1925

Applications of inversion

BOSTON UNIVERSITY

GRADUATE SCHOOL

Thesis

APPLICATIONS OF INVERSION

Submitted by

Marguerite Florence Syner

(S.B., Boston University, 1924)

In partial fulfilment of requirements for

the degree of Master of Arts

1925

APPLICATIONS OF INVERSION

I. Inverse of straight lines.
  (a) Through the centre of inversion.
  (b) Not through the centre of inversion.

II. Inverse of circles.
  (a) Through the centre of inversion.
  (b) Not through the centre of inversion.

III. Inversion used in the construction of the problem of Apollonius.

IV. Relation of inversion to the linkage
  (a) Construction of the straight line by the linkage.
  (b) " " " circle " " " .
  (c) " " " witch " " "
  (d) " " " cissoid " " "
  (e) " " " strophoid " " "

V. The inverse of the curve as a whole.
  (a) The inverse of double points.
  (b) The inverse of triple points.
  (c) The use of these points in harmonic curves and their inverses.

VI. Circular cubics as inverses of conic sections.
  (a) The witch
  (b) The strophoid
  (c) The cissoid

VII. Bicircular quartics as inverses of conic sections.
  Centre of inversion
    (a) Outside of the conic
    (b) At the focus of the conic
    (c) At the centre of the conic
    (d) Any point on the conic

VIII. Construction of varius other curves by inversion.

IX. Inversion in three dimensions.

X. Cyclides.

XI. Inversion applied to the theory of correspondence.

XII. Laguerre Inversion.

APPLICATIONS OF INVERSION

I. Inverse of straight lines.

By the inverse of a point A with respect to the centre of a circle of inversion is meant that $OA \cdot OA' = k$, where O is the centre of inversion and A and A' are inverse points.

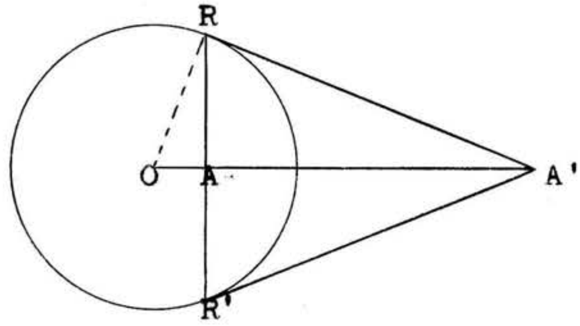


Fig. 1

In Fig. 1 let the circle with centre O be the circle of inversion and let the point A be any point in the circle. Draw a line through A to the centre of the circle. At A erect a perpendicular to the line OA. Let this line meet the circle in the points R and R'. At R and R' draw tangents to the circle, meeting the line OA extended at A'. The triangles OAR and ORA' are similar, therefore $OA:OR::OR:OA'$. $OA \cdot OA'$ is equal to $OR \cdot OR = k$.

The inverse of the centre of the circle of inversion obviously lies at infinity.

The inverse of a line through the centre of inversion is the same straight line, for if the tangents to the circle are drawn at the points where the line cuts the circle, they meet the line through O which is perpendicular to the given line at infinity.

The inverse of a straight line not through the centre of inversion is a circle passing through the centre of inversion.

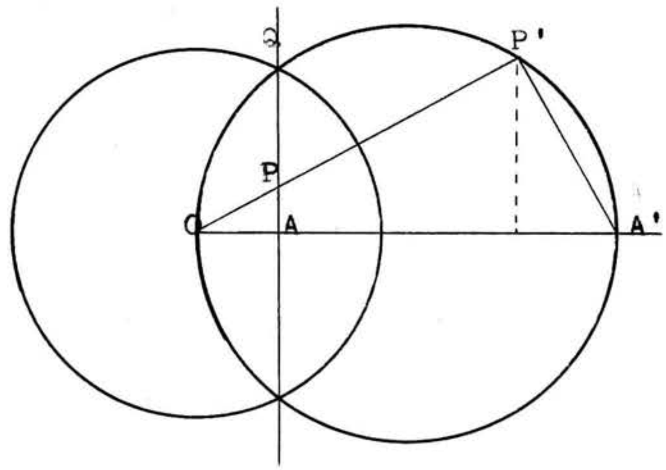


Fig. 2

In Fig. 2 let O be the centre of inversion and P any point on the line AQ and let P' be its inverse. Let OA be the line from O, perpendicular to the line AQ at A, and let the point A' be the inverse of A. Then by the definition of inverse points OA.OA'= k and OP.OP'= k. Therefore OA.OA'= OP.OP' or OA:OP::OP':OA'. The triangles OAP and OP'A' are similar and the angles OAP and APO are equal respectively to the angles OP'A' and OA'P'. But the angles OPA and APP' are supplementary, therefore angles AA'P' and APP' are supplementary. By construction angle OAP is a right angle and since angle PP'A' is equal to the angle OAP it also is a right angle. Angles PAA' and PP'A' are therefore supplementary. The sum of the angles A'AP, APP', PP'A', and P'A'A is thus 360° and the points APP'A' are therefore concyclic. Since angle OP'A' is a right angle the point P' must also lie on a circle whose diameter is OA'. The inverse of the line AQ is therefore a circle whose centre lies on the line OA' and passes through the points O, Q, P', and A'.

That the inverse of a straight line not through the centre of inversion is a circle through the centre of inversion can also be proven analytically. We thus consider inversion as a point-point transformation. In Fig. 2 if we let the circle of inversion have a unit radius and point P have the coordinates (x, y) and the point P' have the coordinates (x', y'), we then have $OP \cdot OP' = 1$ and by similar triangles

$$\frac{x}{x'} = \frac{y}{y'} = \frac{OP}{OP'}$$

dividing $OP \cdot OP' = 1$ by OP'^2 we get

$$\frac{OP}{OP'} = \frac{1}{OP'^2} = \frac{1}{x'^2 + y'^2}$$

therefore, since $OP' = x'^2 + y'^2$

$$\frac{x}{x'} = \frac{y}{y'} = \frac{1}{x'^2 + y'^2}$$

therefore

$$x = \frac{x'}{x'^2 + y'^2} \quad \text{and} \quad y = \frac{y'}{x'^2 + y'^2}$$

By substitution we can find the inverse of any given equation. Thus find the inverse of the straight line $Ax + By + C = 0$. Substituting we get

$$\frac{Ax'}{x'^2 + y'^2} + \frac{By'}{x'^2 + y'^2} + C = 0$$

simplifying and dropping the primes,

$$Cx^2 + Cy^2 + Ax + By = 0$$

The locus of this equation is a circle which passes through the origin. If $C = 0$, the locus is the given line.

The inverse of a system of parallel lines is a system of tangent circles whose centres lie on a line perpendicular to the lines of the system.

In Fig. 3 if each line is inverted, it will give a circle through the centre of inversion and since the lines are parallel all the centres will lie on the same perpendicular line to the given lines and which passes through the centre of inversion. Hence, all the circles will be tangent to each other at O.

To prove this analytically let the equation of the system of the given lines is $x = a$ where a is an arbitrary constant. Substituting the value

$$x = \frac{x'}{x'^2 + y'^2}$$

we get $\dfrac{x'}{x'^2 + y'^2} = a$. By reducing and dropping the primes,

$x^2 + y^2 - \frac{1}{a}x = 0$. This is the equation of a system of circles whose centre lie on the X-axis and which are tangent to each other at the origin.

The inverse of a system of concurrent lines is a system of circles passing through the centre of inversion and through the inverse of the point of concurrency, and having their common chord through these points.

In Fig. 4 we have a system of lines a, b, c concurrent at the point A. The inverse of each line is a circle passing through O. Since the original lines pass through the point A, their inverse circles will necessarily pass through the inverse of the point A which is A'. The line through O, A, and A' is the common chord of the circles.

To prove this analytically, let the system of line be $y = mx + b$, where b is constant and m varies. Substituting

$$\frac{y'}{x'^2 + y'^2} = m\frac{x'}{x'^2 + y'^2} + b$$

reducing and dropping the primes we have

$$x^2 + y^2 + \frac{m}{b}x - \frac{1}{b}y = 0$$

This is the equation of a system of circles passing through the origin and through $(o, \frac{1}{b})$ which is the inverse of (o, b) through which the given lines pass.

By the same method it can be proven that the inverse of a system of concentric circles is a system with two limiting point one at the origin and the other at the inverse of the centre of the concentric circles.

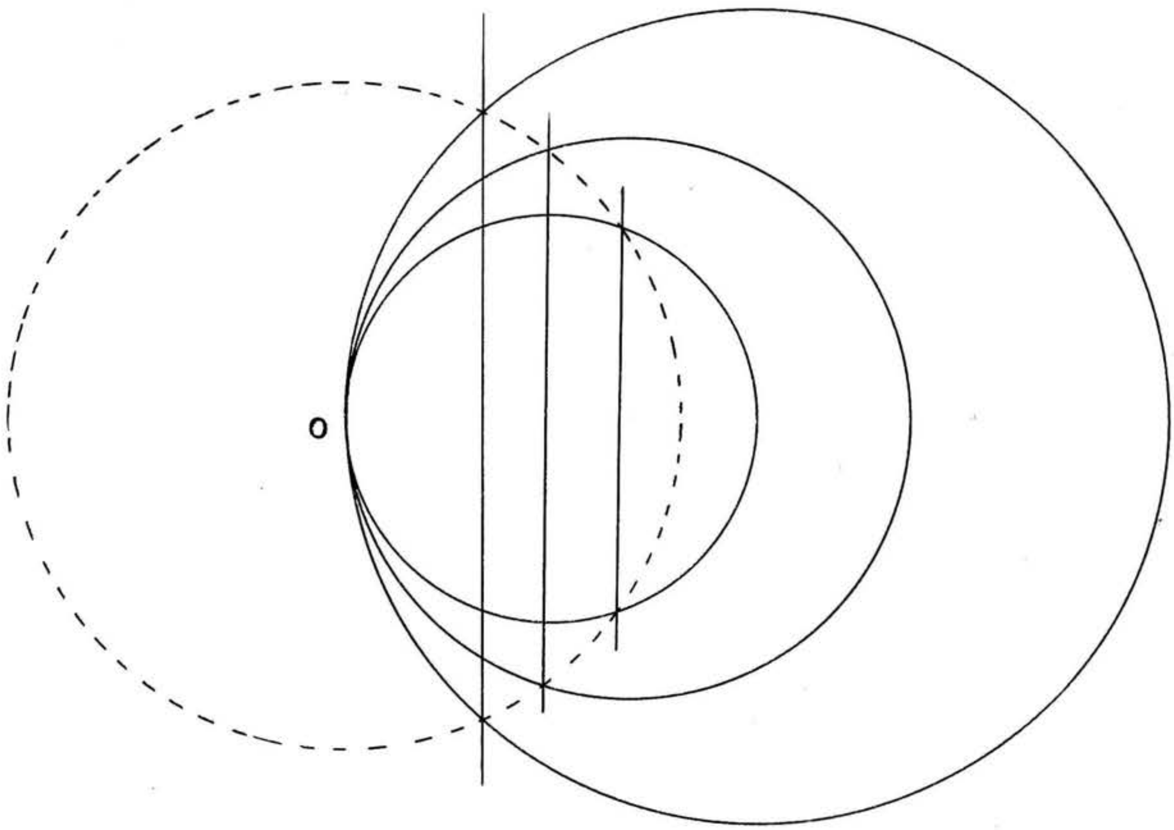


Fig. 3

The Inverse of a Set of Parallel Lines

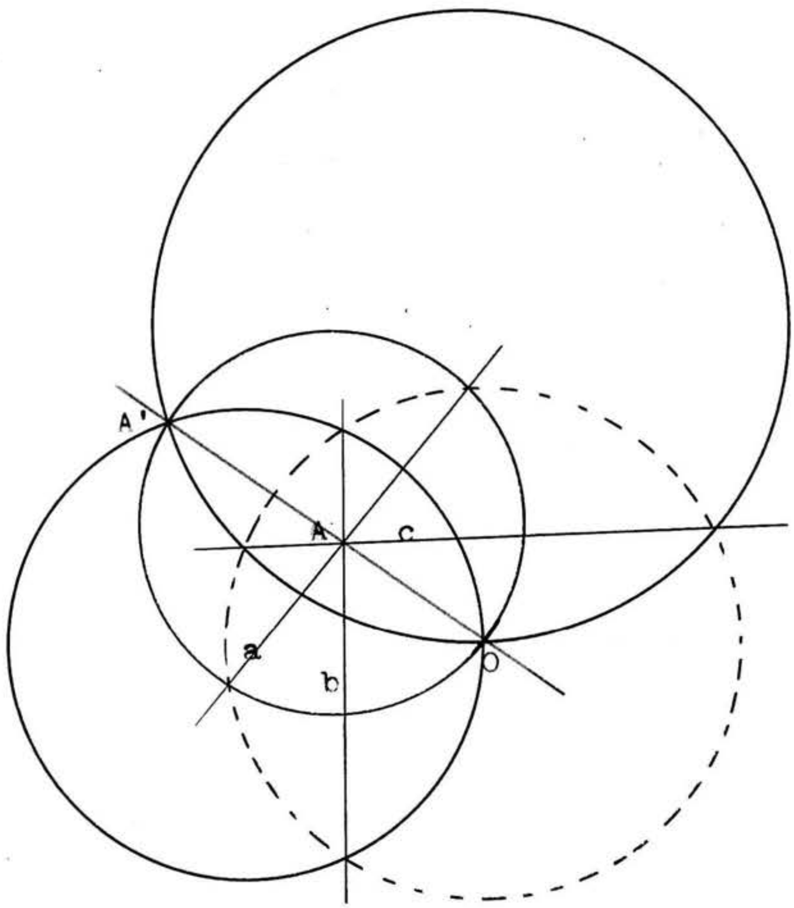


Fig. 4

The Inverse of a Set of Concurrent Lines

II. Inverse of circles.

It has been shown that the inverse of a straight line not through the centre of inversion is a circle through the centre of inversion. Conversely, the inverse of a circle through the centre of inversion is a straight line through the centre of inversion.

The inverse of a circle not through the centre of inversion is a circle.

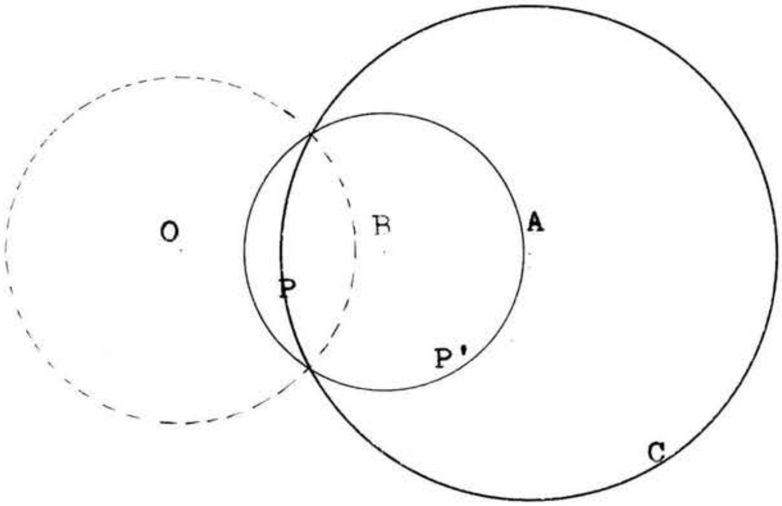


Fig. 5

In Fig. 5 let O be the centre of inversion, and the circle with centre A any given circle. Draw OA the line of centres. Draw any other line from O cutting the given circle in P and C. Find the inverse of the point P. This is P'. Connect A and C. At P' draw a line parallel to AC, meeting the line OA at B. This point B is the centre of the inverse circle. The circle with centre B will pass through the points where the given circle cuts the circle of inversion.

By the definition of inverse points OP.OP' is constant. OP.OC is also constant.

$$(1) \quad OP.OP' = k$$
$$(2) \quad OP.OC = k'$$

dividing (1) by (2)

$$\frac{OP.OP'}{OP.OC} = \frac{k}{k'} = \frac{OP'}{OC} = \frac{k}{k'}$$

by similar triangles

$$\frac{OB}{OA} = \frac{OP'}{OC} \quad \text{and} \quad \frac{BP'}{AC} = \frac{OB}{OA}$$

but since OA, OP', and OC are all constants, OB must be a constant. Hence, BP' must be a constant as all the other terms of the proportion are constants. Therefore B is a fixed point and BP' a constant length. The locus of P' is a circle whose centre is B.

To prove this analytically let $x^2 + y^2 + Dx + Ey + F = 0$ be the equation of the given circle. Substituting we get

$$\frac{x'^2}{(x'^2 + y'^2)^2} + \frac{y'^2}{(x'^2 + y'^2)^2} + \frac{Dx'}{x'^2 + y'^2} + \frac{Ey'}{x'^2 + y'^2} + F = 0$$

multiplying by $x'^2 + y'^2$ and dropping the primes,

$$Fx^2 + Fy^2 + Dx + Ey + 1 = 0$$

The locus is a circle unless $F = 0$, in which case we have an equation of the first degree and its locus is a straight line

Angles are invariant in inversion. Let the equations of the given circles be

$$x^2 + y^2 + D_1x + E_1y + F_1 = 0$$
and
$$x^2 + y^2 + D_2x + E_2y + F_2 = 0$$

then from above the equations of the inverse circles are

$$x^2+y^2+\frac{D_1}{F_1}x+\frac{E_1}{F_1}y+\frac{1}{F_1}=0$$

and
$$x^2+y^2+\frac{D_2}{F_2}x+\frac{E_2}{F_2}y+\frac{1}{F_2}=0$$

but by trigonometry in the original figure $\cos e = \frac{r_1^2+r_2^2-d}{2r_1r_2}$ where r and r are the radii of the circles and d is the length of the line of centres.

$r = \frac{1}{2}\sqrt{D_1^2+E_1^2-4F_1}$ and $r = \frac{1}{2}\sqrt{D_2^2+E_2^2-4F_2}$ and the centres are respectively $(-\frac{D_1}{2}, -\frac{E_1}{2})$ and $(-\frac{D_2}{2}, -\frac{E_2}{2})$. Hence
$$d = \sqrt{\left(\frac{D_2}{2}-\frac{D_1}{2}\right)^2+\left(\frac{E_2}{2}-\frac{E_1}{2}\right)^2} = \frac{1}{2}\sqrt{(D_2-D_1)^2+(E_2-E_1)^2}.$$

Substituting these values in the value of cos e, we get

$$\cos e = \frac{D_1D_2+E_1E_2-2F_1-2F_2}{\sqrt{D_1^2+E_1^2-4F_1}\sqrt{D_2^2+E_2^2-4F_2}}$$

Hence, from the equations of the inverse circles we get

$$\cos e' = \frac{\frac{D_1D_2}{F_1F_2}+\frac{E_1E_2}{F_1F_2}-\frac{2}{F_1}-\frac{2}{F_2}}{\sqrt{\left(\frac{D_1}{F_1}\right)^2+\left(\frac{E_1}{F_1}\right)^2-\frac{4}{F_1}}\sqrt{\left(\frac{D_2}{F_2}\right)^2+\left(\frac{E_2}{F_2}\right)^2-\frac{4}{F_2}}}$$

$$= \frac{D_1D_2+E_1E_2-2F_1-2F_2}{\sqrt{D_1^2+E_1^2-4F}\sqrt{D_2^2+E_2^2-4F_2}}$$

therefore cos e and cos e' are equal. Hence, the angles are invariant.

