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ESSAYS ON BARGAINING THEORY AND APPLICATIONS

by

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ABSTRACT

This dissertation examines the role of negotiations in different institutional settings. In chapter one, I study how voluntary disclosure of information affects outcomes in plea bargaining. A prosecutor negotiates a sentence with a defendant who is privately informed about whether he is guilty or innocent. During negotiations, the prosecutor can investigate for evidence regarding the defendant's type. If prosecutor and defendant do not reach an agreement, they go to trial and obtain payoffs that depend on the prosecutor's evidence. Voluntary disclosure gives rise to endogenous second-order belief uncertainty. A purely sentence-motivated prosecutor might disclose exculpatory evidence. Voluntary disclosure leads to inefficient outcomes as parties might fail to reach an agreement. Mandatory disclosure is socially preferable: there is always agreement, and the defendant is better off if he is innocent and worse off if he is guilty. Furthermore, the prosecutor is better off under mandatory disclosure.

In chapter two, I study how bargaining power affects bargaining outcomes between an entrepreneur and a venture capitalist who provides funds. Entrepreneur and venture capitalist openly disagree about noncontractible future decisions. The contract specifies control rights and cash-flow rights for each party. Noncontractible decisions

are made by the party with control rights. When the entrepreneur has greater bargaining power and the investment value is large, she optimally relinquishes control rights. When the venture capitalist has more bargaining power, she always retains control rights. In general, greater disagreement makes the entrepreneur less likely to retain control rights.

In chapter three, I study a bilateral bargaining model with endogenous recognition probabilities and endogenous surplus. At each period, players exert two types of costly effort: productive effort, which increases the surplus size, and unproductive effort, which affects the probability of being recognized as the proposer. I characterize how differences in the cost of exerting efforts affects outcomes. Advantages in unproductive effort affect the provision of both types of effort, but advantages in productive effort only affect the provision of that effort. Differences in time preferences only affect productive efforts when the probability of recognition is not persistent.

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List of Abbreviations

CDF	....	Cumulative Distribution Function
CEO	....	Chief Executive Officer
E	....	Entrepreneur
IR	....	Individual Rationality
LHS	....	Left Hand Side
MSPE	....	Markovian Subgame Perfect Equilibrium
PBE	....	Perfect Bayesian Equilibrium
RHS	....	Right Hand Side
SPE	....	Subgame Perfect Equilibrium
VC	....	Venture Capitalist

Chapter 1

Voluntary Disclosure of Evidence in Plea Bargaining

1.1 Introduction

1.1.1 Motivation and Results

In many bargaining situations, one party can gather information and keep the outcome private. A real estate agent who inspects a house for sale might decide to conceal information that would increase the price of the house. After reviewing a company, an investor might only reveal information that increases the return rate she is asking for. In plea bargaining (the context I examine in this chapter), a prosecutor who finds exculpatory evidence might hide it from the defendant. I study the disclosure decision of the newly informed party when the disclosure of new information is voluntary. I also examine whether mandatory, instead of voluntary, disclosure of information is socially desirable.

Plea bargaining is a case of special relevance in the US criminal system, in which more than 90% of criminal cases end in plea bargaining instead of a trial.[1] In plea bargaining, a prosecutor and a defendant negotiate for a sentence in order to avoid

[1] See Devers (2011).

trial. During the negotiation, the prosecutor can search for evidence regarding the culpability of the defendant. In many circuit courts in the US, the disclosure of evidence is voluntary during plea bargaining; hence, the prosecutor can hide exculpatory evidence during the negotiation, but is required to disclose it at trial. If the prosecutor wants the judge to assign as high a sentence as possible, is she going to disclose exculpatory evidence? Even if she discloses it, is it socially desirable to impose mandatory disclosure of evidence in plea bargaining?

To answer these questions, I study a dynamic plea-bargaining model between a prosecutor (she) and a defendant (he). The defendant's type can be innocent or guilty, and the defendant is privately informed about his type. The prosecutor has inconclusive default evidence at the beginning of the game, and can decide whether to investigate for new conclusive hard evidence that can be exculpatory if the defendant is innocent or incriminating if the defendant is guilty. The investigation process is not perfect; with some probability, the prosecutor will not find new evidence.

If the prosecutor finds new evidence, she can voluntarily disclose it to the defendant. After the disclosure decision, the prosecutor offers a sentence to the defendant. If the offer is accepted, the game ends; if not, a new period starts. If they do not reach an agreement during a finite number of periods, they go to trial. I model the trial as a rule that assigns a sentence depending on the evidence: Exculpatory evidence sets the defendant free, default evidence leads to a low sentence, and incriminating evidence leads to a high sentence.

The first main result shows that, in some cases, the prosecutor discloses exculpatory evidence. To be specific, if the prosecutor's prior belief about the defendant being guilty is low, when the prosecutor finds exculpatory evidence that exonerates the defendant, she hides it and makes a low offer that is accepted by the innocent defendant. However, if the prior belief is high, in equilibrium, she discloses exculpatory evidence and sets the defendant free.

The prosecutor is able to hide exculpatory evidence because when she investigates for new evidence and disclosure is voluntary, she induces second-order belief uncertainty on the innocent defendant. That is, the defendant does not know what evidence the prosecutor has. This implies that the innocent defendant is willing to accept a positive sentence because of the possibility of the prosecutor showing default evidence at the trial. In order to not reveal the evidence, when the prosecutor has exculpatory evidence she needs the make the same offer that she would have made if she had default evidence. It is optimal for the prosecutor to make that offer when the prior belief is low because it is going to be accepted. However, when the prior belief is high enough, the offer that the prosecutor would have made if she had default evidence is higher than the innocent defendant is willing to accept. So, the prosecutor prefers to disclose the evidence because otherwise the offer will be rejected and they will go to the trial which is costly.

The second main result shows that mandatory disclosure of evidence is socially preferable to voluntary disclosure for the following reasons: First, from a utilitarian point of view, mandatory disclosure of evidence is efficient because an agreement is always reached during the plea bargaining process and the prosecutor never incurs the cost of going to trial. With voluntary disclosure there is a positive probability of going to trial when the prior belief about the defendant being guilty is high enough. Second, the prosecutor is better off with mandatory disclosure of evidence because being able to hide exculpatory evidence has a downside; she cannot extract the full surplus when she has default evidence. This produces a commitment effect; if the prosecutor can ex ante commit to disclose any evidence she found, she would do it. But with voluntary disclosure this is not possible, because if she gets exculpatory evidence she will hide it if she can. Third, in the frequent offers limit—when the length of each period goes to zero—the innocent defendant is better off and the guilty defendant is worse off with mandatory disclosure of evidence.

For both the voluntary and mandatory disclosure of evidence cases, prosecutor and defendant reach an agreement during the first period when the prosecutor's prior belief about the defendant being guilty is low enough. This is because the prosecutor prefers not to investigate and instead to reach an agreement immediately. For higher values of the prior belief, there is a deadline effect. If the disclosure of evidence is voluntary, the deadline effect is as in Spier (1992): The prosecutor and defendant reach an agreement just at the deadline with a high probability compared to other periods, and in some cases, they do not reach an agreement and go to trial. If the disclosure of evidence is mandatory, they also have a higher probability of reaching an agreement at the deadline, but they never go to trial.

Outline: The plan of the chapter is as follows. Section 1.1.2 provides background on the plea bargaining process, and Section 1.1.3 discusses the related literature. Section 1.2 introduces the model. Section 1.3 shows a one-period benchmark. Section 1.4 shows the general result with N periods. Section 1.5 discuss the mandatory disclosure of evidence and policy implications, and Section 1.6 concludes. Appendix A.1 provides extensions of the model, and Appendix A.3.1 contains the proofs.

1.1.2 Plea Bargaining Background

In US criminal law, plea bargaining is the pretrial process in which the prosecutor and the defendant negotiate an agreement such that the defendant pleads guilty in exchange for a lower sentence.[2] This agreement, called a plea bargain, allows the prosecutor and the defendant to avoid a trial and the associated cost and uncertainty. If they do not reach an agreement, the case goes to trial. The role of the prosecutor

[2] In the US system, the judge has to agree with the plea bargain. In this chapter, I assume the judge always agrees with it when the prosecutor and defendant agree.

is to represent society in the criminal case brought against the defendant.

During the trial, all of the evidence must be disclosed. This is because the trial is protected by the Brady Rule, named for *Brady v. Maryland (1963)* , which requires prosecutors to disclose materially exculpatory evidence in their possession to the defendant.[3] The Brady Rule is not always extended to the plea bargaining process. According to Casey (2020), the Brady Rule is applied to the plea bargaining process in the Seventh, Ninth, and Tenth Circuit courts, while it is not applied to pretrial negotiations in the First, Second, Fourth, and Fifth Circuits. State courts are divided in a similar fashion.[4]

Applying the Brady Rule to the plea bargaining process is a policy question that has attracted attention from both scholars and the media.[5] Some arguments in favor of extending the Brady Rule to plea bargaining are related to the knowing and voluntary nature of a guilty plea; failure to disclose materially exculpatory evidence precludes a knowing and voluntary guilty plea. As a consequence, the Brady Rule will likely reduce convictions of innocent defendants. Arguments against hold that extending the Brady Rule will result in higher costs and less efficiency.

The first main result of the chapter shows that even when the Brady Rule does not apply to plea bargaining, the prosecutor might drop cases under certain circumstances. The second main result of the chapter addresses the question of whether the Brady Rule should apply during pretrial negotiations. I show that applying the Brady Rule to pretrial negotiations is desirable, because the prosecutor and the defendant always avoid a costly trial by reaching an agreement. Also, the expected sentence is lower for the innocent and higher for the guilty defendant, while the expected payoff

[3]See *Brady v. Maryland*, 373 U.S. 83, 83 S. Ct. 1194, 10 L. Ed. 2d 215 (1963).

[4]There is no a clear definition in the other Circuits courts.

[5]See Casey (2020); Daughety and Reinganum (2020); or Sanders (2019) for some references. See also a *New York Times* editorial, “Beyond the Brady Rule” https://www.nytimes.com/2013/05/19/opinion/sunday/beyond-the-brady-rule.html

for the prosecutor is higher with mandatory disclosure.

1.1.3 Related Literature

In my model, the prosecutor investigates seeking new evidence, and the voluntary disclosure generates second-order uncertainty on the defendant. Hence, this chapter mainly relates to the literature on pretrial negotiations, bargaining with information arrival, and higher-order uncertainty in bargaining.

Pretrial negotiations: Spier (1992) and Fuchs and Skrzypacz (2013) present pretrial bargaining models with incomplete information and a deadline that includes a rule to assign payoffs. They show that many agreements occur just at the deadline, while in many other cases there is no agreement. Although in my model there is a similar effect regarding many agreements occur at the deadline, I also focus on the disclosure of information. Garoupa and Rizzolli (2011) study a model in which the prosecutor might decide not to investigate before trial and conclude that innocent defendants may be worse off with the Brady Rule at trial. Daughety and Reinganum (2018) present a trial model in which a prosecutor with career concerns can violate the Brady Rule at trial. These papers focus on modeling the trials, while I focus on the pretrial negotiation, and model the trial merely as a rule to assign payoffs.

In the literature on plea bargaining, Landes (1971) examines how the probability of winning at trial affects pretrial negotiations. Grossman and Katz (1983) and Reinganum (1988) study the welfare effects of plea bargaining, depending on the probability of conviction at trial. Baker and Mezzetti (2001) examine a model in which the prosecutor can choose the costly precision of a signal about defendant type. Bjerk (2007) presents a model in which new information can be revealed at trial. Vasserman and Yildiz (2019) present a model in which negotiating parties are optimistic about the decision at trial and anticipate a possible arrival of public information prior to the trial date. None of these papers allow for the possibility of disclosure of information or effects of the Brady Rule during pretrial negotiations.

Bargaining with information arrival: Duraj (2020) considers a bargaining model in which the buyer can choose how accurately she learns about her valuation of a good being traded, and she can disclose the updated valuation. Esö and Wallace (2019) consider a bargaining model in which the value of the good that is being traded is exogenously and privately revealed, and can be disclosed. They show that the possibility of learning might result in a delay in reaching an agreement. Esö and Wallace (2014) analyze the effect of exogenously having verifiable and unverifiable evidence in a one-period bargaining model, and show that the proposer is always better off with verifiable evidence. Hwang and Li (2017) present a model in which the buyer's outside option stochastically arrives and can be disclosed by the seller. If the outside option is private information, the buyer prefers to never reveal it and there is delay in the game. These papers show that each party with new information hides detrimental evidence and discloses beneficial evidence. In my model, the party with new information will disclose not just the beneficial information, but also the detrimental information to the other party. I characterize conditions under which doing so is optimal.

Daley and Green (2020); Fuchs and Skrzypacz (2010); Hwang (2018); Lomys (2017); Ortner (2017); and Ortner (2020) consider variations of the Coase conjecture model with arrival of new information (private or public). They do not consider disclosure of private information. The focus of the present chapter is the possibility of disclosing information and how that affects the efficiency of the bargaining.

Higher-order uncertainty in bargaining: Feinberg and Skrzypacz (2005) study a bargaining model in which one party privately knows his type and the other party has a private belief about the type. This second-order uncertainty is exogenous, and there is no disclosure of information during the bargaining. The authors show that there is delay in the agreement. In my model, the uncertainty is endogenous rather than exogenous, and one party can eliminate the uncertainty of the other party by revealing information.

Friedenberg (2019) studies an alternating-offer bargaining model in which delay in agreement may arise when players face strategic uncertainty—that is, uncertainty about the opponent's play. There is no strategic uncertainty in my model; rather, there is uncertainty in the second-order belief. Also, I focus on the disclosure decision.

1.2 Model

There are two players: a prosecutor (she) and a defendant (he). The prosecutor's only objective is to assign the highest possible sentence to the defendant, regardless of the defendant's innocence, while the defendant wants the lowest possible sentence.[6] The defendant is privately informed of his type α, which can be innocent ($\alpha = I$) or guilty ($\alpha = G$). The defendant's type is unknown to the prosecutor. Let $\theta \in (0, 1)$ denote the prior probability that the prosecutor assigns to $\alpha = G$. The game is divided into two phases: *plea bargaining* and *trial*. The game starts with plea bargaining phase, in which the prosecutor can investigate for new evidence and try to reach an agreement with the defendant to avoid trial. They move to the trial phase only if they fail to reach an agreement before a deadline. The trial is a reduced-form function that assigns a reward to the prosecutor and a punishment to the defendant, depending on

[6]This implies that even though the prosecutor knows the defendant is innocent, she still wants him to have the highest possible sentence. Although this is a simplification, many prosecutors seem to be motivated by high sentences rather than justice. Medwed (2004) notes that many prosecutors resist exonerating the innocent even when prisoners have presented overwhelming proof of their innocence. Also, Keenan et al. (2011) and Garrett (2017) argue that prosecutorial misconduct is a widespread problem in the US,and list cases in which prosecutors suppress exculpatory evidence at the trial. Finally, Pfaff (2017) argues that the criminal justice system provides incentives for prosecutors to seek an overly aggressive punishment, and Alschuler (2015) argues that the plea bargaining process tends to convict more innocent people than trials do.

the evidence the prosecutor has at the time of the trial.

The plea bargaining phase ends at time $T > 0$. This phase is divided into $N \geq 1$ periods, with the length of each period equal to $\Delta = T/N$. The set of evidence that exists in this environment is $y \in \{e, d, h\}$, where $y = e$ stands for *exculpatory evidence*, $y = h$ for *incriminating evidence*, and $y = d$ for *default evidence.*

The prosecutor can investigate for new evidence at the beginning of each period, and she can voluntarily disclose the new evidence. At the end of each period, she makes an offer to the defendant. There is no discount factor or cost of delay during the plea bargaining phase for any player.

The timing within each period n is:

1. Investigation for new evidence: At the beginning of the game, the prosecutor has evidence $y = d$ and a prior belief $Pr(\alpha = G) = \theta \in (0, 1)$. At the start of each period $n = \{1, 2, ..., N\}$, the prosecutor can investigate to obtain more evidence. The probability of getting evidence follows an exponential distribution that depends on the length of each period; the probability of finding new evidence at each period n is equal to $1 - q^{\frac{1}{N}}$, where $q = e^{-\lambda}$ for $\lambda > 0$.[7]

The new evidence depends on the defendant's type: If $\alpha = I$ the investigation's outcome belongs to $y^I \in \{\emptyset, e\}$; if $\alpha = G$ the outcome belongs to $y^G \in \{\emptyset, h\}$. The implication is that after getting $y = e$ or $y = h$, the prosecutor updates her belief to $\theta' = 0$ and $\theta' = 1$, respectively. The prosecutor gets new evidence only once and it replaces default evidence. Note that the probability of finding new evidence is independent of the defendant's type; this implies that if the prosecutor does not get new evidence after the investigation, she does not update her belief about the defendant's type.

[7] Note that $q^{\frac{1}{N}} = e^{-\lambda \frac{\Delta}{T}}$.

The decision to investigate and the outcome of the investigation are private information for the prosecutor. I say that the prosecutor is y-type if she has evidence $y \in \{e, d, h\}$.

2. Disclosure of new evidence: After the outcome is realized, the prosecutor can choose to disclose the new evidence to the defendant. I assume that only new evidence can be disclosed.[8] I also assume that the disclosure of evidence is voluntary during the plea bargaining phase, but it is mandatory at trial. The mandatory case with disclosure of evidence during the plea bargaining phase will be discussed in Section 1.5.

3. Offer: After the prosecutor's decision to disclose or not, the prosecutor makes an offer $x \in \mathbb{R}$ to the defendant. An offer is a sentence that assigns utility $u_D = -x$ to the defendant and utility $u_P = x$ to the prosecutor if it is accepted.[9] If the offer is accepted the game ends, and if the offer is rejected and $n < N$, a new period $n+1$ starts. If the offer is rejected at $n = N$, they go to trial. Figure 1.1 shows the timing of a period n during the plea bargaining phase.

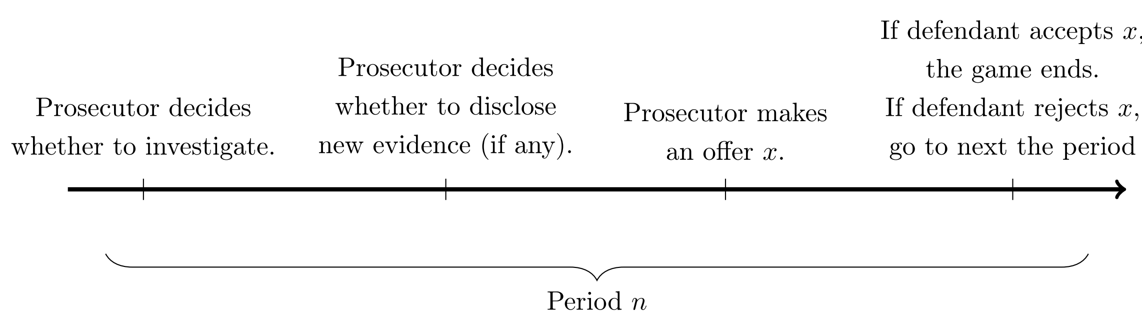

Figure 1·1: Timeline of a period n

This chapter's focus is the plea bargaining phase. For this reason, I model the

[8] I assume evidence is hard; the prosecutor cannot show what she does not have.

[9] The zero-sum nature of the payoff is without loss of generality.

trial as a simple rule that assigns utility to evidence. The defendant's payoff u_D at the trial is

$$u_D = \begin{cases} 0 & \text{if } y = e \\ -d & \text{if } y = d \\ -h & \text{if } y = h. \end{cases}$$

The prosecutor incurs a cost of going to trial, so her payoff is lower than the sentence that is assigned to the defendant. The prosecutor's payoff u_P is

$$u_P = \begin{cases} -c & \text{if } y = e \\ 0 & \text{if } y = d \\ h' & \text{if } y = h, \end{cases}$$

where $0 < c < d < h' < h$.

Assumption 1: to avoid cases in which the prosecutor never investigates for any θ, I consider that getting h is attractive compared to d for at least some values of θ: $h > \left(\frac{1+q}{q}\right)d$.

Histories and strategies. Call $\tilde{y} \in \{\emptyset, e, h\}$ the evidence disclosed by the prosecutor, where $\tilde{y} = \emptyset$ means that the prosecutor has not disclosed any evidence. At any period n before the agreement is reached, the prosecutor's history $h_n^P = \{y_n, \{\tilde{y}_s, x_s\}_{s \leq n}\}$ contains the evidence the prosecutor has, the disclosed evidence, and the offers she has made. The defendant's history $h_n^D = \{\alpha, \{\tilde{y}_s, x_s\}_{s \leq n}\}$ contains his type, the disclosed evidence, and the previous offers. A (pure) strategy for the prosecutor $\sigma^P : h_n^P \to (\{\textit{investigation, no investigation}\}, \tilde{y}_n(y), x_n)$ maps prosecutor's history h_n^P to the decision to investigate or not, a disclosure decision after the outcome of the investigation is realized, and an offer x to the defendant. A strategy for the defendant $\sigma^D : h_n^D \times x_n \to \{\textit{accept, reject}\}$ maps the defendant's history h_n^D to a decision whether to accept or not offer x_n.

Solution concept. An equilibrium is a *perfect Bayesian equilibrium (PBE)* in which the prosecutor only uses pure strategies. For indifference between reaching an agreement at n or $n+1$, I use the tie-breaking rule whereby both prosecutor and defendant prefer to agree in a sentence at n if there is no payoff loss.[10] Also, if the prosecutor gets evidence $y = e$, she is indifferent between disclosing it and offering $x = 0$ and not disclosing it and offering $x = 0$. Both strategies are payoff equivalent and qualitatively the same. I consider that the prosecutor discloses the evidence.

Finally, there are two qualitatively different equilibria for low values of θ. The first is the *investigation equilibrium*, in which the prosecutor investigates for at least the first period. The second is the *no-investigation equilibrium*, in which the prosecutor does not investigate in any period for low values of θ. The same main results of the chapter holds for both equilibria.[11] I select the *investigation equilibrium* to be described in the main part of the chapter, and discuss the *no-investigation equilibrium* in Appendix A.1.1.

1.3 One-Period Benchmark

Before presenting the general result for an arbitrary number of periods, I consider a game with only one period. This benchmark provides intuition for the offer the prosecutor makes at the last period before trial in the general game. I assume the trial is at time T, therefore the probability of finding new evidence is $1 - q$.

I denote the continuation value for the prosecutor as v^P and I define *continuation*

[10] This selection is to rule out equilibria in which prosecutor and defendant delay reaching an agreement with no change in information and payoffs. It can be interpreted as a weak form of players being impatient.

[11] The main results of the chapter are related to higher values of θ, and this multiplicity of equilibria is only for low values of θ.

punishment to be the absolute value of the continuation value for the defendant. I denote the continuation punishment as v^α, with $\alpha \in \{I, G\}$.

The main result of this section is the disclosure decision of the prosecutor and the analysis of the offer she makes. The prosecutor discloses incriminating evidence for any prior belief, and discloses exculpatory evidence when her prior belief is high enough. When the prosecutor discloses the evidence, she offers the same sentence that the defendant will get at trial. When the prosecutor does not disclose exculpatory evidence, she offers a positive sentence, which is accepted for the defendant if he is innocent or guilty.

As explained in the previous section, I restrict attention to equilibria in which the prosecutor always investigates in the first period. Proposition 1 provides the disclosure decision and optimal offer for the prosecutor.

Proposition 1 *In the investigation equilibrium:*

1) Disclosure: If the prosecutor gets evidence $y = h$, she always discloses it. If she gets evidence $y = e$, she discloses it if $\theta > q$.

2) Offers: If $\theta \leq q$, the prosecutor offers $x = dq$ if $y \in \{e, d\}$ and $x = h$ if $y = h$. The defendant accepts the offer. If $\theta > q$, the prosecutor offers $x = 0$ if $y = e$, and $x = h$ if $y = h$. The defendant accepts the offer. The prosecutor offers $x = d$ if $y = d$, and only the guilty defendant accepts the offer.

The prosecutor always discloses $y = h$, because it induces the guilty defendant to accept the offer $x = h$. The defendant accepts $x = h$ because he would receive the same punishment at the trial if he rejects it.

While it is intuitive that the prosecutor always discloses incriminating evidence, her incentive to disclose exculpatory evidence is driven by the second-order belief uncertainty of defendant. To be specific, consider for now that she does not disclose it for any value of θ. The investigation and the nondisclosure of evidence induce second-order belief uncertainty on the innocent defendant, because the innocent defendant does not know whether the prosecutor knows his type or not. The d-type prosecutor believes the defendant is guilty with probability θ, and the e-type prosecutor knows that the defendant is innocent. Assuming no disclosure of exculpatory evidence, the innocent defendant's belief about the prosecutor's type is the following:

$$P^I(d\text{-type prosecutor} \mid \text{nondisclosure}) = q$$
$$P^I(e\text{-type prosecutor} \mid \text{nondisclosure}) = 1 - q$$

The guilty defendant's belief the about prosecutor's type is

$$P^G(d\text{-type prosecutor} \mid \text{nondisclosure}) = 1.$$

Second-order beliefs affect the expected punishment at trial. If there is no disclosure, the expected punishment at trial for the innocent defendant, given these beliefs, is $dq + 0(1 - q)$. The guilty defendant knows for sure that the prosecutor is d-type if there is no disclosure, because the prosecutor always discloses $y = h$; therefore, his expected punishment at trial is $v^G = d$.

If the prosecutor is d-type, she cannot extract all of the surplus from the defendant because of the second-order belief uncertainty. The innocent defendant is not going to accept an offer higher than $x = dq$, and if the prosecutor offers $x = d$ it will only be accepted if the defendant is guilty. Therefore, the d-type prosecutor offers $x = dq$ if $\theta < q$, and offers $x = d$ if $\theta \geq q$.

Suppose now the prosecutor is e-type. The prosecutor is able to hide the exculpatory evidence if she makes the same offer as the d-type prosecutor. This is because the innocent defendant does not know the evidence that prosecutor has, and if the d-type and the e-type make the same offer, the defendant cannot extract information from it.

It is optimal for the e-type prosecutor to make the same offer as the d-type if $\theta \leq q$, because the innocent defendant accepts it. However, it is not optimal to make the same offer as the d-type if $\theta > q$, because the innocent defendant rejects $x = d$ and the prosecutor gets a negative payoff at trial. Hence, if $\theta > q$, the e-type prosecutor must make a lower offer than the d-type prosecutor; and this lower offer reveals her private information.

In equilibrium, the prosecutor discloses $y = e$ if $\theta > q$ and offers $x = 0$, because any other offer will be rejected by the innocent defendant. The intuitive reason is that any offer $x < d$ is not sequentially rational for the d-type prosecutor. Therefore, if the innocent defendant receives an offer $x < d$, he updates his belief about the prosecutor's type to e-type with probability one.

Inefficiency. In equilibrium, if the defendant is innocent and the prosecutor does not find new evidence they go to trial when $\theta > q$. Going to trial is socially inefficient because it is costly for the prosecutor. Hence the voluntary disclosure of evidence is ex ante inefficient when $\theta > q$, because with probability $q(1 - \theta)$ the prosecutor and defendant go to trial.

Commitment Effect. Second-order belief uncertainty allows the prosecutor to hide evidence for $\theta \leq q$; this benefits the e-type prosecutor. However, it has a downside for the d-type prosecutor, because she gets an expected payoff lower than d in equilibrium. This generates a commitment effect for the prosecutor: If she can ex-ante commit to disclose any evidence, she will do it.

The reason is that, with voluntary disclosure of evidence, when $\theta \leq q$ she gets a payoff of dq from having default evidence, no matter the defendant's type. When $\theta > q$, she gets an expected payoff of θd. In both cases, she cannot extract the full surplus from the default evidence. This negative effect of the voluntary disclosure case outweighs the benefit getting a payoff of dq if $y = e$ and $\theta \leq q$. Therefore, for any θ the prosecutor is ex ante better off if she is able to commit to disclose any evidence she receives.

1.4 General Result

The main difference between the case with only one period and the multi-period setting is that in the latter case, the prosecutor can learn about the defendant's type through rejected intermediate offers. I define intermediate offers, which are offers that the prosecutor makes before the last period. In this section I show that the prosecutor updates her prior belief using intermediate offers only for a specific range of θ values. This updating allows her to hide evidence for a larger set of values of prior beliefs compared with the one-period-case benchmark. I also show that the prosecutor prefers to reach an agreement in the first period for low values of θ.

For clearer exposition in this section, I define $\theta^{(n)}$ as the prosecutor's belief about the defendant's being guilty at the beginning of period n, where $\theta = \theta^{(n=1)}$ is the prior belief.

1.4.1 Disclosure Decision and Offers

An important observation for the results in this section is that the relevant period in the decision to hide or reveal exculpatory evidence is the last one before the trial. Because the probability of finding new evidence at each period is $1 - q^{\frac{1}{N}}$, the total probability of finding new evidence by the end of the plea bargaining phase, if the prosecutor investigates during all periods, is $1 - q$. Hence, Proposition 1 implies that if the prosecutor investigates every period, she hides exculpatory evidence in the last period if $\theta^{(n=N)} \leq q$, and discloses it if $\theta^{(n=N)} > q$.

A second important observation is given that the investigation induces second-order belief uncertainty, and results in the d-type prosecutor getting a lower payoff, the prosecutor might prefer not to investigate for low values of θ.

In what follows, I characterize the prosecutor's optimal investigation decision, disclosure decision, and offers. I define the cutoff:

$$\underline{\theta}^N = \frac{dq^{\frac{1}{N}}}{h - dq}.$$

Lemma 1 *Fix $\theta \leq \underline{\theta}^N$. In the investigation equilibrium, the prosecutor investigates in the first period. If $y = h$, she discloses it and offers x=h. The guilty defendant accepts the offer. If $y \in \{e, d\}$ she does not disclose and offers $x = dq^{\frac{1}{N}}$. Both defendant types accept.*

Lemma 1 says the prosecutor and the defendant reach an agreement the first period for $\theta \leq \underline{\theta}^N$. The intuition is the prosecutor prefers to make an offer that is accepted, because if the probability that the defendant is guilty is low enough, the prosecutor is better off not investigating for new evidence the following period, because the risk of finding $y = e$ is higher than finding $y = h$.

If the prosecutor prefers not to investigate the following periods, the way to sustain no-investigation in equilibrium is removing the incentive to deviate to investigate in the next periods. To do that the prosecutor needs to separate the guilty from the innocent, by making an offer the guilty defendant strictly prefers to accept. The highest offer the guilty defendant accepts for sure is the innocent defendant's continuation punishment if there is not going to be more investigation.[12]

Note that the probability that the defendant assigns to the prosecutor be a d-type is decreasing in the number of periods of investigation. If n^* is the number of periods

[12]This is because if this offer is rejected, the prosecutor does not investigate in the following periods, since it is only rejected by the innocent defendant. Therefore, the innocent is indifferent between accepting it and rejecting it. If the offer is higher than the innocent defendant's continuation punishment, the innocent defendant rejects it for sure, and the prosecutor would make a lower offer next period (to avoid trial). Hence, the guilty defendant also rejects it because there is going to be a lower offer next period.

of investigation, then

$$P^I(d\text{-type prosecutor} \mid \text{nondisclosure}) = q^{\frac{n^*}{N}}.$$

This implies that the innocent defendant's continuation punishment is also decreasing in the number of periods of investigation. Hence, if the prosecutor prefers not to investigate, she prefers to reach an agreement at the end of period $n = 1$ by offering $x = dq^{\frac{1}{N}}$.

For $\theta > \underline{\theta}^N$, the prosecutor investigates at every period before the game ends. As I argue above, if $\theta^{(n=N)} \leq q$ the prosecutor hides exculpatory evidence, and if $\theta^{(n=N)} > q$ she discloses it. If the prior belief at that the beginning of the game is $\theta \leq q$, a d-type prosecutor does not need to use the intermediate offers to update her belief about the defendant's type to be able to hide exculpatory evidence.[13] However, if $\theta > q$ at the beginning of the game, the prosecutor can use the intermediate offer to partially skim the guilty from the innocent, update her prior belief such that $\theta^{(n=N)} \leq q$ and therefore be able to hide exculpatory evidence. I show that this is the case for a specific range of θ values.

Define:

$$\bar{\theta}^N = \frac{q}{q^{\frac{N-1}{N}} + q(1 - q^{\frac{N-1}{N}})}.$$

Proposition 2 *For $\theta \in (\underline{\theta}^N, \overline{\theta}^N)$, the prosecutor investigates at any period as long as they have not reached an agreement.*

1) If $y = h$, she discloses it as soon as she gets it and offers $x = h$. The defendant accepts the offer.

2) If $y \in \{e, d\}$, there are two cases: For $\theta \in (\underline{\theta}^N, q]$, the prosecutor hides exculpatory evidence. She makes intermediate offers that are rejected by the defendant at

[13] A d-type prosecutor does not update her belief at the end of a period if the intermediate offer is rejected for sure by both defendant types.

$n < N$, and she offers $x = dq$ that is accepted by both types of the defendant at $n = N$. For $\theta \in (q, \bar{\theta}^N]$, at the end of $n = 1$ the prosecutor offers $x = (1 - q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}} dq$, which is accepted with probability $\mu^G = \frac{\theta - \bar{\theta}}{\theta(1-\bar{\theta})}$ by the guilty defendant and rejected by the innocent defendant. Prosecutor updates her prior belief to $\theta^{(n=2)} = q$. Then, for $2 \leq n < N$ she makes intermediate offers that are rejected by the defendant, and at $n = N$ she offers $x = dq$, which is accepted by the defendant.

Proposition 2 says that the prosecutor is able to hide exculpatory evidence if $\theta \in (\underline{\theta}^N, \bar{\theta}^N]$, and that she uses the intermediate offer to update her belief if $\theta \in (q, \bar{\theta}^N]$ at the end of $n = 1$ in order to expand the range of θ values in which she hides $y = e$ from $(\underline{\theta}^N, q]$ to $(\underline{\theta}^N, \bar{\theta}^N]$. The guilty defendant is indifferent between accepting the offer $x = (1 - q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}} dq$ or rejecting it, because if rejected the prosecutor updates his belief and guilty defendant's continuation punishment is exactly $(1 - q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}} dq$.

At any other period, the intermediate offer is rejected for sure, or used to end the game after a disclosure of evidence. Note that in order to hide exculpatory evidence, the prosecutor needs to wait until the last period to offer $x = dq$.

The reason the prosecutor does not use the intermediate offers at other θ values is because if $\theta \in (\underline{\theta}^N, q]$, the prosecutor is already able to hide exculpatory evidence. Updating her belief within the interval does not change her expected payoff—and decreasing it to be lower than $\underline{\theta}^N$ is not optimal, because the prosecutor is better off investigating every period, since the probability of finding incriminating evidence is high enough at the original θ.

Proposition 3 *Suppose $\theta \in (\bar{\theta}^N, 1]$. The prosecutor investigates at every period as long as they have not reached an agreement. She discloses $y = e$ and $y = h$ as soon as she gets it and offers $x = 0$ and $x = h$ respectively. The defendant accepts it.*

She makes intermediate offers that are rejected by the defendant if $y = d$, and at

$n = N$ she offers $x = d$ if $y = d$ that is accepted by the guilty defendant and rejected by the innocent defendant.

