Interference alignment: capacity bounds and practical algorithms for time-varying channels

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Dissertation

INTERFERENCE ALIGNMENT:  
CAPACITY BOUNDS AND PRACTICAL ALGORITHMS  
FOR TIME-VARYING CHANNELS

by

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MICHAEL FARAG

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ABSTRACT

Wireless communication systems are becoming essential to everyday life. Modern network deployments and protocols are struggling to keep up with these growing demands, due to interference between devices. The recent discovery of interference alignment has shown that, in principle, it may be possible to overcome this interference bottleneck in dense networks. However, most theoretical results are limited to very high signal-to-noise ratios (SNRs) and practical algorithms have only developed for interference alignment via multiple antennas. In this thesis, we develop new capacity bounds for the finite SNR regime by taking advantage of time-varying channel gains. We also explore practical algorithms for parallel single-antenna interference channels, which could arise due to orthogonal frequency-division multiplexing (OFDM).

From the theoretical side, we study the phase-fading Gaussian interference channel. We approximate the capacity region in the very strong interference regime to
within a constant gap. Our coding schemes combines ideas from ergodic and lattice interference alignment. On the practical side, we develop a matching algorithm for pairing together sub-channels for alignment. This algorithm relies on the concept of maximum weight matching from graph theory. Simulations demonstrate that this algorithm outperforms classical techniques when the network is interference limited.
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<td>Channel State Information</td>
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<td>CSIR</td>
<td>Channel State Information at Receiver</td>
</tr>
<tr>
<td>CSIT</td>
<td>Channel State Information at Transmitter</td>
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<tr>
<td>DoF</td>
<td>Degrees-of-Freedom</td>
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<tr>
<td>GDoF</td>
<td>Generalized Degrees-of-Freedom</td>
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<tr>
<td>INR</td>
<td>Interference-to-Noise Ratio</td>
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<tr>
<td>MIMO</td>
<td>Multiple-Input-Multiple-Output</td>
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<td>OFDM</td>
<td>Orthogonal Frequency-Division Multiplexing</td>
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<td>SINR</td>
<td>Signal-to-Interference-and-Noise Ratio</td>
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<td>SISO</td>
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Chapter 1

Introduction

High demand for wireless services has pushed wireless providers to build smaller and denser cells such as the pico cell and femto cell. These kinds of cells have small coverage radii. Their layout is also haphazard compared to the structured layout of macro cells. In such an environment, interference from neighboring cells becomes one of the major limitations on the performance of the network. The traditional ways of dealing with interference, such as orthogonalization, are sub-optimal. For an orthogonalization technique, each user gets $1/K$ of the channel resources where $K$ is the number of users. However, in [Cadambe and Jafar, 2008] Cadambe and Jafar showed that for a time-varying (or frequency selective) $K$-user interference channel, each user can achieve half its interference-free rate at asymptotically high signal-to-noise ratio (SNR). Nazer et al. in [Nazer et al., 2012] show that each user can get half the interference-free rate at any SNR value. These results show that orthogonalization techniques, such as time-division multiplexing (TDM) and frequency-division multiplexing (FDM), are not necessarily optimal for the $K$-user wireless networks. Understanding the optimal way to deal with interference in wireless networks can potentially lead to huge improvement in these networks’ throughputs.

The $K$-user Gaussian interference channel is a classical model to study wireless communication networks. It consists of $K$ transmitter-receiver pairs (or users). Each transmitter has a message to its receiver. Each receiver sees a linear combination of its desired signal and other transmitters signals, contaminated with additive Gaussian
noise. Figure 1.1 shows an example of the $K$-user interference channel.

![Diagram of K-user wireless interference channel](image)

**Figure 1.1:** $K$-user wireless interference channel

However, the capacity region of the $K$-user Gaussian interference channel is a long-standing open problem. While for $K = 2$ users, the capacity region is now known to within one bit [Etkin et al., 2008], the $K > 2$ case has proven considerably more challenging, due to the possibility of interference alignment [Maddah-Ali et al., 2008, Cadambe and Jafar, 2008]. Part of the difficulty arises from the fact that, for $K > 2$ users, the capacity fluctuates rapidly with respect to the channel gains, ultimately leading to discontinuities in the degrees-of-freedom [Etkin and Ordentlich, 2009, Wu et al., 2015, Stotz and Bölcskei, 2015]. This seems to place approximating the capacity region beyond the reach of modern techniques.

We will show, in the first part of the dissertation, that, for *time-varying* Gaussian interference channels, it is possible to derive constant-gap capacity approximations, despite the aforementioned challenges. Our coding strategy combines ideas from the Cadambe-Jafar alignment scheme for time-varying channels [Cadambe and Jafar, 2008], ergodic interference alignment [Nazer et al., 2012], and compute-and-forward [Nazer and Gastpar, 2011].
We will use the generalized degrees-of-freedom (GDoF) as a guideline when studying the capacity of the Gaussian interference channel. GDoF measures the performance of the network at asymptotically high SNR, parameterized by the interference power level.

In the second part of the dissertation, we will provide an opportunistic algorithm that tries to achieve interference alignment over the interference channels. The algorithm is inspired by the ergodic alignment scheme. We will show numerical results of the performance of our algorithm and compare it to traditional techniques.

1.1 Background

Considerable progress was made on the capacity region of the two-user interference channel in the 1970s and 1980s, beginning with the inner and outer bounds of Ahlswede [Ahlswede, 1974], Sato [Sato, 1977], and Carleial [Carleial, 1978]. The capacity region of the Gaussian two-user interference channel was characterized in the very strong regime by Carleial [Carleial, 1975] and in the strong regime by Sato [Sato, 1981] and Han and Kobayashi [Han and Kobayashi, 1981], with the latter also proposing an achievable rate region for all regimes. Costa and El Gamal generalized this strong regime capacity result from Gaussian to discrete memoryless channels [Costa and El Gamal, 1987] and also determined the full capacity for a class of semi-deterministic interference channels [Costa and El Gamal, 1982].

The 2000s saw a renewed interest in the two-user Gaussian interference channel, beginning with the one-bit capacity region approximation due to Etkin et al. [Etkin et al., 2008]. Afterwards, three groups [Motahari and Khandani, 2009, Shang et al., 2009, Annapureddy and Veeravalli, 2009] independently and concurrently discovered that, when the interference is sufficiently weak, treating interference as noise is optimal. Subsequent efforts yielded constant-gap capacity approximations for several two-

The discovery of interference alignment by Motahari et al. [Maddah-Ali et al., 2008] and Cadambe and Jafar [Cadambe and Jafar, 2008] sparked several efforts to characterize the $K$-user capacity region. Using variations on the linear Cadambe-Jafar scheme, the degrees-of-freedom region has been characterized for a large class of time-varying (or frequency-selective) interference networks [Cadambe and Jafar, 2009, Gou and Jafar, 2010, Gou et al., 2012, Ke et al., 2012, Shomorony and Avestimehr, 2014]. There is a rich body of work on linear alignment, and we refer interested readers to [Jafar, 2011b] for a survey.

For static interference channels, linear alignment often corresponds to an overconstrained system of linear equations. However, as first noted by Bresler et al. [Bresler et al., 2010] for many-to-one interference channels, it is possible to align interference on the \textit{signal scale} through the use of lattice codebooks. In [Sridharan et al., 2008], the authors used lattice alignment to approximate the capacity region of symmetric interference channels to within a constant gap in the very strong regime. Lattice alignment was subsequently used by Motahari et al. [Motahari et al., 2014] to show that $K/2$ degrees-of-freedom are achievable (up to a set of channel matrices with measure zero). As noted above, the degrees-of-freedom are discontinuous with respect to the channel gains [Etkin and Ordentlich, 2009, Wu et al., 2015, Stotz and Bölcskei, 2015]. A similar phenomenon appears in the best known approximate capacity characterizations for the two-user X channel [Niesen and Maddah-Ali, 2013] and $K$-user
symmetric interference channel [Ordentlich et al., 2014], and an “outage set” is needed as part of the characterization. We also note that, for $K$-user interference channels, Geng et al. [Geng et al., 2015] have developed sufficient conditions on when treating interference as noise approximates the capacity to within a constant gap [Geng et al., 2015].

Ergodic alignment has been used as a building block to develop achievable strategies for dense interference networks [Jafar, 2011a, Johnson et al., 2011], multi-hop interference networks [Niesen, 2011, Niesen et al., 2013], and multiple-access wiretap channels [Bassily and Ulukus, 2012]. We note that our proof strategy closely resembles the computation alignment scheme proposed by Niesen et al. [Niesen et al., 2013] for approximating the capacity of time-varying multi-hop interference networks.

Recently, there has been an increasing interest in practical approaches to interference alignment. Decentralized algorithms that achieve interference alignment over the $K$-user multiple-input-multiple-output (MIMO) interference channel were proposed in [Gomadam et al., 2008, Peters and Heath, 2009]. In addition, the performance of such algorithms in wireless cellular networks is studied in [Tresch and Guillaud, 2009a, Tresch and Guillaud, 2009b]. Exploiting the backhaul that connects the base stations (BS) in these networks, Ntranos et al. show in [Ntranos et al., 2015] that achieving 1/2 DoF per link is possible over one channel use. The problem of imperfect channel state information (CSI) was addressed in [Nosrat-Makouei et al., 2011] for MIMO interference channels. We will address the problem of huge delays required by most of interference alignment schemes. We develop a scheme that requires relatively few channel realizations and compare its performance to ergodic alignment.
1.2 Contributions

In this section, we go over our main contributions. For our theoretical contributions, we study the fundamental limits of the phase-fading Gaussian interference channel as an example of the time-varying interference channel. On the practical side, we provide an algorithm inspired by ergodic alignment that can be applied on cellular systems.

1.2.1 Phase-Fading Interference Channels in the Very Strong Regime

We determine the capacity region up to a constant gap for the $K$-user phase-fading Gaussian interference channel in the very strong interference regime. Table 1.1 summarizes the results in this area. Note that, for general (not symmetric) static channels, only the DoF are known [Motahari et al., 2014]. Achieving interference alignment in a static general $K$-user interference channel is usually an overconstrained problem. We relax the problem by allowing the phases of the channel gains to be time-varying, holding the magnitudes fixed across time. This makes parameterizing the interference

<table>
<thead>
<tr>
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<tr>
<td>General Static 2-user</td>
<td>Known [Carleial, 1975]</td>
</tr>
<tr>
<td>General Static $K$-user</td>
<td>DoF only [Motahari et al., 2014]</td>
</tr>
<tr>
<td>Symmetric Static $K$-user</td>
<td>Known [Sridharan et al., 2008]</td>
</tr>
<tr>
<td>General time-varying phases $K$-user</td>
<td>Constant Gap This Dissertation</td>
</tr>
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Table 1.1: Summary of results on Gaussian interference channels.
regimes more clear [Sankar et al., 2008]. To characterize the capacity region up to a constant gap, our solution is built up of the following components.

Alignment Scheme

We develop an alignment scheme for the $K$-user interference channel based on lattice codes and ergodic alignment. In this scheme, we exploited the phase variation by selecting the channel realizations to transmit over. We also carefully fixed the beamforming vectors, so that interfering codewords are in integer linear combination at all receivers simultaneously. If all transmitters use the same lattice codebook, all interfering codewords will appear as one codeword due to the property of lattice codes that an integer linear combination of lattice codewords is a lattice codeword.

Constant Gap Result

By taking advantage of the fixed channel magnitudes, we applied our alignment scheme to the $K$-user Gaussian interference channel. By applying the 2-user upper bound in [Etkin et al., 2008] to the channel, we showed that our achievable rate is within a constant gap from the upper bound.

1.2.2 Matching Alignment

Unlike the work we discussed above, here we develop an algorithm inspired by ergodic alignment [Nazer et al., 2012] and study its performance. We propose the matching alignment algorithm as a practical approach to the $K$-user interference channels. We assume that each user is equipped with a single antenna. This makes achieving alignment challenging. Existing schemes require infinitely many channel realizations to achieve perfect alignment [Cadambe and Jafar, 2008, Nazer et al., 2012]. Here, we give up on perfectly achieving alignment in exchange for using a finite number of channel realizations. Unlike ergodic alignment, we do not insist on waiting indefinitely
to find a perfect match for any given channel realization. We try to find the best way to pair available channel realizations using \textit{maximum weight matching} techniques from graph theory.

In general, this pairing technique will not eliminate interference entirely. Therefore, the residue will be treated as noise. We implement the matching alignment algorithm and compare its performance to that of the classical techniques as well as to that of ergodic alignment. Our algorithm outperforms classical techniques when the interfering signals are as strong as the desired signal. Detailed discussion of the results will be provided later.

1.3 Outline

Throughout the dissertation we will use the following notation. We will denote column vectors with lowercase boldface letters (e.g., $\mathbf{x} \in \mathbb{C}^K$) and matrices using boldface uppercase letters (e.g., $\mathbf{H} \in \mathbb{C}^{K \times K}$). Let $\mathbf{x}^T$ denote the transpose of $\mathbf{x}$ and let $\text{diag}(\mathbf{x})$ as well as $\text{diag}(\mathbf{x}^T)$ denote the diagonal matrix formed by placing the elements of $\mathbf{x}$ along the diagonal. For a real scalar $x$, let $\lceil x \rceil$ denote its ceiling (i.e., the smallest integer greater than or equal to $x$), $\lfloor x \rfloor$ denote its floor (i.e., the largest integer less than or equal to $x$), and $\lfloor x \rceil$ denote its rounding to the nearest integer (with ties rounded up). For a complex scalar $x$, let $|x|$ denote its magnitude and $\angle x$ denotes its phase. For a complex vector $\mathbf{x}$, let $\|\mathbf{x}\|$ denote its Euclidean norm. All logarithms are taken with respect to base-2 and we define $\log^+(x) = \max(\log(x), 0)$. Define $\mathbf{I}$ to be the identity matrix and $\mathbf{1}$ to be the all-ones vector.

The rest of the dissertation is organized as follows. In Chapter 2, we review relevant prior work on interference channels. In Chapter 3, we discuss the phase-fading Gaussian interference channel problem and give our scheme as well as our constant gap result for the very strong regime. Matching alignment and its performance will
be discussed in Chapter 4 and we discuss our conclusion and future work in Chapter 5.

1.3.1 Key Assumptions

In Table 1.2, we will list the key assumptions made in this thesis. Part 1 refers to fundamental limits of phase-fading channels in the very strong regime and Part 2 refers to matching alignment algorithm.

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<tr>
<th></th>
<th>Part 1</th>
<th>Part 2</th>
</tr>
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<tbody>
<tr>
<td>CSI available at transmitters (CSIT)</td>
<td>Perfect</td>
<td>Perfect</td>
</tr>
<tr>
<td>CSI available at receivers (CSIR)</td>
<td>Perfect</td>
<td>Perfect</td>
</tr>
<tr>
<td>Magnitudes of channel gains</td>
<td>Fixed across time</td>
<td>Time-varying</td>
</tr>
<tr>
<td>Phases of channel gains</td>
<td>Time-varying</td>
<td>Time-varying</td>
</tr>
<tr>
<td>Number of transmitter antennas</td>
<td>One</td>
<td>One</td>
</tr>
<tr>
<td>Number of receiver antennas</td>
<td>One</td>
<td>One</td>
</tr>
<tr>
<td>Required delay</td>
<td>Infinite</td>
<td>Finite</td>
</tr>
</tbody>
</table>

Table 1.2: Summary of key assumptions.
Chapter 2

Background Review

In this chapter, we will review some essential problems that will serve as background for our work. In the first section, we discuss the 2-user Gaussian interference channel. It will serve as a platform for the discussion of the phase-fading $K$-user Gaussian interference channel in the next chapter. Next, we will review the symmetric $K$-user Gaussian interference channel and achieving its sum capacity up to an outage set using nested lattice codes. We will apply a generalized version of the decoding technique that was developed in [Ordentlich et al., 2014]. Finally, we will give a brief overview of the ergodic alignment concepts which will be used in both the phase-fading part and the matching alignment one.

2.1 2-User Gaussian Interference Channel

In this section, we review prior work on the 2-user Gaussian interference channel. The capacity region of this channel was approximated up to within one bit in [Etkin et al., 2008]. Each transmitter has a message to communicate to its corresponding receiver. See Figure 2.1 for a block diagram of this channel.