If two circles touch each other, the inverse circles will also touch each other. This is shown in Fig. 6 and its proof depends upon the foregoing theorem.

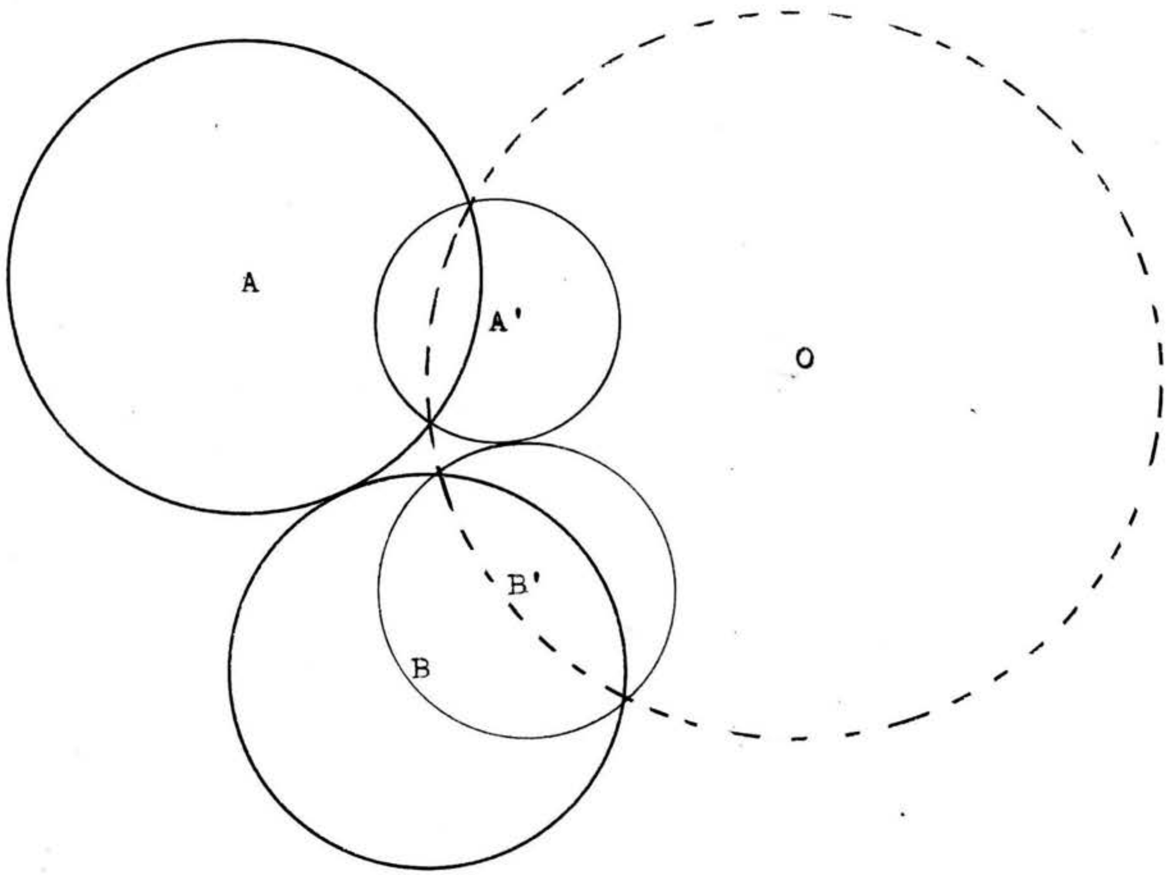


Fig. 6

III. Construction of the problem of Apollonius by inversion.

(1) To construct a circle which will be tangent to three given concurrent circles.

In Fig.7 let the circles with centres A,B,C be the given circles and concurrent at the point O. Let O be the centre of inversion. Since each circle passes through O, their inverses will be straight lines a,b,c respectively. Their are four possible circles which will be tangent to these three lines. In Fig.7 let circle with centre D be the circle internally tangent to the lines. The inverse of this circle will be a circle externally tangent to the three given circles and hence, is one of the required circles.

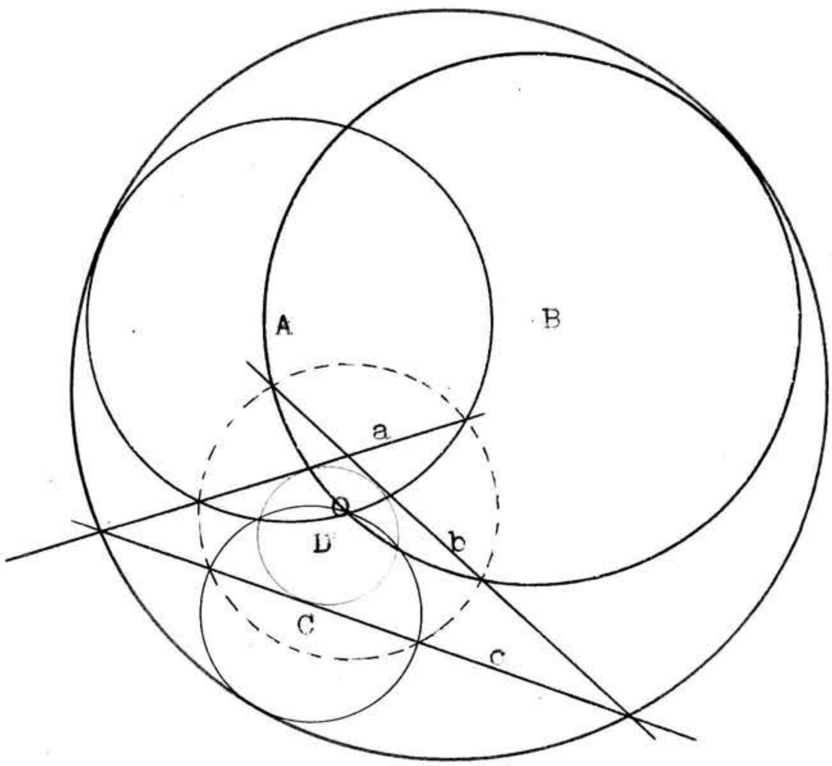


Fig.7

(2) To construct a circle which will be tangent to two given circles and which will pass through a given point.

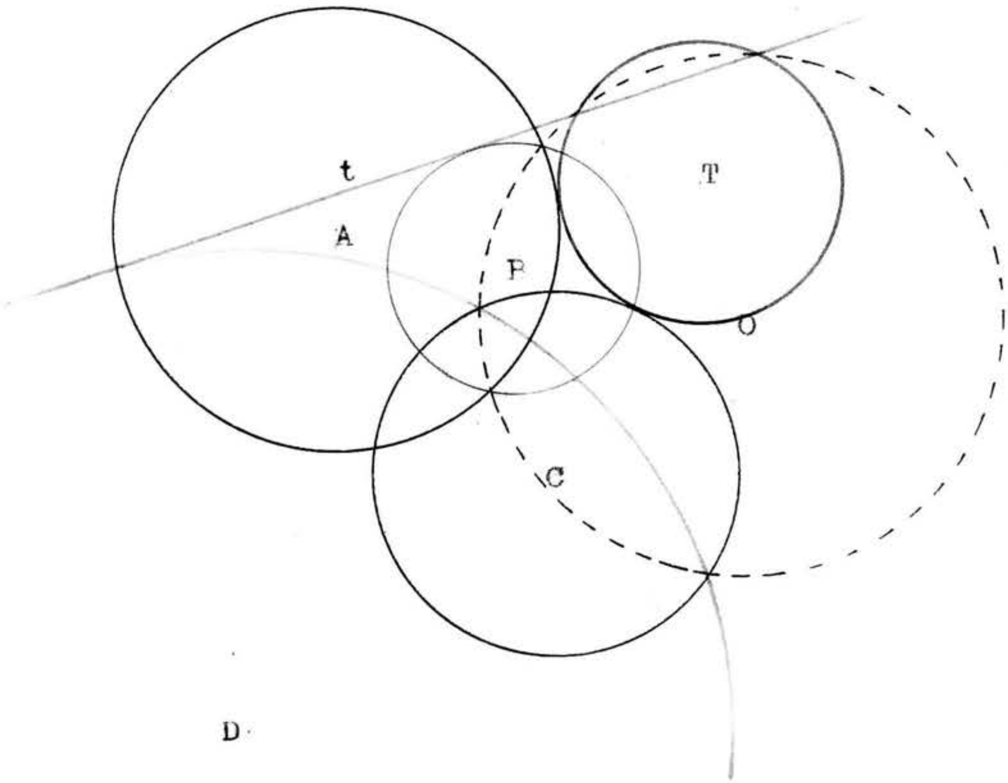


Fig.8

In Fig.8 let the circles with centres A and C be the given circles and O the given point. Take O as the centre of inversion The inverse of the given circles are the circles with centres B and D. Draw any tangent t to the inverse circles.. The inverse of this line will be a circle through the centre of inversion and tangent to the given circles. This is the circle with centre T and it will be externally tangent to the given circles as t was taken as an external tangent. The circle with centre T is the required circle as it is tangent to the given circles and passes through the given point.

(3) To construct a circle which will be tangent to a given circle and to a given line at a given point on the line.

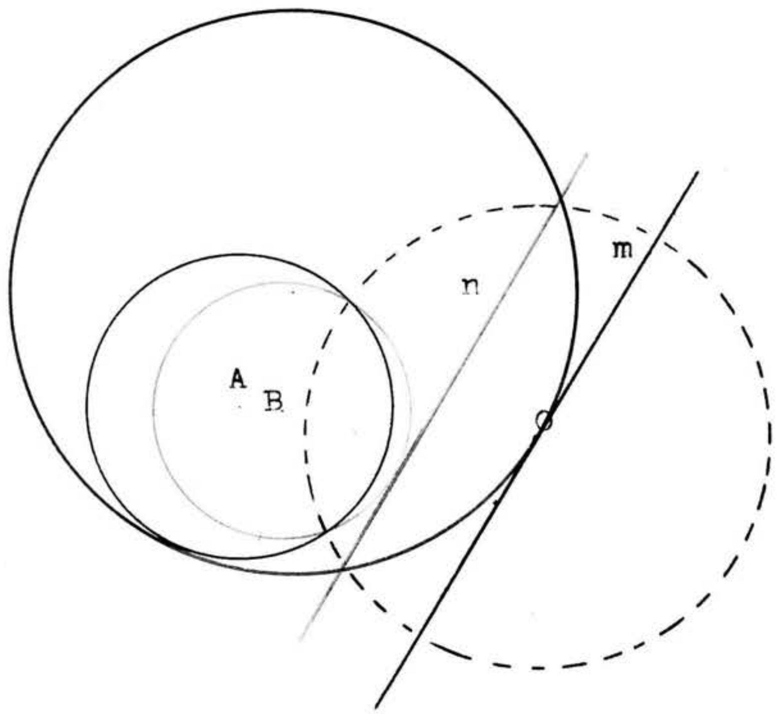


Fig.9

In Fig.9 let the circle with centre A be the given circle. Let O be the given point and m the given line . Take O as the centre of inversion. The inverse of the given circle is the circle with centre B. To this circle draw the tangent line n such that it is parallel to the given line m. The inverse of this line is a circle and it will be tangent to the original circle as angles are invarient in inversion. This circle will also be tangent to the given line at the given point O, as it is the inverse of a straight line and has its centre on a line through O perpendicular to the line m. Hence, this circle is the required circle.

(4) To construct a circle which will be tangent to a given circle and which will pass through two given points.

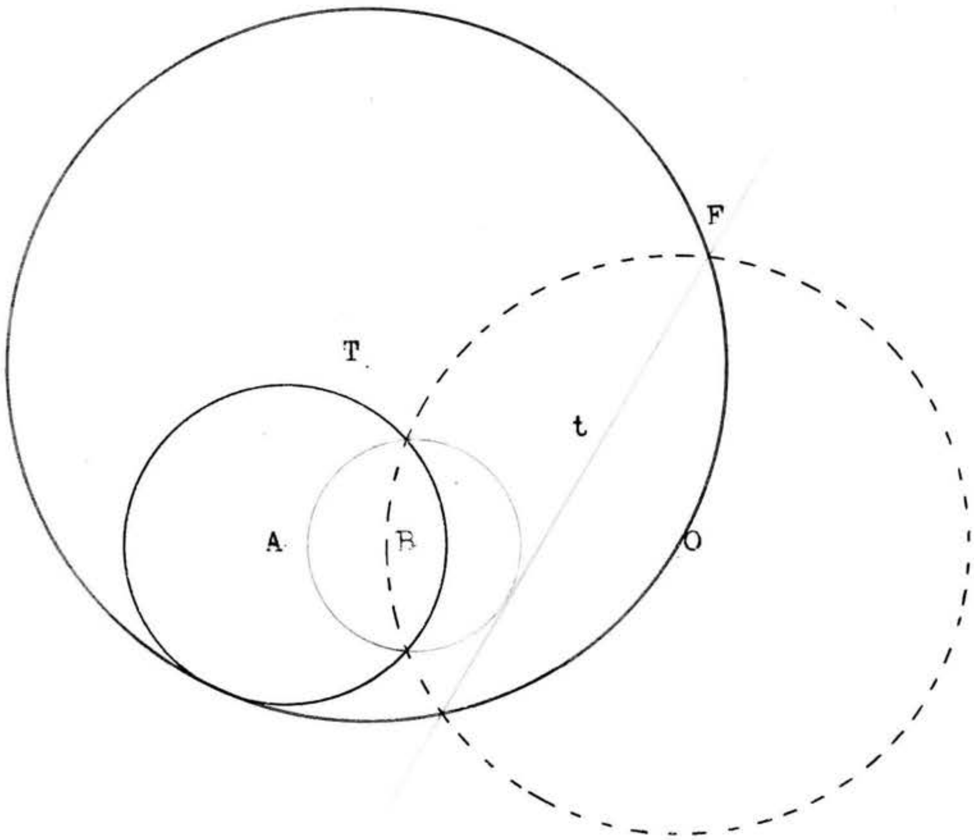


Fig. 10

In Fig. 10 let the circle with centre A be the given circle and given points O and F. With OF as a radius draw the circle of inversion with its centre at O. Find the inverse of the given circle. This is the circle with centre B. To this circle draw a tangent from the point F. The inverse of this tangent t is a circle through O and F and is also tangent to the given circle. Hence, this circle with centre T is the required circle.

IV. Relation of inversion to the Linkage.

The process by which the inverse of any figure may be found depends upon the theory of parallel motion. A method by which perfect parallel motion could be produced was discovered in 1864 by Peaucellier an officer of Engineers in the French army. Other methods of producing parallel motion had been discovered before, but they were imperfect.

Peaucellier's exact parallel motion depended upon a link-work of seven bars moving around two fixed centres. It consisted of a rhomb or diamond formed by four equal links joined to one another. A pair of equal links was joined to two opposite angles of the rhomb and to each other. The point where these two links are joined together is called the fulcrum. The whole structure is called a cell. No matter which way the linkage is moved about the fulcrum, the free angles or the poles will always lie in a straight line with the fulcrum.

The properties of Peaucellier's cell depends upon the distances of the arms or the distances of the fulcrum from the poles. The cell may be made to change its form by closing or opening the diamond. When this is done the lengths of the arms alter, but one increases just as much as the other decreases so that their product remains constant. This product is equal to the difference between the square of either of the links proceeding to the fulcrum and the square of any side of the diamond. It has been demonstrated that when the fulcrum was between the poles a distance of 12 inches, the product of the arms will be 144. When one of the arms was 18 inches, the other was found to be 8 inches: when one was 24 inches, the other was 6 inches, so that the product always remained 144. If these lengths are put in terms of feet, we have the lengths as $1, \frac{3}{2}, 2, 3$ of one arm corresponding to the lengths $1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$ of the other. Thus the lengths of the arms were always inverse to each other. Therefore, if one pole be made to trace any curve, the other pole will trace its inverse.

If one of the poles be made to trace a circle the other pole will describe a circle, for as has been shown before, the inverse of a circle is in general another circle. If, however, the arc described by one of the poles passes through the fulcrum, which will be the centre of inversion, the other pole will describe a straight line. This line will always be perpendicular to the line of centres.

If the two fixed centres are joined by a new link and the centre at F be unfixed, a linkage of eight bars will be formed. This linkage is shown on Pg. 21. With this linkage if the fixed point is focus of a conic, and one of the poles traces the conic, the other pole will trace a limacon. If the fixed point O, or the centre of inversion is the vertex of a conic and one pole traces the conic, the other pole will trace its inverse.

By modifying Peaucellier's cell so that the sum of the arms remain constant instead of their product, a new linkage will be formed which is a quadratic-binomial extractor. If a suitable radius is attached to this cell, a perfect lemniscate can be described. The lemniscate can also be described by a cell similar to Peaucellier's but having one less pair of links. It therefore has 5-bar motion. This is shown on Pg. 22.

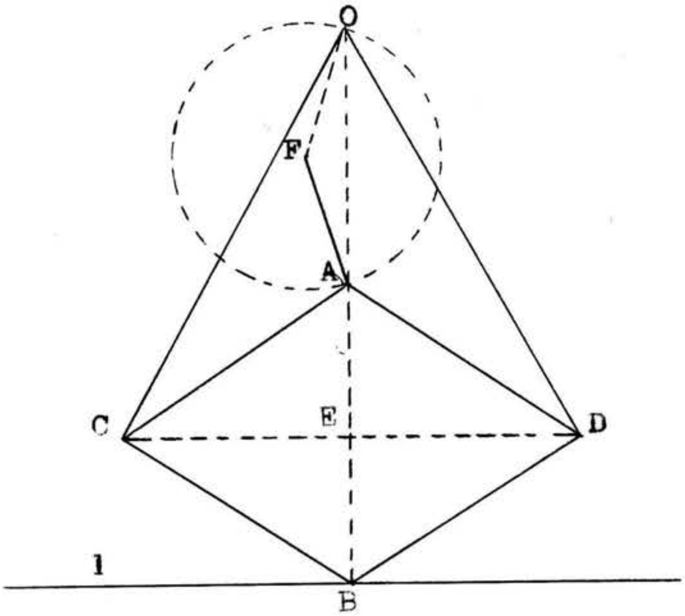
In the Peaucellier's cell, if the fulcrum is set free and in its place one of the poles is fixed as the centre of inversion, then as the liberated fulcrum describes any curve and passes through the fixed pole, the free pole will describe the inverse of the curve. If a second cell is combined with this cell a point may be made to move in a parabola, ellipse, or hyperbola. If one of the poles be made to trace any of these curves, the free pole of the other cell will trace a nodal cubic, or in other words the inverses of the cubics, which are the cissoid, hyper-cissoid, and the hypo-cissoid, respectively. Sylvester uses these terms to describe the nodal cubics, but they are more generally known now as the cissoid, witch, and strophoid.

A simple linkage has been made which will describe these three curves. It consists of radiating bars about fixed centres, and of a bar which moves parallel to itself across the radiating bars. In Fig. 11 we have a pencil of rays proceeding from A, the extremety of a diameter of a circle, and meeting a tangent t to the circle at the other extremety B. If the portion of each ray intercepted between the circle and the tangent be shifted along the ray until one point of it coincides with the centre of the pencil, the other point will trace a cissoid. If everything is kept as above except the tangent which is let moved parallel to itself, and becomes fixed in its new position either further away or nearer to the centre of the pencil, then the curve becomes a witch or a strophoid respectively.

