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Friction in instruments

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THESIS
FRICITION IN INSTRUMENTS

by

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INTRODUCTION

Friction is a very important phenomena in every day life and it is of particular importance to the mechanical engineer. He has at his disposal a vast body of empirical facts from numerous tests which make it possible for him to handle friction for his practical purposes quite well. Difficulties arise however in cases of very high speeds and in cases of highly sensitive and accurate instruments. Very little is known about the physical mechanism which causes friction. It is therefore not surprising that there is often a wide discrepancy among the empirical facts obtained by tests from different laboratories. As a matter of fact, there is no quantitative agreement between any two engineering handbooks on friction (see references given in bibliography). The physicist usually treats mechanics under the assumption that there is no friction.

This thesis proposes to define clearly what friction is, to show how friction affects instruments, and to correlate empirical facts with reasonable physical hypothesis about friction. Present day empirical knowledge makes obsolete the old explanation of sliding friction by Amontons. The purposes of the thesis are to stimulate new research concerning the mechanism of friction and to contribute toward the development of more accurate instruments. The validity of the proposed hypothesis about friction, as of any other hypothesis, must finally be decided by experimental tests.
FRICITION IN INSTRUMENTS

FRICITION

Definition of Friction

Friction is a force opposing a motion. To distinguish between frictional forces and other forces which oppose a motion the following definition will be used:

If the energy \( E \) given by the product of the force \( F \) and the length \( L \) of the path of motion can be recovered, the force is not a friction force

\[
\text{(Reversible process)} \quad \int F \, dL = 0
\]

If the energy is dissipated \( E_d \), the force is called a friction force

\[
\text{(Irreversible process)} \quad E_d = \int F \, dL
\]

It follows from the above definition that all motions which have some kind of friction result in an increase of entropy.

Examples Illustrating the Definition

1. Motion without friction. Motion without friction is reversible -- the energy of the motion is preserved. This definition means that if there is a decrease of kinetic energy, there is an equivalent increase of potential energy. Examples of such motions are:

   a. Motions with constant speed free from any forces.

   b. Motions against elastic forces in which case the elastic member stores the potential energy. Perfectly elastic material is assumed.

   c. Motions in any potential field where the energy is stored in the field. Such a field may be gravitational, electrostatic, magnetic, or a combination of the above mentioned fields.

2. Motion with friction. Motion with friction is associated with a dissipation of energy.
a. **Heat** is developed on the **boundary** between the moving body and the surrounding or supporting material (which may be solid, liquid, or gaseous).

b. **Heat** is developed **inside** an elastic material when it is deformed. This heat causes an elastic hysteresis effect, as shown in the diagram, Figure 1.

![Figure 1 - Elastic Hysteresis](image)

In this example the spring has been stretched and the energy expended is given by the sum of the orange and red areas. On release of the force only that part of the energy represented by the red area is recovered. The orange area represents the dissipated energy.

c. Phenomena described under (a) are ordinary or external friction. Phenomena described under (b) may be called internal friction or elastic hysteresis. There are other phenomena falling within the scope of our definition of friction. If an electrical conductor is moved in a magnetic field, there are eddy currents set up in the conductor. The field of the eddy currents interacts with the existing magnetic field, resulting in a force opposed to the direction of motion. The energy of the eddy currents is dissipated into heat by the resistance of the conductor. Such friction may be called **electrical friction**. In contrast to the mechanical friction about which very little is known, this electrical friction can be treated exactly by electromagnetic field theory by using Maxwell's equations. Even the mechanism of electrical resistivity has been treated theoretically. It may be that such treatments of electrical friction effects may lead to a better understanding of the analogous mechanical friction effect. The inclusion of electro-magnetic effects in our definition of friction means that a magnetic suspension does not eliminate
friction entirely since there are eddy currents induced in the suspended part if it is moved."

d. Whenever a charged particle is accelerated, electromagnetic radiation results which is dissipated into space. This type of radiation, which is also encountered in Cathode rays and in the Betatron, also falls within the scope of our definition for friction. The Betatron is an electron accelerating machine built like an a-c transformer. The electrons whirl around in a circle and as their velocity increases they radiate electromagnetic energy. The limits of the Betatron are reached if the radiated energy is equal to the energy supplied.

e. It has been shown by A. Einstein** that if uncharged particles are accelerated, cylindrical gravitational waves are radiated and dissipate energy.

3. Motion without friction in nature. The question arises whether there exists in nature a motion without friction in the sense of our definition. It is evident that the kind of motion without friction mentioned under 1(a) can only exist in an empty universe and therefore will never be observed. Motion described under 1(b) depends on a perfectly elastic material which does not exist in nature. Motion under 1(c) will have friction if any of the effects mentioned under 2(c), 2(d) or 2(e) are present. In spite of these facts, friction free motions do exist in nature. For example, there are the motions of the planets around the sun and the motions of the electrons around the nucleus in a stable non-radiating atom. One must assume that in these cases, the radiated fields do not transport any energy. Progressive waves always transport energy. Standing waves do not. Therefore, in frictionless cases the fields must be stationary (only time dependent, not space dependent). To summarize, frictionless motions in nature exist only if particles or bodies move on a selected geodesic*** curve which does not give rise to

---

* Magnetic parts have been suspended in vacuum for obtaining extremely high speeds of rotation. Mr. Beams has rotated a 1/32" dia. steel ball with a speed of 23-million rpm, using magnetic suspension in vacuum. See Journal of Applied Physics, volume 17, November 1946, page 886.

** Journal of the Franklin Institute, volume 223, January 1937, #1333-4, page 53.

*** By a geodesic curve there is understood a curve in which there are no forces acting on the particle.
a space dependent change of the field. This last condition gives, in the case of an atom, the Quantum condition.

The Various Kinds of Friction Forces

One usually classifies friction forces into two categories:

a. **Coulomb friction** is a friction force independent of the speed of motion.

b. **Viscous friction** is a friction force proportional to the speed of motion. One usually assumes coulomb friction for boundaries of solid materials, as for instance, in ordinary bearings. In all other cases, one assumes viscous friction.

The difficulties in analyzing friction more exactly arise from the fact that the heat developed by the friction influences the magnitude of the friction. Therefore, one cannot repeat measurements and get the same values each time. The heat also changes the shape and surface quality of the parts. Similar phenomena exist with magnetic effects. Residual magnetic and temperature effects give different values each time one tries to repeat measurements.

Friction in Instruments

Definition of an Instrument

An instrument is a device which converts one physical quantity \( X \) into another \( Y \). Where \( Y \) is utilized to measure or control something as a function of \( X \), \( Y = f(X) \)

Examples:

a. Speed of an automobile into position of a pointer on a scale.

b. Revolutions of an automobile wheel into distance.

c. Temperature into an electrical voltage.

d. Wave form of an electric current into a trace of a line on the face of an oscilloscope.

Classification of Instruments

There are different ways of classifying instruments. One is to classify them into indicating and control instruments. This division however, is only superficial. The control instrument changes the controlled quantity which may be an electrical voltage, air-pressure,
temperature, fluid level, the position of a lever, etc., while the indicating instrument changes the position of a visual pointer which may be mechanical, a light beam, a cathode ray, etc. Thus, an indicating instrument can be considered as a special case of a control instrument in which a visual pointer is being controlled. A better way to classify instruments is according to their physical characteristics -- mechanical, electrical, optical or magnetic or any combination of these characteristics. We shall here restrict ourselves to instruments in which a mechanical motion is present, since only in such instruments is friction of importance. Mechanical motions of solid bodies, fluids, gases, or charged particles may be used. The influence of friction on such mechanical motion will be investigated.

The Ideal Instrument

If the mechanical displacement of the instrument is \( q \), its equivalent mass is \( m \), and the quantity to be measured is \( X \); then it is desirable to have at all times --

\[
q = sX \tag{1}
\]

where \( s \) is the scale factor of the linear instrument.

In order to achieve this ideal condition, one has to convert the quantity \( X \) to be measured into a linear mechanical displacement. For a mechanical displacement a force is required. This force causes an acceleration or a deceleration of the moving mass,

\[
F = m \frac{d^2q}{dt^2}
\]

from which it follows that

\[
mq = \int \int F \, dt \tag{2}
\]

or

\[
q = \frac{1}{m} \int \int F \, (dt)^2
\]

Now we wish to have

\[
q = sX
\]

which means that

\[
sX = \frac{1}{m} \int \int F \, (dt)^2
\]

or

\[
m s \frac{d^2X}{dt^2} = F \tag{3}
\]

In the ideal instrument, the force would be proportional to the second time derivative of the quantity to be measured.
Errors in the Practical Approximation to the Ideal Instrument

The ideal instrument produces a force

\[ F = mS \frac{d^2x}{dt^2} \]  \hspace{1cm} (3)

and this force produces a displacement

\[ q = \frac{1}{m} \int F(dt)^2 \]  \hspace{1cm} (2)

The above equations indicate that an instrument built to approximate ideal instantaneous response would not correct previous errors. All errors in \( F \) would be cumulative, constantly increasing the total error of \( q \) as in any integrating process where the integrand has an error. This kind of instrument is therefore, not practical. A practical instrument does not have cumulative errors but corrects existing errors and gives a known accuracy which is not a function of time.

The Practical Instrument

If one makes the force \( F = S(x-q) \) one gets the following equation of motion:

\[ m \frac{d^2q}{dt^2} = S(x-q) \]  \hspace{1cm} (4)

For constant \( X \) this equation has the solution:

\[ q = X - a \sin \left( \sqrt{\frac{S}{m}} t + \beta \right) \]  \hspace{1cm} (5)

For \( t = 0, \quad x = x_0 \)

\[ \frac{dq}{dt} = v_0 \]

One finds the integration constants

\[ A = \sqrt{(x-x_0)^2 + \left( \frac{v_0^2}{S} \right)} \]

\[ \beta = \tan^{-1} \left( \frac{S}{mv_0} (q-x_0) \right) \]

\( q \) oscillates around the desired value of \( X \). If the amplitude \( A \) of this oscillation becomes large, it is rather inconvenient to make any reading on the instrument. The amplitude becomes large whenever the value \( (X - q) \) becomes large or, in other
words, when there is a rapid change in the quantity to be measured. An example of this phenomena is the pendulum as an instrument to indicate the direction of gravity. For the mathematical pendulum with small amplitudes the above analysis is valid, and for this case, Equation (4) is usually written:

\[ ml \frac{d^2 \phi}{dt^2} = -g \phi \tag{5} \]

where \( \phi \) is the displacement, 
\( g \) acceleration of gravity,
\( l \) length of pendulum,
\( x \) is zero and \( s = \frac{g m}{l} \)

The pendulum is not an exact case of a linear oscillation because the actual equation would be

\[ l \frac{d^2 \phi}{dt^2} = -g \sin \phi \tag{7} \]

however, \( \sin \phi = \phi + \frac{1}{3!} \phi^3 + \ldots \)

for small angles is approximately equal to \( \phi \). Quite generally if the force is a function of \( x \)

\[ F = f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots \tag{8} \]

(Maclaurin series) and, therefore, for small values of \( x \),

\[ F \approx H + CX \]

or any force function can be approximated by a linear function for small values of \( x \).

A magnetic compass indicating the direction of the magnetic pole is an example of the same kind as the pendulum.

In some instruments the force acts only in one direction and there is a spring built into the instrument opposing force. This type of instrument (as, for example, the spring scale) is expressed by the same equation.
In all these instruments it is essential to introduce a damping force which will diminish the vibration amplitude. An investigation of the energy will reveal the need of such a damping force and the most suitable function for this force. Starting with the previous equation,

\[ F = s(x-q) \]  

\[ q = x - A \sin \left( \sqrt{\frac{E}{m}} t + \beta \right) \]

and the equations for the potential energy

\[ V = -s \int dq \]

and the kinetic energy

\[ T = \frac{1}{2} m \left( \frac{dq}{dt} \right)^2 \]

and introducing for convenience

\[ \dot{q} = \frac{dq}{dt} \]

and

\[ \omega^2 = \frac{s}{m} \]

Then one may obtain the equation

\[ \dot{q} = A \omega \cos (\omega t + \beta) \]

\[ T = \frac{1}{2} m A^2 \omega^2 \cos^2 (\omega t + \beta) \]

and

\[ V = -\frac{1}{2} s x^2 + \frac{1}{2} s A^2 \sin^2 (\omega t + \beta) \]

These equations give for time

\[ t_1 = \frac{n \pi - \beta}{\omega}, \quad (n=0,1,2,\ldots) \]

\[ T_1 = \frac{1}{2} m A^2 \omega^2 = \frac{1}{2} A^2 s \]

\[ V_1 = -\frac{1}{2} s x^2 \]

and for time

\[ t_2 = \frac{n \pi - 2\beta}{2\omega} \]

\[ T_2 = 0 \]

\[ V_2 = -\frac{1}{2} s x^2 + \frac{1}{2} s A^2 \]
At the time $t_1$, the kinetic energy $T'$ reaches a maximum, the potential energy $V'$ reaches a minimum, velocity $q'$ reaches a maximum, and $q'$ has the desired value $X$. At the time $t_2$, the kinetic energy $T_2$ is zero, the potential energy $V_2$ reaches a maximum, velocity $q$ is zero and $q$ has the value $X + A$. Its maximum deviation from the desired value.

Now the potential energy brings the moving mass towards the desired position, while the kinetic energy carries it beyond. An effective dampener is required, therefore, to dissipate the kinetic energy. The damping force should be a function of the kinetic energy. This fact indicates that coulomb friction is undesirable, but viscous friction may be desirable.

**The Ideal Dampener**

The ideal dampener should be inactive at time $t_2$, described above, in order to utilize the potential energy to its full extent to bring the moving part into the desired position. Shortly before the desired position is reached at time $t_1$, the dampener should dissipate all the kinetic energy. At time $t_1$, it should again be inactive in order to allow the self-correcting restoring force $F = 5(x - q)$ to play its part. Such a dampener has not yet been built. Utilizing electronic techniques it may be possible to design an electric dampener approaching these ideal conditions.

