (18) is defined everywhere except at the critical points of $\nabla W$ and is bounded in a neighborhood of the critical points $\nabla W$. At these points $f_1 \nabla W$ is locally Lipschitz and continuous but not differentiable because $\lim_{\nabla W \to 0} f_1$ is not defined. If we take $F_2 = \left( \frac{\nabla W^T A(X) \nabla W}{\|\nabla W\|^2} \nabla W - \nabla W^T A(X) \right)$, $f_2 = 1$, and $f_i = 0$, $i = 3 \ldots n$, then the conditions of the Theorem are satisfied, as $< F_2, \nabla W > = 0$. Thus the $n \times (n + 1)/2$ functional degrees of freedom which appear in the Riemannian metric are mostly redundant. At most two functional degrees of freedom have been introduced.

This is a very general class of systems with a particular Lyapunov function because there are really $n$ degrees of freedom in the specification of (16) at each point. One would like, in general, to classify equivalence classes of Lyapunov functions which define systems with a fixed set of equilibria, attractors and saddles and produce convergent dynamics. Two Lyapunov functions would be equivalent if they were Lyapunov for the same set of differential equations. Its unclear how many separate equivalence classes there are for each set of attractors or how to characterize them.

The construction of a smooth $n - 1$ dimensional linearly independent frame which vanishes on the critical points of $\nabla W$, can be carried out in a rapid fashion on almost all of $\mathbb{R}^n$ as follows: Let $(\nabla W_1, \ldots, \nabla W_n)$ be the components of $\nabla W$. Define

$$(19) \quad F_i = \begin{cases} (-\nabla W_2, \nabla W_1, 0, \ldots, 0) & \text{for } i = 2 \\ (\nabla W_1 \nabla W_2, \ldots, \nabla W_{i-1} \nabla W_i W_i - \sum_{j=1}^{i-1} \nabla_j W^2, 0, \ldots, 0) & 3 \leq i \leq n \end{cases}$$

This construction produces a set of $n - 1$ distinct linearly independent, orthogonal vector fields if $\nabla W \neq 0$ or $\nabla_2 W \neq 0$. In general, one cannot expect to produce a continuous linearly independent frame of $n - 1$ vectors which vanishes only at the
critical points of $\nabla W$. This is because it can be shown that the gradient is transverse to a surface homeomorphic to the $n$ dimensional sphere, which is at sufficiently large distance from the origin. If such an orthogonal frame existed and vanished nowhere but at the critical points of $\nabla W$, then one should be able to construct a continuous vector field tangent to the $n$ dimensional sphere. This is only possible on odd dimensional spheres. By starting the above algorithm out on different sets of coordinates it is possible to produce $n - 1$ sets of orthogonal frames, one of which contains a set of $n - 1$ linearly independent vectors at all but the critical points of $\nabla W$.

As an example of what can be shown using this theorem consider the system

\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= -y.
\end{align*}
\]

This system is globally asymptotically stable and all trajectories exponentially converge to the origin. The system has the strict Lyapunov function $H(x, y) = x^2 + y^2$. We can immediately conclude that the system

\[
\begin{align*}
(20) \quad \dot{x} &= \begin{pmatrix} 1 + y^2 & -x^2 \\ -y^2 & 1 + x^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \cosh(x^2 + y^2) \begin{pmatrix} -y \\ x \end{pmatrix}
\end{align*}
\]

is globally asymptotically stable since the matrix in equation (20) is globally positive definite.

**Proof of Theorem X**

The function $W(X)$ as defined in (14) is the sum of nonnegative terms and is bounded below. Moreover, $\lim_{X \to \infty} W(X) = +\infty$ along any line starting from the origin. If $x_1 = 0$ along the chosen ray the $x_i$ must be unbounded for some $i$ in which case, we must have $\lim_{x \to \infty} (x_i - p_i(x_1))^2 = +\infty$. If not, then $x_1$ must be unbounded on the ray, in which case $\lim_{x \to \infty} \int p_i(x_1)dx_1 = +\infty$. The derivative of $W(X)$ on any trajectory of (16) is:

\[
(21) \quad \dot{W} = -f_1 \|\nabla W\|^2 \leq 0.
\]

Because $W$ is unbounded above for $X$ whose norm is sufficiently large, and because $\dot{W} \leq 0$, every trajectory of (16) is bounded. Moreover as $f_1$ is strictly positive, each trajectory must converge to a critical point of $\nabla W$. Furthermore, since $\dot{W}$ is a strict global Lyapunov function for (5), the critical points are the equilibrium points of (5). System (5) has a triangular Jacobian and no zero eigenvalues so it is hyperbolic. Let $F$ be the vector field defined by the right hand side of system (5) and let $H'(W)$ be the Hessian of $W$ at the critical point $E$.

The quadratic form

\[
(22) \quad (X - E)^T \left( H'(W)DF + DFT H'(W) \right) (X - E)
\]

must be negative semi-definite at each of the critical points $E$ of $W$, where $DF$ is the Jacobian of the system $F$.  

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If not, then one can choose a vector \( X - E \) such that the above quadratic form takes on a positive value on the vector \( X - E \). Since

\[
< \nabla W(X), F(X) > = 1/2(X - E)^T \left( H'(W)DF(E) + DF(E)^T H'(W) \right) (X - E) + o(\|X - E\|^2)
\]

(23)

It would follow that the derivative of \( W \) along trajectories of (5) was somewhere positive contradicting Theorem IV.

Proposition IX(i.) implies that for the linear differential equation \( \dot{X} = DF(E)X \) there is a quadratic form \( X^TAX \) whose derivative

\[
X^T \left( DF(E)^T A + ADF(E) \right) X
\]

is negative definite. It follows that for all \( \epsilon > 0 \),

\[
X^T \left( DF(E)^T (H'(W) + \epsilon A) + (H'(W) + \epsilon A)DF(E) \right) X
\]

is negative definite. A simple calculation shows that the Hessian of \( H(W) = \partial^2 W / \partial x_i \partial x_j \) is non-degenerate at the critical points of \( W \). Since \( H'(W) \) is non-degenerate, we can choose an \( \epsilon > 0 \) such that all the eigenvalues of \( H'(W) \) and \( H'(W) + \epsilon A \) have the same sign. By Proposition IX(ii.) the dimension of the subspace spanned by the eigenvectors of \( DF|_E \) whose eigenvalues have negative real parts is the same as the dimension of the space spanned by the eigenvectors of \( H'(W) \) whose eigenvalues have positive real part. The equilibrium points of (16) which are the same as the critical points of \( W \) because \( W \) is a strict Lyapunov function of (16) which are in turn the same as the equilibrium points of (5) (because \( W \) is a strict Lyapunov function for this system as well). If \( E \) is an attractor of (5), then \( E \) is a local minimum for \( W \) and therefore an attractor for (16). If not, then \( E \) is a saddle for \( W \) which implies that there are points in every neighborhood of \( E \) on which \( W \) takes smaller values than \( W \) takes at \( E \). Any such point can never converge to \( E \) on the flow of system (16) since \( W \) is continuous and a strict Lyapunov function for (16).

If (16) can be rewritten as equation (17), then more can be concluded. The Jacobian of system (17) at the equilibrium point \( E \) can be written

\[
J = -A(E)H'(W) + \sum_{i=2}^{n} f_i(E) DF_i(E).
\]

Differentiating the relationship \( < \nabla W, F_i > = 0 \) twice and using the facts \( \nabla W(E) = 0, F_i(E) = 0 \) yields the result

\[
H'(W)DF_i(E) + DF_i(E)^T H'(W) = 0.
\]

Therefore \( J \) can be rewritten in the form \( J = -A(E)H'(W) + Q \) where \( Q^T H'(W) + H'(W)Q = 0 \). The quadratic form \( X^T H'(W)X \) is a strict Lyapunov function for the
linear differential equation $\dot{X} = JX$ since

$$
(24) \quad \frac{d}{dt} X^T H'(W) X = X^T \left( J^T H'(W) + H'(W) J \right) X \\
= X^T \left( -2H'(W)A(E)H'W + Q^T H'(W) + QH'(W) \right) X \\
= -2X^T (H'(W)A(E)H'(W)) X
$$

which is negative definite. Proposition IX(iii) implies that the equilibria of the system $\dot{X} = JX$ are non-degenerate and hyperbolic. Furthermore, the index of this equation is the same as the index $-H'(W)$ at the corresponding equilibrium point by Proposition IX(ii) and in turn, the index of $-H'(W)$ at each equilibrium point $E$ is the same as the index of (5) at this point since $W$ is also a strict Lyapunov function for (5). This completes the proof of the result.