The use of the linkage has come to play a large part in our daily lives. The simple linkage such as Peaucellier's cell of seven bars and having circulo-linear and circulo-circular motion are in constant use in modern machinery. They are used in steamengines, planing machines, construction of maps on the stereographic projection, etc.

Peaucellier's Cell

Fig. a



A and B are inverse points. The arms FO and FA are equal. The point A traces a circle. The inverse point B will trace the line 1

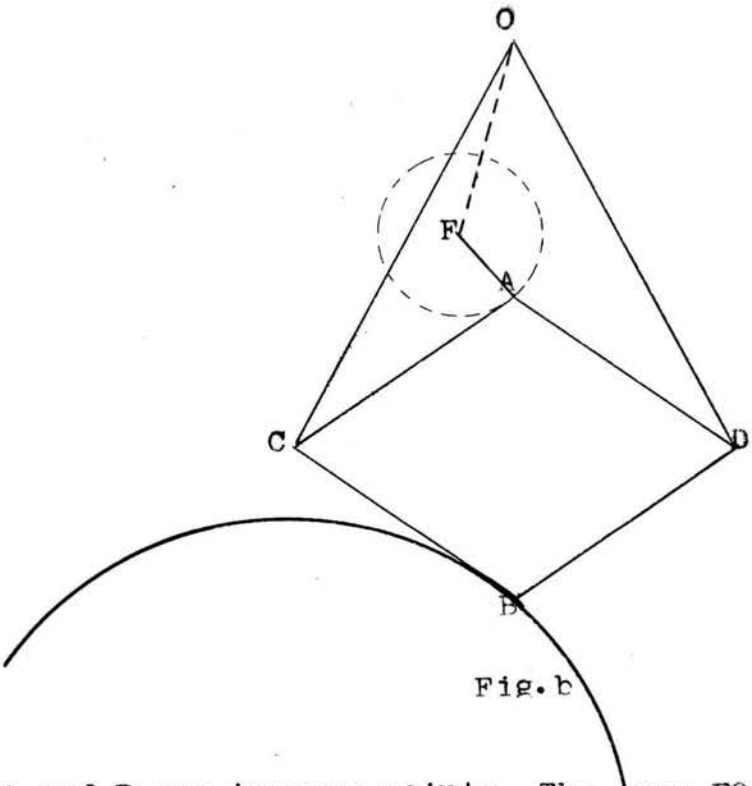


Fig. b

A and B are inverse points. The arms FO and FA not equal. The point A traces a circle not through O. The inverse point B will trace a circle.

A linkage of eight bars. F is joined to O by a link and is now unfixed. Points A and B trace the inverse of each other.

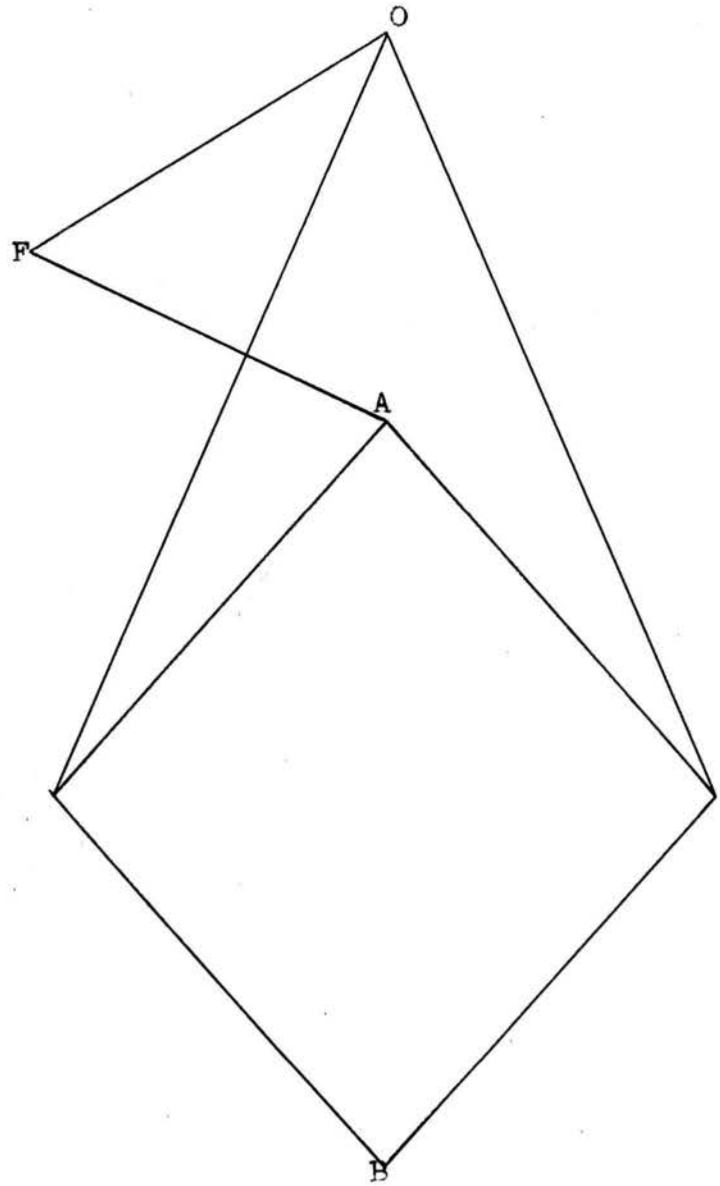


Fig. c

A linkage of 5 bars. Point A is the only fixed point. This linkage will trace a lemniscate at the point B.

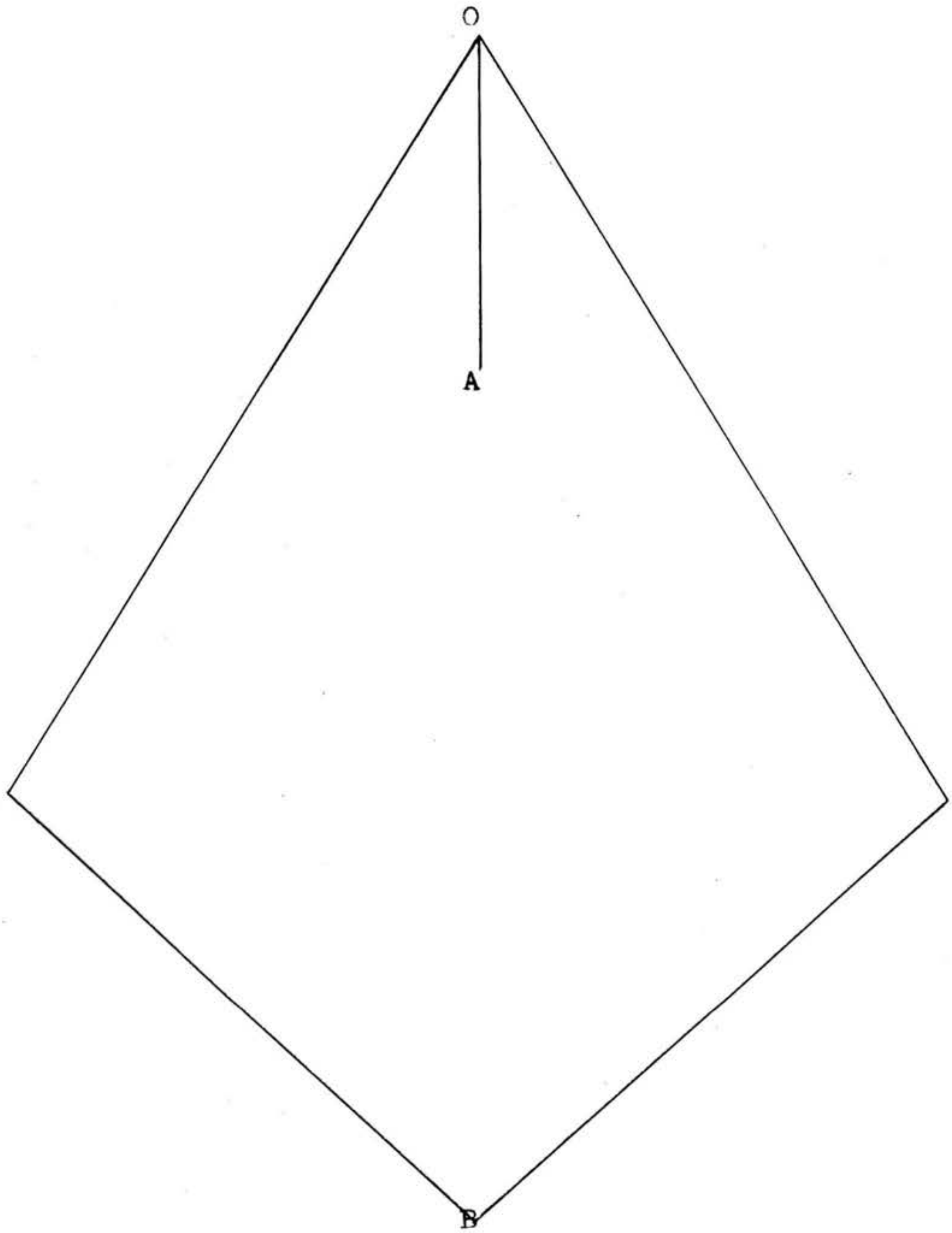


Fig. d

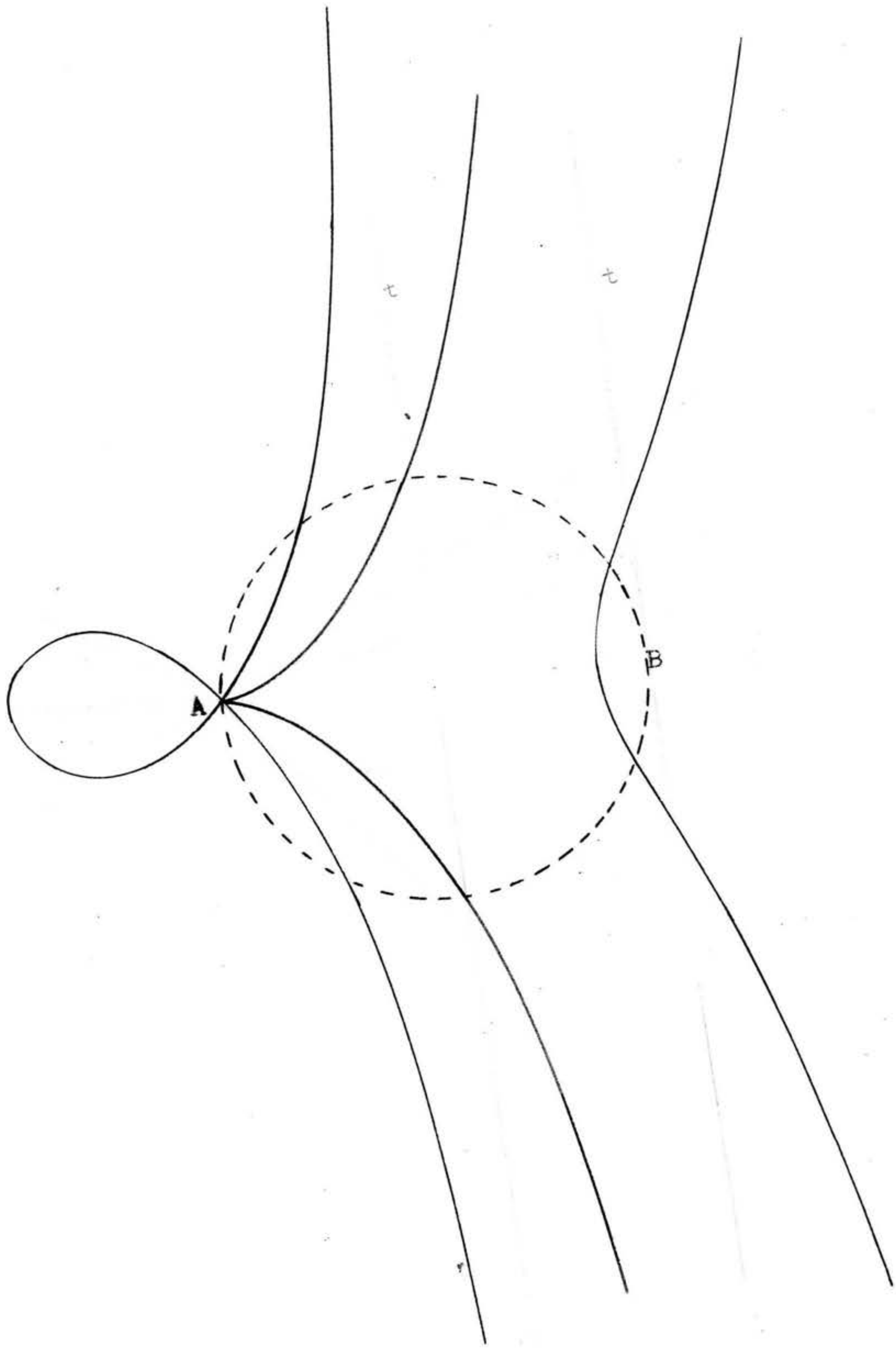


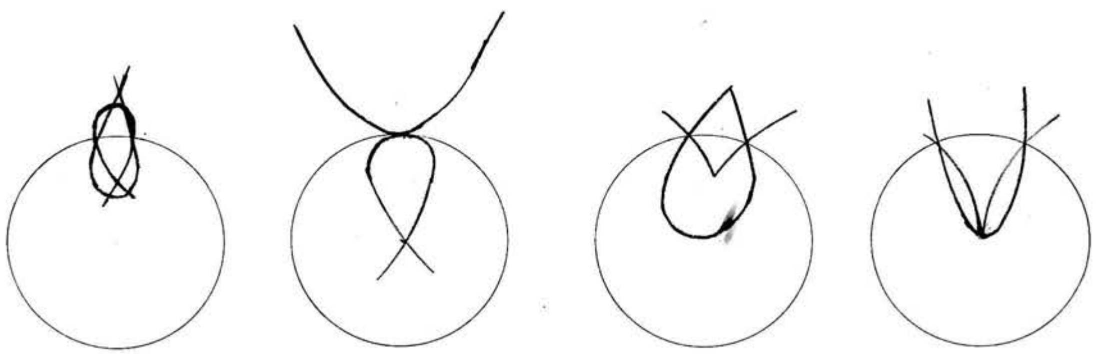
Fig. 11

V. The inverse of the curve as a whole.

In order to study the inverse of curves still further, it is necessary to know something about multiple points, singularities, and points of contact. Of the multiple points the double and triple points are the most common.

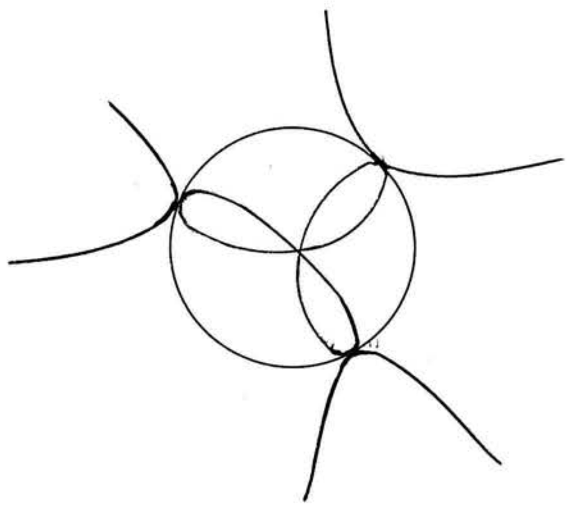
There are three kinds of double points. A double point is that point where the curve cuts itself just once. At this point the curve has two tangents. When the two tangents are real and distinct, the point is called a crunode, or simply a node. When the tangents are imaginary, the point is called an acnode, or a conjugate point. When the tangents are coincident the point is called a spinode or cusp. The limacon furnishes an example of these three kinds of double points, but a study of this curve will be taken up later under Bicircular Quartics.

Nodes and cusps remain invariant in inversion except when these points themselves become the centre of inversion. They then become double points at infinity. This is shown in the figures below.

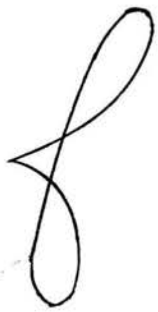


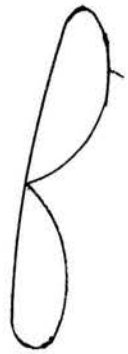
When three branches of the curve pass through the same point, the point is called a triple point. A triple point may be formed in several ways. If in the figure below we let A, B, and C be three nodes and let the nodes coincide, the tangents at A and C to the branch AB coalesce into a single tangent. Also the tangents at A and C to the branch AC, and those at B and C to the branch BC coalesce into two single tangents. The point A therefore becomes a triple point, as the three pairs of tangents at A, B, and C coalesce into three tangents at the point at which the three nodes coincide.

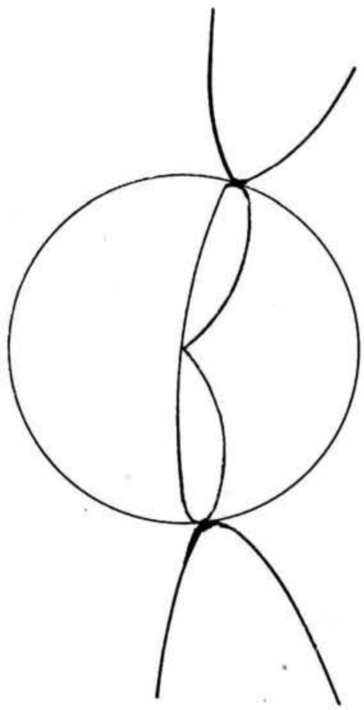
In the following figure can be seen what effect inversion has on the curve when this kind of a triple point is taken as the centre of inversion. Three branches of the curve extend to infinity.



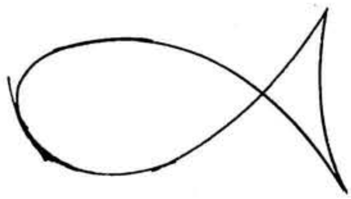
The second kind of triple point is composed of two nodes and a cusp. If the nodes and the cusp unite we have a triple point which consists of a cusp which lies on the curve. In the following figures if this point is taken as the centre of inversion, the curve becomes hyperbolic in shape, having two branches extending to infinity.

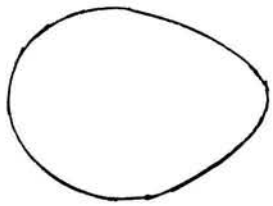


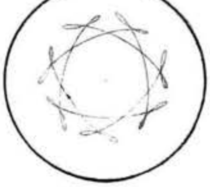




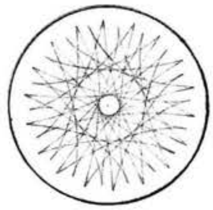
If we have two cusps and a node and let them coincide we have a third kind of a triple point. This point scarcely differs in appearance from a point on the curve. The form of the curve before the double points coalesce, is one which is commonly seen in harmonic curves. A few harmonic curves and their inverses are shown on the following pages.