Proposition 3 says that for $\theta \in (\bar{\theta}^N, 1]$ the prosecutor discloses any evidence as soon as she gets it. The prosecutor does not use intermediate offers to extract information from the defendant.

The prosecutor does not update her belief to be lower than $\bar{\theta}$ using the intermediate offer, because she is better off making a higher offer if $y = d$—which is only accepted by the guilty defendant—than making a lower offer that is accepted by both defendant types. This is because the probability of the defendant's being guilty is high enough. Also, updating her belief within the interval does not change her expected payoff.[14]

1.4.2 Efficiency

The prosecutor and the defendant fail to reach an agreement if $\theta > \bar{\theta}^N$ and $y = d$, because the innocent defendant does not accept the offer the prosecutor makes. In this case the prosecutor goes to trial if the defendant is innocent. Formally, the probability of going to trial is

$$P(trial) = \begin{cases} 0 & \text{if } \theta \leq \bar{\theta}^N \\ q(1-\theta) & \text{if } \theta > \bar{\theta}^N. \end{cases}$$

Recall that that $\bar{\theta}^N$ increases with N. Hence the range of values in which the prosecutor discloses exculpatory evidence is smaller when the negotiation period is divided into more periods. In other words, the equilibrium outcome is less inefficient when N increases.

[14] Also, updating it to be lower than $\underline{\theta}$ is not optimal, because the prosecutor is better off investigating in every period since the probability of finding incriminating evidence is high enough at the original θ.

1.4.3 Frequent-offer Limit

In this subsection I consider the frequent-offer limit, where $N \to \infty$ keeping T constant. The implication is that the probability of finding new evidence at each period is arbitrarily low, because $\Delta \to 0$. Intuitively, the high-frequency limit represents continuous investigation by the prosecutor, whereby she can interrupt the investigation to make an offer at any point.

By definition, the limit values of the θ cutoffs when $N \to \infty$ are

$$\lim_{N\to\infty} \underline{\theta}^N = \frac{d}{h - dq} \equiv \underline{\theta}, \text{ and } \qquad \lim_{N\to\infty} \bar{\theta}^N = \frac{1}{2-q} \equiv \bar{\theta}.$$

The prosecutor ends the game at $t = 0$ for $\theta \leq \underline{\theta}$. For $\theta > \underline{\theta}$, the time at which the game ends depends on the evidence and the value of θ, because the prosecutor uses different strategies depending on θ. In this section I show that the game has a deadline effect—the probability of ending the game has a mass point at the deadline T.

The Path of Agreements.

Deadline effects in pretrial negotiation have been studied in Spier (1992); Fuchs and Skrzypacz (2013); and Vasserman and Yildiz (2019). They have also been observed in experimental studies by Roth et al. (1988) and Güth et al. (2005).[15]

Proposition 4 *There is a deadline effect: The probability of reaching an agreement has a mass point at T. The game ends by T with probability 1 for $\theta \in (\underline{\theta}, \bar{\theta}]$ for both defendant's types, and ends by T with probability 1 for $\theta \in (\bar{\theta}, 1]$ if the defendant is guilty.*

[15] Other papers that find a deadline effect are Cramton and Tracy (1992); Fershtman and Seidmann (1993); and Ma and Manove (1993).

If $\theta \in (\underline{\theta}, \bar{\theta}]$, the prosecutor hides exculpatory evidence; therefore if the defendant is innocent the game ends at time T. Hence the prosecutor ends the game at the deadline. If the defendant is guilty, she ends the game as soon as she gets evidence $y = h$ or at time T if she never gets new evidence; therefore, there is also a deadline effect. Note that if $\theta \in (q, \bar{\theta}]$, the prosecutor makes an initial offer that is rejected by the innocent defendant and accepted by the guilty defendant with probability μ^G. Let τ be the time by which the game ends. The probability that τ is less than t when the defendant is guilty is given by:

If $\theta \in (\underline{\theta}, q]$, then

$$P_G(\tau \leq t) = \begin{cases} 1 - e^{-\lambda \frac{t}{T}} & \text{if } t < T \\ 1 & \text{if } t = T. \end{cases}$$

If $\theta \in (q, \bar{\theta}]$, then

$$P_G(\tau \leq t) = \begin{cases} \mu^G & \text{if } t = 0 \\ 1 - e^{-\lambda \frac{t}{T}}(1 - \mu^G) & \text{if } t \in (0, T) \\ 1 & \text{if } t = T. \end{cases}$$

For $\theta > \bar{\theta}$, the prosecutor reveals any new evidence. Nevertheless, if she does not get new evidence the game ends at T only if the defendant is guilty; if the defendant is innocent, they go to trial. This means that there is a deadline effect only when the defendant is guilty. The probability that the game ends by time τ when the defendant is guilty for $\theta > \bar{\theta}$ is

$$P_G(\tau \leq t) = \begin{cases} 1 - e^{-\lambda \frac{t}{T}} & \text{if } t < T \\ 1 & \text{if } t = T. \end{cases}$$

And for the innocent defendant for $\theta > \bar{\theta}$ it is:

$$P_I\big(\tau \leq t\big) = 1 - e^{-\lambda \frac{t}{T}} \quad \text{for } t \leq T.$$

I consider the trial is at period $T+1$, therefore if the probability of ending the game at or before T is less than one, the game ends at trial at time $T+1$.

Figure 1.2 illustrate the probability of ending the game by time t depending on the defendant's type and the prior belief.

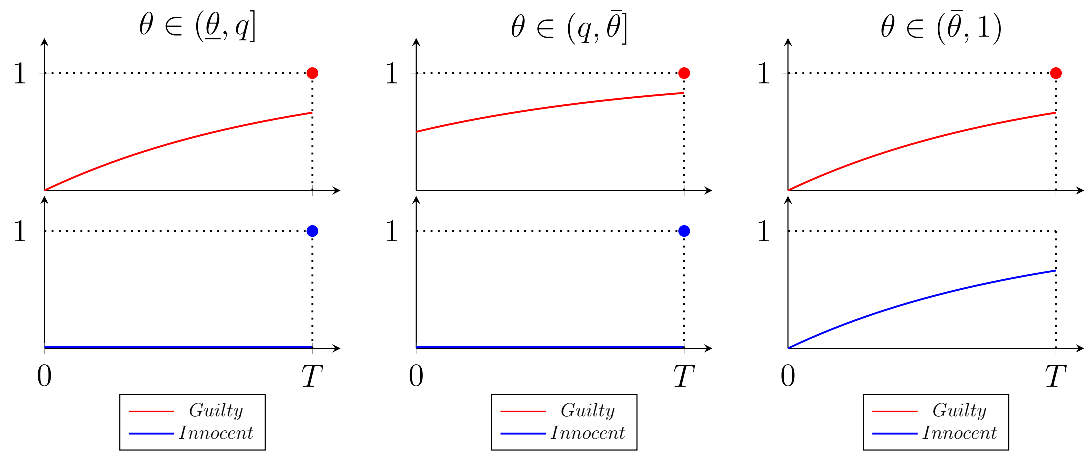

Figure 1·2: Probability of ending the game by time $t \leq T$.
Values: $\mu = 0.5$, $\lambda = 0.8$, $T = 30$, $q = 0.55$.
Left: $\theta \in (\underline{\theta}, q]$, middle: $\theta \in (q, \bar{\theta}]$, right: $\theta \in (\bar{\theta}, 1]$.

Payoffs.

The prosecutor's expected payoff, for a given θ, is weakly increasing in the number of periods for $\theta < \bar{\theta}$. The defendant's expected punishment is also affected by the number of periods for $\theta < \bar{\theta}$; the innocent defendant is weakly worse off if there are more periods of investigation, because the prosecutor investigates during a smaller fraction of the plea bargaining phase if θ is low, and therefore less likely to get $y = e$. In contrast, the guilty defendant benefits from less investigation, because it is less likely to find $y = h$. Prosecutor and defendant are indifferent to the number of period for $\theta \geq \bar{\theta}$.

Corollary 1 *The prosecutor's expected payoff and the innocent defendant's expected punishment is maximized at $N \to \infty$. The guilty defendant's expected punishment is minimized at $N \to \infty$.*

The expected payoff for the prosecutor in the limit case is:

$$v^P = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}] \\ \theta\Big[(1-q)h + qdq\Big] + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}, \bar{\theta}] \\ \theta\Big[(1-q)h + qd\Big] & \text{if } \theta \in (\bar{\theta}, 1) \end{cases}$$

The expected punishment for the innocent defendant and guilty defendant are

$$v^I = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}] \\ dq & \text{if } \theta \in (\underline{\theta}, 1] \end{cases} \quad \text{and} \quad v^G = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}] \\ (1-q)h + qdq & \text{if } \theta \in (\underline{\theta}, \bar{\theta}] \\ (1-q)h + qd & \text{if } \theta \in (\bar{\theta}, 1). \end{cases}$$

As an example, Figure 1.3 compares the cases when the length T of the plea bargaining phase is divided into one, two, and infinite periods. For higher N, the prosecutor is better off for low values of θ because, after investigating in the first period, she can make an offer that both defendant types accept, and this offer is higher if there are more periods. The prosecutor is also better off with more periods if θ belongs to the interval $(q, \bar{\theta}]$, because she can make an intermediate offer that increases the θ threshold in which she can hide evidence and transfer a higher punishment from the guilty defendant to the innocent defendant.

The innocent defendant is worse off for higher N, because the range of θ in which the prosecutor only investigates one period is larger. Also, in that interval the offer the prosecutor makes is higher if N is higher. In the limit case $N \to \infty$, the prosecutor offers $x = d$ because the investigation in the first period is negligible compared with the rest of the periods.

The guilty defendant is better off with less investigation, and he is also better off with more periods because the range of θ values in which the prosecutor discloses exculpatory evidence is smaller; this affects the guilty defendant because if the prosecutor hides exculpatory evidence, she makes a lower effort when $y = d$ than when she discloses it.

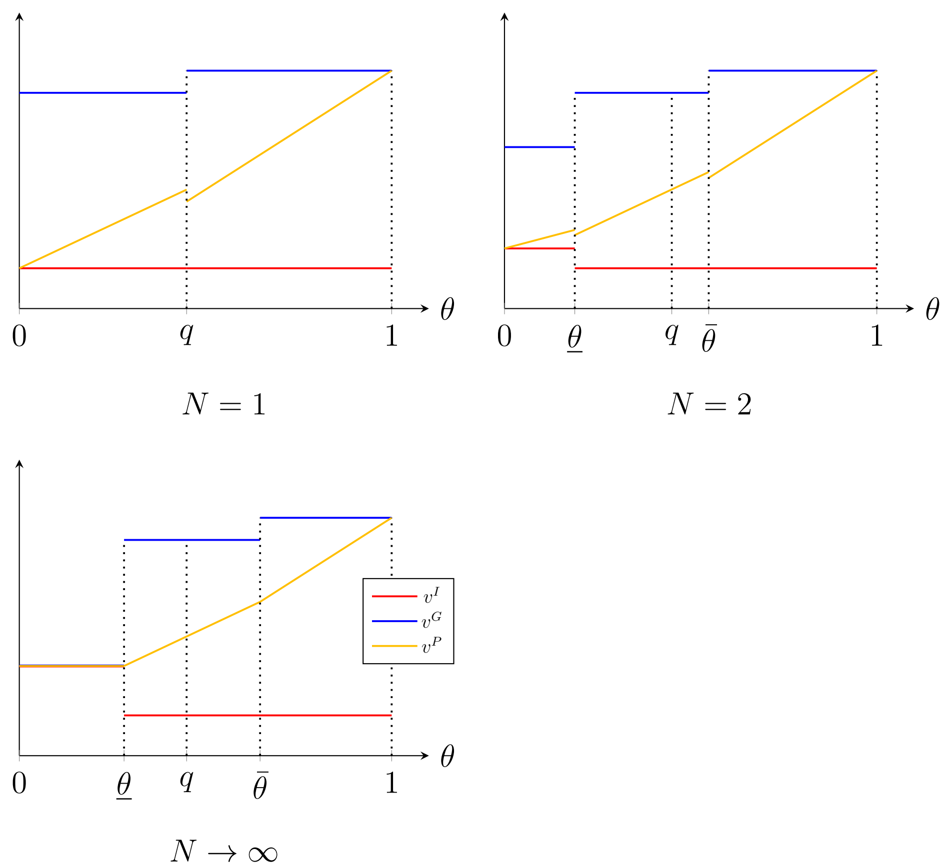

Figure 1·3: Expected payoffs and punishments comparison between $N = 1$, $N = 2$, and $N \to \infty$.
Upper left: $N = 1$, upper right: $N = 2$, bottom: $N \to \infty$.

1.5 Mandatory Disclosure and Policy Implications

The Brady Rule is the legal requirement that the prosecutor must disclose all evidence she has—default, incriminating or exculpatory—to the defendant at trial. The Brady Rule is not always extended to pretrial negotiations; the Fifth Circuit court recently[16] joined the First, Second, and Fourth Circuits by ruling that criminal defendants are not constitutionally entitled to exculpatory evidence prior to entering a guilty plea.[17] The Seventh, Ninth, and Tenth Circuits have ruled that exculpatory evidence must be disclosed before entering a guilty plea. The United States Supreme Court has not ruled on the issue.[18]

In this section, I assume the Brady Rule applies to the pretrial negotiation process as well as the trial. I compare the equilibrium under Brady Rule and under voluntary disclosure of evidence, and I suggest that the Brady Rule should be extended to pretrial negotiations because it improves efficiency. I also show that the outcomes of the Brady Rule case are closer to the objective of assigning a high punishment to the defendant if he is guilty, and set the defendant free if he is innocent.

1.5.1 The Brady Rule: Mandatory Disclosure

The prosecutor does not induce second-order belief uncertainty in the defendant when she investigates, because the defendant knows the evidence the prosecutor has before any offer. Therefore, the prosecutor ends the game if she gets evidence $y = e$ or $y = h$ by offering $x = 0$ and $x = h$, respectively. The trade-off the d-type prosecutor faces each period is whether to investigate.

Define:

$$\underline{\theta}^{BR} = \frac{d}{h}.$$

[16] In 2018, in deciding *Alvarez v. City of Brownsville.*

[17] See Petegorsky (2012); Grossman (2016); and Casey (2020).

[18] See Casey (2020).

Proposition 5 *The following results hold:*

(i) For $\theta \leq \underline{\theta}^{BR}$, the prosecutor does not investigate in the first period. She offers $x = d$ at the end of the first period and the defendant accepts.

(ii) For $\theta > \underline{\theta}^{BR}$, the prosecutor investigates in every period, as long as the game has not ended. If she gets $y = e$ or $y = h$, she offers $x = 0$ and $x = h$, respectively and the defendant accepts the offer. If she does not get new evidence at $n < N$, she offers $x = h$, which is rejected for sure by both defendant types. If she does not get new evidence at $n = N$, she offers $x = d$, which is accepted for sure by both defendant types.

Proposition 5 says that a d-type prosecutor either investigates every period or never investigates. The reason is that for low values of θ, the risk of finding exculpatory evidence is higher than the benefit of finding $y = h$. The opposite is true for high values of θ. Note further that the cutoff and the equilibrium does not depend on the number of periods N.

The d-type prosecutor does not skim the guilty from the innocent using the intermediate offer. If she updates her belief to $\theta' > \underline{\theta}^{BR}$, she gets the same expected continuation payoff. It is neither optimal for her to update her belief to $\theta' \leq \underline{\theta}^{BR}$, because to do so she would offer $x = d$, and she is better off investigating for new evidence.

From Proposition 5, the prosecutor's expected payoff is

$$u^P = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{BR}] \\ \theta\Big((1-q)h + dq\Big) + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}^{BR}, 1). \end{cases}$$

The innocent defendant's and the guilty defendant's expected punishments are

$$u^I = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{BR}] \\ dq & \text{if } \theta \in (\underline{\theta}^{BR}, 1) \end{cases} \quad \text{and} \quad u^G = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{BR}] \\ (1-q)h + qd & \text{if } \theta \in (\underline{\theta}^{BR}, 1). \end{cases}$$

Figure 1.4 shows the payoffs and punishments with mandatory disclosure.

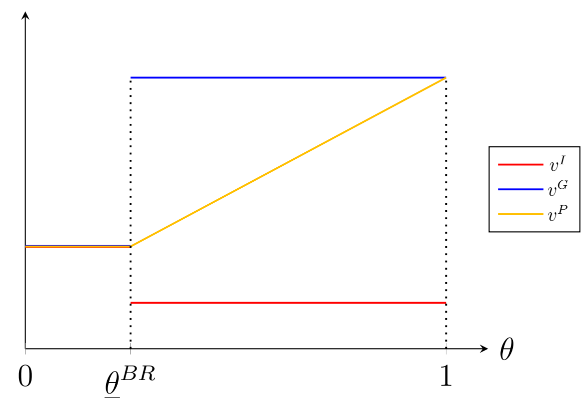

Figure 1·4: Expected payoff and punishment with Brady Rule.

1.5.2 Policy Implications: Voluntary v. Mandatory Disclosure

I first compare the efficiency of the equilibrium when Brady Rule applies to the plea bargaining phase and when it does not apply. I show that mandatory disclosure of evidence is efficient.

Corollary 2 *The equilibrium under Brady Rule is efficient: The prosecutor and the defendant never go to trial.*

Proposition 5 says that the d-type prosecutor makes an offer $x = d$ at $n = N$ that is accepted by the defendant. Hence, the prosecutor and the defendant always reach an agreement in the plea bargaining phase with Brady Rule, which implies that there are no inefficiencies related to going to trial. The policy implication is that under the Brady Rule at the plea bargaining phase there is no inefficiencies, and then it is socially preferable from an utilitarian point of view.

I compare the expected payoffs of the prosecutor and the defendant under Brady Rule and under voluntary disclosure of evidence.

Proposition 6 *The prosecutor is weakly better off with mandatory disclosure of evidence. For $N \to \infty$, the innocent defendant is weakly better off with mandatory disclosure of evidence, while the guilty defendant is weakly worse off.*

The prosecutor is better off with mandatory disclosure of evidence for two reasons: (1) she extracts the full surplus from each defendant type if she investigates, and (2) she investigates for a larger range of prior beliefs.

1. The prosecutor extracts all the surplus under Brady Rule. When the disclosure of evidence is mandatory, the agreement reached by the prosecutor and the defendant is either h if the evidence is incriminating, d if it is default, or zero if it is exculpatory. This implies the prosecutor gets the expected payoff after investigation from each defendant.

When the disclosure of evidence is voluntary, there are two options. First, if the prior belief is below $\bar{\theta}$, the prosecutor hides exculpatory evidence, and second, if the prior belief is above $\bar{\theta}$, she discloses exculpatory evidence.

i) If the prosecutor hides exculpatory evidence, she gets the same expected payoff if the defendant is innocent as compared to the mandatory disclosure of evidence case. This is because the prosecutor and the defendant agree in a sentence equal to dq, which is equal to the expected punishment in the mandatory case. However, the prosecutor gets a lower payoff compared to the mandatory case if the defendant is guilty. This is because in the voluntary case they either agree on a sentence h if the evidence is incriminating, or dq if the prosecutor has default evidence.

ii) If the prosecutor discloses exculpatory evidence, the prosecutor gets the same payoff from the guilty defendant in both the mandatory and voluntary case. This is because when the prior belief is high enough she offers d when she has default evidence and the defendant accepts the offer. However, she gets zero payoff if the defendant is innocent because she either discloses the exculpatory evidence, resulting in a payoff of zero, or her offer of d is rejected by the defendant, ending in a payoff of zero at trial.

2. The prosecutor investigates more under Brady Rule. In both the mandatory and voluntary disclosure cases, the prosecutor prefers not to investigate and reach an agreement at $t = 0$ for lower values of the prior belief. However, the threshold such that she prefers to investigate is lower under Brady Rule. That is, if the disclosure of evidence is mandatory, the prosecutor investigates for a larger range of θ values compared to the voluntary disclosure case.

The prosecutor decides to investigate instead of reaching an immediate agreement if that action gives her an expected payoff higher than d. This happens for lower values of θ in the Brady Rule case compared to the voluntary disclosure of evidence case because the Brady Rule case gives her a higher expected payoff as explained above.

The analysis above implies that there is a commitment effect for the prosecutor: If she could credibly commit at the beginning of the game to disclose all of her evidence, she would do it. Because she cannot commit to disclose evidence when the disclosure is voluntary, she has the incentive to hide exculpatory evidence when she gets it; however, the defendant anticipates this, impeding her ability to extract the full surplus. Therefore, she is better off with mandatory disclosure of evidence.

The comparison of the expected punishments for the defendant depends on the number of periods N. However, in the limit case $N \to \infty$, the innocent defendant is weakly better off with mandatory disclosure and the guilty defendant is worse off with mandatory disclosure.

Considering the limit case, the innocent defendant gets the same expected payoff in both disclosure cases when the prosecutor prefers to investigate. The difference is that the innocent type is worse off when there is no investigation because she gets a punishment of d. In the mandatory disclosure case, the prosecutor investigates more than the voluntary disclosure case. Therefore, the innocent defendant is weakly better off under the Brady Rule.

The guilty defendant is better off with voluntary disclosure of evidence for the range of values in which the prosecutor hides exculpatory evidence because she offers him a lower sentence when she has default evidence. He gets the same punishment in both disclosure cases when the prosecutor discloses exculpatory evidence in the voluntary case. Furthermore, he is weakly better off for low values of the prior belief when the disclosure is voluntary, because he benefits from less investigation.

Figure 1.5 graphically compares the mandatory disclosure case with the voluntary case when $N \to \infty$.

1.5.3 The Path of Agreements

In the Brady Rule case there also is a deadline effect. However, in this case it is the same for both the guilty and the innocent type. The prosecutor and the defendant reach and agreement at the first period when $\theta \leq \underline{\theta}^{BR}$, and an agreement either as soon as the prosecutor gets new evidence or at the deadline is she does not get new evidence if $\theta > \underline{\theta}^{BR}$.

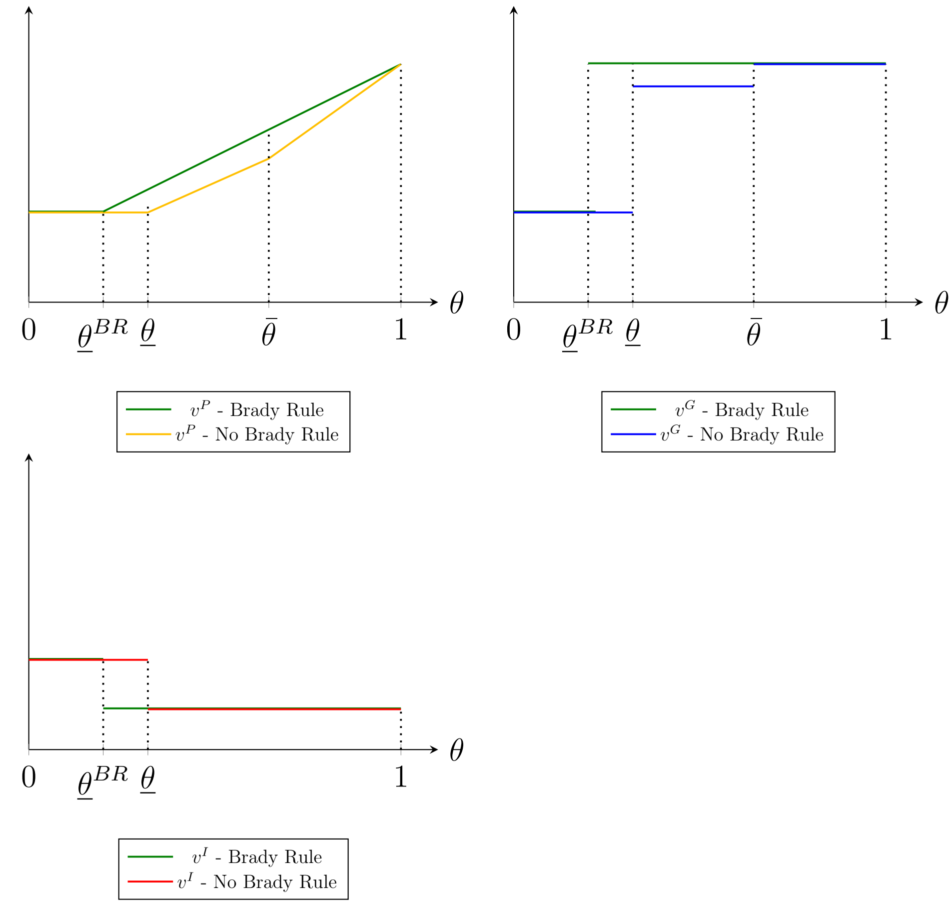

Figure 1·5: Comparison of plea bargaining with no Brady Rule and with Brady Rule.

Considering the limit-offer case, let τ^{BR} denote the time at which the prosecutor and defendant reach an agreement. The probability that the game ends by τ^{BR} when $\theta > \underline{\theta}^{BR}$ is given by

$$P\left(\tau^{BR} \leq t\right) = \begin{cases} 1 - e^{-\lambda \frac{t}{T}} & \text{if } t < T \\ 1 & \text{if } t = T \end{cases}$$

Figure 1.6 graphically shows the path of agreements when $\theta > \underline{\theta}^{BR}$. There are three main differences between the voluntary case and the Brady Rule case. First, in the voluntary case, the probability of ending the game by t is different for the innocent and the guilty types, while in the Brady Rule case is the same for both types. Second, in the voluntary disclosure case, there is a positive probability that there is no agreement at the plea bargaining phase, while under Brady Rule the prosecutor and the defendant always reach an agreement. Third, in the voluntary disclosure of evidence case, the probability of ending the game at t has a mass point at $t = 0$, while under Brady Rule that mass point does not exists.

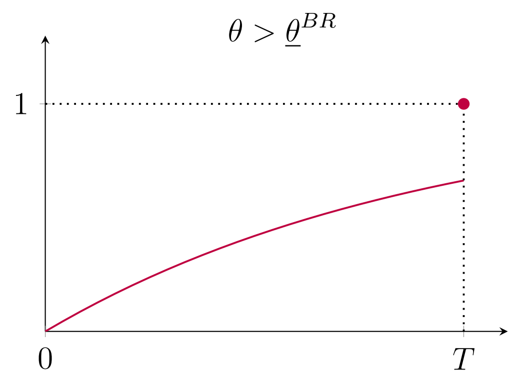

Figure 1·6: Probability of ending the game by time $t \leq T$ for $\theta > \underline{\theta}^{BR}$, for both defendant's type. Values: $\lambda = 0.8$, $T = 30$, $q = 0.55$.

1.6 Concluding Remarks

With voluntary disclosure of evidence in plea bargaining, in equilibrium the prosecutor hides exculpatory evidence when the prior belief about the defendant being guilty is low. However, she discloses the exculpatory evidence when the prior belief about the defendant being guilty is high enough. This means that a prosecutor who is purely sentence-motivated will still disclose exculpatory evidence depending on the prior belief.

Nevertheless, even though there is disclosure of exculpatory evidence when disclosure is voluntary during the plea bargaining phase, the mandatory disclosure protocol during plea bargaining is, from a normative point of view, socially desirable for two reasons: It is efficient in the sense that prosecutor and defendant always reach an agreement before trial, and because the defendant gets a higher sentence if he is guilty and a lower sentence is he is innocent. Finally, I showed that the prosecutor prefers the mandatory disclosure case.

Chapter 2

Venture Capital Contracts under Disagreement

2.1 Introduction

Differences in opinion about the right course of action are very common in early-stage projects, such as those financed by venture capital. A venture capitalist who is planning to invest in a project owned by an entrepreneur does not necessarily agree with the decisions that the entrepreneur considers optimal in the implementation of the project. Therefore, she might condition investment of capital on the provision of control rights of the venture, which gives her the authority to make the decisions.

Conversely, the entrepreneur might not want to relinquish control rights to the venture capitalist because he considers that the decisions that the venture capitalist would make are not optimal for the success of the venture. This disagreement captures the idea of different intuitions or ideas in the context of high uncertainty. I discuss this assumption in Section 2.1.1.

In this chapter I study how the allocation of control rights and cash-flow rights varies depending on which party has the greater bargaining power at the moment when they negotiate the investment contract. The control rights are valuable because decisions are not contractible, and both parties openly disagree about them. The bargaining power of each party is determined by how competitive the venture capital sector is. If it is very competitive, the entrepreneur has the greater bargaining power, and therefore he offers the contract; if it is not competitive, the venture capitalist has the greater bargaining power and she offers the contract.

As the main contribution of this chapter, I show that when the entrepreneur has the greater bargaining power he voluntarily relinquishes control rights when the required investment by the venture capitalist is high enough. When the venture capitalist has the greater bargaining power, she does not relinquish control rights. However, under some conditions the venture capitalist retains the control rights but not the real authority.footnoteIn the sense of Aghion and Tirole (1997): the effective control over decisions.

The difference between control rights and real authority is that control rights allow a player to make decisions, but the real authority allows a player to make her own preferred decision. When the venture capitalist retains control rights but losses the real authority, she is the player that makes the decisions, but she makes the decisions that the entrepreneur considers optimal.

I also show that control and cash flow rights are substitutes when the entrepreneur offers the contract, and complements when the venture capitalist offer the contract.

These results are the consequence of both parties not being symmetric. Besides having the bargaining power as described above, both parties have some informal negotiation power. The venture capitalist provides the investment and consequently she can condition the contract to get a return that compensates her investment. In addition, the entrepreneur provides effort to the project after all the important decisions have been made. This effort decision gives him the possibility of sabotaging the project if he believes it is not worth it in its final form.

I model the non-contractible decision as the *direction of the project*, which represents the strategic decisions that must be made; for example, deciding the target market of the project or which executive members to hire. I consider that there are two possible directions of the project, and each player assigns the highest probability of success to different directions.

The game starts when the party with greater bargaining power offers the investment contract. If it is accepted, the party that has control rights decides the direction of the project. After the direction is chosen, the entrepreneur decides whether to exert costly effort to increase the probability of the project's success. Finally, the outcome of the project is realized, and each party receives what is specified by the cash-flow rights.

The entrepreneur relinquishes control rights when the required investment of the project is high enough. The reason is that the entrepreneur must compensate the venture capitalist for the investment, which can be done in one of two ways: By retaining the control rights and compensating her with a large fraction of cash-flow rights, or by relinquishing the control rights and compensating her with a lower fraction of cash-flow rights.

The intuition is as follows: If the entrepreneur retains the control rights he needs to allocate enough cash-flow rights to the venture capitalist to compensate her for the investment. Considering that the player who does not have the control rights observes a lower expected outcome of the project, the entrepreneur needs to allocate most of the cash-flow rights to the venture capitalist when the required investment is high. In this case the entrepreneur retains a low fraction of the cash-flow rights, which translates to a small fraction of the expected outcome. If the entrepreneur relinquishes the control rights, he needs to allocate a lower amount of cash-flow rights to the venture capitalist, because she observes a higher excepted outcome. Therefore, even though he observes a smaller expected outcome, he is better off by retaining a larger percentage of that expected outcome when the required investment is high.

The entrepreneur's decision to allocate control rights to the venture capitalist is voluntary. That is, although it is feasible for the entrepreneur to keep the control rights and provide enough cash-flow rights to the venture capitalist to induce her to accept the contract, the entrepreneur chooses to relinquish them because he prefers a larger fraction of cash-flow rights.[1]

In this chapter I call *disagreement* to the prior divergence between the players.[2] I show that the entrepreneur is less likely to retain control with more disagreement. However, comparing two levels of disagreement that induce the same allocation of control, the entrepreneur is better off with more disagreement in two extreme cases: when the required return to the venture capitalist is very low or very high.

When the venture capitalist has the bargaining power, she always offers a contract in which she retains control and the entrepreneur accepts the contract. This is because the outside option of the entrepreneur is normalized to zero, and the entrepreneur can always ensure himself a payoff of zero by not making effort. When the project is very unlikely to succeed if the entrepreneur does not exert effort, and the cost of the effort is neither too high nor too low, the venture capitalist retains the control rights, but she chooses the entrepreneur's desired direction. In the sense of Aghion and Tirole (1997), in this case the venture capitalist has the formal authority (the right to make decisions), but the entrepreneur has the real authority (the effective control over decisions).

[1] The importance of voluntarily relinquishing of the control was pointed out by Hellmann (1998).

[2] Considering a direction, the disagreement is the difference between the probability of success that each player assigns to that direction.

2.1.1 Differing Priors Assumption

In the literature there are two ways to justify why a party involved in a joint venture values the control rights. The first one is because that party draws a private and non-transferable benefit from having the control rights. The assumption is that the party in control might have a private agenda, or can have a better career after being in charge of a venture. The second way to justify why each party wants to have the control of the venture is the differing priors assumption (open disagreement). Although the private benefit assumption is the mainstream way to model the valuation for control, I use the differing priors assumption because in the contract between a venture capitalist and an entrepreneur, that approach is more suitable.

1. VC and entrepreneurs might have different "visions". The differing priors assumption captures the idea that different people have different intuitions about the future. An example from Van den Steen (2010a) to support that argument is *"Product design and RD investment decisions for cell phones depend critically what you believe people will be using cell phones for 5 years down the road."* The uncertainty about the future leads to different opinions and visions, which are even more relevant in an early-stage investment project, such as those that are financed by venture capital.

As Van den Steen (2010b) shows, shared beliefs in an organization originate through screening, self-sorting, and manager-directed joint learning. That is, shared beliefs are generated by a relationship built over time. In my model, the venture capitalist and the entrepreneur are both creating a new relationship, and therefore there is no possibility of having a shared belief yet.

Also, although non-mainstream, Van den Steen (2010c) provides a theory of the firm that leads to new perspectives on the firm as a legal entity and on the relationship between the Knightian and Coasian views of the firm. He uses the differing priors assumption to show that the firm is a mechanism to give to the manager interpersonal authority over employees.

2. It allows an easy way to understand the allocation of control depending on who has the greater bargaining power. One of the contributions of this chapter is to explain how ex ante bargaining power affects the allocation of control rights. A model with differing priors helps answer that question in a simple way. When the entrepreneur offers the contract, the tradeoff that he faces is to have a smaller fraction of a larger pie or a larger fraction of a smaller pie. When the venture capitalist offers the contract, the only tradeoff is how to incentivize the entrepreneur to make effort.

The optimal contract that each party offers and the result regarding the voluntary relinquishment of control rights by the entrepreneur are both driven by the differing priors assumption.

3. More intuitive empirical perspective and predictions. First, with the differing priors assumption, the same income is divided by both players. This provides an easier empirical perspective compared to the private benefit assumption, where it is difficult to measure the private benefit.

The downside is that the disagreement is also difficult to measure. However, some papers use a proxy for disagreement (in a broader sense), for example Cumming and Johan (2007) state: *"Our measure for conflict is based on the number of different types of disagreements with the entrepreneur. We asked VCs to report whether they had disagreements with the entrepreneurial firm concerning different matters including strategy, marketing, financial matters, RD, human resources, and product development. Our proxy for conflict is the sum of these potential areas of disagreement; that is, it is a measure of the scope of disagreement."* Ewens et al. (2019) explores, among other things, the role of the bargaining power between the venture capitalist and the entrepreneur.

Three of the predictions of the chapter are a result of the differing priors assumption: 1) Control rights and cash flow rights are substitutes when the entrepreneur has the bargaining power. 2) More disagreement between the venture capitalist and entrepreneur gives rise to a greater allocation of control rights to the venture capitalist. And 3), although the entrepreneur has the bargaining power, he voluntarily relinquishes control rights.