Fix a block length $T$. Transmitter $k$ has a message $\omega_k \in \{1, 2, \ldots, 2^{TR}\}$ and an encoder $\mathcal{E}_k$ that maps $\omega_k$ into a sequence of length $T$ of channel inputs $x_k[1], \ldots, x_k[T]$ that satisfies the power constraint, that is, $\mathcal{E}_k : \{1, 2, \ldots, 2^{TR}\} \to \mathbb{C}^T$ and

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E} [ |x_k[t]|^2 ] \leq P.$$
The relation between the channel inputs and outputs is given by

\[ y_1[t] = h_{1,1}x_1[t] + h_{1,2}x_2[t] + z_1[t] \]  \hspace{1cm} (2.1)
\[ y_2[t] = h_{2,1}x_1[t] + h_{2,2}x_2[t] + z_2[t] \]  \hspace{1cm} (2.2)

where, for \( \ell, k \in \{1, 2\} \), \( \{x_k[t]\} \in \mathbb{C} \) are the channel inputs, \( \{y_{\ell}[t]\} \in \mathbb{C} \) are the channel outputs, \( \{h_{\ell,k}\} \in \mathbb{C} \) are the channel gains and assumed to be fixed during the transmission time and \( \{z_{\ell}[t]\} \) are the noise terms which are i.i.d. \( \mathcal{CN}(0, 1) \).

Receiver \( \ell \) observes a sequence of length \( T \) of channel outputs according to (2.1 - 2.2). The received sequence is mapped to an estimated message \( \hat{\omega}_\ell \) through a decoding function \( D_\ell : \mathbb{C}^T \rightarrow \{1, 2, \ldots, 2^{TR}\} \). The probability of error is

\[ P_e = \mathbb{P} (\{\hat{\omega}_1 \neq \omega_1\} \cup \{\hat{\omega}_2 \neq \omega_2\}) . \]

A symmetric rate \( R \) is achievable if there exists a sequence of encoding and decoding functions such that the probability of error \( P_e \) goes to zero as the block length \( T \) goes to infinity. The symmetric capacity \( C_{\text{sym}} \) of the 2-user interference channel is supremum of all achievable symmetric rates.

The solution of this problem is insightful for two reasons: First, it can be used as
a building block for generating upper bounds on the capacity region of the $K$-user Gaussian interference channel. Second, it gives intuition on how to deal with interference depending on its relative strength. Specifically, it proposes five interference regimes and reveals the optimum strategy for each regime.

We now review these interference regimes for the special case of symmetric gains, that is, $h_{1,1} = h_{2,2} = h_d$ and $h_{1,2} = h_{2,1} = h_c$. Define

$$\text{SNR} = |h_d|^2 P$$
$$\text{INR} = |h_c|^2 P,$$

as the signal-to-noise ratio and the interference-to-noise ratio, respectively. Let the interference level $\alpha$ be

$$\alpha = \log \frac{\text{INR}}{\text{SNR}}.$$

This parameter captures the power of the interfering signal relative to that of the desired signal, it can also be thought of as the ratio of the interference power in dB to the desired signal power in dB.

The generalized degrees-of-freedom (GDoF) was introduced in [Etkin et al., 2008] and measures what fraction of the interference-free channel capacity is achieved at a given $\alpha$. The symmetric GDoF is defined as

$$d_{\text{sym}}(\alpha) = \lim_{\text{SNR} \to \infty} \frac{C_{\text{sym}}}{\log (1 + \text{SNR})}. \quad (2.3)$$

Note that the symmetric capacity $C_{\text{sym}}$ is a function of both $\text{SNR}$ and $\text{INR}$ or, equivalently, a function of $\text{SNR}$ and $\alpha$. The limit in (2.3) is taken as $\text{SNR}$ approaches infinity while $\alpha$ is held fixed. This is why the symmetric GDoF $d_{\text{sym}}(\alpha)$ is a function of $\alpha$ only.
For the two-user Gaussian interference channel, the symmetric GDoF is

\[
d_{\text{sym}}(\alpha) = \begin{cases} 
1 - \alpha & 0 \leq \alpha < \frac{1}{2} \\
\alpha & \frac{1}{2} \leq \alpha < \frac{2}{3} \\
1 - \frac{\alpha}{2} & \frac{2}{3} \leq \alpha < 1 \\
\frac{\alpha}{2} & 1 \leq \alpha < 2 \\
1 & \alpha \geq 2.
\end{cases}
\] (2.4)

It was determined by [Etkin et al., 2008] and used as a guideline for approximating the capacity region to within one bit per user, where the five distinct intervals represent five distinct interference regimes. Figure 2·2 summarizes these results.

![Figure 2·2: Generalized degrees-of-freedom (GDoF) for the symmetric two-user Gaussian interference channel as established by [Etkin et al., 2008]. When the interference level \( \alpha \) is low, then treating interference as noise (TIN) is optimal. When \( \alpha \) is high, it is optimal to decode the interference along with the desired codeword. Time-division multiple-access (TDMA) only achieves 1/2 GDoF, irrespective of \( \alpha \).](image_url)
Noisy Regime, $\alpha \in [0, \frac{1}{2})$

It was proven in [Motahari and Khandani, 2009, Shang et al., 2009, Annapureddy and Veeravalli, 2009] that it is optimal in the interval $\alpha \in [0, \frac{1}{3}]$ to treat interference as noise. That is, each receiver will only try to decode its own codeword while treating the other codeword as noise. For $\alpha \in (\frac{1}{3}, \frac{1}{2})$, treating interference as noise yields a constant gap result.

Weak and Moderately Weak, $\alpha \in [\frac{1}{2}, 1)$

It was shown in [Etkin et al., 2008] that using a special Han-Kobayashi scheme [Han and Kobayashi, 1981], the capacity region can be achieved to within one bit per user. In this scheme, each user splits its message into two messages, private and common. Each message is then encoded independently. The power of the private codeword is chosen so that its interference at the unintended receiver has the same effective power as noise. Each receiver jointly decodes the two common messages and its private one. The private message from the other user is treated as noise.

Strong Regime, $\alpha \in [1, 2)$

In this regime, each receiver is always capable of completely decoding the interfering message, since it has a stronger channel from the interfering user. Thus, each receiver can be viewed as a two-user multiple-access channel and the capacity region is the intersection of the two multiple-access regions [Sato, 1981, Han and Kobayashi, 1981].

Very Strong Regime, $\alpha \in [2, \infty)$

Carleial showed that successive interference cancellation is optimal [Carleial, 1975] in this regime. That is, the interference is so strong that it is optimal to decode it treating the desired signal as noise. Once the interference is decoded, it can be canceled from the received signal. Afterwards, the receiver can have a noisy observation of its own
signal but without the interference. This makes each user able to use the whole channel and achieve its interference-free capacity.

The approximate symmetric capacity to within one bit is given by [Etkin et al., 2008]

\[
C_{\text{sym}} \approx \begin{cases} 
\log \left( \frac{\text{SNR}}{\text{INR}} \right) & 0 \leq \alpha \leq \frac{1}{2} \\
\log (\text{INR}) & \frac{1}{2} \leq \alpha \leq \frac{2}{3} \\
\log \left( \frac{\text{SNR}}{\sqrt{\text{INR}}} \right) & \frac{2}{3} \leq \alpha \leq 1 \\
\log (\sqrt{\text{INR}}) & 1 \leq \alpha \leq 2 \\
\log (\text{SNR}) & \alpha \geq 2.
\end{cases}
\] (2.5)

2.2 K-User Symmetric Gaussian Interference Channel

In [Ordentlich et al., 2014], Ordentlich et al. considered the K-user symmetric interference channel. They approximated the capacity of the channel up to an outage set using nested lattice codes and the compute-and-forward approach. Inspired by their work, we will introduce a new scheme to communicate over the phase-fading interference channels in the next chapter. In this section, we will briefly discuss the problem of the symmetric Gaussian interference channel and the approach proposed in [Ordentlich et al., 2014]. This will give some insight into the scheme that we will be using in the next chapter.

Consider a Gaussian interference channel with K transmitter-receiver pairs. Assume the channel is symmetric, i.e., all direct gains are equal and all cross-gains are equal. Without loss of generality, we can set the direct gains to 1. The channel output
at the $\ell^{th}$ receiver is

$$y_\ell = x_\ell + g \sum_{k \neq \ell} x_k + z_\ell, \quad (2.6)$$

where $x_k$ is the codeword of transmitter $k$ and must satisfy the power constraint

$$\frac{1}{T} \|x_k\|^2 \leq \text{SNR},$$

$z_\ell \in \mathbb{R}^T$ is the additive white Gaussian noise process and $g \in \mathbb{R}$ is the cross-gain. Note that the channel is real and static, that is, the channel gains do not change with time.

For a fixed block length $T$, transmitter $k$ has a message $\omega_k \in \{1, 2, \ldots, 2^{TR}\}$ and encoding function $E_k$ that maps $\omega_k$ into a sequence of channel inputs $x_k$ that satisfies the power constraint, precisely, $E_k : \{1, 2, \ldots, 2^{TR}\} \to \mathbb{R}^T$. Receiver $\ell$ observes a sequence of channel outputs $y_\ell$. It uses its decoding function $D_\ell$ to map the sequence into an estimated message $\hat{\omega}_\ell$, i.e., $D_\ell : \mathbb{R}^T \to \{1, 2, \ldots, 2^{TR}\}$. The error probability is defined as

$$P_e = \mathbb{P} \left( \bigcup_{\ell \in \{1, \ldots, K\}} \{\hat{\omega}_\ell \neq \omega_\ell\} \right).$$

The symmetric rate $R$ is achievable if there exists a sequence of encoders and decoders such that the probability of error can be made arbitrarily small by choosing the block length $T$ long enough. The symmetric capacity $C_{\text{sym}}$ is the supremum of all achievable symmetric rates.

For the given channel, define

$$\text{INR} = g^2 \text{SNR}$$
to be the interference-to-noise ratio and

\[ \alpha = \frac{\log \text{INR}}{\log \text{SNR}} \]

to be the interference level. Note the parameters \( \text{SNR} \), \( \text{INR} \), and \( \alpha \) play the same role as the corresponding parameters in the 2-user channel. The generalized degrees of freedom is defined as before in (2.3). In [Jafar and Vishwanath, 2010], the GDoF curve of the channel was determined to be

\[
d_{\text{sym}}(\alpha) = \begin{cases} 
1 - \alpha & 0 \leq \alpha < \frac{1}{2} \\
\alpha & \frac{1}{2} \leq \alpha < \frac{2}{3} \\
1 - \frac{\alpha}{3} & \frac{2}{3} < \alpha \leq 1 \\
\frac{1}{K} & \alpha = 1 \\
\frac{\alpha}{2} & 1 < \alpha < 2 \\
1 & \alpha \geq 2.
\end{cases}
\tag{2.7}
\]
In [Ordentlich et al., 2014], Ordentlich et al. developed a new decoding scheme and applied it to the symmetric $K$-user Gaussian interference channel. It is built using nested lattice codes where all transmitters use the same codebook. The idea behind the scheme is to exploit the fact that any integer combination of lattice codewords is itself a lattice codeword. According to (2.6), receiver $\ell$ observes

$$y_\ell = x_\ell + g \sum_{k \neq \ell} x_k + z_\ell.$$ 

Since all codewords belong to the same lattice codebook, the sum $\sum_{k \neq \ell} x_k$ is itself lattice codeword which we will denote by $x_{\ell, \text{int}}$. Therefore, receiver $\ell$ effectively sees only two codewords, $x_\ell$ and $x_{\ell, \text{int}}$.

In principle, we could continue to follow the encoding and decoding techniques that were described above for the two-user interference regimes. However, the usual analysis of joint typicality decoding is not directly applicable to nested lattice codes due to the fact that codewords are only pairwise independent. While recent work [Lim et al., 2016] has characterized an achievable region for joint typicality decoding, [Ordentlich et al., 2014] was able to circumvent this issue via compute-and-forward decoding. Consider the strong interference regime where $g > 1$. In the two-user case, each receiver would apply joint typicality decoding to simultaneously recover the desired and the interfering codewords. For $K > 2$ users, each receiver can obtain its desired and effective interfering codewords by first decoding two integer linear combinations via compute-and-forward [Nazer and Gastpar, 2011]

$$a_{1,1}x_\ell + a_{1,2}x_{\ell, \text{int}}$$

$$a_{2,1}x_\ell + a_{2,2}x_{\ell, \text{int}}$$

and then solve for $x_\ell$. Each linear combination is associated with a computation rate. Each receiver can decode an integer combination if the messages rates are
less than the computation rate between the channel and the equation coefficients vectors. This was established in the framework of compute-and-forward [Nazer and Gastpar, 2011]. Let the best computation rate be $R_{\text{comp},1}$ and the second best be $R_{\text{comp},2}$ ($R_{\text{comp},1} > R_{\text{comp},2}$). The lower of the two computation rates is an achievable symmetric rate,

$$R_{\text{sym}} = R_{\text{comp},2}.\quad (2.8)$$

Using this scheme, the symmetric capacity was approximated up to an outage set. The following theorem gives the upper and lower bounds on the symmetric capacity where the above technique is used in the lower bound in the strong, weak, and moderately weak regimes.

**Theorem 1 (Ordentlich et al., 2014, Theorem 1)** The symmetric capacity of the symmetric Gaussian $K$-user interference channel can be lower and upper bounded as follows:

- **Noisy Interference Regime**, $0 \leq \alpha < \frac{1}{2}$,

  $$\frac{1}{2} \log \left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right) - \frac{1}{2} \log (K - 1) \leq C_{\text{sym}} < \frac{1}{2} \log \left(1 + \frac{\text{SNR}}{1 + \text{INR}}\right) + 1.$$

- **Weak Interference Regime**, $\frac{1}{2} \leq \alpha < \frac{2}{3}$,

  $$\frac{1}{2} \log^+ (\text{INR}) - \frac{7}{2} - \log (K) \leq C_{\text{sym}} \leq \frac{1}{2} \log (\text{INR}) + 1.$$

- **Moderately Weak Interference Regime**, $\frac{2}{3} \leq \alpha < 1$,

  $$\frac{1}{2} \log^+ \left(\frac{\text{SNR}}{\sqrt{\text{INR}}}\right) - c - 8 - \log (K) \leq C_{\text{sym}} \leq \frac{1}{2} \log^+ \left(\frac{\text{SNR}}{\sqrt{\text{INR}}}\right) + 1,$$

  for all channel gains except for an outage set of measure $\mu < 2^{-c}$ for any $c > 0$. 

• **Strong Interference Regime**, $1 < \alpha < 2$,

$$
\frac{1}{4} \log^+ (\text{INR}) - \frac{c}{2} - 3 \leq C_{\text{sym}} \leq \frac{1}{4} \log^+ (\text{INR}) + 1,
$$

for all channel gains except for an outage whose measure is a fraction of $2^{-c}$ of the interval $1 < |g| < \sqrt{\text{SNR}}$, for any $c > 0$.

• **Very Strong Interference Regime**, $\alpha \geq 2$,

$$
\frac{1}{2} \log (1 + \text{SNR}) - 1 \leq C_{\text{sym}} < \frac{1}{2} \log (1 + \text{SNR}).
$$

The proof goes as follows. First, a lower bound on the sum of the two computation rates is developed

$$
R_{\text{comp,1}} + R_{\text{comp,2}} > \frac{1}{2} \log \left(1 + \text{SNR} \left(1 + g^2(K-1)\right)\right) - \frac{1}{2} \log(K-1) - 1.
$$

Recall that the symmetric rate is equal to the second best computation rate. According to the above lower bound, lower bounding $R_{\text{comp,2}}$ is equivalent to upper bounding $R_{\text{comp,1}}$. It turn out that the best computation rate $R_{\text{comp,1}}$ can be upper bounded for all values of the channel gains except for the values that belong to an outage set whose measure is a fraction of $2^{-c}$ of measure of the set of all possible values of $|g|$, for any $c > 0$. This means that for a smaller gap, we have a larger outage set and vice versa.

In the next chapter, we will study the phase-fading channel in the very strong regime. The phase-fading channel resembles the static channel that we discussed above because the phase-fading channel gains have fixed magnitudes across time. However, the two channel models differ due to that fact that the channel gains phases change with time. We will take advantage of the time-varying phases to induce alignment that will help characterize the the capacity region in the very strong regime up to a constant gap.
2.3 Ergodic Alignment

Ergodic alignment, which was developed in [Nazer et al., 2012], is a scheme that achieves interference alignment at finite SNR, consider the $K$-user interference channel with time-varying channel gains. The channel model is given by

$$y[t] = H[t]x[t] + z[t],$$

where $y[t] = [y_1[t], \ldots, y_K[t]]$ and $x[t] = [x_1[t], \ldots, x_K[t]]$ are the channel outputs and inputs at time $t$, respectively, $H[t]$ is the channel gain matrix at time $t$ and $z[t] = [z_1[t], \ldots, z_K[t]]$ is the noise vector whose entries are independent and identically distributed (i.i.d.) drawn from a circularly symmetric complex Gaussian distribution with zero mean and unit variance.