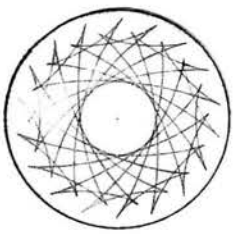


Nodes

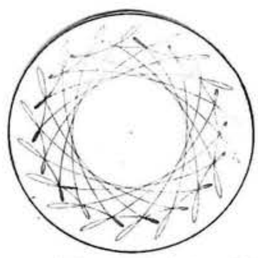


Cusps

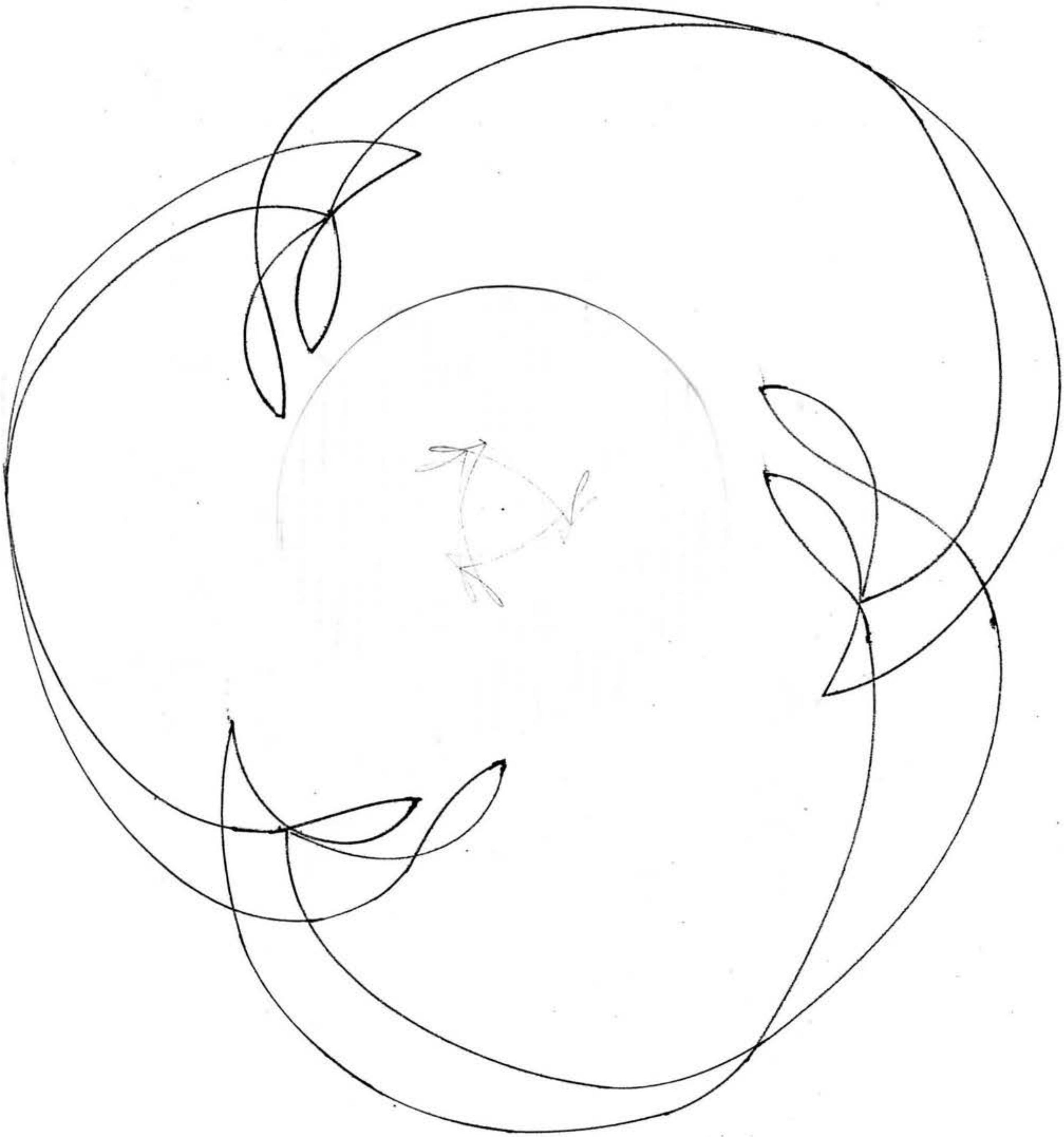
Cusps

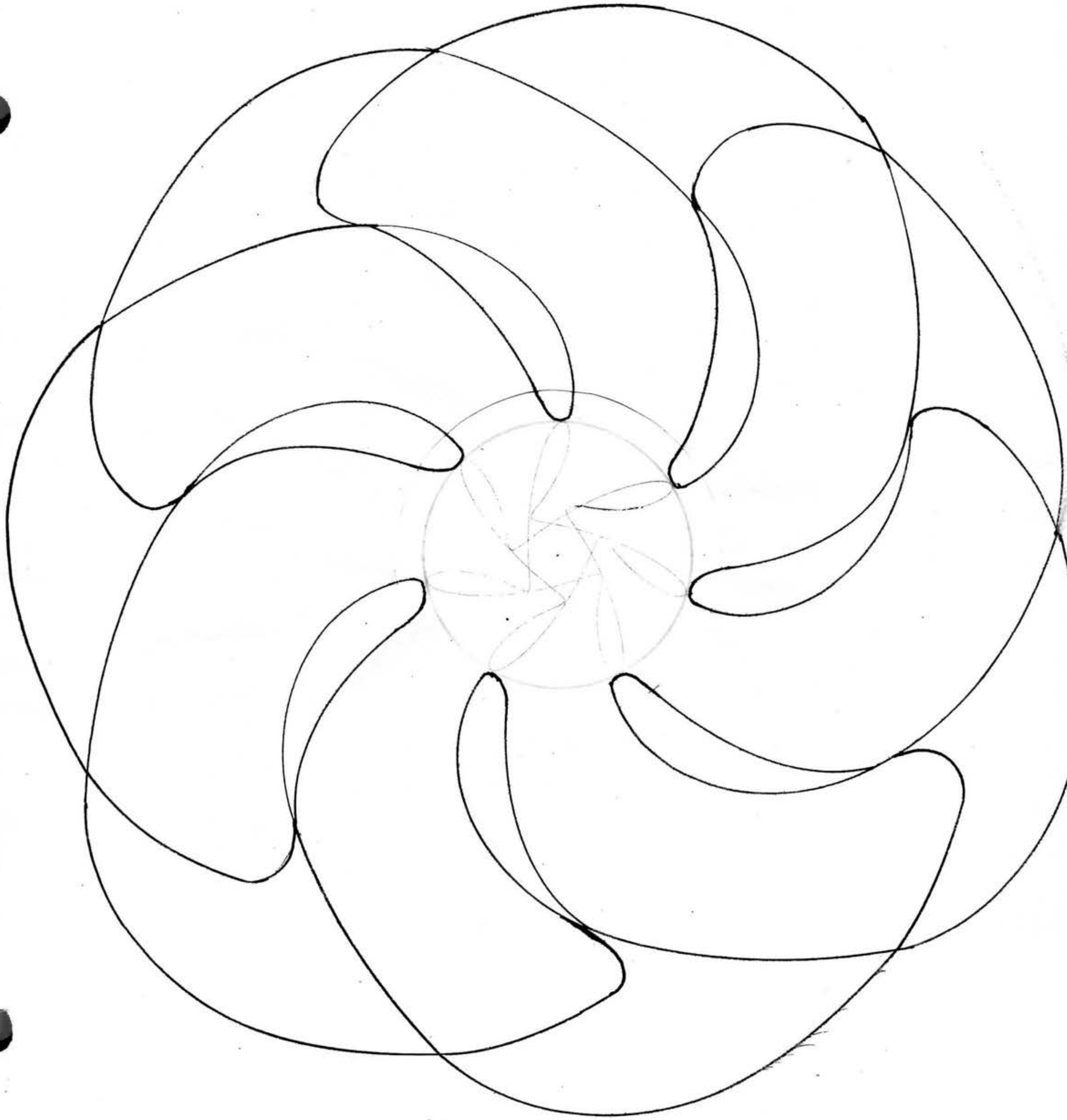


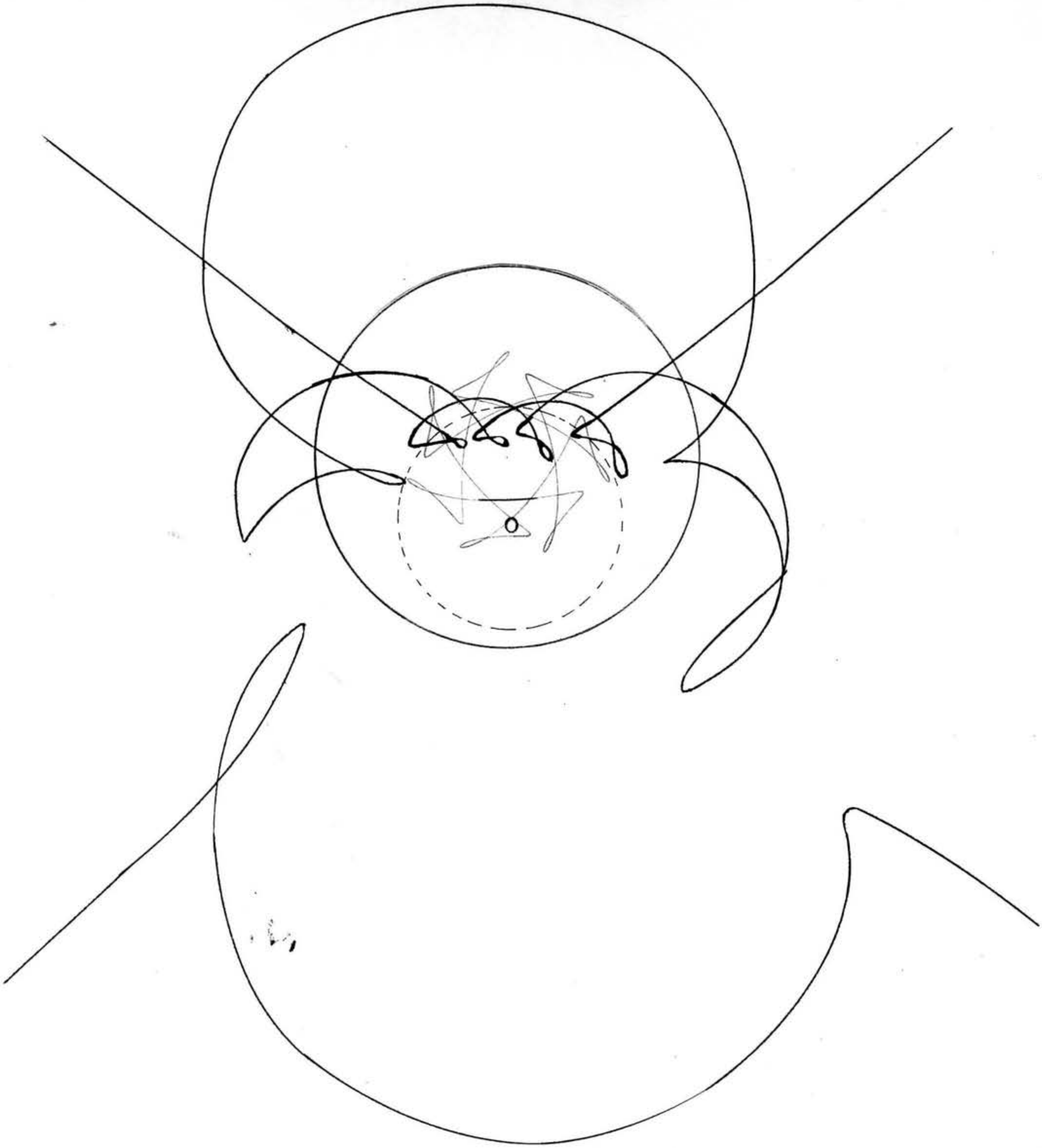
Cusps



Cusps of opposite figure
have become nodes







The Inverse of a Certain Harmonic Curve

VI. Circular cubics as inverses of conic sections.

A circular cubic is a cubic which passes through the circular points at infinity.

The inverse of a conic with respect to a point on the curve is a circular cubic whose asymptote is parallel to the tangent to the conic at the centre of inversion.

In the case of the parabola, letting its vertex be the centre of inversion, we will get a cissoid. This is shown in Fig. 12.

This may also be worked out analytically. If the vertex of the parabola is the origin, its equation is

$$y^2 = 2px$$

substituting the transformation values used before, we get

$$\frac{y'^2}{(x'^2 + y'^2)^2} = \frac{2px'}{x'^2 + y'^2}$$

reducing and dropping the primes,

$$x^3 = y^2\left(\frac{1}{2p} - x\right)$$

If $\frac{1}{2p} = 2a$, we obtain the form of the cissoid usually given.

If the conic is a hyperbola having either vertex as the centre of inversion, the inverse curve is a strophoid. This is shown in Fig. 13. It is often called the logocyclic curve.

The equation of the equilateral hyperbola, when the origin is the right-hand vertex, is

$$x^2 - y^2 + 2ax = 0$$

substituting, we get

$$\frac{x'^2}{(x'^2 + y'^2)^2} - \frac{y'^2}{(x'^2 + y'^2)^2} + \frac{2ax'}{x'^2 + y'^2} = 0$$

Reducing and dropping primes,

$$x(x^2+y^2)+\frac{1}{2a}(x^2-y^2)=0$$

The locus of this equation is the strophoid. If $\frac{1}{2a}$ is replaced by a', when we solve for y^2 we get

$$y^2=x^2\frac{a'+x}{a'-x}$$ which is the form usually given.

When the conic is an ellipse having any point on it as the centre of inversion, the inverse curve is a witch. This is shown in Fig. 14.

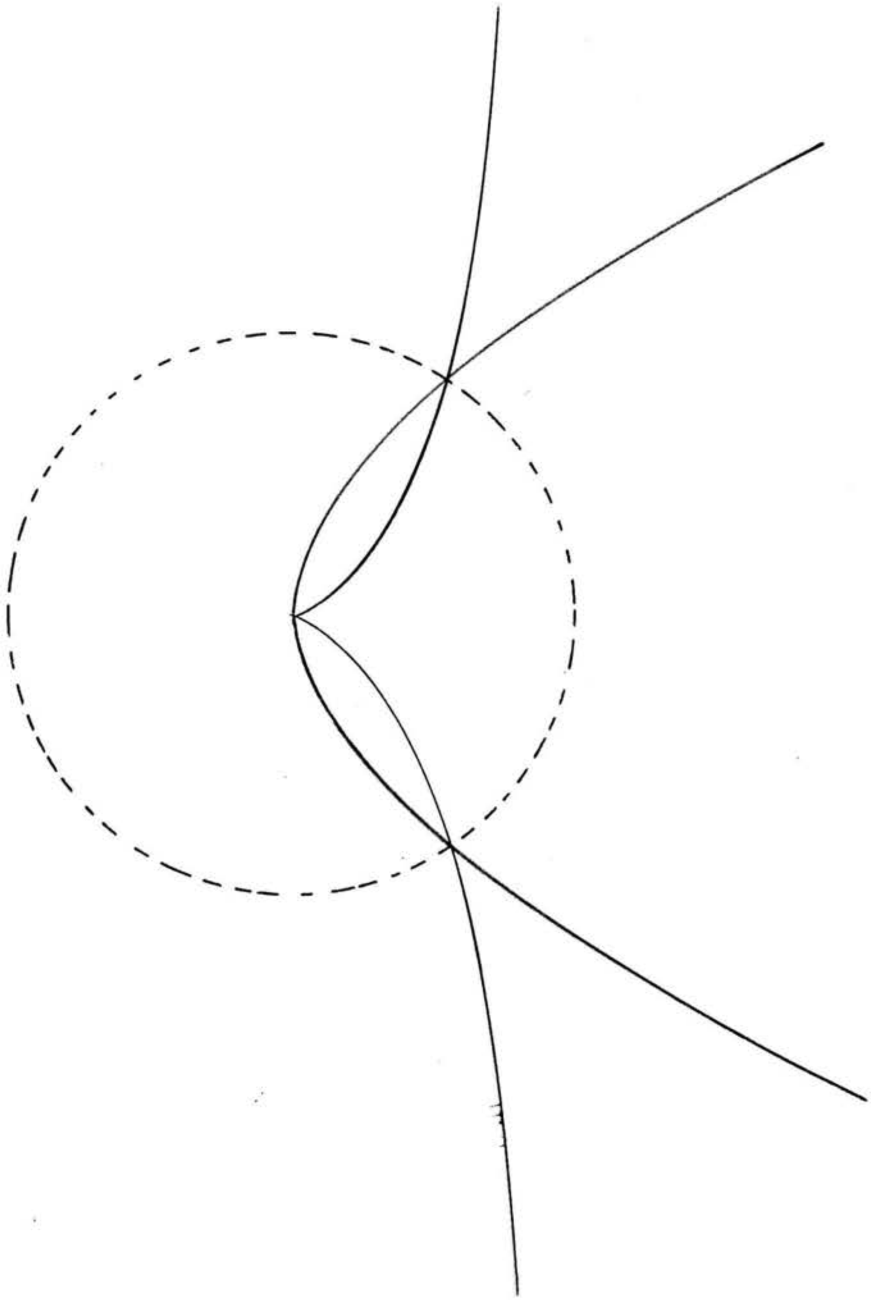


Fig. 12

The inverse of a Parabola. Centre of inversion on the Parabola

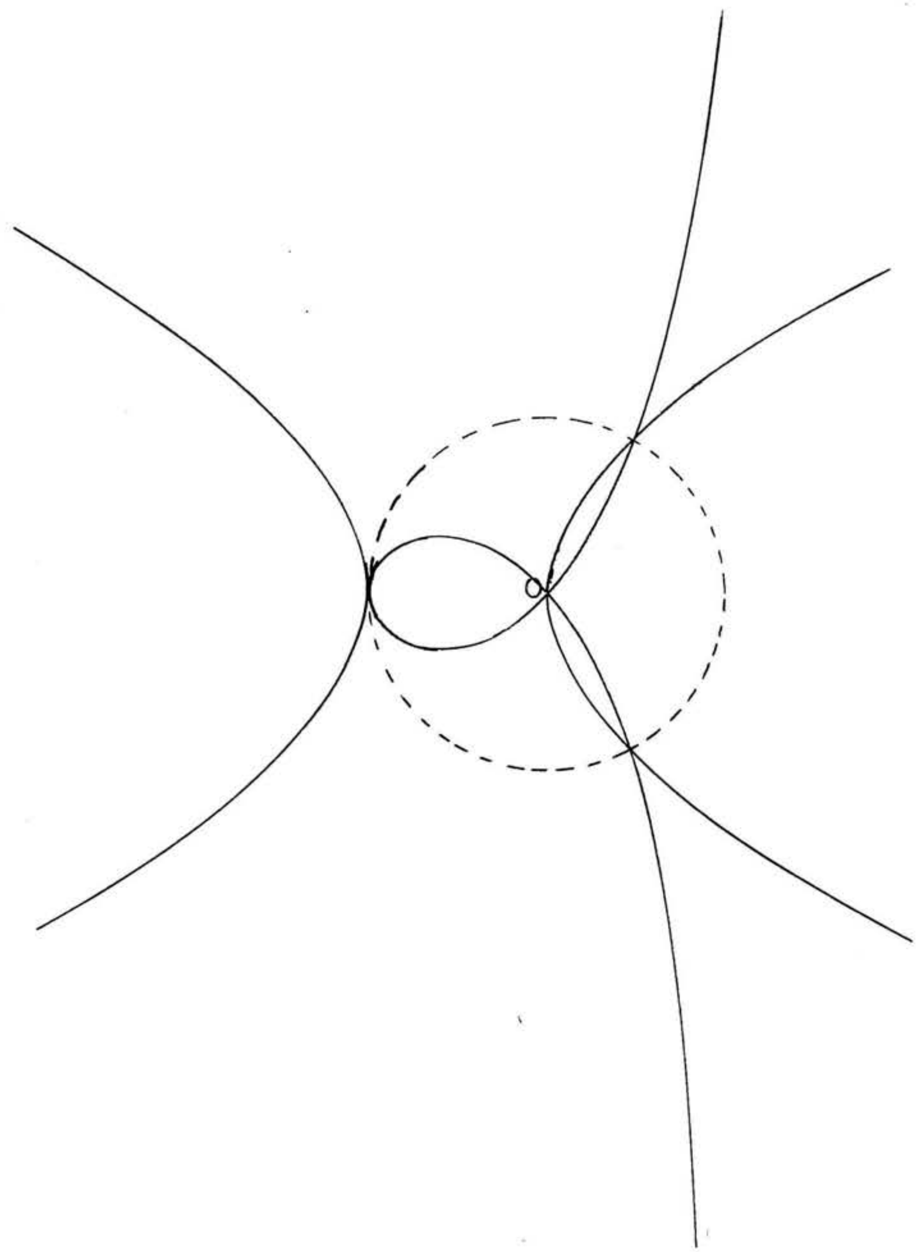


Fig. 13

Inverse of a Hyperbola. Centre of Inversion on the Hyperbola.