### 2.5. Synthesis of Multiple Periodic Orbits.

Using a modification of the above technique, arbitrary numbers of stable periodic orbits of general shape and corresponding unstable periodic orbits, may be derived. The method is to "suspend" the flow of (16) by adding an additional dimension to the system of equations and to require that an appropriate function of two variables (a weakly coercive $H(x_n,x_{n+1})$) remain constant along limit sets of the flow. On these, the derivative is non-zero and is tangent to the level sets of the function $H$. It will follow from further argument (The Poincaré-Bendixson Theorem) that the limit set is a periodic orbit.

**COROLLARY XI.** Suppose that $W(X)$ is defined as in (14) and let $G_1(X), \ldots, G_n(X)$ take on the form of system (17) and . Let $H(x_n,x_{n+1})$ be a weakly coercive function of two variables. Let $f(X)$, be an arbitrary $C^\infty$ function such that either $f(X) > 0$ or $f(X) < 0$ for all $X \neq 0$. Suppose that $x_n(0) \neq 0$ or $x_{n+1}(0) \neq 0$. Then the system,

$$
(25) \quad \dot{x}_i = G_i(x_1, \ldots, x_{n-1}, \log H(x_n,x_{n+1})) \quad i = 1, \ldots, n-1 \\
\dot{x}_n = \frac{HG_n(x_1, \ldots, x_{n-1}, \log H(x_n,x_{n+1}))}{\|\nabla H\|^2} \frac{\partial H}{\partial x_n} - f \frac{\partial H}{\partial x_{n+1}} \\
(26) \quad \dot{x}_{n+1} = \frac{HG_n(x_1, \ldots, x_{n-1}, \log H(x_n,x_{n+1}))}{\|\nabla H\|^2} \frac{\partial H}{\partial x_{n+1}} + f \frac{\partial H}{\partial x_n}
$$

has a periodic orbit for each $\alpha_i, \beta_i$, satisfying $\log H(x_n,x_{n+1}) = x_n(\alpha_i)$, $\log H(x_n,x_{n+1}) = x_n(\beta_i)$, where $\alpha_i, i = 1, \ldots, m$, $\beta_i, i = 1, \ldots, m-1$ are the attractors and the saddlepoints of the equation

$$
\dot{x}_i = G_i(x_1, \ldots, x_n) \quad i = 1, \ldots, n.
$$

The periodic orbits corresponding to the $\alpha_i$ are stable, those corresponding to the $\beta_i$ are unstable. If $H$ is strongly coercive then there is a unique such periodic orbit for each such $\alpha_i, \beta_i$.

**Proof:** Let $u = \log H$, then the system corresponding to (25) (26) takes the form

$$
(27) \quad \begin{align*}
\dot{x}_i &= G_i(x_1, \ldots, x_{n-1}, u), i = 1, \ldots, n-1 \\
\dot{u} &= G_n(x_1, \ldots, x_{n-1}, u)
\end{align*}
$$
which is the same form as (17). It follows that any trajectory of this systems converges to \( \alpha_i, \beta_i \), so that \( \log H \) converges to one of the points \( x_n(\alpha_i), x_n(\beta_i) \). Thus the stability of the limits of the system (25)-(26) follows from the stability of the limits of (17). Consider the \( \omega \) limit set of any trajectory \( \gamma \). Because the system (17) has bounded trajectories and \( H \) is weakly coercive, and any trajectory of (25)-(26) is bounded, it follows that \( \omega(\gamma) \) is closed, bounded and connected. \( H \) is constant on this set. For definiteness, suppose \( H = e^k \neq 0 \). On this set \( \omega(\gamma) \) (25)-(26) become:

\[
\begin{align*}
\dot{x}_i &= 0, \quad i = 1 \ldots n - 1 \\
\dot{x}_n &= -f(a_1, \ldots, a_{n-1}, x_n, x_{n+1}) \frac{\partial H}{\partial x_{n+1}} \\
\dot{x}_{n+1} &= -f(a_1, \ldots, a_{n-1}, x_n, x_{n+1}) \frac{\partial H}{\partial x_n}
\end{align*}
\]

where \( a_i, i = 1, \ldots, n - 1 \) is the projection of some \( \alpha_i, \beta_i \) on the first \( n - 1 \) coordinates. Since \( H \) is weakly coercive and \( f \neq 0 \) this \( \omega \) limit set contains no stationary points. Since (28) holds for points on this set, it follows by the Poincaré-Bendixson Theorem that this set contains a periodic orbit \( \gamma' \). Because \( H \) has no critical points on \( \gamma' \) we can construct an open neighborhood \( \mathcal{N} \) of this orbit such that \( (x', y') \in \mathcal{N} \Rightarrow H(x', y') \neq e^k \). Since this limit set is connected therefore it follows that the \( \omega(\gamma) = \gamma' \) so the limit of each trajectory is a periodic orbit. That the orbits corresponding to the \( \alpha_i \) are attracting and those corresponding to the \( \beta_i \) are repelling follows from the analogous argument of Theorem X. If \( H \) is strongly coercive the function \( f(x,y)(t) = H(tx, ty) \) is strictly monotone and the periodic orbit must be unique, because \( H^{-1}(c) \) is a simple closed curve.

For two dimensional systems the following corollary will be proved by methods analogous to those used in Corollary XI. This result is distinct from Corollary XI because the form is somewhat more general than the prior corollary when applied to the two dimensional case. Note that in this case that there are no convergent coordinates. Such a system when its output feeds forward as input to stable systems can generate periodic behavior of virtually any shape.

**COROLLARY XII.** Let \( H(x,y) \) be a weakly coercive function of two variables. Let \( g(x,y) > 0 \) and let \( f(x,y) > 0 \) or \( f(x,y) < 0 \). Let \( p(z) = \prod_{i=0}^{2n}(z - a_i) \), where \( a_i < a_{i+1}, i = 1 \ldots n - 1 \). Then the system

\[
\begin{align*}
\dot{x} &= -\frac{gH p(\log H)}{||\nabla H||^2} \frac{\partial H}{\partial x} - f \frac{\partial H}{\partial y} \\
\dot{y} &= -\frac{gH p(\log H)}{||\nabla H||^2} \frac{\partial H}{\partial y} - f \frac{\partial H}{\partial x}
\end{align*}
\]

has a periodic orbit for each \( a_i, i = 0 \ldots 2n \). The orbits corresponding to \( \alpha_i \), \( i \) even are attracting, those corresponding to the \( \alpha_i \), \( i \) odd are repelling.

**Proof.**

If \( u = \log H \), then \( \dot{u} = -p(u)g \). Therefore \( \frac{d}{dt} \int p(u) du = -p^2(u)g \leq 0 \) and therefore \( \int p \) is a Lyapunov function for equations (29), (30). Since \( \int p \) is a polynomial of even degree with positive high order coefficient it is bounded below and \( \int p(u) \to \infty \) as
\[ \|u\| \to \infty. \] Therefore \( \log H \) remains bounded and hence \( x, y \) remains bounded on any trajectory. Therefore \( u \) converges to a zero of \( p \) by the LaSalle Invariance Principle because \( \int p(u)du \) is a Lyapunov function for equations (29), (30). The proof of the remainder of the statements of this corollary are analogous to corresponding statements in Corollary XI and so are omitted.

