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Early mathematical formulas and their errors

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Thesis

EARLY MATHEMATICAL FORMULAS AND THEIR ERRORS

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1926
I The development of the symbolized formula.

1 The lack of any formula among the earliest mathematicians.
   (a) Ahmes papyrus contains no general rules of procedure.

2 The development of a formula expressed entirely in words.
   (a) This type of algebra is called "Rhetorical Algebra".
   (b) Use of this form of algebra by Arab, Persian, Italian, and a few Greek mathematicians.

3 The development of a formula with certain recurring operations and processes represented by a symbol.
   (a) Diophantus was the first mathematician to use this form of expression.
   (b) Use of this form of algebra (Syncopated Algebra) by Hindus, western Arabs, and Europeans of Middle Ages.

4 The development of the formula in its modern form—a compact symbolic expression.
   (a) The development of a general symbolism was first necessary.
      (1) The development of a symbol for:
(a) Addition
(b) Subtraction
(c) Multiplication
(d) Division
(e) Equality
(f) Pi

II Various formulas used by early mathematicians and their errors:

1 Formulas for the area of a triangle.
   (a) \( A = \frac{a}{2} \cdot b \)
   (b) \( A = \frac{a + c}{2} \cdot \frac{b}{2} \)
   (c) \( A = \frac{1}{2} (a^2 + a) \)
   (d) \( A = \frac{13}{30} a^2 \)
   (e) \( A = \frac{1}{2} a^2 \)

2 Formulas giving the area of a trapezium
   \[ A = \frac{b_1 + b_2}{2} \cdot c \text{ (Isosceles trapezoid)} \]

3 Formula for the area of a quadrilateral and its diagonal
   \[ A = \frac{a + b}{2} \cdot \frac{c + d}{2} \]
   \[ A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \]
   \[ m = \sqrt{\frac{(ac + bd)(ab + cd)}{ad + bc}} \]
   \[ n = \sqrt{\frac{(ac + bd)(ad + bc)}{ab + cd}} \]

4 Formulas for the measurement of pyramids
   \[ A = \frac{1}{2} B \cdot h \]
\[ V = \frac{1}{2} Bh \]
\[ V = \frac{x}{3} (a^2 + ab + b^2) \]  (Frustum of pyramid)

5 Formulas involving the value of \( \pi \)

(a) Formulas for the area of a circle
\[ A = \left( \frac{8}{9}d \right)^2 \]

(b) Formulas for circumference of circle
\[ c = 3d \quad \text{(Used by Hebrews)} \]
\[ d = \sqrt{\frac{154s + 841}{11}} - 29 \]

(c) Formulas giving the volumes of various solids
\[ V = \left( \frac{16r}{9} \right)^2 h \]  (Volume of a cylinder)
\[ V = \frac{8}{30} d^3 \]  (Volume of a hemisphere)
\[ V = \pi r^2 \sqrt{\pi r^2} \]  (Volume of sphere)
\[ V = \frac{9}{2} (1/2 d)^3 \]
\[ V = \left( \frac{9}{10} \right) \frac{9}{2} (1/2 d)^3 \]  (Volume of sphere)

(d) Formula for the area of surfaces
\[ A = \pi r^2 \]  (Area of surface of sphere)
\[ A = 1/2 (c + a) a \]  (Area of segment of circle)
\[ A = \frac{3}{4} d^2 \]  (Area of a circle)
\[ A = \frac{c^2}{12} \]

(e) Formula involving the trigonometric functions
\[ \sin(n+1)a - \sin na = \sin na - \sin(n-1)a \]
\[ -\sin na \csc a. \]