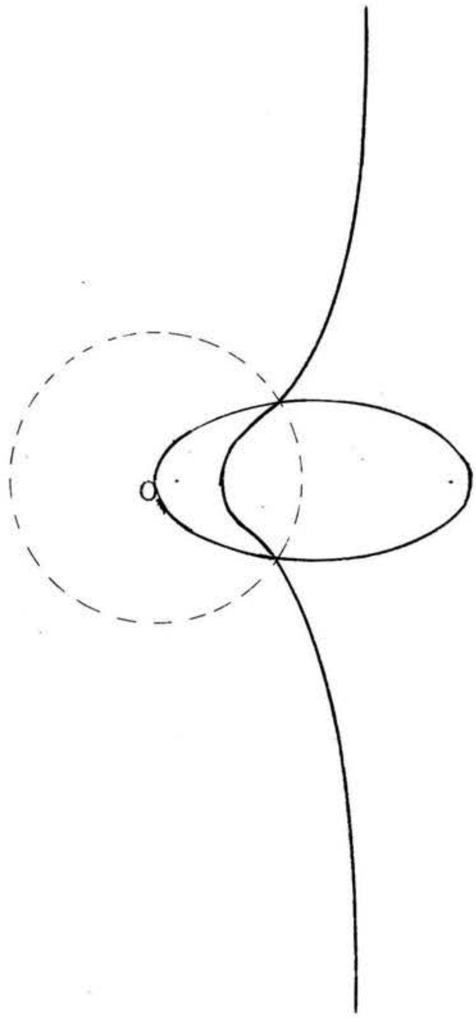


Fig. 14

Inverse of an Ellipse. Centre of Inversion on the Ellipse.

VII. Bicircular quartics as inverses of conic sections.

Bicircular quartics are a class of quartics which possess a pair of nodes at infinity.

The inverse of a conic with respect to any point not on the curve is a bicircular quartic having a third double point at the centre of inversion; and this point will be a node, a cusp or a conjugate point according as the conic is a hyperbola, a parabola or an ellipse. These are shown in Fig. 15, 16, and 17.

When the centre of inversion is the focus of the conic the quartic becomes a cartesian, which is called a limacon when the conic is an ellipse or hyperbola, and a cardiod when the conic is a parabola. A cartesian may be defined as a quartic which has a pair of cusps at the circular points at infinity. ?hen the centre of inversion lies on the curve, the quartic degenerates into a circular cubic, which has already been discussed. When the centre of inversion is the centre of the conic, the inverse curve is a trinodal quartic.

When the conic is an ellipse, the inverse curve takes the form of the limacon as shown in Fig. 18, provided the centre of inversion is at the focus of the ellipse.

The equation of the elippse whose origin is at the focus is

$$(1 - e^2)x^2 + y^2 - 2e^2px - e^2p^2 = 0$$

Substituting the transformation values, we get

$$\frac{(1 - e^2)x'^2}{(x'^2 + y'^2)^2} + \frac{y'^2}{(x'^2 + y'^2)^2} - \frac{2e^2px'}{x'^2 + y'^2} - e^2p^2 = 0$$

Simplifying and clearing of fractions, we have

$$e^2p^2(x^2+y^2)+2e^2px(x^2+y^2)=(1-e^2)x^2+y^2$$

Adding e^2x^2 to both sides and dividing by e^2p^2, we get

$$(x^2+y^2+\frac{1}{p}x)^2=\frac{1}{e^2p^2}(x^2+y^2)$$

If $\frac{1}{p}=a$ and $\frac{1}{e^2p^2}=b$, we get the form of the limacon usually given, that is

$$(x^2+y^2+ax)^2=b^2(x^2+y^2) \quad \text{where } a<b.$$

When the conic is a hyperbola, the inverse takes the form of the curve in Fig. 19. This is often called the hyperbolic limacon.

The same analytic discussion used in finding the inverse of the ellipse still holds good for this case except that here, $a>b$.

When the conic is a parabola, the inverse curve takes the form shown in Fig. 20. This is the cardiod. It is a special case of the limacon.

The analytic discussion is the same as above except that in this case $a=b$.

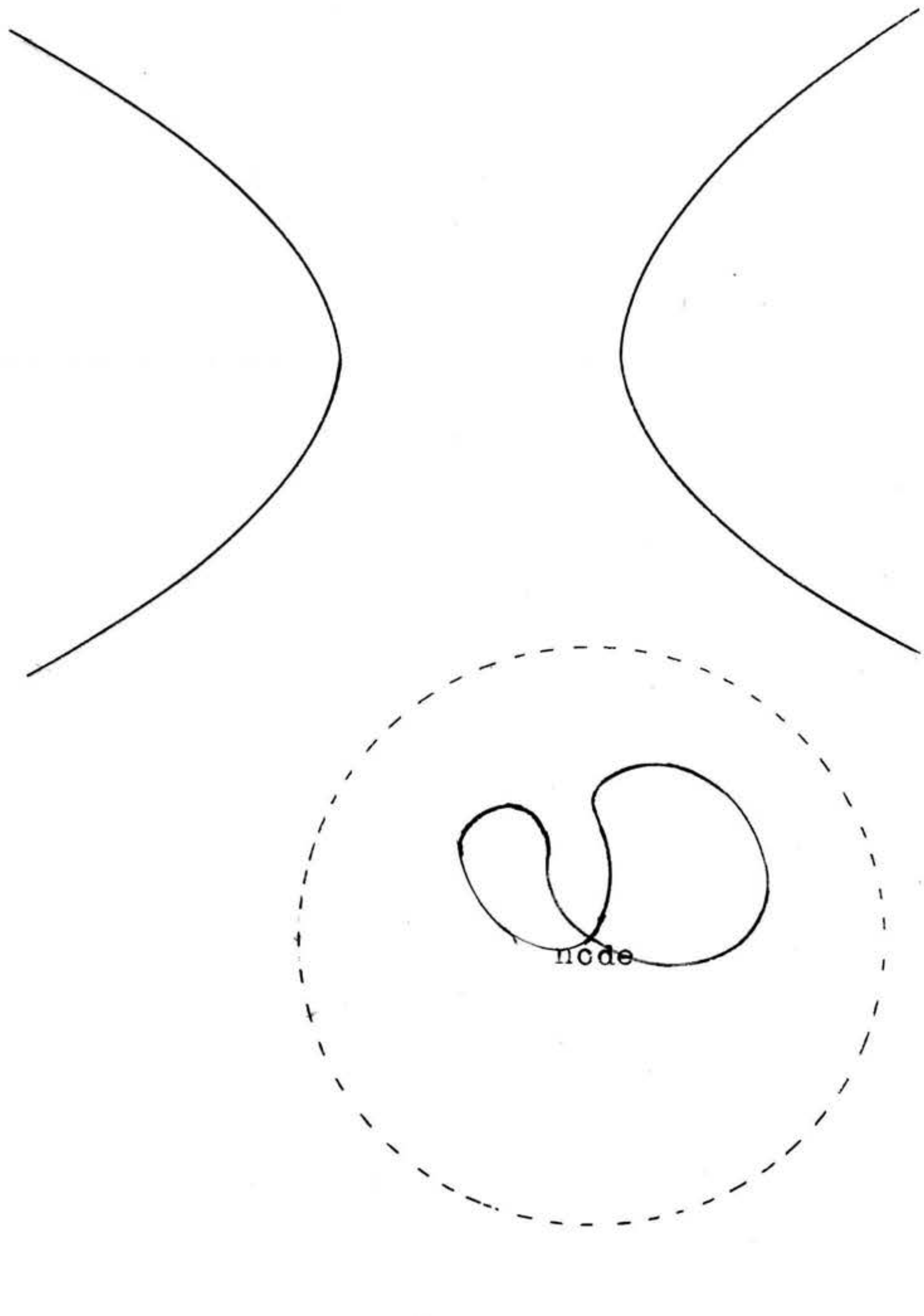


Fig. 15

Inverse of a Hyperbola. Centre of Inversion not on the Hyperbola

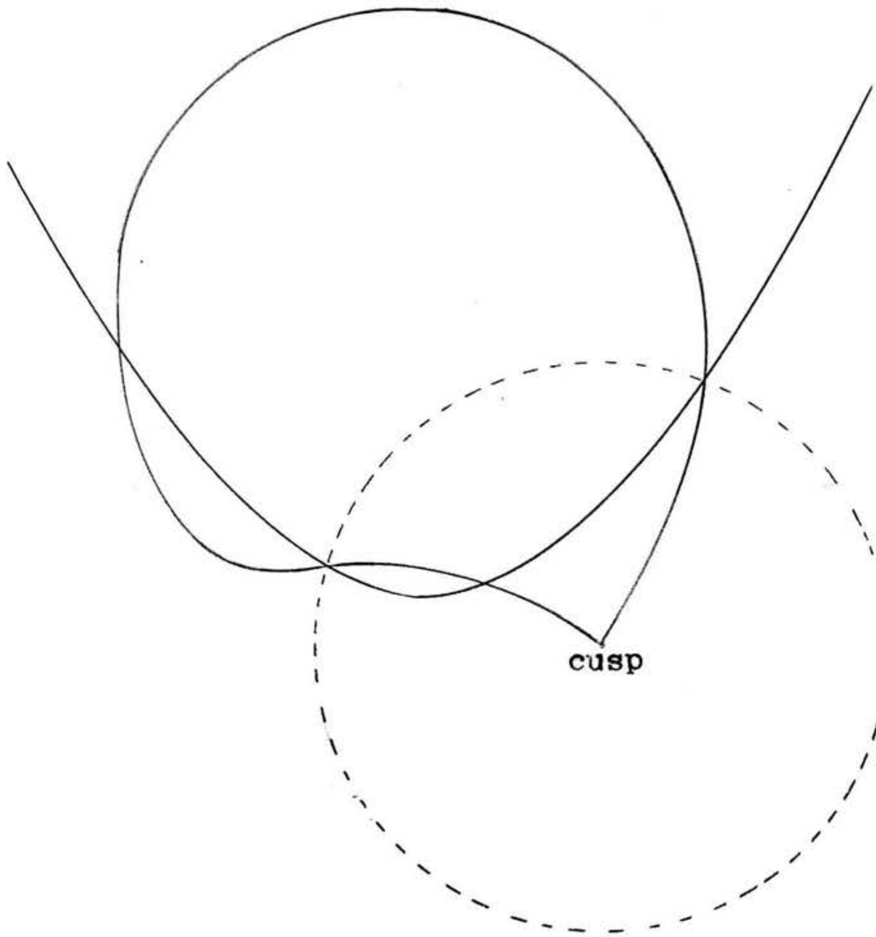


Fig. 16

Inverse of a Parabola. Centre of Inversion not on the Parabola

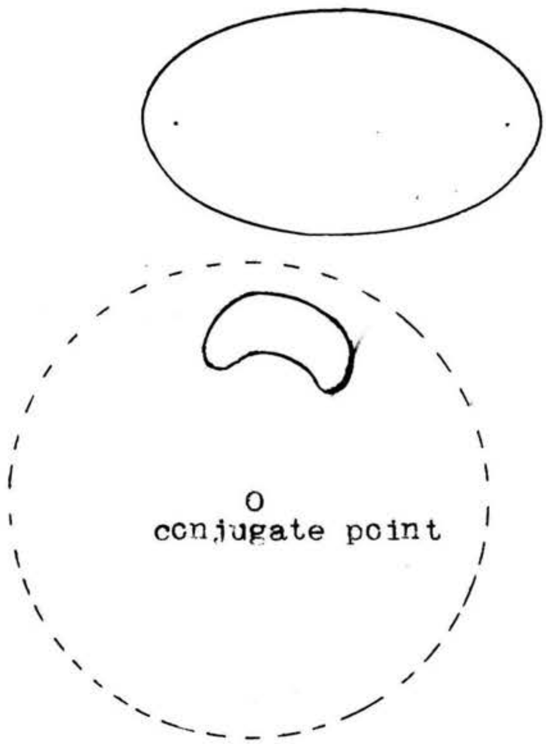


Fig. 17

Inverse of an Ellipse. Centre of Inversion not on the Ellipse.

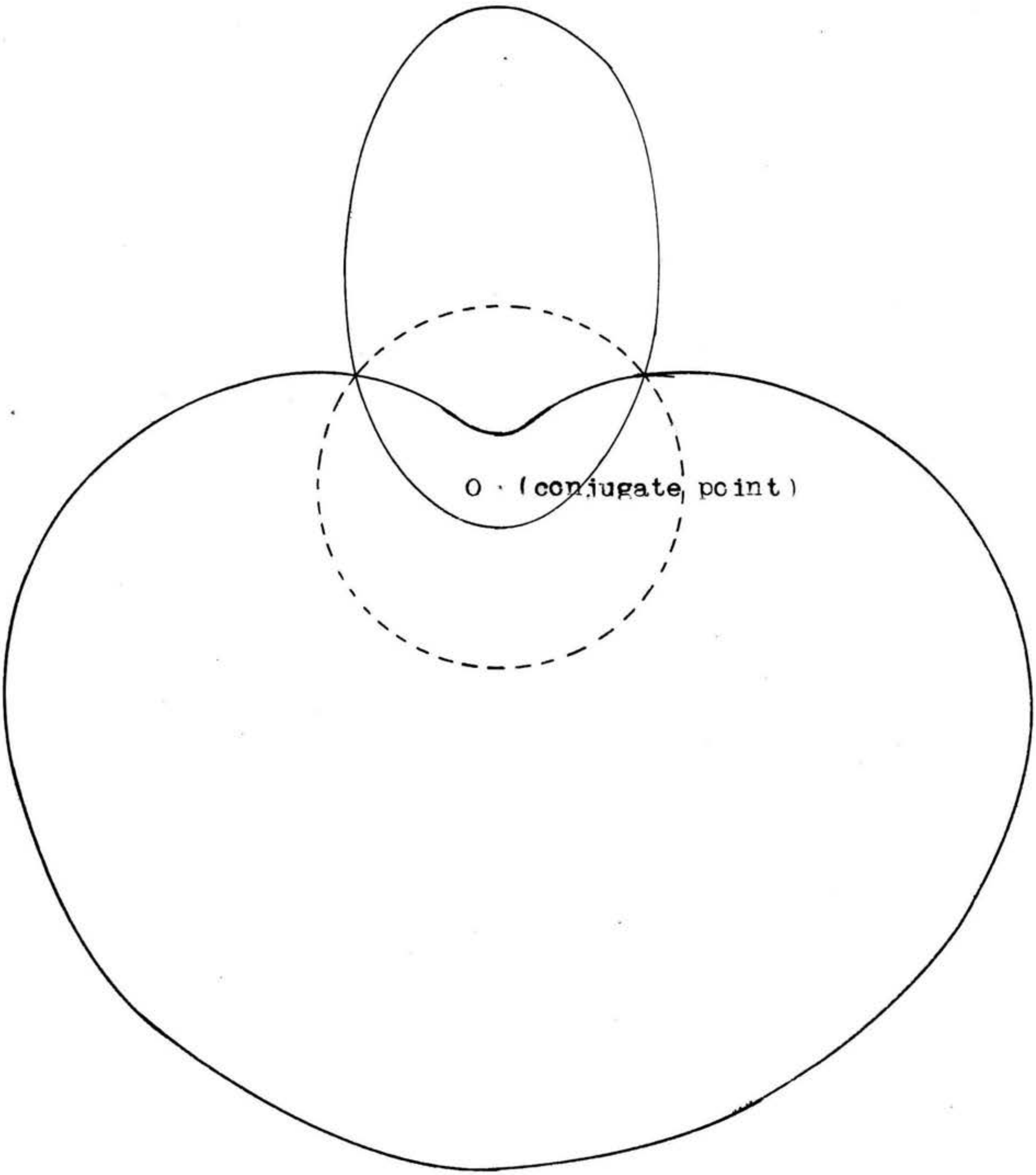


Fig. 18

Inverse of an Ellipse. Centre of Inversion is one of the Foci.

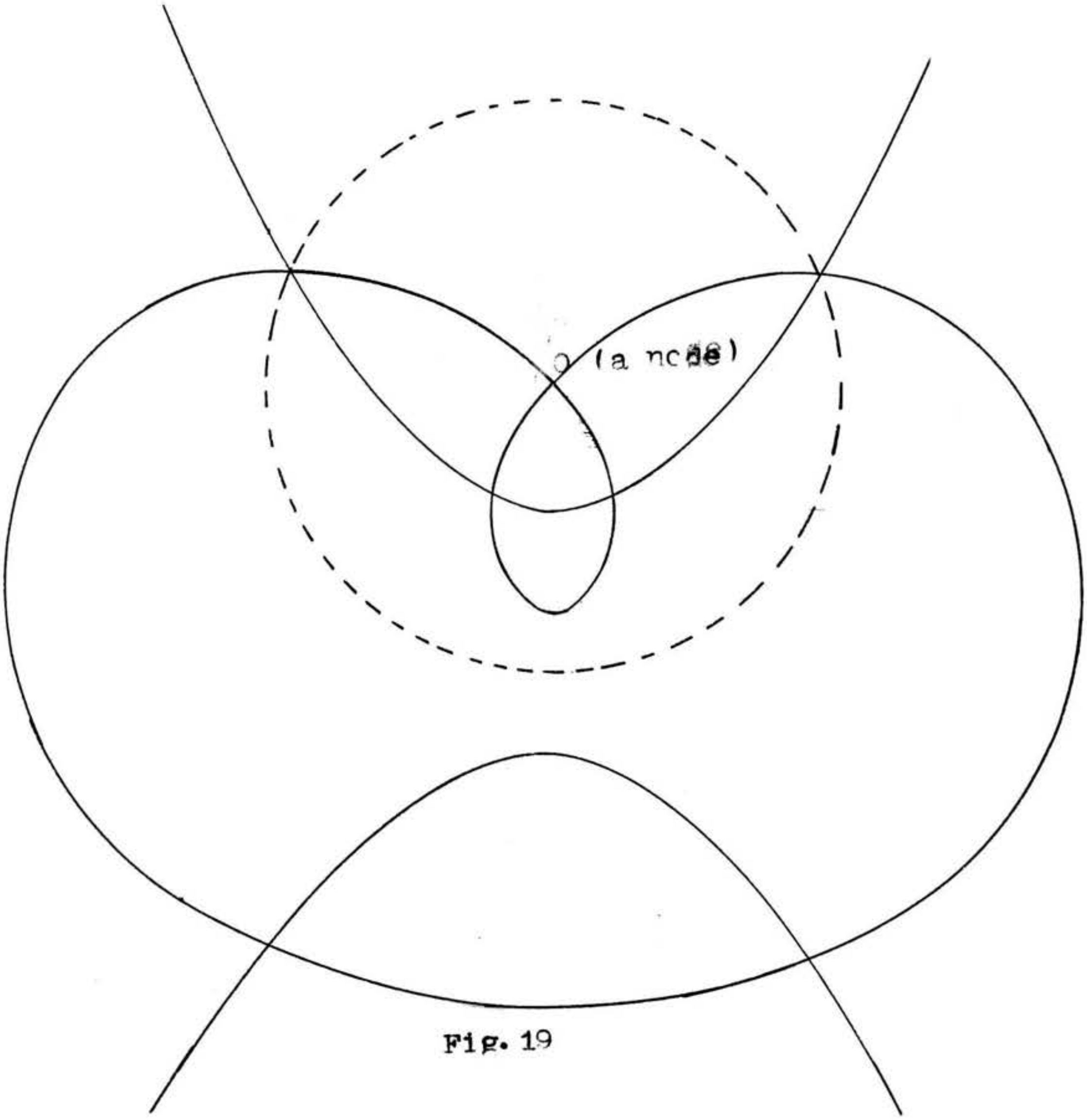


Fig. 19

Inverse of a Hyperbola. Centre of Inversion at one of the Foci.