2.1.2 Related Literature

This chapter relates to the literature of early financing with joint allocation of control rights and cash-flow rights. Townsend (1979), Diamond (1984) and Gale and Hellwig (1985) use a costly state-verification model, in which control rights are understood to be the right to intervene in low payment states. In the present chapter, control rights are related to decisions before the outcome is realized.

Dewatripont and Tirole (1994) present a model in which control rights and cash-flow rights are interrelated. In the present chapter, I consider the separation between control and cash-flow rights. Kaplan and Strömberg (2004) show that financial contracts in which venture capitalists are the financers separate control and cash-flow rights.

Hellmann (1998) presents a model in which the control is the right to appoint a new CEO. In his model, the entrepreneur starts as a CEO, which gives him a private benefit, and the venture capitalist can engage in the costly search for a more productive CEO. In this context, the venture capitalist needs control rights to decide to engage in the costly search. This model shows that control rights are relinquished by the entrepreneur not only when that is the only option to obtain the investment, also when retaining them is feasible. In the present chapter I do not consider private benefit for control, but rather a difference in opinion regarding the optimal decision. I get a similar result on voluntary relinquishing of control, but in this case it is because it is less expensive for the entrepreneur to provide control than cash-flow rights.

The model in Berglof (1994); Marx (1998); Bascha and Walz (2001); Kirilenko (2001); Hellmann (2002); Schmidt (2003); Aghion et al. (2004); Dessein (2005); and Dessi (2005) assume private benefit for having control. In the present chapter the benefit is not private; rather, control is desirable because of the difference of opinion. The models in Bergemann and Hege (1998); Casamatta (2003); Repullo and Suarez (2004); Hellmann (2006); and Cestone (2014) consider some type of moral hazard, and therefore control is valuable to reduce it. Although in my model there is non-contractible effort, the focus is not on the incentive of having control to monitor it, but rather in deciding what action to take in order to incentivize the observable effort provision.

A closely related paper is by Van den Steen (2010a) in which he presents a model of allocation of control rights and cash-flow rights in the presence of disagreement. He assumes Nash bargaining for the allocation of control rights, and does not consider an investment. He shows that cash-flow rights should be allocated to the party in control. The present chapter differs from Van den Steen's in how they negotiate control rights. I consider that each party has the bargaining power at the moment of offering the contract, and my model also induces the difference between having control rights and having real authority, as defined by Aghion and Tirole (1997).

Other papers that introduce disagreement in venture capitalist contracts are by Huang and Thakor (2013) and Jung and Subramanian (2014). In the former, the authors present a model in which an investor and the management can disagree about investment projects in order to propose a theoretical explanation for stock repurchases. The difference with my model is that they focus on how repurchasing varies with disagreement, while I focus on how the allocation of control varies with disagreement. Jung and Subramanian (2014) present a model of disagreement that is understood as different priors about the profitability of a project to show that asymmetric beliefs significantly influence a firm's financial policies. This differs from the present chapter in that I focus on control allocation and not on the optimal financial policy.

Outline: I introduce the model in Section 2.2. I show the allocation of control rights when the entrepreneur has the bargaining power in Section 2.3, and when the venture capitalist has the bargaining power in Section 2.4. Section 2.6 concludes. I present the case in which the control is a continuous variable in Appendix A.2.2. All proofs are in Appendix A.3.2.

2.2 Model

There are two player, an entrepreneur (E) and a venture capitalist (VC). The entrepreneur has a project but needs investment to implement it. The venture capitalist provides the investment. The project is incomplete and they anticipate they will have to make a decision about the direction of the project once the investment contract is signed. This decision is not ex ante contractible and corresponds to the direction $x \in \{L, R\}$. The project yields an output $y = 1$ in case of success and $y = 0$ in case of failure.

After the direction is chosen by the party that has the control rights, the entrepreneur decides whether to exert effort. Formally, the entrepreneur chooses $e \in \{0, 1\}$ with a cost c per unit. The effort is relevant for the outcome; if the entrepreneur does not exerts effort, the probability of success of the project is lower according to each party. Define by $p_i(x)$ the probability of success of the project according to player $i \in \{E.VC\}$ when the direction is $x \in \{L, R\}$. If the entrepreneur does not exert effort, $p_i(x)$ is multiplied by a parameter $\gamma \in [0, 1]$. Formally, the probability of success $P_i(e, x)$ after the effort decision is:

$$P_i(e,x) = \begin{cases} p_i(x) & \text{if } e = 1 \\ \gamma p_i(x) & \text{if } e = 0. \end{cases}$$

Intuitively, the parameter γ represents how important effort is for the outcome. This gives some ex post bargaining power to the entrepreneur; he can sabotage the project if the benefit of exerting effort does not compensate the cost.

The entrepreneur and the venture capitalist openly disagree about the decision with the highest probability of success. The entrepreneur's preferred direction in the sense of probability of success is $x = R$, and the venture capitalist's preferred direction is $x = L$. The probabilities of success according to the entrepreneur are

$$p_E(x) = \begin{cases} \overline{p} & \text{if } x = R \\ \underline{p} & \text{if } x = L, \end{cases}$$

and the probabilities of success according to the venture capitalist are

$$p_{VC}(x) = \begin{cases} \overline{p} & \text{if } x = L \\ \underline{p} & \text{if } x = R, \end{cases}$$

where $\bar{p} > \underline{p}$.

The legal right to choose the direction is given by the control power. Control rights are denoted by $\alpha \in \{0, 1\}$. If $\alpha = 1$ the entrepreneur decides the direction x, and if $\alpha = 0$ the venture capitalist decides it.

The revenue each party receives is a fraction of the outcome, represented by the cash-flow rights. The entrepreneur gets cash-flow rights $\beta \in [0, 1]$, and the venture capitalist gets $1 - \beta$. A contract consists of a set of rights (α, β) for the entrepreneur and $(1 - \alpha, 1 - \beta)$ for the venture capitalist, in exchange for investment $M > 0$. Both parties have an outside option normalized to 0, but the venture capitalist needs to be compensated by the investment.

The parties negotiate the contract through a take-it-or-leave-it offer the party with the bargaining power makes to the other party. If the contract is rejected, the game ends and both parties get a payoff equal to zero. I analyze both cases: when the entrepreneur has the bargaining power and when venture capitalist has it.

The timeline is the following:

1. One of the parties offers a contract $\{(\alpha, \beta), (1-\alpha, 1-\beta)\}$ to the other party.

2. If the contract is accepted, the party with the control rights chooses the direction $x \in \{L, R\}$.

3. The entrepreneur chooses $e \in \{0, 1\}$.

4. The output $y = \{0, 1\}$ is realized. The entrepreneur receives a fraction β and the venture capitalist a fraction $(1-\beta)$.

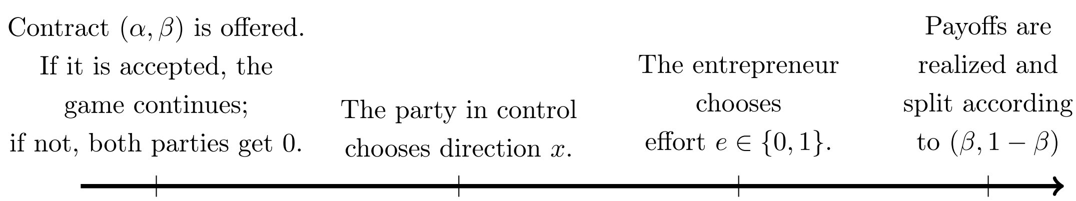

Figure 2·1: Timeline

Assumption 2: I restrict attention to $M \leq \gamma \underline{p}$. That is, I consider cases in which the entrepreneur is not forced to provide control rights to the venture capitalist as the only way to ensure the investment. If the entrepreneur allocates the controls to the venture capitalist it is a voluntary decision.[3]

[3]For values $M > \gamma \underline{p}$ the same results hold regarding the decision of relinquish control rights, however the voluntary nature of the decision does not always hold. In cases of high cost of the effort, the entrepreneur is forced to relinquish control rights to compensate the venture capitalist for the investment.

2.3 The Entrepreneur has the Bargaining Power

In this section I present the contract the entrepreneur offers when he has the bargaining power. The contract corresponds to the allocation of control rights and cash-flow rights that maximizes the entrepreneur's payoff considering the subgame perfect equilibrium the contract induces. The intuition behind the entrepreneur's having the bargaining power is to represent a competitive venture capitalist market.[4]

Note that in this game, both parties have some degree of informal bargaining power. The entrepreneur can sabotage the project if the benefit of exerting effort does not compensate for the cost of it, and the venture capitalist will refuse to participate if the expected payoff she receives is lower than M. These two frictions are considered by the entrepreneur when offering the contract.

There are three qualitatively different options. 1) The entrepreneur keeps control so that he can choose his desired direction. This reduces the venture capitalist's expected payoff, making it necessary to assign more cash-flow rights to the venture capitalist to increase her expected payoff to at least M. 2) The entrepreneur relinquishes control rights in such a way that the venture capitalist prefers to choose the entrepreneur's desired direction. By doing this, the entrepreneur looses the formal control, but keeps the real authority. This contract also requires more cash-flow rights for venture capitalist. 3) The entrepreneur relinquishes control rights and the venture capitalist chooses her desired direction. This increases the venture capitalist's expected payoff, allowing the entrepreneur to keep a larger portion of the control rights.

The tradeoff the entrepreneur faces is to retain control, either formal or real, at the cost of a smaller share of the cash-flow rights, or relinquish control to be able to obtain a larger share of the cash-flow rights.

[4]Most of the literature focuses on this case.

I analyze the subgame perfect equilibrium. I start by describing the decision each player makes at the moment of choosing the direction if they are allowed to do so. The cash-flow rights are parameters of the problem at the moment the party with control rights chooses the direction. At this stage of the game, the only instrument to incentivize the entrepreneur to make effort $e = 1$ is the direction x.

The entrepreneur is going to make effort $e = 1$ if

$$\beta p^E(x) - c \geq \beta \gamma p^E(x) \quad \Leftrightarrow \quad \beta \geq \frac{c}{p^E(x)(1-\gamma)}.$$

This implies that the entrepreneur chooses $e = 1$ if

- $\beta \geq \frac{c}{\underline{p}(1-\gamma)}$ and $x = L$ or $x = R$, or
- $\beta \in \left[\frac{c}{\bar{p}(1-\gamma)}, \frac{c}{\underline{p}(1-\gamma)}\right)$ and $x = R$.

When $\beta < \frac{c}{\bar{p}(1-\gamma)}$, the entrepreneur always chooses $e = 0$.

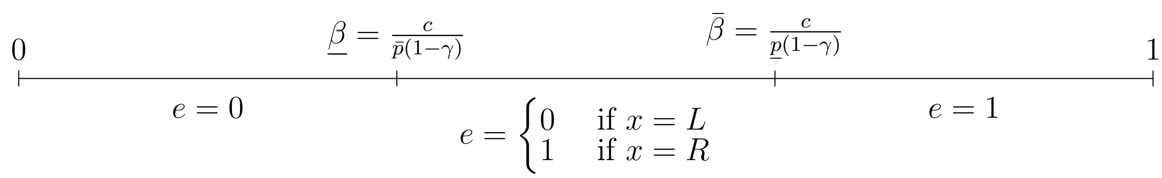

Figure 2·2: Effort made by the entrepreneur depending on β

The entrepreneur is always better off choosing his desired direction $x = R$. The optimal decision for the venture capitalist depends on what effort decision the entrepreneur is going to make. If $\beta \leq \underline{\beta}$, the entrepreneur is never going to make effort regardless of the direction; therefore, the venture capitalist's optimal direction is her desired one, $x = L$. If $\beta > \bar{\beta}$, the entrepreneur is going to choose $e = 1$ regardless of the direction; therefore, the venture capitalist chooses $x = L$. If $\beta \in [\underline{\beta}, \bar{\beta}]$, the entrepreneur will exert effort only if $x = R$. Figure 2.2 depicts the effort decision by the entrepreneur depending on the value of β and on the direction for the case $\beta \in [\underline{\beta}, \bar{\beta}]$.

For the case $\beta \in [\underline{\beta}, \bar{\beta}]$, the direction the venture capitalist chooses will be the one that provides her with a larger expected payoff. The venture capitalist will choose $x = L$ if

$$(1-\beta)\gamma\bar{p} \geq (1-\beta)\underline{p} \quad \Leftrightarrow \quad \gamma \geq \frac{\underline{p}}{\bar{p}} \equiv \bar{\gamma}$$

and $x = R$ if $\gamma < \bar{\gamma}$.

Considering $\beta \in [\underline{\beta}, \bar{\beta}]$, the threshold $\bar{\gamma}$ represents the value that causes the entrepreneur to have real authority, even though he does not have control rights.

If γ is lower than $\bar{\gamma}$, the surplus is reduced by a significant amount if the entrepreneur does not make effort. The entrepreneur prefers not to make effort if the share of the expected surplus is lower than the cost of the effort. In this case he prefers a low amount of surplus over a larger one with an even higher cost. The venture capitalist prefers to choose the entrepreneur's desired direction to increase his surplus and incentivize him to make effort $e = 1$.

If $\gamma \geq \underline{\gamma}$, the effect of the reduction of surplus for the venture capitalist when the entrepreneur does not make effort is lower than the effect of the surplus reduction of the entrepreneur's making effort $e = 1$, but choosing a non-desired direction.

At the moment the entrepreneur offers the contract to the venture capitalist, he considers how β is going to affect his own decisions and the venture capitalist's decision. As described above, there are two possible cases depending on γ:

1. If $\gamma \geq \bar{\gamma}$, the venture capitalist will choose $x = L$ if $\beta \in [\underline{\beta}, \bar{\beta})$.

2. If $\gamma < \bar{\gamma}$, the venture capitalist will choose $x = R$ if she has the control rights and $\beta \in [\underline{\beta}, \bar{\beta})$; this means that both players will choose the same direction.

The functional form of the expected payoffs $U^E(\alpha, \beta, \gamma)$ and $U^{VC}(\alpha, \beta, \gamma)$ can be found in Appendix A.2.1.

When the entrepreneur offers a contract to the venture capitalist, the problem is constrained by offering an expected payoff of at least M. The maximization problem the entrepreneur solves is

$$\begin{aligned} &\max_{\alpha \in \{0,1\}, \beta \in [0,1]} U^E(\alpha, \beta, \gamma) \\ &\text{subject to: } U^{VC}(\alpha, \beta, \gamma) \geq M. \end{aligned} \tag{2.1}$$

The solution of problem (2.1) is the optimal contract the entrepreneur offers to the venture capitalist and the venture capitalist accepts. This solution depends on the parameters of the problem. I group the solution depending on the three relevant parameters for the analysis: γ, M, and c. Proposition 7 is divided into two qualitatively different cases: when $\gamma \geq \bar{\gamma}$ and $\gamma < \bar{\gamma}$.

Proposition 7 *There are three thresholds: $\bar{M}$, $\bar{c}$ and $\underline{c}$.*

1. For $\gamma \geq \bar{\gamma}$. The entrepreneur retains the control rights in the contract if $M \leq \bar{M}$ and relinquishes them otherwise.

2. For $\gamma < \bar{\gamma}$. The entrepreneur retains the control in the contract if $M \leq \bar{M}$. The entrepreneur is indifferent between keeping the control rights or relinquish them if $c \in (\underline{c}, \bar{c}]$ and $M > \bar{M}$. The entrepreneur relinquishes the control rights otherwise.

The contract described in Proposition 7 induces $e = 1$ if $c \leq \bar{c}$. Figure 2.3 presents a graphical interpretation of the results. A different type of contract applies to each area of c and M. The x-axis represents the value of c, from zero to infinity, and the y-axis represents the value of M from zero to $\underline{p}$. The left panel represents the case in which $\gamma \geq \bar{\gamma}$ and the right panel the case in which $\gamma < \bar{\gamma}$.

Intuitively, when the cost of exerting effort c is large enough, it is not optimal for the entrepreneur to make effort; this result is robust to who holds the control rights.

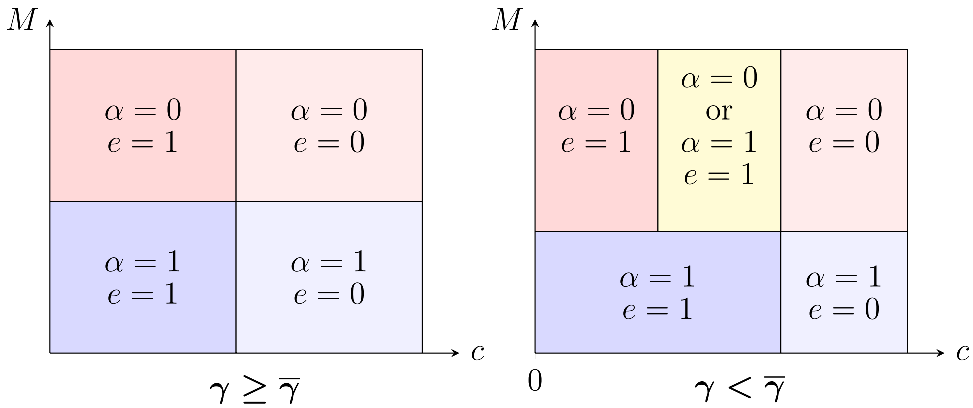

Figure 2·3: Graphical representation of the optimal contract depending on M and c when the entrepreneur offers it. Left: $\gamma \geq \bar{\gamma}$; right: $\gamma < \bar{\gamma}$

.

If $\gamma \geq \bar{\gamma}$, when M is large enough the entrepreneur relinquishes control rights to the venture capitalist; otherwise, he keeps it.

If $\gamma < \bar{\gamma}$, for some values of high enough M, but medium values of c, the entrepreneur is indifferent between retaining control or giving it to the venture capitalist, but chooses cash-flow rights that incentivize the venture capitalist to choose the entrepreneur's desired direction. If M and c are high enough, there is no feasible contract.

The entrepreneur voluntarily relinquishes control rights, because keeping the control an providing more cash-flows rights to the venture capitalist is a feasible option. As Hellmann (1998) points out, the solution for allocation of control rights is trivial if relinquishing them is the only option to obtain a feasible contract. This chapter is not affected by Hellman's comment, because there is a feasible solution whereby the entrepreneur can retain control rights, but he voluntarily relinquishes them.

2.3.1 Effect of Disagreement on the Optimal Contract

To analyze how the allocation of control rights varies with disagreement, I first define the concept of *disagreement* and a *change in disagreement* in a tractable manner.

Definition 2.3.1 *Disagreement is the difference in beliefs about the success probability for each direction:* $\Delta = \bar{p} - \underline{p}$.
An increase (decrease) in Δ *refers to a mean-preserving spread increment (reduction): if* Δ *increases (decreases), then* $\bar{p}$ *increases (decreases) and* $\underline{p}$ *decreases (increases) by the same amount keeping* $\bar{p} + \underline{p}$ *constant.*

Using the above definition for an increment in disagreement, the following proposition shows how the optimal contract varies with the disagreement.

Proposition 8 *If* Δ *increases, the values of* $\bar{M}$, $\bar{c}$ *and* $\underline{c}$ *decrease.*

Proposition 8 says that the entrepreneur is less likely to retain control when there is more disagreement. This is because a higher Δ implies a lower $\underline{p}$, and the share of the outcome the venture capitalist receives depends on $\underline{p}$. Therefore, if $\underline{p}$ is lower, the entrepreneur must provide more cash-flow rights to the venture capitalist to provide an expected payoff of M.

I complement the analysis of how the disagreement affects the optimal contract by showing how the entrepreneur's expected payoff changes with disagreement. I consider only a comparison between qualitatively equal solutions—i.e., changes in disagreement that do not change the optimal decisions of α and e. Proposition 9 states that the answer to who benefits with more disagreement depends on the parameters of the problem.

Proposition 9 *There are thresholds:* $\tilde{M}$ *and* $\underset{\sim}{M}$, *such that* $\tilde{M} > \bar{M} > \underset{\sim}{M}$. *If* Δ *increases:*

1. If the entrepreneur strictly prefers to retain the control rights in the optimal contract, the entrepreneur's payoff increases if $M < \underset{\sim}{M}$ and decreases otherwise.

2. If the entrepreneur strictly prefers to relinquish control rights in the optimal contract, the entrepreneur's payoff increases if $M > \tilde{M}$ and decreases otherwise.

3. If the entrepreneur is indifferent between retaining and relinquishing control rights, the entrepreneur's payoff decreases.

The intuition of Proposition 9 is that when the entrepreneur offers the contract, if c and M are low enough, the entrepreneur retains control $\alpha = 1$ and chooses effort $e = 1$. He is giving a fraction $(1-\beta)$ of $\underline{p}$ to the venture capitalist, while he is keeping a fraction β of $\bar{p}$. If Δ increases, this means $\underline{p}$ decreases; therefore, the entrepreneur needs to increase $(1-\beta)$ to cover the investment cost M. Hence, he gets a lower fraction β of a higher payoff $\bar{p}$. If M is high enough, the entrepreneur's loss for giving up a higher fraction is higher that the benefit of having a higher surplus to split.

If c is high and M is low, the entrepreneur retains control but does not make effort. This case is similar to the one discussed above, except that the surplus is multiplied by γ and the entrepreneur has a cost c that is not affected by Δ. In this case, the entrepreneur is also better off if M is low.

If the entrepreneur is indifferent between relinquishing and keeping the control rights, the analysis is the same as when $\alpha = 1$, with the difference that this contract is optimal when M is higher. In this case, entrepreneur is worse off with an increase of Δ.

Finally, if c and M are high enough, the entrepreneur relinquishes control and does not make effort. The venture capitalist receives a fraction $(1-\beta)$ of $\gamma\bar{p}$; this implies that $(1-\beta)$ decreases if Δ increases. On the other hand, the entrepreneur gets a higher fraction β of a lower payoff $\gamma\underline{p}$. The entrepreneur's gain of a higher fraction is higher than the loss of a lower payoff if M is high. Therefore, if $(\alpha = 0, e = 0)$ is the optimal contract, the entrepreneur is better off when M is high.

2.4 The Venture Capitalist has the Bargaining Power

In this section I present the contract the venture capitalist offers to the entrepreneur when she has bargaining power. The same analysis in the previous section regarding the direction each party chooses and the effort the entrepreneur makes applies to this section.

The entrepreneur accepts any contract with $\beta \in [0,1]$. This is because the entrepreneur's outside option is zero and he can always choose effort $e = 0$ afterward. The problem the venture capitalist solves does not require the individual rationality constraint. The venture capitalist solves the following problem:

$$\max_{\alpha \in \{0,1\}, \beta \in [0,1]} U^{VC}(\alpha, \beta, \gamma). \tag{2.2}$$

The solution of problem (2.2) is stated in Proposition 10. As in the previous case, the solution depends on γ, M, and c.

Define:

$$\underline{\gamma} = \frac{\underline{p}}{\bar{p} + \underline{p}}.$$

Proposition 10 *There are thresholds: $\bar{c}^{VC}$ and $\underline{c}^{VC}$.*

1. For $\gamma \geq \underline{\gamma}$. The venture capitalist retains the control rights.

2. For $\gamma < \underline{\gamma}$. The venture capitalist retains the control right if $c \leq \underline{c}^{VC}$ or $c \geq \bar{c}^{VC}$. For $c \in (\underline{c}^{VC}, \bar{c}^{VC})$, the venture capitalist is indifferent between relinquishing or retaining the control rights.

The contract described in Proposition 10 induces $e = 1$ if $c \leq \bar{c}^{VC}$. A graphical representation of the proposition can be found in Figure 2.4. The result shows that when $\gamma \geq \underline{\gamma}$, the solution only depends on c; the venture capitalist offers $\beta = 0$ to the entrepreneur if c is high enough and offers $\beta = \bar{\beta}$ if c is low enough. Note that $\beta = \bar{\beta}$ is the lowest amount of cash-flow rights that incentivizes the entrepreneur to make effort $e = 1$.

In the case of $\gamma < \underline{\gamma}$, when M is low enough the venture capitalist retains control and offers $\beta = \bar{\beta}$ such that the entrepreneur makes effort $e = 1$ if c is low. If c is high enough she retains control and offers $\beta = 0$. Finally, if c has a medium value, the venture capitalist offers $\beta = \underline{\beta}$ and is indifferent between retaining control and choosing the entrepreneur's desired option to incentivize him to make the effort, or relinquish control.

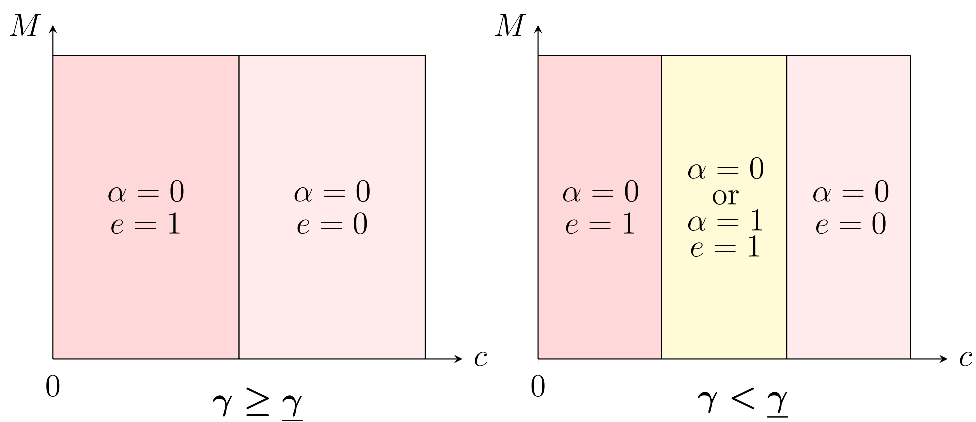

Figure 2·4: Graphical representation of optimal contract depending on M and c when the venture capitalist offers the contract. Left: $\gamma \geq \underline{\gamma}$, right: $\gamma < \underline{\gamma}$

.

The most important feature of the contract that the venture capitalist offer is that she always retains control or is indifferent about it. This difference with the case in which the entrepreneur offers the contract is the absence of an individual rationality constraint for the entrepreneur. The venture capitalist does not need to ensure a minimum payoff for the entrepreneur to accept the contract; therefore, she always offers him the minimum amount that incentivizes him to make effort. Proposition 10 shows that it is always optimal to retain control and offer more cash-flow rights than to offer control and lower cash-flow rights.

2.4.1 Effect of Disagreement on the Optimal Contract

When the venture capitalist offers the contract, in most cases she chooses her desired direction. Therefore, according to the entrepreneur, the expected outcome depends on $\underline{p}$ and the disagreement negatively impacts him.

Proposition 11 *If Δ increases, the values of $\bar{c}^{VC}$ and $\underline{c}^{VC}$ decrease.*

Proposition 11 shows that if Δ increases, the threshold for which the venture capitalist incentivizes $e = 1$ is lower because with a higher Δ, the venture capitalist must give more cash-flow rights to the entrepreneur to compensate for the cost of effort.

To complete the analysis, I compare the payoffs between qualitatively equal solutions when there is an increment in disagreement.

Proposition 12 *There is a thresholds: $\underset{\sim}{c}^{VC}$, such that $\underline{c}^{VC} > \underset{\sim}{c}^{VC}$. If Δ increases:*

1. If the venture capitalist retains the control rights and induces $e = 1$, the venture capitalist's payoff increases if $c < \underset{\sim}{c}^{VC}$ and decreases otherwise.

2. If the venture capitalist retains the control rights and induces $e = 0$, the venture capitalist's payoff increases.

3. If the venture capitalist in indifferent between retaining and relinquishing the control rights, the entrepreneur's payoff decreases.

The intuition of Proposition 12 is that if the contract induces the entrepreneur to not make effort, the entrepreneur gets zero cash-flow rights. The venture capitalist gets $\gamma\bar{p}$, and therefore the venture capitalist benefits if there is more disagreement. In contrast, if she strictly prefers to keep control $\alpha = 0$ and prefers that the entrepreneur makes effort $e = 1$, she offers him $\beta = \bar{\beta}$. Given that β increases if the disagreement increases, lower values of β are better for the venture capitalist. The venture capitalist gets a lower fraction $(1 - \bar{\beta})$ of a higher $\bar{p}$. The gain is higher than the loss if $\bar{\beta}$ is not too high, in which occurs when c is low enough.

If the optimal contract is $(\alpha \in \{0, 1\}, \beta = \underline{\beta}, e = 1)$, the venture capitalist increases her payoff because $(1 - \underline{\beta})$ increases, but also decreases it because $\underline{p}$ decreases. The net effect is an increment in the payoff if $\underline{\beta}$ is not too low.

2.5 Concluding Remarks

This chapter presents a model of the allocation of control rights depending on which party has the bargaining power at the beginning of the game. Both parties openly disagree about the noncontractible optimal decision to make after the contract is signed. The model is able to explain the allocation of control without assuming non-transferable exogenous utility for having it. The present chapter shows that when the entrepreneur has the bargaining power, under certain conditions he relinquishes control rights to the venture capitalist even though by not providing control rights and allocating more cash-flows rights the entrepreneur can satisfy the venture capitalist's individual rationality constraint.

In the case in which the venture capitalist has the bargaining power, she always retains the control rights. The reason for the difference in behavior is given by the individual rationality constraint: The entrepreneur does not need a positive expected return to sign the contract, while the venture capitalist must cover the investment cost. In this case, under some conditions, the entrepreneur has the real authority because the venture capitalist prefers to choose the entrepreneur's desired direction to incentivizes him to make effort, instead of providing him with more cash-flow rights. This result is robust to continuous control rights as shown in Appendix A.2.2.

Finally, considering the allocation of control rights, it is not obvious whether the entrepreneur would have preferred more or less disagreement before signing the contract. If he retains the control, he prefers more disagreement if the investment cost is not too high, and the opposite if he gives up the control. If the venture capitalist is the one offering the contract, she prefers more disagreement if the optimal contract does not induce the entrepreneur to make effort, while she is better off with less disagreement if the contract induces him to make effort.

Chapter 3

Bargaining over an Endogenous Surplus

3.1 Introduction

In many negotiations, the bargaining power of each participant and the surplus that they are trying to divide depends on the actions of the same participants involved in the negotiation. The size of the surplus depends on the individual contribution that each participant makes over time, and the bargaining power on actions related to improving the position in the negotiation.

For example, when legislators discuss a bill, they discuss it and make contributions to it during many sessions before it becomes a law. At the same time, legislators might spend resources on lobbying to gain a larger fraction of the benefits the new law will provide.[1] A factory that negotiates with a supplier to get a customized machine can continue making adjustments to the production line that involves the new machine throughout the negotiation. At the same time, both parties might get the assistance of lawyers to get better conditions in the agreement. Finally, in a business dispute between two partners, both parties can continue growing the firm while negotiating

[1] See Baron and Ferejohn (1987) and Yildirim (2007) for bargaining in legislatures and lobbying respectively.

a settlement. As Yildirim (2007) remarks, they might invest time and resources in getting a better position that does not affect the productivity of the firm.

I study how advantages and characteristics of the participants in the negotiation affect outcomes. I use a dynamic bargaining model in discrete time in which two players simultaneously make two types of efforts: productive effort that increases the value of the surplus, and unproductive effort that increases their probability of being recognized as the proposer. The size of the surplus is increasing in the accumulated amount of efforts of all previous periods. The probability of being recognized as the proposer is higher for the player who makes more unproductive efforts. The player who is elected proposer offers a division of the surplus to the other player. If the offer is accepted, the game ends and each player receives the agreed division. If the offer is rejected, the game moves to a new period in which they can again make both types of efforts.

An important element of the above examples is that the agreements are not reached immediately. Political discussions regarding a bill, negotiations about a product, and agreements in business disputes take time. To account for this, I consider that the cost of the productive effort is convex in each period. This generates the game to be divided into two phases: the contribution and the agreement. In the contribution phase, the players contribute to increase the surplus without reaching an agreement to end the game. In the second part, when the surplus is large enough, they reach an agreement and the game ends.

I characterize the effort dynamic over time. I consider that one player has a specific advantage over the other. I focus in three cases: 1) one player has lower productive costs than the other, 2) one player has lower unproductive costs than the other, and 3) one player is more patient than the other.

The main result shows the interaction between advantages. The player with lower costs of unproductive effort makes higher productive and unproductive effort, even though she does not have a comparative advantage on productive costs. The opposite is not true: A player with lower productive costs makes the same amount of unproductive effort as the other player.

The time preferences do not affect the time when players reach an agreement; both players prefer to end the game at the same time even though one is more patient than the other. However, time preferences have an impact on players' efforts. A more patient player makes more productive effort, but they make the same unproductive effort if the recognition probability is not persistent. If the recognition probability is persistent, a more patient player makes more of both types of effort.

3.1.1 Related Literature

This chapter is related to the literature on bargaining over a non-constant surplus. In most of the literature, the surplus varies exogenously. Merlo and Wilson (1995) consider a bargaining model in which the surplus and the recognition process follow a Markov chain. Eraslan and Merlo (2002) extend the model to allow a k-majority rule instead of unanimity. Ortner (2013) considers a model of stochastic surplus with optimistic agents. In my model the surplus is endogenous.

This chapter is closely related to Che and Sákovics (2004). They study a dynamic holdup problem with linear costs of contributions. If players do not reach an agreement the current period they continue contributing to a common surplus in the next period. They show if players are patient enough, the holdup problem disappear under Markovian strategies. In the present chapter I consider competition for the probability of recognition, and I focus in concave cost that allows to study the contributions over time rather than the case in which agreement is reached the first period.

The literature related to competition for bargaining power, understood as a larger recognition probability, starts with Yildirim (2007). The author allows the agent to exert unproductive costly effort to increase the recognition probability, and examines comparative advantages. Yildirim (2010) study the distribution of surplus when the probability of being the proposer is endogenous. Ali (2015) presents a bargaining model in which the right to propose is sold to the highest bidder in an all-pay-auction. Those papers differ from the present chapter because I focus on the joint decision of increasing the surplus and competing for recognition probabilities, and in how differences in player's characteristics affect outcomes.

In experimental literature, Baranski (2016) studies a bargaining game in which members invest in a common project and then bargain over the distribution. Baranski (2019) compare two bargaining models: endogenous contributions to a common surplus and then bargaining over the surplus, and bargaining first and then the contributions. These papers differ from the present chapter because in my model contributions to the surplus and bargaining are simultaneous actions.[2]

Outline: The rest of the chapter is organized as follows. Section 3.2 introduces the model and Section 3.3 consider a benchmark case with only productive efforts. Section 3.4 analyzes the model with both types of effort, and Section 3.5 concludes. All proofs are in Appendix A.3.3.

3.2 Model

I consider a dynamic bargaining model between two players i and j, where time is discrete and indexed by $t = 0, 1, 2....$ The value of the surplus is endogenous and depends on the efforts each player makes at each period. I denote by x^{t-1} the value of the surplus of the project at the beginning of period t, where at $t = 0$ the value

[2] Other papers that consider contributions to a project that bring payoffs for the players involved are Bonatti and Hörner (2011); Bonatti and Rantakari (2016); Georgiadis (2017); Bowen et al. (2019); and Georgiadis and Tang (2017). Unlike the present chapter, these papers do not focus on simultaneous bargaining and contributions.

of the surplus is x. Each player is risk neutral, and they discount the future returns and costs by $\delta_i, \delta_j \in (0,1)$.