The idea of ergodic alignment is that all users transmit over some channel $H[t_1]$. All transmitters will wait for time slot $t_2$ such that $H[t_2]$ and $H[t_1]$ are complementary. Two channel matrices are complementary if their sum is a diagonal matrix with diagonal entries equal to double those of $H[t_1]$. Therefore,

$$H[t_1] + H[t_2] = 2 \text{diag} (h_{1,1}[t_1], \ldots, h_{K,K}[t_1])$$

where $h_{\ell,k}[t]$ is the entry in the $\ell$th row and $k$th column of $H[t]$. It was shown in [Nazer et al., 2012] that almost all the channel gain matrices can be paired in such manner. All transmitters repeat on $t_2$ what they sent over $t_1$. At this point, receiver $\ell$ has access to two observations of $x_{\ell}[t_1]$, namely

$$y_{\ell}[t_1] = h_{\ell,\ell}[t_1]x_{\ell}[t_1] + \sum_{k \neq \ell} h_{\ell,k}[t_1]x_k[t_1] + z_{\ell}[t_1]$$

$$y_{\ell}[t_2] = h_{\ell,\ell}[t_1]x_{\ell}[t_1] - \sum_{k \neq \ell} h_{\ell,k}[t_1]x_k[t_1] + z_{\ell}[t_2],$$
which is true because $h_{\ell,k}[t_2] = -h_{\ell,k}[t_1]$ for all $k \neq \ell$ and $h_{\ell,\ell}[t_2] = h_{\ell,\ell}[t_1]$ by choice of $H[t_2]$. By adding the two observations, receiver $\ell$ has

$$y_\ell[t_1] + y_\ell[t_2] = 2h_{\ell,\ell}[t_1] + z_\ell[t_1] + z_\ell[t_2],$$

which is an interference-free observation. If the process $\{h_{\ell,k}[t]\}$ has uniform phase and is stationary and ergodic and $\{h_{\tilde{\ell},\tilde{k}}[t]\}$ is independent of $\{h_{\ell,k}[t]\}$ for all $(\tilde{\ell}, \tilde{k}) \neq (\ell, k)$ then receiver $\ell$ can decode if the rate of user $\ell$, denoted by $R_\ell$, is bounded by

$$R_\ell < E \left[ \frac{1}{2} \log (1 + 2|h_{\ell,\ell}|^2P) \right],$$

where $P$ is the average power constraint on the channel input. This rate is more than half the interference-free rate, which leads to $\frac{1}{2}$ DoF. Since $\{h_{\ell,k}\}$ are drawn from continuous distribution, the probability of finding $H[t_2]$ is zero. This is why the channel matrices have to be quantized first and the matching process takes place with respect to the quantized matrices. A detailed analysis of the quantization process is discussed in [Nazer et al., 2012, Chapter III]. One of the main drawbacks of ergodic alignment is its extraordinary delays. The delay required to match every realization scales as $((K-1)P)^{K^2/2}$ which renders the ergodic alignment scheme far from practical.
Chapter 3

Phase-Fading Gaussian Interference Channel in the Very Strong Regime

In this chapter, we will consider the phase-fading Gaussian interference channel. This is a special case of Gaussian interference channels where the channel gain magnitudes are constant across time while the phases are time-varying. We develop a coding scheme and apply it to the phase-fading channel and then characterize the achievable rate region of our scheme in the very strong regime. We show that we achieve the capacity region of the $K$-user channel in the very strong regime up to constant gap. The rest of the chapter is organized as follows. In Section 3.1, we give a formal problem statement. In Section 3.2, we summarize previous work on the very strong interference regime, state our main result, and provide a motivating example. Section 3.3 describes the channel quantization process used in the ergodic alignment scheme. Finally, we present our achievable strategy for $K = 3$ users in Section 3.4 and for $K > 3$ users in Section 3.5.

3.1 Problem Statement

We now provide necessary definitions for a time-varying $K$-user Gaussian interference channel. See Figure 3.1 for a block diagram.

Definition 1 (Messages) Each transmitter (indexed by $k = 1, \ldots, K$) has a message $\omega_k$ that is generated independently and uniformly over $\{1, 2, \ldots, 2^{R_k}\}$ for some rate $R_k \geq 0$. 
Definition 2 (Encoders) Each transmitter has an encoding function

\[ E_k : \{1, 2, \ldots, 2^{TR_k}\} \rightarrow \mathbb{C}^T \]

that maps its message \( \omega_k \) into a sequence of \( T \) complex-valued channel inputs

\[ x_k[1], \ldots, x_k[T] \]

satisfying an average power constraint

\[ \frac{1}{T} \sum_{t=1}^T |x_k[t]|^2 \leq P. \]

Definition 3 (Channel Model) The channel output at time \( t \) at receiver \( \ell \) is

\[ y_{\ell}[t] = \sum_{k=1}^K h_{\ell,k}[t] x_k[t] + z_{\ell}[t] \tag{3.1} \]

where \( h_{\ell,k}[t] \in \mathbb{C} \) denotes the channel gain from transmitter \( k \) to receiver \( \ell \) at time \( t \). Let \( H[t] = \{h_{\ell,k}[t]\} \) denote the channel matrix at time \( t \).

We will focus on the important special case where the magnitudes of the channel gains are constant across time\(^1\), i.e., \( |h_{\ell,k}[t_1]| = |h_{\ell,k}[t_2]| \) for all \( t_1 \) and \( t_2 \). For notational convenience, we will drop the time index when referring to the magnitudes of the channel gains, i.e. instead of writing \( |h_{\ell,k}[t]| \), we will write \( |h_{\ell,k}| \). Furthermore, we assume that the phase of each channel gain \( \angle h[t] \) is i.i.d.\(^2\) across time according to a uniform distribution over \([0, 2\pi)\).

We will assume full channel state information (CSI) is available, i.e., at time \( t \), all transmitters and receivers know \( H[t] \).

\(^1\)Without this assumption, it becomes more challenging to parameterize the very strong regime, due to the inseparability of parallel interference channels. See [Sankar et al., 2008] for more details.

\(^2\)For our achievable scheme, it suffices to assume that each phase sequence is stationary and ergodic, and all sequences are independent of one another. However, the i.i.d. assumption will help simplify our proofs. See [Niesen et al., 2013, §IV] for a sample proof for the stationary ergodic case.
The noise $z_\ell[t]$ is i.i.d. across time according to a circularly symmetric complex Gaussian with mean zero and unit variance, $z_\ell[t] \sim \mathcal{CN}(0, 1)$, and is independent of all channel inputs and channel gains.

**Figure 3.1:** $K$-user Gaussian interference channel with time-varying channel coefficients.

**Definition 4 (Decoders)** Each receiver (indexed by $\ell = 1, \ldots, K$) makes an estimate $\hat{\omega}_\ell$ of its desired message $\omega_\ell$ using a decoding function

$$D_\ell : \mathbb{C}^T \rightarrow \{1, 2, \ldots, 2^{TR_\ell}\}.$$ 

**Definition 5 (Achievable Rates)** A rate tuple $(R_1, \ldots, R_K)$ is achievable if, for any $\epsilon > 0$ and $T$ large enough, there exist encoders and decoders that can attain probability of error at most $\epsilon$,

$$\mathbb{P}\left(\bigcup_{\ell=1}^{K}\{\hat{\omega}_\ell \neq \omega_\ell\}\right) < \epsilon.$$ 

**Definition 6 (Capacity Region)** The capacity region $\mathcal{C}$ is the closure of the set of all achievable rate tuples.
3.2 Main Result

Below, we begin with a brief overview of known capacity results for static very strong interference channels. Afterwards, we will state our constant-gap results for time-varying interference channels. Finally, we provide an example to illustrate some of the key ideas underlying our coding scheme.

3.2.1 Static Very Strong Interference Channels

Consider the static, two-user Gaussian interference channel with channel outputs

\[ y_1[t] = h_{1,1} x_1[t] + h_{1,2} x_2[t] + z_1[t] \]  \hspace{1cm} (3.2)

\[ y_2[t] = h_{2,1} x_1[t] + h_{2,2} x_2[t] + z_2[t] \]  \hspace{1cm} (3.3)

We say that the channel is in the very strong regime if

\[ |h_{1,2}|^2 \geq |h_{2,2}|^2 (1 + |h_{1,1}|^2 P) \]  \hspace{1cm} (3.4)

\[ |h_{2,1}|^2 \geq |h_{1,1}|^2 (1 + |h_{2,2}|^2 P) \]  \hspace{1cm} (3.5)

As shown by Carleial [Carleial, 1975], the capacity region in the very strong regime is the set of rate pairs satisfying

\[ R_1 \leq \log(1 + |h_{1,1}|^2 P) \]  \hspace{1cm} (3.6)

\[ R_2 \leq \log(1 + |h_{2,2}|^2 P) \]  \hspace{1cm} (3.7)

i.e., both users can simultaneously operate at their interference-free capacities. The achievable strategy is for the transmitters to employ i.i.d. Gaussian codebooks. Each receiver decodes the interference first while treating its desired codeword as noise. It then cancels out the interference and proceeds to decode its desired codeword.

The same strategy can be applied for \( K > 2 \) users. However, since each receiver
recover all \( K - 1 \) interfering codewords prior to recovering its desired codeword, the interfering channel gains must be much stronger in order for all users to reach their interference-free capacities. Lattice interference alignment offers the possibility of relaxing these conditions by allowing each receiver to directly decode and cancel the sum of the interfering codewords.

Consider the static, symmetric Gaussian interference channel with channel output

\[
y_{\ell}[t] = x_{\ell}[t] + h \sum_{k \neq \ell} x_k[t] + z_{\ell}[t]
\]  

where \( h \in \mathbb{C} \) is the symmetric cross-channel gain. If all users employ the same lattice codebook, the sum of interfering codewords will correspond to a single effective codeword,

\[
x_{\text{int.}\ell} = \sum_{k \neq \ell} x_k
\]  

where \( x_k = [x_k[1] \cdots x_k[T]]^T \) is the \( \ell \)th codeword. As shown by Sridharan et al. [Sridharan et al., 2008], the receiver can decode the sum of interfering codewords if the rates satisfy

\[
R_\ell < \log \left( \frac{|h|^2 P}{1 + P} \right).
\]  

Thus, if

\[
|h|^2 \geq \frac{(1 + P)^2}{P},
\]  

then each user can operate at its interference-free capacity \( \log(1 + P) \). Note that this is a substantial improvement on the condition for reaching the interference-free
capacity while decoding the interfering codewords individually,

$$|h|^2 \geq \frac{(1 + P)((1 + P)^{K-1} - 1)}{(K-1)P}.$$  \hfill (3.12)

This alignment strategy does not immediately generalize beyond the symmetric case, since it requires that each receiver observes an integer-linear combination of the interfering codewords.

3.2.2 Phase-Fading, Very Strong Interference Channels

Our main result is that, for interference channels with time-varying phase, if all receivers are in the very strong interference regime, the capacity region can be determined to within a constant gap. We define the very strong interference regime by applying Carleial’s condition [Carleial, 1975] to each pair of desired and interfering channel gains. Specifically, we assume that

$$|h_{k,\ell}|^2 \geq |h_{\ell,\ell}|^2(1 + |h_{k,k}|^2P) \quad \forall \ell \neq k.$$  \hfill (3.13)

**Theorem 2** Consider a very strong, phase-fading Gaussian interference channel. Any rate tuple \((R_1, \ldots, R_K)\) satisfying

$$R_\ell < \log(1 + |h_{\ell,\ell}|^2P) - 2K^2 - 1$$  \hfill (3.14)

is achievable.

The achievable strategy combines ideas from ergodic alignment [Nazer et al., 2012, Niesen et al., 2013] and lattice-based interference alignment [Bresler et al., 2010, Sridharan et al., 2008, Ordentlich et al., 2014]. The proof is split into two pieces: Section 3.4 establishes the lower bound for \(K = 3\) users and Section 3.5 generalizes this result to \(K > 3\) users.

Note that, since each user cannot exceed its interference-free capacity, Theorem 2 characterizes the very strong capacity region to within \(2K^2 + 1\) bits per user.
3.2.3 Motivating Example

We now illustrate our alignment strategy with a small example for \( K = 3 \) users. Consider a time slot \( t_1 \) and the channel gains \( h_{\ell,k}[t_1] \). With a slight abuse of notation, we will drop the time index and use \( h_{\ell,k} \) to denote \( h_{\ell,k}[t_1] \). Hence, we have

\[
H[t_1] = \begin{bmatrix}
h_{1,1} & h_{1,2} & h_{1,3} \\
h_{2,1} & h_{2,2} & h_{2,3} \\
h_{3,1} & h_{3,2} & h_{3,3}
\end{bmatrix}.
\] (3.15)

We would like to induce lattice alignment at all three receivers using a linear strategy. This is an overconstrained problem if we limit ourselves to one channel matrix. Instead, we will select a second time slot \( t_2 \) whose channel matrix satisfies

\[
H[t_2] = \begin{bmatrix}
h_{1,1} & h_{1,2} & h_{1,3} \\
h_{2,1} & h_{2,2} & -h_{2,3} \\
h_{3,1} & h_{3,2} & h_{3,3}
\end{bmatrix}.
\] (3.16)

Of course, the probability of such a match occurring within a finite number of time slots is zero. Later, we will carefully argue that, if we allow for a slight mismatch, we can operate at nearly the same rates with a finite blocklength.

We assume that transmitter 1 has two symbols \( s_{1,1} \) and \( s_{1,2} \), transmitter 2 has two symbols \( s_{2,1} \) and \( s_{2,2} \), and transmitter 3 has one symbol \( s_{3,1} \), each with average power \( P \). Let \( x_k = [x_k[t_1] x_k[t_2]]^T \) be the vector of channel inputs from transmitter \( k \). The first channel input is a weighted sum of the symbols and the second is a weighted difference,

\[
x_1 = \begin{bmatrix}
\beta_{1,1}s_{1,1} + \beta_{1,2}s_{1,2} \\
\beta_{1,1}s_{1,1} - \beta_{1,2}s_{1,2}
\end{bmatrix}
\] (3.17)

\[
x_2 = \begin{bmatrix}
\beta_{2,1}s_{2,1} + \beta_{2,2}s_{2,2} \\
\beta_{2,1}s_{2,1} - \beta_{2,2}s_{2,2}
\end{bmatrix}
\] (3.18)

\[
x_3 = \begin{bmatrix}
\beta_{3,1}s_{3,1}
\end{bmatrix}
\] (3.19)

where the scaling coefficients \( \beta_{k,n} \) will be chosen such that the interfering symbols
at each receiver form an integer-linear combinations. The magnitudes are chosen to satisfy \( \frac{1}{2} \leq |\beta_{k,n}| \leq 1 \) so that

- The power constraint is never violated.
- In the worst case, the effective power of each symbol is reduced by a factor of \( 1/4 \), corresponding to a rate loss of 2 bits.

Let \( y_\ell = [y_\ell[t_1] \ y_\ell[t_2]]^T \) be the vector of channel outputs at receiver \( \ell \). To extract its message, each receiver first takes the sum and difference of its observations to obtain the effective channel output vector \( \tilde{y}_\ell = [\tilde{y}_\ell,1 \ \tilde{y}_\ell,2]^T \) where

\[
\tilde{y}_\ell,1 = \frac{1}{2}(y_\ell[t_1] + y_\ell[t_2]) \quad \tilde{y}_\ell,2 = \frac{1}{2}(y_\ell[t_1] - y_\ell[t_2])
\]

Combining terms, we get

\[
\tilde{y}_1 = \left[ h_{1,1}\beta_{1,1}s_{1,1} + h_{1,2}\beta_{2,1}s_{2,1} + h_{1,3}\beta_{3,1}s_{3,1} \right] + \tilde{z}_1
\]
\[
\tilde{y}_2 = \left[ h_{2,1}\beta_{1,1}s_{1,1} + h_{2,2}\beta_{2,1}s_{2,1} \right] + \tilde{z}_2
\]
\[
\tilde{y}_3 = \left[ h_{3,1}\beta_{1,1}s_{1,1} + h_{3,2}\beta_{2,1}s_{2,1} + h_{3,3}\beta_{3,1}s_{3,1} \right] + \tilde{z}_3
\]

where \( \tilde{z}_\ell = [\tilde{z}_\ell,1 \ \tilde{z}_\ell,2]^T \) and

\[
\tilde{z}_\ell,1 = \frac{1}{2}(z_\ell[t_1] + z_\ell[t_2]) \quad \tilde{z}_\ell,2 = \frac{1}{2}(z_\ell[t_1] - z_\ell[t_2])
\]

Notice that, due to the sign flip for \( h_{2,3}[t_2] \) in (3.16), the symbol \( s_{3,1} \) appears in the second effective channel output at receiver 2, rather than the first. This will give our linear scheme the flexibility needed to align interference at all receiver.