**Example**

An automobile has to race from a standing start towards a goal in a minimum time and has to stop exactly at the goal line. For this problem, the skillful driver would use maximum acceleration power of his engine to get started and gain speed and then, at the right point, he would use his maximum braking power - by the traction of the wheels - to stop exactly at the goal line. If traction power of the wheels, horsepower of the engine, gear ratios and, braking torques are known, there exists only one mathematical solution to the problem.

**The Practical Dampener**

The example shows that the dampening force should be a function of kinetic energy, and, therefore, of speed, and it should be zero at zero speed. Recalling the example of the pendulum Equation (7) and the Maclaurin series Equation (8), one can approximate this function by a linear expression, for small values of $(X - q)$. This means that in instruments in which the quantity to be measured does not change very rapidly, it is good design to use a viscous friction dampener where the dampening force is proportional to the velocity. As will be discussed later, such a dampener
introduces dynamic errors but not static errors. Most practical instruments are built for measuring instantaneous values of quantities which do not change too rapidly and therefore they utilize some kind of viscous friction dampener. This may be fluid, air or magnetic. The equation of motion of such instruments will give an indication of the desirable magnitude of such dampening forces.

With the viscous damping force $P$, one gets

$$m \frac{d^2q}{dt^2} = 5(x-q) - P \frac{dq}{dt}. \quad (11)$$

Now it is convenient to bring this equation into a dimensionless form by the following substitutions:

$$\alpha = \frac{P}{2\pi m \omega}, \quad \omega^2 = \frac{5}{m} \quad (12)$$

$$\tau = \omega t$$

$$\frac{d^2q}{dt^2} + 2\alpha \frac{dq}{dt} + q-x = 0 \quad (13)$$

This latter equation has a solution of the form:

$$q-x = e^{\lambda t}$$

This value of $q$ inserted into the equation and divided by $e^{\lambda t}$ gives the following equation for $\lambda$.

$$\lambda^2 + 2\alpha \lambda + 1 = 0$$

The magnitude of $\lambda = \frac{P}{2\sqrt{m\omega}}$ determines the character of the motion. The general solution of the equation is

$$q-x = Ce^{\frac{\lambda t}{2}} + De^{\frac{-\lambda t}{2}} = e^{\frac{0}{2}}[Ce^{\lambda t} + De^{\frac{-\lambda t}{2}}]$$

$$q-x = c_1 e^{\lambda t} + c_2 e^{\frac{-\lambda t}{2}} = e^{\frac{0}{2}}[c_1 e^{\lambda t} + c_2 e^{\frac{-\lambda t}{2}}] \quad (14)$$
When the following quantities are introduced
\[ \mu = \sqrt{\nu^2 - \nu^2} = \nu \lambda, \quad \nu = \sqrt{1 - \nu^2} \]
\[ \tau = c_1 + c_2, \quad \cosh \mu \tau = \frac{1}{2} (e^{\nu \tau} + e^{-\nu \tau}) \]
\[ \kappa = c_1 - c_2, \quad \sinh \mu \tau = \frac{1}{2} (e^{\nu \tau} - e^{-\nu \tau}) \]

one gets
\[ g - x = e^{-\nu \tau} \left[ \cosh \mu \tau + \kappa \sinh \mu \tau \right] \]
and when \( \phi \) defined by \( r = c \sinh \phi \)
is introduced then
\[ q - x = c e^{-\nu \tau} \sinh (\mu \tau + \phi) \]
\[ \frac{dq}{dt} = -\nu c e^{-\nu \tau} \sinh (\mu \tau + \phi) + c e^{-\nu \tau} \cosh (\mu \tau + \phi) \]
For the boundary conditions
\[ t = 0, \quad q = q_0, \quad \frac{dq}{dt} = v_0, \quad \frac{dx}{dt} = 0 \]
one obtains
\[ \tanh \phi = \frac{(q_0 - x)}{v_0 + \alpha (q_0 - x)} \]
\[ c = \frac{1}{\mu} \sqrt{\left[ \frac{v_0}{w} + \alpha (q_0 - x) \right]^2 - \left[ \mu (q_0 - x) \right]^2} \]
\[ q = x + c e^{-\nu \tau} \sinh (\mu \tau + \phi) \]

In the special case where \( q_0 = 0 \)
\[ \frac{v_0}{w} = 0 \]
one gets
\[
\begin{align*}
c &= \frac{1}{\mu} \sqrt{\left(\frac{\lambda \pi}{\mu}\right)^2 - \left(\frac{x}{\mu}\right)^2} = \frac{x}{\mu} \sqrt{\frac{x^2}{\mu^2} - \frac{x^2}{\mu^2}} = \frac{x}{\mu} \\
tgh \, \phi &= \frac{-x \mu}{\lambda x} = \frac{x}{\lambda} \\
sinh \, \phi &= \sqrt{\frac{\lambda^2 \mu^2}{1 - \lambda^2 \mu^2}} = \sqrt{\frac{x^2}{\lambda^2 \mu^2} - \frac{x^2}{\lambda^2 \mu^2}} = \frac{x}{\lambda}
\end{align*}
\]
(see Equation 15)
The following solutions may then be obtained:

for \( \lambda > 1 \), \( \sinh \, \phi = \lambda \)
(overdamped vibration)
\[
q = x \left[ 1 - \frac{1}{\lambda} e^{-\lambda t} \right] \sinh (\mu t + \phi) \tag{20}
\]

Figure 2 - A Periodic Solution

for \( \lambda = 1 \),
(critically damped vibration)
\[
q = x \left[ 1 - \frac{1}{\mu} e^{-\lambda t} \right] \tag{21}
\]

Figure 3 - Case of Critical Dampening

and for \( \lambda < 1 \), \( \sin \, \phi = \lambda \)
(underdamped vibration)
\[
q = x \left[ 1 - \frac{1}{\lambda} e^{-\lambda t} \sin (\mu t + \phi) \right] \tag{22}
\]
It is now desired to establish the optimum value of $\alpha$. What to specify must be decided. Minimum deviation of $q$ from $X$ would appear to be the criterion. But $(q - X)$ is a function of time $t$. One way to avoid this difficulty might be to minimize the average value of $(q - X)$ in the specified time $T$. This process is difficult analytically because $(q - X)$ changes algebraic sign. However, for average value we may substitute root-mean-square, which in this case would be:

$$\sqrt{\int_0^T (q-X)^2 \, dt}$$

Minimizing this value in respect to $\alpha$, one gets,

$$\frac{\partial}{\partial \alpha} \sqrt{\int_0^T (q-X)^2 \, dt} = 0$$

and evaluating this equation, one finds, (using Equation 22)

$$\frac{d}{dx} \sqrt{\int_0^\infty (q-X)^2 \, dt} = 0$$

which yields the value

$$\alpha = \frac{1}{2}$$

(see Appendix A)

This result indicates that our criterion is not suitable for a good instrument. To take the RMS average over a time from 0 to $\infty$ is not of practical significance. If one has a measuring instrument with a step input, one does not take any readings immediately after the step input ($t_1$ should be larger than zero), and one does not watch the instrument an infinite time ($t_2$ should be smaller than infinity).
Another criterion for \( \alpha \) is required. If one introduces the expected accuracy of the measurement and specifies that the vibrations of the instrument be damped to an amplitude corresponding to the expected error of the measurement in a minimum time then one gets, where \( E \) is the error in the measurement in percent (for 1% error \( E = 0.01 \)),

\[
q = X (1 + E) \quad (24)
\]

\[
q - X = \frac{X}{\alpha} e^{-\frac{t}{\alpha}} \sin(\alpha t + \phi) \quad (22)
\]

and therefore,

\[
E = \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \sin(\alpha t + \phi) \quad (25)
\]

With Equation (22) and (15) one can rewrite Equation (25) in the form

\[
E = \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \sin(\alpha t + \phi) \quad (25a)
\]

The maximum values of \( E \) may now be determined as follows

\[
\frac{\partial E}{\partial t} = \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \left[ -\frac{\alpha^2}{\alpha^2 + \lambda^2} \sin(\alpha t + \phi) \right]
\]

and with \( \alpha^2 + \lambda^2 = 1 \)

\[
\frac{\partial E}{\partial t} = \frac{1}{\alpha} e^{-\frac{t}{\alpha}} \left[ -\sin(\lambda t) \right] = 0
\]

Therefore, for maximum \( E \) one has

\[
\lambda t = n \pi \quad \text{where} \quad n = 0, 1, 2, 3, \ldots \ldots
\]

We are here interested in the value of the first maximum which gives

\[
\lambda t = \pi \frac{n}{\pi}
\]

This expression inserted into equation (25a) gives

\[
E = e^{-\frac{\pi t}{\pi}} = e^{-\left(\frac{\pi t}{\pi - \alpha^2}\right)} \quad (26)
\]

This relationship between the expected error \( E \) and the optimum damping factor \( \alpha \) can be plotted graphically. (See Figure 5 and Appendix B).
This design makes the amplitude of the first sine wave of vibration equal to the expected error. If the damping is smaller than specified, it takes a larger time until the instrument quiets down so that an accurate reading is possible. If the damping is larger, it takes more time for the instrument to reach the desired position (see Figure 5).

For example, an instrument having 4-1/2% error permits a reading in the shortest possible time if the damping is 0.7 of critical damping.

The effect of Coulomb friction must also be taken into account if the instrument has any bearings which have this kind of friction. This friction-force $F$ is constant but always counteracting the motion. It is

$$ F_{\text{Coulomb}} = \begin{cases} +F & \text{for } t \leq \pi \frac{L}{C} \\ -F & \text{for } t \leq 2\pi \frac{L}{C} \end{cases} $$

This function developed into a Fourier series gives

$$ \frac{1}{T} F(\sin t + \frac{1}{6} \sin 3t + \frac{1}{5} \sin 5t + \ldots) $$

and the equation of motion is in dimensionless form similar to Equation (13).

$$ \frac{d^2 q}{dt^2} + 2\alpha \frac{dq}{dt} + q - x + \frac{4F}{\pi^2} (\cdots) = 0 $$

The solution of this equation with the same boundary conditions as above,

$$ t=0, \quad q_0=0 \quad \frac{dq_0}{dt}=0 $$

is similar to Equation (21), if one assumes that the friction terms are small.

$$ q(x) = -\frac{x}{L} e^{-x} \sin(xt + \phi) + \frac{4F}{\pi^2} (\cdots) $$

This series converges very rapidly. We are again interested in the amplitude equation similar to Equation (25a).
FIG. 5. OPTIMUM DAMPING FACTOR $\alpha$
FOR SPECIFIED ERROR $\varepsilon$

$$\varepsilon = \frac{-dtc}{\sqrt{1 - \alpha^2}}$$

RESULTING VIBRATION IF DESIGN CONDITION IS FOLLOWED.
\[
\frac{\partial E}{\partial x} = \frac{1}{x} e^{-\alpha x} \sin \omega t + \frac{4F}{\pi x s} \left[ \cos \omega t + \frac{1}{s} \sin \omega t \right] + \frac{4F}{\pi x s} \left[ \sin \omega t + \frac{1}{s} \sin \omega t \right] = 0
\]
\[
E = 1 \frac{e^{-\alpha x}}{x} \left[ \sin \omega t + \frac{1}{s} \sin \omega t \right]
\]

This equation shows that in this case the magnitude \( x \) of the quantity to be measured plays a role.

For small \( x \) one finds the viscous damping \( d \) plays only a minor role.

\[
E \approx \frac{4F}{\pi x s}
\]

For large \( x \) one finds that the Coulomb friction plays a minor role and Equation (28) is an approximate solution.

For values of \( x \) in between large and small, one can find the optimum value of \( d \) by comparison of the energy dissipated in Coulomb friction and viscous friction and adjusting the viscous friction so that the energy dissipated in the case of combined Coulomb and viscous friction is equal to the energy dissipated in pure viscous friction.

So far we have considered only a step input \( x \). If \( x \) is a function of \( t \) the results may be different. As long as \( x \) is a linear function of the above derived results still hold true, because in this case, the solution of the differential equation is the same and the error calculation is the same also. If \( x = f(t) \) is a nonlinear function, however, one arrives at different results. As shown in Equation (8), only for small values of \( x \) is a linear treatment always a good approximation. Therefore, the above analysis is restricted to small values of \( x \) or to the measurements of slowly changing quantities.

Undesirable Effects of Friction

If friction is present in a moving instrument part, the friction always acts against the direction of motion. Therefore, if the motion is reversed, the friction force is also reversed, the direction of the error caused by friction is reversed, and the reading of the instrument differs from the previous reading by an amount corresponding to double the friction force. If it is viscous friction, this force is zero at a static condition, and, therefore, at a static condition, the instrument is at least accurate; however, if it is Coulomb friction, the friction at rest may even be larger than in the moving condition and the error introduced by the friction may be considerable.

In some cases, this friction even introduces secondary effects, as, for instance, in free gyroscopes. The friction torque on one gimbal axis causes a precession \( \varphi \) about the other gimbal axis, because a frictional torque \( T \) produces a change in angular momentum \( L \).

...
\[ \frac{d\vec{H}}{dt} = \vec{T} \]

Calling the angle between the axis of rotation of the spinning gyro wheel and the axis of rotation of the table on which the gyro is mounted, one gets:

\[ d\vec{H} = H \sin \theta \frac{d\theta}{dt} \]

\[ T' = H \sin \theta \frac{d\theta}{dt} \]

If the gyroscope is mounted on an oscillating table, the friction torque reverses during each oscillation and the precessional errors therefore reverse also. If however, the gyro is mounted on a rotating platform, the friction torque does not reverse and the precession continues in one direction until \( \theta \) approaches zero, or in other words, until the gyro axis is in line with the axis of rotation of the platform.

The friction on the spinning gyro wheel axis is not important because the energy dissipated by this friction is supplied by the driving motor. As a matter of fact, if there were no friction on this axis no driving motor would be required and since the driving motor keeps the speed constant, it exactly compensates the frictional effects acting on this axis.

**Empirical Facts**

**Sliding Friction**

Friction may depend on any of the following physical conditions:

- a. Surface roughness
- b. Surface lubrication
- c. Oil viscosity
- d. Oil pressure in bearing
- e. Oil film thickness
- f. Speed of motion
- g. Bearing area
- h. Normal load on bearing surface
- i. Temperature
- j. Hardness of surfaces in contact
Spinning-Gyro-Wheel

Speed of Rotation held constant by MOTOR.

