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Space-variant Fourier analysis: the exponential chirp transform *

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Abstract

Space-variant sensing is the architectural basis of all higher vertebrate visual systems (Schwartz, 1994). One evident motivation for this is that the space-complexity of the human visual system is reduced by up to four orders of magnitude (Royer and Schwartz, 1990) via the use of space-variant architecture (for a given ratio of field width to maximum resolution). This observation has obvious practical advantages for application in machine vision. Unfortunately, the practical application of space-variant image architectures is obstructed by the difficulty of performing common image processing operations in a domain of varying pixel size and connectivity. Despite some recent progress in this area (e.g. see (Wallace et al., 1994)) it has so far been impossible to apply familiar frequency domain image processing techniques directly to space-variant images.

In this paper we focus on a particular space-variant map, the log-polar map, which has been shown to model the primate visual system and which has been applied to machine vision contexts by a number of investigators during the past two decades. Associated with the log-polar map is an exponential chirp transform which allows frequency domain estimation in the log-polar plane, while preserving an aspect of the shift-invariant properties of the usual Fourier transform. (Note that the familiar Mellin transform, which is a Fourier transform applied to the log-polar FREQUENCY domain, is a related, but very different approach. Specifically, the Mellin transform is a shift-invariant form of image processing which, per se, has absolutely nothing to do with foveal vision).

We demonstrate application of the exponential chirp with several simple template matching examples, and show that aspects of shift, size and rotation invariance are provided, while still preserving the underlying space-variant architecture of the sensor. We describe three different algorithms for computing the exponential chirp transform of an image. Somewhat surprisingly, we show that by combining the exponential chirp with the Mellin transform, it is possible to evaluate the exponential chirp transform with the same computational complexity as the FFT. Thus, the favorable space-complexity of the log-polar architecture may be joined with the computational complexity of the FFT. Moreover, the favorable symmetry properties of the Mellin transform and log-polar mapping are combined, using the methods of this paper, with a foveal image architecture, to provide a form of invariant template matching (using frequency domain convolution) at rates which are several orders of magnitude faster than is possible with conventional space-invariant image formats. We suggest that the methods outlined in this
paper provide a practical means of performing machine vision on log-polar image formats.
1 Introduction

1.1 Space-variant vision, log-polar mapping and the spatial architecture of biological vision

It was suggested by one of us (Schwartz, 1977) that the two-dimensional map function provided by the complex logarithm provides an approximate model of the representation of the visual field in the primate visual cortex. In more recent work it has been shown that a related conformal mapping, found numerically by a combination of computer-aided brain flattening and other anatomical and computational techniques, is in fact an accurate model of the primate spatial architecture. Error bars in the range of 5-10% have been placed on these estimates (Schwartz, 1994). Since no other models of two-dimensional human topography have been presented with error analysis (and, in fact, all one-dimensional analysis to date also lack error analysis), we suggest that conformal map models, together with the earlier, and somewhat less accurate log-polar model, provide a good working basis for machine vision applications of novel biological architectures for image processing.

The log-polar (also known as the complex logarithmic) map is defined as follows:

\[ w = K \log(z + a) \]  

In this equation, \( K \) is an experimental constant which is only of relevance to the biological scale factor of a particular map, and will be dropped in the following discussion (see (Schwartz, 1994) for a review of estimates of this parameter). The constant \( a \) is a real parameter, which provides the region of quasi-linear mapping near \( z = 0 \), and which is generally believed to lie in the range of 0.3 to 0.7 degrees (see (Wilson et al., 1990) and (Schwartz, 1994) for discussion and reviews of this issue).

In the context of image processing, the log-polar coordinate transform must be joined with an algorithm for image warping, which expresses the space-variance of pixel size, as well as pixel location. This can be stated as follows. We associate each pixel \( W \) from the range of the log-polar warp (using \( f \) to represent the log-polar mapping) a set of domain pixels \( f^{-1}(W) \in Z \), where \( Z \) represents the set of domain pixels which would come,
commonly, from a conventional video sensor. Formally, using the notation $\Xi z$ to represent the location of a domain pixel,

$$f^{-1}(W) = Z \mid \log(\Xi z + a) \in W$$  \hspace{1cm} (1.2)

Thus, each pixel $W$ represents the support of a small group of pixels in the domain, whose size increases with increasing distance from the origin. The location of these pixels is expressed by the complex logarithmic coordinate change of equation (1.1).

Figures 7 and 9 of this paper show some examples of the distorted image warp associated with these equations. A more extensive discussion of the general conformal image warp problem is provided in (Frederick and Schwartz, 1990).

Equations (1.1) and (1.2) summarize our current state of knowledge about the structure of the large-scale spatial layout of the primate visual field\footnote{We do not address other issues of biological functional architecture here, such as columnar structure, receptive field structure, etc. See the recent review (Schwartz, 1994) for a discussion of some of these other issues.}. They establish the spatial data structure for human vision, (to the extent that it is currently understood), and provide both opportunity and challenge for computer vision applications, specifically:

**Opportunity** Up to four orders of magnitude of image compression can be provided (and is believed to be provided in human vision (Rojer and Schwartz, 1990)) by the use of space-variant architectures. Secondarily, the particular form of the image warp associated with the complex logarithm function is known to be related to size and rotation invariance in a fundamental way. A large body of literature has been published over the past two decades around this latter observation, but there has been relatively little practical success to date in exploiting this feature of log-polar architecture.

**Challenge** The challenge, which can be appreciated by viewing the extreme distortion of image structure caused by the image warp of equations (1.1) and (1.2) as shown in figure 7 and 9, is that it is very difficult to perform any image processing in log-polar coordinates.
The intrinsic space-variance of the map tends to obstruct all common methods of image processing, and especially, Fourier transform based methods. The principle result of this paper is to provide a solution to this latter problem, and to demonstrate that a form of generalized Fourier transform, which we call the exponential chirp algorithm, can be used to achieve shift, size and rotation invariant processing on space-invariant data at complexity that is comparable to that of the FFT, while at the same time retaining the favorable properties of space-variant vision.

The following results will now be presented:

- We describe an algorithm for estimating the Fourier transform of an image which has been remapped by an arbitrary map function, in the range coordinates of the map function. This will be illustrated using the log-polar map function, and one and two-dimensional examples will be illustrated.