Let \( h \) be a complex scalar. The following basic quantization properties will be used frequently in our proof:

- If \( |h| \geq 1 \), then \( \frac{1}{2} \leq \frac{||h||}{|h|} \leq 1 \).
• If $|h| \geq \frac{1}{2}$, then $\frac{1}{2} \leq \frac{|h|}{|h|} \leq 1$.

We now proceed to set the scaling coefficients, beginning with setting $\beta_{1,2} = 1$ and aligning interference for $\tilde{y}_{2,2}$. If $|h_{2,3}| \geq |h_{2,1}|$, we choose the integer coefficient and scaling parameter as follows:

$$b_2 = \left\lfloor \frac{|h_{2,3}|}{|h_{2,1}|} \right\rfloor \quad (3.20)$$

$$\beta_{3,1} = \frac{b_2 h_{2,1}}{h_{2,3}}. \quad (3.21)$$

The resulting effective channel consists of the desired symbol plus an integer-linear combination of two interfering symbols plus noise,

$$\tilde{y}_{2,2} = h_{2,2} \beta_{2,2} s_{2,2} + h_{2,1} (s_{1,2} + b_2 s_{3,1}) + \tilde{z}_{2,2}. \quad (3.22)$$

On the other hand, if $|h_{2,3}| < |h_{2,1}|$, we choose

$$b_2 = \left\lceil \frac{|h_{2,1}|}{|h_{2,3}|} \right\rceil \quad (3.23)$$

$$\beta_{3,1} = \frac{h_{2,1}}{b_2 h_{2,3}} \quad (3.24)$$

to obtain

$$\tilde{y}_{2,2} = h_{2,2} \beta_{2,2} s_{2,2} + h_{2,3} \beta_{3,1} (b_2 s_{1,2} + s_{3,1}) + \tilde{z}_{2,2}. \quad (3.25)$$

Now, we choose $\beta_{2,1}$ as a function of $\beta_{3,1}$ in order to align interference for $\tilde{y}_{1,1}$. Again, we carefully distinguish two cases:

• If $|h_{1,2}| \geq |h_{1,3}||\beta_{3,1}|$, the integer coefficient, scaling parameter, and resulting
effective channel are

\[
b_1 = \left\lfloor \frac{|h_{1,2}|}{|h_{1,3}| |\beta_{3,1}|} \right\rfloor
\]

\[
\beta_{2,1} = \frac{b_1 h_{1,3} \beta_{3,1}}{h_{1,2}}
\]

\[
\tilde{y}_{1,1} = h_{1,1} \beta_{1,1} s_{1,1} + h_{1,3} \beta_{3,1} (b_1 s_{2,1} + s_{3,1}) + \tilde{z}_{1,1}.
\]

- Otherwise, if \(|h_{1,2}| < |h_{1,3}| |\beta_{3,1}|\), we set

\[
b_1 = \left\lceil \frac{|h_{1,3}| |\beta_{3,1}|}{|h_{1,2}|} \right\rceil
\]

\[
\beta_{2,1} = \frac{h_{1,3} \beta_{3,1}}{b_1 h_{1,2}}
\]

\[
\tilde{y}_{1,1} = h_{1,1} \beta_{1,1} s_{1,1} + h_{1,2} \beta_{2,1} (s_{2,1} + b_1 s_{3,1}) + \tilde{z}_{1,1}.
\]

Next, we set \(\beta_{1,1}\) to align interference for \(\tilde{y}_{3,1}\) as follows:

- If \(|h_{3,1}| \geq |h_{3,2}| |\beta_{2,1}|\), the integer coefficient, scaling parameter, and resulting effective channel are

\[
b_3 = \left\lfloor \frac{|h_{3,1}|}{|h_{3,2}| |\beta_{2,1}|} \right\rfloor
\]

\[
\beta_{1,1} = \frac{b_3 h_{3,2} \beta_{2,1}}{h_{3,1}}
\]

\[
\tilde{y}_{3,1} = h_{3,3} \beta_{3,1} s_{3,1} + h_{3,2} \beta_{2,1} (b_3 s_{1,1} + s_{2,1}) + \tilde{z}_{3,1}.
\]

- Otherwise, if \(|h_{3,1}| < |h_{3,2}| |\beta_{2,1}|\), we set

\[
b_3 = \left\lceil \frac{|h_{3,1}|}{|h_{3,2}| |\beta_{2,1}|} \right\rceil
\]

\[
\beta_{1,1} = \frac{h_{3,2} \beta_{2,1}}{b_3 h_{3,1}}
\]

\[
\tilde{y}_{3,1} = h_{3,3} \beta_{3,1} s_{3,1} + h_{3,1} \beta_{1,1} (s_{1,1} + b_3 s_{2,1}) + \tilde{z}_{3,1}.
\]
Finally, we note that there is only one interfering symbol in $\tilde{y}_{1,2}$ and no desired symbol in $\tilde{y}_{3,2}$. Thus, we are free to set $\beta_{2,2} = \beta_{3,2} = 1$. Overall, each desired symbol is observed over an effective channel output along with the integer-linear combination of (at most) two interfering symbols plus noise. Since this scheme sends 5 symbols over 2 channel uses, we will see a rate loss factor of $\frac{5}{6}$ per user (after time sharing). By repeating this strategy over many channel realizations and sending lattice codewords over the resulting effective channels, we can achieve the rates $R_\ell = \frac{5}{6} \log(1 + \frac{1}{4}|h_{\ell,\ell}|^2)$, where the factor of 1/4 stems from the fact that $|\beta_{k,n}| \geq 1/2$. (See Section 3.4.3 for details on the lattice coding scheme.) In Section 3.4, we will generalize this motivating example so that each user can send $(N - 1)$ symbols over $N$ channel uses, corresponding to a rate loss factor of $\frac{N-1}{N}$ per user. By taking $N \to \infty$, we can eliminate the rate loss factor entirely. Then, in Section 3.5, we will extend this scheme to $K$ users.

3.3 Channel Quantization

In order to establish a constant-gap result, our scheme carefully matches up channel matrices to create opportunities for lattice alignment. This matching process is carried out using a variation on the ergodic alignment technique from [Nazer et al., 2012]. As noted earlier, the probability that a particular channel matrix occurs (corresponding to a perfect match) is zero. The key idea is to quantize the channel matrices into a finite number of bins, and then match up time slots based on quantized channel matrices. Overall, this allows us to match up nearly all channel matrices that occur within a large, finite blocklength.

We begin by noting that it suffices to quantize the channel phases since the magnitudes are fixed. We quantize the phase by dividing the interval of possible phases $[0, 2\pi)$ into $\nu N \max_{\ell,k} |h_{\ell,k}|$ equal-sized segments, and representing each segment by
its midpoint. Let \( Q \) denote the set of phase quantization points, \( Q : [0, 2\pi) \to Q \) the quantization function, and \( \hat{h}_{\ell,k}[t] = |h_{\ell,k}| \exp(jQ(\angle(h_{\ell,k}[t]))) \) the quantized version of \( h_{\ell,k}[t] \). This quantizer has the following useful properties:

- For any \( h_{\ell,k}[t] \in \mathbb{C} \), the quantization error is at most
  \[
  |\hat{h}_{\ell,k}[t] - h_{\ell,k}[t]| \leq \frac{\pi}{\nu N}. \tag{3.26}
  \]

- For any quantized channel gain \( \hat{h}_{\ell,k} \in Q \), multiplying by an \( N^{\text{th}} \) root of unity yields another quantization point,
  \[
  \exp\left(\frac{j2\pi n}{N}\right) \hat{h}_{\ell,k} \in Q, \tag{3.27}
  \]
  for all \( n \in \mathbb{Z} \).

Denote the quantized channel matrix at time \( t \) by

\[
\hat{H}[t] = \{\hat{h}_{\ell,k}[t]\}_{\ell,k} \tag{3.28}
\]

and let \( \hat{H} \) denote the set of all possible quantized channel matrices. By construction, \( \hat{H} \) is finite with size \( |\hat{H}| = Q^{K^2} \).

The goal of the matching process is to create groups of \( N \) channel matrices that are amenable to lattice alignment. To this end, we will split the blocklength \( T \) into \( N \) consecutive subblocks, each of length \( T/N \).\(^3\) We will then choose \( T \) large enough so that each subblock is strongly typical. This will later allow us to match up all but a vanishing fraction of the channel matrices. The following lemma makes this precise.

**Lemma 1** For any parameters \( \nu, N \in \mathbb{N} \), and \( \gamma > 0 \), and for \( T \) large enough, we have that the empirical distribution of quantized channel matrices is close to the true

\(^3\)Note that we have tacitly assumed that \( T \) is divisible by \( N \) and will do so throughout the paper.
distribution within all subblocks. Specifically, we have that

\[
\sum_{t=(n-1)(T/N)+1}^{nT/N} 1 \{ \hat{H}[t] = \hat{H} \} \geq \frac{1 - \gamma}{|\mathcal{H}|} \frac{T}{N}
\]  

(3.29)

across all subblocks \( n = 1, \ldots, N \) and quantized channel realizations \( \hat{H} \in \hat{\mathcal{H}} \) with probability at least \( 1 - \gamma \).

**Proof 1** For a given \( n \in \{1, \ldots, N\} \), it follows from standard typicality arguments [1, §2.4] that, for \( T \) large enough, (3.29) is satisfied for all \( \hat{H} \in \hat{\mathcal{H}} \) with probability at least \( 1 - \frac{\gamma}{N} \). Using a union bound, we find that (3.29) holds for all \( n \) and \( \hat{H} \in \hat{\mathcal{H}} \) with probability at least \( 1 - \gamma \).

![Figure 3-2: Phase quantization for a single channel gain with \( N = 4 \), \( \nu = 3 \), and max_{\ell,k} |h_{\ell,k}| = 1.9. The number of quantization bins is \( \nu N \max_{\ell,k} \left| h_{\ell,k} \right| = 24. \)](image)

3.4 Proof of the Lower Bound in Theorem 2 for \( K = 3 \) Users

The alignment scheme of Cadambe and Jafar [Cadambe and Jafar, 2008] admits a simpler form for \( K = 3 \) users. Since our coding strategy uses a variation on this scheme as a building block, it is simpler to start with the proof for \( K = 3 \) users. This will help us generate intuition for the \( K > 3 \) case. Our strategy consists of
three components: channel matching, Fourier modulation and alignment, and lattice coding.

3.4.1 Channel Matching

Our matching scheme groups together time slots whose channel matrices are well-suited for lattice alignment. The goal is to show that all but a vanishing fraction of time slots can be matched up.

We start by splitting the time slots into $N$ consecutive subblocks, each of length $T/N$, with $N$ to be specified later. Specifically, the $n^\text{th}$ subblock refers to the time slots $t = (n-1)T/N + 1$ to $t = nT/N$. We employ the quantization scheme from Section 3.3 for some $\nu \in \mathbb{N}$ to be specified later. For any $\gamma > 0$, we know from Lemma 1 that, for $T$ large enough, the number of quantized channel realizations satisfies (3.29) for all subblocks $n = 1, \ldots, N$ and quantized channel realizations $\hat{H} \in \hat{\mathcal{H}}$ with probability at least $1 - \gamma$. If, for some $n$ and $\hat{H}$, (3.29) does not hold, then we declare an error.

For the remainder of this section, we condition on the event that (3.29) holds for all $n = 1, \ldots, N$ and $\hat{H} \in \hat{\mathcal{H}}$.

For each subblock $n = 1, \ldots, N$ and quantized channel realization $\hat{H} \in \hat{\mathcal{H}}$, we will designate the first $\frac{1-\gamma}{|\hat{H}|} T$ time slots$^4$ time slots satisfying $\hat{H}[t] = \hat{H}$ for all $\hat{H} \in \hat{\mathcal{H}}$ as useable, and ignore the rest. The total number of such useable time slots is $(1-\gamma)T$, which will impose a rate loss factor of $1 - \gamma$.

We will match each useable time slot in the first subblock with a unique time slot in each of the remaining $N-1$ subblocks. Overall, this will produce exactly $(1-\gamma)T/N$ non-overlapping groups of $N$ timeslots. Initialize $t_1 = 1$ and $B_{\hat{H}} = 0$ for all $\hat{H} \in \hat{\mathcal{H}}$, then run the following procedure:

1. Set $\hat{H} = \hat{H}[t_1]$. If $B_{\hat{H}} < \frac{1-\gamma}{|\hat{H}|} T$, then increment $B_{\hat{H}}$ by 1 and go to Step 2.

$^4$Since $T$ will eventually be taken to infinity, we can safely assume that this quantity is integer-valued.
Otherwise, go to Step 4.

2. For each subblock $n = 2, \ldots, N$, find the first unused time slot $t_n$ whose quantized channel matrix satisfies

$$
\hat{\mathbf{H}}[t_n] = \begin{bmatrix}
\hat{h}_{1,1}[t_1] & \hat{h}_{1,2}[t_1] & \hat{h}_{1,3}[t_1] \\
\hat{h}_{2,1}[t_1] & \hat{h}_{2,2}[t_1] & \omega^{-(n-1)}\hat{h}_{2,3}[t_1] \\
\hat{h}_{3,1}[t_1] & \hat{h}_{3,2}[t_1] & \hat{h}_{3,3}[t_1]
\end{bmatrix}
$$

(3.30)

where

$$
\omega \triangleq \exp\left(-\frac{j2\pi}{N}\right).
$$

(3.31)

3. Group the resulting time slots $t_1, \ldots, t_N$ together and mark them as “used” for the remainder of the matching procedure.

4. If $t_1 < T/N$, then increment $t_1$ by 1 and loop back to Step 1. Otherwise, terminate the procedure.

Recall that, by design, multiplying a quantization point by an $N$th root of unity results in another quantization point. Since there are exactly $(1 - \gamma) T$ useable time slots in each subblock with quantized channel realization $\hat{\mathbf{H}}$, then the procedure above places all useable time slots into matched groups.

**Remark 1** Note that our scheme only requires instantaneous CSI at the transmitters. In other words, each transmitter can determine what to transmit at time $t$ using only knowledge of $\mathbf{H}[t], \mathbf{H}[t-1], \ldots, \mathbf{H}[1]$ (or their quantized values).

### 3.4.2 Fourier Modulation and Alignment

As discussed earlier, inducing lattice alignment over a single channel matrix with a linear strategy is infeasible. Our scheme takes $N$ matched time slots and produces $N - 1$ lattice-aligned effective channels at each receiver. The key idea is to exploit the phase offset for $\hat{h}_{2,3}[t_n]$ to change the effective channels over which the symbols
from transmitter 3 symbols arrive at receiver 2. This symbol index shift provides the flexibility needed to induce lattice alignment, just as in the motivating example.

To transform the phase offsets into a symbol index shift, we will modulate our symbols using the Discrete Fourier Transform (DFT). Define the DFT matrix

$$W \triangleq \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2}
\end{bmatrix}$$ (3.32)

as well as the inverse DFT matrix

$$W^{-1} = \frac{1}{N} \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\
1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{-(N-1)} & \omega^{-(2(N-1))} & \cdots & \omega^{-(N-1)^2}
\end{bmatrix}$$ (3.33)

We also define the following matrix whose diagonal contains the $N$th roots of unity,

$$F \triangleq \text{diag} \left( [1 \omega^{-1} \omega^{-2} \cdots \omega^{-(N-1)}] \right).$$ (3.34)

From the time-shifting property of the DFT [Vetterli et al., 2014, Table 3.7], it follows that

$$W^{-1}FW \begin{bmatrix}
s_1 \\
\vdots \\
s_{N-1} \\
s_N
\end{bmatrix} = \begin{bmatrix}
s_2 \\
\vdots \\
s_N \\
s_1
\end{bmatrix}. \quad (3.35)$$

We now describe our alignment scheme. Consider any group of matched time slots
$t_1, \ldots, t_N$ and define

$$x_k = \begin{bmatrix} x_k[t_1] \\ \vdots \\ x_k[t_N] \end{bmatrix}, \quad y_\ell = \begin{bmatrix} y_\ell[t_1] \\ \vdots \\ y_\ell[t_N] \end{bmatrix}, \quad z_\ell = \begin{bmatrix} z_\ell[t_1] \\ \vdots \\ z_\ell[t_N] \end{bmatrix}$$

(3.36)

$$D_{\ell,k} = \text{diag} \left( [h_{\ell,k}[t_1], \ldots, h_{\ell,k}[t_N]] \right)$$

(3.37)

We can then concisely express the $\ell^{th}$ channel output as

$$y_\ell = \sum_{k=1}^{3} D_{\ell,k} x_k + z_\ell.$$

It will be useful to express the channel outputs in terms of the quantized channel gains. Specifically, by defining

$$\hat{D}_{\ell,k} = \text{diag} \left( [\hat{h}_{\ell,k}[t_1], \ldots, \hat{h}_{\ell,k}[t_N]] \right)$$

(3.38)

$$\hat{z}_\ell = \sum_{k=1}^{3} (D_{\ell,k} - \hat{D}_{\ell,k}) x_k + z_\ell,$$

(3.39)

we can write

$$y_\ell = \sum_{k=1}^{3} \hat{D}_{\ell,k} x_k + \hat{z}_\ell.$$

The $k^{th}$ transmitter has $N - 1$ complex symbols $s_{k,1}, \ldots, s_{k,N-1}$ to convey over these $N$ matched time slots. For notational convenience, we also define $s_{k,N} = 0$. Each symbol has an associated scaling coefficient $\beta_{k,n} \in \mathbb{C}$ that will be chosen to induce lattice alignment. Define

$$s_k = \begin{bmatrix} s_{k,1} \\ \vdots \\ s_{k,N} \end{bmatrix}, \quad \beta_k = \begin{bmatrix} \beta_{k,1} \\ \vdots \\ \beta_{k,N} \end{bmatrix}.$$
To generate the channel input vector, we scale the symbols and then take the DFT,
\[ x_k = W \text{diag}(\beta_k)s_k. \]

In order to meet the power constraint, each symbol is assumed to have average power \( P/N \).