Fig. 6. Free Gyroscope
k. Properties of material in shaft and bearing and their combined action

l. Bearing misalignment

m. Rigidity of bearing surface

One can find many contradictory statements about friction in the literature. These contradictions are due to the fact that not all the above mentioned thirteen variables have been taken into account.

If one separates the operating conditions of the bearings into a number of well defined conditions, the empirical facts can be brought on a common basis. It must be recognized that actual operating conditions may often be a combination of the extreme conditions which will be defined in the following section.

**Operating Condition I (Oil Film Separation)**

For this condition the pressure in the oil film is sufficiently large to carry the full load and there is a substantial oil film present. This condition can be achieved by pumping oil under pressure into the journal. As was shown by Beauchamp Tower*, high oil pressures are developed in journals under running conditions without using any external pumps, by proper design of the bearing. It is important in this case to have no oil holes in the high pressure area of the journal. According to Stribeck, a perfect oil film is formed in a lubricated bearing if

\[ p \geq 20 \sqrt{v} \]

\( p \) load per unit area \( (\text{lbs/inch}^2) \)

\( v \) velocity \( (\text{feet/min}) \)

whenever \( v \geq 50 \text{ ft per minute} \).

Under these conditions of complete surface separation, friction is independent of bearing material, rigidity of bearing surface, and bearing load.

At high surface speeds over 500-feet per minute, the friction is proportional to temperature, proportional to the fourth root of the velocity, inversely proportional to the thickness of the oil film.

*Proceedings Institute of Mechanical Engineers, 1863, pages 632-659
At medium surface speeds between 50 and 500-feet per minute, the friction is proportional to bearing area, proportional to oil viscosity, proportional to the square root of the velocity, inverse proportional to the thickness of the oil film. In bearings operating in this speed range and with sufficient clearance, the thickness of the oil film varies directly with the oil viscosity. In this case, therefore, the two effects cancel and friction is independent of oil viscosity and of oil film thickness.

At low speeds under 50-feet per minute, the friction is proportional to the area, viscosity, speed and inverse of the thickness of the separating oil film.

(This statement is in agreement with Newton's law for fluid friction.)

Note: The influence of the surface roughness may be neglected if it can be assumed that the surface speed does not reach values where the surface boundary gives rise to eddies in the fluid but stays within the limits of viscous or laminar flow.

In engineering practice, for computing friction it is customary to use a coefficient of friction, multiplied by the normal load. This practice seems to be in contradiction with the above stated empirical fact that the fluid friction is independent of load.

The following tables taken from engineering handbooks for friction coefficients for journal bearings will clarify this matter.

(Machinery's Handbook, The Industrial Press, 1945, page 504, journal friction for lard oil, bath lubrication)

<table>
<thead>
<tr>
<th>Load (- lbs/inch)</th>
<th>100</th>
<th>153</th>
<th>205</th>
<th>310</th>
<th>415</th>
<th>520</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity 157 ft/min</td>
<td>0.0042</td>
<td>0.0027</td>
<td>0.0020</td>
<td>0.0014</td>
<td>0.0012</td>
<td>0.0009</td>
</tr>
<tr>
<td>velocity 451 ft/min</td>
<td>0.0090</td>
<td>0.0052</td>
<td>0.0042</td>
<td>0.0029</td>
<td>0.0021</td>
<td>0.0017</td>
</tr>
</tbody>
</table>

Multiplying this friction coefficient with the load per square inch one gets:

<table>
<thead>
<tr>
<th>Velocity</th>
<th>Friction Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>157 ft/min</td>
<td>0.42 0.415 0.41 0.43 0.50 0.47</td>
</tr>
<tr>
<td>451 ft/min</td>
<td>0.90 0.8 0.86 0.90 0.87 0.88</td>
</tr>
</tbody>
</table>
which is practically a constant, substantiating the statement that friction in the bearing is independent of load. Furthermore, the example shows that a speed increase by a factor of 3 increases the friction approximately by a factor of $\sqrt{3} = 1.7$

Other values for friction coefficients in Machinery's Handbook give similar results, though not all values are constant over the entire load range, as they are in the case of the lard-oil lubricated bearing. Of course, if there is any rupture of the continuous oil film, any seizing of the bearing, or any change in oil viscosity, such multiplications do not result in a constant.

Another example is taken from Kent's Mechanical Engineers' Handbook, 1939, John Wiley and Sons, pages 8 - 29.

<table>
<thead>
<tr>
<th>Coefficients of Friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load</td>
</tr>
<tr>
<td>velocity</td>
</tr>
<tr>
<td>252 ft/min</td>
</tr>
<tr>
<td>419 ft/min</td>
</tr>
</tbody>
</table>

Here the multiplication of load with coefficient gives:

\[
\begin{align*}
252 \text{ ft/min} & \quad 0.62 \quad 0.65 \quad 0.62 \\
419 \text{ ft/min} & \quad 0.78 \quad 0.84 \quad 0.87
\end{align*}
\]

Again speed increase of 1.6 gives a friction increase of $\sqrt{1.6} = 1.25$.

The influence of load on a journal bearing with complete oil film separation has been reported by many experimenters. The results of these tests differ, however. The effect of the load on a journal bearing depends on the design of the bearing and its lubrication. The load may change the thickness of the oil film thereby effecting the friction indirectly. The oil film thickness can be measured easily by electric capacitance methods. The oil is a dielectric and with the shaft and journal forms an electric capacitor. The capacitance is proportional to the thickness of the oil film. If the load causes a decrease of the oil film thickness, it increases the friction accordingly.

Another influence of the load may be a change of the viscosity of the lubricant. Some lubricants are less viscous under load. This influence tends to reduce friction under load.

Because under some operating conditions, outlined below, the friction depends on the load and because the techniques for measuring
the oil film thickness of a bearing under operating conditions had not until recently been developed, there is confusion in the literature regarding the dependence of friction on the load in a fully lubricated journal bearing. Actually, the load has no direct effect but only secondary effects on fluid friction.

Of course, a large load requires a larger bearing area than a smaller load and in this way affects friction which is proportional to bearing area. The load affects the design of the bearing, but once the bearing is built its friction is practically independent of load, unless the load causes a change in the oil film thickness or in the viscosity of the lubricants.

Operating Condition II (Boundary Layer Lubrication)

For this condition there is lubrication present to cover the surfaces but the lubricant has not enough pressure to carry the load. This condition is defined as the case of boundary lubrication.

Under these conditions friction is independent of speed of motion, independent of bearing area, proportional to normal load and depends on surface roughness and other surface properties.

Although the friction is independent of the speed of motion, one observes frequently a somewhat higher friction at rest than under moving conditions.

![Inclined Plane](image)

**Figure 7 - Inclined Plane**

If one places a body of weight $W$ on an inclined floor with an angle of inclination $\alpha$ and resolves its weight into a normal component $N$ and a sliding component $F$, one finds

\[
F = W \sin \alpha \\
N = W \cos \alpha
\]

from which

\[
\frac{F}{N} = \tan \alpha
\]

If the coefficient of friction is $\mu$ and the frictional force $F = \mu N$, from which it follows that if a body on an inclined plane starts to slide down by its own weight, then

\[
\mu = \tan \alpha
\]

since the sliding component is equal to the frictional resistance. This relation serves as the simplest means to measure the coefficient
of friction between various surfaces.

The coefficient of friction for the various metals which are used in bearings varies between 0.07 and 0.11 under moving conditions and between 0.11 and 0.15 under stationary conditions. These coefficients of friction cannot be given more accurately without specifying surface roughness, structure and hardness, lubricant used, and the combination of metals used in journal and shaft.

The proper combination of metals is discussed in the following section.

Operating Condition III (Dry Friction)

The following quotation taken from The Physics and Chemistry of Surfaces, by N. K. Adam, Oxford University Press, 3rd edition, 1941, pages 174 and 222, shows clearly that dry friction does not apply in instrument bearings.

"Unless very special precautions are taken a solid surface is nearly always contaminated with foreign matter in a thin layer. If a solid surface is left unprotected in ordinary air for a short time its surface usually becomes coated with a film of greasy material. Even if these films are only one molecule thick, they may profoundly alter the properties of the surface.

"The friction between metals as ordinarily observed, is the friction between metals covered with absorbed gases, usually oxygen, or even thin oxide film. The coefficient of friction is usually less than unity for such surfaces. If the metals are thoroughly de-gassed by heating in a vacuum, the coefficient of friction rises to very high values, often to 5 or 6, and, in the case of gold, to nearly thirty. In view of the extremely high friction found with really clean metals, it is fortunate, for engineering, that metals are not found with perfectly clean surfaces in practice."

For further details on dry friction, see Hans Ernst and M. Eugene Merchant, "Surface Friction of Clean Metals", proceedings on Friction and Surface Finish, M.I.T., June 1940.

This paper relates the surface roughness and the static internal slip stress to the coefficient of friction of pure dry surfaces. The static internal slip stress is expressed as a function of temperature, melting point, density and latent heat of fusion. It must be emphasized that the coefficient of friction theoretically calculated seems to agree with experiments only in the case of mixed pairs of clean metals forming solid solutions at room temperature. For pairs mutually insoluble at room temperature the observed friction coefficient for clean metals is lower than the
theoretically calculated one. These metal combinations possess anti-scoring properties and are therefore particularly suitable for bearing applications.

Rolling Friction

Rolling friction in ball and roller bearings follows the same law of friction as sliding friction with boundary lubrication (Operating Condition II). The coefficient for rolling friction for ball bearings is 0.001 to 0.003. The coefficients for roller bearings are slightly higher, about 0.002 to 0.009. In both cases these values apply only to well-made, first class bearings; the lower value applies to the highly loaded bearings, and the larger value to oversize bearings run with less load than their capacity.

Summary of Empirical Facts

In spite of the breakdown of friction into various categories, only approximate numerical values for friction have been given. There is so much variation in the values given by different experimenters that more accurate values cannot be given at present. This fact is the result of a number of different factors, as for example:

1. Sometimes conditions of oil film lubrication, boundary lubrication and dry friction have been mixed up in a set of figures giving friction coefficients at various speeds and loads.

2. The influence of the lubricant on the friction coefficient.

3. The influence of bearing clearances.

4. The influence of temperature on friction.

5. There is always some elastic deformation of the shaft and the journal and this deformation can result in a misalignment seriously affecting local bearing pressures (load per area) and may even lead to permanent deformation of the bearing surfaces, (scoring, scratching, etc.).

6. The surface properties of a given material may vary widely due to:
   a. The formation of metal oxide or metal soap films,
   b. Different surface roughness (only very recently has an attempt been made to evaluate this factor numerically),
   c. Hardness of the surface,
   d. Grain structure and grain direction.
As an example of grain direction, consider, for instance, wooden bearings where this influence is more evident than in metals.

As an example of grain structure, consider a skating rink. If the ice is covered with snow, then the friction for the skater is increased. Usually the snow is brushed off the ice when good skating is desired. If the snow-covered ice were smoothed with a lawn roller, a smooth surface, as read with a profilometer which indicates surface roughness, would be obtained, however, good skating would not result. In spite of the fact that chemically the surface is the same, the grain structure of ice formed by compressing snow is entirely different from the grain structure of ice formed by freezing water. The friction on the ice formed by freezing water is much smaller than the friction on the ice obtained by compression of snow. Unfortunately, most metal finishing processes striving for dimensional accuracy leave a compressed amorphous layer on the surface corresponding to the compressed snow on the natural ice and do not expose the original crystal structure. Only the method of superfinishing, developed in recent years,* exposes the unbroken crystal structure.

The skating rink is also an excellent illustration for the problem of bearing seizure. If a block of ice is slid over the frozen surface, there is always the danger present that the block of ice will freeze to the surface and become solidly bonded to it, as a result of the fact that these similar materials are soluble in each other. This phenomena has already been discussed under dry friction.

* M.I.T. Conferences on Friction and Surface Finish, June 1940
Theories About the Mechanism of Friction

A useful theory about friction must make it possible to predict the friction in bearings from the conditions present. From the empirical facts it is evident that different mechanisms must act for the different operating conditions. In operating case 1, the case in which the shaft is supported by a lubrication film under pressure, the friction theory is a part of hydrodynamics. Operating case 2, with boundary lubrication as well as rolling friction, seems to be the most difficult one to account for by a theory in agreement with empirical facts. The case of dry friction (operating case 3) is only of interest as a case to be avoided.

1. Mechanism of Sliding Friction with Separation of Sliding Surfaces by Oil Film Under Pressure

Sliding friction with separation of sliding surfaces by oil film under pressure is a case of fluid motion where viscosity is predominant. For the simplest case consider a shoe moving on a plane surface, and assume that the shoe is very long normal to the direction of motion. This condition causes the motion of the oil in the central region of the shoe, at least, to be in one plane. This two-dimensional theory was first developed by O. Reynolds.*

The general differential equations of fluid motion are known as the Navier-Stokes equations: (valid for an incompressible fluid in laminar flow, see appendix C for their derivation).

\[
\begin{align*}
\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial x} + b \frac{\partial a}{\partial y} + c \frac{\partial a}{\partial z} &= X - \frac{1}{S} \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2} \right) \\
\frac{\partial b}{\partial t} + a \frac{\partial b}{\partial x} + b \frac{\partial b}{\partial y} + c \frac{\partial b}{\partial z} &= Y - \frac{1}{S} \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} + \frac{\partial^2 b}{\partial z^2} \right) \\
\frac{\partial c}{\partial t} + a \frac{\partial c}{\partial x} + b \frac{\partial c}{\partial y} + c \frac{\partial c}{\partial z} &= Z - \frac{1}{S} \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right)
\end{align*}
\]

where \(a, b, c\) are velocity components in \(x, y, z\) directions of Cartesian Coordinates, \(X, Y, Z\) are body forces for unit volume in direction of coordinate axes (as for instance, gravity acting on the fluid),

\[p\] is pressure per unit area,

\[S\] is mass density,

\[\mu\] is coefficient of viscosity.