- We show that this algorithm allows template matching to be performed in a size, rotation and shift invariant manner. Unlike the Mellin-transform (Kellman and Goodman, 1977; Sheng and Arsenault, 1986b), which has similar symmetry properties, the present method is fully space-variant, and thus possesses the favorable space-complexity properties of the map function (e.g. log-polar) which is being utilized. In other words, the Mellin transform is NOT performed on a "foveal" architecture, while the present method does have this property.

- We describe three different algorithms for computing the exponential chirp transform, with complexity $O(N^4)$, $O(N^3 \log N)$ and $O(N^2 \log N)$, where $N$ is the rank of the (transformed) image. For typical log-polar images, formed from a standard 512x512 TV image, $N$ is usually in the range of 25-50 (Wallace et al., 1994), and see figures 7 and 9. Because of the small image size typical of log-polar images (i.e. the advantageous space-complexity of this format), and the fast exponential chirp algorithm presented in this paper, extremely high rates of template matching can be realized. We estimate that roughly 50-100 matches/second (on current single processor DSP architectures,
such as the TI320C40) are possible\textsuperscript{2}. This allows template matching to be performed using inexpensive DSP support at rates which formerly have been possible only with optical processing.

- To illustrate the properties of the exponential chirp transform, we show several simple examples of shift and size invariant pattern recognition using the exponential chirp transform.

2 The generalized chirp transform

In one-dimensional signal processing applications, the term "chirp transform" generally refers to an algorithm for the computation of the discrete Fourier transform (in general a z-transform), for uniformly sampled sequences, which has some advantages over the traditional FFT (Rabiner and Gold, 1975). In the present paper, we outline a generalization of this method to estimate the Fourier transform for nonuniform sampled sequences.

The basic idea, as outlined in the introduction, is that we wish to estimate the Fourier transform in the coordinates range of some mapping function.\textsuperscript{3} Of particular interest is the log-polar map function. Since the Fourier transform is defined in terms of integration, an immediate solution to this problem is provided by transforming the signal via the map function, and then using the Jacobian of the transformation to re-write the Fourier integration in the new coordinates. Then, we re-interpret the combination of the transformed Fourier kernel and the Jacobian of the map function as a new kernel, in the transformed coordinates. This new kernel is not space-invariant, like the usual trigonometric kernel. However, newly defined convolution, in terms of this somewhat strange kernel,

\textsuperscript{2}The C40 DSP performs a size 64 FFT in 100 micro-seconds. Performing a 64x64 2D FFT on the C40 takes roughly 7 milliseconds.

\textsuperscript{3}For N-dimensional band-limited functions there exists more than one nonuniform-sampling which allows the correct reconstruction of a given band-limited function (Zayed, 1993). We will not discuss in detail the restrictions of such mapping functions (which generates a non-uniform sampling), but we imply here that it is invertible and belonging to the class $C^1$ along all the domain of interest. The reader interested in more details on the reconstruction of multidimensional band-limited signals should consult (Feichtinger and Grochenig, 1992).
has precisely the same symmetry properties as the original Fourier kernel. Specifically, shift in the domain leads to a phase change in the range. This is the property that was desired.\textsuperscript{4}

Such a kernel may be termed a generalized “chirp” kernel, in analogy to the quadratic dependence of the original chirp kernel on coordinate variables (Oppenheim and Shafer, 1975). However, we emphasize that the generalized chirp has an arbitrary space-variance, which is determined entirely by the chosen map function. When the map function used is the log-polar map, the chirp behavior is exponential in space. For other map functions, there would be an analogous “generalized” chirp transform. In the present paper we will discuss a particular instance of the 1D and 2D discrete generalized chirp transform for the case of a map function which is the real logarithm and the complex logarithm, respectively, and show three different algorithms for computing it, with complexity of the form $O(N^4), O(N^3 \log N)$, and $O(N^2 \log N)$. Following a discussion of anti-aliasing, we will show examples of the use of the exponential chirp algorithm to perform template matching in simple optical character recognition applications.

2.1 One-dimensional case

Given a function $f(x)$ and an invertible transformation $w : x \rightarrow \xi$ the Fourier transform of $f(x)$ is:

$$F(\kappa) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi j x \kappa) \, dx$$

It becomes, (using the Jacobian) in the $\xi$ space:

$$F'(\kappa) = \int_{-\infty}^{+\infty} f(x(\xi)) \frac{\partial x(\xi)}{\partial \xi} \exp(-2\pi j x(\xi) \kappa) \, d\xi$$

(2.1)

The kernel is:

\textsuperscript{4}In other work, (Bonmassar and Schwartz, 1996) we have derived this same argument using Lie Group methods. The present argument yields the same result, and is much simpler to present.
The associated integral transform:
\[ \int_{-\infty}^{+\infty} f(\xi) K_T(\xi, \kappa) d\xi \quad (2.3) \]
is "invariant" up to a phase, under translation in the \( x \) domain. Note, however, that this invariance should really be termed "quasi-invariance", since the underlying map is space-variant. Thus, as the object is shifted away from the center-of-fixation of the map, its high frequency structure is filtered by the falling resolution of the map function. Thus, the representation is invariant, up to the frequency components which survive the under-sampling that occurs when shifting away from the origin. Also, it is evident that careful anti-aliasing must be applied, which will be discussed below.

2.1.1 One-dimensional example: logarithmic mapping

We will now illustrate the general one-dimensional case with the log-polar mapping. Following (Rojer and Schwartz, 1990), we will consider the one-dimensional transformation: \(^5\)
\[ \xi(x) = \begin{cases} 
\log(x + a) & : x \geq 0 \\
2 \log(a) - \log(-x + a) & : x < 0 
\end{cases} \quad (2.4) \]
for which the kernel, as in eq.(2.2) is:
\[ \exp[\xi - 2 \pi j (\exp(\xi) - a) \kappa] : \xi \geq \log(a) \]
\[ a^2 \exp[-\xi - 2 \pi j (a - a^2 \exp(-\xi)) \kappa] : \xi < \log(a) \quad (2.5) \]
The kernel found is reminiscent of a "chirp" with exponentially growing

\(^5\)This represents a logarithmic mapping in which the singularity at the origin is removed by defining two separate branches, using some finite and positive "a" to provide a linear map for \( ||x|| < a \) and becomes smoothly logarithmic for \( ||x|| >> a \).
Figure 1: Example of one-dimensional chirp kernel (real part) for log-polar mapping, without anti-aliasing filtering.