Thus if \( p(u) = u(u - 1)(u - 2) \), \( H(x, y) = x^2 + y^2 \), \( g = 1 \), \( f = 7x^2 + 3y^2 + 2 \) substituting into the formulas (29) and (30) shows that the system

\[
\begin{align*}
\dot{x} &= - \left[ \log(x^2 + y^2)(\log(x^2 + y^2) - 1)(\log(x^2 + y^2) - 2) \right] x \\
&\quad - (7x^2 + 3y^2 + 1)y \\
\dot{y} &= - \left[ \log(x^2 + y^2)(\log(x^2 + y^2) - 1)(\log(x^2 + y^2) - 2) \right] y \\
&\quad + (7x^2 + 3y^2 + 1)x 
\end{align*}
\]

has three periodic circular orbits. The first and the third with radii 1, \( e^2 \) are stable and the second with radius \( e \) is unstable. Because \( f \) is not constant along the orbits the velocity varies along these trajectories and the period of each of the two orbits is distinct. Note the derivatives of (31), (32) are not defined at the equilibrium point at the origin even though it is clear that the origin is an unstable equilibrium. The phase portrait of this system is schematized in figure 4 below.
AN EXAMPLE SYSTEM WITH THREE PERIODIC ORBITS

Fig. 4. This figure is a schematic of the flow of (31), (32) above. The outer and inner circle are attracting periodic orbits, the middle circle is repelling. The origin is an unstable equilibrium.
These results are just examples of what can be proved. Multiple “suspensions” as in corollary XI will produce invariant tori. Note the logarithm in (25), (26) is here used for simplicity, any strictly monotone function \( f : (0, \infty) \to (-\infty, \infty) \), will work as well. It is also possible to mix periodic orbits with stable equilibria by choosing an equation with a more complicated expression than (25), (26), (29), (30) above. Note that (25), (26), (29), (30) are not necessarily defined when \( x_n, x_{n+1} = 0 \). However, the proofs above shows that no trajectory has any such point as an \( \omega \) limit point unless it has \( x_n, x_{n+1} = 0 \) or \( x, y = 0 \) as initial data, respectively. It can be shown using smoothness and coercivity of \( H \) that \( x_n, x_{n+1} = 0 \) , \( \dot{x}, \dot{y} = 0 \) under these circumstances provided that \( (0, 0) \) is a non degenerate critical point of \( H \). It is an open problem to synthesize a system which has arbitrary periodic orbits as attractors in an easily computable fashion, directly by a differential equation. However this can be done via transformations from the systems (25),(26) defined above.

2.6. Fitting Procedures for This Class of Models. Theorem X and Corollary XI fix a large class of systems with the same stable dynamics. By continuously indexing a subclass of systems within this class by a finite set of parameters one can potentially fit trajectories of data to systems in this model class. To fix ideas of the earlier sections a specific simple example will be discussed. No proofs will be given because the numerical stability and rates of convergence of algorithms of this class is a topic for another paper. These techniques have been discussed in Pearlmutter (Pearlmutter, 1988) and are just representative of how fits can be constructed using trajectories to constrain model systems such as those in equation (17).

We choose a two dimensional system with two attractors \( \alpha_1 = (1, 3), \alpha_2 = (3, 5) \) and a saddle \( \beta_1 = (b_1, b_2), 1 < b_1 < 3 \). The coordinates of the saddle are chosen as free parameters. Then the original system corresponding to (5) is

\[
\begin{align*}
\dot{x} &= -(x-1)(x-b_1)(x-3) \\
\dot{y} &= -(y-p(x)) 
\end{align*}
\]

where

\[ p(x) = 3\frac{(x-3)(x-b_1)}{(2)(b_1-1)} + b_2\frac{(x-1)(x-3)}{(b_1-1)(b_1-3)} + 5\frac{(x-1)(x-b_1)}{2(3-b_1)} \]

Our Lyapunov function for this case is:

\[
V(x, y) = 1/2((y-p(x))^2 + q(x)) + r(x)
\]

where

\[ q(x) = \int (x-1)(x-b_1)(x-3)s^2(x)dx, \]

\[ s(x) = \frac{(x-1)(2b_2-5b_1+5)-(x-b_1)(2b_1+4)+(x-3)(3b_1+2b_2-9)}{2(b_1-1)(b_1-3)}, \]
and
\[ r(x) = K \int (x - 1)(x - b_1)(x - 3)dx. \]

Then
\[ \frac{\partial V}{\partial y} = -(y - p(x)) \]
\[ \frac{\partial V}{\partial x} = (y - p(x))s(x) - 1/2(x - 1)(x - b_1)(x - 3)(s^2(x) + K). \]

Construct a new differential equation:
\[ \begin{align*}
\dot{x} &= -(a + (bx + cy))^2 \frac{\partial V}{\partial x} + (ex + fy + g) \frac{\partial V}{\partial y} \\
\dot{y} &= -(a + (bx + cy))^2 \frac{\partial V}{\partial y} + -(ex + fy + g) \frac{\partial V}{\partial x}.
\end{align*} \]

Using Theorem X it follows that this nine parameter family of differential equations converges to one of the points (1, 3), (b_1, b_2), (3, 5) and has (1, 3), (3, 5) as attractors. The only constraints on the parameters are, \( K > 0, a > 0, 1 < b_1 < 3 \). Generically they can be chosen such that \( K \geq 0, a \geq 0 \) at a cost of complicating the argument of Theorem X considerably. Now suppose we are given a trajectory \( \gamma(t) \) converging to one of the attractors, for example (3,5). Choose a time interval \( T \) and attempt to minimize the following function of the nine parameters \( P = (a, b, c, e, f, g, b_1, b_2, K) \) by first choosing arbitrary initial values.

\[ E(P) = 1/2 \int_0^T (x(t, P) - \gamma_x(t))^2 + (y(t, P) - \gamma_y(t))^2 dt \]

The simplest way (but not perhaps the most efficient) is to minimize \( E \), by the method of gradient descent. Accordingly
\[ \nabla_P E = \int_0^T (x(t, P) - \gamma_x(t)) \frac{\partial x(t, P)}{\partial P} + (y(t, P) - \gamma_y(t)) \frac{\partial y(t, P)}{\partial P} dt \]

\( \frac{\partial x(t, P)}{\partial P} \), and \( \frac{\partial y(t, P)}{\partial P} \) are not explicitly given but obey the variational equation.

\[ \begin{align*}
\frac{d}{dt} \frac{\partial x(t, P)}{\partial P} &= \left| \begin{array}{c}
\frac{\partial V_x'}{\partial x} \\
\frac{\partial V_y'}{\partial x}
\end{array} \right| \frac{\partial x(t, P)}{\partial P} + \left| \begin{array}{c}
\frac{\partial V_x'}{\partial y} \\
\frac{\partial V_y'}{\partial y}
\end{array} \right| \frac{\partial x(t, P)}{\partial P} \\
\frac{d}{dt} \frac{\partial y(t, P)}{\partial P} &= \left| \begin{array}{c}
\frac{\partial V_x'}{\partial x} \\
\frac{\partial V_y'}{\partial x}
\end{array} \right| \frac{\partial y(t, P)}{\partial P} + \left| \begin{array}{c}
\frac{\partial V_x'}{\partial y} \\
\frac{\partial V_y'}{\partial y}
\end{array} \right| \frac{\partial y(t, P)}{\partial P} \]

where the vector field \( V_x', V_y' \) is the right hand side of (36). and the initial data is
\[ \frac{\partial x(0, P)}{\partial P} = \frac{\partial y(0, P)}{\partial P} = 0. \]

Once one calculates \( \nabla_P E \) by integrating (36), (38), and (39), one updates \( P \) by the equation
\[ P_{n+1} = P_n - \mu \nabla_P E. \]
Its also possible to use a Newton's method for finding a minimum of \( E \) but the variational equations for \( \frac{\partial^2 x}{\partial P \partial P}, \frac{\partial^2 y}{\partial P \partial P} \) are quite complex.

For such a system it would be revealing to apply the techniques of Chiang et al (1988) to see how the stable manifold of each equilibrium point was varying as a function of the given parameters. It would also be useful to have a continuous method for updating the parameters of (36). But as \( E(P) \) is a global measure of goodness of fit, its hard to see how this can be done while staying on the same trajectory.

3. General Stable Dynamics. In this section a method is given which enables the construction of vector fields with stable dynamics whose Morse indices take a more general form than those of (5). A strict Lyapunov function is given for this system as well. The method of showing that a given function is a Lyapunov function appears to be novel. Its first necessary to specialize the notion of Lagrange interpolation to the problem at hand.

3.1. A System Converging to a Specified Set of Equilibria in \( \mathbb{R}^n \).

**Lemma XIII.** For any set of values \( a_i, i = 1, \ldots, n \) there is a set of polynomials \( \delta_i(x), i = 1, \ldots, n \) such that:

i. \( \delta_i(x) \geq 0 \) and 0 only at \( a_j, j \neq i \)

ii. \( \delta_i(a_j) = 1, \) if \( j = i \)

iii. \( \delta_i'(a_j) = 0, \) \( j = 1, \ldots, n \)

iv. \( \min_{x} \max_{i} \delta_i(x) > 2^{-2(n+1)} > 0 \)

**Proof**

Let

\[
\delta_i(x) = \prod_{j=1}^{n} \left( \frac{x-a_j}{a_i-a_j} \right)^2 \left( 1 + c_{1i}(x-a_i) + c_{2i}(x-a_i)^2 \right)
\]

where \( c_{1i}, c_{2i} \) is yet to be determined. Condition ii. follows immediately from the form of \( \delta_i \). Because \( \delta_i \) is a product of squares of the form \( (x-a_j), j \neq i, \delta_i'(a_j) = 0, \) for \( j \neq i \) setting \( \delta_i'(a_i) = 0, \) fixes

\[
c_{1i} = -2 \sum_{j=1}^{n} \frac{1}{a_i-a_j}
\]

Fixing \( c_{2i} > c_{1i}^2/4 \) makes the last factor a sum of squares and hence non-negative which proves iv.