*Philosophical Transactions, Royal Society, 1886.
If these equations are applied to our special two-dimensional case and the origin of the coordinate system is placed on the edge of the moving shoe, X-axis in the direction of motion, the Y-axis at right angle to the surfaces, (see Figure 8) then the c and all derivatives of c are zero and all derivatives in respect to Z are zero.

For a steady state condition, partial derivatives in respect to t are zero. The equations can be rewritten as

\[ \frac{a}{\partial x} + b \frac{\partial a}{\partial y} - X = - \frac{1}{s} \frac{\partial b}{\partial y} + \frac{\mu}{s} \left( \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \right) \]

\[ a \frac{\partial b}{\partial x} + b \frac{\partial b}{\partial y} - Y = - \frac{1}{s} \frac{\partial a}{\partial y} + \frac{\mu}{s} \left( \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} \right) \]

The left hand side of the equations represent the inertia forces and body forces; the right hand side, the fluid forces. If a lubricating oil is used, the left hand side of the equations may be neglected because the viscous and pressure forces are much larger than the body and inertia forces.

One therefore gets

\[ \frac{\partial p}{\partial x} = \mu \left( \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \right) \]

\[ \frac{\partial p}{\partial y} = \mu \left( \frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial y^2} \right) \]

Assuming a thin oil film one can set

\[ \frac{\partial b}{\partial y} \approx 0, \quad b \approx 0. \]

and all derivatives of b are zero. Furthermore, the change of a in the flow direction x is very slow as compared with its change in the y direction,

\[ \frac{\partial^2 a}{\partial x^2} \ll \frac{\partial^2 a}{\partial y^2} \]
Therefore,\[ \frac{\partial p}{\partial x} = \mu \frac{f^2}{y^2} \] which can be integrated easily.

\[ \mu \frac{da}{dy} = \frac{dp}{dx} y + c_1 \]

\[ \mu a = \frac{1}{2} \frac{dp}{dx} y^2 + c_1 y + c_2. \]

Boundary conditions are:

for \( y = 0 \), \( a = v \) sliding velocity,

which yields \( c_2 = \mu v \)

for \( y = h \), \( a = 0 \) (\( h \) is the thickness of oil film),

which yields \( c_1 = -(\frac{1}{2} h \frac{\partial p}{\partial x} + \frac{\mu v}{h}) \)

\( h \) is a variable dependent on \( x \) and independent from \( y \).

Hence:

\[ a = \frac{1}{2\mu} \frac{dp}{dx} (y^2 - h^2) + \frac{v}{h} (h - y) \] \hspace{1cm} (34)

and

\[ \frac{\partial a}{\partial y} = \frac{1}{\mu} \frac{dp}{dx} (y - \frac{h}{2}) - \frac{v}{h}. \]

The force of friction on the upper surface per unit area is

\[ c_1 = \mu \left( \frac{da}{dy} \right)_{y=h} = \frac{h}{2} \frac{dp}{dx} - \mu \frac{v}{h} \] \hspace{1cm} (35)
Similarly, the force of friction on the lower surface is

\[
T_2 = \mu \left( \frac{da}{dy} \right)_{y=0} = -\frac{h}{2} \frac{dp}{dx} - \mu \frac{v}{h} 
\]  
(36)

If the quantity of fluid moving through the gap between the two sliding surfaces is designated \( Q \), then

\[
Q = \int_0^h a \, dy = \int_0^h \left[ \frac{1}{2} \mu \frac{dp}{dx} (y^2 - h^2) + \frac{v}{h} (h - y) \right] \, dy
\]

\[
= \frac{1}{2} \mu \frac{dp}{dx} \left[ \frac{h^3}{3} - h \frac{h^2}{2} \right] + \frac{v}{h} (h^2 - h^2)
\]

\[
= -\frac{h^3}{12} \frac{dp}{dx} + \frac{vh}{2}
\]

and, therefore,

\[
\frac{dp}{dx} = 12 \mu \left( \frac{v}{2h^2} - \frac{Q}{h^3} \right) \]  
(37)

\[
p = 12 \mu \int_0^x \left[ \frac{v}{2h^2} - \frac{Q}{h^3} \right] \, dx + p_0 \]  
(37a)

\[
p = 6 \mu v \int_0^x \frac{dx}{h^2} - 12 \mu Q \int_0^x \frac{dx}{h^3} + p_0 \]  
(37b)

We will now apply this result to the simple case of a slightly inclined sliding shoe of length \( L \) which has a clearance \( h_0 \) at one end and a clearance \( h_1 \) at the other end; furthermore, we will assume that there is atmospheric pressure \( p_0 \) at each end.
The value of $h$ as a function of $x$ is, in this case,

$$h = h_0 + \frac{x f}{l} \quad (38)$$

where

$$f = h_1 - h_0 \quad (39)$$

![Figure 8 Sliding Shoe](image)

From Equation (38) it follows that:

$$\int_0^x \frac{dx}{h^2} = \left. \frac{-l^2}{2d(x+l)h_0} \right|_0^x = -\frac{l}{2d h_0} + \frac{l^2}{2d(x+l)h_0},$$

and

$$\int_0^x \frac{dx}{h^3} = \left. \frac{-l^3}{2d(x+l)h_0^2} \right|_0^x = -\frac{l^2}{2d h_0^2} + \frac{l^3}{2d(x+l)h_0^2}.$$

From Equation (38)

$$dx + l h_0 = -h l$$

These values inserted into Equation (37b) give

$$p = 6 \mu l \left[ \frac{Q}{d h_0^2} - \frac{Q}{d h^2} - \frac{v}{d h_0} + \frac{v}{d h} \right] + p_0 \quad (40)$$

For $h = h_0$, Equation (40) gives $p = p_0$ as required.

Furthermore, for $h = h_0$ it is required that $p = p_0$ and therefore, the expression in the bracket is zero.

The equation for $Q$ is, therefore,

$$Q = \frac{h_0 h_1}{h_0 + h_1} v \quad (41)$$
This value of \( Q \) inserted into Equation (40) gives together with Equation (39) the equation for \( \rho \)

\[
\rho = \rho_0 + \frac{6 \mu L v}{h_i - h_o} \left[ -1 + \frac{1}{h_i + h_o} \left( \frac{h_o h_i}{h^2} - \frac{h_i}{h_o} \right) \right]^{\frac{1}{2}}
\]

To find the maximum pressure

\[
\frac{d\rho}{dh} = 0 = \frac{6 \mu L v}{h_i - h_o} \left[ -1 + \frac{2 h_o h_i}{h^3 (h_i + h_o)} \right]
\]

from which together with Equation (38), it follows that

\[
\begin{align*}
\rho_M &= \frac{2 h_o h_i}{h_i + h_o} = \frac{d}{d} X + h_o \\
X_M &= \frac{L}{\delta} \left( -h_o + \frac{2 h_o h_i}{h_i + h_o} \right) = \frac{L h_o}{h_i + h_o}
\end{align*}
\]

This value inserted into Equation (42) gives the value of maximum pressure.

\[
\rho_M = \rho_0 + \frac{6 \mu L v}{h_i - h_o} \left[ - \frac{h_o + h_i}{2 h_o h_i} - \frac{1}{h_i + h_o} \left( \frac{h_o + h_i}{h_i + h_o} \right) \right]^{-\frac{1}{2}}
\]

\[
= \rho_0 + \frac{6 \mu L v}{h_i - h_o} \left[ - \frac{h_o + h_i}{h_i + h_o} - \frac{h_i}{h_o (h_i + h_o)} \right]^{-\frac{1}{2}}
\]

\[
\rho_M = \rho_0 + \frac{3}{2} \mu L v \left( \frac{h_o + 3 h_i}{h_o h_i (h_i + h_o)} \right)^{-\frac{1}{2}}
\]

If now \( h_o \to h_i \),

then \( \rho \) very large \( \to \infty \)

and

\[
X \approx \frac{L}{2}
\]

If we have a solid wedge instead of an oil film wedge, we get mathematically similar results, namely, that the normal force approaches infinity as the wedge angle approaches zero. However, in this case, an infinitesimal normal movement of the surfaces will make the normal force disappear completely. Conditions are different in the case of a fluid wedge. The infinite pressure is only in the center of the fluid pad and does not disappear.
with a small normal displacement of the surfaces, because the fluid wedge is deformable and adjusts itself to the new conditions.

In practice, however, as \( p \to \infty \), in other words, the oil loses its viscosity under very high pressure. Thus, if an oil which does not lose viscosity under high pressure is obtained, an oil film bearing for extremely heavy loads can be built, (the new molybdenum-compound greases approach this condition); but with ordinary lubricating oil one has to keep \( h_0 \neq h \), to obtain suitable pressures for maintaining a lubricating film to support the load.

It is of interest to evaluate the supported load: 

\[
L = \int_{x=0}^{x=L} \rho \, dx
\]

and since, from Equation (38)

\[
\int dx = \frac{L}{\delta} \, dh
\]

\[
\begin{align*}
  x=0 & \text{ gives } h=h_0 \\
  x=L & \text{ gives } h=h_1
\end{align*}
\]

Then

\[
L = \int_{h=h_0}^{h=h_1} \frac{L}{\delta} \, \rho \, dh
\]

and, from Equation (42),

\[
L = \left[ \rho_0 + \frac{6 \mu L v}{h_1-h_0} \left( \frac{1}{h_0} + \frac{1}{h+ho} \frac{h}{h_0} \right) \right] \frac{L}{\delta}
\]

\[
+ \frac{6 \mu L v}{(h_1-h_0)^2} \left[ \frac{h_0 h_1}{h_0 + h_0 \left( \frac{1}{h_0} - \frac{1}{h_1} \right) + nh h_0 - nh h_1} \right]
\]

and \( \lambda = \frac{h_0}{h_1} \) (39)

The equation for \( L \) may be written

\[
L = \rho_0 L + \frac{6 \mu L v}{h_0} \frac{\ln \lambda}{(\lambda - 1)}
\] (43)
If we let \( \lambda \rightarrow 1 \) then we need
\[
\lim_{\lambda \rightarrow 1} \left( \frac{h \ln \lambda}{\lambda - 1} \right) = \frac{1}{1} = 1
\]
and, therefore, for \( h_0 = h_1 \),
\[
L = \beta_o L + \frac{6}{h_0} \mu L^2 \nu.
\]
This result is the load per unit width supported by a parallel surface bearing if \( \mu \) is a constant and the bearing is infinitely wide and of length \( L \).

The friction may be evaluated from Equations (35) and (36):
\[
\tau_1 = \frac{h}{2} \frac{d\beta}{dx} - \mu \frac{v}{h}
\]
(35)
\[
\tau_2 = -\frac{h}{2} \frac{d\beta}{dx} - \mu \frac{v}{h}
\]
(36)
\[
\frac{d\beta}{dx} = \frac{d\beta}{dh} \frac{dh}{dx},
\]
and, from Equation (42) and (38),
\[
\frac{d\beta}{dx} = \frac{6 \mu L \nu}{h_i - h_0} \left[ -\frac{1}{h^2} + \frac{2 h_0 h_1}{h^2 (h_0 + h_1)} \right] \frac{h_i - h_0}{L}
\]
which makes
\[
\tau_1 = \mu \nu \left[ -\frac{1}{h} + \frac{6 h_0 h_1}{h^2 (h_0 + h_1)} - \frac{3}{h} \right] = 2 \mu \nu \left[ -\frac{3 h_0 h_1}{h^2 (h_0 + h_1)} - \frac{2}{h} \right]
\]
\[
\tau_2 = \mu \nu \left[ -\frac{6 h_0 h_1}{h^2 (h_0 + h_1)} + \frac{3}{h} \right] = 2 \mu \nu \left[ -\frac{3 h_0 h_1}{h^2 (h_0 + h_1)} - \frac{1}{h} \right].
\]
The friction forces $F$ are (from Equation 38)

on the upper surface
$$F_1 = \int_{x=0}^{x=l} T_1 \, dx = \frac{F}{\delta} \int_{h_0}^{h_1} T_1 \, dh$$

on the lower surface
$$F_2 = \int_{x=0}^{x=l} T_2 \, dx = \frac{F}{\delta} \int_{h_0}^{h_1} T_2 \, dh$$

Thus, the equation for the upper force is
$$F_1 = \frac{2 \nu \alpha l}{h_1 - h_0} \left[ \frac{3 h_0 h_1}{h_0 + h_1} \left( -\frac{1}{h_0} + \frac{1}{h_1} \right) - 2 \ln \frac{h_0}{h_1} \right],$$
or, from Equation (39),
$$F_1 = -2 \nu \alpha l \left[ \ln \frac{3}{2 h_0} + \frac{2 \lambda}{h_0 (1-\lambda)} \ln 2 \right]. \quad (44)$$

Then, if $h_0 = h_1$ and $\lambda = 1$
$$F_1 = -2 \nu \alpha l \left[ \ln \frac{3}{2 h_0} - \frac{2}{h_0} \right] = \nu \alpha l \frac{2 h_0}{h_0}$$

The equation for the lower force is
$$F_2 = \frac{2 \nu \alpha l}{h_1 - h_0} \left[ \frac{3 h_0 h_1}{h_0 + h_1} \left( \frac{1}{h_0} + \frac{1}{h_1} \right) - \ln \left( \frac{h_0}{h_1} \right) \right],$$
or, from Equation (39),
$$F_2 = 2 \nu \alpha l \left[ \ln \frac{3}{2 h_0} - \frac{1}{h_0} \frac{\ln \lambda}{\lambda - 1} \right]. \quad (45)$$

Then if $h_0 = h_1$ and $\lambda = 1$
$$F_2 = 2 \nu \alpha l \left[ \ln \frac{3}{2 h_0} - \frac{1}{h_0} \right] = \nu \alpha l \frac{2 h_0}{h_0}$$

The work dissipated in the oil film is
$$W = v (F_1 - F_2) = 2 \nu \alpha l \left[ \ln \frac{3}{2 h_0} + \frac{2 \lambda}{h_0 (1-\lambda)} \ln \frac{h_0}{h_1} \right.$$}
$$+ \frac{3}{h_0 + h_1} \frac{\ln \lambda}{\lambda - 1} \right].$$
This limiting case of perfectly parallel surfaces is of course unrealistic. However, it is instructive to show what happens when this case is approached. There is an infinite pressure built up in the center of the sliding surface and zero energy dissipation in the oil film. Actually, the shoe is not infinitely long and therefore, the pressure does not build up as indicated; the oil escapes sideways. To analyze this condition would require a three-dimensional treatment of the oil flow. Such a treatment is beyond the scope of this paper. In a journal bearing the theory has to be applied to the cylindrical surface, and the clearance between the shaft and journal has to be taken into account. The treatment is mathematically complicated. It has been done by A. Sommerfeld* for a full journal bearing as well as for a half journal bearing. The dependence of friction on the velocity as derived by Sommerfeld agrees well with the empirical results given above and with the experiment by Striebeck.