Figure 2: One-dimensional example of a kernel (real part) with anti-aliasing filtering.
frequency and magnitude. Figure 2 illustrates the one-dimensional kernel, with anti-aliasing filtering. Aliasing must be carefully handled, due to the rapidly growing frequency of the kernel.

2.2 Two-dimensional case

Given a function $f(x, y)$ and an invertible and differentiable transformation $w : (x, y) \rightarrow (\xi, \eta)$, the Fourier transform of $f(x, y)$ in the $(\xi, \eta)$ space is given by the following integral transform:

$$\int_{-\infty}^{+\infty} \int f(x(\xi, \eta), y(\xi, \eta)) K_T(\xi, \eta, k, h) \, d\xi \, d\eta$$  \hspace{1cm} (2.6)

We define the two-dimensional exponential chirp kernel as $K_T(\xi, \eta, k, h)$:

$$K_T(\xi, \eta, k, h) = |J(\xi, \eta)| \cdot \exp[-2\pi j (k \cdot x(\xi, \eta) + h \cdot y(\xi, \eta))]$$ \hspace{1cm} (2.7)

where $J(\xi, \eta)$ is the Jacobian of the transformation. Note that we have achieved the objective of finding a kernel such that we can compare test data appearing at different positions in the image plane, using convolutions and other image processing operations performed only in the transformed coordinates. For the case of interest here, in which the transformation is a log-polar mapping, this allows us to work with the log-polar image directly, despite the fact that an object is grossly deformed in size and shape in the log-polar plane, when it moves in the image plane. Moreover, since the log-polar plane is orders of magnitude smaller than the image plane, we can reap the benefits of the space-variant architecture and avoid the penalty of the lack of simple shift invariance. We consider here the following two-dimensional log-polar (or complex logarithmic) transformation (Rojer and Schwartz, 1990): $w = \log(z + a)$, where "$a$" is a positive definite parameter. The transformation between spaces can therefore be written:

\[ w = \log(z + a), \]

6The complex log transformation requires a branch cut, which is taken in this case to divide the plane into two parts ($\text{Real}(z) > 0$ and $\text{Real}(z) < 0$). Note that this is identical to (and in fact was motivated by) the anatomy of the brain: the two sides of this mapping are in direct correspondence with the two hemispheres of the brain. The visual cortex, which is of the form of a complex logarithmic mapping, is divided in this way for similar reasons.
Figure 3: Parametric Plot of the 1D exponential chirp kernel with $a = 1$ and $\kappa = 1$, the points represent the discrete version of the kernel. The plot, performed on the Eulero plane, shows the spiraling behaviour of the 1D ECT.
\[
\xi = \log \sqrt{(x + a)^2 + y^2}
\]
\[
\eta = \arctan \frac{y}{x + a}
\]

so we can express \( x \) and \( y \) as:

\[
x = \exp(\xi) \cos(\eta) - a
\]
\[
y = \exp(\xi) \sin(\eta)
\] (2.9)

Eq. (2.6) becomes:

\[
\iint_{D} f(\xi, \eta) \exp(2\xi) \exp[-2\pi j \{h(\exp(\xi) \cos(\eta) - a) + h \exp(\xi) \sin(\eta)\}] \, d\xi \, d\eta
\] (2.10)

where \( D \equiv \{(\xi, \eta) : -\infty \leq \xi < +\infty \text{ and } \frac{-3\pi}{2} \leq \eta < \frac{\pi}{2}\} \).
Figure 4 illustrates the kernel in two dimensions, its behavior is exponential both in frequency and amplitude along the $\xi$ axis and is oscillating along the $\eta$ axis.

3 2D discrete generalized chirp transform

We next show that the ECT has a simple discrete form that can be applied directly to the log-map representation of the image $f_{i,j}$. If we denote with $(x_0, y_0)$ the position of the fovea and with $R$ the length of the radius of the visual field, using the same notation as in (Bonmassar and Schwartz, 1994), the log-map representation of an image $I(x,y)$, filtered to avoid aliasing (see section 5), is:

$$f_{n,m} = \hat{I}(x_0 + x_{n,m}, y_0 + y_{n,m})$$

where
\[ x_{n,m} = [\exp(\xi_n) \cos(\eta_m) - a] \]
\[ y_{n,m} = [\exp(\xi_n) \sin(\eta_m)] \] 

(3.1)

and

\[ \xi_n \in [0, \frac{1}{N-1} \log(R + 1), \frac{2}{N-1} \log(R + 1), \ldots, \frac{N-2}{N-1} \log(R + 1), \log(R + 1)] \]
\[ \eta_m \in \left[ -\frac{3}{2} \pi, -\frac{3M-4}{2M} \pi, \frac{M-8}{2M} \pi, \frac{M-4}{2M} \pi \right] \]

(3.2)

\[ n = 1, 2, \ldots, N \quad m = 1, 2, \ldots, M \]

Therefore, the discrete exponential chirp transform (DECT) can be written as:

\[ S_{k,h} = \sum_{n=1}^{N} \sum_{m=1}^{M} f_{n,m} \exp[2\xi_n - 2\pi j\{\exp(\xi_n) \cdot \cos(\eta_m) - a\} \frac{k}{N} + \exp(\xi_n) \sin(\eta_m) \frac{h}{M}] \] 

(3.3)

\[ k = 1, 2, \ldots, N \quad h = 1, 2, \ldots, M \]

The one-dimensional kernel in figure 3 gives a good insight into the discrete transform: the real and imaginary components (in the horizontal and vertical-axes respectively) form a set of points along a spiral with increasing magnitude.
4 Algorithms for the discrete exponential chirp transform

We now demonstrate three algorithms for computing the DECT.