The \( \ell \)th receiver creates \( N \) effective channels by taking the inverse DFT:
\[
\tilde{y}_\ell = W^{-1}y_\ell \\
= \sum_{k=1}^{3} W^{-1} \hat{D}_{\ell,k} W \text{diag}(\beta_k)s_k + \tilde{z}_\ell
\]
where \( \tilde{z}_\ell = W^{-1}\hat{z}_\ell \) is a mixture of Gaussian noise and quantization noise. Using the quantization error bound (3.26), the variance\(^5\) of each element of \( \tilde{z}_\ell \) is at most
\[
\text{Var}(z_{k,n}) \leq \frac{1}{N} \left( 1 + \frac{\pi^2 KP}{\nu^2 N^2} \right).
\]

From (3.30), the quantized channel gains satisfy
\[
\hat{D}_{\ell,k} = \begin{cases} 
\hat{h}_{\ell,k}I & (\ell, k) \neq (2, 3), \\
\hat{h}_{2,3}F & (\ell, k) = (2, 3).
\end{cases}
\]

Define \( \hat{h}_{\ell,k} = \hat{h}_{\ell,k}[t_1] \). For \( n = 1, \ldots, N - 1 \), we can use (3.35) to express the effective channel outputs as
\[
\begin{align*}
\tilde{y}_{1,n} &= \hat{h}_{1,1}\beta_{1,n}s_{1,n} + \hat{h}_{1,2}\beta_{2,n}s_{2,n} + \hat{h}_{1,3}\beta_{3,n}s_{3,n} + \tilde{z}_{1,n} \\
\tilde{y}_{2,n} &= \hat{h}_{2,1}\beta_{1,n}s_{1,n} + \hat{h}_{2,2}\beta_{2,n}s_{2,n} + \hat{h}_{2,3}\beta_{3,n+1}s_{3,n+1} + \tilde{z}_{2,n} \\
\tilde{y}_{3,n} &= \hat{h}_{3,1}\beta_{1,n}s_{1,n} + \hat{h}_{3,2}\beta_{2,n}s_{2,n} + \hat{h}_{3,3}\beta_{3,n}s_{3,n} + \tilde{z}_{3,n}.
\end{align*}
\]

\(^5\)Strictly speaking, the variance is evaluated with respect to both the randomness of the noise and the randomness of the codebooks.
where \( \tilde{y}_{\ell,n} \) and \( \tilde{z}_{\ell,n} \) are the \( n \)-th entries of \( \tilde{\mathbf{y}}_\ell \) and \( \tilde{\mathbf{z}}_\ell \), respectively.

To determine the scaling and integer coefficients as well as resulting lattice-aligned effective channels, we start by setting \( \beta_{3,1} = 1 \) and \( n = 1 \), and then run the following procedure:

1. Align interference in \( \tilde{y}_{1,n} \):

   If \( |\hat{h}_{1,2}| \geq |\hat{h}_{1,3}| \beta_{3,n} \), set

   \[
   b_{1,n} = \left\lfloor \frac{\hat{h}_{1,2}}{|\hat{h}_{1,3}| \beta_{3,n}} \right\rfloor \quad \beta_{2,n} = \frac{\hat{h}_{1,3} \beta_{3,n}}{|\hat{h}_{1,2}|} \frac{b_{1,n} \hat{h}_{1,3} \beta_{3,n}}{|\hat{h}_{1,2}|} \\
   \tilde{y}_{1,n} = \hat{h}_{1,1} \beta_{1,n} s_{1,n} + \hat{h}_{1,3} \beta_{3,n} (b_{1,n} s_{2,n} + s_{3,n}) + \tilde{z}_{1,n}.
   \]

   Otherwise, if \( |\hat{h}_{1,2}| < |\hat{h}_{1,3}| \beta_{3,n} \), set

   \[
   b_{1,n} = \left\lceil \frac{\hat{h}_{1,3} \beta_{3,n}}{|\hat{h}_{1,2}|} \right\rceil \quad \beta_{2,n} = \frac{\hat{h}_{1,3} \beta_{3,n}}{b_{1,n} \hat{h}_{1,2}} \frac{\hat{h}_{1,3} \beta_{3,n}}{b_{1,n} \hat{h}_{1,2}} \\
   \tilde{y}_{1,n} = \hat{h}_{1,1} \beta_{1,n} s_{1,n} + \hat{h}_{1,2} \beta_{2,n} (s_{2,n} + b_{1,n} s_{3,n}) + \tilde{z}_{1,n}.
   \]

2. Align interference in \( \tilde{y}_{3,n} \):

   If \( |\hat{h}_{3,1}| \geq |\hat{h}_{3,2}| \beta_{2,n} \), set

   \[
   b_{3,n} = \left\lfloor \frac{\hat{h}_{3,1}}{|\hat{h}_{3,2}| \beta_{2,n}} \right\rfloor \quad \beta_{1,n} = \frac{\hat{h}_{3,2} \beta_{2,n}}{|\hat{h}_{3,1}|} \frac{\hat{h}_{3,2} \beta_{2,n}}{|\hat{h}_{3,1}|} \\
   \tilde{y}_{3,n} = \hat{h}_{3,3} \beta_{3,n} s_{3,n} + \hat{h}_{3,2} \beta_{2,n} (b_{3,n} s_{1,n} + s_{2,n}) + \tilde{z}_{3,n}.
   \]

   Otherwise, if \( |\hat{h}_{3,1}| < |\hat{h}_{3,2}| \beta_{2,n} \), set

   \[
   b_{3,n} = \left\lceil \frac{\hat{h}_{3,2} \beta_{2,n}}{|\hat{h}_{3,1}|} \right\rceil \quad \beta_{1,n} = \frac{\hat{h}_{3,2} \beta_{2,n}}{b_{3,n} |\hat{h}_{3,1}|} \frac{\hat{h}_{3,2} \beta_{2,n}}{b_{3,n} |\hat{h}_{3,1}|} \\
   \tilde{y}_{3,n} = \hat{h}_{3,3} \beta_{3,n} s_{3,n} + \hat{h}_{3,1} \beta_{1,n} (s_{1,n} + b_{3,n} s_{2,n}) + \tilde{z}_{3,n}.
   \]
3. Align interference in $\tilde{y}_{2,n}$:

If $|\hat{h}_{2,3}| \geq |\hat{h}_{2,1}| |\beta_{1,n}|$, set

$$b_{2,n} = \left\lceil \frac{|\hat{h}_{2,3}|}{|\hat{h}_{2,1}| |\beta_{1,n}|} \right\rceil, \quad \beta_{3,n+1} = \frac{b_{2,n} \hat{h}_{2,1} \beta_{1,n}}{\hat{h}_{2,3}},$$

$$\tilde{y}_{2,n} = \hat{h}_{2,2} \beta_{2,n} s_{2,n} + \hat{h}_{2,1} \beta_{1,n} (s_{1,n} + b_{2,n} s_{3,n+1}) + \tilde{z}_{2,n}.$$

Otherwise, if $|\hat{h}_{2,3}| < |\hat{h}_{2,1}| |\beta_{1,n}|$, set

$$b_{2,n} = \left\lfloor \frac{|\hat{h}_{2,1}| |\beta_{1,n}|}{|\hat{h}_{2,3}|} \right\rfloor, \quad \beta_{3,n+1} = \frac{\hat{h}_{2,1} \beta_{1,n}}{b_{2,n} \hat{h}_{2,3}},$$

$$\tilde{y}_{3,n} = \hat{h}_{3,3} \beta_{3,n} s_{3,n} + \hat{h}_{2,3} \beta_{2,n+1} (b_{2,n} s_{1,n} + s_{3,n+1}) + \tilde{z}_{2,n}.$$

4. If $n < N - 1$, increment $n$ by 1 and loop back to Step 1. Otherwise, terminate the procedure.

For completeness, we also set $\beta_{1,N} = \beta_{2,N} = 1$.

Overall, the $\ell$th receiver observes $N - 1$ effective channels of the form

$$\bar{y}_{\ell,n} = g_{\ell,n} s_{\ell,n} + g_{\text{int},\ell,n} s_{\text{int},\ell,n} + \tilde{z}_{\ell,n} \quad (3.45)$$

where

$$\frac{1}{2} |h_{\ell,\ell}| \leq |g_{\ell,n}| \leq |h_{\ell,\ell}| \quad (3.46)$$

$$\frac{1}{2} \min_{k \neq \ell} |h_{\ell,k}| \leq |g_{\text{int},\ell,n}| \leq \min_{k \neq \ell} |h_{\ell,k}|, \quad (3.47)$$

and $s_{\text{int},\ell,n}$ is an integer-linear combination of (at most) two interfering symbols. From the perspective of each receiver, each symbol appears in exactly one effective channel (if we ignore its small contribution to the quantization error). The phases of $g_{\ell,n}$ and $g_{\text{int},\ell,n}$ are independent and uniform over $\mathcal{Q}$. We will ignore the $N$th channel at each
receiver. Next, we will show how to code over these effective channels.

3.4.3 Lattice Coding

The users select their codewords from nested lattice codebooks so that integer-linear combinations of codewords are themselves codewords. This allows us to use the compute-and-forward scheme to first decode the integer-linear combination of interfering codewords, cancel out its contribution to the channel observation, and finally decode the desired codeword.

We will code across each quantized channel realization separately. Using the compute-and-forward framework [Nazer and Gastpar, 2011], the $k^{th}$ user selects a nested lattice codebook $C_\ell$ with blocklength $\frac{(1-\gamma)T}{|\hat{H}|N}$, power $P/N$, and rate

$$\hat{R}_k = \log(1 + |h_{k,k}|^2 P) - 3 - \log\left(1 + \frac{\pi^2 KP}{v^2 N^2}\right).$$

(3.48)

It then splits its message $m_k$ into $(N-1)|\hat{H}|$ equal-rate submessages $m_{k,n}^{(\hat{H})}$, one for each subchannel $n \in \{1, \ldots, N-1\}$ and quantized channel realization $\hat{H} \in \hat{H}$. The submessage $m_{k,n}^{(\hat{H})}$ is then mapped to the corresponding nested lattice codeword $s_{k,n}^{(\hat{H})} \in C_\ell$.

For each quantized channel realization $\hat{H} \in \hat{H}$, we group together all $\frac{(1-\gamma)T}{|\hat{H}|N}$ useable time slots from the first subblock for which $\hat{H}[t] = \hat{H}$. Using the modulation and alignment scheme described above, the $k^{th}$ transmitter sends the codewords $s_{k,1}^{(\hat{H})}, \ldots, s_{k,N-1}^{(\hat{H})}$ over these time slots. It then repeats these codewords over the matched time slots in the remaining $N-1$ subblocks, again following the modulation and alignment scheme. Note that the average transmit power is at most $(N-1)P/N$, which meets the power constraint.

The $\ell^{th}$ receiver observes its desired codewords over $N-1$ effective channels of
the form
\[
\tilde{y}_{\ell,n}^{(\hat{H})} = g_{\ell,n}^{(\hat{H})} s_{\ell,n}^{(\hat{H})} + g_{\text{int},\ell,n}^{(\hat{H})} s_{\text{int},\ell,n}^{(\hat{H})} + \tilde{z}_{\ell,n}^{(\hat{H})}
\]  \hspace{1cm} (3.49)

where \(|g_{\ell,n}^{(\hat{H})}|\) and \(|g_{\text{int},\ell,n}^{(\hat{H})}|\) satisfy (3.46) and (3.47), respectively, \(s_{\text{int},\ell,n}^{(\hat{H})}\) is an integer-linear combination of two interfering codewords, and \(\tilde{z}_{\ell,n}^{(\hat{H})}\) is the effective noise, which consists of both channel noise and quantization error, and satisfies (3.43).

Before recovering its desired codeword \(s_{\ell,n}^{(\hat{H})}\), the \(\ell\)th receiver first decodes the interference while treating the desired codeword as noise. It follows from [Nazer and Gastpar, 2011, Theorem 3] and [? Lemma 5] that the integer-linear combination \(s_{\text{int},\ell,n}^{(\hat{H})}\) can be decoded successfully from \(\tilde{y}_{\ell,n}^{(\hat{H})}\) if the rates of all interfering codebooks satisfy

\[
\tilde{R}_\ell < \log \left( \text{SINR}_{\text{int},\ell,n}^{(\hat{H})} \right) \quad \ell \neq k
\]  \hspace{1cm} (3.50)

where

\[
\text{SINR}_{\text{int},\ell,n}^{(\hat{H})} \triangleq \frac{|g_{\text{int},\ell,n}^{(\hat{H})}|^2 P}{|g_{\ell,n}^{(\hat{H})}|^2 P + \frac{1}{N} \left( 1 + \frac{\pi^2 K_P}{\nu^2 N^2} \right)} \geq \min_{k \neq \ell} \frac{1}{\frac{1}{4} |h_{\ell,k}|^2 P}{|h_{\ell,\ell}|^2 P + \left( 1 + \frac{\pi^2 K_P}{\nu^2 N^2} \right)}
\]  \hspace{1cm} (3.51)

where the inequality follows from (3.46) and (3.47). From the very strong interference condition (3.13), we know that the rate of the \(k\)th user satisfies

\[
\tilde{R}_k < \log \left( 1 + \frac{\min_{k \neq \ell} |h_{\ell,k}|^2 P}{|h_{\ell,\ell}|^2 P + 1} \right) - 3 - \log \left( 1 + \frac{\pi^2 K_P}{\nu^2 N^2} \right)
\]  \hspace{1cm} (3.53)

\[
< \log \left( \frac{\min_{k \neq \ell} |h_{\ell,k}|^2 P}{|h_{\ell,\ell}|^2 P + 1} \right) - 2 - \log \left( 1 + \frac{\pi^2 K_P}{\nu^2 N^2} \right)
\]  \hspace{1cm} (3.54)

\[
= \log \left( \frac{\min_{k \neq \ell} \frac{1}{4} |h_{\ell,k}|^2 P}{|h_{\ell,\ell}|^2 P + 1} \right) - \log \left( 1 + \frac{\pi^2 K_P}{\nu^2 N^2} \right)
\]  \hspace{1cm} (3.55)

\[
< \log \left( \text{SINR}_{\text{int},\ell,n}^{(\hat{H})} \right),
\]  \hspace{1cm} (3.56)
and thus the interference can be decoded with vanishing probability of error.

Next, the $\ell^{th}$ receiver forms the effective channel $\tilde{y}_{\ell,n}(\hat{H}) - g_{\text{int},\ell,n}s_{\text{int},\ell,n}$ and attempts to decode $s_{\ell,n}(\hat{H})$. Since

$$\tilde{R}_\ell < \log \left( 1 + \frac{1}{1 + \frac{\nu^2 K P}{\nu^2 N^2}} \right) \leq \log \left( 1 + \frac{|g_{\ell,n}(\hat{H})|^2 P}{1 + \frac{\nu^2 K P}{\nu^2 N^2}} \right), \quad (3.57)$$

decoding succeeds with vanishing probability of error.

Summing across all quantized channel realizations and normalizing by the total number of time slots $T$, we find that the achievable rate for the $k^{th}$ transmitter is

$$R_k = \frac{1 - \gamma}{|\mathcal{H}|N} \sum_{n=1}^{N-1} \sum_{\hat{H} \in \mathcal{H}} \tilde{R}_k. \quad (3.58)$$

Taking the quantization parameter $\nu$ to zero, we get that

$$\lim_{\nu \to 0} R_k = \frac{1 - \gamma}{N} \sum_{n=1}^{N-1} \left( \log(1 + |h_{k,k}|^2 P) - 3 \right) \quad (3.59)$$

Taking the number of subblocks $N$ to infinity and then the typicality parameter $\gamma$ to zero, we find that the rates

$$R_k = \log(1 + |h_{k,k}|^2 P) - 3 \quad (3.60)$$

are achievable for $k = 1, \ldots, K$.