Inverse of a Parabola. Centre of Inversion at the Focus.

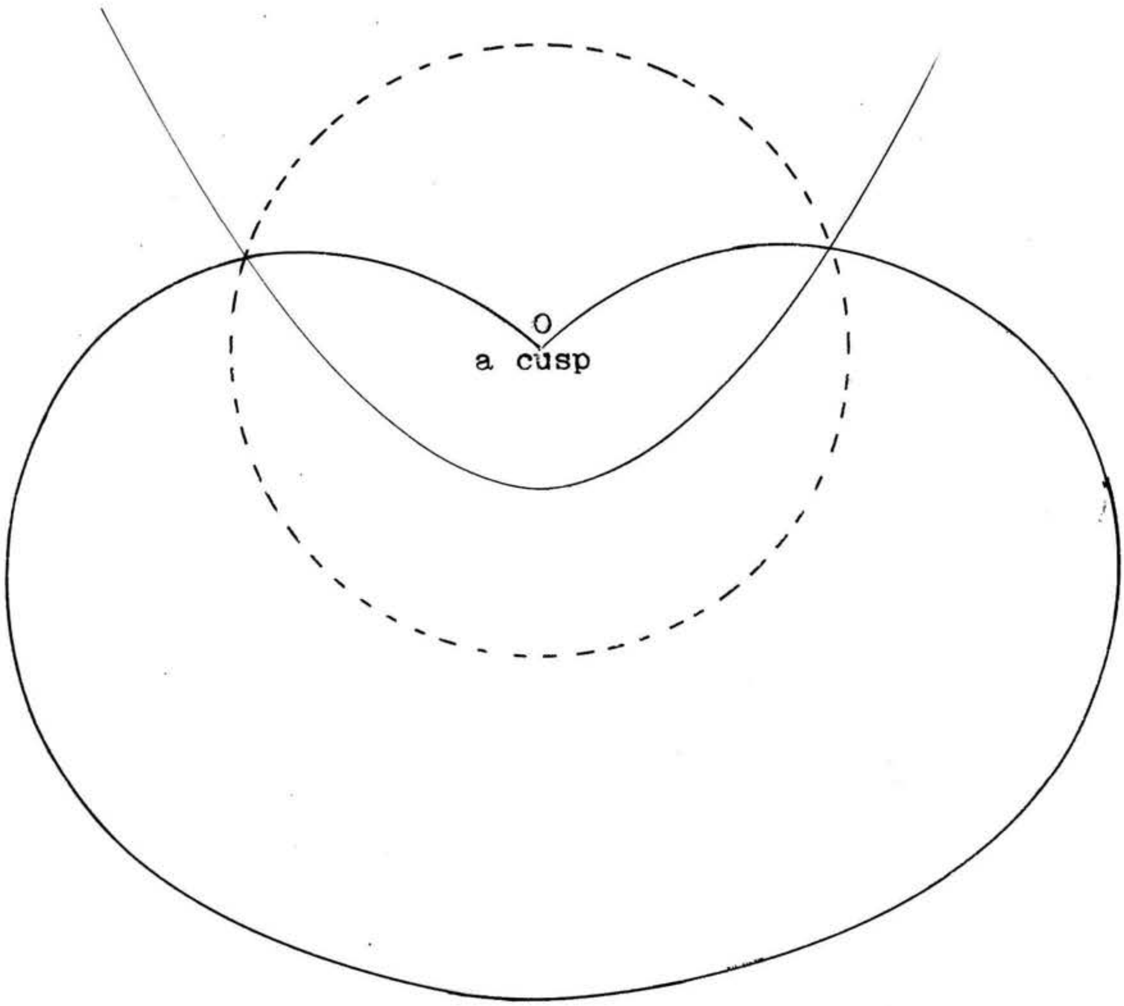


Fig. 20

When the centre of inversion is the centre of a hyperbola, the inverse curve is a lemniscate.

The equation of the equilateral hyperbola is

$$x^2 - y^2 = a^2.$$

Substituting the values of x and y, we get

$$\frac{x'^2}{(x'^2 + y'^2)} - \frac{y'^2}{(x'^2 + y'^2)} = a^2$$

Reducing and dropping the primes,

$$(x'^2 + y^2)^2 = \frac{1}{a}\,(x^2 - y^2).$$

The locus of this curve is a lemniscate. If $\frac{1}{a} = a'$, we get the form of the equation usually given,

$$(x^2 + y^2)^2 = a'\,(x^2 - y^2).$$

The form of the curve is shown in Fig. 21.

When the centre of inversion is the centre of an ellipse, the inverse curve is the Cassinian Oval.

The equation of the ellipse is

$$x^2 - y^2 = a^2 b^2$$

Substituting and simplifying, we get

$x'^2 - y'^2 = a^2 b^2 (x'^2 + y'^2)^2$ or dropping primes,

$$x^2 - y^2 = a^2 b^2 (x^2 + y^2)^2$$

The locus of this curve is the Cassinian Oval. Its form is shown in Fig. 22.

Inverse of a hyperbola. Centre of inversion at the centre of the hyperbola.

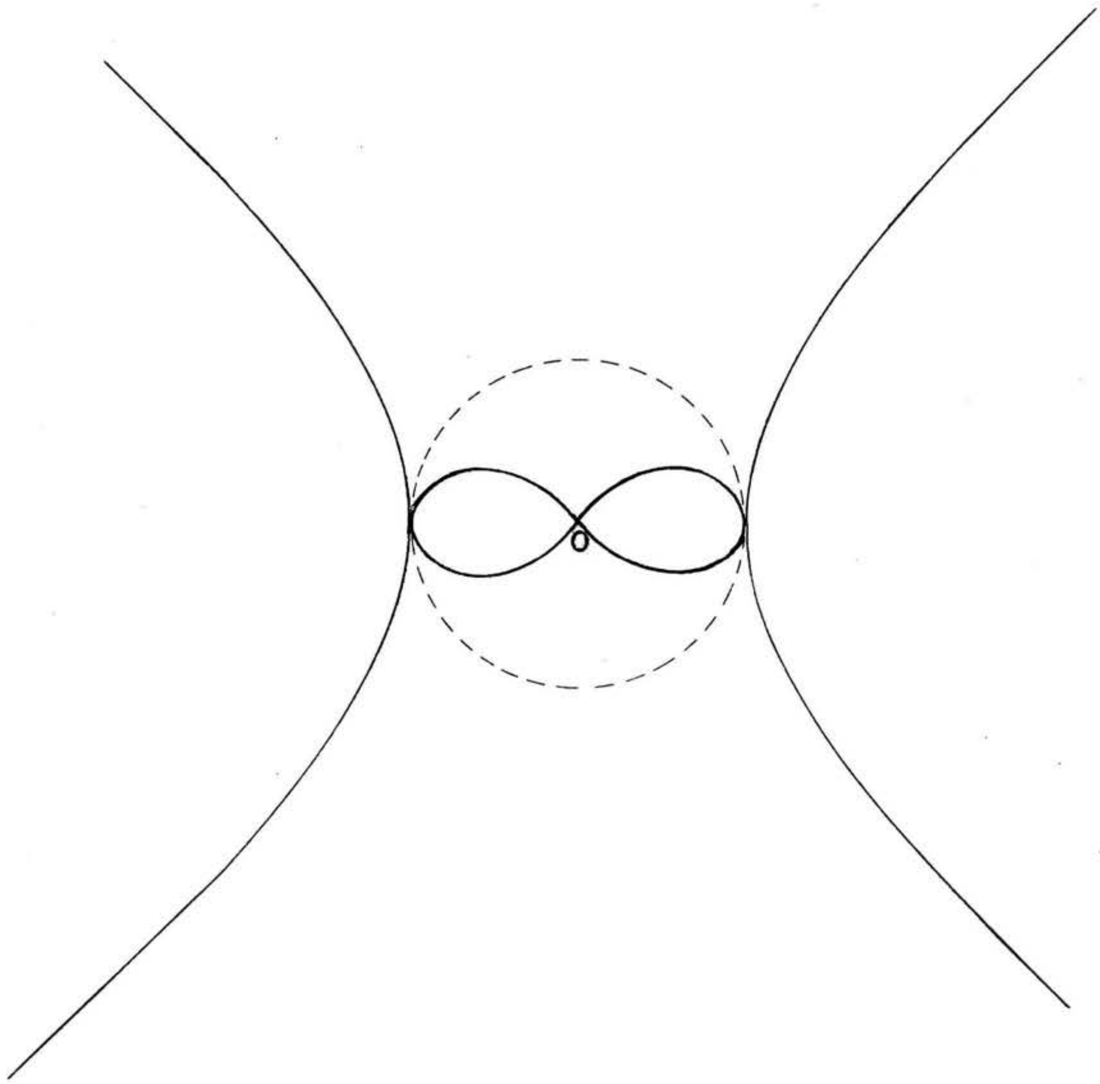


Fig. 21

Inverse of an Ellipse. Centre of Inversion at centre of Ellipse.