4.1 $O(N^4)$: the DECT

The simplest method is direct: we simply generate the DECT kernels, and then to perform the discrete transform of the image. Since there are $O(N^2)$ frequency components to compute, this "direct" method has complexity $O(N^4)$, (where $N$ is the rank of the image, i.e. 512 for a 512x512 image). Since $N$, for usual practical applications of the logmap, is in the range of 64 (e.g. see (Sandini and Dario, 1989; Wallace et al., 1994; Bederson et al., 1992; Engel et al., 1994)), the high complexity of this direct approach is not completely prohibitive. However, it is possible to achieve better results, as follows.

4.2 $O(N^3 \log N)$ and Rader's Chirp Algorithm

Many fast versions of various transforms have been recently presented (e.g., (Liu and Chiang, 1995; Ferrari, 1995; Strain, 1992; Greengard and Strain, 1991; Yang et al., 1993; Kelly and Madisetti, 1993)). In the present section we adapt an efficient algorithm, first described by Rader (Rader, 1968; Clellan and Rader, 1979), that expresses the Fourier transform in terms of a correlation by using the multiplicative property of the power law. This method computes the ECT by re-writing the original transform in terms of a mixture of convolution and correlation, which are then evaluated efficiently with the use of the FFT (Blahut, 1984; Elliot and Rao, 1982; Nussbaumer, 1982). The major difficulty in using the traditional approach (2D FFT) is that the exponential chirp kernel is clearly not separable in terms of $\xi$ and $\eta$:

$$K_T(\xi, \eta, k, h) = e^{2\pi j (\xi h \sin(\eta) + \eta k \cos(\eta))}$$  \hspace{1cm} (4.1)

\footnote{Storing the kernels into memory requires order $O(N^4)$ of memory locations.}
Therefore, the 2D DECT transformation cannot be performed by using 1D transforms of rows and columns, as done in the two-dimensional FFT. However, since the exponential chirp kernel is "partially" separable (the parts with \( h \) and \( k \) can be computed separately), we can develop a scheme for computing fast parts of the transform. In order to simplify the discussion, the parameter "\( a \)" is set to zero, and the chirp is presented in a continuous space. The discrete case, and the case of non-zero \( a \), follows in a straightforward way.

Considering positive frequencies \( k \), the ECT can be rewritten as:

\[
S^+(\hat{k}, h) = \int \int_{\mathcal{D}} f_{\xi, \eta} e^{2\xi - 2\pi j (\xi \cos(\eta) + \eta \sin(\eta))} d\xi d\eta
\]

(4.2)

where: \( \hat{k} = \log(k) \), we can therefore introduce the following function \( F(\hat{k}, h, \eta) \):

\[
F(\hat{k}, h, \eta) = \left( f_{\xi, \eta}^* e^{2\xi + 2\pi j \eta k} \right) \sigma_\xi e^{-2\pi j e^\xi \cos(\eta)}
\]

(4.3)

where \( \sigma_\xi \) indicates the operation of 1D cross-correlation done over the variable \( \xi \) and \( f_{\xi, \eta}^* \) is the complex conjugate of \( f_{\xi, \eta} \). Finally eq. (4.2) becomes:

\[
S^+(\hat{k}, h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\hat{k}, h, \eta) d\eta
\]

(4.4)

The complexity of eq.(4.4) using FFTs is \( O(N^3 \log(N)) \), since we have to compute eq.(4.3) for every \( \eta \) and \( k \) performing 1D correlations\(^8\). This procedure is evidently more convenient than a straightforward computation of the transform which requires order \( O(N^4) \) operations. A similar procedure can be applied for negative frequencies, equation (4.2) becomes:

\[
S^- (\hat{k}, h) = \int \int_{\mathcal{D}} f_{\xi, \eta} e^{2\xi - 2\pi j (-\xi \cos(\eta) + \eta \sin(\eta))} d\xi d\eta
\]

(4.5)

The key idea here is that, similar to eq.(4.3), we can write the following

\(^8\)Storing the kernels into memory requires order \( O(N^3) \) of memory locations.
convolution:

\[
G(\tilde{k}, h, \eta) = \left( f_{\xi, \eta} e^{2\xi - 2\pi j e^\xi \sin(\eta) \hat{h}} \right) \ast e^{-2\pi j e^\xi \cos(\eta)} \tag{4.6}
\]

\(k = e^{-\hat{k}}\) and \(\ast\) indicates the 1D convolution operator done over the variable \(\xi\). This time the ECT can be computed efficiently by convolutions or correlations involving \(\hat{\eta}\) in the following way:

\[
\hat{S}(k, \hat{\eta}) = \int \int f_{\xi, \eta} e^{2\xi - 2\pi j \left( e^\xi \cos(\eta) k + h_0 \frac{e^\xi}{\xi} \cos(\eta - \hat{h}) - h_0 \frac{e^\xi}{\xi} \cos(\eta + \hat{h}) \right)} d\xi d\eta \tag{4.7}
\]

where: \(h = h_0 \sin(\hat{h})\). We can therefore introduce the following function \(H(k, \hat{\eta}, \xi)\):

\[
H(k, \hat{\eta}, \xi) = \left( \left( f_{\xi, \eta} e^{2\xi + 2\pi j e^\xi k \cos(\eta)} \right) \ast_{\eta} e^{h_0 \pi j e^\xi \cos(\eta)} \right) \ast_{\eta} e^{-h_0 \pi j e^\xi \cos(\eta)} \tag{4.8}
\]

Therefore the chirp transform can be written as:

\[
\hat{S}(k, \hat{\eta}) = \int_{-\infty}^{\infty} H(k, \hat{\eta}, \xi) d\xi \tag{4.9}
\]

Finally, it is useful to note that the above substitutions for both \(k\) and \(h\) introduce non-uniform frequency sampling. The exponential substitution introduces a logarithmic compression of one frequency axis, and the trigonometric substitution introduces a non-uniform (trigonometric) sampling of the other axis.

4.3 \(O(N^2 \log N)\): The fast exponential chirp (FECT) algorithm

We now show the surprising and important result that it is possible to compute the DECT, but with a logmap in frequency, at the same cost as the conventional FFT, i.e. at \(O(N^2 \log N)\) complexity\(^9\).