### 3.5 Proof of the Lower Bound in Theorem 2 for $K > 3$ Users

The achievability proof for $K > 3$ users employs the same basic components as in the $K = 3$ user case: channel matching, Fourier modulation and alignment, and lattice coding. The main difference is that, following the alignment scheme of Cadambe and Jafar [Cadambe and Jafar, 2008], we need a much larger set of beamforming vectors. We will match up time slots based on quantized channel matrices, using the
quantization scheme in Section 3.3. Recall that this results in two sources of rate loss. First, some time slots are ignored since their quantized channel matrices do not find a match. Second, the alignment and equalization is carried out imperfectly due to quantization error. However, as $\gamma$ tends to 0 and $\nu$ and blocklength $T$ tend to infinity, both sources of rate loss vanish. Since the analysis of these rate loss terms can be directly inferred from the $K = 3$ user case, we will assume hereafter that $\hat{h}_{\ell,k}[t] \approx h_{\ell,k}[t]$ (i.e., the quantization parameter $\nu$ is very large) and that all but a $o(1)$ fraction of time slots are useable (i.e., the typicality parameter $\gamma$ is very small). This will allows us to streamline the proof.

### 3.5.1 Channel Matching

Let $M$ be a natural number that will be specified later and define

$$N = (M + 1)^{K^2} \quad (3.61)$$

$$d_{\ell,k} = (M + 1)^{(k-1)K+(\ell-1)}. \quad (3.62)$$

We will divide the $T$ channel realizations into $N$ consecutive subblocks, each of length $T/N$. As before, the $n^{th}$ subblock refers to the time slots from $t = (n - 1)T/N + 1$ to $t = nT/N$. Initialize $t_1 = 1$ and search for the first unmatched time slots $t_2, \ldots, t_N$ in subblocks $n = 2, \ldots, N$, respectively, that satisfy

$$h_{\ell,k}[t_n] = \omega^{-(n-1)d_{\ell,k}} h_{\ell,k}[t_1] \quad \forall \ell, k \in \{1, \ldots, K\}. \quad (3.63)$$

These time slots $t_1, \ldots, t_N$ are set aside as a matched pair, $t_1$ is incremented by 1, and we repeat the search.\(^{6}\) This process continues until $t_1 = T/N$ and will match up nearly all time slots (if the sequence of channel realizations is typical).

---

\(^6\)A vanishing fraction of time slots will not be matched, and should be ignored as in Section 3.4.1.
3.5.2 Fourier Modulation and Alignment

Consider any group of matched time slots \(t_1, \ldots, t_N\) and define the associated channel input, output, and noise vectors according to (3.36) as well as the diagonal channel gain matrices according to (3.37). Due to the matching condition (3.63), the diagonal channel gain matrices satisfy

\[
D_{\ell,k} = h_{\ell,k} F^{d_{\ell,k}}
\]

(3.64)

where \(h_{\ell,k} \triangleq h_{\ell,k}[t_1]\) and \(F\) is defined in (3.34).

The codeword symbols will be carried along the following set of beamforming vectors:

\[
\mathcal{V} = \left\{ \left( \prod_{\ell,k} \beta_{\ell,k,i_{\ell,k}} F^{d_{\ell,k} i_{\ell,k}} \right) 1 : i_{\ell,k} \in \{0, 1, \ldots, M - 1\} \right\}
\]

(3.65)

where the scaling coefficients \(\beta_{\ell,k,i_{\ell,k}}\) will be used later on to create lattice-aligned subchannels. It is shown in [Niesen et al., 2013, p.3826] that the \(M K^2\) beamforming vectors in \(\mathcal{V}\) are orthogonal to one another. Moreover, the values of the indices \(\{i_{\ell,k} : \ell, k \in \{1, \ldots, K\}\}\) that identify each beamforming vector can be uniquely determined from the sum \(\sum_{\ell,k} d_{\ell,k} i_{\ell,k}\).

Each transmitter has \(M K^2\) complex symbols \(\{s_{k,v} : v \in \mathcal{V}\}\) to convey over the \(N\) matched time slots. As implied by our notation, each beamforming vector is scaled by the associated complex symbol, and the transmitter sends the sum of the resulting vectors,

\[
x_k = \sum_{v \in \mathcal{V}} v s_{k,v}.
\]

(3.66)
The channel output vector is thus
\[ y_\ell = \sum_{k=1}^{K} D_{\ell,k} x_k + z_\ell \]  
(3.67)
\[ = \sum_{k=1}^{K} \sum_{v \in V} D_{\ell,k} v s_{k,v} + z_\ell \]  
(3.68)
\[ = \sum_{k=1}^{K} \sum_{v \in V} h_{\ell,k} F_{d_{\ell,k}} v s_{k,v} + z_\ell. \]  
(3.69)

Due to the $F_{d_{\ell,k}}$ terms, the set of received beamforming vectors
\[ \tilde{V} = \left\{ \left( \prod_{\ell,k} \beta_{\ell,k,i_{\ell,k}} F_{d_{\ell,k}i_{\ell,k}} \right) 1 : i_{\ell,k} \in \{0, 1, \ldots, M\} \right\} \]  
(3.70)
is larger than the transmitted set $V$. As before, the $N = (M + 1)K^2$ vectors in $\tilde{V}$ are orthogonal to one another and their indices can be uniquely determined from $\sum_{\ell,k} d_{\ell,k} i_{\ell,k}$.

For each vector $\tilde{v} \in \tilde{V}$, the $k^{th}$ receiver forms the effective channel output
\[ \tilde{y}_{k,\tilde{v}} = \frac{1}{\beta_{\tilde{v}}} \tilde{v}^* y_k \]  
(3.71)
\[ = \frac{1}{\beta_{\tilde{v}}} \sum_{k=1}^{K} \sum_{v \in V} h_{\ell,k} \tilde{v}^* F_{d_{\ell,k}} v s_{k,v} + \tilde{z}_{k,\tilde{v}} \]  
(3.72)
where
\[ \beta_{\tilde{v}} = \prod_{\ell,k} \beta_{\ell,k,i_{\ell,k}} \]  
(3.73)
is the product of the scaling coefficients for the indices $\tilde{i}_{\ell,k}$ that identify $\tilde{v}$ and
\[ \tilde{z}_{k,\tilde{v}} = \frac{1}{\beta_{\tilde{v}}} \tilde{v}^* z_k. \]  
(3.74)
Since $||\tilde{v}|| = \beta_{\tilde{v}} \sqrt{N}$, we have that $\tilde{z}_{k,\tilde{v}} \sim \mathcal{CN}(0, 1/N)$. 
Due to the orthogonality of the beamforming vectors, only one symbol from each transmitter will be observed in this effective channel. For \( k = 1, \ldots, K \), define

\[
\tilde{v}_k = \frac{\beta_{\ell,k}\tilde{i}_{\ell,k}-1}{\beta_{\ell,k}\tilde{i}_{\ell,k}} F - d_{\ell,k} \tilde{v}, \tag{3.75}
\]

taking \( \beta_{\ell,k,-1} = 0 \). The effective channel output can be expressed as

\[
\tilde{y}_k = \frac{1}{\tilde{\beta}_\ell} \sum_{k=1}^{K} h_{\ell,k}\tilde{v}^* F^{d_{\ell,k}} \tilde{v}_k s_{k,\tilde{v}} + \tilde{z}_{k,\tilde{v}} + \sum_{k \neq \ell} h_{\ell,k} \frac{\beta_{\ell,k}\tilde{i}_{\ell,k}-1}{\beta_{\ell,k}\tilde{i}_{\ell,k}} s_{k,\tilde{v}} + \tilde{z}_{k,\tilde{v}}. \tag{3.76}
\]

Note that each symbol will appear in exactly one effective channel output at each receiver.

We now select the scaling coefficients to induce lattice alignment. First, we set \( \beta_{\ell,\ell,i_{\ell,\ell}} = 1 \) for \( \ell = 1, \ldots, K \) and \( i_{\ell,\ell} = 0, \ldots, M \) to obtain

\[
\tilde{y}_{k,\tilde{v}} = \beta_{\ell} \left( h_{\ell,\ell} s_{\ell,\tilde{v}} + \sum_{k \neq \ell} h_{\ell,k} \frac{\beta_{\ell,k}\tilde{i}_{\ell,k}-1}{\beta_{\ell,k}\tilde{i}_{\ell,k}} s_{k,\tilde{v}} \right) + \tilde{z}_{k,\tilde{v}}. \tag{3.77}
\]

Next, we define

\[
k_{k,\min} = \arg\min_{k \neq \ell} |h_{\ell,k}| \tag{3.79}
\]

to be the index of the weakest interferer observed at receiver \( \ell \), and set \( \beta_{\ell,k_{\ell,\min},i_{\ell,\ell}} = 1 \) for \( \ell = 1, \ldots, K \) and \( i_{\ell,\ell} = 0, \ldots, M \) as well. Now, for each \( \ell \in \{1, \ldots, K\} \) and \( k \in \{1, \ldots, K\} \setminus \{\ell, k_{\ell,\min}\} \), we set

\[
\beta_{\ell,k,0} = 1, \tag{3.80}
\]
and recursively select integer and scaling coefficients for $i_{\ell,k} = 1, \ldots, M$ as follows:

$$
\beta_{\ell,k,i_{\ell,k}} = \frac{h_{\ell,k} \beta_{\ell,k,i_{\ell,k} - 1}}{h_{\ell,k_{\min}} b_{\ell,k,i_{\ell,k}}}, \\
(3.82)
$$

This yields the following effective channels:

$$
\bar{y}_{k,\tilde{v}} = \beta_{\tilde{v}} \left( h_{\ell,\ell} s_{\ell,\tilde{v}} + h_{\ell,k_{\min}} \sum_{k \neq \ell} b_{\ell,k,i_{\ell,k}} s_{k,\tilde{v}} \right) + \tilde{z}_{k,\tilde{v}},
$$

and guarantees that

$$
\frac{1}{2K^2} \leq |\beta_{\tilde{v}}| \leq 1.
(3.83)
$$

Overall, the $\ell^{th}$ receiver has $M K^2$ desired symbols, each of which is seen across an effective channel associated with a unique beamforming vector from $\tilde{\mathcal{V}}$. By indexing these effective channels and symbols with index $n = 1, \ldots, M K^2$, we obtain effective channels of the form

$$
\bar{y}_{\ell,n} = g_{\ell,n} s_{\ell,n} + g_{\text{int},\ell,n} s_{\text{int},\ell,n} + \tilde{z}_{\ell,n}
(3.84)
$$

where

$$
\frac{1}{2K^2} |h_{\ell,\ell}| \leq |g_{\ell,n}| \leq |h_{\ell,\ell}|
(3.85)
$$

$$
\frac{1}{2K^2} \min_{k \neq \ell} |h_{\ell,k}| \leq |g_{\text{int},\ell,n}| \leq \min_{k \neq \ell} |h_{\ell,k}|,
(3.86)
$$

and $s_{\text{int},\ell,n}$ is an integer-linear combination of (at most) $K - 1$ interfering symbols. From the perspective of each receiver, each symbol appears in exactly one effective channel. At each receiver, we will ignore the $N - M K^2$ effective channels that do not carry a desired symbol. In the next section, we evaluate the rate attainable over
these effective channels via lattice coding.

### 3.5.3 Lattice Coding

The lattice coding scheme for $K > 3$ users is very similar to that for $K = 3$ users. We summarize the key differences below, noting that we have intentionally omitted the effect of channel quantization.

Using the compute-and-forward framework [Nazer and Gastpar, 2011], the $k$th user selects a nested lattice codebook $C_\ell$ with power $P/N$ and rate

$$\tilde{R}_k = \log(1 + |h_{k,k}|^2 P) - 2K^2 - 1.$$  \hspace{1cm} (3.87)

It then splits its message into equal-rate submessages according to the total number of subchannels (which will depend on the channel quantization process). Each submessage is then mapped to a nested lattice codeword.

For a given channel realization $H$, we group together all useable time slots from the first subblock for which $H[t] = H$. Using the modulation and alignment scheme described above, the $k$th transmitter sends the codewords $s^{(H)}_{k,1}, \ldots, s^{(H)}_{k,MK^2}$ over these time slots. It then repeats these codewords over the matched time slots in the remaining $N-1$ subblocks, again following the modulation and alignment scheme. Note that the average transmit power is at most $\frac{MK^2P}{(M+1)K^2}$, which meets the power constraint.

The $\ell$th receiver observes its desired codewords over $MK^2$ effective channels of the form

$$\tilde{Y}_{\ell,n}^{(H)} = g^{(H)}_{\ell,n} s_{\ell,n}^{(H)} + g^{(H)}_{\text{int},\ell,n} s_{\text{int},\ell,n}^{(H)} + \tilde{Z}_{\ell,n}^{(H)}$$  \hspace{1cm} (3.88)

where $|g^{(H)}_{\ell,n}|$ and $|g^{(H)}_{\text{int},\ell,n}|$ satisfy (3.85) and (3.86), respectively, $s_{\text{int},\ell,n}^{(H)}$ is an integer-linear combination of at most $K - 1$ interfering codewords, and $\tilde{Z}_{\ell,n}^{(H)}$ is the effective noise of variance $1/N$. 

As before, the $\ell$th receiver first decodes the interference while treating the desired codeword as noise. Using [Nazer and Gastpar, 2011, Theorem 3] and [? , Lemma 5], the integer-linear combination $s_{\text{int}, \ell, n}^{(H)}$ can be decoded with vanishing probability of error from $\tilde{y}_{\ell, n}^{(H)}$ if the rates of all interfering codebooks satisfy

$$\tilde{R}_\ell < \log \left( \text{SINR}_{\text{int}, \ell, n}^{(H)} \right) \quad \ell \neq k$$

(3.89)

where

$$\text{SINR}_{\text{int}, \ell, n}^{(H)} \triangleq \frac{|g_{\text{int}, \ell, n}^{(H)}|^2 P}{|g_{\ell, n}^{(H)}|^2 P + \frac{1}{N}}$$

(3.90)

$$\geq \frac{1}{2^{2K^2}} \frac{\min_{k \neq \ell} |h_{\ell,k}|^2 P}{|h_{\ell,\ell}|^2 P + 1}$$

(3.91)

and the inequality follows from (3.85) and (3.86). From the very strong interference condition (3.13), we know that the rate of the $k$th user satisfies

$$\tilde{R}_k < \log \left( 1 + \frac{\min_{k \neq \ell} |h_{\ell,k}|^2 P}{|h_{\ell,\ell}|^2 P + 1} \right) - 2K^2 - 1$$

(3.92)

$$< \log \left( \frac{\min_{k \neq \ell} |h_{\ell,k}|^2 P}{|h_{\ell,\ell}|^2 P + 1} \right) - 2K^2$$

(3.93)

$$= \log \left( \text{SINR}_{\text{int}, \ell, n}^{(H)} \right),$$

(3.94)

which implies the interference is recovered successfully.

The $\ell$th receiver then forms the effective channel $\tilde{y}_{\ell, n}^{(H)} - g_{\text{int}, \ell, n}^{(H)} s_{\text{int}, \ell, n}^{(H)}$ and attempts to decode $s_{\ell, n}^{(H)}$. Since

$$\tilde{R}_\ell < \log \left( 1 + \frac{1}{2^{2K^2}} |h_{\ell,\ell}|^2 P \right) \leq \log \left( 1 + |g_{\ell, n}^{(H)}|^2 P \right),$$

(3.95)

decoding succeeds with vanishing probability of error.
Overall, the achievable rate for the $k^{th}$ transmitter is

$$R_k = \frac{MK^2}{(M + 1)K^2 \tilde{R}_k}. \quad (3.96)$$

By sending the parameter $M$ to infinity, it follows that the rates

$$R_k = \log(1 + |h_{k,k}|^2 P) - 2K^2 - 1 \quad (3.97)$$

are achievable for $k = 1, \ldots, K$. 
Chapter 4

Matching Alignment

For a $K$-user interference channel, ergodic interference alignment offers higher rates and degrees-of-freedom (DoF) compared to orthogonal transmission strategies such as time division multiplexing (TDM), frequency division multiplexing (FDM) [Nazer et al., 2012]. The basic idea underlying ergodic alignment is that, after transmitting symbols over a channel matrix $H$, the transmitters buffer their symbols, wait until a complementary channel matrix $H_C$ occurs, and then retransmit their symbols. Specifically, we say that a channel matrix is complementary if its diagonal elements are (nearly) equal of those in $H$ while its off-diagonal elements are (nearly) the additive inverse of those in $H$. Thus, if the receivers sum up their observations, they can obtain interference-free observations of their desired signals. Overall, this enables each user can reach half its interference-free capacity at any SNR (corresponding to $K/2$ total DoF).