III Approximation formulas used by mathematicians of Middle Ages.
I Formulas approximating the roots of numbers:

\[
\sqrt[3]{A} = \sqrt[3]{a^2 + b} = a + \frac{b}{2a}
\]

\[
\sqrt[n]{A} = 1/a \sqrt[3]{Aa^2}
\]

\[
\sqrt[n]{A} = \sqrt[3]{a^3 + b} = a + \frac{b}{3a(a+1)}
\]

\[
\sqrt[n]{n} = b + \frac{a(n-b^3)}{n + a(n-b^3)}
\]

\[
\sqrt[n]{an + b} = a + \frac{b}{(a+1)^n - bn}
\]

2 Formulas approximating the roots of equations:

(a) Double false position formula:

\[
x = \frac{f_1g_2 - f_2g_1}{f_1 - f_2}
\]

(b) Single false position formula:

\[
x = \frac{g(f - b) - gf}{f - b}
\]

(c) Solution of the quadratic involves many different formulas—all of which give an accurate result, although negative roots are often neglected.

(d) The importance of the false position methods in the solution of higher degree equations.

IV Conclusion.
The term formula is applied when we express any of the fundamental truths of mathematics in the form of a comprehensive, symbolized statement. In the light of this modern interpretation, the formula has had a gradual evolution.

In the very earliest stages of mathematical history, the formula did not exist. The Ahmes papyrus, the most ancient mathematical manuscript known to us, does not contain any general rules of procedure, but merely a statement of results obtained in the computation of various problems.\(^{(1)}\) For example, if we select at random any of the examples in the papyrus, we find a statement of this nature—"Reckon the amount of corn contained in a circular container of diameter nine khets and height ten khets." There then follows the particular method of solving this problem. The reader is told to:

\[
\text{Subtract } \frac{1}{9} \text{ of } 9 \text{—namely } 1. \text{ Remainder } = 8. \\
\text{Multiply } 64, \text{ ten times } = 640 = \text{ volume in cu. cubits.}
\]

\(^{(1)}\) James Gow "A Short History of Greek Math"—pl6.  
\(^{(2)}\) Rhind Papyrus—p80. Ex. no. 40.
Add $\frac{1}{2}$ of $640 = 960$ = volume in khan.

Take $\frac{1}{20}$ of $960 = 48$.

Forty-eight then represents the number of hekat of corn contained in the receptacle.

Other examples of this same type follow, but there is no general rule of procedure drawn. Each example must be worked out separately without any suggestion of a general formula. Cajori says that "the principal defect of Egyptian arithmetic was the lack of a simple comprehensive symbolism—a defect which not even the Greeks could remedy." (1)

In the next stage of mathematical history, we find the development of a general formula expressed entirely in words. This represents an advance in mathematics, however, for it shows that people had come to recognize the power of a generalized principle for the solution of problems of the same type. This division of Algebra in which no symbols are used, has been named Rhetorical Algebra by Nesselmann. (2)

In the mathematical works of the early Arabs, we find this type of Algebra used exclusively. Hence their equations were written out in the


Ball--"History"--p103
form of words. One of Al Khowarazmi's equations is represented in its Latin translation as: (1)

"Census et quinque radices aequantur viginti quatuor."

In its modern symbolic form this equation would be written: \( x^2 + 5x = 24 \). The Arabs, however, are not the only peoples whose mathematical works come under the head of Rhetorical Algebras. In addition to the Eastern Arabs, we find the Persians, the early Italian writers, the Greek mathematicians Iamblichus and Thymaridas, Regiomontanus and even Leonardo expressing their rules and equations in statements wholly devoid of symbolism.

As civilization progressed mathematicians began to search for a method of expressing their mathematical principles in a more concise form. However, the transition from an algebra expressed entirely in words to a completely symbolized algebra was slow. The Arabic notation had not as yet permeated the West and what symbols were used were not generally adopted.