\(^9\)Storing the kernels into memory requires order \(O(N^2)\) memory locations.
The ordinary ECT can be written:

\[ S(k, h) = e^{2\pi j k} \int_D f_{\xi, \eta} e^{2\pi j \xi (\cos(\eta) + \alpha \cdot h \cdot \sin(\eta))} \, d\xi \, d\eta \quad (4.10) \]

The key idea is to introduce a log-polar mapping in frequency. This change in coordinates surprisingly allows us to re-write the full two-dimensional exponential chirp algorithm in separable form, and thus achieve full FFT performance. In addition, this logmap in frequency brings us into the context of the Mellin-Fourier transform, well known for its properties of scale and rotation invariance. Note, however, that we are not performing the usual Mellin-Fourier transform here, but have retained the space-variance of the original coordinate transform! The coordinate transform to be done is:

\[ k(\gamma, \theta) = \exp(\gamma) \cos(\theta) - b \]
\[ h(\gamma, \theta) = -\exp(\gamma) \sin(\theta) \quad (4.11) \]

This change of coordinates transforms the ECT to the following correlation:

\[ \tilde{S}(\gamma, \theta) = e^{2\pi j \gamma k(\gamma, \theta)} \int_D (f_{\xi, \eta} e^{2\pi j \xi \cos(\gamma + \alpha \cdot h \cdot \sin(\gamma + \alpha \cdot h \cdot \sin(\eta)))} e^{-2\pi j \xi \cos(\gamma + \alpha \cdot h \cdot \sin(\eta))}) \, d\xi \, d\eta \quad (4.12) \]

The above correlation can therefore be computed very efficiently by using FFTs, but the following issues must be taken into account in the implementation. First, since eq.(4.12) expresses a 2D linear correlation, zero-padding must be introduced to avoid circular computation. Second, the anti-aliasing filtering (see next section) can be performed directly on the kernel by using an extension of the Fermi function in 2D. Finally the storage in memory of matrices which are fixed by the size of the image in the logmap, substantially reduces the overall number of computations (by eliminating the evaluation of transcendental functions). In appendix A the numerical implementation is discussed in detail.
5 Anti-aliasing filtering

When a function is not sampled at a sufficiently high rate, fold-over (or aliasing) errors occur in the reconstructed frequency domain function (see for upper-bound aliasing error, (Papoulis, 1966; Stickler, 1967; Jr., 1968; Splettstosser, 1979), and for examples of alias-free sampling, (Shapiro and Silverman, 1960; Beutler, 1970)). The kernel of the DECT, in eq.(3.3), is clearly a complicated function in which the frequency of the real and imaginary part grow exponentially with $\xi$. As both magnitude and phase of the DECT increase exponentially, the frequency grows rapidly, and aliasing will occur. Therefore anti-aliasing filtering must be introduced: the kernel must be windowed (in the $(\xi, \eta)$ plane) when it reaches excessive high frequency according to the classical Shannon-Whittacker-Kotel’nikov sampling theorem (Jerry, 1977). This can be done by using an ideal filter, as shown in this section, and in appendix B a smoother version of the same anti-aliasing filtering is presented. Due to the space-variant nature of the sampling, a careful analysis of anti-aliasing must be supplied.

Following the definition of instantaneous frequency (Pol, 1946), we define the space-variant 2D frequencies $f_\xi f_\eta$ of the complex kernel:

$$\exp[2\pi j g(\xi, \eta)]$$

as the partial derivatives of the phase function $g(\xi, \eta)$ with respect to $\xi$ and $\eta$. When applied to the chirp these result in (ignoring the smooth real part of the kernel):

$$\left| \frac{\partial g(\xi, \eta)}{\partial \xi} \right| = |k \exp(\xi) \cos(\eta) + h \exp(\xi) \sin(\eta)|$$

$$\left| \frac{\partial g(\xi, \eta)}{\partial \eta} \right| = |h \exp(\xi) \cos(\eta) - k \exp(\xi) \sin(\eta)|$$

In order to anti-alias, we must filter out the set of samples of the chirp that do not satisfy the following inequality:

$$\log(\frac{N+1}{N-1}) \geq N _\xi f_\xi$$

$$\frac{2\pi}{M} \geq N _\eta f_\eta$$

where $N_\xi$ and $N_\eta$ are Nyquist factors greater than 2, $N$ and $M$ are the
Figure 5: Magnitude of the DECT of a sinusoidal grating ($f=3/64$) in the original undistorted space. Notice that the origin is located in the center and that the magnitude of the DECT correctly generates two spikes in the location of the (positive and negative) frequency of the grating.
length of the vectors \( \xi_n \) and \( \eta_m \) and \( R \) is the radius of the visual field. The effectiveness of the anti-aliasing filtering and the correct values for \( N_\xi \) and \( N_\eta \) were found using sinusoidal gratings images (in the uniform sampling domain), see figure 5. The effect of anti-aliasing filtering on the 2D DECT kernel is shown in figure 6.\(^\text{10}\) From here-on we include this anti-aliasing filter in discussion of the DECT.

The images used are sampled non-uniformly (Jerry, 1977; Papoulis, 1977; Kramer, 1959; Clark et al., 1985; Stark, 1979), after storing them in the usual uniform sampled fashion and after non-uniform filtering to avoid aliasing.

If \( \mathcal{W} \) is the spectral support of our image \( \mathcal{I} \) (\( \mathcal{W} \) is a region of \( \mathbb{R}^2 \) over which the Fourier transform of \( \mathcal{I} \) is non-vanishing), then our sampling is error-free if the replications, introduced by the sampling, of the spectral support \( \mathcal{W} \) do not overlap. We will therefore apply a local (space-variant) filtering, so that we can sample the 2D signal without aliasing at the local Nyquist rate.\(^\text{11}\) According to our non-uniform sampling, regions near the fovea which are sampled with a higher sampling frequency require a less abrupt low-pass filtering than regions in the periphery which are sampled much more coarsely. The filtering is done using a position-dependent averaging or Gaussian filter before sampling non-uniformly.