At the beginning of each period, both players simultaneously choose productive efforts (e_i^t, e_j^t), and unproductive efforts $(\varepsilon_i^t, \varepsilon_j^t)$. The productive efforts (e_i^t, e_j^t) increase the value of the surplus to

$$x^t = l(\sum_{\tau=0}^{t-1}(e_i^\tau + e_j^\tau) + e_i^t + e_j^t),$$

where $\sum_{\tau=0}^{t-1}(e_i^\tau + e_j^\tau)$ is the sum of all previous productive efforts and $l(\cdot)$ is a strictly increasing, concave, and two-times-differentiable function. These assumptions imply that the surplus is not decreasing over time and that the increment is anonymous—i.e., it does not depend on who made it—so one unit of effort e_i makes the same contribution as one unit of effort e_j.

After the surplus increases to x^t, one of the players is elected the proposer. The probability of being the proposer for each player depends on unproductive efforts $(\varepsilon_i^t, \varepsilon_j^t)$.

$$p(\varepsilon_i, \varepsilon_j) = \begin{cases} p_i > p_j & \text{if } \varepsilon_i > \varepsilon_j \\ p_i = p_j & \text{if } \varepsilon_i = \varepsilon_j \\ p_i < p_j & \text{if } \varepsilon_i < \varepsilon_j, \end{cases}$$

where $\frac{\partial p_i(\varepsilon_i,\varepsilon_j)}{\partial \varepsilon_i} > 0$ and $\frac{\partial^2 p_i(\varepsilon_i,\varepsilon_j)}{\partial (\varepsilon_i)^2} < 0$. This means that the player who exerts more unproductive effort is elected the proposer with a higher probability. If both players exert the same amount of effort, the probability of being elected proposer is 1/2 for each player. Note that the probability of being recognized as the proposer only depends on the effort made in the current period t. In Section 3.5, I extend the model to persistent recognition.

The player elected proposer makes an offer to the other player. An offer is an allocation of the surplus by the proposer to her opponent and herself. At each period

t, the offer are non negative values (x_i^t, x_j^t) such that $x_i^t + x_j^t = x^t$. If the responder accepts the proposed allocation the game ends and the payoffs are realized. If the responder rejects the allocation, a new period $t+1$ starts.

The cost of the productive efforts in period t are $c_{e,i}f(e_i)$ and $c_{e,j}f(e_j)$, where $c_{e,i}$ and $c_{e,j}$ are constants and $f(\cdot)$ is a convex and two-times-differentiable function.

The cost of the unproductive efforts is $c_{\varepsilon,i}\varepsilon_i$ and $c_{\varepsilon,j}\varepsilon_j$, where $c_{\varepsilon,i}$ and $c_{\varepsilon,j}$ are constants.

Strategies and solution concept. I focus on Markovian strategies—i.e., strategies that only depend on payoff-relevant information. In each period t the only payoff-relevant information on previous actions is summarized in the size of the surplus x^{t-1}. The efforts are defined as Markovian strategies, where they only depend in the state variable x^{t-1}. For the bargaining part of the period, after efforts are made, I consider an action $a_i^t(x^t)$ for all i as

$$a_i^t(x^t) \in \begin{cases} \text{offer } (x_i^t, x_j^t) & \text{if player } i \text{ is the proposer,} \\ \text{accept or reject} & \text{if not,} \end{cases}$$

A strategy for player i is a Markovian strategy $(e_i(x^{t-1}), \varepsilon_i(x^{t-1}))$ for the effort and a sequence of actions $a_i^t(x^t)$. A strategy profile is a Markovian subgame perfect Nash equilibrium (Markovian SPNE) if it is a Markovian perfect equilibrium in each period t. The solution concept is Markovian SPNE.

Proposition 13 *There exists a unique Markovian SPNE.*

Proposition 13 provides a preliminary result for Section 3.3 and 3.4. It implies that the equilibrium efforts and the decision of ending the game are unique.

3.3 Benchmark: Only Productive Effort

I first analyze a benchmark model in which players only make productive efforts. The probabilities of being elected proposer are exogenous: p_i and $p_j = 1 - p_i$.

Since the last part of each period is a take-it-or-leave-it offer, the first step is defining the optimal actions for the proposer and the responder. If player i is the proposer, player j rejects any offer lower than her continuation value $\delta_j V_i^t(x^t)$. Thus, if proposer i wants to end the game she needs to offer at least j's continuation value. Optimality requires that the offer is exactly the continuation value. The value of i's payoff is $x^t - \delta_j V_i^t(x^t)$ and j's value is $\delta_j V_i^t(x^t)$.

In period t, the proposer prefers to end the game and splits the surplus if

$$x^t - \delta_j V_j^t(x^t) \geq \delta_i V_i^t(x^t).$$

This means that if the remaining surplus after she pays the rival player's continuation value (to be indifferent) is equal to or larger than her own continuation value, they reach an agreement. Within a period, players are playing a subgame perfect equilibrium (SPE) and the proposer does not consider the sunk cost of the effort made in the current period. Note that this condition is the same for both players. The previous condition can be written as

$$S^t \equiv x^t - \delta_i V_i^t(x^t) - \delta_j V_j^t(x^t) \geq 0. \tag{3.1}$$

S^t represents the extra amount over her own continuation value the proposer will receive. The continuation value is the minimum amount a player is willing to receive to agree to end the game. The value S^t is called the "prize" of period t and is the source of bargaining, since players are actually bargaining over who gets it. If the prize is greater than zero after efforts are made—i.e. condition (3.1) is satisfied—both players agree to end the game and divide the surplus.

The expected value for player i if the efforts are (e_i^t, e_j^t) and the game ends is $\left[p_i\left(x^t - \delta_j V_j^t(x^{t+1})\right) + (1-p_i)\delta_i V_i^t(x^t) - c_i(e_i^t)\right]$. This can be written as $p_i S^t + \delta_i V_i^t(x^t) - c_i(e_i^t)$. If under the same efforts, condition (3.1) is not satisfied the expected value is $\delta_i V_i(x^t) - c_i(e_i^t)$. This implies that the optimal expected values at the beginning of period t for players i and j are given by the following maximization problem.

$$\begin{aligned} V_i(x^{t-1}) &= \max_{e_i^t} \left\{ \max\left\{\delta_i V_i(x^t) - c_i(e_i^t),\ \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t)\right\}\right\} \\ V_j(x^{t-1}) &= \max_{e_j^t} \left\{ \max\left\{\delta_j V_j(x^t) - c_j(e_j^t),\ \delta_j V_j^t(x^t) + p_j S^t - c_j(e_j^t)\right\}\right\} \end{aligned} \tag{3.2}$$

Subject to: $x^t = l(l^{-1}(x^{t-1}) + e_i^t + e_j^t)$.

Lemma 2 *The continuation value $V_i(x)$ for each player exists and is unique. Furthermore, $V_i(x)$ is increasing, concave, and differentiable.*

Since $S^t = x^t - \delta_i V_i^t(x^t) - \delta_j V_j^t(x^t)$, equation (3.2) can be seen as a system of functional equations for functions $V_i^t(x)$ and $V_j^t(x)$. These functions are unique, strictly increasing, and concave in x and differentiable. The uniqueness is an application of the contracting mapping theorem and is strictly increasing and concave because x is strictly increasing and concave. It is differentiable because x and $c(\cdot)$ are differentiable functions.

Note that the game will end when, for optimal efforts, $S(x^t) \geq 0$. This implies that $V_i(x^{t-1}) = \delta_i V_i(x^t) - c_i(e_i^t)$ until the last period, in which $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t)$.

Lemma 3 *S^t gets a value larger or equal than zero in finite time.*

Intuitively, if the optimal path of effort is different from zero and gives $S(x^t) < 0$ for all t, then players are facing a path of infinite costs. In this case it is optimal to deviate to a constant path of null efforts; then the game becomes a simple Rubinstein bargaining game in which the surplus does not increase and the discounted sum of continuation values is equal to the discounted value of the surplus and then $S(x) > 0$. This means that the minimum payoff for each player comes for exerting null efforts and playing a Rubinstein game; this implies that $S(x^t) > 0$. Then the game always ends at some finite t.

The above result does not imply that the game always ends at period $t > 0$. If the value of the surplus is big enough at the beginning of the game, then the game ends at $t = 0$.

The game can be understood as a two-phase game:

1. *Construction phase:* Players do not want to end the game, because they prefer to increase the surplus.

2. *Ending phase:* Both players want to end the game, because waiting an additional period does not report net benefits.

3.3.1 Optimal Efforts and the Dynamic of the Game

If $S(x^t) < 0$ the game will not end at t; however, it will end in a finite time. Denote by $t + s$ the time the game ends. Note that s is a function of the continuation efforts path. Since the game will end at $t + s$, the value $\delta_i V_i(x^t) - c_i(e_i^t)$ can be written as $\delta_i^s(\delta_i V_i(x^{t+s}) + p_i S^{t+s}) - \sum_{\tau=0}^{s} \delta^\tau c_i(e_i^{t+\tau})$, where the time s and the optimal sequence $\{e_i^\tau, e_j^\tau\}_{\tau=t+1}^{s}$ are functions of the efforts made at t. Then, the first-order conditions can be written as

$$\text{For } S(x^t) \geq 0: \qquad \frac{\partial x^t}{\partial e_i^t}\left[\delta_i \frac{\partial V_i^t(x^t)}{\partial x^t} + p_i \frac{\partial S^t}{\partial x^t}\right] = \frac{\partial c_i(e_i^t)}{\partial e_i^t}$$

$$\text{For } S(x^t) < 0: \qquad \frac{\partial x^{t+s}}{\partial e_i^t}\left[\delta_i^s \frac{\partial V_i^{t+s}(x^{t+s})}{\partial x^{t+s}} + p_i \frac{\partial S(x^{t+s})}{\partial x^{t+s}}\right] = \sum_{\tau=0}^{t+s} \delta_i^\tau \frac{\partial c_i(e_i^{t+\tau})}{\partial e_i^t}.$$

The first-order condition for the case $S(x^t) \geq 0$ shows that the optimal condition is to exert effort until the marginal cost is equal to the marginal benefit. The marginal benefit is composed of an increment in the expected prize and an increment in the continuation value. The first-order condition for the case $S(x^t) < 0$ contains the same idea: It is optimal to exert effort until the discounted marginal benefit is equal to the discounted path of costs until the game ends.

Proposition 13 implies that there is a unique sequence of equilibrium efforts $\{e_i, e_j\}_{t=0}^{\infty}$. This can be seen for the FOCs: The left-hand side is a decreasing function of effort and the right-hand side is increasing in effort.

Lemma 4 *The optimal sequence of effort for each player is decreasing over time. Furthermore, each effort $e_i(x)$ is a decreasing and convex function of x, where x is the size of the surplus at the beginning of the period.*

For the intuition of Lemma 4, consider a best-response sequence $\{e_j\}$ for player j. Then the game can be understood as being to increment the value of x from x^t to x^{t+s} at the lower cost, both in time delay δ_i and $c_i(\cdot)$. Then, when x is *small* it is less expensive to make increments given the concavity of x. When x is *large* it is more expensive, because to make the same increments as before the effort must be larger. It is optimal to choose a decreasing path of equilibrium effort.

The main result is the interaction between both players and how they use their comparative advantages.

Proposition 14 *The following equilibrium results hold:*

1. *If $c_i = c_j$, $\delta_i > \delta_j$ and $p_i = p_j$ the efforts are $e_i(x) > e_j(x)$.*

2. *If $c_i < c_j$, $\delta_i = \delta_j$ and $p_i = p_j$ the efforts are $e_i(x) > e_j(x)$.*

3. *If $c_i = c_j$, $\delta_i = \delta_j$ and $p_i > p_j$ the efforts are $e_i(x) > e_j(x)$.*

Proposition 14 compares the equilibrium efforts given conditions on the cost, discount factors, and recognition probabilities. If the cost function is the same for both players and they have the same chances of being elected proposer, the more patient player exerts more effort in equilibrium. This is because the future benefits are more important for the more patient player. Her continuation value will be larger than her competitor's continuation value, and considering that the continuation value is the minimum amount the player must receive to agree to finish the game, the patient player has more incentives to exert more effort and ensure herself a larger payment. Also, she will have a lower adjusted marginal cost.

The second result says that if both players are equally patient and have the same recognition probabilities, but player i is more efficient (lower costs), then player i exerts more effort on the equilibrium path. The reason is that player i can increase the continuation value at a lower cost compared with player j. This means than she will get a larger minimum payment for herself. Note that these results also apply for the periods before the game ends.

The last result says that if the players are equally patient and efficient, but one has a larger recognition probability, she will contribute more in equilibrium. Since her chances to be the proposer are larger, she expects to get a larger share of the surplus; then it is optimal for her to contribute more in equilibrium.

3.4 Productive and Unproductive Effort

Now I consider the full model. As in the benchmark case of only productive effort, at the end of each period there is a take-it-or-leave-it offer. The subgame after the proposer is chosen is the same as the benchmark model. The player who is elected proposer in period t proposes an allocation the other player agrees with if her payment is larger than her continuation value. This means:

$$x^t - \delta_j V_j^t(x^t) \geq \delta_i V_i^t(x^t).$$

The previous expression can be written as

$$S(x^t) \equiv x^t - \delta_i V_i^t(x^t) - \delta_j V_j^t(x^t) \geq 0.$$

This condition is a similar to (3.1). The only difference is that in this case $V_i^t(x^t)$ is the optimal continuation value with respect to both efforts. The same results regarding two phases of the game hold in this model, which implies that the game can be separate in the *construction phase* and the *ending phase.*

The expected value for player i if the efforts are $(e_i^t, \varepsilon_i^t, e_j^t, \varepsilon_j^t)$ and the game ends is $\left[p_i\left(x^t - \delta_j V_j^t(x^{t+1})\right) + (1-p_i)\delta_i V_i^t(x^t) - c_{e,i}(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)\right]$, which can be written as: $p_i S(x^t) + \delta_i V_i^t(x^t) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. If, under the same efforts, condition $S(x^t) \geq 0$ is not satisfied, the expected value is $\delta_i V_i(x^t) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. This implies that the optimal expected values at the beginning of period t for player i and j are given by the following maximization problem.

$$V_i(x^{t-1}) = \max_{e_i^t, \varepsilon_i^t}\left\{ \max\left\{\delta_i V_i(x^t) - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t,\ \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t\right\}\right\}$$
$$V_j(x^{t-1}) = \max_{e_j^t, \varepsilon_i^t}\left\{ \max\left\{\delta_j V_j(x^t) - c_j(e_j^t) - c_{\varepsilon,i}\varepsilon_i^t,\ \delta_j V_j^t(x^t) + p_j S^t - c_j(e_j^t) - c_{\varepsilon,j}\varepsilon_j^t\right\}\right\}$$
Subject to: $x^t = l(l^{-1}(x^{t-1}) + e_i^t + e_j^t)$.
$$\tag{3.3}$$

The same result for $V_i(x)$ in Section 3.3 applies. The game ends in finite time, and at the beginning of the game there will be a *construction phase* in which $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$. In the last period of the game (*ending phase*), the optimal value will be $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$.

In the *construction phase* the optimal unproductive effort is zero, because both players know the surplus will not be divided and thus who is the proposer is not relevant. In the last period, being the proposer becomes valuable and both players will make positive unproductive effort given by the condition

$$S^{t+1}\frac{\partial p_i(\varepsilon_i^t, \varepsilon_j^t)}{\partial \varepsilon_i^t} = c_{\varepsilon,i}(\varepsilon_i^t). \tag{3.4}$$

The optimal unproductive effort ε_i is given by the amount of effort such that the marginal cost is equal to the marginal benefit. The marginal benefit is measured as the marginal increment of the expected prize given a change in the probability of winning it. Note that the only objective of the unproductive effort is to improve the chance of winning the prize.

The following results explain the advantages of the heterogeneity of both players.

Proposition 15 *Consider t^* to be the last period of the game. Then the following equilibrium results hold:*

1. *If $c_{\varepsilon,i} < c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$ and $\delta_i = \delta_j$ efforts are $\varepsilon_i^t = \varepsilon_j^t = 0$ for all $t < t^*$, $\varepsilon_i^t > \varepsilon_j^t > 0$ for $t = t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

2. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$ and $\delta_i > \delta_j$ efforts are $\varepsilon_i^t = \varepsilon_j^t = 0$ for all $t < t^*$, $\varepsilon_i^t = \varepsilon_j^t > 0$ for $t = t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

3. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} < c_{e,j}$ and $\delta_i = \delta_j$ efforts are $\varepsilon_i^t = \varepsilon_j^t = 0$ for all $t < t^*$, $\varepsilon_i^t = \varepsilon_j^t > 0$ for $t = t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

The first result of the above proposition explains that the player who has a comparative advantage in unproductive cost will exert more unproductive effort and have a larger recognition probability. This advantage implies a larger contribution to the joint surplus, even though they are symmetric in other elements. Since the recognition probability is larger, the probability of winning the prize is larger. Hence, she contributes more because she can get a larger share of the surplus.

The second result shows that differences in the discount factor do not play any role in the decision regarding unproductive effort. This is because the players exert unproductive effort only when $S^t \geq 0$, and there is no discount at that moment. Finally, the last result shows that if the only difference is given by the cost of the productive effort, it only affects the productive effort. Since there are no other differences, both players made the same unproductive effort and the recognition probability is the same for both.

The unproductive effort is null until the condition $S(x) \geq 0$ is satisfied, because the objective of that effort is to increase the chance of winning the prize, so the only moment in which the players want to increase their probability of being recognized the proposer is in the period in which they will decide the shares of the surplus, which means at the end of the game.

3.4.1 Persistent Unproductive Effort

In many settings, the effort of gaining bargaining power persists over time; for example, the effort made by the lobbyist is still important after it is made. If a player makes effort and increases her bargaining skills, it is unlikely she will lose that skill in the next period. To include this extension, I modify the recognition process in the following way.

The new element is that the probability of recognition $p(h_t)$ depends on the history of unproductive efforts. The assumptions about the recognition probabilities are $\frac{\partial p_i(h_t)}{\partial \varepsilon_i^s} > 0$ and $\frac{\partial^2 p_i(h_t)}{\partial (\varepsilon_i^s)^2} < 0$ $\forall i, s \leq t$. This means that the effort made in any period is going to increment the recognition probability in future periods. Furthermore, the recognition probabilities are symmetric. This means that $\frac{\partial p_i(h_t)}{\partial \varepsilon_i^s} = -\frac{\partial p_i(h_t)}{\partial \varepsilon_j^s}\ \forall i, s \leq t$.

The rule $p(h_t) = \{p_i(h_t), p_j(h_t)\}$ is given by

$$p(h_t) = \begin{cases} p_i(h_t) > p_j(h_t) & \text{if } \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_i^\tau > \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_j^\tau \\ p_i(h_t) = p_j(h_t) & \text{if } \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_i^\tau = \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_j^\tau \\ p_i(h_t) < p_j(h_t) & \text{if } \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_i^\tau < \sum_{\tau=0}^{t} \gamma_\tau \varepsilon_j^\tau, \end{cases}$$

where $\gamma_\tau \in [0, 1]$ is the weight assigned to the effort made in period τ.

Since the recognition is persistent, the productive and unproductive effort now depends on the history of productive efforts and the current value of the surplus. A strategy for player i is productive and unproductive efforts $(e_i(x^{t-1}, h_t), \varepsilon_i(x^{t-1}, h_t))$ and a sequence of actions $a_i^t(x^{t-1}, h_t)$. A strategy profile is a Markovian subgame perfect Nash equilibrium (Markovian SPNE) if it is a Markov perfect equilibrium in each period t. I call it Markovian because even though the equilibrium deepens on the history of unproductive efforts, it still only depends on the current value of the surplus. The solution concept is Markovian SPNE.

In this case the value of the prize becomes $S(x^t, h_t) = x^t - \delta_i V_i^t(x^t, h_t) - \delta_j V_j^t(x^t, h_t)$. As in the previous case, the expected value for player i if the efforts are $(e_i^t, \varepsilon_i^t, e_j^t, \varepsilon_j^t)$ and the game ends is $\left[p_i\left(x^t - \delta_j V_j^t(x^t, h_{t+1}\right) + (1 - p_i)\delta_i V_i^t(x^t, h_{t+1}) - c_{e,i}(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)\right]$. This can be written as $p_i S(x^t, h_t) + \delta_i V_i^t(x^t, h_{t+1}) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. If, under the same efforts, condition $S(x^t, h_t) \geq 0$ is not satisfied, the expected value is $\delta_i V_i(x^t, h_{t+1}) - c_i(e_i^t) - c_{\varepsilon,i}(\varepsilon_i^t)$. This implies that the optimal expected values at the beginning of period t for players i and j are given by the following maximization problem.

$$
\begin{aligned}
V_i(x^{t-1},h_t) &= \max_{e_i^t,\varepsilon_i^t}\left\{ \max\left\{\delta_i V_i(x^t,h_{t+1}) - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t,\right.\right.\\
&\qquad\qquad \left.\left.\delta_i V_i^t(x^t,h_{t+1}) + p_i(h^t)S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t\right\}\right\}\\
V_j(x^{t-1},h_t) &= \max_{e_j^t,\varepsilon_i^t}\left\{ \max\left\{\delta_j V_j(x^t,h_{t+1}) - c_j(e_j^t) - c_{\varepsilon,i}\varepsilon_i^t,\right.\right.\\
&\qquad\qquad \left.\left.\delta_j V_j^t(x^t,h_{t+1}) + p_j(h^t)S^t - c_j(e_j^t) - c_{\varepsilon,j}\varepsilon_j^t\right\}\right\}
\end{aligned}
\tag{3.5}
$$

Subject to: $x^t = l(l^{-1}(x^{t-1}) + e_i^t + e_j^t)$.

The same results for $V_i(x,h)$ apply. The game ends in finite time, and at the beginning of the game there will be a *construction phase* in which $V_i(x^{t-1}) = \delta_i V_i^t(x^t) - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$, and in the last period of the game (*ending phase*) the optimal value will be $V_i(x^{t-1}) = \delta_i V_i^t(x^t) + p_i(h^t)S^t - c_i(e_i^t) - c_{\varepsilon,i}\varepsilon_i^t$.

The productive effort path of equilibrium is given by the same first-order condition as in the baseline model and the same results apply. For the unproductive effort, this case is different from the non-persistent case. In the *construction phase* and in the last period the optimal unproductive effort is positive, because being the proposer is valuable for both players and depends on the complete path of efforts. The optimal conditions are

$$
\begin{aligned}
&\text{For } S(x^t,h_t)\geq 0: && \left[p_i(h^t)\frac{\partial S(x^t,h_t)}{\partial \varepsilon_i^t} + S(x^t,h_t)\frac{\partial p_i(h^t)}{\partial \varepsilon_i^t}\right] + \delta_i\frac{\partial V_i(x^t,h_{t+1})}{\partial \varepsilon_i^t} = c_{\varepsilon,i}\\
&\text{For } S(x^t,h_t) < 0: && \delta_i^s\left[p_i(h^{t+s})\frac{\partial S(x^{t+s},h_{t+s})}{\partial \varepsilon_i^t} + S(x^{t+s},h_{t+s})\frac{\partial p_i(h^{t+s})}{\partial \varepsilon_i^t}\right] +\\
& && \delta_i^{s+1}\frac{\partial V_i(x^{t+s},h_{t+s+1})}{\partial \varepsilon_i^t} = \sum_{\tau=0}^{t+s}\delta_i^\tau c_{,i}\frac{\partial_i^{t+\tau}}{\partial_i^t}.
\end{aligned}
$$

The optimal unproductive effort ε_i is given by the effort whereby the marginal cost is equal to the marginal benefit. The marginal benefit is measured as the marginal increment of the continuation value and the expected prize, given a change in the probability of winning it. Note that the only objective of the unproductive effort is to improve the chance of winning the prize.

Proposition 16 *There exists a unique Markovian SPNE. Consider t^* to be the last period of the game. Then the following equilibrium results hold*

1. *If $c_{\varepsilon,i} < c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$, and $\delta_i = \delta_j$, efforts are $\varepsilon_i^t > \varepsilon_j^t > 0$ for $t \leq t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

2. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} = c_{e,j}$, and $\delta_i > \delta_j$, efforts are $\varepsilon_i^t > \varepsilon_j^t > 0$ for $t \leq t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

3. *If $c_{\varepsilon,i} = c_{\varepsilon,j}$, $c_{e,i} < c_{e,j}$, and $\delta_i = \delta_j$, efforts are $\varepsilon_i^t = \varepsilon_j^t > 0$ for $t \leq t^*$ and $e_i^t > e_j^t$ for all $t \leq t^*$.*

The first result of Proposition 16 shows that having lower cost of unproductive effort means that the agent exert in more effort to increase recognition probability, and because of that the player will contribute more to the project. The reason is that since the recognition probability is larger, the probability of winning the prize is also larger; thus the expected share of surplus is larger.

The second result shows that being more patient increases both productive and unproductive effort. Since unproductive effort has a persistent effect, the more patient player will exert more unproductive effort. She perceives larger benefits and lower adjusted marginal costs, because she places more importance on future cost reduction, given more unproductive effort today.

Similar to unproductive effort, the contribution to the surplus is larger for the more patient player because the increment in the future expected prize (reinforced for a larger recognition probability) and the reduction in future cost are more important for her than for her rival player.

Finally, if the only difference between the players is given by the cost of productive effort, there will be no difference in unproductive effort and the recognition probability will be the same for both players. This means that the only difference will be given by more productive effort by agent with lower cost of productive effort.

An important element of the game is how the persistence of the recognition probability works, The more intuitive way is to assume that the effort made two periods ago is less important than the effort made one period ago; This means that $\frac{\partial p_i(h_t)}{\partial \varepsilon_i^s} \geq \frac{\partial p_i(h_t)}{\partial \varepsilon_i^{s'}}$ $\forall i, s' < s \leq t$. In other words, the marginal increment in the recognition probability at time t by the effort made in s is greater than or equal to the effort made in s' with $s' < s$. Note that under the previous assumption, $\gamma_\tau \geq \gamma_{\tau'}$ $\forall \tau \geq \tau'$.

The dynamics of the unproductive efforts under the previous assumption are explained in the following lemma.

Lemma 5 *Unproductive efforts are increasing over time if γ_t is constant or increasing in t.*

Lemma 5 says if that recent unproductive efforts are more important to the recognition probability than efforts more distant in time, both players exert an increasing sequence of unproductive effort over time until the game is over.

Exerting an increasing amount of productive effort is intuitive. Since the recognition process is persistent, it is optimal to exert a positive amount of effort in each period; otherwise the player loses the advantage over her rival. The effort is increasing, since more recent efforts are more important and at the beginning of the game the surplus is low, and thus the increment in expected benefit will be low. In latter periods the benefits is larger, and thus there are incentives to gain bargaining power.

In the opposite case, if more recent efforts are less important than efforts at the beginning of the game, the dynamics will depend on the specification of the problem. A particular example is the case in which players are equally patient ($\delta_i = \delta_j$) and the weight of each unproductive effort is discounted by $\gamma \in (0, 1)$; this means $\gamma_\tau = \gamma^\tau$.

Lemma 6 *If $\delta_i = \delta_j = \delta$, under the natural specification of the recognition probability:*[3]

$$p_i(h_t) = f\left(\sum_m^t \gamma^m \varepsilon_i^m, \sum_m^t \gamma^m \varepsilon_j^m\right)$$

- *If $\gamma > \delta$, then ε is increasing over time.*
- *If $\gamma = \delta$, then ε is constant over time.*
- *If $\gamma < \delta$, then ε is decreasing over time.*

Lemma 6 says that if the "discount factor" for the efforts is bigger than the discount factor of benefits and cost, then the unproductive effort is increasing over time. The intuition is that even though efforts at the beginning of the game are more important, later efforts are still important—and since players are not very patient, as time progresses the value of the future payment increases.

3.5 Concluding Remarks

This chapter contributes to the literature of bargaining over an endogenous surplus. The model considers contributions to a common project in which the shares of the surplus created cannot be decided beforehand. A player might decide to contribute more and make more effort to gain bargaining power; however, this is costly. The tradeoff is how much effort to allocate to contribute to the surplus and to gain bargaining power.

In equilibrium, every player has a veto power given by the possibility of rejecting the proposed allocation of the shares of the surplus. This implies that even in absence of a formal mechanism, they can ensure a minimum payment that induces them to contribute to the surplus.

The optimal effort is decreasing on the size of the surplus. Also, a more patient player contributes more than a less patient player. This is because the more patient player has a

[3] An example of this natural probability function is $p_i(h_t) = \frac{\sum_m^t \gamma^m \varepsilon_i^m}{\sum_m^t \gamma^m \varepsilon_i^m + \sum_m^t \gamma^m \varepsilon_j^m}$.

larger continuation value, since future payoffs are more valuable for her. A more efficient player—i.e., with lower costs—also contributes more in equilibrium than a more inefficient player.

The player with lower unproductive cost exerts more unproductive effort, and she also exerts more productive effort than the other player. She makes more productive effort because her comparative advantages in recognition probability mean that she has a larger probability of being recognized as the proposer, and thus she will get a larger expected share of the surplus. This generates incentives for her to contribute more to the surplus. This analysis is true for both the persistent recognition and the not-persistent recognition case.

In the case of persistent recognition probability, the more patient player will make more productive and unproductive efforts. Since current efforts affect future benefits and costs, being more patient has advantages for recognition probability, and since the more patient player will have a bigger chance of being elected proposer, she also will exert more productive effort.

If the recognition probability is persistent, in cases in which more recent efforts in the current period are more important than more distant in time efforts, the unproductive effort is increasing over time. This is increasing, because at the beginning of the game the surplus is low; thus there is no need to exert high effort, because the value of the expected payment is low. However, when the surplus is larger, every extra increment in bargaining power is valuable, since players compete for the surplus generated and thus she makes more effort.

Appendix A

Appendices

A.1 Chapter 1: Additional Results

A.1.1 No-investigation Equilibria

In the main text of the chapter, I described the equilibrium in which there is always investigation in the first period. In this section I show the class of equilibria in which the prosecutor does not investigate at any period for low values of θ. These results do not change the main findings of the chapter.

The reason why the no-investigation equilibrium cannot be supported as an equilibrium for high values of θ is because the incentives to deviate to get $y = h$ are increasing with θ. On the other hand, if θ is low, the prosecutor might prefer not to investigate if the innocent defendant accepts $x = d$ with positive probability.

Note that if the innocent defendant accepts $x = d$ with probability 1, the prosecutor will deviate to investigate, because she benefits from finding $y = h$, and she is not affected by finding $y = e$ because she can offer $y = d$ and the defendant will accept. The reason a no-investigation equilibrium can exist is that the innocent defendant accepts offer $x \leq d$ with probability lower than 1. This induces the prosecutor to not deviate because of the possibility of getting $y = e$ and getting a negative payoff at the trial.

I define $\mu_n^I(d)$ as the probability that the innocent defendant accepts $x = d$ at period n. For each sequence $\{\mu_1^I(d), \mu_2^I(d), ...\mu_{N-1}^I(d), \mu_N^I(d)\}$ of probabilities of accepting $x = d$ at n, I define:

$$\underset{\sim}{\theta}^N = \begin{cases} \frac{d\tilde{\mu}^I_{1,..,N}(d)}{h-d\left(1-\tilde{\mu}^I_{1,..,N}(d)\right)} & \text{if } \mu_N^I(d) \in \left[0, \frac{c}{d+c}\right) \\ \frac{\left(1-\mu^I_{1,..,N}(d)\right)c}{h-d+\left(1-\mu^I_{1,..,N}(d)\right)c} & \text{if } \mu_N^I(d) \in \left[\frac{c}{d+c}, 1\right], \end{cases}$$

where $\tilde{\mu}^I_{1,..,N}(d) = \mu_N^I(d)\prod_{j=1}^{N-1}\left(1-\mu_j^I(d)\right)$ is the probability that the innocent defendant accepts $x = d$ at $n = N$, and $\mu^I_{1,..,N}(d) = 1 - \prod_{j=1}^{N}\left(1-\mu_j^I(d)\right)$ is the probability that the innocent defendant accepts $x = d$ between $n = 1$ and $n = N$.

Proposition 17 *For $\theta \leq \underset{\sim}{\theta}^N$, the prosecutor never investigates and offers $x = d$ at the end of each period. The guilty defendant accepts it the first period and the game ends, and the innocent defendant accepts it with probability $\mu_n^I(d)$ at each period n.*

The guilty defendant does not deviate because $x = d$ is the best offer he can receive. The innocent defendant does not deviate because he is indifferent between accepting d or getting d at the trial. The prosecutor does not deviate to offer another x because it is going to be rejected for sure, and she does not deviate to investigate because her expected payoff for the deviation is lower than no-investigation.

Note that $\underset{\sim}{\theta}^N = 0$ if $\mu_n^I(d) = 1$ for any n. The intuition is that if the innocent defendant accepts $x = d$ for sure at some period, the prosecutor deviates to investigating.

For $\theta > \underset{\sim}{\theta}^N$, the equilibrium is the same as that for the investigation equilibrium. Note that $\underset{\sim}{\theta}^N < \underline{\theta}^N$ for each N.

The expected payoff for the prosecutor for all θ is

$$u^P = \begin{cases} \theta d + (1-\theta)\mu^I_{1,..,N}(d)d & \text{if } \theta \in (0, \underset{\sim}{\theta}^N] \\ \theta\Big[(1-q^{\frac{1}{N}})h + q^{\frac{1}{N}} dq^{\frac{1}{N}}\Big] + (1-\theta)dq^{\frac{1}{N}} & \text{if } \theta \in (\underset{\sim}{\theta}^N, \underline{\theta}^N] \\ \theta\Big[(1-q)h + qdq\Big] + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}^N, \bar{\theta}^N] \\ \theta\Big[(1-q)h + qd\Big] & \text{if } \theta \in (\bar{\theta}^N, 1). \end{cases}$$

The expected punishment for the innocent defendant and the guilty defendant are

$$u^I = \begin{cases} d & \text{if } \theta \in (0, \underset{\sim}{\theta}^N] \\ dq^{\frac{1}{N}} & \text{if } \theta \in (\underset{\sim}{\theta}^N, \underline{\theta}^N] \\ dq & \text{if } \theta \in (\underline{\theta}^N, 1] \end{cases} \quad \text{and} \quad u^G = \begin{cases} d & \text{if } \theta \in (0, \underset{\sim}{\theta}^N] \\ (1-q^{\frac{1}{N}})h + q^{\frac{1}{N}} dq^{\frac{1}{N}} & \text{if } \theta \in (\underset{\sim}{\theta}^N, \underline{\theta}^N] \\ (1-q)h + qdq & \text{if } \theta \in (\underline{\theta}^N, \tilde{\theta}^N] \\ (1-q)h + qd & \text{if } \theta \in (\tilde{\theta}^N, 1). \end{cases}$$

The innocent defendant is worse off compared with the case in which there is investigation the first period, because with no-investigation he gets her highest possible punishment if θ is small enough. On the other hand, the guilty defendant is better off when there is no-investigation because he only gets d as punishment.