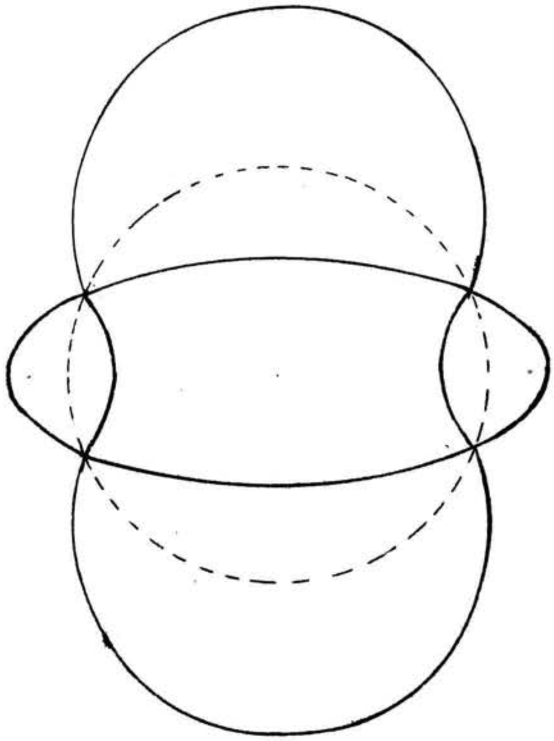


Fig. 22

Cassinian Oval

VIII. Construction of various other curves by inversion.

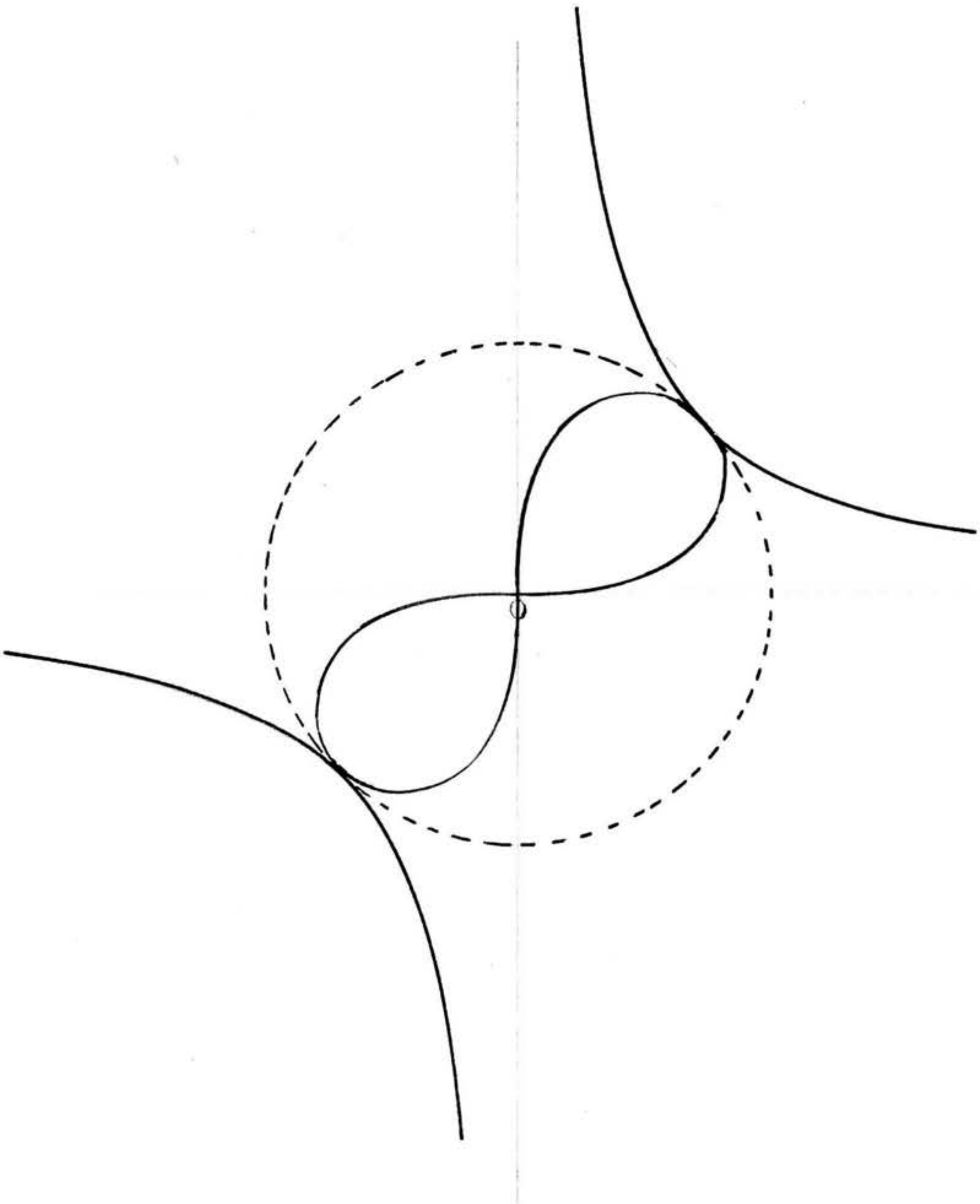


Fig. 23

Inverse of the curve $y = \frac{1}{x}$

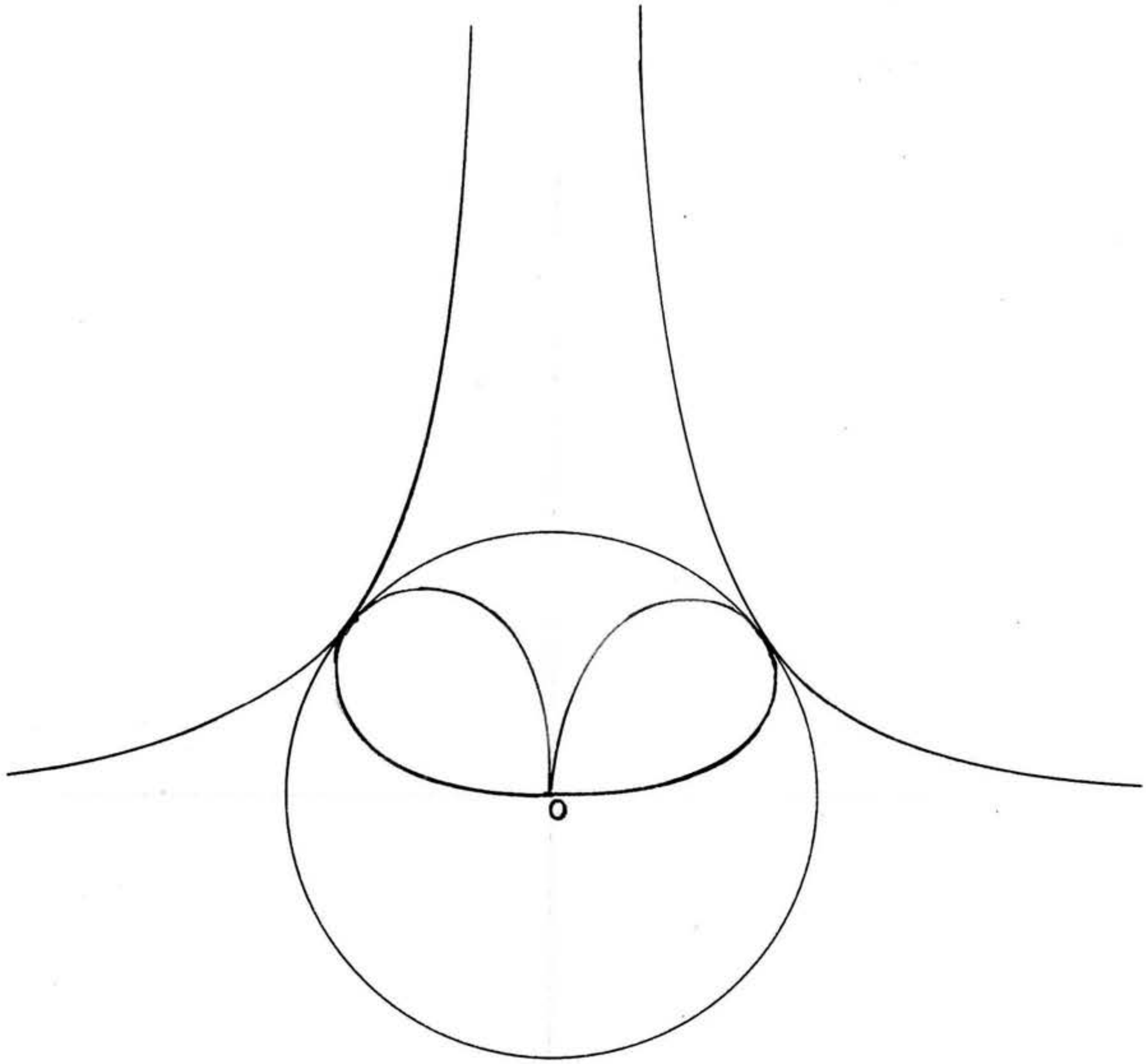


Fig. 24

Inverse of the curve $y = \frac{1}{x^2}$

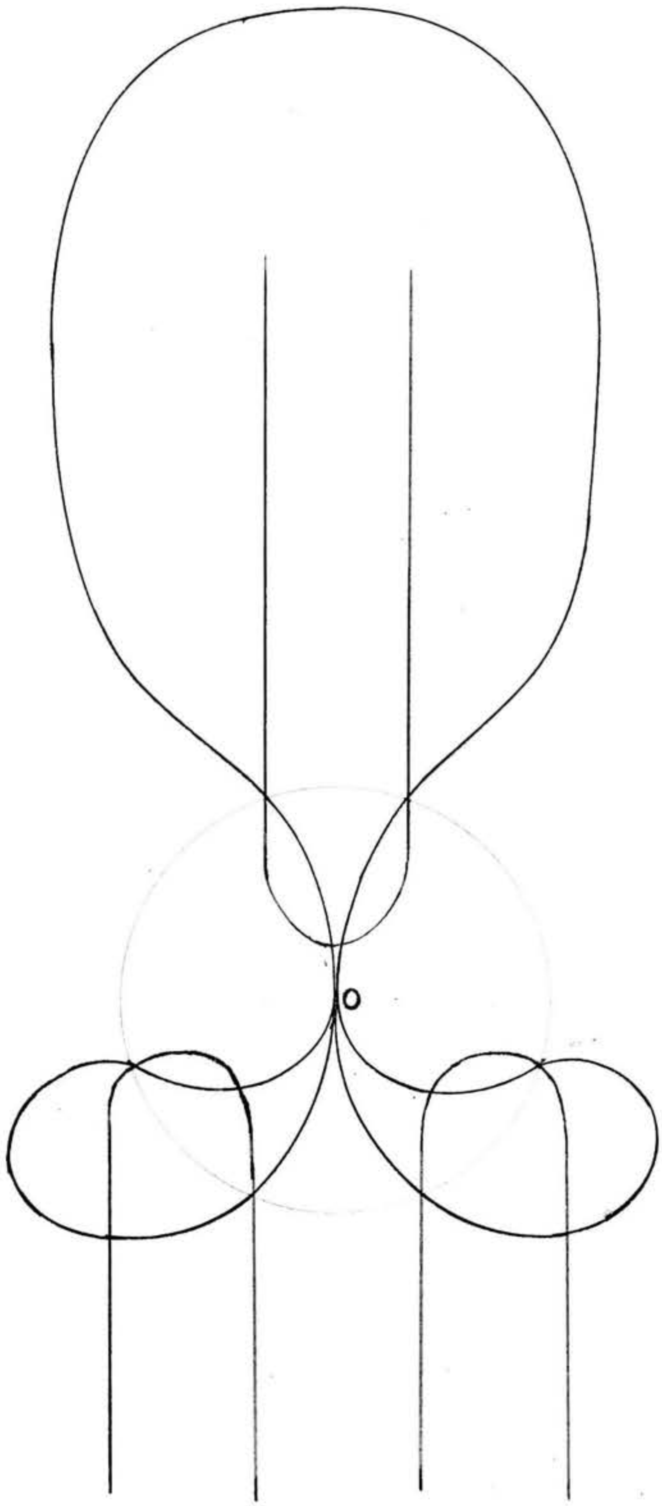


Fig. 25

The Inverse of a Section of the Curve $y = \sec(x)$

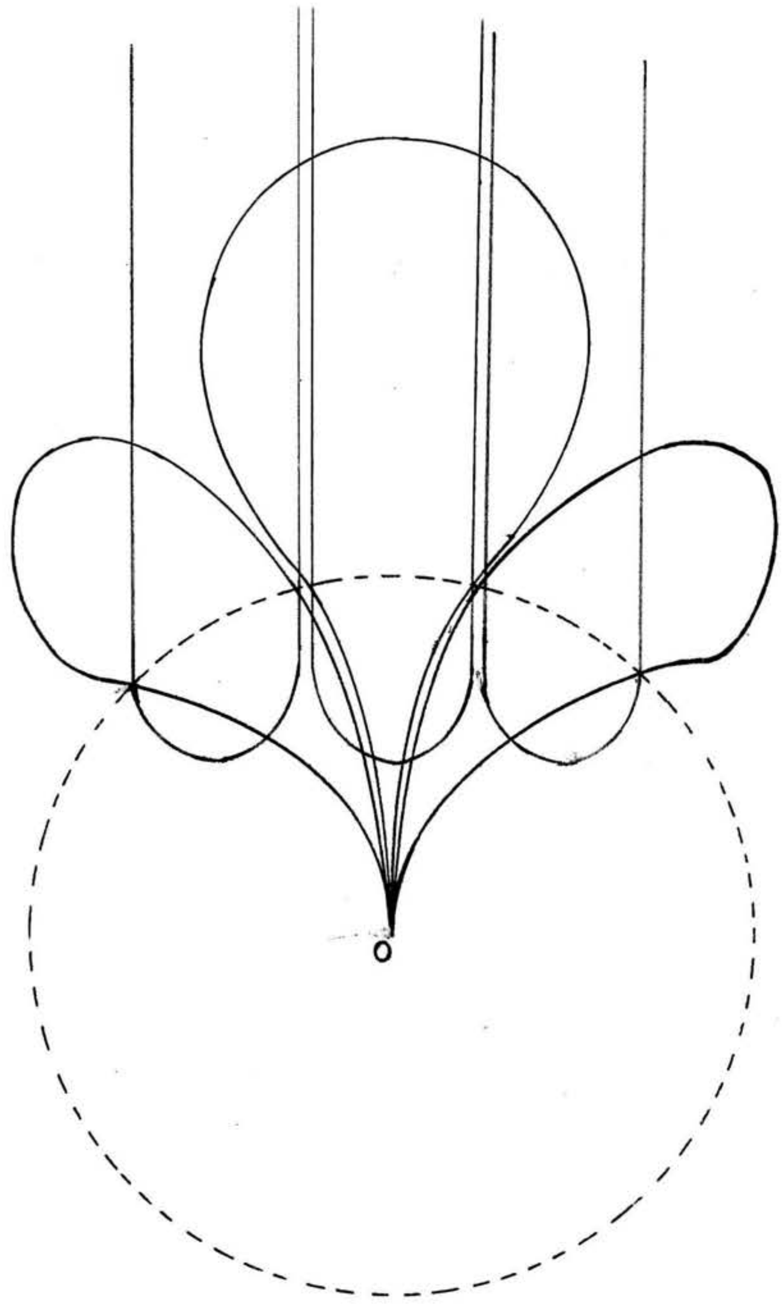


Fig. 26

The Inverse of a Section of the curve $y = \sec^2(x)$

IX. Inversion in three dimensions.

Inversion as has been applied to two dimensions can well be carried over into three dimensions. In the figure on Pg.54 let the sphere with centre O be the sphere of inversion, and let the sphere with centre A be any other sphere which passes throug O. The inverse of this sphere then becomes the plane through the points where the given sphere and the sphere of inversion intersect each other .

The equation of a sphere through the centre of inversion is

$$Ax^2 + Ay^2 + Az^2 + Bx + Cy + Dz = 0$$

The transformation values in this case are

$$\frac{x'}{x'^2+y'^2+z'^2},\ \frac{y'}{x'^2+y'^2+z'^2},\ \frac{z'}{x'^2+y'^2+z'^2},$$ for x, y, and z .

substituting these values, we get

$$Ax'^2 + Ay'^2 + Az'^2 + Bx'(x'^2+y'^2+z'^2) + Cy'(x'^2+y'^2+z'^2) + Dz'(x'^2+y'^2+z'^2) = 0$$

simplifying and dropping primes, we get

$$A + Bx + Cy + Dz = 0$$

This is the equation of a plane.

In the same way we may prove that the inverse of a sphere not through the centre of inversion is another sphere.

Let the equation of this sphere be

$$Ax^2 + Ay^2 + Az^2 + Bx + Cy + Dz + E = 0$$

Substituting, we have

$$Ax'^2 + Ay'^2 + Az'^2 + Bx'(x'^2+y'^2+z'^2) + Cy'(x'^2+y'^2+z'^2) +$$
$$Dz'(x'^2+y'^2+z'^2) + E(x'^2+y'^2+z'^2)^2 = 0$$

Simplifying and dropping the primes, we get

$$A + Bx + Cy + Dz + Ex^2 + Ey^2 + Ez^2 = 0$$

This is the equation of a sphere not through the origin.

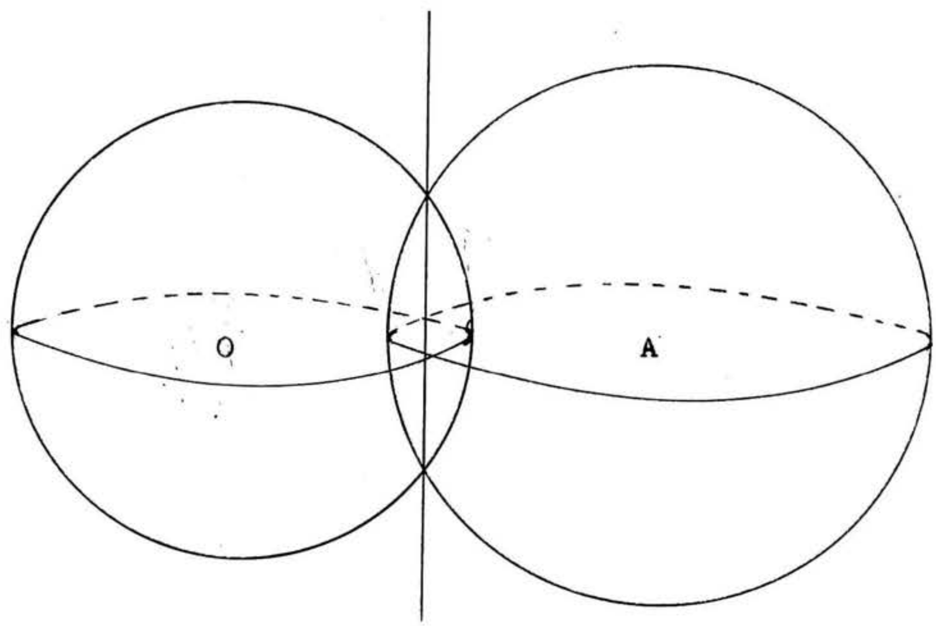


Fig. 27

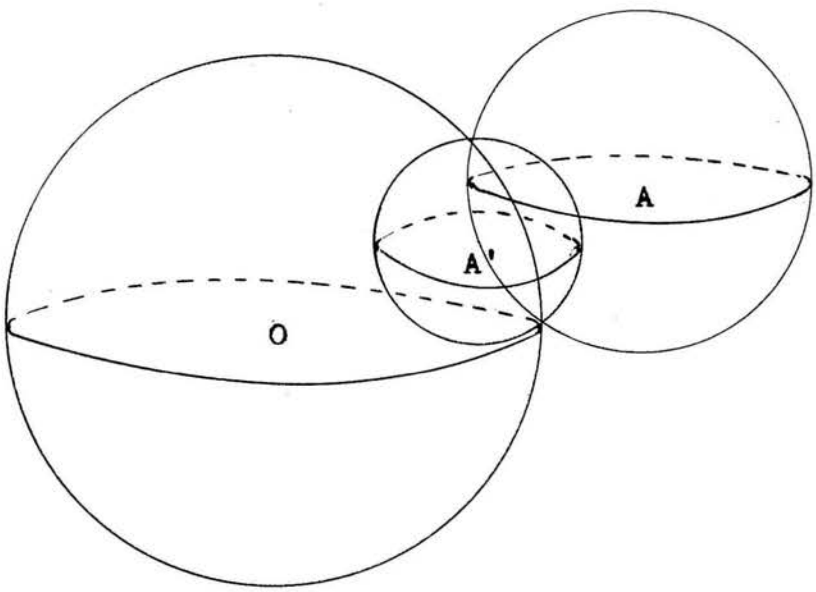


Fig. 28

X. Cyclides.

A cyclide is the envelope of a sphere whose centre moves on a fixed quadric F, and which cuts a fixed sphere J orthogonally. Thus the cyclide is its own inverse with respect to the sphere J: for any sphere which cuts J orthogonally is its own inverse in respect to it, so that the generating sphere not being changed by inversion, neither is the envelope.

A cyclide is also defined by the equation

$$u_0(x^2+y^2+z^2) + u_1(x^2+y^2+z^2) + u_2=0$$

where u_0 is a constant, u_1 = a polynomial of the first degree, and u_2 = a polynomial of the second degree in x, y, and z.

If $u_0 \neq 0$ the surface is of the fourth degree and represents a bi-quadratic surfave with the imaginary circle at infinity as a double curve.

If $u_0=0$, the equation is of the third degree and represents a cubic surface passing through the imaginary circle at infinity

If u_0 and u_1 are both equal to zero, the surface degenerates into a quadric surface or plane.

The Dupin's cyclide is a special case of the cyclide. It is defined as the envelope of afamily of spheres which are tangent to three fixed spheres. If the centres of the spheres do not lie on a straight line, they can be made to do so by inversion. In the plane of the centres of the three spheres, draw a circle orthogonal to the three spheres and take any point on this circle as the centre on inversion. The circle then goes into a straight line and will pass through the centre of the transformed spheres.

Hence, the surface enveloped by spheres tangent to the original three spheres is inverted into a surface enveloped by spheres tangent to three spheres whose centres lie on a straight line. This inverse surface is known as Dupin's cyclide. Therefore, any Dupin's cyclide is the inverse of a ring surface formed by revolving a circle about an axis not in its plane.

In Fig. 28 let three given spheres be the spheres with centres A, B, and C. The circle with centre Z is the circle in the plane of their centres and cuts them orthogonally. Let the circle of inversion have its centre O at any point on this circle. Since the orthogonal circle now passes through the centre of inversion, its inverse will be a straight line. The inverse of the three original spheres will be three other spheres with centres A', B', and C'. These centres will lie on the straight line which is the inverse of the orthogonal circle. Let the sphere with centre H be a sphere tangent to the three given spheres. Its inverse is the sphere with centre H' and is tangent to the spheres whose centres lie on the straight line. The envelope of this sphere formed by revolving it around the line B'A'C', is the Dupin cyclide.

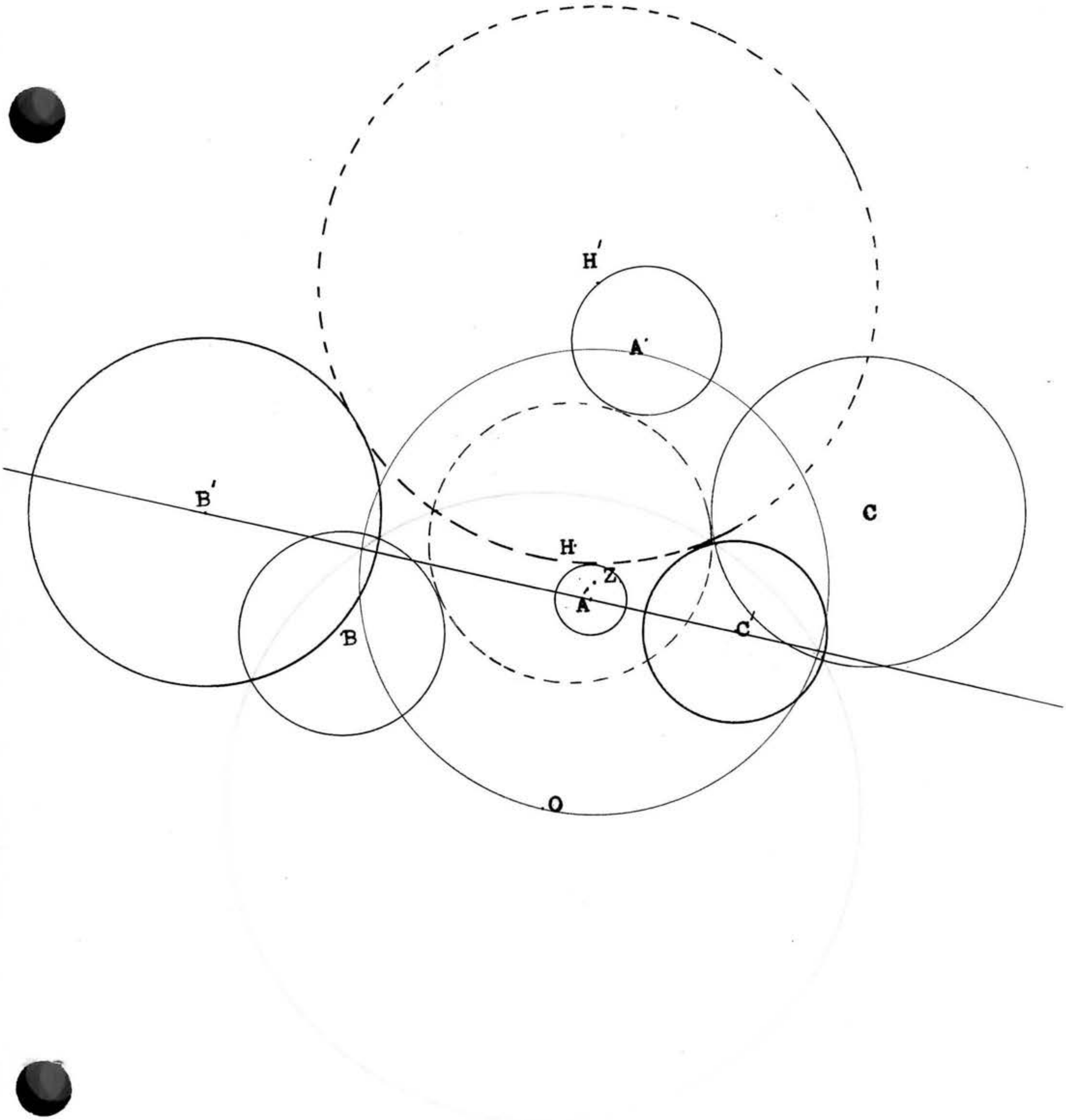


FIG. 28

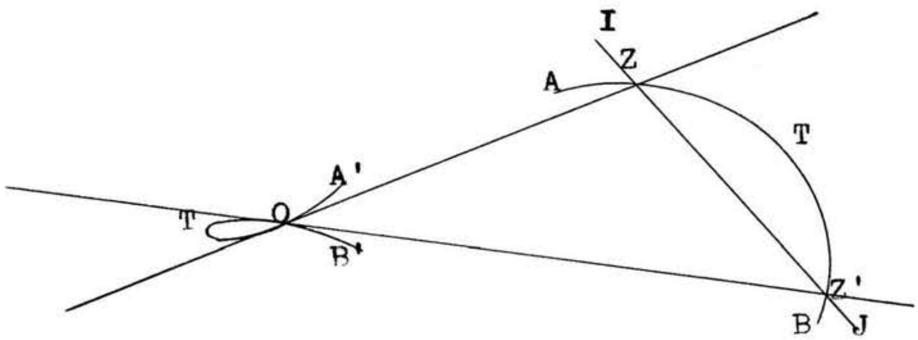
XI. Inversion applied to the Theory of Correspondence.