6 The matching filter

The log-map is an example of a non-uniform sampling in which the sampling rate varies over the plane of the visual field. Since we are primarily interested in shift invariance we want to concentrate on the particular object features which do not change for different space positions: a band of low-frequency components. In order to locate shifted objects we use the phase of the DECT (denoted as \( S\{\} \), see (Bonmassar and Schwartz, 1994)) and modify the following classic phase only filter (POF) (see (Horner and

\(^\text{10}\) A more general filtering can be done introducing a gradual attenuation rather than a sudden cut as shown in figure 6, multiplying the chirp kernel by a smooth filter, as similarly done for the short time Fourier transform (STFT). See appendix B.

\(^\text{11}\) This type of filtering is very common for the log-polar mapping, see (Weiman, 1990; van der Spiegel et al., 1989)
Figure 6: The anti-aliasing effect on the 2D-DECT kernel (real part).
P.D.Gianino, 1984)), as similarly done in (Kumar and Bahri, 1989):

\[ F_{\phi}(k, h) = \exp[j \phi(k, h)] \]  (6.1)

to the following band-pass phase-only filter (BP-POF):

\[ F_{\phi}^{BP}(k, h) = A(k, h) \exp[j \phi(k, h)] \]  (6.2)

where \( A(k, h) \) is a zero-phase ideal band-pass filter. Given two images \( I_1(x, y) \) and \( I_2(x, y) \), one of which represents the template to match, the phase function \( \phi(k, h) \)\(^{12} \) domain is the following:

\[
Z_1(k, h) = S \{ I_1(x, y) \} = X_1 + jY_1 \\
Z_2(k, h) = S \{ I_2(x, y) \} = X_2 + jY_2
\]

\[ \phi(k, h) = \arctan \left( \frac{X_1 Y_2 - X_2 Y_1}{X_1 X_2 + Y_1 Y_2} \right) \]  (6.3)

This method has been further improved to perform the detection and discrimination of any letter of the alphabet, by shaping the filter \( A(k, h) \) according to the spectrum of the template that has to be detected by setting:

\[ A(k, h) = 1 \]

if the spectrum of the template (\( \| H_T(k, h) \|^2 \))

\[ \| H_T(k, h) \|^2 \geq \mu \]

where \( \mu \) is a positive number. This improvement increases the S.N.R. (signal to noise ratio) of generally all the detections, but it is particularly significant when the spectrum matrix \( \| H_T(k, h) \|^2 \) is sparse. In the POF all phases are assigned to the same magnitude (i.e. one). If a point in

\(^{12}\)The phase function \( \phi(k, h) \) was computed with uniform sampling in frequency using the DECT (e.g. the classical POF (Horner and P.D.Gianino, 1984)) and with nonuniform sampling in frequency using the FECT (application of a matching filter in the Mellin-Fourier domain, e.g. (Chen et al., 1994)).
frequency has null (or a very small) spectrum component then no phase can be associated with it (Oppenheim and Shafer, 1975), since the phase component of such point would be a random number in a $2\pi$ interval (in radians). For this reason we compute the phase-only filter just along points which have a lower-bounded spectrum magnitude.

7 Results

The images of the letters in Figure 7 form an array of 256 by 256; the fovea is always centered in the center of the image with $R = 128$; the log-map images are 64 by 26 with parameter $a = 10.186$; therefore the polar log-map introduces a compression factor of approximately 40 in this
case. The image of letters was initially filtered with a non-uniform mean-filter, as follows. The image was divided into sectors with exponentially growing radii, and the polar-log representation was given by taking the average value in every sector. The logmap of two images, the one with the two letters and the one with template (letter to match) were transformed according to eq. (3.3). Finally, we compare the two transforms using the BP-POF matching filter with frequency cut-off @0.2 - 0.4 as applied to original images; the Nyquist factor chosen was $N_\xi = 3.5$ and $N_\eta = 3.0$. 
Figure 8: Result of matching the letter E (left, the letter F in the right), using the discrete exponential chirp transform together with the band-pass filtered phase only filter, in the presence of the “distractor” F (left, the letter E in the right). All matching was performed directly in the space-variant (polar-log) plane.

The results in figure 8 show that the DECT is capable of finding a shifted copy of the letter ‘F’, even in the presence of the similar-looking letter ‘E’ and vice-versa. Figure 8 presents the outputs of a BP-POF matching filter implemented using an ideal IIR filter, shaped with the spectrum of the template as described in the previous paragraph. A second example shows the simultaneous size and shift invariant properties of the FECT (Fast Exponential Chirp Transform, see appendix A)\textsuperscript{13}. In this case, a small F, not centered in the image plane, is used as a template to match a large F, at a different, non-central fixation. The BP-POF is used now as a matching algorithm, to detect shifts\textsuperscript{14}. All computations

\textsuperscript{13} The transform is simultaneously size, rotation and shift invariant, although rotation is not shown in this example, since the FECT performs a transformation in the range of the Mellin transform.

\textsuperscript{14} In the Mellin-Fourier Transform scalings and rotations are simple shifts, therefore the
Figure 9: A simultaneous shift and size change is applied to the letter E. On top the visual field is shown. On the bottom are shown the respective space-variant, log plane representations.

Figure 10: Response of the BP-POF applied to the ECT in the Mellin-Fourier domain, the horizontal axis represents the scale factor and the vertical axis represent the rotation angle between the two letters "E" in the previous figure.
are performed in space-variant coordinates. A third example shows the detection of a shifted letter F camouflaged by white noise. The image plane is shown in figure 11. The BP-POF correlator, applied using the DECT for this problem, is shown in figure 12. The frequency information present in the DECT, with uniform sampling in frequency, can be extracted reconstructing the image using the inverse FFT, as shown in figure 13.

BP-POF is a shift-detector computed using FFTs (Kumar and Bahri, 1989).
8 Conclusions

In the present paper, we have shown the solution to an apparently contradictory goal: to combine the space-variant imaging properties of a mapping, with the properties of the Fourier transform. The underlying idea is quite simple: we have transposed the usual trigonometric kernel via a map function, and defined a new kernel by combining these transposed kernels with the Jacobian of the map. The resulting kernel, which we call the Exponential Chirp (in the case that the map function is the complex logarithm), provides a form of Fourier transform, but one which inherits the "foveating" properties of the complex logarithm. This method is illustrated both for one-dimensional applications (i.e. signal processing) and two-dimensional applications (i.e. image processing). Space-variant anti-aliasing, and three algorithms for performing the ECT are described, which vary in computational complexity and in sampling strategy:

1. $N^4$, with a log sampling in $h$ space and a uniform sampling in frequency.