It is not clear whether the prosecutor is better or worse off with the no-investigation equilibrium compared with the investigation equilibrium. The trade-off for $\theta \leq \underset{\sim}{\theta}$ is:

1. If there is investigation the first period, the prosecutor can find evidence $y = h$. If not, she offers $x = q^{\frac{1}{N}} d$, which is lower than d, but it is accepted with probability 1 by both players.

2. If there is no investigation in every period, the prosecutor offers $x = d$, but it is accepted with a probability lower than 1 by the innocent defendant.

If N is low, to investigate the first period is costly for the prosecutor; the extreme case is $N = 1$, in which the offer the prosecutor makes is reduced to $x = dq$. Nevertheless, if N is large, to investigate only one period generates a small decrease in the offer the prosecutor can make. In other words, intuitively the prosecutor is going to be better off in the investigation equilibrium when N is large. This is captured by Lemma 7.

Lemma 7 *Consider a sequence of probability of acceptance $\mu^I_n(d)$ for all n. For $\theta \leq \underset{\sim}{\theta}^N$ there is a N^* such that for $N \geq N^*$ the prosecutor is better off investigating in the first period than never investigating.*

Figures A.1 and A.2 illustrate how payoffs change depending on $\mu^I(d)$ and N.

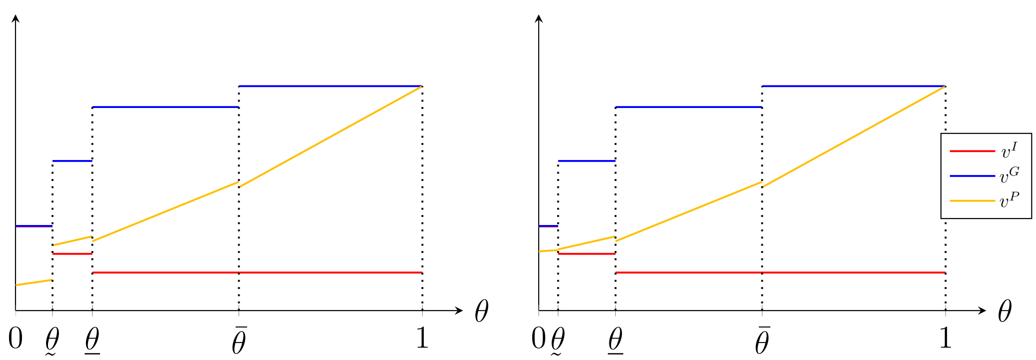

Figure A·1: Expected payoffs and punishments with no-investigation equilibrium and $N = 2$. The left panel corresponds to $\mu^d = 0.3$ and the right panel to $\mu^d = 0.7$.

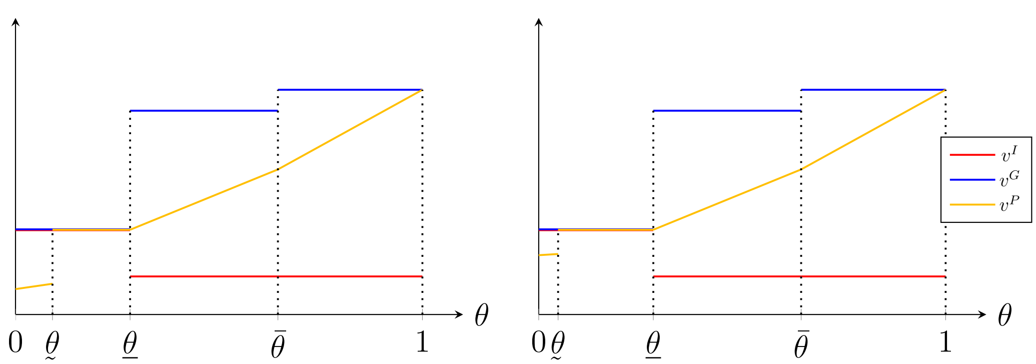

Figure A·2: Expected payoffs and punishments with no-investigation equilibrium and $N \to \infty$. The left panel corresponds to $\mu^d = 0.3$ and the right panel to $\mu^d = 0.7$.

A.1.2 Public Investigation

This extension shows that if the investigation decision is public information, the prosecutor does not investigate for low values of θ. The equilibrium with public and private investigation decisions coincides when $N \to \infty$ and T is fixed.

If the prosecutor decides not to investigate, she does not induce uncertainty in the defendant; he knows the prosecutor has evidence $y = d$.

Define:

$$\underline{\theta}^{Public} = \frac{d}{h - dq}.$$

Lemma 8 *If $\theta \leq \underline{\theta}^{Public}$ the prosecutor does not investigate, and she offers $x = d$ at the end of period $n = 1$. The defendant accepts it and the game ends at the first period.*

First, note that as in the case of private investigation, the defendant's continuation punishment is decreasing in the number of periods of investigation; therefore the prosecutor either investigates every period as long as there is no agreement, or she does not investigate in any period. Then the question is for which values she does not investigate. If $\theta > \bar{\theta}$, she always investigate because $\theta\big((1-q)h + dq\big) > d$. If $\theta \leq \bar{\theta}$, she does not investigate for $\theta \leq \underline{\theta}^{Public}$.

The prosecutor's expected payoff is

$$u^P = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{Public}] \\ \theta\Big[(1-q)h + qdq\Big] + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}^{Public}, \tilde{\theta}^N] \\ \theta\Big[(1-q)h + qd\Big] & \text{if } \theta \in (\tilde{\theta}^N, 1) \end{cases}$$

and the defendant's expected punishments are

$$u^I = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{Public}] \\ dq & \text{if } \theta \in (\underline{\theta}^{Public}, 1] \end{cases} \quad \text{and} \quad u^G = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{Public}] \\ (1-q) \cdot h + q \cdot dq & \text{if } \theta \in (\underline{\theta}^{Public}, \tilde{\theta}^N] \\ (1-q) \cdot h + q \cdot d & \text{if } \theta \in (\tilde{\theta}^N, 1) \end{cases}$$

The prosecutor's payoff is weakly better with public investigation. The payoffs coincide at the limit as $N \to \infty$. The reason the prosecutor is better off with public investigation is because it is credible that she is not going to investigate for low values of θ, and therefore the innocent defendant is willing to accept $x = d$.

A.1.3 Inconclusive Evidence

In this section I show that the same intuitions in the baseline model are extended to the case in which the evidence is not conclusive regarding the type of defendant. I consider a simplified version of the model, in which the prosecutor investigates and there is only one period of negotiation.

Consider further that the probability of finding new evidence is $1 - q^G$ if the defendant in guilty, and $1 - q^I$ if the defendant is innocent, with $q^G > q^I$. This assumption implies that if the prosecutor does not find new evidence, the posterior belief about the defendant's being guilty is higher than the prior belief. Lastly, consider that the new evidence found is $y = h$ with probability π^G if the defendant is guilty and π^I if the defendant is innocent, with $\pi^G > \pi^I$.

If the prosecutor finds evidence $y = h$, she discloses it and offers $x = h$, which is accepted by both defendant types. If the prosecutor does not find new evidence, there is no disclosure. The defendant's second-order belief, depending on his type, is

$$P^G(\text{d-type} \mid \text{no disclosure}) = 1 - q^G$$

$$P^I(\text{d-type} \mid \text{no disclosure}) = 1 - q^I.$$

Then the continuation punishment for each defendant type at trial is $v^G = dq^G$ and $v^I = dq^I$. In this model, if the prosecutor does not find new evidence, she updates her belief to

$$P(\alpha = G \mid y = d) = \frac{q^G\theta}{q^G\theta + q^I(1-\theta)} \equiv \theta^d.$$

The optimal offer the d-type prosecutor makes depends on θ^d. If the prosecutor offers the guilty defendant's continuation punishment, only the guilty defendant accepts it. Both defendant types accept it if the offer is equal to the innocent defendant's continuation punishment. Therefore, the prosecutor offers $x = dq^G$ if $\theta^d > \frac{q^I}{q^G} \equiv \bar{\theta}$.

If the prosecutor finds evidence $y = e$, her posterior belief is

$$P(\alpha = G|y = e) = \frac{(1-q^G)\pi^G\theta}{(1-q^G)\pi^G\theta + (1-q^I)\pi^I(1-\theta)} \equiv \theta^e.$$

Consider the case in which $\theta^d > \bar{\theta}$ and $\theta^e < \bar{\theta}$. In this case the d-type prosecutor makes a high offer to the defendant, but the e-type prefers to make a low offer. The low offer is going to be rejected, because the defendant will know that prosecutor has evidence $y = e$ because she is playing a strictly dominated strategy for the d-type.

The two candidates for optimal strategy for the e-type prosecutor are to disclose $y = e$ and offer $x = 0$ or make the high offer. The latter case is preferred if $\theta^e dq^G + (1-\theta^e)(-c) \geq 0$ or $\theta^e \geq \frac{c}{dq^G+c}$.

Define:

$$\tilde{\theta} \equiv \frac{c}{dq^G + c}.$$

The prosecutor is going to disclose evidence $y = e$ if the following conditions hold:

$$\theta^d > \bar{\theta},\ \theta^e < \bar{\theta},\ \text{and}\ \theta^e < \tilde{\theta}$$

Figure A.3 shows the conditions when the prosecutor discloses exculpatory evidence.

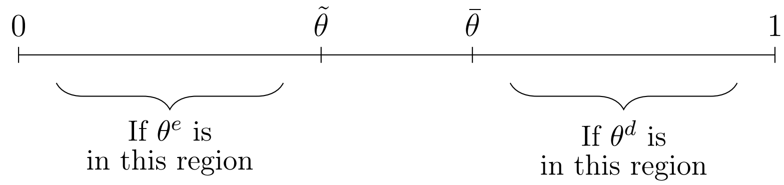

Figure A·3: Conditions of disclosure of exculpatory evidence.

Now, in terms of the prior belief θ, the prosecutor discloses exculpatory evidence if the prior belief is either not too high or too low.[1]

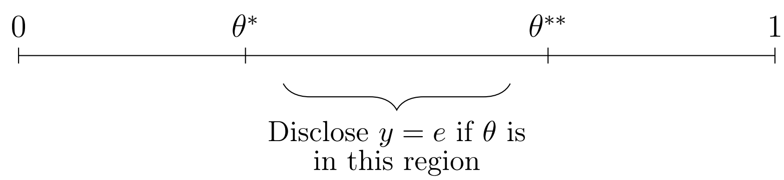

Figure A·4: Conditions on θ of disclosure of exculpatory evidence.

Figure A.4 shows that, for the inconclusive case, the prior belief cannot be too high to disclose exculpatory evidence, because in that case the prosecutor still will make a high offer if she gets exculpatory evidence.

If the prosecutor does not get new evidence, and $\theta^d > \bar{\theta}$, the prosecutor makes an offer that only the guilty defendant accepts. Therefore, if the defendant is innocent he does not reach an agreement and they go to trial. If the disclosure of evidence is mandatory, they always reach an agreement.

---

[1] $\theta^* = \frac{q^I\bar{\theta}}{q^G - \bar{\theta}(q^G - q^I)}$ and $\theta^{**} = \frac{(1-q^I)\pi^I\tilde{\theta}}{(1-q^G)\pi^G - \tilde{\theta}((1-q^G)\pi^G - (1-q^I)\pi^I)}$

A.2 Chapter 2: Additional Results

A.2.1 Utility Functions

The utility functions $U^E(\alpha, \beta, \gamma)$ and $U^{VC}(\alpha, \beta, \gamma)$ are different depending on if $\gamma \geq \bar{\gamma}$ or $\gamma < \bar{\gamma}$.

1. If $\gamma \geq \bar{\gamma}$, the venture capitalist will choose $x = L$ if $\beta \in [\underline{\beta}, \bar{\beta})$. The expected payoffs are

For the entrepreneur:

$$U^E(\alpha, \beta) = \begin{cases} \alpha\Big[\beta\gamma\bar{p}\Big] + (1-\alpha)\Big[\beta\gamma\underline{p}\Big] & \text{if } \beta < \underline{\beta} \\ \alpha\Big[\beta\bar{p} - c\Big] + (1-\alpha)\Big[\beta\gamma\underline{p}\Big] & \text{if } \beta \in [\underline{\beta}, \bar{\beta}) \\ \alpha\Big[\beta\bar{p} - c\Big] + (1-\alpha)\Big[\beta\underline{p} - c\Big] & \text{if } \beta \geq \bar{\beta} \end{cases} \tag{A.1}$$

For the venture capitalist:

$$U^{VC}(\alpha, \beta) = \begin{cases} \alpha\Big[(1-\beta)\gamma\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\gamma\bar{p}\Big] & \text{if } \beta < \underline{\beta} \\ \alpha\Big[(1-\beta)\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\gamma\bar{p}\Big] & \text{if } \beta \in [\underline{\beta}, \bar{\beta}) \\ \alpha\Big[(1-\beta)\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\bar{p}\Big] & \text{if } \beta \geq \bar{\beta}, \end{cases} \tag{A.2}$$

where $\alpha \in \{0, 1\}$.

2. If $\gamma < \bar{\gamma}$, the venture capitalist will choose $x = R$ if she has the control rights and $\beta \in [\underline{\beta}, \bar{\beta})$; this means that both players will choose the same direction. The expected payoffs are

For the entrepreneur:

$$U^E(\alpha,\beta)=\begin{cases}\alpha\Big[\beta\gamma\bar{p}\Big]+(1-\alpha)\Big[\beta\gamma\underline{p}\Big] & \text{if } \beta<\underline{\beta}\\ \beta\bar{p}-c & \text{if } \beta\in[\underline{\beta},\bar{\beta})\\ \alpha\Big[\beta\bar{p}-c\Big]+(1-\alpha)\Big[\beta\underline{p}-c\Big] & \text{if } \beta\geq\bar{\beta}\end{cases} \tag{A.3}$$

For the venture capitalist:

$$U^{VC}(\alpha,\beta)=\begin{cases}\alpha\Big[(1-\beta)\gamma\underline{p}\Big]+(1-\alpha)\Big[(1-\beta)\gamma\bar{p}\Big] & \text{if } \beta<\underline{\beta}\\ (1-\beta)\underline{p} & \text{if } \beta\in[\underline{\beta},\bar{\beta})\\ \alpha\Big[(1-\beta)\underline{p}\Big]+(1-\alpha)\Big[(1-\beta)\bar{p}\Big] & \text{if } \beta\geq\bar{\beta},\end{cases} \tag{A.4}$$

where $\alpha \in \{0,1\}$.

A.2.2 Continuous Control Rights

In many cases, control rights are not totally hold by only one party. The decision is made through ex post bargaining, in which control rights represent ex post bargaining power and not the absolute authority. In this section I modify the model to include the possibility of a stylized negotiation regarding the optimal direction.

I model the negotiation by considering that control rights are a continuous variable $\alpha \in [0,1]$. The interpretation is that control rights are the recognition probability of being elected proposer. This means that the entrepreneur is recognized as the proposer with probability α and the venture capitalist with probability $1-\alpha$. Figure A.5 shows the extra step in the timeline in which the proposer is elected.

When the entrepreneur offers the contract, the problem he solves is the same as the one described if α is discrete (problem (2.1)), considering a continuous $\alpha \in [0,1]$. Utility

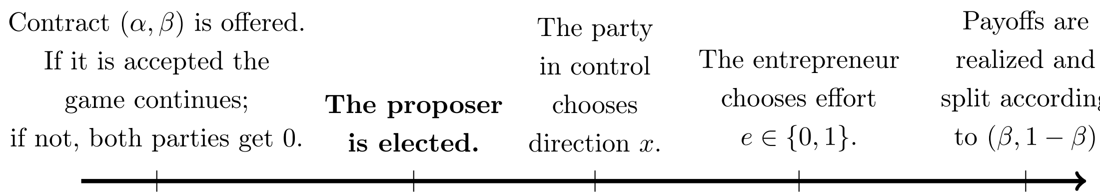

Figure A·5: Timeline

functions are given by equations (A.1)-(A.4).

$$\begin{aligned} &\max_{\alpha \in [0,1], \beta \in [0,1]} U^E(\alpha, \beta, \gamma) \\ &\text{subject to: } U^I(\alpha, \beta, \gamma) \geq M. \end{aligned} \tag{A.5}$$

The solution of this problem is that entrepreneur in general allocates $\alpha \in (0, 1)$, that means that at least he relinquishes some control right. The specific value of α depends on the parameters of the problem and can be found in the proof of Proposition 18 in Appendix A.3.2. Proposition 18 describes the efforts that the contract induces.

Proposition 18 *The contract that the entrepreneur offers has the following characteristics:*

1. If c is high enough, the entrepreneur chooses $e = 0$.

2. If c is low and M is high enough, the entrepreneur chooses $e = 1$.

3. If $\gamma \geq \bar{\gamma}$ and c and M are low enough, the entrepreneur chooses $e = 1$ only if he is elected as "decision maker."

4. If $\gamma < \bar{\gamma}$ and c and M are low enough, the entrepreneur chooses $e = 1$.

In the case 4 of Proposition 18, the venture capitalist chooses the entrepreneur's desired direction. That means that the entrepreneur is indifferent in an allocation of any $\alpha \in [0, 1]$. The same qualitative analysis of the characteristics of the optimal contract when α is discrete applies to this case. Interestingly, the result in which the entrepreneur chooses to relinquish control rights when there is a feasible contract with $\alpha = 1$ still exists when the control rights are continuous.

Corollary 3 *In the optimal contract the entrepreneur offers positive control rights to the investor, even though $\alpha = 1$ is a feasible solution.*

The result stated in Corollary 3 is important because it shows that the voluntary provision of control rights is not implied by the nature of a discrete election, and is a general result regarding the nature of the game.

If the venture capitalist has the bargaining power at the beginning of the game, the problem she solves is the same as when control rights are discrete (problem (2.6)). Utility functions are given by equations (A.1)-(A.4).

$$\max_{\alpha \in \{0,1\}, \beta \in [0,1]} U^I(\alpha, \beta, \gamma). \tag{A.6}$$

The solution of problem (A.6) is given by Proposition 19.

Proposition 19 *If the venture capitalist offers the contract, the optimal contract is the same as the case in which $\alpha \in \{0, 1\}$.*

Proposition 19 says that if the venture capitalist has the bargaining power at the beginning of the game, the contract she offers is the same contract as in the discrete case. The intuition is the same: Given that the venture capitalist does not need to secure a minimum payoff for the entrepreneur, she always retains control and chooses the direction in order to incentivize the entrepreneur to work if needed.

A.3 Proofs

A.3.1 Chapter 1

Proposition 1. This result is implied by Proposition 2 and 3.

Lemma 1. This result is implied by Proposition 2 and 3.

Proposition 2 and Proposition 3. The formal description of the equilibrium for a general N is the following:

If $\theta \leq \left(0, \underline{\theta}\right]$:

- **Prosecutor:** only investigates at $n = 1$. If $y = h$, she discloses it and offers $x = h$. If $y \in \{e, d\}$, she does not disclose it and offers $x = dq^{\frac{1}{N}}$ for all n.
- **Guilty defendant:** If disclosure of $y = h$; $\mu^G(x) = 1$ if $x \leq h$, and $\mu^G(x) = 0$ otherwise. If no-disclosure of $y = h$; $\mu^G(x) = 1$ if $x \leq d$, and $\mu^G(x) = 0$ otherwise for all n.
- **Innocent defendant:** If disclosure of $y = e$; $\mu^I(x) = 1$ if $x \leq 0$, and $\mu^I(x) = 0$ otherwise. If no-disclosure of $y = e$; $\mu^I(x) = 1$ if $x \leq dq^{\frac{1}{N}}$ and $\mu^I(x) = 0$ otherwise for all n.

If $\theta \in \left(\underline{\theta}, q\right]$:

- **Prosecutor:** investigates every period. If $y = h$, she discloses it and offers $x = h$. If $y \in \{e, d\}$, she does not disclose it and offers $x = h$ if $n \in \{1, ..., N-1\}$, and $x = dq$ if $n = N$.
- **Guilty defendant:** If disclosure of $y = h$; $\mu^G(x) = 1$ if $x \leq h$, and $\mu^G(x) = 0$ otherwise. If no-disclosure of $y = h$; $\mu^G(x) = 1$ if $x \leq (1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq$, and $\mu^G(x) = 0$ otherwise for $n \in \{1, ..., N-1\}$, and $\mu^G(x) = 1$ if $x \leq d$, and $\mu^G(x) = 0$ otherwise if $n = N$.
- **Innocent defendant:** If disclosure of $y = e$; $\mu^I(x) = 1$ if $x \leq 0$, and $\mu^I(x) = 0$ otherwise. If no-disclosure of $y = e$; $\mu^I(x) = 1$ if $x \leq dq^{\frac{n}{N}}$ and $\mu^I(x) = 0$ otherwise for all n.

If $\theta \in \left(q, \bar{\theta}\right]$:

- **Prosecutor:** investigates every period. If $y = h$, she discloses it and offers $x = h$. If $y \in \{e, d\}$, she does not disclose it and offers $x = (1 - q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}} dq$ if $n = 1$, $x = h$ if $n \in \{1, ..., N-1\}$, and $x = dq$ if $n = N$.
- **Guilty defendant:** If disclosure of $y = h$; $\mu^G(x) = 1$ if $x \leq h$, and $\mu^G(x) = 0$ otherwise. If no-disclosure of $y = h$; $\mu^G(x) = 1$ if $x < (1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}} dq$, $\mu^G(x) = \frac{\theta - \bar{\theta}}{\theta(1-\bar{\theta})}$ if $x = (1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}} dq$ and $\mu^G(x) = 0$ otherwise for $n \in \{1, ..., N-1\}$, and $\mu^G(x) = 1$ if $x \leq d$, and $\mu^G(x) = 0$ otherwise if $n = N$.
- **Innocent defendant:** If disclosure of $y = e$; $\mu^I(x) = 1$ if $x \leq 0$, and $\mu^I(x) = 0$ otherwise. If no-disclosure of $y = e$; $\mu^I(x) = 1$ if $x \leq dq^{\frac{n}{N}}$ and $\mu^I(x) = 0$ otherwise for all n.

If $\theta \in \left(\bar{\theta}, 1\right)$:

- **Prosecutor:** investigates every period. If $y = h$ or $y = e$, she discloses it and offers $x = h$ or $x = 0$ respectively. If $y = d$, she offers $x = h$ if $n \in \{1, ..., N-1\}$, and $x = d$ if $n = N$.
- **Guilty defendant:** If disclosure of $y = h$; $\mu^G(x) = 1$ if $x \leq h$, and $\mu^G(x) = 0$ otherwise. If no-disclosure of $y = h$; $\mu^G(x) = 1$ if $x \leq (1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}} dq$, and $\mu^G(x) = 0$ otherwise for $n \in \{1, ..., N-1\}$, and $\mu^G(x) = 1$ if $x \leq d$, and $\mu^G(x) = 0$ otherwise if $n = N$.
- **Innocent defendant:** $\mu^I(x) = 1$ if $x \leq 0$, and $\mu^I(x) = 0$ otherwise for all n.

Remark: if the prosecutor hides exculpatory evidence, the innocent defendant rejects $x = d$ with probability one. Also, his belief about the prosecutor type is $P^I(d - \text{type} \mid x < d) = 0$ and $P^I(d - \text{type} \mid x = d) = 1$. This implies if the prosecutor deviates and offer $x \in (0, d)$, the innocent defendant rejects the offer because she is better off going to trial.

Proof: note that for every period n it is not possible to have $y = h$ at the beginning of the period on the equilibrium path. This is because the prosecutor discloses it and offer $x = y$ at the end of the period that she gets it. Also, for every period after the first one $n \leq 2$ it is not possible that the prosecutor's belief about the defendant's type belongs to the interval $(0, \underline{\theta}^N]$ or $(\bar{\theta}, \tilde{\theta}^N]$. This is because on the equilibrium path, in the first period the prosecutor either ends the game if $(0, \underline{\theta}^N]$ or updates her belief to $\theta' = \bar{\theta}$ if $\big(\bar{\theta}, \tilde{\theta}^N]$.

For this section consider the following notation: y_f^n represents the evidence that the prosecutor has after the investigation at period n.

I. Last period before trial ($n = N$): Suppose $n = N$. Following on-path strategies the prosecutor can only have $y^N \in \{e, d\}$ at the beginning of the period. If $y_f^N = h$; to disclose it and to offer $x^N = h$, and $\mu^G(h) = 1$ is an equilibrium. The guilty defendant's continuation punishment if he rejects $x = h$ is $-v^G = h$, so he is indifferent.

Note if $\mu^G(h) < 1$, the prosecutor best response is to offer $x = h - \epsilon$, with arbitrarily small ϵ. The guilty defendant is strictly better off accepting than rejecting, therefore he accepts $\mu^G(x = h - \epsilon) = 1$. This equilibrium is payoff equivalent to accept $x = h$ with probability 1 if $\epsilon \to 0$.

Disclosing $x = h$ is the best response. The prosecutor's continuation value if she discloses $x = h$ is $v^P = h$. If the prosecutor does not disclose $y = h$, the guilty defendant's belief about the prosecutor's type is $P_G(\text{d}-\textit{type} \mid \textit{no-disclosure}\ ) = 1$, it implies the guilty defendant does not accept anything higher than $x = d$ that gives the prosecutor a continuation value $v^P = d$.

Consider $\theta \in (\underline{\theta}^N, \bar{\theta}^N]$. If $y_f^N = d$, the continuation punishments are $-v^G = d$ and $-v^I = dq$. The prosecutor's optimal offer is either $x = -v^I$ such that both defendant's types accept it $\mu^I(dq) = \mu^G(dq) = 1$, or $x = -v^G$ such that only the guilty defendant accepts it $\mu^I(d) = 0, \mu^G(d) = 1$. The prosecutor is better off offering the innocent defendant's continuation punishment, because it brings her an expected payoff of: qd that is larger than θd when $\theta \leq \bar{\theta}$.

If $\mu^I(dq) < 1$, the prosecutor's best response is to offer $x = dq - \epsilon$, with small ϵ. The innocent defendant is strictly better off accepting than rejecting, therefore he accepts with probability $\mu^G(dq - \epsilon) = 1$. This equilibrium is payoff equivalent to accept $x = dq$ with probability 1 if $\epsilon \to 0$.

If the outcome of the investigation is $y_f^N = e$, disclosing it gives the prosecutor a continuation payoff of $v^P = 0$, because the innocent defendant's continuation punishment is zero. If she does not disclose it she can offer $x = dq$ that is accepted by the innocent defendant.

Consider now $\theta \in (\tilde{\theta}^N, 1)$. Suppose $y_f^N = d$. Continuation punishments are $-v^G = d$ and $-v^I = dq$. The prosecutor's optimal offer is either $x = -v^I$ such that both defendant's types accept it with $\mu^I(dq) = \mu^G(dq) = 1$, or $x = -v^G$ such that only guilty defendant accepts it $\mu^I(d) = 0, \mu^G(d) = 1$. The prosecutor is better off offering the guilty defendant's continuation punishment, because it brings her an expected payoff of: θd instead of dq when $\theta > \bar{\theta}$. Note this implies the prosecutor offers $x = d$ and guilty defendant accepts it.

Note if $\mu^G(d) < 1$, the prosecutor's best response if to offer $x = d - \epsilon$, with small ϵ. The guilty defendant is strictly better off accepting than rejecting, therefore he accepts $\mu^G(d - \epsilon) = 1$. This equilibrium is payoff equivalent to accept $x = d$ with probability 1 if $\epsilon \to 0$.

If $y_f^N = e$, disclosing it is an equilibrium. It cannot be an equilibrium where a e-type prosecutor can successfully hide evidence and get a conviction higher than zero. If the prosecutor has evidence $y = d$ she will offer $x = d$ because every other offer is strictly dominated. Therefore if there is no disclosure and the innocent defendant gets an offer $x \in (0, d)$, he will update his belief about the prosecutor type to $P_I(y = d \,|x \in (0, d)) = 0$, because otherwise she would have offered $x = d$, and then the innocent defendant will reject the offer.

If the prosecutor does not disclose e and offers $x = d$, the equilibrium has to be such that the innocent defendant rejects the offer with a high probability. Note that at the interim period after the no-disclosure and before the offer, the innocent defendant's belief about the prosecutor's type is $P_I(y = d \mid \textit{no-disclosure}\ ) = 1$, therefore she is indifferent between accepting or rejecting $x = d$. It cannot be an equilibrium a probability of accepting $x = d$ at $n = N$ such that $\mu^I(d)d - (1 - \mu^I(d))c > 0$, because if that is the case the prosecutor will never disclose $x = e$, and then the innocent defendant's continuation punishment will be $-v^I = dq$, and then he will deviate to $\mu^I(d) = 0$. Therefore the equilibrium is such that $\mu^I(d)d - (1-\mu^I(d))c \leq 0$ or $\mu^I(d) \leq \frac{c}{c+d}$, that induces the prosecutor to disclose $x = e$. Every $\mu^I(d) \leq \frac{c}{c+d}$ is payoff equivalent to $\mu^I(d) = 0$.

Note that it cannot be that $x = d - \epsilon$ is accepted with probability 1, because if that is the case, the prosecutor can hide evidence $y = e$. This is known by the innocent defendant and his continuation punishment would be $-v^I = -dq$, therefore she rejects every offer higher than dq. However, if prosecutor offer $x = dq$, the defendant knows that the prosecutor has evidence $y = e$ and rejects any offer higher than $x = 0$. This is a contradiction. The equilibrium must be that innocent defendant probability of accepting x is $\mu^I(x) = 0$ if $x > 0$. If the innocent defendant updates his belief to $P_I(y = d \mid x < d) = 0$ when receiving an offer $x < d$, therefore he has no incentives to deviates to accept $x > 0$.

The prosecutor does not disclose and offers $x = 0$ is also an equilibrium, however is payoff equivalent and qualitatively the same that to disclose $y = e$, because at the moment of getting the offer $x = 0$, the innocent defendant knows that prosecutor has evidence $y = e$, therefore is a revelation trough signaling.

Note if $\mu^I(x = 0) < 1$, the prosecutor best response is to offer $x = -\epsilon$, with small ϵ. The innocent defendant is strictly better off accepting than rejecting, therefore he accepts $\mu^G(x = -\epsilon) = 1$. This equilibrium is payoff equivalent to accept $x = 0$ with probability 1 if $\epsilon \to 0$.

Finally, note that if at the beginning of the period $y^N = d$, the prosecutor will investigates, otherwise $v^P = dq$ instead if $v^P = \theta(1 - q^{\frac{1}{N}})h + (1 - \theta(1 - q^{\frac{1}{N}}))dq$ if she investigates and $\theta \leq \bar{\theta}$. And $v^P = d$ instead if $v^P = \theta((1 - q^{\frac{1}{N}})h + q^{\frac{1}{N}}dq$ if she investigates and $\theta > \bar{\theta}$.

II. Intermediate periods ($1 < n < N$):

Suppose $n \in (1, N)$. Following on-path strategies the prosecutor can only have $y^n_f \in \{e, d\}$. Suppose the prosecutor investigates and gets evidence $y = h$; to disclose it and offer $x^n = h$, and $\mu^G(x = h) = 1$ is an equilibrium. The guilty defendant's continuation punishment if he rejects $x = h$ is $-v^G = h$, so he is indifferent.

Note if $\mu^G(h) < 1$ and there is a rejection, the prosecutor can offer $x = h$ again at period $n+1$. If this repeats until $n = N$, the prosecutor best response is offer $x = h - \epsilon$, with small ϵ. The guilty defendant is strictly better off accepting than rejecting, therefore he accepts $\mu^G(x = h - \epsilon) = 1$. This equilibrium is payoff equivalent to accept $x = h$ with probability 1 if $\epsilon \to 0$.

Case $\theta \in (\underline{\theta}^N, \bar{\theta}]$: Suppose now $y_f^n = d$. The continuation punishments are $-v^G = (1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq$ and $-v^I = dq$. The prosecutor's continuation payoffs is $v^P = \theta\big((1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq\big) + (1-\theta)dq$.

The highest offer such that both defendant types accept is $qd^{\frac{n}{N}}$. The guilty defendant accepts it with probability 1. The innocent defendant also accepts it with probability 1; if the innocent defendant rejects it, the next period prosecutor's belief will be $\theta = 0$, therefore starting next period prosecutor is not going to investigate anymore, it implies innocent defendant's expected punishment is $qd^{\frac{n}{N}}$. Note the guilty defendant is indifferent between to accept it and to reject it, so he does not deviate. This is the offer that generates the highest payoff for the prosecutor such that both defendant types accept. This is not a profitable deviation from offering x such that both defendant types reject, because $\theta\big((1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq\big) + (1-\theta)dq$ is higher than $qd^{\frac{n}{N}}$ when $\theta > \underline{\theta}^N$. Any higher offer is rejected by the innocent defendant because $dq^{\frac{n}{N}}$ is the highest continuation punishment that the innocent defendant can have, that it is reached when the prosecutor does not investigates any further period.

The highest offer that only the guilty defendant accepts is his highest continuation punishment, that is reached when the prosecutor investigates all the remaining periods. The value is equal to: $x = (1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq$. If the prosecutor makes that offer, it cannot be an equilibrium that the guilty defendant accepts it with probability 1; this is because if that happens the prosecutor will update her belief to $\theta = 0$ if the defendant deviates. Therefore, it cannot be an equilibrium. If the prosecutor accepts with probability $\mu^G < 1$ such that if there is a rejection and the prosecutor updates her belief to a θ such that the optimal action after a rejection includes less investigation, then the prosecutor will deviates and always rejects the offer. It can be that the prosecutor accepts with probability $\mu^G < 1$ such that the prosecutor still investigates every period in the continuation game, in this case the prosecutor is indifferent between to make the offer $x = (1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq$, to make a higher offer such that both types reject, or offer $x = (1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq$ and the guilty defendant rejects it. All those strategies are payoff equivalent. Therefore I consider the case when the prosecutor makes an offer $x > (1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq$ such that both types reject.

If $y_f^n = e$ the prosecutor equilibrium strategy is mimic a d-type prosecutor. The relevant deviation to check this is an equilibrium is not to mimic the d-type prosecutor. If the prosecutor discloses $x = e$ or offers $x < -v^G$, the innocent defendant updates $P_I(y = d \mid x < -v^G) = 0$, that gives a lower expected continuation payoff for the prosecutor. Therefore, to mimic a d-type prosecutor is an equilibrium.

The prosecutor investigates at the beginning of n is an equilibrium, because it gives him an expected payoff of $\theta\Big(1-q^{\frac{N+1-n}{N}}\Big)h + \Big(1-\theta(1-q^{\frac{N+1-n}{N}})\Big)dq$ that is higher than the one shot no-investigation payoff $\theta(1-q^{\frac{N-n}{N}})h + (1-\theta(1-q^{\frac{N-n}{N}}))dq$.

<u>Case $\theta \in (\tilde{\theta}^N, 1)$:</u> Consider $\theta \in (\tilde{\theta}^N, 1)$. Suppose $y_f^n = d$. The continuation punishments are $-v^G = (1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}d$ and $-v^I = dq$. The prosecutor continuation value is $v^P = \theta\Big((1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}))d\Big)$.