We restrict ourselves here to an explanation of the mechanism of fluid friction, which can be much more easily understood in terms of the simple example treated above rather than in terms of an actual practical case. The conditions in the example would be similar to those in thrust bearings. (Mitchell and Kingsbury bearings use pivoted segments so that the ratio \( \lambda = h_0/h_1 \) is self-adjusting with the operating condition.)

![Diagram of Kingsbury Thrust Bearing](image)

In the Kingsbury bearing the segments are pivoted at the center, as shown in Figure 9, while in the Mitchell bearing, the segments are pivoted off-center below the point of maximum pressure, as given by Equation (42a). This kind of support, however, restricts the application of the bearing for rotation in one direction.

**Application of the Oil-Film Supported Bearing to Instruments**

The oil-film supported bearing is unable to support any load (beyond that of atmospheric pressure) at very slow speeds (Equation 43), which means that at low speeds complete oil film separation is lost and the condition of boundary lubrication or dry friction predominates. Thus, although the oil-film bearing described is excellent for high speed machinery, it does not function as described in instruments where the velocity is normally zero.

It is therefore desirable to investigate types of fluid-film bearings in which the load supporting pressure in the fluid-film does not depend on the bearing velocity but is built up by an external pump.

In such bearings, a high viscosity is not required to build up sufficient pressure in the oil film to support the load. Therefore, a fluid with a very low viscosity can be chosen, in which case the frictional resistance force, according to Equations (44) and (45), becomes very low too. An ideal fluid for such a bearing is air. The analysis of an air bearing is an entirely different one from that of an oil-film bearing. Air is compressible and has a very low viscosity. It is therefore essential first to review the thermodynamic principles of air flow.
FLOW OF GASES

GLOSSARY

A  Area (feet)$^2$

$C_p$  Specific heat at constant pressure (BTU/lbs)

$C_v$  Specific heat at constant volume (BTU/lbs)

D  Diameter (inch)

g  Acceleration of gravity in ft per sec$^2$ (32.17)

H  Enthalpy (total heat) (heat content) (BTU/lbs)

h  Gap between surfaces (inch)

J  Joule's constant (-778) ft lb./BTU

$k = \frac{C_p}{C_v} = 1.4$ for diatomic gases

p  Pressure (lbs/sq. ft.)

R  Specific gas content $R = \frac{pV'}{T} = J (C_p - C_v)$ (feet)

t  Time (sec)

T  Temperature in degrees (Rankine) (Absolute temperature on the Fahrenheit scale)

V  Volume (cubic ft.)

$V'$  Volume of 1 lb. (cubic ft. for lbs.)

v  Velocity (feet/sec)

$w = \frac{Av}{V'}$  flow in lbs per sec

$S = \frac{1}{V'}$  density

Simple algebra shows that $\frac{kR}{J(k-1)} = C_p$
If rapid flow is assumed, the adiabatic law of expansion due to Poisson,
\[ \frac{p_1}{p_2} = \left( \frac{V_1}{V_2} \right)^k \]
is valid.

This law combined with Clapeyron's equation of state,
\[ \frac{p_1 V_1}{p_2 V_2} = \frac{T_1}{T_2} \]
gives
\[ \frac{T_1}{T_2} = \left( \frac{V_2}{V_1} \right)^{k-1} = \left( \frac{p_1}{p_2} \right)^{\frac{k-1}{k}} \]  
(47)

Now the specific heat at constant pressure is defined by
\[ \left( \frac{\partial H}{\partial T} \right)_{p=\text{constant}} = C_p \]
and therefore
\[ \Delta H = H_1 - H_2 = \int_{T_1}^{T_2} C_p \, dT = \frac{KRT_1}{J(k-1)} \left[ 1 - \left( \frac{T_2}{T_1} \right)^{\frac{k-1}{k}} \right] \]  
(48)

If the initial velocity is assumed to be \( v_i = 0 \), then
the energy equation is
\[ WJ (H_1 - H_2) = \frac{1}{2} V_i^2 \frac{W}{J} \]
The last two equations combined give
\[ v_2 = \sqrt{2g \frac{K}{K-1} RT_1 \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}} \right]} \]  
(49)

and
\[ W = \frac{P_2 V_2}{V_2} = \frac{P_2 V_i}{(P_2) \sqrt{\frac{KRT_1}{P_1}}} \]  
(50)

\[ W = \rho_1 \rho_2 \sqrt{2g \frac{K}{K-1} RT_1 \left[ \left( \frac{p_2}{p_1} \right)^{\frac{2}{K}} - \left( \frac{p_2}{p_1} \right)^{\frac{k+1}{K}} \right]} \]
We now wish to investigate the maximum gas flow if the pressure is varied and everything else is kept constant. Accordingly, set

$$\frac{\partial (W')}{\partial p_2} = 0,$$

which gives

$$\left(\frac{1}{\beta_1^{\frac{2}{k}}} \left(\frac{2}{K}\right) - \left(\frac{1}{\beta_1^{\frac{1}{K+1}}} \left(\frac{2}{K}\right) \right) = 0,$$

and thence

$$p_2^* = p_1 \left(\frac{2}{k+1}\right)^{k+1} = p_c$$  \hspace{1cm} (51)

We will call the critical pressure $p_c$. Any pressure $p_2 \leq p_c$ will result in the same flow rate, which means that the flow velocity has an upper limit. By combining Equations (49) and (51) we may find this upper limit:

$$v_{\text{max}} = \sqrt{\frac{2g}{k-1} \cdot \frac{k+1}{k+1} - \frac{2}{k+1}} = \sqrt{\frac{2g}{k+1} \cdot \frac{k}{RT}}$$  \hspace{1cm} (52)

from which it follows

$$w_{\text{max}} = \frac{H_2 v_2}{(p_1^{\frac{k}{k+1}} \cdot RT)} = H_2 p_1 \left(\frac{2}{k+1}\right)^{k+1} \sqrt{\frac{2g}{k+1}}$$  \hspace{1cm} (53)

Thus, for diatomic gases where $k=1.4$ with $g=32.17$

$$w_{\text{max}} = 3.88 \frac{H_2 p_1}{\sqrt{RT}} \left[\frac{\text{LBS}}{\text{SEC}}\right]$$  \hspace{1cm} (53a)

and for air

where $R=53.3$

$$w_{\text{max}} = 0.532 \frac{H_2 p_1}{\sqrt{RT}} \left[\frac{\text{LBS}}{\text{SEC}}\right]$$  \hspace{1cm} (53b)

This formula is valid for the discharge of air whenever
\[ \frac{p_2}{p_1} \leq \left( \frac{2}{k+1} \right)^{k-1} = \left( \frac{2}{2.4} \right)^{0.53} = 0.53 \]

and when the velocity of approach can be neglected \( (v_1^2 \leq v_2^2) \) and the shape of the discharge nozzle is well designed. Discharge through a hole in a thin plate is only 55-percent of the value given by the formula, due to the fact that the cross-sectional area of the flowing stream converges just after it leaves the thin plate orifice. However, discharge through a well rounded orifice is 99-percent of the value given by the formula.

**Application to Air Bearings:**

The simplest type of air bearing to analyze is a circular pad support, as shown in Figure 10.

![Figure 10 Circular Pad Air Bearing](image)

**Table:**

<table>
<thead>
<tr>
<th>Supported load</th>
<th>( L ) (lbs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pad diameter</td>
<td>( D ) inches</td>
</tr>
<tr>
<td>Nozzle diameter</td>
<td>( D_0 ) inches</td>
</tr>
<tr>
<td>Bearing gap</td>
<td>( h ) inches</td>
</tr>
<tr>
<td>Air supply pressure</td>
<td>( \frac{p_3}{\text{in}^2} ) lbs</td>
</tr>
<tr>
<td>Air pressure in bearing</td>
<td>( \frac{p_2}{\text{in}^2} ) lbs</td>
</tr>
<tr>
<td>Atmospheric air pressure</td>
<td>( \frac{p_0}{\text{in}^2} ) lbs</td>
</tr>
</tbody>
</table>

Assuming that \( p_2 > \frac{p_3}{0.53} \),

and \( p_1 > \frac{p_2}{0.53} \).
which conditions in air bearings are generally satisfied, one can use Equation (53) for this analysis.

Now

\[ A_2 = \frac{\pi D_0^2}{4} \]
\[ A_3 = \pi D h \]

because in an equilibrium condition the load is compensated by the air pressure

\[ L = \frac{\pi D^2}{4} p_2 \]

and therefore,

\[ p_2 = \frac{4L}{\pi D^2} \]

Assuming adiabatic expansion of the air from the supply line into the bearing and with an air temperature \( T \) equal to room temperature one gets an air flow

\[ W_2 = 0.532 \frac{A_2 P_1}{V_{T_1}} = 0.532 \frac{\pi D_0^2}{4} \frac{P_1}{V_{T_1}} \]

Assuming that this air which has expended to the pressure \( p_2 \) and cooled to a temperature \( T' \) will be warmed up to the room temperature \( T \) inside the bearing pad before it escapes through the gap \( h \) into the atmosphere, one gets for the air flow:

\[ W_2 = 0.532 \frac{A_3 P_2}{V_{T_2}} = 0.532 \frac{\pi D h}{V_{T_2}} \frac{4L}{\pi D^2} \]

\[ = 0.532 \frac{h}{D} \frac{4L}{V_{T_2}} \]

Equating \( W_2 = W_{23} \) one gets:

\[ 0.532 \frac{\pi D_0^2}{4} \frac{P_1}{V_{T_1}} = 0.532 \frac{h}{D} \frac{4L}{V_{T_2}}, \]
from which it follows that

$$D_0 = \sqrt{\frac{16hL}{\pi p}}$$  \hspace{1cm} (54)

The necessary nozzle diameter $D_0$, for this equation, is obtained when the air supply pressure $p$, the bearing load $L$, and the bearing clearance $h$ are known. In order to avoid ordinary friction, $h$ should be larger than 0.0005-inch in smooth surface bearings. Therefore, the formula gives the minimum nozzle diameter for satisfactory bearing operation. Of course, for reasons of economy in air consumption, it is desirable to make $D_0$ a minimum, in accordance with satisfactory bearing operation.

This analysis can be used for spherical bearings provided that the pad diameter is small compared to the sphere diameter.

In spherical bearings, sometimes, two pairs of opposed pads are used in order to provide omni-directional stability. In such cases, if there is sufficient clearance between the sphere and its pad support, the condition on the load supporting pad is essentially the same. The gap in the opposed pad adjusts itself to such a large dimension that the air pressure inside it is very nearly equal to the atmospheric pressure $p_3$.

In case this clearance is not provided, the air pressure of the opposed pad increases the bearing load on the load-carrying pad and, in an extreme case, may cancel its load carrying capacity.

If the sphere is not exactly spherical, or if the pads are not exactly normal to the surface of the sphere, then the line of action of the force acting on the sphere as a result of the air of two opposed pads forms a couple exerting a torque on the sphere. This torque may be opposed to the direction of rotation, as in ordinary friction, or it may be in the direction of rotation being equivalent to negative friction (which also causes errors in instruments). This inherent torque has been found from experience to range from 5 dyne centimeters to as much as 500 dyne centimeters and is much larger than the torque due to the viscosity of the air, which has been neglected in the analysis. Attempts to eliminate torques due to geometrical inaccuracies by the use of self-aligning bearing pads have been unsuccessful because of friction in the spherical self-aligning supports of the pads. In the use of an air
bearing for the self-aligning support of the main bearing pad, conditions become so complicated that each test gives different results. One must not overlook the fact that the air contains impurities like dust, etc., which may disturb the symmetry of the air flow.

In a journal bearing of length \( l \) and diameter \( D \), the conditions are somewhat different from those of a circular pad. As an example, let us analyze a cylindrical bearing with 4 pads. The area of support analogous to the circular pad is

\[
A_2 = \frac{l \pi D}{4}
\]

The area of leakage is

\[
A_3 = \frac{1}{4} \pi D h
\]

The 4 pads are adjacent to each other so that there is no leakage from the flat side of each pad (see Figure 11).

Therefore, as before

\[
L = A_2 \rho_2
\]

\[
\rho_2 = \frac{4L}{\pi DL}
\]

\[
W_{12} = 0.532 \frac{A_2 \rho_1}{VT} = 0.532 \frac{\pi D^2 \rho_1}{4VT}
\]

\[
W_{23} = 0.532 \frac{A_3 \rho_2}{VT} = 0.532 \frac{\pi D h \rho_2}{4VT} \frac{4L}{\pi DL}
\]

\[
= 0.532 \frac{hL}{VT}
\]

With \( W_{12} = W_{23} \),

one gets

\[
\frac{\pi D_0^2}{4} \rho_1 = \frac{hL}{L}
\]
and therefore
\[ D_0 = 2 \sqrt{\frac{hL}{2\pi\rho}} \]  

The above remarks in regard to the torque resulting from the geometrical inaccuracy of the bearing apply also of course to the journal bearing.

In the analysis, the width of the rim on the bearing pads has been neglected. The rim width reduces the effective area of the bearing surface somewhat. The width should be large enough to prevent scoring and scratching of the bearing surfaces in the case where the air flow is turned off. Furthermore, the rim width serves to equalize the inaccuracies of the bearing surfaces, and provides an air flow more nearly symmetrical than a sharp edge would.