2. $N^3 \log(N)$, with a uniform sampling in space, and a polar (either radial or angular) sampling in frequency.

3. $N^2 \log(N)$, with a log-polar sampling in both space and frequency: the fast ECT.
The third algorithm above, which can be performed at a rate which we estimate to be about 50 frames/second for practical forms of image log-polar maps on a single low cost DSP (e.g. TI320C31) provide the possibility of performing shift, size and rotation invariant template matching in low-cost machine vision systems at rates which are comparable to optically based template matching!

The practical importance of this work is twofold:

- A space-variant generalization of the Mellin transform is disclosed
- A speed-up of image template matching via several orders of magnitude is provided.

These points will now be briefly discussed.

The Mellin transform, in two-dimensions, may be viewed as the following two transformations: a two-dimensional power spectrum, followed by a complex log mapping centered on the origin of the 2D frequency space. This transform is shift invariant (due to the initial power spectrum), and converts rotation and size scaling to shift (due to the complex log second stage). The Mellin-Fourier transform is then obtained by applying a final stage of Fourier Analysis. The Mellin-Fourier transform thus obtained by Fourier transform, complex log, and Fourier transform is fully invariant to size, shift and rotation. The Mellin-Fourier transform has been described many times over the past three decades (Brousil and Smith, 1967; Casasent and Psaltis, 1976; Schwartz, 1977; Kellman and Goodman, 1977; Sheng and Arsenault, 1986a). One reason for interest in it is that opto-electronic methods for constructing this transform have been developed (Casasent and Psaltis, 1976), offering the possibility of high speed performance. Furthermore, the presence of the log-polar mapping (in a spatial sense) in primate vision, has motivated the suggestion that the Mellin-Fourier transform may play some role in biological vision (Cavanagh, 1978). Unfortunately, this biological application appears to be precluded by the strong space-variance which is evident in biological vision, and which is in fact the basis for modeling the primate system using a space-variant mapping! The Mellin-Fourier transform, being space-invariant, cannot provide a model for primate vision (Schwartz, 1981). And, the very space-invariance which has attracted attention to the Mellin-Fourier transform also precludes its application in active vision systems: there is no need to move the fixation
Figure 14: Block diagram of the proposed filter for detection of shifted templates. The image \((I(x_n, y_m))\) is filtered nonuniformly \((\tilde{I}(x_n, y_m))\) before log-polar sampling \((f_{n,m})\) to avoid aliasing. The resulting image \((f_{n,m})\) is transformed by using the DECT (with uniform sampling in frequency). Same procedure is applied to the matching template. The two resulting transforms \((S_f(k, h)\) and \(S_t(k, h))\) are used to compute the difference of phase, which is needed for the phase only matching filter. The filter output gives the location of the match between image and template.
point of a non-foveating system which is space-invariant! Another obstacle to exploitation of the Mellin-Fourier transform is its computational cost. The initial Fourier transform is performed at full image size, which has a complexity of $N^2 \log(N)$, but for an $N$ of perhaps $N = 512$. By comparison, the ECT has a complexity, for the most efficient of our algorithms, also of $N^2 \log(N)$, but for $N$ of perhaps 64, corresponding to the same 512x512 input image. Put another way, the Mellin-Fourier transform does not benefit from the space-complexity of the log map, which is performed in frequency space, not in the original large image space. And, were it possible to extend this work to much higher resolution sensors, the ECT space-complexity would increase even more dramatically, considering the $10^4$ space-complexity advantage estimate for human vision (Rojer and Schwartz, 1990).

Also, in our experience, the size scaling of the Mellin-Fourier transform approach does not work well, for the following reason. When a small target is to be matched, the energy of the match is essentially proportional to the product of the area of the template and the area of the target. Small targets have small energy, and it is a problem to locate the match, unless the target and the template do not differ by a large change in scale.

The Exponential Chirp provides a solution to these problems. First, it is space-variant, since it inherits the “foveating” property of the log-map, or whatever space-variant function is to be used. Therefore, a system which was based on the ECT would have the correct form of varying resolution that is known to occur in biological visual systems. In fact, the generality of our method allows any map function to be applied with this approach, should some alternative to the log-polar map be desired. Moreover, a machine vision system which used active vision techniques would be able to provide effective exploitation of the properties of the ECT. A target which appeared far off the “fovea” of the system could be matched by frequency domain techniques, such as the phase-only filter, the cepstrum, or cross-correlation. A far eccentric target would provide a weak and ambiguous match, due to the fall in resolution of the ECT, thus providing a form of “attentional” mechanism, i.e. a rapid, but low confidence inference of target presence. A camera movement, however, would foveate, by bringing the target into a more foveal location, provide increasingly better template matches by providing increased resolution from the target, correlated with the fixed template. Since the more centrally fixated target has more “de-
tail" as well as more "energy", the DECT solves the problem of poor signal to noise ratio that is inherent in the Mellin-Transform application. There are many research problems which remain to be solved in this work. The very basic form of template matching demonstrated in this paper must be adapted and extended to practical image processing tasks. The suggestion of biological relevance must be pursued, in the form of detailed models which have yet to be developed. However, this combination of active vision techniques and space-variant template matching, in a geometrically invariant context, provide an exciting set of possibilities, both for further research into the nature of biological vision, and for the construction of high-performance machine vision systems.
Figure 15: Block diagram of the proposed filter for detection of rotated and scaled templates. The image ($I(x_n, y_m)$) is filtered nonuniformly ($\tilde{I}(x_n, y_m)$) before log-polar sampling ($f_{nm}$) to avoid aliasing. The resulting image ($f_{nm}$) is transformed by using the FECT (with log-polar sampling in frequency). Same procedure is applied to the matching template. The two resulting transforms ($|S_f(k, h)|$ and $|S_t(k, h)|$) are transformed using the Fast Fourier Transform (FFT) and the results ($\hat{S}_f(k, h)$ and $\hat{S}_t(k, h)$) are used to compute the difference of phase, which is needed for the phase only matching filter. Rotations and scalings (i.e. rotations and scalings in frequency) are transformed into shifts under the log-polar distortion, therefore the filter output gives a spike in the location of the total rotation and scaling between the original image and the template.
9 Appendices