The equilibrium strategy for the prosecutor is to offer $x > (1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}d$ such that is rejected by both defendant types. If she deviates to offer x such that both types accepts, she has to offer $qd^{\frac{n}{N}}$ as analyzed above. This is not a profitable deviation because $qd^{\frac{n}{N}}$ is lower than $\theta\Big((1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}))d\Big)$ when $\theta > \tilde{\theta}^N$:

$$\begin{aligned}
\theta\Big((1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}))d\Big) &\geq \tilde{\theta}^N\Big((1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}))d\Big) \\
&> \tilde{\theta}^N\Big((1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}))dq\Big) + (1-\tilde{\theta})dq \\
&> qd^{\frac{n}{N}}
\end{aligned}$$

If the prosecutor offer x such that guilty defendant accepts with probability $\mu^G(x) < 1$ and innocent defendant rejects it, and prosecutor updates her belief to θ' such that she does not investigates at least one period in the continuation game, then the guilty defendant will always rejects and the prosecutor will be worse off. If θ' is such that $\theta' \in (\underline{\theta}^N, \bar{\theta}]$ the prosecutor is worse off, because expected payoff will $\theta\big((1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}} dq\big) + (1-\theta)dq$ that is lower than $\theta\big((1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}} d\big)$ for $\theta > \tilde{\theta}$. If $\mu^G > 0$ induces a $\theta' > \tilde{\theta}$, the expected payoff is the same than to offer $x > (1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}} d$ and get rejected by both defendant types. Therefore, there is no profitable deviation.

If $y_f^n = e$, the equilibrium strategy for the prosecutor is to disclose it and offer $x = 0$. If the prosecutor deviates to no-disclosure and offer $x = 0$, it brings the same payoffs than disclose it and offer $x = 0$, therefore is not a profitable deviation. If the prosecutor does not disclose and offer $x \in (0, d]$ the innocent defendant will reject it, because he will updated his belief to $P_I(y = d \mid \text{no-disclosure and } x \in (0, d]) = 0$ given that a d-type the prosecutor never offers less than d.

The prosecutor investigates at the beginning of n is an equilibrium, because it gives him an expected payoff of $\theta\Big((1 - q^{\frac{N+1-n}{N}})h + q^{\frac{N+1-n}{N}} d\Big)$ that is higher than the one shot no-investigation payoff $\theta\Big((1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}} d\Big)$.

III. First period ($n = 1$):

Suppose the prosecutor investigates and gets evidence $y = h$. To disclose it and offer $x = h$, and $\mu^G(h) = 1$ is the equilibrium. There is no profitable deviation: the guilty defendant continuation punishment if he rejects $x = h$ is $-v^G = h$, so he is indifferent.

Note if $\mu^G(h) < 1$ and realization of random decision is rejection, the prosecutor can offer $x = h$ again at later periods $n > 1$. If this repeats until $n = N$, the prosecutor best response if offer $x = h - \epsilon$, with small ϵ. The guilty defendant is strictly better off accepting than rejecting, therefore he accepts $\mu^G(x = h - \epsilon) = 1$. This equilibrium is payoff equivalent to accept $x = h$ with probability 1 if $\epsilon \to 0$.

To disclose $x = h$ is best response. The prosecutor continuation value if she discloses

$x = h$ is $v^P = h$. If the prosecutor does not disclose $x = h$, $P_G(y = d \mid \textit{no-disclosure}\ ) = 1$, it implies the guilty defendant does not accept anything higher than $x = d$. In that case the prosecutor makes an offer that is rejected for sure. At $n + 1$ the prosecutor discloses $y = h$. If the prosecutor never discloses, her continuation value $v^P = d$ that is lower than h. As before, the prosecutor is indifferent between disclose $x = h$ at n or at $n + 1$; I assume the prosecutor discloses it as soon as she gets it.

Case $\theta \in (0, \underline{\theta}^N]$: The equilibrium is to investigate the first period and offer $x = dq^{\frac{1}{N}}$ if $y \in \{e, d\}$. Both defendant's type accept. The prosecutor expected continuation payoff at the moment of making the offer is $dq^{\frac{1}{N}}$. The innocent defendant and the guilty defendant expected punishment are: $-v^I = dq^{\frac{1}{N}}$ and $-v^G = dq^{\frac{1}{N}}$.

The guilty defendant does not deviate to rejection because if the prosecutor observes a rejection she will updates her belief to $\theta' = 0$ and never investigates and offer $x = dq^{\frac{1}{N}}$. Therefore, the guilty defendant is not better off. The same applies to the innocent defendant, if there is a rejection the prosecutor is not going to investigates and she will offer $x = dq^{\frac{1}{N}}$ next period.

Note that it is not possible to have a different belief than $\theta' = 0$ when there is a rejection, because if $\theta' > 0$ the prosecutor will investigates at least one more period, that bring an expected payoff of at least $(1 - q^{\frac{1}{N}})h + q^{\frac{1}{N}} dq^{\frac{2}{N}}$ to her. This is larger than $dq^{\frac{1}{N}}$, therefore the guilty defendant will reject with probability one. Thus, it is not possible to have $\theta' > 0$.

The prosecutor does not deviates to offer less than $x = dq^{\frac{1}{N}}$ because it brings her a lower payoff. If $x > dq^{\frac{1}{N}}$ the innocent defendant will reject it because $dq^{\frac{1}{N}}$ is his highest expected punishment when there is no further investigation, so he will never accepts something larger. It cannot be that the guilty defendant accepts it with probability one, because in that case $\theta' = 0$ and the prosecutor will decrease the offer in later periods, therefore the guilty defendant is better of rejecting it. If the guilty defendant accepts it with probability $\mu^G < 1$ such that there is investigation is future periods the prosecutor is worse off, because if there is at least one more investigation her payoff will be at most $v^P = \theta\Big((1 - q^{\frac{1}{N}})h + q^{\frac{1}{N}} dq^{\frac{2}{N}}\Big) +$

$(1-\theta)dq^{\frac{2}{N}}$ that is lower than $dq^{\frac{1}{N}}$ when $\theta \leq \tilde{\theta}^N$. Therefore, there is not profitable deviation.

The prosecutor investigates is an equilibrium, because otherwise she gets $dq^{\frac{1}{N}}$ instead of $\theta\Big((1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}dq^{\frac{1}{N}}\Big)(1-\theta)dq^{\frac{1}{N}}$ that is larger than $dq^{\frac{1}{N}}$ when $\theta \leq \tilde{\theta}^N$.

Case $\theta \in (\underline{\theta}^N, \bar{\theta}]$: The analysis of this case is the same one that $\theta \in (\underline{\theta}^N, \bar{\theta}]$ when $n \in (1, N)$.

Case $\theta \in (\bar{\theta}, \tilde{\theta}^N]$: The equilibrium is to investigate the first period and offer $x = (1-q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}}dq$ if $y \in \{e, d\}$. The guilty defendant accepts with probability $\mu^G(x) = \frac{\theta-\bar{\theta}}{\theta(1-\bar{\theta})}$. The innocent defendant rejects it. The prosecutor updates belief to $\theta' = \bar{\theta}$.

The guilty defendant does not deviate because if he rejects she gets the same expected payoff. If the innocent defendant accepts he is worse off.

The prosecutor does not deviate; if $x < (1-q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}}dq$ the guilty defendant accepts for sure but payoff is lower. Also, it cannot be an equilibrium, because if guilty defendant accepts for sure, then the prosecutor does not investigates anymore because $\theta' = 0$; therefore the guilty defendant deviates to rejection. If $x \in [(1-q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}}dq, (1-q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}}d)$ the guilty defendant rejects given that the continuation punishment when $\theta = \bar{\theta}$ is lower. If $\theta' > \bar{\theta}$ then the prosecutor accepts; however it is not profitable for the prosecutor when $\theta \leq \tilde{\theta}$. Also, it cannot be an equilibrium because if the guilty defendant accepts for sure, then the prosecutor does not investigates anymore because $\theta' = 0$; therefore the guilty defendant deviates to rejection. If $x \geq (1-q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}}d)$ and $\theta = \bar{\theta}$ if rejection; the guilty defendant always rejects and the prosecutor is worse off.

If the prosecutor delays the offer x such that $\theta = \bar{\theta}$, for some values of θ she will be indifferent but for others she will be worse. Suppose the prosecutor delays the offer x to period $n > 1$, at n there were n investigations, so the offer that makes the guilty defendant indifferent is $x = (1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq$. The prosecutor is willing to make this offer if:

$$\theta\Big((1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}dq\Big) + (1-\theta)dq \geq \Big((1-q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}d\Big)$$

$$\iff \quad \theta \leq \frac{q^{\frac{N-n}{N}}}{q^{\frac{N-n}{N}} + q(1-q^{\frac{N-n}{N}})} \equiv \tilde{\theta}'$$

Note that $\tilde{\theta}' < \tilde{\theta}^N$ for $n > 1$. This implies if the prosecutor waits until period n, she is not going to skim the guilty defendant if $\theta \in (\tilde{\theta}', \tilde{\theta}^N]$. For values $\theta \in (\bar{\theta}, \tilde{\theta}']$ the prosecutor gets the same payoff making the offer x at the first period or waiting until n. For values $\theta \in (\tilde{\theta}', \tilde{\theta}^N]$ the prosecutor is worse off waiting until n, because her payoff making the offer at the first period is: $\theta\Big((1 - q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}} dq\Big) + (1-\theta)dq$ that is larger than wait until n, where the payoff is $\Big((1 - q^{\frac{N-}{N}})h + q^{\frac{N-1}{N}} d\Big)$.

In conclusion, the prosecutor does not deviates and offer $x = (1 - q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}} dq$ at period 1.

Investigation the first period is an equilibrium, because otherwise her payoff is: $\theta\Big((1 - q^{\frac{N-1}{N}})h + q^{\frac{N-1}{N}} dq\Big) + (1-\theta)dq$ instead off $\theta\Big((1-q)h + qdq\Big) + (1-\theta)dq$ from the ex ante perspective at the beginning of period 1.

Case $\theta \in (\tilde{\theta}^N, 1)$: The analysis of this case is the same one that $\theta \in (\tilde{\theta}^N, 1)$ when $n \in (1, N)$.

Proposition 4. For $\theta \in (\underline{\theta}, \bar{\theta}]$, the prosecutor makes an offer that is accepted by both defendant's type at $t = T$, therefore the game ends for sure at T. Note that at $T - \epsilon$ the game ends only if the prosecutor gets $y = h$, it happens with probability $1 - e^{-\lambda\frac{T-\epsilon}{T}}$ if $\theta \in (\underline{\theta}, q]$, and $1 - e^{-\lambda\frac{T-\epsilon}{T}}(1 - \mu^G)$ if $\theta \in (q, \bar{\theta}]$.

For $\theta \in (\bar{\theta}, 1]$, at $t = T$ the d-type prosecutor makes an offer that is rejected by the innocent defendant, therefore there is no mass point. If the defendant is guilty, he accepts the offer that the d-type prosecutor makes at $t = T$. Note that at $T - \epsilon$ the game ends if the defendant is guilty only if the prosecutor gets $y = h$, it happens with probability $1 - e^{-\lambda\frac{T-\epsilon}{T}}$ if $\theta \in (\bar{\theta}, 1]$.

Proposition 5. If $\theta \leq \frac{d}{h}$: At period n the prosecutor expected payoff is: $v^P = d$. If the prosecutor one-shot deviates at n, her payoff will be:

$$\theta((1 - q^{\frac{1}{N}})h + q^{\frac{1}{N}} d) + (1-\theta) q^{\frac{1}{N}} d$$

This is larger than d if: $\theta > \frac{d}{h}$, but $\theta \leq \frac{d}{h}$ by assumption. It is not a profitable deviation.

If $\theta > \frac{d}{h}$: At period n the prosecutor expected payoff is:

$$\theta((1 - q^{\frac{N-n+1}{N}})h + q^{\frac{N-n+1}{N}}d) + (1-\theta)q^{\frac{N-n+1}{N}}d$$

If the prosecutor one-shot deviates at n, her payoff will be:

$$\theta((1 - q^{\frac{N-n}{N}})h + q^{\frac{N-n}{N}}d) + (1-\theta)q^{\frac{N-n}{N}}d$$

The deviation payoff is larger than no-deviation if $\theta < \frac{d}{h}$, however $\theta > \frac{d}{h}$ by assumption. Therefore, there is no profitable deviation.

Proposition 6. Considering the best case of always-investigation $N \to \infty$, the prosecutor's payoffs are:

$$u^P = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}] \\ \theta\Big[(1-q)h + qdq\Big] + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}, \tilde{\theta}] \\ \theta\Big[(1-q)h + qd\Big] & \text{if } \theta \in (\tilde{\theta}, 1) \end{cases}$$

while the Brady Rule payoff are:

$$u^P = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{BR}] \\ \theta\Big((1-q)h + dq\Big) + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}^{BR}, 1) \end{cases}$$

Note $\underline{\theta}^{BR} < \underline{\theta}$ given that $\frac{d}{h} < \frac{d}{h-dq}$. For $\theta \leq \underline{\theta}^{BR}$ the prosecutor payoff are the same. For $\theta \in (\underline{\theta}^{BR}, \underline{\theta}]$ the prosecutor payoff for no-investigation is d that is larger than the prosecutor payoff of always-investigation $\theta\Big((1-q)h + dq\Big) + (1-\theta)dq$ when $\theta > \underline{\theta}^{BR}$.

For $\theta \in (\underline{\theta}, \tilde{\theta}]$, the prosecutor payoff with Brady Rule is: $\theta\Big((1-q)h + dq\Big) + (1-\theta)dq$

that is larger than $\theta\Big[(1-q)h + qdq\Big] + (1-\theta)dq$. For $\theta \in (\tilde{\theta}, 1)$, Brady Rule payoff is $\theta\Big((1-q)h + dq\Big) + (1-\theta)dq$ that is larger than $\theta\Big((1-q)h + dq\Big) + (1-\theta)dq$ when $\theta < 1$.

The innocent defendant expected payoff for the innocent defendant under no Brady Rule is:

$$u^I = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}] \\ dq & \text{if } \theta \in (\underline{\theta}, 1] \end{cases}$$

and with Brady Rule is:

$$u^I = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{BR}] \\ dq & \text{if } \theta \in (\underline{\theta}^{BR}, 1) \end{cases}$$

For $\theta < \underline{\theta}^{BR}$ the expected payoff are the same. For $\theta \in (\underline{\theta}^{BR}, \underline{\theta}]$ under Brady Rule expected payoff is dq that is smaller than no Brady Rule expected payoff d. For $\theta \in (\underline{\theta}^{BR}, 1)$ the expected payoff are the same.

The guilty defendant expected payoff for the innocent defendant under no Brady Rule is:

$$u^G = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}] \\ (1-q)\cdot h + q \cdot dq & \text{if } \theta \in (\underline{\theta}, \tilde{\theta}] \\ (1-q)\cdot h + q \cdot d & \text{if } \theta \in (\tilde{\theta}, 1) \end{cases}$$

and with Brady Rule is:

$$u^G = \begin{cases} d & \text{if } \theta \in (0, \underline{\theta}^{BR}] \\ (1-q)h + qd & \text{if } \theta \in (\underline{\theta}^{BR}, 1) \end{cases}$$

For $\theta < \underline{\theta}^{BR}$ the expected payoff are the same. For $\theta \in (\underline{\theta}^{BR}, \underline{\theta}]$ under Brady Rule the expected payoff is $(1-q)h + qd$ that is larger than no Brady Rule expected payoff d.

For $\theta \in (\underline{\theta}, \tilde{\theta}]$, the guilty defendant expected payoff under Brady Rule is $(1-q)h + qd$ that is larger than expected payoff with no Brady Rule $(1-q)h + qdq$. For $\theta \in (\tilde{\theta}, 1)$, the expected punishment are the same.

Proposition 18. Call $\mu_n^I(x)$ the probability that the innocent defendant accepts the offer x at period n. The prosecutor expected payoff of no-deviation is:

$$\theta d + (1-\theta) d\mu^I_{1,..,N}$$

where $\mu^I_{1,..,N}$ is the probability of accepting $x = d$ at any period between 1 and N. This reflects that the prosecutor always offers $x = d$ if she has evidence $y = d$.

For $\mu^I_N \in [0, \frac{c}{d+c})$: If the prosecutor one-shot deviates at period $n = 1$, her payoff will be:

$$\theta\Big((1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}d\Big) + (1-\theta)\Big((1-q^{\frac{1}{N}})d\mu^I_{1,..,N-1} + q^{\frac{1}{N}}d\mu^I_{1,..,N}\Big)$$

where $\mu^I_{1,..,N-1}$ is the probability the prosecutor accepts d in any of the periods from 1 to $N-1$. This probability reflects the fact that the prosecutor best strategy if she gets $y = e$ at period $n = 1$ is offer $x = d$ every period until $n = N-1$. If the defendant-defendant rejects $x = d$ at $n = N-1$, the prosecutor discloses $y = e$ at N.

The prosecutor is not going to deviate after period $n = 1$, because if the game has not ended it is because the defendant is innocent. Therefore, investigating is a strictly dominated strategy.

The prosecutor is better off deviating if:

$$\theta > \frac{d(\mu^I_{1,..,N} - \mu^I_{1,..,N-1})}{h - d(1-\mu^I_{1,..,N} + \mu^I_{1,..,N-1})} \qquad \Longleftrightarrow \qquad \theta > \frac{d\tilde{\mu}^I_1}{h - d(1-\tilde{\mu}^I_1)}$$

where $\tilde{\mu}^I_n = \mu^I_{n,..,N} - \mu^I_{n,..,N-1} = (1-\mu^I_n)(1-\mu^I_{n+1})\cdots(1-\mu^I_{N-1})\mu^I_N$ is the probability of the innocent defendant accepts $x = d$ at period $n = N$. Then $\frac{d\tilde{\mu}^I_n}{h-d(1-\tilde{\mu}^I_n)} = \underline{\theta}^N$, therefore, if $\theta \leq \underline{\theta}^N$ the prosecutor is better off no deviating.

Note that deviations after first period are also not profitable, because the prosecutor deviates at n if $\theta^n > \frac{d\tilde{\mu}_n^I}{h-d(1-\tilde{\mu}_n^I)}$, and note that $\underline{\theta}^N < \frac{d\tilde{\mu}_n^I}{h-d(1-\tilde{\mu}_n^I)}$ because $\tilde{\mu}_1^I < \tilde{\mu}_n^I$, therefore the prosecutor does not deviates for $\theta \leq \underline{\theta}^N$.

For $\mu_N^I \in [\frac{c}{d+c}, 1]$: If the prosecutor one-shot deviates at period n, her payoff will be:

$$\theta\Big((1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}d\Big) + (1-\theta)\Big((1-q^{\frac{1}{N}})\Big(d\mu_{1,..,N}^I - c(1-\mu_{1,..,N}^I)\Big) + q^{\frac{1}{N}}d\mu_{1,..,N}^I\Big)$$

In this case the prosecutor offers $x = d$ in every period, include period $n = N$, even if she gets $y = e$.

The prosecutor is not going to deviate after period $n = 1$, because if the game has not ended it is because the defendant is innocent. Therefore, investigating is a strictly dominated strategy.

The prosecutor is better off deviating if:

$$\theta > \frac{(1-\mu_{1,..,N}^I)c}{h-d+(1-\mu_{1,..,N}^I)c}$$

Note $\frac{(1-\mu_{1,..,N}^I)c}{h-d+(1-\mu_{1,..,N}^I)c} = \underline{\theta}^N$, therefore, if $\theta \leq \underline{\theta}^N$ the prosecutor is better off no deviating.

Lemma 8. Expected payoffs are:

$$u^P = \begin{cases} \theta d + (1-\theta)\mu^I(d)d & \text{if } \theta \in (0, \underline{\theta}] \\ \theta\Big[(1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}dq^{\frac{1}{N}}\Big] + (1-\theta)dq^{\frac{1}{N}} & \text{if } \theta \in (\underset{\sim}{\theta}, \underline{\theta}^N] \\ \theta\Big[(1-q)h + qdq\Big] + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}^N, \tilde{\theta}^N] \\ \theta\Big[(1-q)h + qd\Big] & \text{if } \theta \in (\tilde{\theta}^N, 1) \end{cases}$$

$$u^P = \begin{cases} \theta\Big[(1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}dq^{\frac{1}{N}}\Big] + (1-\theta)dq^{\frac{1}{N}} & \text{if } \theta \in (0, \underline{\theta}^N] \\ \theta\Big[(1-q)h + qdq\Big] + (1-\theta)dq & \text{if } \theta \in (\underline{\theta}^N, \tilde{\theta}^N] \\ \theta\Big[(1-q)h + qd\Big] & \text{if } \theta \in (\tilde{\theta}^N, 1) \end{cases}$$

If $\theta > \underline{\theta}^N$ payoffs are the same.

<u>For $\mu_N^I \leq \frac{c}{c+d}$:</u> The prosecutor payoff when $\theta \leq \underset{\sim}{\theta}$ is:

$$\theta d + (1-\theta)\mu^I(d)d$$

Note if N increases $\underset{\sim}{\theta}$ will change, because there are more probability of accepting $x = d$ μ_n^I. Given $\mu_n^I < 1 \ \forall n \in [0, N]$, the value of $\underset{\sim}{\theta}$ is always lower than 1, no matter the number or periods. This implies $\theta d + (1-\theta)\mu^I(d)d < d$. Therefore, for any sequence $s = \{\mu_1^I, \mu_2^I, ...\}$ of extra probabilities of acceptance when N increases, $\lim_{N\to\infty} \theta d + (1-\theta)\mu^I(d)d = d - \upsilon_s$. Therefore, given that:

$$\lim_{N\to\infty} \theta\Big[(1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}dq^{\frac{1}{N}}\Big] + (1-\theta)dq^{\frac{1}{N}} = d$$

Therefore $\exists N^*$ such that $\forall N \geq N^*$, $\theta\Big[(1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}dq^{\frac{1}{N}}\Big] + (1-\theta)dq^{\frac{1}{N}} \geq d - \upsilon_s$.

<u>For $\mu_N^I > \frac{c}{c+d}$:</u> The prosecutor's payoff when $\theta \leq \underset{\sim}{\theta}$ is:

$$\theta d + (1-\theta)\mu^I(d)d$$

$\underset{\sim}{\theta}$ is the probability of being accepted at some period between the first one and the last one. $\lim_{N\to\infty} \underset{\sim}{\theta} \leq 1$. If the limit is 1, then $\lim_{N\to\infty} \theta d + (1-\theta)\mu^I(d)d = d$, but $\lim_{N\to\infty} \underset{\sim}{\theta} = 0$. The prosecutor always gets the payoff of no-investigation.

If $\lim_{N\to\infty} \underset{\sim}{\theta} < 1$, then $\lim N \to \infty \theta d + (1-\theta)\mu^I(d)d = d - \upsilon_S$ and $\lim_{N\to\infty} \underset{\sim}{\theta} > 0$. Therefore $\exists N^*$ such that $\forall N \geq N^*$, $\theta\Big[(1-q^{\frac{1}{N}})h + q^{\frac{1}{N}}dq^{\frac{1}{N}}\Big] + (1-\theta)dq^{\frac{1}{N}} \geq d - \upsilon_s$.

Lemma 9. For values of $\theta > \underline{\theta}^{\text{Public}}$, the proof of Lemma 9 is the same than the proof of Proposition 2 and 3 in Appendix A.3.1.

For $\theta < \underline{\theta}^{\text{Public}}$, the prosecutor payoff is $v^P = d$. If the prosecutor deviates at any period $n \in [1, N]$, the defendant is going to observes it and therefore her payoff is going to be:

$$\theta\Big((1 - q^{\frac{1}{N}})h + q^{\frac{1}{N}} dq^{\frac{1}{N}}\Big) + (1 - \theta)dq^{\frac{1}{N}}$$

The best strategy for the prosecutor is to disclose $y = h$ and hide $y = e$. On the other hand, the innocent defendant continuation punishment if there is only one investigation is $q^{\frac{1}{N}} d$.

The Prosecutor is better off deviating if: $\theta > \frac{d}{h - dq^{\frac{1}{N}}}$. However $\frac{d}{h - dq^{\frac{1}{N}}} > \frac{d}{h - dq}$, therefore the prosecutor is worse off deviating.

Lemma 10. The prosecutor's expected payoff with mandatory disclosure of evidence is:

$$v^P_{BR} = \theta\Big[(1 - q^G)\pi^G h + dq^G\Big] + (1 - \theta)\Big[(1 - q^I)\pi^I h + dq^I\Big]$$

1. If $\theta \leq \theta^*$, the prosecutor hides exculpatory evidence and offer dq^I if $y \in \{e, d\}$. The prosecutor payoff is:

$$v^P_{\theta \leq \theta^*} = \theta\Big[(1-q^G)\big(\pi^G h + (1-\pi^G)dq^I\big) + q^G dq^I\Big] + (1-\theta)\Big[(1-q^I)\big(\pi^I h + (1-\pi^I)dq^I\big) + q^I dq^I\Big]$$

Note that $(1 - q^G)\pi^G h + dq^G > (1 - q^G)\big(\pi^G h + (1 - \pi^G)dq^I\big) + q^G dq^I$ for $q^G > q^I$, and $(1 - q^I)\pi^I h + dq^I > (1 - q^I)\big(\pi^I h + (1 - \pi^I)dq^I\big) + q^I dq^I$.

2. If $\theta \in (\theta^*, \theta^{**}]$, the prosecutor discloses exculpatory evidence and offer dq^G if $y = d$. The

prosecutor payoff is:

$$v^P_{\theta\in(\theta^*,\theta^{**}]} = \theta\Big[(1-q^G)\big(\pi^G h\big) + q^G dq^G\Big] + (1-\theta)\Big[(1-q^I)\big(\pi^I h\big)\Big]$$

Note that $(1-q^G)\pi^G h + dq^G > (1-q^G)\big(\pi^G h\big) + q^G dq^G$ for $q^G > q^I$, and $(1-q^I)\pi^I h + dq^I > (1-q^I)\big(\pi^I h\big)$.

3. If $\theta > \theta^{**}$, the prosecutor discloses exculpatory evidence and offer dq^G if $y \in \{e, d\}$, that is rejected by the innocent defendant. The prosecutor payoff is:

$$v^P_{\theta>\theta^{**}} = \theta\Big[(1-q^G)\big(\pi^G h + (1-\pi^G)dq^G\big) + q^G dq^G\Big] + (1-\theta)\Big[(1-q^I)\big(\pi^I h + (1-\pi^I)(-c)\big)\Big]$$

Note that $(1-q^G)\pi^G h + dq^G > (1-q^G)\big(\pi^G h + (1-\pi^G)dq^G\big) + q^G dq^G$ for $q^G > q^I$, and $(1-q^I)\pi^I h + dq^I > (1-q^I)\big(\pi^I h + (1-\pi^I)(-c)\big)$.

A.3.2 Chapter 2

Proposition 7. The optimal contract that entrepreneur offers depend on M and c in the following way:

- If $\gamma \geq \bar{\gamma}$, entrepreneur offers $(\alpha = 1, \beta = 1 - \frac{M}{\underline{p}}, e = 1)$ for c and M such that:

  $$M \leq \bar{p} - \tfrac{c}{(1-\gamma)}\tfrac{\bar{p}}{\underline{p}} \quad \text{and} \quad M \leq \tfrac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}, \text{ or}$$

  $$M \geq \bar{p} - \tfrac{c}{(1-\gamma)}\tfrac{\bar{p}}{\underline{p}}, \quad M \leq \underline{p} - \tfrac{c}{(1-\gamma)}\tfrac{\underline{p}}{\bar{p}} \quad \text{and} \quad M \leq \tfrac{\bar{p}\underline{p}}{\bar{p}^2-\underline{p}^2}(\bar{p} - \gamma\underline{p} - c)$$

  offer is $(\alpha = 1, \beta = 1 - \frac{M}{\gamma\underline{p}}, e = 0)$ if:

  $$M \geq \underline{p} - \tfrac{c}{(1-\gamma)}\tfrac{\underline{p}}{\bar{p}} \quad \text{and} \quad M \leq \gamma\tfrac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$$

  offer is $(\alpha = 0, \beta = 1 - \frac{M}{\bar{p}}, e = 1)$ if:

  $$M \geq \underline{p} - \tfrac{c}{(1-\gamma)}\tfrac{\underline{p}}{\bar{p}} \quad \text{and} \quad M \leq \bar{p} - \tfrac{c}{(1-\gamma)}\tfrac{\bar{p}}{\underline{p}}$$

  and offer is $(\alpha = 0, \beta = 1 - \frac{M}{\gamma\bar{p}}, e = 0)$ otherwise.

- If $\gamma < \bar{\gamma}$, entrepreneur offers $(\alpha = 1, \beta = 1 - \frac{M}{\underline{p}}, e = 1)$ for c and M such that:

  $$M \leq \bar{p} - \tfrac{c}{(1-\gamma)}\tfrac{\bar{p}}{\underline{p}} \quad \text{and} \quad M \leq \tfrac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}, \text{ or}$$

  $$M \geq \bar{p} - \tfrac{c}{(1-\gamma)}\tfrac{\bar{p}}{\underline{p}}, \quad M \leq \underline{p} - \tfrac{c}{(1-\gamma)}\tfrac{\underline{p}}{\bar{p}} \quad \text{and} \quad M \leq \tfrac{\bar{p}\underline{p}}{\bar{p}^2-\underline{p}^2}(\bar{p} - \gamma\underline{p} - c)$$

  offer is $(\alpha = 1, \beta = 1 - \frac{M}{\gamma\underline{p}}, e = 0)$ if:

  $$M \geq \underline{p} - \tfrac{c}{(1-\gamma)}\tfrac{\underline{p}}{\bar{p}} \quad \text{and} \quad M \leq \gamma\tfrac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$$

  offer is $(\alpha = 0, \beta = 1 - \frac{M}{\bar{p}}, e = 1)$ if:

  $$M \geq \underline{p} - \tfrac{c}{(1-\gamma)}\tfrac{\underline{p}}{\bar{p}} \quad \text{and} \quad M \leq \bar{p} - \tfrac{c}{(1-\gamma)}\tfrac{\bar{p}}{\underline{p}}, \text{ or}$$

  $$M \leq \underline{p} - \tfrac{c}{(1-\gamma)} \quad \text{and} \quad M \leq \tfrac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$$

offer is $(\alpha \in \{0,1\}, \beta = 1 - \frac{M}{\underline{p}}, e = 1)$ if:

$$M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}, \quad M \geq \underline{p} - \frac{c}{(1-\gamma)}, \quad M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}} \quad \text{and} \quad M \geq \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}, \text{ or}$$

$$M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}} \quad , \quad M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}} \quad \text{and} \quad M \geq \frac{\bar{p}\underline{p}}{\bar{p}^2-\underline{p}^2}(\bar{p} - \gamma\underline{p} - c)$$

offer is $(\alpha = 0, \beta = 1 - \frac{M}{\gamma\bar{p}}, e = 0)$ if:

$$M \geq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}, \quad M \geq \gamma\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}} \quad \text{and} \quad M \leq \gamma\bar{p}$$

Part 1: consider $\gamma \geq \bar{\gamma}$. Suppose first entrepreneur keeps control $\alpha = 1$, he always chooses his desired direction. Suppose further entrepreneur plans to choose effort $e = 1$; VC's IR is: $\beta\underline{p} \geq M$. If IR binds then $\beta = 1 - \frac{M}{\underline{p}}$. This is a SPE if:

- $\beta \geq \underline{\beta}$ or $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$
- it is feasible: $1 - \frac{M}{\underline{p}} \geq 0$ or $\underline{p} \geq M$ that holds by assumption.

Suppose now E plans to choose effort $e = 0$; VC's IR is: $\beta\gamma\underline{p} \geq= M$. If IR binds then $\beta = 1 - \frac{M}{\gamma\underline{p}}$. This is a SPE if:

- $\beta \leq \underline{\beta}$ or $M \geq \gamma\underline{p} - \frac{c}{(1-\gamma)}\frac{\gamma\underline{p}}{\bar{p}}$
- it is feasible: $1 - \frac{M}{\gamma\underline{p}} \geq 0$ or $\gamma\underline{p} \geq M$.

Note that if $M > \gamma\underline{p}$, $e = 0$ is not feasible, while $e = 1$ is feasible only if $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$.

If $M \leq \gamma\underline{p}$ and $M > \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$ only $e = 0$ is feasible. If $M \leq \gamma\underline{p}$ and $M \geq \gamma\underline{p} - \frac{c}{(1-\gamma)}\frac{\gamma\underline{p}}{\bar{p}}$ only $e = 1$ is feasible.

The only parameters under which $e = 0$ and $e = 1$ are both feasible are when $M \leq \gamma\underline{p}$, and $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$ and $M \geq \gamma\underline{p} - \frac{c}{(1-\gamma)}\frac{\gamma\underline{p}}{\bar{p}}$. In this case $\Pi^{\alpha=1}_{e=1} \geq \Pi^{\alpha=1}_{e=1}$ if:

$$(1 - \frac{M}{\underline{p}})\bar{p} - c \geq (1 - \frac{M}{\gamma\underline{p}})\gamma\bar{p} \quad \iff \quad c \leq \bar{p}(1-\gamma)$$

The set of parameter when $e = 0$ and $e = 1$ are both feasible always satisfies $c \leq \bar{p}(1-\gamma)$, therefore in that area $e = 1$ is always optimal.

Suppose now E gives up control $\alpha = 1$. Suppose E plans to choose effort $e = 1$; VC's IR is: $\beta\bar{p} \geq= M$. If IR binds then $\beta = 1 - \frac{M}{\bar{p}}$. This is a SPE if:

- $\beta \geq \bar{\beta}$ or $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$.
- it is feasible: $1 - \frac{M}{\bar{p}} \geq 0$ or $\bar{p} \geq M$ that holds by assumption.

Suppose now E plans to choose effort $e = 0$; VC's IR is: $\beta\gamma\bar{p} \geq= M$. If IR binds then $\beta = 1 - \frac{M}{\gamma\bar{p}}$. This is a SPE if:

- $\beta \leq \bar{\beta}$ or $M \geq \gamma\bar{p} - \frac{c}{(1-\gamma)}\frac{\gamma\bar{p}}{\underline{p}}$
- it is feasible: $1 - \frac{M}{\gamma\bar{p}} \geq 0$ or $\gamma\bar{p} \geq M$ that holds by assumption.

Note that if $M > \gamma\bar{p}$, $e = 0$ is not feasible, while $e = 1$ is feasible only if $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$.

If $M \leq \gamma\bar{p}$ and $M > \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$ only $e = 0$ is feasible. If $M \leq \gamma\bar{p}$ and $M \geq \gamma\bar{p} - \frac{c}{(1-\gamma)}\frac{\gamma\bar{p}}{\underline{p}}$ only $e = 1$ is feasible.

The only parameters under which $e = 0$ and $e = 1$ are both feasible are when $M \leq \gamma\bar{p}$, and $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$ and $M \geq \gamma\bar{p} - \frac{c}{(1-\gamma)}\frac{\gamma\bar{p}}{\underline{p}}$. In this case $\Pi^{\alpha=0}_{e=1} \geq \Pi^{\alpha=0}_{e=1}$ if:

$$(1 - \frac{M}{\bar{p}})\underline{p} - c \geq (1 - \frac{M}{\gamma\bar{p}})\gamma\underline{p} \quad \iff \quad c \leq \underline{p}(1-\gamma)$$

The set of parameter when $e = 0$ and $e = 1$ are both feasible always satisfies $c \leq \underline{p}(1-\gamma)$, therefore in that area $e = 1$ is always optimal.