To show iv., choose \( c_{2i} \) sufficiently large so that \( 1 + c_{1i}(x-a_i) + c_{2i}(x-a_i)^2 > 1/2. \) If \( x > a_n, \delta_n(x) > 1/2 \) since \( |x-a_j| > |a_n-a_j|, j = 1, \ldots, n-1, \) similarly if \( x < a_1 \delta_1(x) > 1/2. \) Of course if \( x = a_i, \delta_i(x) = 1. \) Finally if \( a_i < x < a_{i+1}, \) and \( x \geq (a_i + a_{i+1})/2, \)

\[
\delta_i(x) \geq 1/2 \prod_{j=i+1}^{n} > (1 - 1/2(\frac{a_{i+1}-a_i}{a_j-a_i}))^2 > 2^{-2(n-i+1)}.
\]
Likewise if \( x \leq (a_i + a_{i+1})/2 \),

\[
\delta_{i+1}(x) > 1/2 \prod_{j=1}^{i}(1 - 1/2(\frac{a_{i+1} - a_i}{a_{i+1} - a_j}))^2 > 2^{-(2i+1)}.
\]

The worst case of these inequalities yields \( iv \). Much better bounds are available for this case. Its hard however to make these bounds independent of the configuration of the set of values \( a_i \).

We now choose a set of points in "special position" in \( R^n \) and construct a differential equation which converges to one of these points. The choice of points and system is noteworthy because sets of equilibria with a very large class of indices consistent with the Morse Inequalities can be constructed in this manner. The system is notationally complex but the idea behind the construction is simple. One builds up the dynamics in each higher dimension by "interpolating" one dimensional systems which vanish on a given set of values \( v_1, \ldots, v_n \). The interpolating functions vanish on all the projected coordinates of the entire set of points in the next lower dimension save one point \( p \). The given values are the coordinate values such that \( (p, v_1), \ldots, (p, v_n) \) are all points in the projection of the original set of points.

This iterative feedforward decomposition allows one to isolate the lower dimensional dynamics from those one dimension higher and allows one to precisely control the dynamics of the system in an iterative fashion. This is perhaps best clarified by an example. Suppose we wish to synthesize a two dimensional system with two attractors \( A = \{(-1, 0), (1, 0)\} \), two saddles \( S = \{(0, -1), (0, 1)\} \), and a repeller at the origin. This system is not minimal and cannot be synthesized by the techniques already discussed. The construction described in theorem XIV below can be specialized to this case to obtain the following system.

\[
\begin{align*}
\dot{x} &= -(x - 1)x(x + 1) \\
\dot{y} &= -\frac{x^2(x - 1)^2}{4}(1 + 3(x + 1) + 3(x + 1)^3)y \\
&\quad - \frac{x^2(x + 1)^2}{4}(1 + -3(x - 1) + 3(x - 1)^3)y \\
&\quad - (x^2 - 1)^2y(y + 1)(y - 1)
\end{align*}
\]

The phase portrait of this example is schematized in figure 5.
A NON-MINIMAL STABLE SYSTEM

FIG. 5. A simple example of the non-minimal construction. This is a schematic of the phase portrait of (40), (41). The y axis as well as the x axis and the lines $x = 1$ and $x = 2$ are invariant sets for this system. The origin $(0,0)$ is a repeller. The two non-zero points on the x axis are attractors and the corresponding points on the y axis are saddles. This construction is impossible using the minimal system of equations (5) above.
We now show all systems exemplified by (40), (41) above are stable and calculate their Morse Indices.

**THEOREM XIV.** Let \( Z = \alpha_1, \ldots, \alpha_m \), be a set of distinct points in \( \mathbb{R}^n \), and such that for each \( 0 \leq i < n \), and each \( \alpha \in \pi_i(Z) \), \( \pi_{i+1}(\pi_i^{-1}(\alpha) \cap Z) \) contains an odd number of points. For each \( \alpha_i \) define

\[
\text{ind}_j(\alpha_i) = \begin{cases} 
1 & x_j(\pi_j(\pi_{j-1}^{-1}(\alpha_i)) \cap Z)) = (a_1, \ldots, a_{2k-1}, x_j(\alpha_i), \ldots, a_{2l+1}) \\
0 & \text{Otherwise}
\end{cases}
\]

In other words, rank order all elements whose first \( j - 1 \) coordinates are the same as \( \alpha_i \), by the size of the \( j \)th coordinate and define \( \text{ind}_j(\alpha_i) = 0 \) if \( \pi_j(\alpha_i) \) has odd rank in this set and \( \text{ind}_j(\alpha_i) = 1 \) otherwise. Define

\[
\text{ind}(\alpha_i) = \sum_{j=1}^n \text{ind}_j(\alpha_i).
\]

For each \( i \), one can choose \( \lambda_i \) such that \( < \lambda_i, a > = d_a \neq < \lambda_i, b > = d_b \), as in Lemma I. Choose a set of interpolation functions \( \{ \delta_{d_a}(a) \in \pi_i(Z) \} \) as defined in Lemma XIII. Then define

\[
\delta_{(i+1)a}(X) = \delta_{d_a}(< \lambda_i, \pi_i(X) >)
\]

(\( \delta_{d_a} \) is defined to be 1). The system of differential equations:

\[
y_i = - \sum_{a \in \pi_i-1(Z)} \delta_{d_a} \prod_{z \in \pi_i(\pi_{i-1}^{-1}(a) \cap Z)} (y_i - x_i(z))
\]

converges to \( [x_1(\alpha_i) \ldots x_n(\alpha_i)]^T \) for some \( \alpha_i \). This differential equation is hyperbolic and \( \text{ind}(\alpha_i) \) is the dimension of the unstable manifold of the point \( \alpha_i \).

**Proof**

First we show that the system (43) must have convergent dynamics and furthermore must converge to one of the points in \( Z \). To see this inductively apply Theorem II. Certainly the result must hold for one dimensional systems because the form of the first equation of (43) is the same as the first equation of (5). So inductively suppose that the first \( i \) equations converge to one of the points \( \pi_i(Z) \). Now consider the \((i + 1)st\) differential equation. This equation must have bounded orbits independent of the convergence of the first \( i \) coordinates. This is because the right hand side of the differential equation is a sum of products of non-negative functions (at least one of which is positive), with polynomials of odd degree whose high order coefficients are negative by construction. Therefore if \( y_i^2 \) is sufficiently large \( \frac{d}{dt} y_i^2 < 0 \). For fixed \( y_1, \ldots, y_{i-1} \), the \( i \)th equation has the Lyapunov function,

\[
V^*(y_i) = \sum_{a \in \pi_{i-1}(Z)} \delta_{d_a} \int_{x \in \pi_{i}(\pi_{i-1}^{-1}(a) \cap Z)} (y_i - x_i(z))dy_i.
\]
This equation (44) has derivative

\[ \dot{V}^*(y_i) = - \left( \sum_{a \in \pi_{i-1}(Z)} \delta^*_a \prod_{z \in \pi_i(\sigma^{-1}_a \cap Z)} (y_i - x_i(z)) \right)^2 \]

which is the square of the derivative of the ith equation of (43). Note \((y_1, \ldots, y_{i-1})\) converges to a point of \(a' \in \pi_{i-1}(Z)\) by hypothesis. But then \(\delta^*_a = 1\), and \(\delta^*_a = 0\), for \(a \neq a'\). In this case \(\frac{d}{dt} V^*(y_i) = 0\) iff \((a, y_i) \in \pi_i(Z)\), so by induction and Theorem II all trajectories of the system converge to a point \(\alpha_i\) of \(Z\).

The assertion about the hyperbolicity of the system (43) follows from its form, because (43) is feedforward and has a lower triangular Jacobian which has non-zero on diagonal terms at each equilibria. Each of the on diagonal terms of the Jacobian take the form

\[ D_{ii}(\alpha_j) = - \prod_{z \in \pi_i(\sigma^{-1}_a \cap Z)} (x_i(\alpha_j) - x_i(z)) \neq 0 \]

By construction \(\text{ind}_k(\alpha_j) = (\text{sign}(D_{kk}(\alpha_j)) + 1)/2\). where

\[ \text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{otherwise.} \end{cases} \]

This fact and the hyperbolicity of the system completes the proof.