Appendix A. The Fast Exponential Chirp Transform (FECT) algorithm.
The first step of the FECT algorithm is to create the two following vectors for the convolution operation (following the notation in eq(4.12)):

\[ \tilde{\xi}_{n+i} = \xi_n + \xi \]  \hspace{1cm} (A.1)
\[ \tilde{\eta}_{m+l} = \eta_m + \eta_l \]

where \( n, i = 1, ..., N \) and \( m, l = 1, ..., M \). We are interested in the fast transform expressed in eq. (B.3). Note that even if it was derived assuming \( b = 0 \), it still holds in practice with a "small" \( b \). In the implementation of the FECT the value of 0.1 for both \( \sigma_x \) and \( \sigma_y \) was used. Since DECT is calculated, as we have mentioned, between \([-\frac{1}{2}, \frac{1}{2}]\) we chose \( \gamma = \xi - \xi_N - \log(2) \) in eq.(B.3). The FECT can be easily implemented following these steps:

- **Step 1:** computation of the first term of the cross-correlation function:
  \[ P_{n,m} = f_{n,m}^* \exp(2\xi_n - 2\pi j \exp(\xi_n) \cos(\eta_n)) \]  \hspace{1cm} (A.2)
  \( n = 1, 2, ..., N \) \( m = 1, 2, ..., M \)

- **Step 2:** computation of the second term (see Appendix B):
  \[ Q_{i,l} = \exp(-\pi j \exp(\tilde{\xi}_i - \xi_N) \cos(\tilde{\eta}_l)) \cdot \]  \hspace{1cm} (A.3)
  \[ \frac{1}{1 + \exp\left(\frac{-\pi N}{\log(R+1)} + \frac{N}{N} \exp(\tilde{\xi}_i - \xi_N - \log(2)) \cos(\tilde{\eta}_l)\right)} \]
  \[ \frac{1}{1 + \exp\left(\frac{-\pi M}{\log(R+1)} + \frac{N}{N} \exp(\tilde{\xi}_i - \xi_N - \log(2)) \sin(\tilde{\eta}_l)\right)} \]
  \( i = 1, 2, ..., 2N \) \( l = 1, 2, ..., 2M \)

- **Step 3:** computation of the shift term:
  \[ S_{n,m} = \exp(\pi j a(\exp(\xi_n - \xi_N) \cos(\eta_n) - b)) \]  \hspace{1cm} (A.4)

\[ ^{15} \text{As done for other proofs in this paper, we will show the complete algorithm for just one hemifield (} x \geq 0 \text{), the computation of the other hemifield is straightforward.} \]
• **Step 4:** finally the fast transform is given by:

\[
((P^{zp}) \circ Q)^{e,hw} \cdot S
\]  

where "zp" stands for zero-padding (to the dimensions of \(H\)), "e" is extract (to the dimensions of \(S\)), "hw" stands for horizontal wrap (since \(\eta = -\theta\)) and the product ",," is intended term by term. The above correlation is obviously done using 2D-FFTs.

**Appendix B. The 2D Fermi-function.** The condition (5.3) can be tested using the following Fermi function, that has been extended to become a 2D function, which is equal to one for \(x \text{ and } y \to +\infty\) and to zero for \(x \text{ or } y \to -\infty\):

\[
\phi(x, y) = \frac{1}{(1 + e^{-\frac{x}{\sigma_x}})(1 + e^{-\frac{y}{\sigma_y}})}
\]  

(B.1)

where \(\sigma_x\) and \(\sigma_y\) are two parameters which control the slope of the sigmoidal function, the anti-aliasing filtering can be done by simply multiplying the kernel by the following:

\[
\phi\left(\frac{\log(R+1)}{N-1} - N\xi f_{\xi}; \frac{2\xi}{M} - N\eta f_{\eta}\right)
\]  

(B.2)

This function can be incorporated in the fast chirp transform, and eq. (4.12) giving the following cross-correlation \((b = 0)\):

\[
e^{2\pi i j h(\gamma, \theta)} \int_{D} \left(f_{\xi,\eta} e^{2\xi} e^{-2\pi i (\gamma + \xi) \cos(\theta + \eta)} \right) \phi\left(\frac{\log(R+1)}{N-1} - N\xi e^{(\gamma + \xi)} \cos(\theta + \eta), \frac{2\xi}{M} + N\eta e^{(\gamma + \xi)} \sin(\theta + \eta)\right) d\xi d\eta
\]  

(B.3)

**Appendix C. A more general form for the ECT** A certain number of different log-polar mappings for image processing, have been introduced during the past decades (e.g. see (Weiman, 1989; Chen et al., 1994; Asselin and Arsenault, 1994; Rojer and Schwartz, 1990)). It is therefore interesting
to consider a more general form for the log-polar transformation:

\[(\xi, \eta) = (a \log(g(x, y)), c \arctan(\chi(x, y))) \]  
(C.1)

where \(\chi(x, y)\) and \(g(x, y)\) are two arbitrary \(C^1\) functions. The inverse mapping is given by:

\[(x, y) = (f(\exp(\frac{\xi}{a}, \tan(\frac{\eta}{c})), g(\exp(\frac{\xi}{a}, \tan(\frac{\eta}{c})))) \]  
(C.2)

where \(f\) and \(g\) are two functions for which \(x = f(\chi, \varrho)\) and \(y = g(\chi, \varrho)\). The more general expression for the ECT given by eq. (2.7) is:

\[K_T(\xi, \eta, k, h) = \frac{e^{\xi'} \sec^2(\eta')}{ac} \cdot (f_\xi(e^{\xi'}, \tan(\eta')) g_\eta(e^{\xi'}, \tan(\eta')) - g_\xi(e^{\xi'}, \tan(\eta')) f_\eta(e^{\xi'}, \tan(\eta'))) \cdot \exp[-2\pi j (k \cdot f(e^{\xi'}, \tan(\eta')) + h \cdot g(e^{\xi'}, \tan(\eta')))] \]  
(C.3)

where \(\xi' = \frac{\xi}{a}\) and \(\eta' = \frac{\eta}{c}\).

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References


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