Now, comparing $\alpha = 1$ with $\alpha = 0$:

- If $M > \gamma\underline{p}$, $M > \gamma\bar{p} - \frac{c}{(1-\gamma)}\frac{\gamma\bar{p}}{\underline{p}}$, and $M > \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$ the only feasible contract is $(\alpha = 0, e = 0)$.

- If $M > \gamma \underline{p}$, $M \leq \gamma \bar{p} - \frac{c}{(1-\gamma)} \frac{\gamma \bar{p}}{\underline{p}}$, and $M > \underline{p} - \frac{c}{(1-\gamma)} \frac{\underline{p}}{\bar{p}}$ there is no feasible contract including $\alpha = 1$. Therefore the optimal contract is $(\alpha = 0, e = 1)$.

- If $M \leq \gamma \bar{p} - \frac{c}{(1-\gamma)} \frac{\gamma \bar{p}}{\underline{p}}$, and $M \leq \underline{p} - \frac{c}{(1-\gamma)} \frac{\underline{p}}{\bar{p}}$ the two possible optimal contracts are $(\alpha = 1, e = 1)$ and $(\alpha = 0, e = 1)$. $(\alpha = 1, e = 1)$ is optimal if:

$$\Pi_{e=1}^{\alpha=1} \geq \Pi_{e=0}^{\alpha=1} \quad \Longleftrightarrow \quad \Big(1 - \frac{M}{\underline{p}}\Big)\bar{p} - c \geq \Big(1 - \frac{M}{\bar{p}}\Big)\underline{p} - c \quad \Longleftrightarrow \quad M < \frac{\bar{p}\underline{p}}{\bar{p} + \underline{p}}$$

- If $M > \gamma \bar{p} - \frac{c}{(1-\gamma)} \frac{\gamma \bar{p}}{\underline{p}}$, and $M \leq \underline{p} - \frac{c}{(1-\gamma)} \frac{\underline{p}}{\bar{p}}$ the two possible optimal contracts are $(\alpha = 1, e = 1)$ and $(\alpha = 0, e = 0)$. $(\alpha = 1, e = 1)$ is optimal if:

$$\Pi_{e=1}^{\alpha=1} \geq \Pi_{e=0}^{\alpha=0} \iff \Big(1 - \frac{M}{\underline{p}}\Big)\bar{p} - c \geq \Big(1 - \frac{M}{\gamma \bar{p}}\Big)\gamma \underline{p} \iff M < \frac{\bar{p}\underline{p}}{\bar{p}^2 - \underline{p}^2}(\bar{p} - \gamma \underline{p} - c)$$

- If $M \leq \gamma \underline{p}$, and $M > \underline{p} - \frac{c}{(1-\gamma)} \frac{\underline{p}}{\bar{p}}$ the two possible optimal contracts are $(\alpha = 1, e = 0)$ and $(\alpha = 0, e = 0)$. $(\alpha = 1, e = 0)$ is optimal if:

$$\Pi_{e=1}^{\alpha=0} \geq \Pi_{e=0}^{\alpha=0} \quad \Longleftrightarrow \quad \Big(1 - \frac{M}{\gamma \underline{p}}\Big)\gamma \bar{p} \geq \Big(1 - \frac{M}{\gamma \bar{p}}\Big)\gamma \underline{p} \quad \Longleftrightarrow \quad M < \gamma \frac{\bar{p}\underline{p}}{\bar{p} + \underline{p}}$$

<u>Part 2:</u> consider $\gamma < \bar{\gamma}$. If E chooses $\alpha = 1$, the analysis of Part 1 holds. Suppose E gives up control $\alpha = 0$; if E plans to choose a contract such that VC chooses her preferred direction and E chooses effort $e = 1$, then VC's IR is: $\beta \bar{p} \geq M$. If IR binds then $\beta = 1 - \frac{M}{\bar{p}}$. This is a SPE if:

- $\beta \geq \bar{\beta}$ or $M \leq \bar{p} - \frac{c}{(1-\gamma)} \frac{\bar{p}}{\underline{p}}$.

- it is feasible: $1 - \frac{M}{\bar{p}} \geq 0$ or $\bar{p} \geq M$ that holds by assumption.

If E plans to choose a contract such that VC chooses E's preferred direction and E chooses effort $e = 1$, then VC's IR is: $\beta \underline{p} \geq M$. If IR binds then $\beta = 1 - \frac{M}{\underline{p}}$. This is a SPE if:

- $\beta \geq \underline{\beta}$ or $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$.
- $\beta \leq \bar{\beta}$ or $M \geq \bar{p} - \frac{c}{(1-\gamma)}$.
- it is feasible: $1 - \frac{M}{\underline{p}} \geq 0$ or $\underline{p} \geq M$ that holds by assumption.

Suppose now E plans to choose effort $e = 0$; VC's IR is: $\beta\gamma\bar{p} \geq M$. If IR binds then $\beta = 1 - \frac{M}{\gamma\bar{p}}$. This is a SPE if:

- $\beta \leq \underline{\beta}$ or $M \geq \gamma\bar{p} - \frac{c}{(1-\gamma)}\gamma$
- it is feasible: $1 - \frac{M}{\gamma\bar{p}} \geq 0$ or $\gamma\bar{p} \geq M$ that holds by assumption.

Note that if $M > \gamma\bar{p}$, $M \geq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$, and $M \geq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, there is no feasible contract. If $M \leq \gamma\bar{p}$, and $M \geq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, $(\alpha = 0, e = 0)$ is the only feasible contract.

If 1) $M \geq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$ and $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$; and 2) $M \leq \underline{p} - \frac{c}{(1-\gamma)}$ and $M \geq \gamma\bar{p}$; and 3) $M \leq \gamma\bar{p} - \frac{c}{(1-\gamma)}\gamma$ and $M \leq \underline{p} - \frac{c}{(1-\gamma)}$, the contract $(\alpha = 0, e = 1)$ is the only feasible.

If $M \geq \gamma\bar{p} - \frac{c}{(1-\gamma)}\gamma$ and $M \leq \gamma\bar{p}$, the feasible contracts are $(\alpha = 0, e = 1)$ and $(\alpha = 0, e = 0)$. $(\alpha = 0, e = 1)$ is optimal if:

$$\Pi_{e=1}^{\alpha=0} \geq \Pi_{e=0}^{\alpha=0} \iff \Big(1 - \frac{M}{\bar{p}}\Big)\underline{p} - c \geq \Big(1 - \frac{M}{\gamma\bar{p}}\Big)\gamma\underline{p} - c \iff c \leq \underline{p}(1-\gamma)$$

which always holds.

If $M \geq \underline{p} - \frac{c}{(1-\gamma)}$, $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, and $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$, optimal contract is $(\alpha = 0, e = 1)$ where VC chooses her desires direction, or $(\alpha \in \{0,1\}, e = 1)$ where VC chooses E's desired direction.$(\alpha = 0, e = 1)$ where VC chooses her desires direction is optimal if:

$$\Pi_{e=1}^{\alpha=0} \geq \Pi_{e=1}^{\alpha\in\{0,1\}} \iff \Big(1 - \frac{M}{\bar{p}}\Big)\underline{p} - c \geq \Big(1 - \frac{M}{\underline{p}}\Big)\bar{p} - c \iff M \geq \frac{\bar{p}\underline{p}}{\bar{p} + \underline{p}}$$

If $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, and $M \geq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$, and $M \geq \gamma\bar{p}$ optimal contract is $(\alpha = 0, e = 1)$, $(\alpha \in \{0,1\}, e = 1)$ is the only feasible contract.

If $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, and $M \geq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$, and $M < \gamma\bar{p}$ optimal contract is $(\alpha = 0, e = 0)$ or $(\alpha \in \{0,1\}, e = 1)$ where VC chooses E's desired direction. $(\alpha = 0, e = 0)$ is optimal if:

$$\Pi_{e=0}^{\alpha=0} \geq \Pi_{e=1}^{\alpha\in\{0,1\}} \iff \Big(1-\frac{M}{\gamma\bar{p}}\Big)\gamma\underline{p} \geq \Big(1-\frac{M}{\underline{p}}\Big)\bar{p}-c \iff M \geq \frac{\bar{p}\underline{p}}{\bar{p}^2-\underline{p}^2}(\bar{p}-\gamma\underline{p}-c)$$

Then,

- $(\alpha = 0, e = 1)$ is optimal if (1) $M \leq \underline{p} - \frac{c}{(1-\gamma)}$ or if (2) $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$ and $M \leq \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$, or (3) $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$ and $M \geq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$.
- $(\alpha \in \{0,1\}, e = 1)$ is optimal if (1) $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, $M \geq \underline{p} - \frac{c}{(1-\gamma)}$, $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$, and $M \geq \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$, or (2) $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, $M \geq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$, and $M \geq \frac{\bar{p}\underline{p}}{\bar{p}^2-\underline{p}^2}(\bar{p} - \gamma\underline{p} - c)$
- $(\alpha = 0, e = 0)$ is optimal if (1) $M \geq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, $M \geq \underline{p} - \frac{c}{(1-\gamma)}$ and $M \leq \gamma\bar{p}$, or (2) $M \leq \underline{p} - \frac{c}{(1-\gamma)}$ and $M \leq \gamma\bar{p}$, $M \leq \frac{\bar{p}\underline{p}}{\bar{p}^2-\underline{p}^2}(\bar{p} - \gamma\underline{p} - c)$, and $M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$.

Now, under the parameter where $\alpha = 1$ and $\alpha = 0$ are both feasible, consider the following comparison:

- If $M \leq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$, $(\alpha = 1, e = 1)$ is better than $(\alpha = 0, e = 1)$ if

  $$\Pi_{e=1}^{\alpha=1} \geq \Pi_{e=1}^{\alpha=0} \iff \Big(1-\frac{M}{\underline{p}}\Big)\bar{p}-c \geq \Big(1-\frac{M}{\bar{p}}\Big)\underline{p}-c \iff M < \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$$

  $(\alpha = 1, e = 1)$ is better than $(\alpha \in \{0,1\}, e = 1)$ if

  $$\Pi_{e=1}^{\alpha=1} \geq \Pi_{e=1}^{\alpha\in\{0,1\}} \iff \Big(1-\frac{M}{\underline{p}}\Big)\bar{p}-c \geq \Big(1-\frac{M}{\underline{p}}\Big)\bar{p}-c$$

  which is always true. $(\alpha = 1, e = 1)$ is better than $(\alpha = 0, e = 0)$ if

  $$\Pi_{e=1}^{\alpha=1} \geq \Pi_{e=0}^{\alpha=0} \iff \Big(1-\frac{M}{\underline{p}}\Big)\bar{p}-c \geq \Big(1-\frac{M}{\gamma\bar{p}}\Big)\gamma\underline{p} \iff M < \frac{\bar{p}\underline{p}}{\bar{p}^2-\underline{p}^2}(\bar{p}-\gamma p-c)$$

- If $M \geq \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{p}}{\bar{p}}$ and $M \leq \gamma\underline{p}$, $(\alpha = 1, e = 0)$ is better than $(\alpha = 0, e = o)$ if

$$\Pi_{e=0}^{\alpha=1} \geq \Pi_{e=0}^{\alpha=0} \iff \left(1 - \frac{M}{\gamma\underline{p}}\right)\gamma\bar{p} \geq \left(1 - \frac{M}{\gamma\bar{p}}\right)\gamma\underline{p} \iff M < \gamma\frac{\bar{p}\underline{p}}{\bar{p} + \underline{p}}$$

Proposition 8. Consider an small increment of $\Delta = \bar{p} - \underline{p}$ to $\Delta + 2\epsilon = (\bar{p} + \epsilon) - (\underline{p} - \epsilon)$ such that $\bar{p} + \underline{p}$ remains constant. From Proposition 7 the values $\bar{M}$, $\bar{c}$ and $\underline{c}$ decrease with Δ.

Proposition 9. consider an small increment of $\Delta = \bar{p} - \underline{p}$ to $\Delta + 2\epsilon = (\bar{p} + \epsilon) - (\underline{p} - \epsilon)$ such that $\bar{p} + \underline{p}$ remains constant. If entrepreneur offers the contract:

1. If the optimal contract is $(\alpha = 1, e = 1)$, if Δ increases, entrepreneur's payoff increases if $M < \frac{\underline{p}^2}{\bar{p} + \underline{p}}$.

2. If the optimal contract is $(\alpha \in \{0, 1\}, e = 1)$, if Δ increases, entrepreneur's payoff decreases.

3. If the optimal contract is $(\alpha = 1, e = 0)$, if Δ increases, entrepreneur's payoff increases if $M < \gamma\frac{\underline{p}^2}{\bar{p} + \underline{p}}$.

4. If the optimal contract is $(\alpha = 0, e = 0)$, if Δ increases, entrepreneur's payoff increases if $M > \gamma\frac{\bar{p}^2}{\bar{p} + \underline{p}}$.

Proof:

- If optimal contract is $\alpha = 1, \beta = 1 - \frac{\underline{p}}{M}$, such that entrepreneur makes effort $e = 1$, entrepreneur payoff is $\Pi = (1 - \frac{\underline{p}}{M})\bar{p} - c$, therefore, if Δ increases to $\Delta + 2\epsilon$:

$$\left(1 - \frac{M}{(\underline{p} - \epsilon)}\right)(\bar{p} + \epsilon) - c > \left(1 - \frac{M}{\underline{p}}\right)\bar{p} - c \iff M < \frac{\underline{p}(\underline{p} - \epsilon)}{\bar{p} + \underline{p}}$$

Note $\frac{\underline{p}(\underline{p}-\epsilon)}{\bar{p}+\underline{p}} < \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$. Therefore, if the optimal contract is $\alpha = 1, \beta = 1 - \frac{\underline{p}}{M}$, it is possible that entrepreneur is better off or worse off depending on M.

- If optimal contract is $\alpha \in \{0,1\}, \beta = 1 - \frac{\underline{p}}{M}$, the same above analysis applies. Because $\frac{\underline{p}(\underline{p}-\epsilon)}{\bar{p}+\underline{p}} < \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$, and $\alpha \in \{0,1\}, \beta = 1 - \frac{\underline{p}}{M}$ is only possible when $M > \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$, entrepreneur is always worse off.

- If optimal contract is $\alpha = 0, \beta = 1 - \frac{\gamma\bar{p}}{M}$, such that entrepreneur does not make effort, entrepreneur payoff is $\Pi = (1 - \frac{\gamma\bar{p}}{M})\gamma\underline{p}$, therefore, if Δ increases to $\Delta + 2\epsilon$:

$$\left(1 - \frac{M}{\gamma(\bar{p}+\epsilon)}\right)\gamma(\underline{p}-\epsilon) > \left(1 - \frac{M}{\gamma\bar{p}}\right)\gamma\underline{p} \quad \Longleftrightarrow \quad M > \gamma\frac{\bar{p}(\bar{p}+\epsilon)}{\bar{p}+\underline{p}}$$

  Note $\gamma\frac{\bar{p}(\bar{p}+\epsilon)}{\bar{p}+\underline{p}} > \gamma\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$. Therefore, if the optimal contract is $\alpha = 0, \beta = 1 - \frac{\gamma\bar{p}}{M}$, entrepreneur is better off or worse off depending on M.

- If optimal contract is $\alpha = 1, \beta = 1 - \frac{\gamma\underline{p}}{M}$, such that entrepreneur does not make effort, entrepreneur payoff is $\Pi = (1 - \frac{\gamma\underline{p}}{M})\gamma\bar{p}$, therefore, if Δ increases to $\Delta + 2\epsilon$:

$$\left(1 - \frac{M}{\gamma(\underline{p}-\epsilon)}\right)\gamma(\bar{p}+\epsilon) > \left(1 - \frac{M}{\gamma\underline{p}}\right)\gamma\bar{p} \quad \Longleftrightarrow \quad M < \gamma\frac{\underline{p}(\underline{p}-\epsilon)}{\bar{p}+\underline{p}}$$

  Note $\gamma\frac{\underline{p}(\underline{p}-\epsilon)}{\bar{p}+\underline{p}} < \gamma\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$. Therefore, if the optimal contract is $\alpha = 0, \beta = 1 - \frac{\gamma\underline{p}}{M}$, entrepreneur is better off or worse off depending on M.

Proposition 10. The optimal contract that venture capitalist offers depend on M and c in the following way:

- If $\gamma \geq \frac{\underline{p}}{\bar{p}+\underline{p}}$, venture capitalist offers $(\alpha = 0, \beta = \bar{\beta}, e = 1)$ for c and M such that:

$$M \leq \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}} \quad \text{and} \quad c \leq \underline{p}(1-\gamma)^2$$

  and offer $(\alpha = 0, \beta = 0, e = 0)$ otherwise.

- If $\gamma < \frac{\underline{p}}{\bar{p}+\underline{p}}$, venture capitalist offers $(\alpha = 0, \beta = \bar{\beta}, e = 1)$ for c and M such that:

$$M \le \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}} \quad \text{and} \quad c \le (1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$$

venture capitalist offers $(\alpha \in \{0,1\}, \beta = \underline{\beta}, e = 1)$ for c and M such that:

$$M \le \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{\underline{p}}}{\bar{p}} \quad \text{and} \quad c \ge (1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}} \quad \text{and} \quad c \le (\underline{p} - \gamma\bar{p})(1-\gamma)\frac{\bar{p}}{p}$$

venture capitalist offers $(\alpha = 0, \beta = 0, e = 0)$ for c and M such that:

$$M \le \gamma\bar{p} \quad \text{and} \quad c \ge (\underline{p} - \gamma\bar{p})(1-\gamma)\frac{\bar{p}}{p}$$

Proof:

Part 1: consider $\gamma \ge \underline{\gamma}$. Suppose VC keeps control $\alpha = 0$ and wants to incentive E to exert effort $e = 1$. Therefore $\beta = \bar{\beta} = \frac{c}{\underline{p}(1-\gamma)}$. To be a feasible β it has to be $\beta \le 1$ or $c \le \underline{p}(1-\gamma)$. Also the expected payoff has to be at least M, or $M \le \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$.

If VC wants E to not make effort $e = 0$, the optimal β is $\beta = 0$. This is a feasible contract if the payoff is at least M, or $\gamma\bar{p} \ge M$ that always is true because $\gamma\bar{p} \ge \underline{p}$.

Both $e = 1$ and $e = 0$ are feasible contract when $M \le \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$ and $\gamma\bar{p} \ge M$. In this case $e = 1$ is better if:

$$\Pi^{\alpha=0}_{e=1} \ge \Pi^{\alpha=0}_{e=0} \quad \Longleftrightarrow \quad \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}} \ge \gamma\bar{p} \quad \Longleftrightarrow \quad c \le \underline{p}(1-\gamma)^2$$

Note the area $M \le \underline{p}$ and $c \le \underline{p}(1-\gamma)^2$ is a subset of the area $M \le \bar{p} - \frac{c}{(1-\gamma)}\frac{\bar{p}}{\underline{p}}$. Therefore $\Pi^{\alpha=0}_{e=1} \ge \Pi^{\alpha=0}_{e=0}$ if $c \le \underline{p}(1-\gamma)^2$ and $\Pi^{\alpha=0}_{e=1} < \Pi^{\alpha=0}_{e=0}$ otherwise.

Suppose now VC gives up control $\alpha = 1$ and wants to implement $e = 1$. Optimal β is $\beta = \underline{\beta}$. This is a SPE if $\underline{\beta} \le 1$ or $c \le \bar{p}(1-\gamma)$, and $\Pi^{\alpha=1}_{e=1} = (1-\bar{\beta})\underline{p} \ge M$, or $M \le \underline{p} - \frac{c}{(1-\gamma)}\frac{\underline{\underline{p}}}{\bar{p}}$. If VC wants to implement $e = 0$, optimal β is $\beta = 0$, and $\Pi^{\alpha=1}_{e=0} = \gamma\underline{p}$ has to be at least M.

If $M \le (1-\bar{\beta})\underline{p}$ and $M \le \gamma\underline{p}$ both $e = 1$ and $e = 0$ are feasible. $e = 1$ is optimal if:

$$\Pi^{\alpha=1}_{e=1} \ge \Pi^{\alpha=1}_{e=0} \quad \Longleftrightarrow \quad (1-\bar{\beta})\underline{p} \ge \gamma\underline{p} \quad \Longleftrightarrow \quad \bar{p}(1-\gamma)^2$$

Note the combination of M and c such that $M \leq \gamma \underline{p}$ and $c \leq \bar{p}(1-\gamma)$ is a subset of $M \leq \underline{p} - \frac{c}{(1-\gamma)} \frac{\underline{p}}{\bar{p}}$.

Summing up, if $M > \gamma \underline{p}$ and $M \leq \underline{p} - \frac{c}{(1-\gamma)} \frac{\underline{p}}{\bar{p}}$, $e = 1$ is optimal. There is no feasible contract with $M > \gamma \underline{p}$ otherwise.

If $M \leq \gamma \underline{p}$ and $c \leq \bar{p}(1-\gamma)$, optimal contract is $e = 1$. If $c > \bar{p}(1-\gamma)$ optimal contract is $e = 0$.

Considering options $\alpha = 0$ and $\alpha = 1$. If $c \leq \underline{p}(1-\gamma)$, $\Pi_{e=1}^{\alpha=0}$ is better than $\Pi_{e=1}^{\alpha=1}$ if:

$$\Pi_{e=1}^{\alpha=0} \geq \Pi_{e=1}^{\alpha=1} \quad \Longleftrightarrow \quad c \leq (1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$$

Note that $(1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}} > \underline{p}(1-\gamma)^2$ is $\gamma \geq \bar{\gamma}$, therefore $\Pi_{e=1}^{\alpha=0} \geq \Pi_{e=1}^{\alpha=1}$ in that range of value.

If $c \in [\underline{p}(1-\gamma), \bar{p}(1-\gamma)]$, $\Pi_{e=0}^{\alpha=0}$ is better than $\Pi_{e=1}^{\alpha=1}$ if:

$$\Pi_{e=0}^{\alpha=0} \geq \Pi_{e=1}^{\alpha=1} \quad \Longleftrightarrow \quad c \geq (\underline{p} - \gamma\bar{p})(1-\gamma)\frac{\bar{p}}{\underline{p}}$$

Note $(\underline{p} - \gamma\bar{p})(1-\gamma)\frac{\bar{p}}{\underline{p}} < \underline{p}(1-\gamma)^2$ if $\gamma \geq \bar{\gamma}$, therefore $\Pi_{e=0}^{\alpha=0} \geq \Pi_{e=1}^{\alpha=1}$ in that range of parameters.

Part 2: Consider $\gamma < \frac{\underline{p}}{\bar{p}+\underline{p}}$. If VC keeps the control $\alpha = 0$, the same analysis about optimal conditions under which contract is optimal applies. However in this case $(\underline{p} - \gamma\bar{p})(1-\gamma)\frac{\bar{p}}{\underline{p}} > \underline{p}(1-\gamma)^2 > (1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$.

Considering when contracts are feasible, $(\alpha = 0, e = 1)$ is optimal for $c \leq (1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}-\underline{p}}$,$(\alpha \in \{0,1\}, e = 1)$ is optimal for $c \in \left[(1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}-\underline{p}}, (\underline{p} - \gamma\bar{p})(1-\gamma)\frac{\bar{p}}{\underline{p}}\right]$ and $(\alpha = 0, e = 0)$ for $c > (\underline{p} - \gamma\bar{p})(1-\gamma)\frac{\bar{p}}{\underline{p}}$.

Considering options $\alpha = 0$ and $\alpha = 1$ (same as Part 1). Note if $c \leq \underline{p}(1-\gamma)^2$, $\Pi_{e=1}^{\alpha=0} \geq \Pi_{e=1}^{\alpha=1}$ if $c \leq (1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}$, which always holds.

Note that $\Pi_{e=1}^{\alpha\in\{0,1\}}$ is always equal to $\Pi_{e=1}^{\alpha=1}$, and $\Pi_{e=0}^{\alpha=0} > \Pi_{e=0}^{\alpha=1}$ always.

Proposition 11. consider an small increment of $\Delta = \bar{p} - \underline{p}$ to $\Delta + 2\epsilon = (\bar{p}+\epsilon) - (\underline{p}-\epsilon)$ such that $\bar{p}+\underline{p}$ remains constant. From the values in Proposition 10 $\bar{c}^{VC}$ and $\underline{c}^{VC}$ decrease with Δ.

Proposition 12. Comparative statics: consider an small increment of $\Delta = \bar{p} - \underline{p}$ to $\Delta + 2\epsilon = (\bar{p} + \epsilon) - (\underline{p} - \epsilon)$ such that $\bar{p} + \underline{p}$ remains constant. If the venture capitalist offers the contract:

1. If optimal contract is $(\alpha = 0, \beta = \bar{\beta}, e = 1)$, if Δ increases, venture capitalist's payoff increases if $\beta < \frac{\underline{p}}{\bar{p}+\underline{p}}$.
2. If optimal contract is $(\alpha \in \{0, 1\}, \beta = \underline{\beta}, e = 1)$, if Δ increases, venture capitalist's payoff increases if $\underline{\beta} > \frac{\bar{p}}{\bar{p}+\underline{p}}$.
3. If optimal contract is $(\alpha = 0, \beta = 0, e = 0)$, if Δ increases, venture capitalist's payoff increases.

Proof:

- If $\gamma \geq \frac{\underline{p}}{\bar{p}+\underline{p}}$ and $c < \underline{p}(1-\gamma)^2$, or if $\gamma < \frac{\underline{p}}{\bar{p}+\underline{p}}$ and $c < \frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}(1-\gamma)$, venture capitalist gives $\beta = \bar{\beta}$ to the entrepreneur. Venture capitalist is better off if:

$$\left(1 - \frac{c}{\underline{p}(1-\gamma)}\right)\bar{p} < \left(1 - \frac{c}{(\underline{p}-\epsilon)(1-\gamma)}\right)(\bar{p}+\epsilon) \iff \frac{c}{(\underline{p}-\epsilon)(1-\gamma)} < \frac{\underline{p}}{\bar{p}+\underline{p}}$$
$$\iff \bar{\beta}^{\epsilon} < \frac{\underline{p}}{\bar{p}+\underline{p}}$$

- If $\gamma \geq \frac{\underline{p}}{\bar{p}+\underline{p}}$ and $c > \underline{p}(1-\gamma)^2$, or if $\gamma < \frac{\underline{p}}{\bar{p}+\underline{p}}$ and $c > (\underline{p}-\gamma\bar{p})(1-\gamma)\frac{\bar{p}}{\underline{p}}$, venture capitalist gives $\beta = 0$ to the entrepreneur. Venture capitalist increases her payoff from $\gamma\bar{p}$ to $\gamma(\bar{p}+\epsilon)$.

- If $\gamma < \frac{\underline{p}}{\bar{p}+\underline{p}}$ and $c \in [\frac{\bar{p}\underline{p}}{\bar{p}+\underline{p}}(1-\gamma), (\underline{p}-\gamma\bar{p})(1-\gamma)\frac{\bar{p}}{\underline{p}}$, $\beta = \underline{\beta}$ venture capitalist gives $\beta = \underline{\beta}$ to the entrepreneur. Venture capitalist is better off if:

$$\left(1-\frac{c}{\bar{p}(1-\gamma)}\right)\underline{p} < \left(1-\frac{c}{(\bar{p}+\epsilon)(1-\gamma)}\right)(\underline{p}-\epsilon) \iff \frac{c}{(\bar{p}+\epsilon)(1-\gamma)} > \frac{\bar{p}}{\bar{p}+\underline{p}}$$
$$\iff \underline{\beta}^{\epsilon} > \frac{\bar{p}}{\bar{p}+\underline{p}}$$

Proposition 17. The optimal contract has the following characteristic:

- If c is high enough, entrepreneur keeps $\beta \leq \underline{\beta}$. Entrepreneur chooses $e = 0$.

$$\beta = \min\left\{1-\left(\frac{M}{\gamma(\bar{p}+\underline{p})}\right)^{\frac{1}{2}}, \underline{\beta}\right\} \quad \text{and} \quad \alpha = \frac{1}{\bar{p}-\underline{p}}\left[\bar{p}-\frac{M}{\gamma}\frac{1}{1-\beta}\right]$$

- If c is low, and M is high enough, entrepreneur keeps $\beta \geq \bar{\beta}$. He chooses $e = 1$

$$\beta = \max\left\{1-\left(\frac{M}{\bar{p}+\underline{p}}\right)^{\frac{1}{2}}, \bar{\beta}\right\} \quad \text{and} \quad \alpha = \frac{1}{\bar{p}-\underline{p}}\left[\bar{p}-M\frac{1}{1-\beta}\right]$$

- If $\gamma \geq \bar{\gamma}$, and c and M are low enough, entrepreneur keeps $\beta \in [\underline{\beta}, \bar{\beta}]$. Entrepreneur choose $e = 1$ only if he is elected as proposer.

$$\alpha = \min\left\{\frac{1}{\gamma\bar{p}-\underline{p}}\left[\gamma\bar{p}-\left(\frac{M\gamma(\bar{p}^2-\underline{p}^2)}{(\bar{p}-\gamma\underline{p})-c}\right)^{\frac{1}{2}}\right], 1\right\} \quad \text{and} \quad \beta = 1-\frac{M}{(1-\alpha)\gamma\bar{p}+\alpha\underline{p}}$$

- If $\gamma < \bar{\gamma}$, and c and M are low enough, entrepreneur keeps $\beta \in [\underline{\beta}, \bar{\beta}]$. Investor chooses entrepreneur's desired direction when she is the proposer. Entrepreneur always makes effort.

$$\beta = 1-\frac{M}{\underline{p}} \quad \text{and} \quad \alpha \in [0,1]$$

Proof: This proof is divided in 2 sections, depending on the value of γ.

Section 1: $\gamma \geq \overline{\gamma}$. This proof is divided in 4 parts, the first three are the optimal contract for each cutoff β, and the last one is the proof of the Proposition 17:

<u>Part 1:</u> If $\beta < \underline{\beta}$ Entrepreneur solves:

$$\max_{\alpha\in[0,1],\beta\in[0,\underline{\beta}]} \alpha\Big[\beta\gamma\bar{p}\Big] + (1-\alpha)\Big[\beta\gamma\underline{p}\Big]$$
$$\text{subject to: } \alpha\Big[(1-\beta)\gamma\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\gamma\bar{p}\Big] \geq M$$

The solution of this problem only exists if $(1-\underline{\beta})\gamma\bar{p} \geq M$. The optimal contract depends on the parameters of the problem. The first case is when $\gamma\underline{p} < M$, where $\alpha = 1$ is not possible because even if taking $\beta = 0$ it is not possible to satisfies venture capitalist's IR constraint. Then optimal solution requires some control rights to Investor.

An important feature of the contract is that venture capitalist's IR restriction always binds if the solution exists. This is because the entrepreneur can always increase α or β and be better off.

Lemma 9 *Optimal contract is given by:*

$$\beta = \min\left\{1 - \Big(\frac{M}{\gamma}\frac{\bar{p}-\underline{p}}{\bar{p}\bar{p}-\underline{p}\underline{p}}\Big)^{\frac{1}{2}}, \underline{\beta}\right\}$$
$$\alpha = \frac{1}{\bar{p}-\underline{p}}\Big[\bar{p} - \frac{M}{\gamma}\frac{1}{1-\beta}\Big]$$

Proof. The objective function is concave, the solution is obtained using first order conditions. The feasibility $\alpha \in [0,1]$ and $\beta \in [0,\underline{\beta}]$ is guaranteed given the conditions. ■

The second case when $\gamma\underline{p} \geq M$ gives the possibility of $\alpha = 1$. Note first if $(1-\underline{\beta})\gamma\underline{p} > M$ then venture capitalist's IR never binds and optimal contract is $\alpha = 1$ and $\beta = \underline{\beta}$.

If $(1-\underline{\beta})\gamma\underline{p} \leq M$ the optimal solution is expressed in the following lemma.

Lemma 10 *If $(1-\underline{\beta})\gamma\underline{p} \leq M$, optimal contract has the following form:*

- *If* $1 - \left(\frac{M}{\gamma}\frac{\bar{p}-\underline{p}}{\bar{p}\bar{p}-\underline{p}\underline{p}}\right)^{\frac{1}{2}} < \underline{\beta}$, *then:*

$$\begin{aligned}
\alpha =& \min\left\{\frac{1}{\bar{p}-\underline{p}}\Big[\bar{p} - \Big(\frac{\gamma}{M}\frac{\bar{p}\bar{p}-\underline{p}\underline{p}}{\bar{p}-\underline{p}}\Big)^{\frac{1}{2}}\Big], 1\right\}\\
\beta =& 1 - \frac{M}{\gamma}\left[\bar{p} - \alpha(\bar{p}-\underline{p})\right]
\end{aligned}$$

- *If* $1 - \left(\frac{M}{\gamma}\frac{\bar{p}-\underline{p}}{\bar{p}\bar{p}-\underline{p}\underline{p}}\right)^{\frac{1}{2}} \geq \underline{\beta}$*, then:*

$$\begin{aligned}
\beta =& \underline{\beta}\\
\alpha =& \frac{1}{\bar{p}-\underline{p}}\Big[\bar{p} - \frac{M}{\gamma}\frac{1}{1-\beta}\Big]
\end{aligned}$$

Proof. Use first order condition to solve the problem only considering IR and then adjust if $\alpha > 1$ or $\beta > \underline{\beta}$. ■

Part 2: If $\beta \in [\underline{\beta}, \bar{\beta}]$ the entrepreneur solves:

$$\begin{aligned}
&\max_{\alpha\in[0,1],\beta\in[\underline{\beta},\bar{\beta}]} \alpha\Big[\beta\bar{p} - c\Big] + (1-\alpha)\Big[\beta\gamma\underline{p}\Big]\\
&\text{subject to: } \alpha\Big[(1-\beta)\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\gamma\bar{p}\Big] \geq M
\end{aligned}$$

The solution of this problem only exists and binds if $(1-\underline{\beta})\gamma\bar{p} \geq M \leq (1-\bar{\beta})\underline{p}$. I will only consider that case.

Lemma 11 *Optimal contract is given by:*

$$\begin{aligned}
\alpha &= \min\left\{\frac{1}{\gamma\bar{p}-\underline{p}}\left[\gamma\bar{p} - \left(\frac{M\gamma(\bar{p}\bar{p}-\underline{p}\underline{p})}{(\bar{p}-\gamma\underline{p})-c}\right)^{\frac{1}{2}}\right], 1\right\}\\
\beta &= 1 - \frac{M}{\gamma\bar{p} - \alpha(\gamma\bar{p}-\underline{p})}
\end{aligned}$$

Proof. The objective function is concave, the solution is obtained using first order conditions. The feasibility $\alpha \in [0,1]$ and $\beta \in [\underline{\beta}, \bar{\beta}]$ is guaranteed given the conditions. ■

Part 3: If $\beta > \bar{\beta}$ the entrepreneur solves:

$$\max_{\alpha \in [0,1], \beta \in [\bar{\beta}, 1]} \alpha \Big[\beta \bar{p} - c\Big] + (1-\alpha)\Big[\beta \underline{p} - c\Big]$$
$$\text{subject to: } \alpha\Big[(1-\beta)\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\bar{p}\Big] \geq M$$

The solution of this problem only exists if $\bar{p} \geq M$. As in the previous cases, the optimal contract depends on the parameters of the problem. If $(1-\bar{\beta})\underline{p} < M$, where $\alpha = 1$ is not possible because even if taking $\beta = \bar{\beta}$ it is not possible to satisfies venture capitalist's IR constraint. Then optimal solution requires some control rights to venture capitalist.