By means of quadric inversion, correspondence can be produced. The correspondence of point to point can be formed by means of a fixed fundamental conic, the base, and the fixed origin O. Points that are collinear with the origin and conjugate with respect to the base are said to be inverse. If for the fundamental conic and the origin we take a circle and its centre, the points are the ordinary inverse points with regard to a circle. Hence, the process is simply circular inversion generalized. The points have a one to one correspondence, that is for a point P in a plane there corresponds definitly a point P'. It is necessary in this work to bring in the circular points at infinity or I and J. If P lies at O, its inverse will lie on the line IJ: if P be at I, its inverse will lie on OI: if P lies at J, its inverse is on OJ.

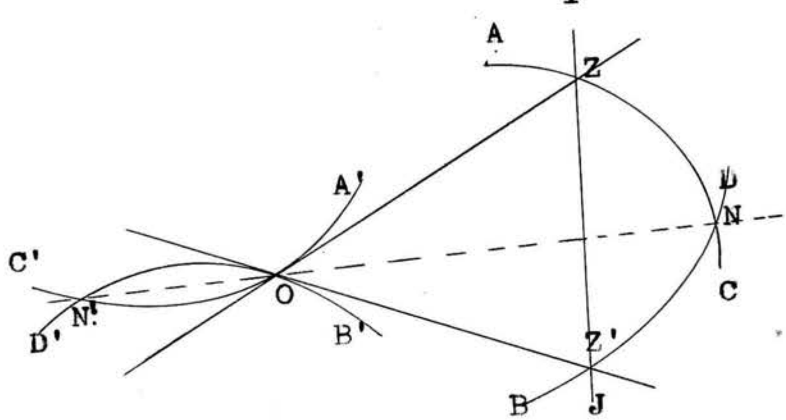
In general, the inverse of a straight line is a conic passing through O, I, and J. If, however, the line passes through O, the conic becomes a degenerate conic, composed of the given line and the line at infinity. Similarly, when a curve passes through O, the line infinity will be part of the inverse curve.

We can now study the effect of inversion on singularities. We have seen that a point inverts into a single point. For the ordinary point, three consecutive points are collinear; but if these three points and O, I, J lie on a conic, then their inverses are collinear, and on the inverse curve there is an inflection. Similarly, an inflection may be lost by inversion and it will be lost unless the inflectional tangent passes through a principal point.

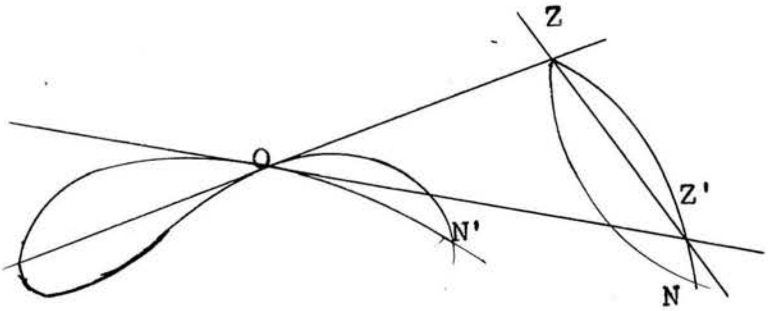
If the curve cuts IJ at n points other than I and J, the inverse has at O a multiple point of order n. In the figure below an ordinary branch cuts IJ at Z, Z'. The inverse has a loop with a node at O, the two tangents being OZ, OZ'. If OZ be tangent at Z the inverse has OZ as an inflectional tangent at O.



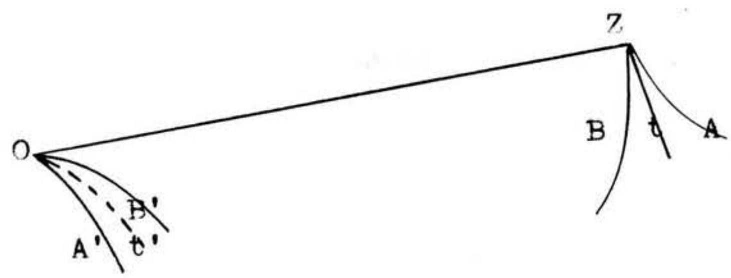
If two branches of the curve cut IJ at Z and Z' and continues to form a node N just off the line IJ, there will be in the inverse curve, two nodes, one at N' and the other at O. As the node approaches IJ. Z. Z' becoming coincident, these two nodes become consecutive on the line OZ which is the common tangent to the two branches. This is shown below.



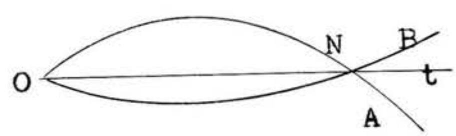
If the curve cutting the line IJ at Z and Z' has a node at N, there will be a loop at O, followed by a node at N', not on the tangent OZ. This is shown below.



If there is a cusp at Z on IJ, the inverse will be a cusp, but each branch of the curve will have their concavities in the opposite direction from the original curve. The original cuspidal tangent will become part of the conic, separating the two branches that form the cusp in the inverse, and will curve in the same direction as the branches of the conic.

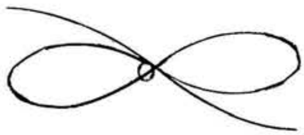


If the original cuspidal tangent passes through O, the inverse will have a cusp at O and a node on the cuspidal tangent.



If the three points Z, Z', Z'' on IJ be made to come together, there will be in the inverse curve a triple point at O. The three points may be made to come together by taking IJ as a tangent to a node. The inverse curve will have a branch and a cusp, the two having the same tangent.

The three Z's may be made to come together if we have a cusp and IJ as its tangent. There will be at O, a triple point formed by the union of two nodes and a point of inflection. This is shown below. Of the three tangents at this point, two will be coincident.



We have just used inversion as a method for analysing singularities. We can now use it to investigate the properties of the curve as a whole.

If a conic be inscribed in a triangle, the lines joining the points of contact to the opposite vertices are concurrent. Let the triangle be OIJ and the point of concurrency be M. Invert with respect to the conic that touches OI, OJ at I and J and which passes through M. There is now a cusp at O, with OM as tangent and also cusps at I and J. Since the original conic does not cut $O, I,$ or J, the inverse curve does not cut IJ, OI, or OJ except at the points O, I, J. The inverse is therefore a quartic with three cusps. This is shown in Fig. 29.

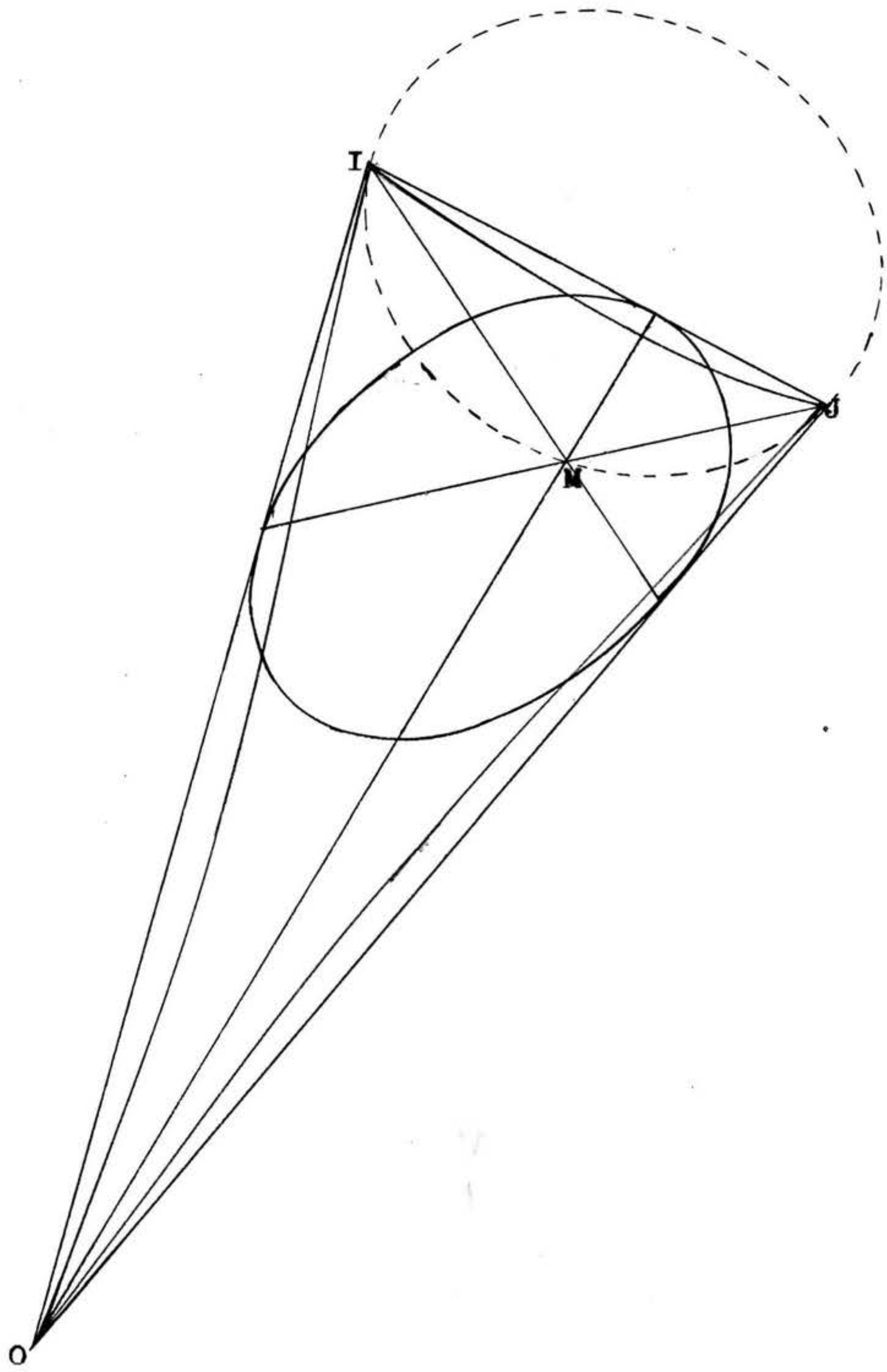


Fig. 29

If the conic cuts the sides of the triangle in three pairs of real points, the inverse curve is a trinodal quartic, as in Fig. 30.

If the intersections are external, the inverse takes the form shown in Fig. 31.

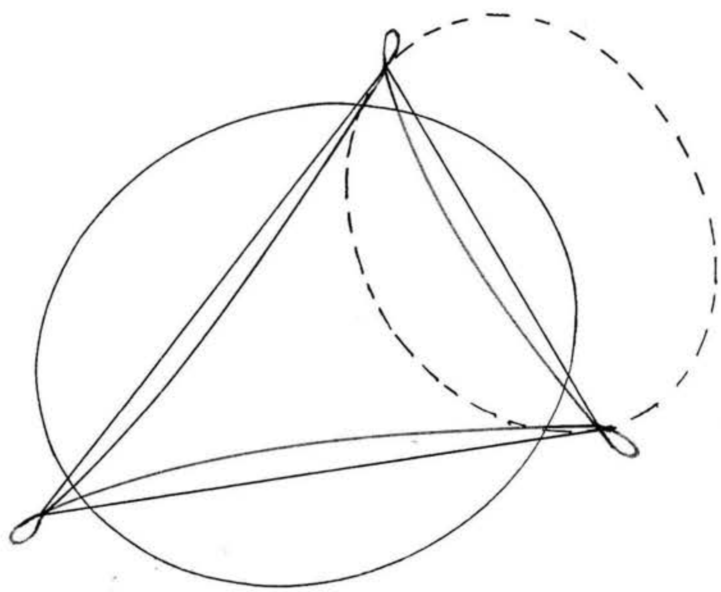


Fig. 30

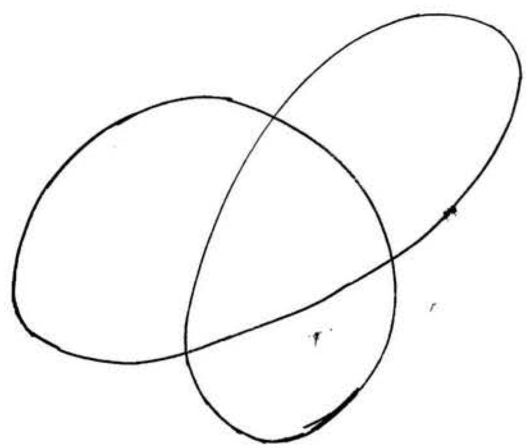


Fig. 31

XII. Laguerre Inversion

For lack of space there is no place here for a lengthy discussion on Laguerre Inversion, but some of its interseting properties are given below.

In this inversion we deal with oriented lines and circles. Suppose l is an oriented line: other lines not parallel may be transformed. Corresponding oriented lines shall be concurrent on l, and the product of the tangents of the halves of their angles with l shall be a given constant. Such a transformation shall be called a Laguerre Inversion.

Some of the transformations of the two inversions are given below:

Laguerre Inversion	Inversion
Oriented line to oriented line.	Point to point.
Oriented circle to oriented circle.	Circle to circle.
An oriented circle properly tangent to corresponding oriented lines anallagamatic.	Circle through two mutually inverse points anallagmatic.
Proper tangency of oriented circles invariant.	Tangency of circles invariant.
Common proper tangential segment of two oriented circles invariant.	Angle of intersection of two circles invariant.

SUMMARY

We may now summerize the theorms of inversion:

1. A straight line not through the centre of inversion is transformed into a circle through the centre of inversion.

2. A straight line through the centre of inversion is transformed into itself.

3. A circle not through the centre of inversion is transformed into a circle not through the centre of inversion.

4. A circle through the centre of inversion is transformed into a straight line not through the centre of inversion.

5. A conic is transformed in general into a curve of fourth order through the circle points at infinity.

6. A conic through the centre of inversion is transformed into a curve of the third order through the circle points.

By means of inversion, the problem of Apollonius is reduced to a simple construction. The long preliminary constructions which are necessary when the problem is constructed by means of Euclidean Geometry are no longer necessary. When the circles degenerate into lines and points the method of construction of the tangent circle by means of inversion still holds good.

The linkage has been shown to be a mechanical instrument for constructing a straight line without means of a straight edge. The length of the arms of the fulcrum determine the nature of the inverse curve. That is whether the arms are of equal or unequal length. Since these arms are determined by the distance from the fulcrum to the centre of inversion and to one of the inverse points of the linkage which is made to traverse a certain curve, the moving arm will pass through the centre of inversion provided the arms are equal. If the given curve to be traced is a circle the inverse curve will be a straight line. If, however, the given circle does not pass through the centre of inversion, the inverse curve will likewise be a circle. The reason for this depends upon the nature of inversion. By mounting a linkage with a suitable radius, one point may be made to traverse an ellipse, hyperbola, or parabola, while the inverse point will then trace a witch, strophoid, or cissoid. Other linkages have been constructed which will trace other well known curves but they are all based upon inversion.

By taking the centre of inversion at a double or triple point, the inverse curve becomes an entirely different looking curve. Inversion can be used as a method for constructing curves having these multiple points. Thus if we take any point on the ellipse as the centre of inversion, the inverse curve is a witch. In this case the centre of inversion becomes a conjugate point. In the hyperbola, the centre of inversion becomes a node and in the parabola, the origin becomes a cusp. If the centre of inversion is at the centre of the conic, it still is a multiple point but of a slightly different nature. If the conic is a hyperbola, the inverse curve is a lemniscate. The centre of inversion is therefore a bi-flecnode. When the conic is an ellipse the centre of inversion still remains a conjugate point. When the centre of inversion is at the focus of a conic, we have this point as a conjugate point, a cusp, or a node, provided the given curve is an ellipse, a parabola, or a hyperbola respectively

In version can be used to study the nature of singularities and to determine the properties of a curve as a whole. This is done by taking O, the centre of inversion, on the conic and also I and J, the circular points at infinity.

Inversion may be thought of as a point-point transformation in which there is only a one-to-one relationship. By using the transformation formula $x = \frac{x'}{x'^2 + y'^2}$, $y = \frac{y'}{x'^2 + y'^2}$, all the geometric constructions thus far made can be shown to be true when worked out analytically.

The properties of inversion hold true for the third dimension. The difference being that instead of circles we are dealing with spheres and in place of lines we are using planes. Tha class of surfaces known as cyclides can well be studied by means of inversion. Since in general, the centre of three given spheres do not lie on a straight line, they can be made to do so by inversion. This enables us to get an easier construction for spheres tangent to the three given circles and hence, the properties of the cyclides can be studied to a better advantage.

The properties of Laguerre Inversion is similar to the ordinary inversion, but in this inversion the orientation of the line and circle are taken into consideration, and play an important part in the laws of this inversion.

BIBLIOGRAPHY

Modern Pure Geometry by R. Lachlan Macmillan, N.Y. 1893
Theory of inversion---inverse points, lines, circles, corresponding properties of inverse figures, power relations of inverse figures, inversion applied to coaxal circles.

Modern Analytic Geometry
by Charlotte Angas Scott
Macmillan CO. N.Y. 1894
Construction for inverse points, circular inversion, quadric inversion including an analysis of singularities by inversion, inverse of the curve as a whole, effect of deficiency on inversion, effect of inversion on double points and double lines, effect of inversion on the focus, inversion in correspondence.

An Elementary Treatise on Cubic and Quartic Curves
by A.B. Basset
London--George Bell&Sons. 1901
Cubic curves including their inverses and harmonic properties. Quartic curves---triple points, quartics as inverses of conics including the limacon, lemniscate and cardiod.

Mathematical Papers Vol. III
by James Joseph Sylvester
Cambridge--University Press 1909
Mechanical conversion of Motion by means of the linkage. Construction of conics by the linkage and their dependence upon inversion.

Higher Plane Curves
by George Salmon Third Edition
Dublin: Hodges, Foster, & Figgis, Grafton St. 1897
Characteristics of inverse curves; inversion applied to obtain focal properties: inversion a case of quadric transformation.

Geometry of Three Dimensions
by George Salmon Fourth Edition
Inverse Dublin: Hodges, Figgis, &CO., Grafton St. 1882
Inverse surfaces: inversion applied to obtaining focal properties: study of cyclides by inversion.

A Treatise on the Circle and the Sphere
by Julian Lowell Coolidge
Oxford: at the Claredon Press 1916
Inverse of the line and circle; inversion applied to anallagmatic curves and to the centre of similitude; re-

lations between mutually inverse circles. Inversion in three dimensions. Inversion in pentaspherical space; a study of cyclides by inversion, especially the Dupin cyclide. Laguerre inversions: inversion in linear complex.

Greek Mathematics Vol. I.
by Sir Thomas Little Heath
Oxford: at the Clarendon Press 1921
Early use of the form of the lemniscate in Greek Astronomy.

Higher Geometry
by Frederick S. Woods
Ginn and Company 1922
Inversion as a point-point transformation; quadric inversion; inversion in tetracyclical coordinates; inversion in three dimensions with a discussion of cyclides.

Plane Algebraic Curves
by Harold Hilton
Oxford: Clarendon Press 1920
Inverse of conics when the centre of inversion is the centre of the conic, when it is on the conic, when it is the focus of a conic, and when it is entirely without the conic.

Projective Geometry Geometry with Applications to Engineering
by Peter Field
D. Van Nostrand Co. New York 1923
Linkages: Kempe's Translator, Sylvester's Pantograph, Peaucellier's Inversor, The Inversor Pantograph, The Stork's Beak, Affine Transformation, and Ritter' Perspectograph.

The Elements of Analytic Geometry
by Percy F. Smith and Arthur Sullivan Gale
Ginn and Co. Boston 1904
Equations of an inversion, analytic discussion of the inversion of conic sections, angles invariant under inversion, inversion of systems of straight lines, concentric circles, and orthogonal circles.