2. Mechanism of Sliding Friction with Boundary Layer Lubrication

Sliding friction with boundary layer lubrication is the case which occurs most frequently in bearings. It even occurs in the pressurized oil-film lubricated bearing in the starting condition where the oil flow pressure depends on the bearing velocity (Equation 42).

The theories of Amontons and Coulomb which explain the mechanism of sliding friction in terms of surface roughness are not in agreement with the experimental facts. Surfaces of the same roughness but of different material can show different coefficients of friction. Furthermore, extremely smooth surfaces have higher friction than slightly uneven surfaces. The latter case is usually one of dry friction, adhesion combined with welding, and is not of interest for bearing applications. The first important empirical fact about sliding friction with boundary layer lubrication is that the friction is independent of the bearing area and proportional to the bearing load. This fact can be very easily accounted for by the following explanation.

The actual contact between two surfaces has no relation to the area of the surfaces but is proportional to the normal load. The method of measuring the hardness of materials by penetrating a hardened steel sphere under load and measuring the area of surface penetration suggests an equation for determining the area of actual contact of two surfaces under load:
\[ H_0 = \frac{L}{H} \]
\[ A_0 = \frac{L}{H} \]

where:

- \( A_0 \) = actual contact area (inch\(^2\))
- \( L \) = load in lbs
- \( H \) = hardness in lbs/inch\(^2\)

Since friction is proportional to load, then, according to the above equation, it is proportional to actual area of contact and is inversely proportional to surface hardness. It is a well-known empirical fact that hard bearing surfaces give low friction coefficients.

Hardened steel surfaces and jewels make excellent bearings. A good grade babbit contains nickel and antimony, and therefore as discussed before, the actual bearing surface is a hard surface. After the running in period, the soft matrix of lead and tin is not in sliding contact with the shaft. The actual bearing surface is therefore nickel and antimony, which is very hard. The same is true for phosphor bronze bearings. It must be borne in mind that the area of contact is formed by a deformation of the surfaces. However, this deformation is not entirely permanent, but is partly an elastic deformation. When the bearing load is decreased, the area of actual contact decreases also.

A second important empirical fact about sliding friction under boundary lubrication conditions is that the friction force is independent of the speed of motion, in other words, the energy dissipated by friction is proportional to the speed of motion. The following analogies may help to explain this mechanism:

(1) In playing the violin, the force required to move the bow is independent of speed. The bow excites vibration in the string. The frequency of this vibration is independent of the speed of the bow. The kinetic energy of the linear motion of the bow is partially converted into the energy of the vibrating string which is dissipated as acoustical energy.

(2) In blowing an organ pipe, a similar phenomena takes place. The kinetic energy of the linearly moving air stream is partially converted into the energy of a vibrating air column.

(3) If wind blows through telephone wires, a part of the kinetic energy of the linear motion of the air is converted into the vibrating energy of the wires. The phenomena of singing telephone wires is well known.
To summarize these analogies - whenever there is a system of mass and elasticity, vibrations may be excited by linear motions of other masses which are moving very close to the excitable system.

On the basis of the evidence from the above empirical facts a theory for sliding friction with boundary lubrication may now be proposed. The crystal lattices of solid bodies are held in place by elastic forces and therefore form a vibrating system inside the solid body similar to that of a violin string. A linear motion of another crystal lattice very close to the first one can excite vibrations in the crystal lattice of the first. These vibrations are dissipated as heat according to the kinetic theory of heat. An analysis of the actual forces which excite the vibrations would be concerned with forces between the molecules, known as Van der Waals forces. Such an analysis is beyond the scope of this paper.

A theory for the friction between fluids would be required to omit the elastic forces and therefore has no vibrating system; energy dissipation would then be by inelastic collisions. Fluid friction would then be proportional to the velocity of motion.

It should be emphasized that the analogies given above are incomplete. Actually, the explanation of the excitation of violin strings requires taking into account the friction between the bow and the string. The explanation of the singing telephone wires requires hydrodynamic theory or turbulent flow. A better analogy to the excited vibration of the crystal lattice may be the vibration from an ordinary playing card when it is held against a fast rotating ratchet wheel. The ratchet wheel in this case represents the crystal lattice of one surface and the playing card, one single section of the crystal lattice of the second surface.

This theory for the sliding friction between solids with boundary layer lubrication has the following implications: -- The friction must be proportional to the actual surface contact area, \( A_0 \), which is proportional to the sum of the lengths of all crystal lattices vibrating. The friction to start a motion must be larger than the friction to sustain a motion, because, in striking, the energy to excite the vibrations has to be supplied, while in moving, only the energy to maintain the vibrations has to be supplied and this energy is equivalent to the energy dissipated by friction.

By using X-ray diffraction techniques it may be possible to detect the vibrations of the crystal lattices which have been assumed. It may also be possible to supply the energy for the vibrations of the crystal lattices externally, for instance by a magnetostriction oscillator, and thereby to reduce the friction considerably.

For substantiation of the proposed theory, it would be possible to establish a relationship between the coefficient of friction and
the hardness, density, elastic hysteresis and attracting forces between the 2 surfaces. However, the information available at the present time on elastic hysteresis and surface attraction forces is too meager to check any such relationships by empirical facts.

**Devices for Maintaining Boundary Layer Lubrication (Avoiding Dry Friction)**

In continuous sliding action, it is always possible that in the course of time the boundary layer of lubrication gets squeezed out and the condition of dry friction (case 3) appears. This condition must be avoided in well-designed bearings because it leads to a rapid destruction of the bearing surfaces. Various devices are in use to avoid dry friction. The simplest one is to drip oil occasionally through an oil hole into the bearing. A better device is one which drips oil constantly to the bearing while the bearing is in motion. In larger bearings, oil grooves may be properly arranged to distribute this oil over the entire bearing surface. In a more refined device, a large ring, which rides over the shaft in the center of the bearing and drags through an oil reservoir picks up some of the oil in the reservoir and drips some of this oil on the shaft as it rides over it (see Figure 12).

![Figure 12 Oil-Ring-Bearing](image-url)
In some cases, an endless chain or belt is used in place of the ring. In other bearings, capillary action is used to assure continuous lubrication. This action is accomplished by a wick in contact with the rotating shaft.

Some bearings are manufactured from metal powders and a combustible powder compressed together. After compression, they are fired in a furnace. The metal powder welds together while the combustible powder burns out, leaving a porous metal which because of capillary action, soaks up oil readily.

In other bearings, a composition of a hard and a soft material is used, forming an alloy with a reasonably soft matrix carrying hard compound crystals. After a short running-in period, the surface of the soft material is slightly lower than the surface of the hard material which acts as the actual bearing surface. The small space between the soft material and the shaft can then be filled with oil by capillary action. Bearing bronzes and babbit are bearings of this kind. The soft matrix is formed from tin or lead, while the hard crystals are either copper, antimony, nickel or silver. These bearings have the additional advantage that the plastic deformation of the matrix, after a short running-in period, will be such as to conform exactly to the surface of the shaft or axle.

A still different method of providing continuous lubrication is to mix into a bearing material some lubricant like graphite, which will be exposed to the surface as the bearing wears. These bearings are called oilless bearings. They may be mixtures of graphite with bronze, or wood (maple), the pores of which have been filled with grease, usually in a pressure tank. Cast iron, which contains a lot of graphite, acts to some extent like an oilless bearing.

3. Mechanism of Rolling Friction

It is well known that in many cases substitution of a rolling motion for sliding motion can reduce the coefficient of friction considerably. This fact was recognized with the invention of the wheel. A sled works very satisfactorily in winter on ice and snow; but in summer, the friction between the ground and the sled is very large and the force necessary to pull the sled gets very large. If wheels are put on the sled, it is usually said that rolling friction has replaced sliding friction, and that therefore less force is required to pull the sled. Actually,
however, the sliding motion has been transferred to another surface. Instead of the sled sliding on the ground, the wheels are sliding around a stud. However, the stud and the wheel bushing can be made from some suitable material and kept well lubricated so that force required to pull the sled with wheels is much smaller than that required to pull it without wheels. Now, to eliminate the sliding motion between the wheel bushing and the stud, balls or rollers can be put around the stud to obtain a roller bearing. Again the sliding motion has only been transferred to another surface. Instead of the wheel bushings sliding on the stud, the balls or rollers are sliding against each other. This is a well-known fact, and at high speeds, this sliding causes scoring of the ball or roller surfaces. Accordingly, a retainer, usually made from bronze or laminated plastic, is often used to keep the balls or rollers separated. Now, the sliding motion is between the balls and the retainer. This surface is under lower load than the surface on which the balls roll. Again, the friction has been reduced, because the friction is proportional to the load. This is as far as the present art has progressed. A further refinement in a device for overcoming friction would be the use of rolling motion in the ball separator, as shown in Figures 13 and 14. In these bearings, all sliding motion would be changed into rolling motion.

For example, in the ball bearing design illustrated in Figure 13, the two rows of balls are used which are kept separated by concave spherical rollers which turn in opposite direction from that of the balls. The ball races are, as is customary, concave spherical also.

For another example, in the tapered roller bearing design illustrated in Figure 14, there are small tapered separating rollers between the larger load carrying rollers. The apex of the cones of all the rollers is in one point. The separating rollers are kept in position by means of four rings which roll over their end-flanges. The smaller separating rollers roll in opposite direction from that of the larger load carrying rollers. Two of the rings rotate in the same direction as that of the bearing, and two in the opposite direction. The rolling action of the tapered rollers causes their correct axial location so that all cone apexes originate from the same point.

Friction forces in such bearings would still be caused by the following phenomena:

- Unevenness of the rolling surface.
- Deformation of the material.
- Air friction.
- Combined sliding and rolling motion.
Unless the bearings rotate at extremely high velocity, the effect of air friction can be neglected. In a well-manufactured bearing, the frictional effect caused by the surface unevenness of the rolling surfaces can also be neglected.

The effect of combined rolling and sliding action takes place, for instance, on the faces of the large rollers shown in Figure 14. It also takes place on the surface of rolling balls when there is more than a point contact, in which case, the surface speed of the ball and of the contact area cannot match everywhere in the contact area. Due to elastic deformation under load, there is always a certain contact area and not a point contact. The contact area is, in a first order approximation, proportional to normal load.

Elastic deformation does not directly contribute to the friction because the potential energy of the elastic deformation is recovered at the subsequent elastic expansion. However, elastic hysteresis does contribute to the friction. The elastic hysteresis of chrome-nickel steels as used in ball bearings is extremely small and therefore not a major factor in ball bearing friction. However, elastic deformation indirectly causes sliding friction in the following manner.

Under a compression load a convex surface gets smaller and a concave surface increases. Therefore, two matching surfaces, one convex, the other concave, under a change in load must slide on each other to some extent (see Figure 15).

The conclusion that rolling friction is essentially caused by sliding friction now seems warranted. This conclusion is borne out by the fact that roller friction follows the same empirical laws as the sliding friction with boundary layer lubrication with the exception that the coefficient of friction is smaller.
If straight lines are scribed equidistant from each other on the surface of the rollers and also on the surface of the two plates, then, under load in the contact area, the distance between the lines on the plates will be increased and the distance between the lines on the rollers will be decreased. As a result, when the roller is in motion, there is a sliding motion in the contact area because of the opposed change in the line spacing.

The friction between the sliding surfaces and the distance through which the sliding takes place determines the amount of the rolling resistance. The area of the sliding surfaces depend on the hardness of the surfaces in contact and their normal pressure. The distance through which the sliding takes place depends on the difference of elastic deformation of the surfaces in contact. If a cylindrical roller were to roll on a cylindrical roller of the same size and material the deformation of the surfaces in contact would be identical, and no sliding motion and therefore no rolling resistance due to sliding friction would result. Unfortunately, this arrangement is not practical.

By using elastic theory the amount of sliding which takes place in a ball bearing can be determined from the load, the modulus of elasticity and the dimensions of the bearing, and the hardness of the surfaces. With such an analysis it may be possible to establish the coefficient of rolling friction for a bearing of given dimensions where the modulus of elasticity, hardness, and coefficient of sliding friction of the materials are known. Such an analysis, however, is not simple, and empirical facts on rolling friction are not well enough established to check the validity of the analysis.

For ball bearings it is important to use close grained steels of good elastic qualities with low elastic hysteresis and hard, smooth rolling surfaces.

A comparison between ball and roller bearings follows. Theoretically, balls operate with point contact while rollers operate with line contact. Ball bearings are therefore preferable for very low friction, and roller bearings for very high loads. This is valid for conventional bearings as well as for the proposed new bearings shown in Figures 13 and 14.

Rollers have sliding motion on their end faces. If in a cylindrical roller bearing the rollers have a slight taper due to inaccuracies in machining or due to uneven wear then considerable end thrust can be developed and as a result considerable sliding friction on their faces. In ball bearings this difficulty is avoided. In tapered roller bearings tapers of all rolling surfaces have to originate from one single point since otherwise sliding motion on the rolling surface would occur.
The size of balls or rollers is usually determined by the requirements for minimizing the outside diameter of the bearing. Commercial bearings have usually between seven and sixteen rollers or balls. The load capacity of balls is proportional to the cube of their diameter. For bearings with high load capacity, only a few rollers with large diameter are used, and for bearings with lower load capacity, many rollers with small diameter are used. The number of balls determines the ball diameter in any given shaft for commercial applications. The rolling speed is given by the diameter of the inner ring or race, the size of which in turn is determined by strength and stiffness requirements for the shaft. The ball diameter has no influence on the rolling speed. The speed of the ball container is half of the rolling speed. This speed causes a centrifugal acceleration acting on the balls or rollers and increases their load at high speeds. In high speed application, therefore, the size of the balls or rollers must be kept small. Multiple row ball bearings or roller bearings are used for high speeds and for very high speeds sleeve bearings are preferred.

The following analysis will help to clarify the above conclusions.