An important feature of the contract is that venture capitalist's IR restriction always binds if the solution exists. This is because the entrepreneur can always increase α or β and be better off.

Lemma 12 *Optimal contract is given by:*

$$\beta = \max\left\{1 - \left(M \frac{\bar{p} - \underline{p}}{\bar{p}\bar{p} - \underline{p}\underline{p}}\right)^{\frac{1}{2}}, \bar{\beta}\right\}$$
$$\alpha = \frac{1}{\bar{p} - \underline{p}}\Big[\bar{p} - M\frac{1}{1-\beta}\Big]$$

Proof. The objective function is concave, the solution is obtained using first order conditions. The feasibility $\alpha \in [0,1]$ and $\beta \in [\bar{\beta}, 1]$ is guaranteed given the conditions. ■

When $(1-\bar{p})\underline{p} \geq M$ gives the possibility of $\alpha = 1$. The optimal solution is expressed in the following lemma.

Lemma 13 *If $(1-\bar{p})\underline{p} \geq M$, optimal contract has the following form:*

$$\alpha = \min\left\{\frac{1}{\bar{p} - \underline{p}}\Big[\bar{p} - \Big(\frac{1}{M}\frac{\bar{p}\bar{p} - \underline{p}\underline{p}}{\bar{p} - \underline{p}}\Big)^{\frac{1}{2}}\Big], 1\right\}$$
$$\beta = 1 - M\big[\bar{p} - \alpha(\bar{p} - \underline{p})\big]$$

Proof. Use first order condition to solve the problem only considering IR and then adjust if $\alpha > 1$. ■

Part 4: Proof of Proposition 17 when $\gamma \geq \bar{\gamma}$:

Call $\tilde{\alpha} = \min\left\{\frac{1}{\gamma\bar{p}-\underline{p}}\left[\gamma\bar{p} - \left(\frac{M\gamma(\bar{p}\bar{p}-\underline{p}\underline{p})}{(\bar{p}-\gamma\underline{p})-c}\right)^{\frac{1}{2}}\right], 1\right\}$ and $\tilde{\beta} = 1 - \frac{M}{\gamma\bar{p}-\alpha(\gamma\bar{p}-\underline{p})}$.

Call c_1 to the c such that: $(\bar{p}-\underline{p})\gamma\left[1 - \left(\frac{M}{(\bar{p}-\underline{p})}\right)^{\frac{1}{2}}\right] = \tilde{\beta}\Big[\gamma\underline{p} + \tilde{\alpha}(\bar{p}-\gamma\underline{p})\Big] - \tilde{\alpha}c.$

Call $c_2 = (\bar{p}+\underline{p})\left[(1-\gamma) - \left(\frac{M}{(\bar{p}+\underline{p})}\right)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}\right]$. Then $\tilde{c} = \max\{c_1, c_2\}$.

Call $\tilde{M}$ the M such that: $\tilde{\beta}\Big[\gamma\underline{p} + \tilde{\alpha}(\bar{p}-\gamma\underline{p})\Big] - \tilde{\alpha}c = (\bar{p}-\underline{p})\left[1 - \left(\frac{M}{(\bar{p}-\underline{p})}\right)^{\frac{1}{2}}\right] - c.$

$U_1^E \geq U_2^E$ if and only if $(\bar{p}-\underline{p})\gamma\left[1 - \left(\frac{M}{(\bar{p}-\underline{p})}\right)^{\frac{1}{2}}\right] \geq \tilde{\beta}\Big[\gamma\underline{p} + \tilde{\alpha}(\bar{p}-\gamma\underline{p})\Big] - \tilde{\alpha}c$ and $U_1^E \geq U_3^E$ if and only if $c \geq (\bar{p}+\underline{p})\left[(1-\gamma) - \left(\frac{M}{(\bar{p}+\underline{p})}\right)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}\right]$.

On the other hand, if $c < \tilde{c}$, then $U_2^E \geq U_3^E$ if and only if $\tilde{\beta}\Big[\gamma\underline{p} + \tilde{\alpha}(\bar{p}-\gamma\underline{p})\Big] - \tilde{\alpha}c \leq (\bar{p}-\underline{p})\left[1 - \left(\frac{M}{(\bar{p}-\underline{p})}\right)^{\frac{1}{2}}\right] - c$

Section 2: $\gamma < \overline{\gamma}$**.** This proof is also divided in 4 parts, the first three are the optimal contract for each cutoff β, and the last one is the proof of the Proposition 17 when $\gamma < \bar{\gamma}$:

Part 1: If $\beta < \underline{\beta}$. This is the same proof that Part 1 of the proof of Proposition 17.

Part 2: If $\beta \in [\underline{\beta}, \bar{\beta}]$ the entrepreneur solves:

$$\max_{\alpha\in[0,1],\beta\in[\underline{\beta},\bar{\beta}]} \alpha\Big[\beta\bar{p} - c\Big] + (1-\alpha)\Big[\beta\bar{p} - c\Big]$$

$$\text{subject to: } \alpha\Big[(1-\beta)\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\underline{p}\Big] \geq M$$

α does not play any role given both player are going to choose $x = R$. Optimal β is given by venture capitalist's IR constraint. If $\underline{p}(1-\bar{\beta}) > M$, then optimal β is $\bar{\beta}$ and venture capitalist's IR does not bind. If $\underline{p}(1-\bar{\beta}) - \leq M$, then optimal $\beta = 1 - \frac{M}{\underline{p}}$.

Part 3: If $\beta > \underline{\beta}$. This is the same proof that Part 3 of the proof of Proposition 17.

Part 4: Proof of Proposition 13 when $\gamma < \bar{\gamma}$. Call $\tilde{c} = \max\Big\{\Big(1 - \frac{M}{\underline{p}}\Big)\bar{p} - (\bar{p} + \underline{p})\gamma\Big[1 - \Big(\frac{M}{\gamma(\bar{p}+\underline{p})}\Big)^{12}\Big], (\bar{p}+\underline{p})\Big[(1-\gamma) - \Big(\frac{M}{(\bar{p}+\underline{p})}\Big)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}\Big]\Big\}$ and $\tilde{M}$ the M such that: $M\frac{\bar{p}}{\underline{p}} - \big(M(\bar{p}+\underline{p})\big)^{\frac{1}{2}} - \underline{p} = 0$.

$U_1^E \geq U_2^E$ iff $c \geq \Big(1 - \frac{M}{\underline{p}}\Big)\bar{p} - (\bar{p} + \underline{p})\gamma\Big[1 - \Big(\frac{M}{\gamma(\bar{p}+\underline{p})}\Big)^{12}\Big]$, and $U_1^E \geq U_3^E$ iff $c \geq (\bar{p}+\underline{p})\Big[(1-\gamma) - \Big(\frac{M}{(\bar{p}+\underline{p})}\Big)^{\frac{1}{2}}(1-\gamma)^{\frac{1}{2}}\Big]$.

On the other hand, if $c < \tilde{c}$, then $U_2^E \geq U_3^E$ if and only if $M\frac{\bar{p}}{\underline{p}} - \big(M(\bar{p}+\underline{p})\big)^{\frac{1}{2}} - \underline{p} \leq 0$.

Proposition 18. This proof is divided in 2 sections, depending on the value of γ.

Section 1: $\gamma \geq \overline{\gamma}$**.** This proof is divided in 4 parts, the first three are the optimal contract for each cutoff β, and the last one is the proof of the Proposition 17 for $\gamma \geq \overline{\gamma}$:

Part 1: If $\beta < \underline{\beta}$ the problem that venture capitalist solves is:

$$\max_{\alpha\in[0,1],\beta\in[0,\underline{\beta}]} \alpha\Big[(1-\beta)\gamma\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\gamma\bar{p}\Big]$$
$$\text{subject to: } \alpha\Big[\beta\gamma\bar{p}\Big] + (1-\alpha)\Big[\beta\gamma\underline{p}\Big] \geq 0$$

Given any β will motivate the entrepreneur to exert effort, venture capitalist keeps all the rights. It implies $\alpha^* = 0$ and $\beta^* = 0$.

Part 2: If $\beta \in [\underline{\beta}, \bar{\beta}]$ the problem the venture capitalist solves is:

$$\max_{\alpha\in[0,1],\beta\in[\underline{\beta},\bar{\beta}]} \alpha\Big[(1-\beta)\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\gamma\bar{p}\Big]$$
$$\text{subject to: } \alpha\Big[\beta\bar{p} - c\Big] + (1-\alpha)\Big[\beta\gamma\underline{p}\Big] \geq 0$$

Note $\underline{\beta}\gamma\underline{p} \geq 0$. Then optimal contract is $\alpha^* = 0$ and $\beta^* = \underline{\beta}$.

Part 3: If $\beta > \bar{\beta}$ the problem venture capitalist solves is:

$$\max_{\alpha\in[0,1],\beta\in[\bar{\beta},1]} \alpha\Big[(1-\beta)\underline{p}\Big] + (1-\alpha)\Big[(1-\beta)\bar{p}\Big]$$
$$\text{subject to: } \alpha\Big[\beta\bar{p} - c\Big] + (1-\alpha)\Big[\beta\underline{p} - c\Big] \geq 0$$

Note $\beta = \bar{\beta}$ and $\alpha = 0$ gives $U^E = \beta\underline{p} - c = c\frac{\gamma}{1-\gamma} > 0$. then optimal contract is $\alpha^* = 0$ and $\beta^* = \bar{\beta}$.

Part 4: Call $U^I_{\beta<\underline{\beta}} \equiv U^I_1$, $U^I_{\underline{\beta}<\beta<\bar{\beta}} \equiv U^I_2$ and $U^I_{\beta>\bar{\beta}} \equiv U^I_3$. Consider the following comparisons:

- $U^I_1 > U^I_2$ iff $c > 0$. Then always $U^I_1 > U^I_2$.
- $U^I_2 > U^I_3$ iff $c > (1-\gamma)^2\frac{\bar{p}\underline{p}}{\bar{p}-\gamma\underline{p}} \equiv \tilde{c}$.
- $U^I_1 > U^I_3$ iff $c > \underline{p}(1-\gamma)^2 \equiv c_M$.

Note $c_M < \tilde{c}$. Comparing each case:

- If $c \leq c_M$: $U^I_1 > U^I_2$, $U^I_2 < U^I_3$ and $U^I_1 < U^I_3$, then U^I_3 is the optimal.
- If $c \in (c_M, \tilde{c}]$: $U^I_1 > U^I_2$, $U^I_2 < U^I_3$ and $U^I_1 > U^I_3$, then U^I_1 is the optimal.
- If $c > \tilde{c}$: $U^I_1 > U^I_2$, $U^I_2 > U^I_3$ and $U^I_1 > U^I_3$, then U^I_1 is the optimal.

Section 2: $\gamma < \overline{\gamma}$**.** This proof is divided in 4 parts, the first three are the optimal contract for each cutoff β, and the last one is the proof of the Proposition 17 for $\gamma < \overline{\gamma}$:

Part 1: If $\beta < \underline{\beta}$. This is the same proof than Part 1 of the proof of Proposition 17 for $\gamma \geq \bar{\gamma}$.

Part 2: If $\beta \in (\underline{\beta}, \bar{\beta}]$ the problem venture capitalist solves is:

$$\max_{\alpha\in[0,1],\beta\in[\underline{\beta},\bar{\beta}]} (1-\beta)\underline{p}$$
$$\text{subject to: } \beta\bar{p} - c \geq 0$$

then any α is optimal, given no matter who decides, the decision will always be $x = R$. Note $\frac{c}{\bar{p}} < \underline{\beta}$, this implies optimal control rights are $\beta^* = \underline{\beta}$.

Part 3: If $\beta > \underline{\beta}$. This is the same proof than Part 3 of the proof of Proposition 17 for $\gamma \geq \bar{\gamma}$.

Part 4: Call $U^I_{\beta<\underline{\beta}} \equiv U^I_1$, $U^I_{\underline{\beta}<\beta<\bar{\beta}} \equiv U^I_2$ and $U^I_{\beta>\bar{\beta}} \equiv U^I_3$. Consider the following comparisons:

- $U^I_1 > U^I_2$ iff $c > \bar{p}(1-\gamma)\left(1-\gamma\frac{\bar{p}}{\underline{p}}\right) \equiv \bar{c}$.
- $U^I_2 > U^I_3$ iff $c > (\bar{p}-\underline{p})(1-\gamma)\frac{\bar{p}\underline{p}}{\bar{p}\bar{p}-\underline{p}\underline{p}} \equiv \underline{c}$.
- $U^I_1 > U^I_3$ iff $c > \underline{p}(1-\gamma)^2 \equiv c_M$.

Note if $\gamma < \underline{p}\frac{\bar{p}-\underline{p}}{\bar{p}\bar{p}-\underline{p}\underline{p}}$ then $\underline{c} < c_M < \bar{c}$. Comparing each case:

- If $c \leq \underline{c}$: $U^I_1 < U^I_2$, $U^I_2 < U^I_3$ and $U^I_1 < U^I_3$, then U^I_3 is the optimal.
- If $c \in (\underline{c}, c_M]$: $U^I_1 < U^I_2$, $U^I_2 > U^I_3$ and $U^I_1 < U^I_3$, then U^I_2 is the optimal.
- If $c \in (c_M, \bar{c}]$: $U^I_1 < U^I_2$, $U^I_2 > U^I_3$ and $U^I_1 > U^I_3$, then U^I_2 is the optimal.
- If $c > \bar{c}$: $U^I_1 > U^I_2$, $U^I_2 > U^I_3$ and $U^I_1 > U^I_3$, then U^I_1 is the optimal.

If $\gamma \in \left[\underline{p}\frac{\bar{p}-\underline{p}}{\bar{p}\bar{p}-\underline{p}\underline{p}}, \bar{\gamma}\right]$ then $\bar{c} < c_M < \underline{c}$. Comparing each case:

- If $c \leq \bar{c}$: $U^I_1 < U^I_2$, $U^I_2 < U^I_3$ and $U^I_1 < U^I_3$, then U^I_3 is the optimal.
- If $c \in (\bar{c}, c_M]$: $U^I_1 > U^I_2$, $U^I_2 < U^I_3$ and $U^I_1 < U^I_3$, then U^I_3 is the optimal.
- If $c \in (c_M, \underline{c}]$: $U^I_1 > U^I_2$, $U^I_2 < U^I_3$ and $U^I_1 > U^I_3$, then U^I_1 is the optimal.
- If $c > \underline{c}$: $U^I_1 > U^I_2$, $U^I_2 > U^I_3$ and $U^I_1 > U^I_3$, then U^I_1 is the optimal.

A.3.3 Chapter 3

Proposition 13. This result is implied by Lemma 2 and 3 in addition to showing that the sequence of equilibrium efforts $\{e_i, e_j\}_{t=1}^{\infty}$ is unique.

Fix a sequence $\{e_2\}_{t=0}^{\infty}$.

$$V_i(x^{t-1}) = \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i),\ \beta_i V_i(x^t) - z_i(e_i, x^t) \right\} \right\} \tag{A.7}$$

- $\delta_i V_i(x^t) - c_i(e_i)$ is strictly concave, because $\delta_i V_i(x^t)$ is concave and $-c_i(e_i)$ is strictly concave.

  Note $\lim_{e_i \to 0} \delta_i V_i(x^t) - c_i(e_i) = \delta_i V_i(x^t)$ and $\lim_{e_i \to \infty} \delta_i V_i(x^t) - c_i(e_i) = -\infty$, then the maximum value exists and is unique.

- $\beta_i V_i(x^t) - z_i(e_i, x^t)$ is strictly concave for the relevant values of e_i. $\beta_i V_i(x^t)$ is concave considering $-z_i(e_i, x^t) = -c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)]$ it's clear that $-c_i(e_i)$ is strictly concave and $x^t - \delta_j V_j(x^t)$ is concave for values of e_i such that $x^t - \delta_j V_j(x^t) > 0$.

  Note $\lim_{e_i \to 0} \delta_i V_i(x^t) - c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)] = \delta_i V_i(x^t) + p_i[x^t - \delta_j V_j(x^t)]$ and $\lim_{e_i \to \infty} \delta_i V_i(x^t) - c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)] = -\infty$, then the maximum value exists and is unique.

Then for each fixed sequence $\{e_j\}_{t=0}^{\infty}$ (A.7) has an unique maximizer.

Lemma 2. I am going to prove first that the continuation value $V_i(x)$ for each player exist and is unique. The value of $V_i(x)$ for player i in period t is:

$$\begin{aligned} V_i(x^{t-1}) &= \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t),\ p_i[x^t - \delta_j V_j(x^t)] + p_j[\delta_i V_i(x^t)] \right\} - c_i(e_i) \right\} \\ &= \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i),\ p_i[x^t - \delta_j V_j(x^t)] + p_j[\delta_i V_i(x^t)] - c_i(e_i) \right\} \right\} \end{aligned}$$

Then,

$$V_i(x^{t-1}) = \max_{e_i} \left\{ \max \left\{ \delta_i V_i(x^t) - c_i(e_i),\ p_j \delta_i V_i(x^t) - c_i(e_i) + p_i[x^t - \delta_j V_j(x^t)] \right\} \right\} \quad \text{(A.8)}$$

In the same way, the value of $V_j(x)$ for player j in period t is:

$$V_j(x^{t-1}) = \max_{e_j} \left\{ \max \left\{ \delta_j V_j(x^t) - c_j(e_j),\ p_i \delta_j V_j(x^t) - c_j(e_j) + p_j[x^t - \delta_i V_i(x^t)] \right\} \right\} \quad \text{(A.9)}$$

Note equation (A.1) and (A.2) generates a system of functional equations. Replacing the value of x^{t-1} for x and x^t for y, the system can be written as:

$$V_i(x) = \max_{e_i} \left\{ \max \left\{ \delta_i V_i(y) - c_i(e_i),\ p_j \delta_i V_i(y) - c_i(e_i) + p_i[y - \delta_j V_j(y)] \right\} \right\}$$
$$V_j(x) = \max_{e_j} \left\{ \max \left\{ \delta_j V_j(y) - c_j(e_j),\ p_i \delta_j V_j(y) - c_j(e_j) + p_j[y - \delta_i V_i(y)] \right\} \right\}$$

Subject to: $y = l(l^{-1}(x) + e_i + e_j)$

From the point of view of i, considering a fix strategy e and $V_j(\cdot)$, i's problem is:

$$V_i(x) = \max_{y \in [x, \infty)} \left\{ \max \left\{ \delta_i V_i(y) - k_i(x, y),\ \beta_i V_i(y) - z_i(x, y) \right\} \right\}$$

where $e_i = l^{-1}(y) - l^{-1}(x) - e_j$, $\beta_i = p_j \delta_i \in (0, 1)$ and $z_i(x, y) = c_i(e_i) - p_i[y - \delta_j V_j(y)]$.

To show that $V_i(\cdot)$ is unique I will show that is a contraction mapping, and then is unique. First step is show that Blackwell's sufficient conditions hold.

Define the operator T by:

$$T(f)(x) = \max_{y \in [x, \infty)} \left\{ \max \left\{ \delta_i f(y) - k_i(x, y),\ \beta_i f(y) - z_i(x, y) \right\} \right\} \quad \text{(A.10)}$$

(a) *Monotonicity* consider $f, g \in B(x)$ where $B(x)$ is the set of continuous and bounded functions. Suppose $f(x) \leq g(x)$ for all x. Then:

$$
\begin{aligned}
&\max\Big\{\delta_i f(y) - k_i(x,y),\ \beta_i f(y) - z_i(x,y)\Big\} \\
&\quad \leq \max\Big\{\delta_i g(y) - k_i(x,y),\ \beta_i g(y) - z_i(x,y)\Big\} \\
&\max_{y\in[x,\infty)}\left\{\max\Big\{\delta_i f(y) - k_i(x,y),\ \beta_i f(y) - z_i(x,y)\Big\}\right\} \\
&\quad \leq \max_{y\in[x,\infty)}\left\{\max\Big\{\delta_i g(y) - k_i(x,y),\ \beta_i g(y) - z_i(x,y)\Big\}\right\} \\
&T(f)(x) \leq T(g)(x)
\end{aligned}
$$

(b) *Discounting*

$$
\begin{aligned}
T(f+a)(x) &= \max_{y\in[x,\infty)}\left\{\max\Big\{\delta_i[f(y)+a] - k_i(x,y),\ \beta_i[f(y)+a] - z_i(x,y)\Big\}\right\} \\
&= \max_{y\in[x,\infty)}\left\{\max\Big\{\Big[\delta_i f(y) - k_i(x,y)\Big] + \delta_i a,\ \Big[\beta_i f(y) - z_i(x,y)\Big] + \beta_i a\Big\}\right\} \\
&\leq \max_{y\in[x,\infty)}\left\{\max\Big\{\Big[\delta_i f(y) - k_i(x,y)\Big] + \delta_i a,\ \Big[\beta_i f(y) - z_i(x,y)\Big] + \delta_i a\Big\}\right\} \\
&= T(f)(x) + \delta_i a
\end{aligned}
$$

Then T is a contraction, and using the contraction mapping theorem exists an unique continuous and bonded function that satisfies (3). Then V_i and V_j are unique.

Now to show that $V_i(x)$ is increasing, concave and differentiable for all i:

Increasing: Consider $x < x'$. Note that,

$$
\begin{aligned}
k_i(x,y) &= c_i(l^{-1}(y) - l^{-1}(x) - e_j) \\
z_i(x,y) &= c_i(l^{-1}(y) - l^{-1}(x) - e_j) - p_i(y - \delta_j V_j(y))
\end{aligned}
$$

increasing x the same y can be chosen at a lower cost. Then $f(x)$ is increasing. Formally,

$$
\begin{aligned}
T(f)(x) &= \max_{y\in[x,\infty)} \left\{ \max\left\{ \delta_i f(y) - k_i(x,y),\ \beta_i f(y) - z_i(x,y) \right\} \right\} \\
&< \max_{y\in[x,\infty)} \left\{ \max\left\{ \delta_i f(y) - k_i(x',y),\ \beta_i f(y) - z_i(x',y) \right\} \right\} \\
&= \max_{y\in[x',\infty)} \left\{ \max\left\{ \delta_i f(y) - k_i(x',y),\ \beta_i f(y) - z_i(x',y) \right\} \right\} \\
&= T(f)(x')
\end{aligned}
$$

Then $V_i(x)$ is strictly increasing in x.

Concave: Note that $V_i(x)$ is bounded by the next period x. Since x is concave and $V_i(x)$ is strictly increasing, then $V_i(x)$ is concave.

Differentiable: Since $k_i(\cdot)$ and $z_i(\cdot)$ are continuously differentiable, using Benveniste and Scheinkman (1979)'s theorem, then V_i is differentiable.

Lemma 3. Suppose $S(x) \geq 0$ is never going to be satisfied. The maximization problem of each agent will be:

$$
\max_{e_i} \left\{ \lim_{\tau\to\infty} \delta_i^\tau V_i(x^\tau) - c_i(e_i) \right\}
$$

First note that $V_i(x) \geq 0$. If player i expect a negative continuation value then she can change her strategy to make effort 0 in each subsequent period. It will ensure her a non negative payoff given by her bargaining power and the value of the surplus. Second, since x is a convex function of the efforts, and $V_i(x)$ is bounded for the expected value of x in the next period, then $V_i(x)$ can not increase faster than x. Then $0 \leq \lim_{\tau\to\infty} \delta_i^\tau V_i(x^\tau) \leq \lim_{\tau\to\infty} \delta_i^\tau x^\tau = 0$ implies $\lim_{\tau\to\infty} \delta_i^\tau V_i(x^\tau) = 0$.

The maximization problem then becomes:

$$
\max_{e_i} \left\{ -c_i(e_i) \right\}
$$

and the optimal effort is 0 for both players. Now the game become the usual Rubinstein such that in each period $x^t = \delta_1 V_1(x^t) + \delta_2 V_2(x^t)$ since the value of x will not increase. Contradiction.

Lemma 4. First, $e_i(x)$ is a strict monotone function of x: consider x and $\hat{x}$ such that $x \neq \hat{x}$ and $e_i(x) = e_i(\hat{x})$. Then, since $e_i(\cdot)$ is optimal, $V_i(x) = V_i(\hat{x})$ using optimal strategies. But since $V_i(\cdot)$ is a strict monotone function, it implies $x = \hat{x}$. So, $e_i(\cdot)$ is strict monotone function.

Second, $e_i(x)$ is decreasing and convex: using the optimal condition:

$$\frac{\partial x^t}{\partial e_i(x^{t-1})}\left[p_i \frac{\partial S(x^t)}{\partial x^t} + \delta_i \frac{\partial V_i(x^t)}{\partial x^t}\right] = c_i \frac{\partial f(e_i(x^{t-1}))}{\partial e_i(x^{t-1})}$$

if $S(x^t) \geq 0$, or:

$$\frac{\partial x^t}{\partial e_i(x^{t-1})} \delta_i \frac{\partial V_i(x^t)}{\partial x^t} = c_i \frac{\partial f(e_i(x^{t-1}))}{\partial e_i(x^{t-1})}$$

if $S(x^t) < 0$.

Note the left hand side (LHS) is decreasing in effort and right hand side (RHS) is increasing in effort. Consider $\hat{x}^{t-1}$ such that $\hat{x}^{t-1} < x^{t-1}$, then the LHS curve shifts down (and the slope increases) and then the new equilibrium effort is lower.

Proposition 14.

Result 1. The first order condition can be iterated and then the optimal condition for effort at t when the game is going to end at period $t+s$ is:

$$\sum_{\tau=s}^{\infty} \delta_i^{\tau} p_i \frac{\partial S^{t+\tau}}{\partial x^{t+\tau}} \frac{\partial x^{t+\tau}}{\partial e_i(x^{t-1})} = \sum_{\tau=0}^{\infty} \delta_i^{\tau} c_i \frac{\partial f(e_i(x^{t+\tau}))}{\partial e_i(x^{t-1})}$$

Previous expression can be written as a function of e_i^t and δ_i for player i (symmetrically for player j)

$$LHS_i(e_i^t, \delta_i) = RHS_i(e_i^t, \delta_i)$$

Note that for any value of e $LHS_i(\cdot, \delta_i)$ is larger than $LHS_j(\cdot, \delta_j)$ since $\delta_i > \delta_j$. On the other hand, $RHS_i(\cdot, \delta_i)$ is lower than $RHS_j(\cdot, \delta_j)$ since $\frac{\partial e_i(x^{t+\tau})}{\partial e_i(x^t)}$ is negative, and $\delta_i > \delta_j$.

Since LHS_i is decreasing in e because S is concave, and RHS_i is increasing in e because effort is decreasing and convex, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

Result 2. Now, optimal expression can be written as:

$$LHS_i(e_i^t) = RHS_i(e_i^t, c_i)$$

For any value of e $LHS_i(\cdot)$ is equal than $LHS_j(\cdot)$. On the other hand, $RHS_i(\cdot, c_i)$ is lower than $RHS_j(\cdot, c_j)$ because$c_i < c_j$.

Since LHS_i is decreasing and RHS_i is increasing in e, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

Result 3. Last, optimal expression can be written as:

$$LHS_i(e_i^t, p_i) = RHS_i(e_i^t)$$

For any value of e $LHS_i(\cdot, p_i)$ is larger than $LHS_j(\cdot, p_j)$ because $p_i > p_j$, and $RHS_i(\cdot)$ is equal than $RHS_j(\cdot)$.

Since LHS_i is decreasing and RHS_i is increasing in e, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

Proposition 15.

Result 1. Optimal expression for ε is:

$$S^{t+1}\frac{\partial p_i(\varepsilon_i^t, \varepsilon_j^t)}{\partial \varepsilon_i^t} = c_{\varepsilon,i}(\varepsilon_i^t)$$

Note the left hand side function of ε is the same for both players. Since it is a decreasing function because p is concave and since the right hand side is constant, then the equilibrium exists, it is unique and , so since $c_{\varepsilon,i} < c_{\varepsilon,j}$ and $\varepsilon_i > \varepsilon_j$. It implies $p_i > p_j$ so using Result 3 of Proposition 14: $e_i(x) > e_j(x)$.

Result 2 and 3. Note that $c_{\varepsilon,i} = c_{\varepsilon,j}$ optimal expression for ε gives the same result for both players. And since optimal expression for e does not play any role in the value of ε then $\varepsilon_i = \varepsilon_j$ and then $p_i = p_j$. So, Result 2 and 3 correspond to Result 1 and 2 of Proposition 14.

Proposition 16.

Result 1. Optimal expression for ε is:

$$\sum_{\tau=s}^{\infty} \hat{\delta}_\tau \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} S(x^{t+\tau}) = \sum_{\tau=0}^{\infty} \delta_i^\tau c_{\varepsilon,i} \frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$$

This expression can be written as:

$$LHS_{\varepsilon,i}(\varepsilon_i^t) = RHS_{\varepsilon,i}(\varepsilon_i^t, c_{\varepsilon,i})$$

$LHS_{\varepsilon,i}(\cdot)$ is the same for i and j for ε. $RHS_{\varepsilon,i}(\cdot, c_{\varepsilon,i})$ is lower for player i because $c_{\varepsilon,i} < c_{\varepsilon,j}$. And since $LHS_{\varepsilon,i}$ is decreasing because p is concave and $RHS_{\varepsilon,i}$ is increasing because ε is concave, then the equilibrium exists, it is unique and $\varepsilon_i^t > \varepsilon_j^t$. It implies $p_i(h_t) > p_i(h_t)$, and then $e_i(x) > e_i(x)$ using Result 3 of Proposition 15.

Result 2. Optimal expression for ε can be written as:

$$LHS_{\varepsilon,i}(\varepsilon_i^t) = RHS_{\varepsilon,i}(\varepsilon_i^t, \delta_i)$$

$LHS_{\varepsilon,i}$ is the same function for both players, $RHS_{\varepsilon,i}(\cdot, \delta_i)$ is lower for player i because $\delta_i > \delta_j$ and because $\frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$ is negative. Since $LHS_{\varepsilon,i}$ is decreasing and $RHS_{\varepsilon,i}$ is increasing, the solution exists, is unique and $\varepsilon_i^t > \varepsilon_j^t$. It implies $p_i(h_t) > p_i(h_t)$.

The optimal expression for e can be written as:

$$LHS_{e,i}(e_i^t, \delta_i, p_i) = RHS_{e,i}(e_i^t, \delta_i)$$

For any value of e $LHS_i(\cdot, \delta_i, p_i)$ is larger than $LHS_j(\cdot, \delta_j, p_j)$ since $\delta_i > \delta_j$ and $p_i > p_j$. On the other hand, $RHS_i(\cdot, \delta_i)$ is lower than $RHS_j(\cdot, \delta_j)$ since $\frac{\partial e_i(x^{t+\tau})}{\partial e_i(x^t)}$ is negative, and $\delta_i > \delta_j$.

Since LHS_i is decreasing in e and RHS_i is increasing in e, the equilibrium exists and it is unique. Also $e_i(x) > e_j(x)$.

<u>Result 3.</u> Since there are not comparative advantages that affect ε, then $\varepsilon_i = \varepsilon_j$. It implies $p_i = p_j$. Then using Result 2 of Proposition 15 the effort are $e_i(x) > e_j(x)$.

Lemma 5. Optimal condition for ε is:

$$\sum_{\tau=s}^{\infty} \hat{\delta}_\tau \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} S^{t+1+\tau} = \sum_{\tau=0}^{\infty} \delta_i^\tau c_{\varepsilon,i} \frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$$

$$\sum_{\tau=s}^{\infty} \hat{\delta}_\tau \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} S(x^{t+\tau}) = \sum_{\tau=0}^{\infty} \delta_i^\tau c_{\varepsilon,i} \frac{\partial \varepsilon_i^{t+\tau}}{\partial \varepsilon_i^t}$$

if the current period is t and the game will finish at $t+s$.

<u>Case 1:</u> Recent efforts are more important (γ_t increasing in t): $\frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} < \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}}$. LHS for optimal condition in if the current period is $t+1$ is:

$$\begin{aligned} LHS_{t+1} &= \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}} S(x^{t+\tau}) \\ &> \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) \\ &> \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) = LHS_t \end{aligned}$$

Note that $\hat{\delta}_\tau \leq \hat{\delta}_{\tau'}$ $\forall \tau > \tau'$.

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t$ $\forall i$.

Case 2: Recent efforts are equally important than later efforts (γ_t increasing in t): $\frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^t} = \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}}$.

LHS for optimal condition in if the current period is $t+1$ is:

$$\begin{aligned} LHS_{t+1} &= \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t+1}} S(x^{t+\tau}) \\ &= \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau-1} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) \\ &> \sum_{\tau=s}^{\infty} \hat{\delta}_{\tau} \frac{\partial p_i(h_{t+\tau})}{\partial \varepsilon_i^{t}} S(x^{t+\tau}) = LHS_t \end{aligned}$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t$ $\forall i$.

Lemma 6. Note:

$$\frac{\partial p_i(h_{t+\tau})}{\varepsilon_i^t} = \frac{\partial f(\sum_{m=0}^{t+\tau} \gamma^m \varepsilon_i^m, \sum_{m=0}^{t+\tau} \gamma^m \varepsilon_j^m)}{\partial \sum_{m=0}^{t+\tau} \gamma^m \varepsilon_i^m} \gamma^t = \varphi(h_{t+\tau})\gamma^t$$

then the optimal condition for ε is:

$$\sum_{\tau=s}^{\infty} \delta^\tau \varphi(h_{t+\tau})\gamma^t S(x^{t+\tau}) = \sum_{\tau=0}^{\infty} \delta_i^\tau \frac{\partial c_{\varepsilon,i}(\varepsilon_i^{t+\tau})}{\partial \varepsilon_i^t}$$

if the current period is t and the game will finish at $t+s$.

Result 1: $(\gamma > \delta)$ LHS for optimal condition in if the current period is t is:

$$LHS_t = \sum_{\tau=s}^{\infty} \delta^{\tau} \varphi(h_{t+\tau}) \gamma^t S(x^{t+\tau}) = \delta \gamma^t \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau})$$
$$< \gamma^{t+1} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau}) = LHS_{t+1}$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t \ \forall i$.

Result 2: $(\gamma = \delta)$. LHS for optimal condition in if the current period is t is:

$$LHS_t = \sum_{\tau=s}^{\infty} \delta^{\tau} \varphi(h_{t+\tau}) \gamma^t S(x^{t+\tau}) = \delta \gamma^t \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau})$$
$$= \gamma^{t+1} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau}) = LHS_{t+1}$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} = LHS_t$ the value of ε is $\varepsilon_i^{t+1} = \varepsilon_i^t \ \forall i$.

Result 3: $(\gamma < \delta)$. LHS for optimal condition in if the current period is t is:

$$LHS_t = \sum_{\tau=s}^{\infty} \delta^{\tau} \varphi(h_{t+\tau}) \gamma^t S(x^{t+\tau}) = \delta \gamma^t \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau})$$
$$> \gamma^{t+1} \sum_{\tau=s}^{\infty} \delta^{\tau-1} \varphi(h_{t+\tau}) S(x^{t+\tau}) = LHS_{t+1}$$

Since LHS of the optimal condition is decreasing in ε and RHS is increasing, the solution exists, is unique and since $LHS_{t+1} > LHS_t$ the value of ε is $\varepsilon_i^{t+1} > \varepsilon_i^t \ \forall i$.

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