Next it is shown that there is a system of the form (43) whose index set matches a general given set of Morse indices. The class of allowable indices is specifically characterized. It will be seen that many such systems can be constructed with the same set of indices.

**COROLLARY XV.** Let \((m_0, m_1, \ldots, m_n)\) be a sequence of non-negative indices consistent with the Morse Inequalities in \(\mathbb{R}^n\) for a manifold with boundary whose homology groups satisfy the conditions (13). Let \(\{e_i|i = 1, \ldots, n\}\) be a set of non-negative constants. Then there is a system of the form (43) with a set of equilibria whose Morse indices take the form

\[ (1, \ldots, 0) + e_1(1, 1, 0, \ldots, 0) + e_2(0, 1, 1, 0) \ldots + e_n(0, 0, \ldots, 1, 1). \]

The \(e_i\) are restricted such that if \(e_m > 0\), then \(e_{m-k} > 0\), for \(k = 1, \ldots, m-1\).

All possible sets of Morse indices consistent with (13) are generated by using arbitrary nonnegative \(e_i\).

For example, the set of indices \((2,2,1)\) satisfy the above conditions but \((1,2,2)\) do not, and both satisfy the Morse inequalities (12) under the conditions (13). Thus in two dimensions there is a system of the form (43) with two attractors, two saddles and a sink with convergent dynamics. It appears impossible to construct such a system of the form (43) with an attractor, two repellers and a sink, even though there are many such systems with convergent dynamics.
Proof

The proof is by induction. Certainly the theorem is true for any one-dimensional system since any one-dimensional system of the form (43) takes the form of the first equation of (5). Assume the theorem is true for dimension $n-1$. Let $(i_0, \ldots, i_n)$ be a set of indices consistent with the conditions (47). Then the set of indices $(i_0, i_1, \ldots, i_{n-1}-i_n)$ are a set of indices consistent with the Morse inequalities. Assume the theorem is true for dimension $n-1$. Let $(i_0, \ldots, i_n)$ be a set of indices consistent with the conditions (47). Then the set of indices $(i_0, i_1, \ldots, i_{n-1}-i_n)$ are a set of indices consistent with the conditions (47) and hence the Morse inequalities. There are two cases to consider. First suppose $i_n = 0$. Choose any system $\dot{Y} = F(Y)$ of the form (43) with the set of indices $(i_0, i_1, \ldots, i_{n-1})$. Let $Z$ be the set of equilibrium points of this system. For each equilibrium $a \in Z$ in $R^{n-1}$, choose a point $(a, y_a) \in R^n$. As in Theorem XIV choose a set of functions $\delta_a, a \in Z$, such that $\delta_a \geq 0$, at least one $\delta^*_a > 0$, $\delta^*_a(a) = 1$, and $\delta^*_a(a') = 0$ for $a \neq a'$, $a, a' \in Z$. Then the system

$$
\dot{y}_i = F_i(y_1, \ldots, y_{n-1}), i = 1, \ldots, n-1
$$
$$
\dot{y}_n = -\sum_{a \in Z} \delta^*_a(y_n - y_a)
$$

is of the form (43) so by Theorem XIV converges to one of the $(a, p_a)$. Moreover, for each point $(a, p_a)$, $\text{ind}_n(a, p_a) = 0$, and this completes the construction in this case, since the remainder of the indices of all the points $(a, p_a)$ are the same as the indices of the corresponding points $a \in Z$ for (48).

If $i_n > 0$, then $i_{n-1} > i_n$ by hypothesis. As above choose any system $\dot{Y} = F(Y)$, which has the indices $(i_0, \ldots, i_{n-1}-i_n)$, and choose $\delta^*_a$, as in the previous paragraph. Fix an equilibrium point $a^*$ of index $n-1$. For each each $a \neq a^*$, choose a point in $R^n, (a, p_a)$. For $a^*$ choose a sequence of points $(a^*, p_0), \ldots, (a^*, p_{2i_n})$, such that $p_i < p_{i+1}, i = 0, \ldots, 2i_n - 1$. Consider the system

$$
\dot{y}_i = F_i(y_1, \ldots, y_{n-1}), i = 1, \ldots, n-1
$$
$$
\dot{y}_n = -\sum_{a \in Z-a^*} \delta^*_a(y_n - p_a) - \delta^*_a \prod_{i=0}^{2i_n}(y_n - p_i)
$$

This system also has the form (43) so Theorem XIV applies to show that the system converges to one of the points $(a, p_a), (a^*, p_i)$, for $i = 0 \ldots 2i_n$. Then $\text{ind}_n(a, p_a) = 0$, $\text{ind}_n(a^*, p_i) = 0$, $i$ even, 1 otherwise. The remainder of the indices $\text{ind}_k$ are defined by (50). By summing the number of equilibria with indices $1$ to $n$, we see that the system (50)–(51) has the indices $(i_0, \ldots, i_n)$ which completes the proof.

3.2. Lyapunov Function and Fits for This Class of Equations. One would like to construct a strict Lyapunov function for this class of systems. This is because one can use the Lyapunov function to produce a general class of models that have the same equilibria with the same indices for this class of systems. This construction is analogous to Theorem X. It will be shown is this section that there are constants $M_1, \ldots, M_n$ such that on any bounded domain the function

$$
U(Y) = \sum_{i=1}^{n} M_i \sum_{a \in \text{ind}_{a+i}(Z)} \delta^*_a R_a(y_i)
$$
where

\begin{equation}
R_{a}(y) = \int_{z \in \pi_{i}(\pi_{i-1}^{-1}(z) \cap \Omega)} \prod_{x_{i} \in \pi_{i-1}^{-1}(z)} \{ (y_{i} - x_{i}(z)) \} \, dy_{i}.
\end{equation}

is a strict Lyapunov function for the system (43). Furthermore the flow of \(-\nabla U\) can be seen to enter the domain. This result is strictly weaker than Proposition VIII because each of the Lyapunov functions are locally defined. There is no guarantee that this construction can be made globally, although I conjecture that this is the case. The method of proof is to show that the derivative of the \(i^{th}\) term in the sum.

\begin{equation}
\frac{d}{dt} T_{i}(Y) = \frac{d}{dt} \sum_{a \in \pi_{i-1}(z)} \delta_{ia} R_{a}(y).
\end{equation}

takes the form

\begin{equation}
\dot{T}_{i}(Y) = -\dot{y}_{i}^{2} + \sum_{j,k<i} A_{jk}(Y) \dot{y}_{j} \dot{y}_{k}
\end{equation}

for some polynomial functions of the variables \(Y\) and where the \(\dot{y}_{k}\) is the \(k^{th}\) function \(Q_{k}\) of the right hand side of (43).

There are then theorems which can be applied to show that for some set of constants \(M_{i}, i = 1, \ldots, n\), the derivative of equation (52) is a globally negative definite function of the derivatives \(\dot{y}_{k}\). This will then be used to show that the Hessian of \(U(Y)\) is non-degenerate at each of the equilibrium points of (43). This will enable us to equate the index of (43), and the index of the Hessian of \(U, H(U)\), at each of the equilibrium points of equation (43). The analogue of Theorem X for the class of systems built from the Lyapunov function \(U(Y)\) will then be obtained.

Given the form of (55), the theorem which is applied to show (52) is a negative definite functions of the derivatives \(\dot{y}_{k}\) is the Gershigorin Circle Theorem. The form of this theorem used is adapted from Noble & Daniel (1977).

**Theorem XVI (Gershigorin Circle Theorem).** Let \(A\) be a symmetric matrix, and suppose for each index \(i\)

\begin{equation}
A_{ii} + \sum_{j \neq i} |A_{ij}| \leq -\mu
\end{equation}

Then \(A\) is negative definite and moreover \(<X^{T}AX> \leq -\mu||X||^{2}\)

For an example of a proof, see Noble & Daniel (1977).

To show that \(\dot{T}_{i}(Y)\) takes the form (55) we apply the Hilbert's Nullstellensatz. This famous theorem gives conditions for a power of a polynomial to be a linear combination of a set of other polynomials with polynomial coefficients. A version of this theorem is stated below which is adapted from van der Waerden (1950).