Unfortunately, the ergodic alignment scheme requires extraordinarily long delays in order to match up a significant fraction of channel matrices and approach the promised rates. In fact, the ergodic alignment scheme was originally proposed to show that interference alignment was theoretically possible at any SNR, rather than as a roadmap towards a practical scheme. In this chapter, we take the opposite approach: instead of waiting for a “perfect” match for every channel matrix, attempt to find the best possible pairings amongst the available channel matrices. In particular, we consider a frequency-selective interference channel and propose a matching alignment
scheme that pairs together matrices across orthogonal sub-channels.

Unlike the ergodic alignment scheme, our proposed matching alignment scheme does not lead to a simple expression for the achievable rates. Instead, we cast the problem of finding the best pairings of sub-channels as a maximum weight matching problem over an undirected, bipartite graph. The vertices of this graph correspond to sub-channels and the edges correspond to the sum rate attainable by pairing together the linked sub-channels. It is well-known that polynomial-time algorithms exist for the maximum weight matching problem [Nemhauser and Wolsey, 1988, Ch. III.2]. Using these algorithms, we investigate the performance of our scheme numerically and demonstrate its advantages over conventional strategies such as frequency-division and treating interference as noise. We also investigate the average number of channel realizations needed to approach the ideal performance represented by ergodic alignment.

4.1 Problem Statement

We will consider a $K$-user, single-antenna Gaussian interference channel whose bandwidth is divided into $N$ orthogonal sub-channels, perhaps due to the use of orthogonal frequency-division multiplexing (OFDM). Note that this is the same scenario as that considered in the initial work of Cadambe and Jafar [Cadambe and Jafar, 2008]. We will assume that coding is carried out across $T$ time slots.

**Definition 7 (Messages)** Transmitter $k$ (for $k = 1, \ldots, K$) has a message $\omega_k$ that is generated independently and uniformly over $\{1, 2, \ldots, 2^{NTR_k}\}$.

**Definition 8 (Encoders)** This transmitter has an encoder $E_k : \{1, 2, \ldots, 2^{NTR_k}\} \rightarrow \mathbb{C}^{NT}$ that maps its message $\omega_k$ into a sequence of $T$ channel inputs $(x_k[1], \ldots, x_k[T]) = E_k(\omega_k)$ across the $N$ sub-channels where $x_k[t] = [x_k[1][t] \cdots x_k[N][t]]^T$ and $x_k[m][t]$ is the input to the $m^{th}$ sub-channel during time slot $t$. We impose a power constraint across
each sub-channel,

$$\frac{1}{TN} \sum_{t=1}^{T} \sum_{m=1}^{N} |x_k^{[m]}[t]|^2 \leq P.$$ \hfill (4.1)

**Remark 2** Notice that this power constraint precludes the use of power allocation across sub-channels (e.g., waterfilling).

**Definition 9 (Channel Model)** For $m = 1, \ldots, N$ and $\ell = 1, \ldots, K$, the output of the $m^{th}$ sub-channel at the $\ell^{th}$ is given by

$$y_{\ell}^{[m]}[t] = \sum_{k=1}^{K} h_{\ell,k}^{[m]}[t] x_k^{[m]}[t] + z_{\ell}^{[m]}[t] \hfill (4.2)$$

where $h_{\ell,k}^{[m]}$ is the channel gain from transmitter $k$ and $z_{\ell}^{[m]}[t]$ is independent and identically distributed (i.i.d.) $\mathcal{CN}(0,1)$ noise. See Fig 4·1. For simplicity, we will assume that all transmitters and receivers have full channel state information.

---

![Diagram of the K-user interference channel.](image)

**Figure 4·1:** Diagram of the $K$-user interference channel.

Let $y_{\ell}[t] = [y_{\ell}^{[1]}[t] \cdots y_{\ell}^{[N]}[t]]^T$ denote the vector of the $\ell^{th}$ receiver’s observations across all sub-channels at time $t$. 
Definition 10 (Decoders) For $\ell = 1, \ldots, K$, the $\ell^{th}$ receiver has a decoding function $D_{\ell} : \mathbb{C}^{NT} \rightarrow \{1, 2, \ldots, 2^{NTR_{\ell}}\}$ that it uses to make an estimate

$$\hat{\omega}_{\ell} = D_{\ell}(y_{\ell}[1], \ldots, y_{\ell}[T])$$

of its desired message $\omega_{\ell}$.

Definition 11 (Achievable Rate Tuples) We say that a rate tuple $(R_1, \ldots, R_K)$ is achievable if, for any $\epsilon > 0$ and $T$ large enough, there exist encoders and decoders that can attain probability of error at most $\epsilon$, $\mathbb{P}\left(\bigcup_{\ell=1}^{K}\{\hat{\omega}_{\ell} \neq \omega_{\ell}\}\right) < \epsilon$. We say that a sum rate $R$ is achievable if there is at least one achievable rate tuple satisfying $\sum_{m=1}^{K} R_m = R$.

Throughout the chapter, we will focus on linear encoding and decoding strategies across sub-channels coupled with standard i.i.d. Gaussian codebooks across time. Hence, for notational convenience, we will drop the time index $t$ for the remainder of the chapter. For example, we assume that $x_k^{[m]}$ refers to all inputs from the $k^{th}$ transmitter into the $m^{th}$ sub-channel.

4.2 Classical Techniques

4.2.1 Orthogonalization

One natural strategy is to avoid interference and assign only one user to each orthogonal sub-channel (e.g., frequency-division multiplexing). Since we have imposed a power constraint on each sub-channel, the sum-rate optimal orthogonalization strategy is to activate the user with the largest channel magnitude. Overall, this leads to the following sum rate:

$$R_{\text{FDM}} = \frac{1}{N} \sum_{m=1}^{N} \max_{\ell} \log \left(1 + |h_{\ell,m}^{[\ell]}|^2 P\right). \quad (4.3)$$
4.2.2 Treat Interference as Noise

Another possibility is for every user to use every sub-channel simultaneously (at full power) and then treat the resulting interference as noise at the receivers. This strategy can achieve the following sum rate:

\[
R_{\text{TIN}} = \frac{1}{N} \sum_{m=1}^{N} \sum_{\ell=1}^{K} \log \left( 1 + \frac{|h_{\ell,k}^{[m]}|^2 P}{1 + \sum_{k \neq m} |h_{\ell,k}^{[m]}|^2 P} \right). \tag{4.4}
\]

In certain regimes corresponding to low interference strength, it is known that treating interference as noise is optimal [Annapureddy and Veeravalli, 2009, Motahari and Khandani, 2009, Shang et al., 2009].

4.3 Analysis of Ergodic Alignment in Infinite Bandwidth

Recall the ergodic interference alignment scheme discussed in Section 2.3. Note that the analysis of the ergodic alignment can be carried out over frequency sub-channels as independent channel uses compared to time slots in the previous discussion. In this case, frequency selective channels are assumed to provide the necessary channel variation to make matching possible. The issue of the delay would translate to bandwidth limitation. This means that if ergodic alignment is to be carried over frequency sub-channels instead of time slots, the number of required sub-channels will scale with \((KP)^{K^2/2}\). This enormous bandwidth requirement is one of the main reasons that makes ergodic interference alignment far from practical. The ability of ergodic alignment to achieve high rates at finite SNR motivates our proposed scheme. However, we will back off from the huge sub-channels requirement by assuming finite number of sub-channels. We will try to pair sub-channels in a similar way to ergodic alignment and explore the rates we can achieve.
4.4 Matching Alignment Scheme

As discussed above, our scheme is closely related to ergodic interference alignment [Nazer et al., 2012] with one important distinction: instead of waiting for a perfect match for each channel realization, we try to find the best pairings among the available $N$ sub-channels.

The reason we take this path is that, as argued in [Nazer et al., 2012, Sec. IV-A], the expected delay to find a (near) perfect match for a given sub-channel is extremely high and scales roughly as $(KP)^{K^2/2}$. As a result, the ergodic alignment scheme is often viewed as impractical. Our proposed matching alignment scheme is designed to overcome this delay bottleneck. Given a finite set of $N$ sub-channels, it aims to find the set of $N/2$ sub-channel pairings that yields the highest sum rate. See Figure 4·2. Specifically, we calculate the sum rate attainable over a potential sub-channel pairing via the signal-to-interference-and-noise ratios (SINRs) of each user using linear signaling strategies combined with i.i.d. Gaussian codebooks. We then take advantage of classical maximum weight matching algorithms to determine the optimal set of pairings. Below, we give detailed expressions for the achievable sum rate. Afterwards, we discuss how to optimize the linear scaling parameters at the receivers and transmitters.

4.4.1 Sum Rate

We begin by evaluating the sum rate available for matching two arbitrary sub-channels $m, n \in \{1, \ldots, N\}, m \neq n$. We set $x_k^{[m]}$ to be a scaled version of the symbol $s_k$ that transmitter $k$ intends to send to receiver $k$ where $s_k$ is assumed to be an element of a codeword drawn from an i.i.d. Gaussian codebook with power $P$. More formally,

$$x_k^{[m]} = v_k^{[m]} s_k$$
Figure 4.2: Channel bandwidth is divided into $N$ sub-channels with the focus on an arbitrary time slot.

where $v_k^{[m]}$ is a power scaling parameter and $|v_k^{[m]}| \leq 1$ for all $k = 1, \ldots, K$ and $m = 1, \ldots, N$. Let $\mathbf{v}^{[m]} = [v_1^{[m]}, \ldots, v_K^{[m]}]^\top$ and $\mathbf{V} = [\mathbf{v}^{[1]}, \ldots, \mathbf{v}^{[N]}]$. To mimic the idea of ergodic interference alignment, we will repeat the symbol $s_k$ over sub-channel $v$,

$$x_k^{[n]} = v_k^{[n]} s_k.$$  

Receiver $\ell$ will linearly mix the two sub-channel outputs, $y_\ell^{[m]}$ and $y_\ell^{[n]}$, that is

$$\tilde{y}_\ell = u_\ell^{[m]} y_\ell^{[m]} + u_\ell^{[n]} y_\ell^{[n]}$$

where $u_\ell^{[m]}$ and $u_\ell^{[n]}$ are scaling parameters. Let $\mathbf{u}^{[m]} = [u_1^{[m]}, \ldots, u_K^{[m]}]^\top$ and $\mathbf{U} = [\mathbf{u}^{[1]}, \ldots, \mathbf{u}^{[N]}]$. Define

$$S^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]}) = |u_\ell^{[m]} h_{\ell, \ell} v_\ell^{[m]} + u_\ell^{[n]} h_{\ell, \ell} v_\ell^{[n]}|^2 \mathbf{P},$$  

$$I^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, \mathbf{v}^{[m]}, \mathbf{v}^{[n]}) = \sum_{k \neq \ell} |u_\ell^{[m]} h_{\ell,k} v_k^{[m]} + u_\ell^{[n]} h_{\ell,k} v_k^{[n]}|^2 \mathbf{P}.$$
If each receiver treats interference as noise, the rate of the sub-channel pair \( \{m, n\} \) is given by
\[
R_{\ell}^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]}) = \frac{1}{2} \log \left( 1 + \text{SINR}^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]}) \right)
\] (4.5)

where
\[
\text{SINR}^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]}) = \frac{S^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]})}{|u_\ell^{[m]}|^2 + |u_\ell^{[n]}|^2 + I^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]})}.
\]

Let the sum rate be
\[
R_{\Sigma}^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]}) = \sum_{\ell=1}^{K} R_{\ell}^{[m,n]}(u_\ell^{[m]}, u_\ell^{[n]}, v_\ell^{[m]}, v_\ell^{[n]}),
\]
where \( R_{\Sigma} \) is a function of the pair \( \{m, n\} \), the power scaling parameters \( v^{[m]}, v^{[n]} \) and the receiver scaling parameters \( u^{[m]} \) and \( u^{[n]} \). An optimal choice of these parameters is the one that maximizes the sum of such rate over all available sub-channel pairs.

More formally, let
\[
R_{\text{MA}}^{[m,n]} = \max_{v^{[m]}, v^{[n]}} \left( \max_{u^{[m]}, u^{[n]}} \left( R_{\Sigma}^{[m,n]}(u^{[m]}, u^{[n]}, v^{[m]}, v^{[n]}) \right) \right)
\] (4.6)
be the maximum sum rate that can be achieved when the two sub-channels \( m \) and \( n \) are paired. We define a pairing scheme \( \mathcal{M} \) to be a subset of all possible pairs \( \{m, n\} \) where \( m, n \in \{1, \ldots, N\} \), such that if \( \{m, n\}, \{m', n'\} \in \mathcal{M} \) then \( m \neq m', n' \) and \( n \neq m', n' \). Let \( \mathcal{U} \) be the set of all possible pairing schemes \( \mathcal{M} \). Now, the problem can be written as the following three-fold optimization problem
\[
R = \max_{\mathcal{M} \in \mathcal{U}} \left( \frac{1}{N} \sum_{\{m,n\} \in \mathcal{M}} R_{\text{MA}}^{[m,n]} \right)
\] (4.7)
where \( R \) is the average sum rate for a given set of \( N \) sub-channels.
We will try to solve this problem piece by piece. We have a maximization over the receiver parameters $\mathbf{u}$, a maximization over transmitter parameters $\mathbf{v}$, and the maximization over the pairing scheme $\mathcal{M}$. We will deal with these problems as follows.

### 4.4.2 Receiver Optimization

If the symbol $s_\ell$ is transmitted over sub-channels $m$ and $n$, then the receiver obtains two noisy observations of it, namely $y_{\ell|m}$ and $y_{\ell|n}$. Then, the estimate that maximizes the SINR is the MMSE estimator, and the receiver scaling parameters are the MMSE coefficients. The estimated symbol at the receiver side $\hat{s}_\ell$ is given by

$$\hat{s}_\ell = \mathbb{E}\left[ s_m | y_{\ell|m}, y_{\ell|n}, h_{\ell|m}, h_{\ell|n}, v_{[m]}, v_{[n]} \right]$$

$$= u_{\ell|m} y_{\ell|m} + u_{\ell|n} y_{\ell|n}$$

by plugging in the optimal values of $u_{\ell|m}, u_{\ell|n}$ in (4.5) we get

$$R_{\ell|m,n}(v_{[m]}, v_{[n]}) = \log \left( 1 + P g_{\ell,m}^* \left( I_2 + P \sum_{k \neq \ell} g_{\ell,k}^* g_{\ell,k} \right)^{-1} g_{\ell,m} \right)$$

(4.8)

where $g_{\ell,m} = \left[ v_{k|m} h_{\ell,k|m}, v_{k|n} h_{\ell,k|n} \right]^T$ and $I_n$ is the identity matrix of size $n$. We get the following sum rate

$$R_{\Sigma|m,n}(v_{[m]}, v_{[n]}) = \sum_{m=1}^{K} R_{\ell|m,n}(v_{[m]}, v_{[n]})$$

(4.9)

Hence, the maximum sum rate (4.6) reduces to

$$R_{MA}^{[m,n]} = \max_{V_m, V_n} \left( R_{\Sigma|m,n}(V_m, V_n) \right)$$

(4.10)
4.4.3 Transmitter Optimization

Simultaneous maximization of the transmitter and receiver scaling parameters corresponds to a non-convex problem. As a workaround, we have chosen to numerically approximate the maximum sum rate. Specifically, for each possible pairing, we will randomly sample transmitter scaling parameters according to a uniform distribution and use the maximum sum rate attained across these realizations.

4.4.4 Maximum Weight Matching

At this point, we have worked out the sum rate of each two sub-channels if paired. To find the optimal way to pair the sub-channels, we will pose the problem as a maximum weight matching problem, which is a well-known problem from the combinatorial optimization literature [Nemhauser and Wolsey, 1988, Ch. III.2]. Matching is defined to be a subset of edges in which no two edges share the same node. Maximum weight matching is defined over weighted undirected graph. It is the problem of finding the matching with the largest weight. There exists a polynomial-time algorithm to solve the problem. The main idea of the solution is to use the Hungarian algorithm, which solves the same problem when defined over a bipartite graph [Papadimitriou and Steiglitz, 1982, Chapter 11] then generalize it to solve the problem over a general graph (not bipartite). The Hungarian algorithm poses the bipartite maximum weight matching as a linear program. To generalize the solution for general graphs, the idea of shrinking a blossom is applied. The time complexity of the maximum weight matching is $O(n^3)$ where $n$ is the number of nodes in the graph.

4.4.5 Sub-channels pairing

To find the best possible pairing of sub-channels, we will pose the problem as a maximum weight matching problem. We will build our graph as follows, each sub-channel represents a node in the graph. Every node is connected to all other nodes
with weighted undirected edges. Let the weight of the edge connecting node \( m \) to \( n \) be \( w(m, n) \). We will apply two different weight functions and we will compare their results on the same plot together with the traditional techniques as well as the ergodic alignment.