In this analysis the symbols will be:

\[ n_s = \text{(rev/min) of shaft} \]

\[ D = \text{dia of inner race (inch)} \]

\[ d = \text{dia of ball or roller (inch)} \]

\[ v_c = \text{velocity of ball container (cage) (inch/min)} \]

\[ v_r = \text{rolling velocity (inch/min)} \]

\[ n_r = \text{(rev/min) of roller} \]

\[ n_c = \text{(rev/min) of cage} \]

Figure 17 Ball Bearing Kinematics
The rolling action of the roller in the stationary housing demands that
\[ v_c = \frac{n_r \pi d}{2} \quad (55) \]

Furthermore, it requires that
\[ v_r = 2v_c \quad (56) \]

The rolling action of the roller on the rotating shaft demands that
\[ v_c = \frac{\pi D n_s}{2} - \frac{\pi d n_r}{2} \quad (57) \]

Equating Equations (55) and (57) one finds that
\[ v_c = \frac{n_r \pi d}{2} = \frac{\pi D n_s}{2} - \frac{\pi d n_r}{2} \]

from which it follows that
\[ n_r = \frac{D}{2d} n_s \quad (58) \]

and this result with Equation (55) gives
\[ v_c = \frac{\pi D n_s}{4} \]

from which one can determine the centrifugal acceleration,
\[ a = \frac{2v_c^2}{(D + d)} \frac{1}{12 \times 3600} \left( \frac{\text{feet}}{\text{sec}^2} \right) \]

This acceleration acts on the mass of the ball or roller and causes an additional load. The bearing has to be strong enough to carry this additional load. This additional load together with the high speed may result in considerable heat development in the bearing as a result of rolling friction. This heat can lead to a damage of the bearing surfaces if it is not dissipated rapidly enough.

**Instruments Without Mechanical Bearings**

All the various kinds of mechanical bearings discussed above have the undesirable property that they develop torques which are added to or subtracted from the torques developed by the quantity being measured. These additional torques lead to inaccuracies of the measured quantities. For highly accurate instruments it is therefore desirable to avoid the use of mechanical bearings. Instruments without bearings are well known,
examples of such instruments are:

- Mercury thermometer
- Mercury barometer
- Cathode ray oscilloscope
- Spectrograph
- Microphone
- Thermocouple
- Geiger counter.

These instruments are not free of friction, as was indicated in our definition of friction. However, their friction is of a much smaller order of magnitude than that of any instruments equipped with a mechanical pointer suspended in mechanical bearings. This fact is responsible for the high accuracies obtainable by optical and electrical instruments, which can be designed easily without requiring mechanically moved parts.

Much of the advance in modern instrumentation is due to the elimination of mechanically moved parts. The analytical chemist uses a spectroscope instead of a weighing scale when he requires a highly sensitive instrument. The modern tendency is to base measuring standards on spectroscopy. The distance of the mercury lines has been proposed as the standard of length and the frequency of the ammonia line, as the standard of time.

The magnetic compass needle mounted on a jewel bearing has been replaced by the flux gate compass which has no moving parts. In principle, the flux gate is a transformer. When an outside magnetic field is present there is a second harmonic induced in the secondary (Figure 18). The amplitude of this signal is proportional to the strength of the earth's magnetic field. Such a transformer with three legs will give the discretion of magnetic north by a vector addition of the measured magnetic fields in the three legs.
Figure 18 Principle of Flux Gate

Other examples of instruments without mechanical bearings are:

a. The integrating accelerometer which consists of an electrolytic solution in which the weight of the deposited metal is proportional to the integrated acceleration.

b. The Bendix convectron which indicates the direction of gravity by the difference of electrical resistance of two
Figure 19 Convectron

c. An angular rate of rotation indicator, designed by the author. This instrument consists of a gas-filled tube where the Coriolis-acceleration acting on the moving ions is converted into an electrical voltage.

As outlined under the heading "Classification of Instruments", this thesis is concerned primarily with instruments which have a mechanical motion of mechanical parts. The mechanical motion of ions or gases, as mentioned in examples (a), (b) and (c) above, will not be investigated further.

There are instruments, however, which have moving mechanical parts but do not have mechanical bearings. These instruments will now be discussed. The moving mechanical part of this kind of instrument will be called the sensitive element. This sensitive element is under the influence of gravity and therefore has to be suspended somehow. Instead of using mechanical bearings the following methods of suspension may be used:

1. Suspension in a Fluid

The forces of gravity can be compensated when the sensitive element is enclosed in a container the specific gravity of which is exactly equal to the specific gravity of the fluid in which this container floats. Minor adjustments of the specific gravity of the supporting fluid are possible by temperature regulation. The fluid is in a larger outer container. Small jewel bearings prevent physical contact of the inner and outer container. These bearings do not add much friction because they do not carry any loads but merely assure the central position of the inner container.

An example of such an instrument is the Schuler gyro-horizon, 1929 model. This instrument consists of a gyro mounted in a spherical bowl that floats in a liquid contained in an outer
bowl (Figure 20). The inner bowl is kept centered by a pointed rod extending upward from the bottom of the outer bowl. The outer bowl is carried on gimbals (not shown). A plane mirror is normal to the spin axis. The center of gravity of the gyro with the inner bowl is at such a distance below the center of buoyancy that the period of precession of the spin axis is about 84 minutes, which is the period of a mathematical pendulum with a length equal to the radius of the earth.

Another ingenious instrument using fluid suspension is the German Anschutz Gyro Compass, 1926 model. The gyro wheels are enclosed in a hermetically sealed spherical globe that floats in an outer spherical shell filled with an electrolyte consisting of acidulated water and glycerine. Both globes are made of thin metal. The outside of the gyro globe and the inside of the spherical shell are each covered by a thin layer of hard rubber vulcanized on the metal shell. There are several electrodes inlaid in the insulating hard rubber coatings, so as to be flush with the spherical surface. The weight of the liquid displaced by the gyro-globe is nearly equal to its own weight. Any tendency of the gyro-globe to move from the central position within the surrounding spherical shell is prevented by the magnetic force of repulsion developed by the interaction of the magnetic field about an alternate current carrying coil located inside at the bottom of the gyro globe and the magnetic field of eddy currents induced in the lower part of the outer spherical shell. The coil produces a conical repelling field directed toward the center of the gyro-globe. The current for the motors of the gyro and the above mentioned coil is conducted into the gyro globe through the electrolyte by suitably spaced electrodes. Lubrication of the gyro is accomplished by wicks which dip into an oil reservoir contained inside the gyro-globe. The gyro globe is filled with hydrogen which
prevents any oxidation of the oil or the insulating material and improves the heat transfer inside the inner globe. Motion of the inner globe relative to the outer shell is detected by the difference of electrical resistance caused by the displacement of suitably connected electrodes. A servo follow-up system keeps the outer shell aligned with the inner globe so that only small displacements of the current carrying electrodes can occur.

As discussed under the heading, "Undesirable Effects of Friction", viscous friction encountered in fluid suspension results in an accurate instrument under static conditions. The main objections against the type of fluid suspensions discussed above are the difficulties of repair and maintenance.

2. Magnetic Suspension

If the sensitive element contains some magnetic material it is possible to compensate the forces of gravity by a magnetic field. A small displacement of a mechanical part suspended according to this principle changes the force exerted by the magnetic field considerably while the gravitational force remains substantially unaltered. Consequently, any displacement disturbs the equilibrium condition. It is necessary to add a servo system which regulates the strength of the magnetic field in such a way that a displacement of the suspended part results in a change of the magnetic field returning the part back to its original position. This can be done by utilizing either magnetic induction or a photoelectric system for detecting the displacement and automatically controlling the magnetic field which keeps the part suspended. Such systems have been used to achieve very high speeds of rotation of parts suspended in vacuum*. These magnetic suspensions seem not to be of practical use for instruments, however.

3. Elastic Suspension

Elastic suspension is in common use for sensitive instruments. Usually the sensitive element is suspended from a wire or string. Examples of such instruments are the galvanometer and the torsion scale. The friction of this type of suspension is caused by elastic hysteresis which is the internal molecular friction inside the elastic material which is used for the suspension wire. If quartz fibers or wires of chrome nickel steel (0.14C, 4.7Ni, 1.0Cr) are used, this internal friction is extremely small. Elastic hysteresis is defined as the ratio of the total work for deformation to the elastic work for

*Journal of Applied Physics, Vol. 17, p 886, November 1946
deformation. It is very time-consuming to measure elastic hysteresis directly by experiment based on this definition. Of course, in these tests it is necessary to stay below the elastic limit of the material. Some data on elastic hysteresis are in Volume 5 of Handbuch the Experimentalphysik*. There it is reported that under repeated stresses the elastic hysteresis tends to go to zero. A general survey of existing information has been conducted by F. C. Thomson**.

A new device to measure elastic hysteresis quickly and accurately has been constructed by Guy Dawance. The device is based on Equation (22) of this thesis. The specimen is mechanically vibrated by means of a pendulum. The phase difference between the vibration of the specimen and the vibration of the pendulum is measured by an optical system involving a mirror linked to the specimen and to the pendulum. This phase difference permits a determination of the damping factor of the specimen, by means of Equation (22)***. The disadvantages of elastic suspension are:

a. The possible motion of the sensitive element is limited. In order to obtain enough motion the suspension wire has to be made rather long.

b. The instruments are very much disturbed by outside vibrations and therefore require special mounting on a base, which is free from vibrational disturbances.

These disadvantages restrict the use of such instruments to laboratories and exclude their practical use in moving vehicles or in factories.

**Servo Follow-up System**

It is possible to overcome the disadvantage of the limited motion in elastic suspensions by the addition of a servo follow-up system****. The principle of such a system is as follows (Figure 22).

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***Groupement Recherches Aeronautiques, Note Technique No. 57, 1946, by Guy Dawance.
All four springs have same spring constant. Two opposed springs are tension springs. Two opposed springs are compression springs.
FIG. 22.  INERTIA WHEEL (SCHEMATIC)
Suppose, for example, that a flywheel is suspended in ordinary bearings, as shown (or in some other manner), and the housing in which these bearings are mounted is not stationary but is mounted on ball bearings and equipped with a gear which is driven by an electric motor. By means of an electrical pick-up device, which in Figure 22 is a potentiometer, but which may be replaced by an inductive, capacitive or strain gage pick-up, the displacement between the flywheel (sensitive element) and the housing can be measured. This signal can then be amplified and fed to the motor driving it in a direction so as to reduce the measured displacement to zero. By this system the relative motion between the sensitive element and the housing will be kept small. Therefore, the energy dissipated by friction in the suspension bearings of the flywheel will be very much reduced because there is no energy dissipated when there is no relative motion. Now, since only a limited motion is necessary in these bearings, they could be replaced by some kind of elastic suspension. This elastic suspension could be made much more compact and stable than the ordinary suspension of a galvanometer because it would require only very minute displacement. Such a suspension is shown in Figure 21 and takes the place of the flywheel bearing shown in Figure 22. In applications in which it is not desirable to have an elastic restoring force, two of the springs in Figure 21 can be tension springs and the two other ones can be compression springs of exactly the same spring constant. This principle is applicable to the gimbal suspension of a free gyroscope (Figure 6) with servo follow-up on each gimbal axis.

There are problems encountered with a servo follow-up system such as that shown in Figure 22 -

1. It is possible that the follow-up housing will be driven by the servo motor into oscillations of increasing amplitude. It is then necessary to introduce somewhere in the system sufficient damping in order to stabilize the system. The physical explanation for this oscillation is very simple. Let us assume that the flywheel has been displaced by a fixed amount. This displacement results in a displacement of the potentiometer thereby giving a voltage which is amplified and fed to the servo motor causing the housing to rotate and to follow up the displacement of the flywheel. As this happens the potentiometer voltage diminishes when it reaches zero however, the inertia of the housing causes the rotation to continue although the motor is no longer driving. This overshooting of the desired zero position gives a new voltage to the potentiometer causing a rotation in the opposite direction and the same process is repeated. If the energy supplied by the amplifier during one cycle is larger than the total damping (friction) of the system, an increase of the vibration amplitude is obtained and vice versa. Electrical dampening into the system is usually introduced into the system where further damping is required. How this should be done ideally has been discussed under the heading, "Ideal Dampener". A
practical way to damp these oscillations may be obtained in the following manner.

a. A generator is connected to the servo motor. This generator gives a voltage proportional to the speed of the housing relative to its base (turntable). This voltage is mixed with the voltage of the potentiometer in the amplifier. This arrangement functions as follows: if the potentiometer gives the command for the motor to turn clockwise and the motor is already turning clockwise, the mixer subtracts from the potentiometer voltage a voltage corresponding to the speed of the motor. This means that where the motor happens to turn very fast and only a small correction of position is required, the generator voltage being subtracted may be larger than the potentiometer voltage, in which case the servo motor is reversed and acts as an electric brake bringing the system to rest near the desired position.

b. It is also possible to obtain a voltage proportional to the relative velocity between the flywheel and the housing by use of an electrical network (Figure 23) which gives the derivative of the potentiometer voltage. This signal can be used in a similar manner to that of the above mentioned generator voltage.

Figure 23 Lead-Network

In analysis of these two types of electrical dampeners, Herbert Harris* shows that for high static sensitivity the generator device is preferable.

If a constant speed (of the turntable) is superimposed on the system, the generator voltage is constant. Then if the constant speed is changed to another value, there is a shift between the position of the housing and the flywheel. Introducing a high pass filter (RC network) prevents the passage of such constant voltages but allows the higher frequency stabilizing signals to pass (Figure 24).

2. The other problem encountered in the follow-up servo system under discussion is caused by the friction in the bearings of the follow-up housing. At a first approximation this friction does not matter because the energy dissipated in these bearings is supplied by the amplifier. However, this friction has an undesirable secondary effect. It resists an increase in speed between the housing and the base (turntable) and it assists a decrease of this speed. Consequently, it takes more time to increase the speed than it takes to decrease it. Now during increasing the speed of the housing there is relative motion between the flywheel and the housing which motion transmits through the flywheel bearing a frictional force assisting the existing motion of the flywheel, while during decreasing speed of the housing the frictional force transmitted through the flywheel bearing counteracts the existing motion of the flywheel. Accordingly, the time during which the speed of the flywheel is increased by the friction in its bearings is longer than the time during which it is decreased. The net effect is an increase in the existing speed of the flywheel, because the magnitude of the friction is independent of the relative speed. This speed increase corresponding to a negative friction has been experimentally observed; an example of the potentiometer voltage recorded during this experiment is shown in Figure 25a.