**Theorem XVII (Hilbert's Nullstellensatz).** Let \(f_{1}, f_{2}, \ldots, f_{n}\) all be polynomials in \(n\) variables \(y_{1}, \ldots, y_{n}\) with real coefficients. Suppose that a polynomial \(g\) vanishes on the set of points (possibly with complex coordinates) where every function \(f_{1}, \ldots, f_{n}\)
vanishes. Then \( g^\rho = \sum_{i=1}^{n} e_i f_i \) for some integer exponent \( \rho \) and real polynomial functions \( e_i \). Moreover, \( \rho = 1 \) if the ideal generated by the set of functions \( f_1, \ldots, f_n \) is the ideal of polynomial functions which all vanish at a fixed set of points.

For a proof of Theorem XVII see for example, van der Waerden (1950). All the functions constructed in the Nullstellensatz are specifically computable but the complexity of such a computation is very high in general.

Now we turn to show that \( T_i(Y) \) along the flow of the differential equation takes the form (55). This amounts to showing that

\[
\frac{\partial T_i}{\partial y_j} = \begin{cases} 
0 & j > i \\
-\dot{y}_i & j = i \\
\sum_{k < i} t^{i}_{jk} \dot{y}_k & j < i
\end{cases}
\]  

(57)

for some polynomial functions \( t^i_{jk}, T_i \) is independent of \( y_j, j > i \) so for \( j > i \), \( \frac{\partial T_i}{\partial y_j} = 0 \). It has already been shown in equations (44), (45), that \( \frac{\partial T_i}{\partial y_j} = -\dot{y}_i \). Furthermore for \( j < i \),

\[
\frac{\partial T_i}{\partial y_j} = \sum_{a \in \pi_{i-1}(Z)} \left( \frac{\partial}{\partial y_j} \delta^*_a \right) R_a(y_i).
\]  

(58)

From the definition of \( \delta^*_a \) in equation (42) and Lemma XIII-iii. above, \( \frac{\partial T_i}{\partial y_j} = 0 \) if \( \pi_{i-1}(Y) \in \pi_{i-1}(Z) \). But by construction, the set of points on which the right hand side of the first \( i - 1 \) equations of (43) vanishes (see argument of Theorem XIV) is the set of points \( (\pi_{i-1}(Z)) \), the projection of the set of zeros of the entire equation (43) onto the first \( i - 1 \) coordinates. It follows immediately from the Nullstellensatz that \( \left( \frac{\partial T_i}{\partial y_j} \right)^\rho \) for some \( \rho \geq 1 \) takes the form defined in (57). Also by the Nullstellensatz for each case of \( \frac{\partial T_i}{\partial y_j} \rho = 1 \) and equation (57) will hold if the ideal generated by the first \( i \) functions \( Q_1, \ldots, Q_i \) of the right hand side of (43) is the entire polynomial ideal of functions which vanish on the finite set \( \pi_i(Z) \). This must be checked because in many other cases \( \rho > 1 \) occurs, making the argument of Theorem XX below impossible. The simplest such example is the case \( g = x_1 - a, n = 1, f_1 = (x_1 - a)^2 \).

To accomplish this we first generalize Lagrange Interpolation to \( R^n \), where ideals of functions vanishing on a fixed set are interpolated rather than a polynomial function of one variable. Note that there are many more degrees of freedom possible in this multidimensional interpolation than in the standard one dimensional Lagrange Interpolation. The fact that the \( Q_i \) generate the entire polynomial ideal is a a simple corollary of this result.

**Theorem XVIII (Generalized Lagrange Interpolation).**

Let the \( \lambda_i, i = 1, \ldots \) be chosen as in (42) of Theorem XIV. For \( \alpha_j \in \pi_j(Z), \vec{y} \in R^{i+1} \) define

\[
\Delta_{\alpha_j}(\vec{y}) = \prod_{\beta \in \pi_j(Z), \beta \not= \alpha_j} \frac{\langle \lambda_j, \vec{y} - \beta \rangle}{\langle \lambda_j, \alpha_j - \beta \rangle}
\]  

(59)
and for $y^j \in \mathcal{R}^{i+1}$ define

$$
\omega_{a_j}(y^j) = \Delta_{a_j} \left( \pi_j(y^j) \right) \prod_{(\alpha_j, \gamma) \in \pi_{j+1}(Z)} \left( x_{j+1}(y^j) - \gamma \right).
$$

Let $\lambda_0 = 1$, $\Delta_\emptyset = 1$, and $\pi_0(Z) = \emptyset$. For each $j$, let $\{c_{j\alpha} | \alpha \in \pi_j(Z)\}$ be an arbitrary set of non-zero constants. Then the set of polynomial functions $P_1, \ldots, P_k$ where

$$
P_k = \sum_{\omega \in \pi_k(Z)} c_{(k-1)\omega} \omega_
$$
generate the entire ideal of functions which vanish on $\pi_k(Z)$.

**Proof**

Surely $P_1, \ldots, P_k$ are each in the ideal of functions which vanish on $\pi_k(Z)$ so we only have to show that these functions generate the entire ideal. The proof of this is by induction. If $n = 1$, then $p_1 = c_0 \prod \pi_1(Z)(y - \gamma)$ which is a generator of the polynomial ideal of functions which vanish on $\pi_1(Z)$. Assume therefore that the theorem is true for $n = 1 \ldots k-1$. We first show that $\omega_\alpha$ is in the ideal generated by the $P_1 \ldots P_k$, for each $\alpha \in \pi_k(Z)$. To do this multiply $P_k$ by $\Delta_\alpha$, then

$$
P_k \Delta_\alpha(y) = c_{(k-1)\omega} \Delta^2_\alpha(y) \prod_{(\alpha, \gamma) \in \pi_k(Z)} (x_k(y) - \gamma) + \sum_{\alpha' \neq \alpha} c_{(k-1)\omega'} \Delta_\omega \omega_{\alpha'}
$$

The second term in the summand vanishes on $\pi_k(Z)$ so is in the ideal generated by the $P_1 \ldots P_{k-1}$ by the inductive hypothesis. Using the definition of $\Delta_\alpha(y)$ in equation (59), and the fact that $\lambda_k - 1, \alpha \neq \lambda_k - 1, \alpha'$, for $\alpha \neq \alpha', \alpha, \alpha' \in \pi_{k-1}(Z)$ it follows that

$$
\Delta^2_\alpha(y) = \Delta_\alpha(y) \Delta_\alpha(y - \alpha + \alpha) = \Delta_\alpha + \Delta_\alpha < \lambda_k - 1, (y - \alpha) > S
$$

for a fixed polynomial function $S$ in $x_1(y), \ldots, x_{k-1}(y)$. It follows therefore that the first summand in (62) can be written as $c_{(k-1)\omega} \omega_{\alpha}$ plus a polynomial which vanishes on $\pi_{k-1}(Z)$. This polynomial is also in the ideal generated by the $P_1 \ldots P_{k-1}$ by the inductive hypothesis. Therefore, $\omega_\alpha$ for each $\alpha \in \pi_k(Z)$ is the ideal generated by $P_1 \ldots P_{k-1}$.

Now let $S'$ be any polynomial which vanishes on $\pi_k(Z)$. For each $\alpha \in \pi_k(Z)$ let

$$
S'_\alpha(y) = S(\alpha, x_k(y), \ldots, x_n(y)).
$$

By the division algorithm and the hypothesis that $S'$ vanishes on $\pi_k(Z)$,

$$
S'_\alpha = T_\alpha(x_k(y), \ldots, x_n(y)) \prod_{(\alpha, \gamma) \in \pi_k(Z)} (x_k(y) - \gamma)
$$

for a fixed polynomial function $T$. Consider

$$
R = S' - \sum_{\alpha \in \pi_k(Z)} T_\alpha \omega_{\alpha}.\]
For each \( \alpha \in \pi_{k-1}(Z) \),

\[
R(\alpha, x_k(\bar{y}), \ldots x_n(\bar{y})) = S' - T_{\alpha} \prod_{(\alpha, \gamma) \in \pi_k(Z)} (x_k(\bar{y}) - \gamma) \\
= S' - S'_{\alpha} \\
= 0.
\]

Therefore \( R \) is in the ideal generated by the functions vanishing on \( \pi_{k-1}(Z) \) and by the inductive hypothesis, in the ideal generated by \( p_1, \ldots p_{k-1} \). This completes the proof, since each \( \omega_{\alpha} \) is in the ideal generated by \( p_1, \ldots p_k \) which are our interpolation functions.

Note that the set of \( \{ \omega_{\alpha} | \alpha \in \pi_k(Z) \} \), for some \( k \) are a multidimensional analogue of Lagrange Interpolation Polynomials.