**Matching Alignment Weight Function**

This weight function requires that all the sub-channels are paired. The weight of an edge corresponds to the maximum sum rate attainable by pairing the two nodes connected by that edge, i.e., the rate we have from (4.10),

\[
w(m, n) = R_{\text{MA}}^{[m,n]}\]

**Best of All Weight Function**

We do not insist on pairing each sub-channel. However, we only pair sub-channels if they are better paired, otherwise, we use FDM or TIN whichever better. More precisely, for each sub-channel \( m \) we define

\[
R_{\text{TIN}}^{[m]} = \sum_{\ell=1}^{K} \log \left( 1 + \frac{|h_{\ell,\ell}|^2 P}{1 + \sum_{k \neq \ell} |h_{\ell,k}|^2 P} \right) \\
R_{\text{FDM}}^{[m]} = \max_{\ell} \log \left( 1 + |h_{\ell,\ell}|^2 P \right)
\]

to be the TIN rate and the orthogonalization rate respectively. Therefore, the best rate among TIN and orthogonalization techniques is given by

\[
R_{s}^{[m]} = \max \left( R_{\text{TIN}}^{[m]} , R_{\text{FDM}}^{[m]} \right)
\]

hence, we assign the weights as follows

\[
w(m, n) = \max \left( R_{\text{MA}}^{[m,n]} , R_{s}^{[m]} + R_{s}^{[n]} \right)
\]
To decide how the sub-channels are to be paired, we will input our graph into a maximum weight matching algorithm. See Figure 4·3. The output is the matching with the largest possible sum of weights. The edges selected in the output matching are connecting the sub-channels that will be paired. For example, if the edge that connects node 2 and node 5 is selected in the matching, this means that all transmitters will send over sub-channel 5 the symbols that were sent over sub-channel 2.

Figure 4·3: An example of a maximum weight matching $\mathcal{M}^*$ where $\sum_{(m,n) \in \mathcal{M}^*} w(m,n) = 8$. Each edge weight represents the maximum sum rate for pairing the associated vertices (sub-channels). Note that edges that belong to the same matching, have the same color.

4.5 Pair-wise Upper Bound

We will display the results of different techniques on the same plot along with a pair-wise upper bound. The upper bound assumes that each receiver is observing only
two signals, its own signal and the most harmful interfering signal. Define

$$\text{UB}(\ell; k) = \frac{1}{N} \sum_{m} \log \left( 1 + \frac{|h_{\ell,\ell}[m]|^2 + |h_{\ell,k}[m]|^2}{\min \left( 1, \frac{|h_{\ell,\ell}[m]|^2}{|h_{\ell,k}[m]|^2} \right) P} \right).$$

Using the result from [Nazer et al., 2012, Appendix A], we get that

$$R_{\ell} + R_{k} \leq \min \left( \text{UB}(\ell; k), \text{UB}(k; \ell) \right), \quad (4.11)$$

for $\ell \neq k$ where $R_{\ell}$ the rate of user $\ell$. Note we take the minimum of the two terms since UB($\ell; k$) is not symmetric. Define the set of all possible pairs of users

$$S_{K} = \{ (\ell, k) | \ell, k \in \{1, \ldots, K\}, \ell \neq k \},$$

thus we can generate the upper bounds as in (4.11) for all possible pairs in $S_{K}$. By adding all these bounds, we get a bound on the sum rate

$$(K - 1) \sum_{\ell=1}^{K} R_{\ell} \leq \sum_{(\ell,k) \in S_{K}} \min \left( \text{UB}(\ell; k), \text{UB}(k; \ell) \right).$$

Finally, we have

$$\sum_{\ell=1}^{K} R_{\ell} \leq \frac{1}{(K - 1)} \sum_{(\ell,k) \in S_{K}} \min \left( \text{UB}(\ell; k), \text{UB}(k; \ell) \right). \quad (4.12)$$

4.6 User Selection

4.6.1 User Selection Problem

So far we considered a $K \times K$ interference channel. A more general case is that the network has $\tilde{K}$ users and the goal is to pick only the best $K$. In this section, we will consider two scenarios of this problem. Throughout the rest of the chapter, we assume that $\tilde{K} > K$. 
In this case, we have $\tilde{K}$ transmitters and $\tilde{K}$ receivers. Each transmitter has a message for its corresponding receiver. This can be thought of as $\tilde{K} \times \tilde{K}$ interference channel. However, only $K$ pairs are allowed to transmit over any given sub-channel. Yet, we allow that different pairs operate on different sub-channels.

$K \times  \tilde{K}$ Network

Another interesting case comes from the cellular network, specially the downlink case. In this scenario, each resource block is allocated to one out of a number of active receivers in the range of the transmitter. This is done through a scheduler algorithm in the transmitter. We will use this example to motivate the second network setup.

We assume we have a fixed number of transmitters, say $K$, and $\tilde{K}$ receivers. We want to select $K$ out of the $\tilde{K}$ active receivers to communicate with the transmitters; one receiver for each transmitter. This will create a $K \times K$ interference channel which we will apply the matching alignment scheme. Therefore, the solution must find the best possible subset of active users and find the best possible sum rate for the selected subset.

4.6.2 Algorithm for User Selection

To select the best set of active users, we will modify the way we assign the weights of the edges of the graph which is input to the maximum weight matching algorithm. We will exhaustively search all the subsets of active users of size $K$ to maximize the sum rate. Precisely, let

$$\mathcal{K} = \{\{k_{\ell_1}, \ldots, k_{\ell_K} \} : k \in \{1, \ldots, \tilde{K}\}, k_m \neq k_n \text{ if } m \neq n\}$$  \hspace{1cm} (4.13)
to be the set of all possible subsets of size $K$ of the active users. Then the edge connecting node $m$ and node $n$ has a weight

$$w(m, n) = \max_{\mathcal{K}} R_{\text{MA}}^{[m,n]},$$

where $R_{\text{MA}}^{[m,n]}$ is calculated as in (4.10).

**Remark 3** $R_{\text{MA}}^{[m,n]}$ is a function of the selected users since it is a function of the channel gains.

With the weights of all edges fixed, we run the maximum weight matching algorithm as before.

**Remark 4** This modification changes the complexity of the algorithm to be

$$O \left( \tilde{K}^K N^2 + N^3 \right),$$

since for each edge, we check $\tilde{K}^K$ potential edges, and select the best before running the maximum weight matching algorithm.

We show the performance of this solution compared to orthogonalization technique as in Sec. 4.2.1 and a modified TIN technique. The modified TIN will search for the best $R_{\text{TIN}}^{[m,n]}$ over the set $\mathcal{A}$ as defined in (4.13).

### 4.7 Numerical Results

We now investigate the performance of our algorithm numerically and compare it to classical strategies such as FDM and TIN as well as the idealized performance represented by ergodic alignment. For maximum weight matching, we employed the MATLAB code provided in [Saunders, 2013]. It is based on the idea of shrinking a blossom which is discussed in [Nemhauser and Wolsey, 1988, Ch. III.2] and its complexity is $O(n^3)$ [Saunders, 2013, Rantwijk, 2008] where $n$ is the number of nodes.
4.7.1 Abstract Channel Model

In Figure 4.4, we investigate the average sum rate for a 3-user interference channel with $N = 50$ Rayleigh faded sub-channels, i.e., $\{h_{\ell,k}^{[m]}\}$ are drawn i.i.d. $\mathcal{CN}(0, 1)$. The plot shows the upper bound from [Nazer et al., 2012], the idealized ergodic interference alignment rate

$$R_{EA} = \frac{1}{N} \sum_{m=1}^{N} \left[ \sum_{\ell=1}^{K} \frac{1}{2} \log \left( 1 + 2 |h_{\ell,\ell}^{[m]}|^2 P \right) \right],$$

the FDM (orthogonalization) rate which is given by (4.3), and the TIN rate which is given by (4.4). Notice that matching alignment offers a significant gain over FDM and TIN, but does not match the slope of ergodic alignment. While the sum rate is a good indication on the overall performance of the network, it does not capture how fair these techniques are to the users of the network. Figure 4.5 displays the average symmetric rate against the SNR (in dB) as a measure of the performance of the network if fairness is enforced, that is, all users are forced to operate at the same rate. Note that, we did not change the algorithm, i.e., every technique is still trying to maximize the sum rate. The upper bound we use is generated using (4.12) and plugging in $R_\ell = R_{sym}$ for all $\ell$, hence we get

$$R_{sym} \leq \frac{1}{K(K-1)} \sum_{(\ell,k) \in S_K} \min \left( \text{UB}(\ell,k), \text{UB}(k,\ell) \right).$$

Next, we examine how many sub-channels are paired and how many sub-channels used FDM and the same for TIN to generate the “Best of All” curve in Figure 4.4. This indicates the percentage of sub-channels for which each scheme was the best.

From Figure 4.6, in the 3-user case and when we have 50 sub-channels, matching alignment is most useful for SNR level below 11 dB. This range will increase if we increase the available sub-channels $N$, because increasing $N$ improves the performance
Figure 4.4: Performance of different schemes under Rayleigh fading for $K = 3$ and $N = 50$.

of matching alignment. Note that although the average number of sub-channels that used orthogonalization technique is more than those which used matching alignment for $\text{SNR} > 12$, this was not reflected as a gain for “Best of All” over “Matching Alignment”. This is due to that both matching alignment rate and orthogonalization rate are close to each other in this regime and therefore, picking the best of them is not significantly better than the other.

Figure 4.7 shows the average rate of all edges that used matching alignment against SNR in dB and on the same figure we have the average rate of all edges that used
orthogonalization and that used TIN.

This motivates looking into the relation between the SNR and the number of sub-channels $N$ required to attain a certain performance. We cannot use rate as measure of performance in this case because it depends directly on SNR. A natural choice to measure the performance could be the ratio of the rate of matching alignment to that of ergodic alignment. Figure 4.8 gives the relations between SNR and the number of required sub-channel to attain 90% of ergodic alignment rate. We can see that the relation between $N$ and SNR in dB is almost linear (for $\text{SNR} > 2.5 \text{ dB}$) indicating that
Figure 4.6: The average count of each scheme for $K = 3$ and $N = 50$.

to keep the performance of matching alignment at certain level, we need on average 10 sub-channels for every 1 dB increase in SNR.

Another interesting case is the weak interference case where the interference power is less than the signal power. More precisely, we have the channel gains as follows $\{h^{[m]}_{\ell,k}\}$ are i.i.d. drawn from $\mathcal{CN}(0, \gamma)$ if $k \neq \ell$ and from $\mathcal{CN}(0, 1)$ otherwise. Figure 4.9 shows the performance of the different schemes for the mentioned channel model at $\gamma = 0.25$. We can see that both TIN and matching alignment perform better than the first case. The gap between infinite horizon ergodic alignment and matching is around 2.5 dB in the high SNR range ($\text{SNR} > 15$ dB) which is less than the previous
Figure 4.7: The average rate of selected edges that used each technique $K = 3$ and $N = 50$.

4.7.2 User Selection Results

Here we discuss the results of the user selection problem as discussed in Section 4.6.1. We run the algorithm discussed in Section 4.6.2. Below is the performance of matching alignment compared to the classic techniques of the two scenarios discussed earlier.
Results of $\tilde{K} \times \tilde{K}$ Network

Here we assume all signals suffer the same attenuation and hence the SNR and INR at all receivers are equal in power. We run the algorithm for $\tilde{K} = 10$, $K = 3$ and $N = 50$. Figure 4.10 shows that the matching alignment still outperforms classical techniques, however, the dB gain is less than previous case.

Results of $K \times \tilde{K}$ Network

We now show the performance of the matching alignment with user selection as discussed in Section 4.6.2. The result is for a cellular network where the distance between neighboring transmitters is 10 meters which is a realistic assumption when considering a picocell setup. The number of cells is 3 which makes $K = 3$ and the number of users in each cell is 5 which makes $K = 3$ and $\tilde{K} = 15$. The path loss between transmitter $k$ and receiver $\ell$ is given by

$$PL(\ell,k) = 128.1 + 37.6 \log\left(\frac{\text{Distance}(\ell,k)}{1000}\right)$$

where the Distance($\ell,k$) is the distance between transmitter $k$ and receiver $\ell$ measured in meters [3GPP TR36.814 (V9.0.0), 2010]. We assume the cells are in the form of squares. For each cell, we randomly allocate 5 receivers in a square of side 10 meters, where the center of the square is the transmitter. For each transmitter $k \in \{1, \ldots, K\}$ and for each receiver $\ell \in \{1, \ldots, \tilde{K}\}$, the channel gains $\{h_{\ell,k}^{[m]}\}$ are i.i.d. drawn from $\mathcal{CN}(0,-PL)$. Figure 4.11 shows the performance of the algorithm under these parameters when transmitter SNR = 70 dB which leads to around 30 dB of SNR at the receiver side. We can see that matching alignment outperforms the best of TIN and orthogonalization schemes. However, Figure 4.12 shows the opposite when the separation between the base stations is 50 meters while all other parameters are kept fixed.
Figure 4.8: Average required number of sub-channels $N$ versus SNR to hold the rate of matching alignment fixed at 90% of the ergodic alignment rate for the three user case.
Figure 4-9: Performance of the different schemes for the weak interference case where $K = 3$, $N = 50$ and $\gamma = 0.25$. 
Figure 4.10: Performance of the user selection algorithm in a $\tilde{K} \times \tilde{K}$ network where $\tilde{K} = 10$, the number of users allowed to transmit over any given sub-channel $K = 3$ and the number of the sub-channels $N = 50$. 
Figure 4.11: Performance of matching alignment in a cellular network setup where $K = 3, \tilde{K} = 15$, $N = 50$ and BS separation 10 meters.
Figure 4.12: Performance of matching alignment in a cellular network setup where $K = 3, \tilde{K} = 15$, $N = 50$ and BS separation 50 meters.
Chapter 5
Conclusions

In this dissertation, we studied the problem of the wireless interference channel from different angles. First, we studied the fundamental capacity limits of the phase-fading $K$-user Gaussian interference channel. We developed an alignment scheme that combines ideas from ergodic alignment and compute-and-forward. The alignment scheme uses carefully matched channel realizations along with power allocation to make all the interfering codewords form an integer linear relation at all receivers simultaneously. Using lattice codebook, each receiver sees one interference codeword only which is the integer linear combination of all the interfering codewords. We can achieve the outer bound of the capacity region of the $K$-user interference channel in the very strong regime up to a constant gap by using lattice decoding.

On another hand, we studied the Gaussian interference channel under more practical conditions. To do so, we developed the matching alignment algorithm, which is inspired by the ergodic alignment scheme. Matching alignment does not assume an infinite horizon and is not allowed to wait for a perfect match for each channel realization. Instead, matching alignment takes a more pragmatic approach. It tries to find the best match among already given finite-size set of channel realization using graph matching from a combinatorial optimization point-of-view. We studied the performance of this matching alignment algorithm numerically. We applied it to the Gaussian interference channel in different SNR regimes and in different interference regimes. We showed that we can have some dB gains in over traditional techniques.
5.1 Future Directions

The alignment scheme developed in the first part of the dissertation can be applied in all interference regimes. This motivates to better understand the performance of the scheme in the strong regime which can lead to characterizing the symmetric capacity in the strong regime. Another future direction is to try to understand how our alignment scheme behaves in the weak and moderately weak regimes and whether it can achieve the outer bounds (up to a constant gap) or not.

In addition, matching alignment has a good potential to be applied in real life scenarios, studying the performance of it when applied with practical communications conditions will be very interesting.
3GPP TR36.814 (V9.0.0) (Mar. 2010). Evolved universal terrestrial radio access (e-utra); further advancements for e-utra physical layer aspects.


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  – Developed an algorithm for multi-user wireless channels that pairs frequency sub-channels to maximize signal-to-interference-and-noise ratio. The algorithm uses general graph matching and power scaling techniques.
  – Developed a scheme for the $K$-user interference channel to achieve the capacity region in when the interference is very strong using nested lattice codes.

• Teaching Assistant Sept. 2013 - May 2014
  – Held recitation sessions for undergraduate students in probability theory in electrical and computer engineering.

Alexandria University, Engineering Mathematics and Physics Department

• Teaching Assistant Feb. 2008 - July 2011
  – Held discussion sessions for undergraduate engineering students for mathematical engineering courses such as ordinary/partial differential equations, vector analysis, analytical geometry.
– Supervised a group of two other teaching assistants.
– Helped in grading exams.

RESEARCH INTEREST

• Information Theory.
• Multi-user Wireless Communication.

PUBLICATIONS


VOLUNTEER EXPERIENCE

**Boston University Academy**

• Robot Design Judge at FIRST LEGO League Dec. 2014
  – Assessed and judged the robot design of middle school teams in FIRST LEGO League held at BU Academy.