By addition of a torque motor fed by a constant current power supply it was possible to eliminate this undesirable effect. The current in the torque motor is adjusted to compensate the friction torque of the bearing between the housing and the turntable (Figure 22). The recording voltmeter gave the following result (Figure 25b).
Figure 25b Potentiometer Voltage (Flywheel at Constant Speed) with Servomotor and Constant Torque Motor Driving

By the use of such a device the errors of the instrument due to bearing friction are reduced to a level corresponding to the noise level of the servo system.

A servo system can also be used to obtain a mechanical recording of a measurement by the use of the null system of measurement which is very accurate. This system uses an instrument which indicates a zero reading when and only when two quantities, a standard quantity and the quantity being measured, are equal. Examples of this system are a balance scale, a bridge circuit, and a temperature recording potentiometer. In the last example, the voltage of the potentiometer is compared with the voltage of a thermocouple. The potentiometer is linked to a recording mechanism and adjusted by means of a servomotor which is controlled by the voltage difference between the potentiometer and the thermocouple in such a way that the voltage difference is kept near zero. Such a system using electronic amplification can be built as a highly accurate and sensitive temperature recorder.
Findings and Conclusions

Friction force $F_0$ is defined by the equation $F_0 = \frac{1}{L} \oint Fdl$, where $F$ is the total force applied and $L$ is the length of the path through which the motion takes place. This definition implies that only a reversible process has zero friction.

Friction forces are classified into viscous friction, which depends upon the speed of motion, and Coulomb friction, which is independent of the speed of motion.

An instrument is defined as a transducer. Classification of instruments is discussed. The ideal instrument is defined as the instrument which at all times gives instantaneous correct values of measurement. In such an instrument the directive force has to be proportional to the second time derivative of the quantity to be measured. It is shown that any design approximating the ideal instrument would have cumulative errors. A more practical instrument, which has a restoring force is analyzed. It is shown that such an instrument leads to vibrations at the output. Thereby, the need of a damping force is established. The requirements for the ideal dampener which would give maximum speed of response and no vibrations are discussed. The equations for the instrument with a practical viscous dampener are set up. The optimum values of the viscous dampener are derived as a function of the instrument error, which is given by other factors. (Figure 5)

The undesirable effects of Coulomb friction and its application to free gyroscopes is discussed. Empirical facts about sliding friction in bearings are presented. These facts are ordered into three operating conditions, namely, fluid film operation, boundary layer lubrication and
dry friction. Empirical facts about rolling friction are gathered and summarized.

Theories about the mechanisms of friction are discussed. The oil film bearing is treated by hydrodynamic theory. The application of the oil film bearing to instruments is found impractical. The air-bearing is treated by thermodynamic theory. The mechanism of sliding friction with boundary layer lubrication is investigated. It is shown that the actual area of contact of two plane surfaces which are not geometrically perfect is determined by the hardness of the material. A new hypothesis of sliding friction is advanced in which sliding friction is caused by vibration of the crystal-lattice of the surfaces in contact. Two experimental tests for this hypothesis are suggested. One is based on observation of a surface under sliding motion by means of X-ray diffraction techniques; the other on an observation of friction forces when the surfaces are excited with a supersonic mechanical vibration. The mechanism of rolling friction is explained as sliding friction. New designs for low friction ball and roller bearings are suggested.

It is outlined that highly accurate instruments do not use mechanical bearings. Some optical and electrical instruments are discussed. Fluid suspension, magnetic suspension, and elastic suspension of a mechanical part are described. Servo systems to reduce friction in mechanical instruments and to make instruments more practical are discussed.
**APPENDIX A**

**ERR MS ERROR OF AN INSTRUMENT AS A FUNCTION OF THE DAMPING FACTOR**

\[ q - x = -x e^{-d t} \left( \frac{d}{\lambda} \sin \lambda t + \cos \lambda t \right) \]  
\[ (q - x)^2 = x^2 e^{-2d t} \left( \frac{d^2}{\lambda^2} \sin \lambda t + \cos \lambda t \right)^2 \]  
\[ \int_0^\infty (q - x)^2 dt = x^2 e^{-2d t} \left[ \frac{\sin \lambda t (-2d \sin \lambda t - 2d \cos \lambda t)}{4d^2 + 4 \lambda^2} + \frac{2 \lambda}{(4d^2 + 4 \lambda^2)^{1/2}} \right] \]  
\[ + \frac{\cos \lambda t (2d \sin \lambda t + 2d \cos \lambda t)}{4d^2 + 4 \lambda^2} + \frac{2 \lambda}{(4d^2 + 4 \lambda^2)^{1/2}} \]  
\[ + \left[ \frac{-2d \sin \lambda t - 2d \cos \lambda t}{2 (4d^2 + 4 \lambda^2)^{1/2}} \right] \frac{2 \lambda}{\lambda} \]  
\[ = x^2 \left\{ -\frac{d^2}{4d^2 + 4 \lambda^2} + \frac{2d}{4d^2 + 4 \lambda^2} + \frac{2 \lambda}{2d^2 + 2 \lambda^2} \right\} \]  
\[ = x^2 \left\{ -\frac{d}{4} - \frac{d}{2} - \frac{\lambda}{4d} - \frac{\lambda}{4d} \right\} \]  

\[ = x^2 \left\{ -\frac{d}{4} - \frac{d}{2} - \frac{\lambda}{4d} - \frac{\lambda}{4d} \right\} \]  

\[ \frac{d}{dt} \int_0^\infty (q - x)^2 dt = -x^2 \left[ 1 + \frac{d}{4d^2 + \lambda^2} \right] = 0 \]  
\[ 4d^2 = 1 \]  
\[ d = \frac{1}{2} \]
## APPENDIX B

**TABULATION OF FORMULA:** \[ e^{-\frac{\alpha}{\lambda}} \]

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \alpha^2 )</th>
<th>( \lambda = (1 - \alpha^2) )</th>
<th>( \lambda )</th>
<th>( \frac{\alpha^2}{\lambda} )</th>
<th>( e )</th>
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<tbody>
<tr>
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<td>0</td>
<td>1.000</td>
<td>1.00</td>
<td>0</td>
<td>100%</td>
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<tr>
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<td>0</td>
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<td>1.00</td>
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<td>94</td>
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<td>0.002</td>
<td>0.998</td>
<td>1.00</td>
<td>0.13</td>
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<td>0.99</td>
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<td>0.98</td>
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<td>0.910</td>
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<td>0.840</td>
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<td>0.640</td>
<td>0.80</td>
<td>2.36</td>
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<td>0.360</td>
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<td>0.53</td>
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<td>1.000</td>
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</tbody>
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APPENDIX C

DERIVATION OF NAVIER-STOKES EQUATIONS FOR FLUID FLOW

In accordance with d'Alambert's principle, the forces acting on a fluid element are:

1. The inertia forces caused by accelerations.
2. The body forces (as for example, forces of gravity).
3. The pressure gradient existing in the fluid.
4. The viscous forces resisting the motion.

Therefore, the equilibrium condition for the fluid element is:

\[ \rho \ddot{s} = \rho F - \nabla p + \mathbf{C}, \]  

where \( \rho \) is mass density

\( \ddot{s} \) acceleration vector (components \( x, y, z \))

\( F \) body force vector (components \( X, Y, Z \))

\( \nabla p \) pressure gradient (components \( \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \))

\( \mathbf{C} \) viscous shear force (components \( C_x, C_y, C_z \)).

Written in rectangular coordinates \( (x, y, z) \) the equation is

\[ \rho \ddot{x} = \rho X - \frac{\partial p}{\partial x} + C_x \]
\[ \rho \ddot{y} = \rho Y - \frac{\partial p}{\partial y} + C_y \]
\[ \rho \ddot{z} = \rho Z - \frac{\partial p}{\partial z} + C_z \]  

(c-2)

With \( \mathbf{v} \) the velocity of the fluid (components \( a, b, c \)) one has,

\[ \frac{ds}{dt} = \mathbf{v} \]
\[ \frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = c, \]
Equations \( (c-1), (c-2) \) and \( (c-3) \) combined give:

\[
\frac{\partial v}{\partial t} + a \frac{\partial v}{\partial x} + b \frac{\partial v}{\partial y} + c \frac{\partial v}{\partial z} = f - \frac{1}{\rho} \frac{\partial p}{\partial t} + \frac{1}{\rho} \tau \tag{c-4}
\]

which when written in coordinate components is

\[
\begin{align*}
\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial x} + b \frac{\partial a}{\partial y} + c \frac{\partial a}{\partial z} &= X - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \tau x \\
\frac{\partial b}{\partial t} + a \frac{\partial b}{\partial x} + b \frac{\partial b}{\partial y} + c \frac{\partial b}{\partial z} &= Y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{1}{\rho} \tau y \\
\frac{\partial c}{\partial t} + a \frac{\partial c}{\partial x} + b \frac{\partial c}{\partial y} + c \frac{\partial c}{\partial z} &= Z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \tau z
\end{align*}
\]

\[S\]

Shear force:

Coefficient of friction: \( \mu \)

In an incompressible fluid, the shear force is the product of the area of the surface.

*In a compressible fluid an additional friction term resisting a volume change must be added.
Consider now the volume element $dV = dx \, dy \, dz$

$$S_{xy} = \mu \frac{\partial (u_\alpha)}{\partial y} \, dy \, (dx \, dz)$$

and in an analogous way

$$S_{xx} = \mu \frac{\partial (u_\alpha)}{\partial x} \, dx \, (dy \, dz),$$

$$S_{xz} = \mu \frac{\partial (u_\alpha)}{\partial z} \, dz \, (dy \, dx).$$

Therefore,

$$T_{xx} = \frac{S_{xx}}{dx \, dy \, dz} = \mu \frac{\partial^2 u_\alpha}{\partial x^2},$$

$$T_{xy} = \frac{S_{xy}}{dx \, dy \, dz} = \mu \frac{\partial^2 u_\alpha}{\partial y \partial x},$$

$$T_{xz} = \frac{S_{xz}}{dx \, dy \, dz} = \mu \frac{\partial^2 u_\alpha}{\partial z \partial x}$$

and therefore,

$$T_y = T_{xx} + T_{xy} + T_{xz} = \mu \nabla^2 u_\alpha$$

which, when inserted into Equation (c-4) with analogous expressions for

$$\frac{T_y}{c} = \mu \nabla^2 c$$

for

$$\frac{T_z}{c} = \frac{\mu \nabla^2 c}{c}$$

gives

$$\frac{\partial u_\alpha}{\partial t} + a \frac{\partial u_\alpha}{\partial x} + b \frac{\partial u_\alpha}{\partial y} + c \frac{\partial u_\alpha}{\partial z} = X - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu \nabla^2 u_\alpha}{\rho}$$

$$\frac{\partial b}{\partial t} + a \frac{\partial b}{\partial x} + b \frac{\partial b}{\partial y} + c \frac{\partial b}{\partial z} = Y - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu \nabla^2 b}{\rho}$$

$$\frac{\partial c}{\partial t} + a \frac{\partial c}{\partial x} + b \frac{\partial c}{\partial y} + c \frac{\partial c}{\partial z} = Z - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu \nabla^2 c}{\rho}$$

or

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + a \frac{\partial F}{\partial x} + b \frac{\partial F}{\partial y} + c \frac{\partial F}{\partial z} = F - \frac{1}{\rho} \frac{\partial p}{\partial t} + \frac{\mu \nabla^2 F}{\rho}.$$
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ABSTRACT

Friction is defined as a force acting against the direction of motion and resulting in a dissipation of energy in an irreversible manner. The only frictionless motions in nature exist if particles or bodies move on a selected geodesic curve (a geodesic curve is a curve in which there are no forces acting on the particle) which does not give rise to a space dependent change of the field.

Coulomb friction is independent from the speed of motion while viscous friction is proportional to the speed of motion. An instrument is defined as a device which transduces one physical quantity into another. Instruments may be classified into control - or measuring instruments. A better way to classify instruments is according to the physical qualities which are transduced. This thesis restricts itself to instruments in which a mechanical motion is present. An ideal instrument is defined as giving accurate and instantaneous readings. In such an instrument the directive force acting on the pointer should be proportional to the second time derivative of the quantity to be measured. Such an instrument has cumulative errors and is therefore impractical. The practical instrument requires a restoring force when the pointer is off the desired point. Such a restoring force causes vibrations of the pointer. Therefore, it is necessary to add a dampening force. The ideal dampener would bring the pointer to rest on the desired point in a minimum time. It would require a complicated computer. In practical instruments a dampener using viscous friction is used. The optimum value of the viscous dampener depends on the instrument error, which is caused by other effects. A formula relating optimum viscous dampening and instrument error is given.
The static errors of instruments caused by Coulomb friction are shown. The effect of Coulomb friction on gyro-gimbal suspension is discussed. Empirical facts about friction in bearings are collected. They are classified into the following categories: oil film separation, boundary layer lubrication, dry surfaces and ball or roller bearings.

The effect of surface grain structure and grain direction on the sliding friction is demonstrated in practical examples. The mechanism of sliding friction with oil film separation is analyzed using hydrodynamic theory. The mechanism of air bearings is treated using thermodynamic theory. Oiling devices to avoid dry friction are described. A hypothesis for the mechanism of friction in surfaces with boundary layer lubrication is made, according to which such friction is caused by a vibration of the crystal-lattice. This hypothesis could be tested experimentally by X-ray diffraction or by artificially excited vibrations with magnetostriction oscillators which should effect the coefficient of friction. It is shown that the major part of rolling friction in bearings is caused by sliding friction. Designs for low friction ball and roller bearings are suggested. The effect of centrifugal force on high speed ball bearings is discussed. Instruments without bearings are described. The high sensitivity of optical and electrical instruments is pointed out. Mechanical instruments using fluid suspension, magnetic suspension and elastic suspension are described. A servo follow-up motion as a means to reduce friction is described. The accuracy of a null point system of measurement is pointed out and its application in conjunction with servo systems shown.