**Corollary XIX.** The first \( i \) functions \( Q_1, \ldots, Q_i \) of the right hand side of equation (43) generate the entire ideal of polynomial functions by the the functions which vanish on the finite set of points \( \pi_i(Z) \).

**Proof**

The proof is again by induction. Surely since each \( Q_i, i = 1 \ldots k \) vanishes on the set of points \( \pi_i(Z) \), the ideal generated by these polynomials is contained in the polynomial ideal of functions which vanish on \( \pi_i(Z) \). It is shown that a \( P_j, j = 1 \ldots n \) defined in equation (61) of Theorem XVIII is in the ideal generated by the \( Q_j, j = 1 \ldots n \). For \( n = 1 \), \( Q_1 = -P_1 \). As an inductive hypothesis, assume that for \( n < k \), the ideal generated by the \( Q_i, i = 1 \ldots n \) contains \( P_j, j = 1 \ldots n \) and therefore is the ideal of polynomials which vanish on \( \pi_n(Z) \). Note that

\[
Q_k(\bar{y}) = \sum_{\alpha \in \pi_{k-1}(Z)} \Delta^2_{\alpha}(1 + c < \lambda_k, (\pi_{k-1}(\bar{y}) - \alpha) > + c' < \lambda_k, (\pi_{k-1}(\bar{y}) - \alpha) >)^2 \prod_{(\alpha, \gamma) \in \pi_k(Z)} (x_k(\bar{y}) - \gamma)
\]

where \( \Delta_{\alpha} \) is defined as in equation (59) and \( c, c' \) are constants. By equation (63) and the definition of \( \pi_{k-1}(Z) \), \( Q_k \) can be rewritten as \( \sum_{\alpha \in \pi_{k-1}(Z)} \omega_{\alpha} + P \) where \( P \) vanishes on \( \pi_{k-1}(Z) \). Theorem XVIII then completes the proof of this corollary.

**Theorem XX.** On any sufficiently large open domain \( D \) in \( \mathbb{R}^n \) there are positive constants \( M_1, M_2 \ldots M_n \) such that the function \( U(Y) \) defined in equation (52) is a strict Lyapunov function for the system of differential equations (43). Moreover the Hessian of \( U \) is non-degenerate at each of the equilibrium points \( Z \) of (43) and the index of \( \Pi(U) \) at this point is the same as the index of (43). \( D \) and the constants \( M_i, i = 1 \ldots n \) may be chosen so that \(-\nabla U\) enters \( \partial D \).

**Proof**

The argument of Theorem XVIII and Corollary XIX and the discussion above have established that \( \tilde{T}_i(Y) = -Q_i^2 + \sum_{j, k, i} A_{jk}(Y)Q_jQ_k \) where \( A \) is a matrix of polynomial functions of \( Y \), and \( Q_i \) is the \( i \)th function of the vector field (43). Choose \( D = [l_1, L_1] \times \ldots \times [l_n, L_n] \) to be a box so large so that \( l_i < -2 \min x_i(Z), L_i > 2 \max x_i(Z) \).

The method of proof is to show that there are constants \( M_1, \ldots, M_n \) so that \( \frac{d}{dt} U(Y) = \sum_{i=1}^n M_i \tilde{T}_i(Y) = \sum_{j, k} B_{jk} Q_jQ_k \) is a negative definite function of the functions \( Q_i \) on \( D \). It will follow immediately that \( U(Y) \) is a strict Lyapunov function for
First note that the only term in the sum (52) which depends on \( y_n \) is \( T_n \). Moreover,

\[-y_n \partial U / \partial y_n = -y_n \partial T_n / \partial y_n < 0,\]

when \( y_n = l_n \) or \( y_n = L_n \), by the choice of \( l_n, L_n \), since \( y_n T_n / \partial y_n \) evaluated at each of these points is a sum of products of positive values with non-negative values at least one of which is positive by Lemma XIII-iv. To construct a \( U \) whose derivative along trajectories is negative definite we inductively apply Gershigorin's criterion starting with \( i = n \) and working downwards. Choose the \( \mu = 1 \) in (56) and choose \( M_n = 1 \). Since \( \tilde{T}_n(Y) = -Q_n^2 + \sum_{j,k < n} A_{jk}(Y)Q_jQ_k \), equation (56) is satisfied for the index \( i = n \). Suppose \( M_n, \ldots, M_{n-k} \) have already been chosen so that if \( \frac{d}{dt} \sum_{i=n-k}^{n} M_i T_i(Y) = \sum_{ij} A_{ij}^{(n-k)}Q_iQ_j \) then

\[(64) \quad A_{ii}^{(n-k)} + \sum_{j \neq i} |A_{ij}^{(n-k)}| < -1\]

for \( i \geq n - k \) and \( -y_j \sum_{i=n-k}^{n} M_i \partial T_i / \partial y_j < 0 \) for \( j \geq n - k \) and \( y_j = l_j \), or \( y_j = L_j \).

\[
\sum_{ij} A_{ij}^{(n-k)}Q_iQ_j = \sum_{i,j \neq (n-k-1)} A_{ij}^{(n-k)}Q_iQ_j + \sum_j \left( A_{(n-k-1)j}^{(n-k)} + A_{(n-k-1)j}^{(n-k)} \right) Q_j Q_{n-k-1}.
\]

Since the \( A_{ij}^{(n-k)}(Y) \) are polynomials in the variables \( y_1, \ldots y_n \), they are bounded on \( D \). Hence, for \( K > 0 \) (sufficiently large) it follows that

\[(65) \quad -K + \max_{D} A_{(n-k-1)(n-k-1)}^{(n-k)}(Y) + \sum_{j \neq n-k-1} \max_{D} |A_{j(n-k-1)}^{(n-k)}| \leq -1.\]

Also, note that the only terms in the sum dependent on \( y_{n-k-1} \) are the functions which are multiples of \( T_i \), for \( i \geq (n-k-1) \). As was observed for \( T_n \) above,

\[-y_{n-k-1} \partial T_{n-k-1} / \partial y_{(n-k-1)} < 0\]

if

\[y_{n-k-1} = l_{n-k-1}, \text{ or } y_{n-k-1} = L_{n-k-1} \text{ and } Y \in D.\]

It follows that for \( K' > 0 \) sufficiently large, and \( y_{n-k-1} = l_{n-k-1} \) or \( y_{n-k-1} = L_{n-k-1} \),

\[(66) \quad -y_{n-k-1} \left( \sum_{i=n-k}^{n} M_i \partial T_i / \partial y_{n-k-1} + K' \partial T_{n-k-1} / \partial y_{(n-k-1)} \right) < 0.\]

Choose \( M_{n-k-1} \) to be the maximum of \( K \) and \( K' \). This choice can be made consistently for each \( k \leq n \) at each inductive step without any modification in the values \( M_j \), for \( j \geq n - k \). This is because \( \tilde{T}_i \) contains no terms \( A_{ki}(Y)Q_kQ_l \), for \( k > i \) and \( l > i \) and only the diagonal term \(-Q_i^2\) for \( k = i \) or \( l = i \). Hence any choice of values of the \( M_i \)'s which satisfy (64) for matrices \( A^k \), and index \( i \) satisfies (64) for \( A^k, k' \leq k \).
It follows from the inductive choice of $M_i, i = 1 \ldots n$, that $U$ is a strict Lyapunov function for the flow (43). The estimate $\frac{d}{dt} U = \langle \nabla U, Q \rangle \leq -\|Q\|^2$, shows that $H(U)$ is non-degenerate at the critical points $Z$. In fact, for every $\epsilon > 0$ there is a neighborhood of each equilibrium point $E$ of (43) such that

$$(X^T - E^T) [H(U(E))DQ(E)] (X - E) \leq -\frac{(1 - \epsilon)\|X - E\|^2}{\|DQ^{-1}\|^2}$$

and since (43) is hyperbolic $DQ(E)$ and therefore $H$ is non-degenerate at each of the equilibria $E$. As in the argument of Theorem X the agreement of the indices of (43) and $-H(U)$ at $E$ follows from the non-degeneracy of the Hessian of $U$ at $E$.

It would be nice to have simple a-priori bounds for the permissible values for the $M_i$. This is an important open problem.

The following analogue of Theorem X is stated as an example of what might be proved for this class of systems.

**Theorem XXI.** Choose a $C^\infty$ Frame of Vector Fields $F_2, \ldots F_n$ such that for each $i, \langle F_i, \nabla U \rangle = 0$ and $F_i$ where $U$ is defined as in equation (52) and $F_i$ vanishes at the critical points of $U$. Choose arbitrary $C^\infty$ scalar functions $f_i, i = 2, \ldots n$ and choose $A(X)$ to be a $C^\infty$ Riemannian metric, and choose initial data in $D$ where $D$ is defined as in Theorem XX.

Then the system of differential equations.

$$(67) \quad \dot{X} = -A(X)\nabla U + \sum_{i=2}^{n} f_i(X)F_i(X)$$

Either converges to an equilibrium point of (43) or the system exits $D$. In all cases, the system must converge to a set of equilibria. In $D$, moreover, the index of each of the critical points of this system is the same as the index of the corresponding equilibrium point of (43). If $f_i = 0$, and $A(X)$ is diagonal on $\partial D$ then any trajectory which starts in $D$ converges to an equilibrium point of (43).

**Proof**

The fact that all trajectories of (67) are convergent follows from the fact that $U$ is a strict Lyapunov function of (67) is bounded below and $\lim_{\|X\| \to \infty} U = \infty$. The proof of the assertion about the agreement of indices is analogous to the corresponding statements in Theorem XIV and so is omitted. If the final condition holds then $\partial D$ is an entrance set so the conclusion follows. The rest of the conclusions follow from the fact that $U$ is a strict Lyapunov function for (67) and that $U$ has critical points in $D$ which match those of (43).

**Appendix**

**Proof of Proposition IX**

i. The form of $\dot{V} = X^T(A^T S + SA)X$ is invariant over changes of coordinates. We
therefore change variables from the \( X \) to \( U = (u_1, \ldots, u_n) \) where the components \( U_\sim = (u_1, \ldots, u_r) \) are coefficients relative to vectors \( U_\sim \), which span \( A_\sim \), and the remainder of the coefficients \( U_\sim = (u_{r+1}, \ldots, u_n) \) are coefficients relative to vectors \( U_{r+1}, \ldots, U_n \) which span \( A_\sim \). In the new coordinate system the system \( \dot{X} = AX \) may be written:

\[
\dot{U} = \begin{bmatrix} A^+ & 0 & U_+ \\ 0 & A^- & U_- \end{bmatrix}
\]

where all of the eigenvalues of \( A^+ \) have positive real parts and all of the eigenvalues of \( A^- \) have negative real parts. Under these conditions there exists unique negative definite \( S^+ \) and positive definite \( S^- \) such that

\[
S^+ A^+ + A^+ T S^+ = -I_r
\]

and

\[
S^- A^- + A^- T S^- = -I_{n-r}
\]

where \( I_k \) denotes the \( k \) dimensional identity matrix (For proof see Hale (1980), for example). Then,

\[
S' = \begin{bmatrix} S^+ & 0 \\ 0 & S^- \end{bmatrix}
\]

has been constructed so that \( U^T S' U \) is a strict Lyapunov function for the system (68). Transform the system back to the \( X \) coordinates and also transform the quadratic form \( S' \) to the \( X \) coordinates to obtain a \( S \) which satisfies condition (i).

ii. Given a \( X^T S X \) such that \( X^T (A^T S + SA) X \) is negative definite, transform the system \( \dot{X} = AX \) to \( U \) coordinates \((u_1, \ldots, u_n)\), used in i. above, and the quadratic form \( S \), to the form \( U^T S' U \). Then,

\[
U^T S' U = [U_+, U_-] \begin{bmatrix} S^{++} & R \\ R^T & S^{--} \end{bmatrix} \begin{bmatrix} U_+ \\ U_- \end{bmatrix}
\]

Since \( \frac{d}{dt} U^T S' U \bigg|_{U_+} = U_+^T (S^{++} A^+ + A^+ T S^{++}) U_+ \) is negative definite, and the eigenvalues of \( A^+ \) all have positive real parts, \( S^{++} \) must be negative definite. Also since \( \frac{d}{dt} U^T S' U \bigg|_{U_-} = U_-^T (S^{--} A^- + A^- T S^{--}) U_- \) is negative definite, \( S^{--} \) must be positive definite as shown by Hale (1980). The definiteness of \( S' \) can now be computed by computing the signs of the determinants of the principal minors of \( S' \). The first \( r \) minors are principal minors of \( S^{++} \), a negative definite matrix, and start out negative and alternate in sign. The remaining \( n - r \) principal minors are matrices of the form

\[
\begin{vmatrix} S^{++} & R_k \\ R_k^T & S^{--} \end{vmatrix}
\]

where \( S^{--}_k \) is the \( k \)th principal minor of \( S^{--} \),

\[37\]
It follows that the determinant of this minor takes on the value,
\[ \det(S^{*+})\det(S_k^{*+} - R_k^T(S^{*+})^{-1}R_k), \]
(Rao, 1973) the second matrix being the sum of a positive definite and positive semi-definite matrix and, therefore, positive definite. It follows that the second determinant in the product is always positive, and hence that there are no more alternations in sign. It follows that \( S^* \) must have \( r \) negative eigenvalues and \( n - r \) positive eigenvalues which completes the proof of \( \text{ii} \). The proof that the number of alternations in sign of the determinants of the principal minors of a symmetric matrix determines the dimension of the eigenspaces whose eigenvectors have positive and negative real parts is standard in linear algebra. See for example, Gantmakher (1959).

\( \text{iii} \). By hypothesis \( \frac{d}{d\lambda} X^T S X = X^T (B^T S + SB) X \) is negative definite, and therefore the matrix \( B^T S + SB = -P^T P \) for some non-singular matrix \( P \). Multiply both sides by \( (P^T)^{-1} P^{-1} \) to obtain a matrix \( B' \), similar to \( B \), and a quadratic form \( S' \), similar to \( S \) such that \( (B')^T S' + S' B = -I \). Now choose an orthogonal matrix \( Q \) such that \( Q^T S' Q = D \) for some non-singular diagonal matrix \( D \). Pre- and post-multiply by \( Q^T Q = I \) to obtain a matrix \( B'' \) similar to \( B' \) such that

\[ D(B'') + (B'')^T D = -I. \]

One now solves equation (69) by expanding and equating coefficients to show that any \( B'' \) of the form, \( B'' = D^{-1}(-I + E) \) is a solution to (69) where \( E \) is skew-symmetric \( (E^T = -E) \). Any such matrix \( B'' \) must be hyperbolic. If \( B'' \) were singular it would follow that there is a non-zero vector \( Y \), such that \( B'' Y = 0 \), but then as \( D^{-1} \) is non-singular it follows that \( (-I + E) Y = 0 \). Therefore \( Y = EY \), but since \( E \) is skew-symmetric,

\[ 0 < \|Y\|^2 = Y^T Y = Y^T EY = 0 \]

which is a contradiction. If \( B'' \) had a pure imaginary eigenvalue, then there would be a pair of non-vectors \( Y, Z \) such that \( B'' Y = aZ \) and \( B'' Z = -aY \) for some non-zero constant \( a \). We obtain the pair of equations:

\[ \begin{align*}
Y &= EY - aDZ \\
Z &= EZ + aDY
\end{align*} \]

Taking the dot product of (70) with \( Y \) and (71) with \( Z \) we obtain

\[ \begin{align*}
0 < \|Y\|^2 &= -aY^T DZ \\
0 < \|Z\|^2 &= aZ^T DY,
\end{align*} \]

since \( E \) is skew-symmetric. Since \( D \) is diagonal the left hand side of (72) is the negative of (73). This contradiction shows that \( B'' \) and hence \( B \) has no eigenvectors with zero real part.
One can now show that $B'' = D^{-1}(-I + E)$ has the same number of eigenvectors with positive real parts as does $-D$. Consider the function $B''(\epsilon) = -D^{-1} + \epsilon D^{-1} E$. When $\epsilon = 0$, $B''(0) = -D^{-1}$, $B''(1) = B''$. There are $n$ distinct eigenvalues of $B''(0)$ counting multiplicity which are the entries of $-D^{-1}$. $D^{-1}$ has the same number of positive and negative eigenvalues as does $D$. Each of these eigenvalues are a continuous function of $\epsilon$. If one of the number of eigenvalues with negative real part was different from $-D^{-1}$, by continuity at least one of the eigenvalues of $B''(\epsilon')$ must have zero real part for some $\epsilon', 0 < \epsilon' < 1$. But $B''(\epsilon') = D^{-1}(I + E')$, where $E' = \epsilon' E$ is skew-symmetric. No such matrix may have an eigenvalue with zero real part. This contradiction establishes the result.

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