Diophantus is generally regarded as the

(1) Fink "History of Math."--p89.
first mathematician of note to introduce a system of abbreviations for those operations and quantities which reoccur often in mathematical expressions. Diophantus employed a symbol to represent an unknown quantity, but as he had only one such symbol, he could use but one unknown in any given expression. (1) In every case his symbols are merely abbreviations for words which would denote the process. "Syncopated Algebra" is the name applied to this form of algebra by Isselmann. Cajori defines "Syncopated Algebras" as those "in which everything is written out in words except that the abbreviations are used for certain frequently recurring operations and ideas." (2) We find very many formulas written in this form by the western Arabs, Hindus, and Europeans of the Middle Ages. In fact, "it may be said that European Algebra did not advance beyond this stage until the close of the sixteenth century". (3) Even Cardan, in his "Ars Magna", published in 1545, uses this notation, and we find this illustration of his manner of ex-

(1) Ball "History of Mathematics"--p105.
(2) Cajori from Isselmann p302--306.
(3) Ball "History of Mathematics"--p103.
pressing mathematical facts.

**Cubus p 6 rebus aequalis 20**

In this equation p is an abbreviation for the Latin plus, and rebus, the word used to designate the unknown. Hence this equation would be represented in modern symbols as $x^3 + 6x = 20$. Again, $R. w. cu. R 108 p 10$ would represent

$$\sqrt[3]{\sqrt{10r} + 10}$$  (1)

Mathematicians were by this time awakened to the possibilities and advantages of a completely symbolized formula. The invention of the printing press made possible the wide dissemination of mathematical works, and in addition stimulated thought toward the discovery of a general symbolic notation. Yet we must not suppose that this change took place immediately. In fact, it has often taken more than a hundred years before one form of symbol could be agreed upon by even our most eminent Mathematicians for one of the fundamental processes. The Greeks, Hindus, and some of the European writer had indicated addition by juxtaposition. Still later the abbreviation p for the Latin plus had been used. The sign $+$ was probably a ligature for the Latin word 'et'. At any

(1) Young "Fundamental Concepts"--p235
rate it was gradually introduced by German and English mathematicians until, in 1630, it was quite generally recognized as a symbol for addition.

There were many different methods of expressing the process of subtraction. In early mathematical history, the Hindus had used a dot to denote minus, and the Italian algebraists had used the first letter of the Latin minus. The present symbol for minus is thought to have been obtained from the bar which was often used in old manuscripts to denote the omission of words of letters. (1) When this modern symbol was finally introduced, it was not used exclusively to denote subtraction. Instead, we find this symbol representing division, ratio, and proportion indifferently. Although plus and minus signs were used by Widman in his "Mercantile Arithmetic", published in 1489, Vieta, however, was the first well known mathematician to employ these signs consistently throughout his work. (2) It was not until the middle of the seventeenth century that

(1) Ball "History of Math"--p207
(2) Ball "History"--p195.
they were recognized as well known symbols for these processes.

A symbol to express multiplication was slower in being developed. The sign $\times$ was introduced in 1631 by Oughtred.(1) About this same time, the dot was introduced to denote the same process. At a somewhat later date Leibnitz used the sign $\div$ to represent multiplication and $\times$ for division. The sign $\times$ is still used to denote the process of multiplication altho the dot is now preferred in algebra in order to avoid confusion with the letter x.

The sign for division has had a much slower evolution. In the first place, the process of division was carried out by a series of subtractions and hence no abbreviation for this process was necessary. The early Arabs had indicated division in the form of a fraction as $a/b$ or $a-b$. Oughtred had used a dot to denote division and still later the sign $\div$ had been introduced by Leibnitz.(2) Probably our modern symbol for division is a combination of the horizontal line for division and the symbol of proportion. The sign was not general-

(1) Smith "History of Math"—vol. 2 p404
(2) Ball "History of Math"—p241.
ly known till 1668 when Pell introduced it in England in his translation of Rahn's work. (1)

The symbol for equality was first introduced by Robert Recorde in his book entitled "The Whetstone of Wit". He says that he used two parallel bars to represent equality because two things cannot be more equal. Until 1600, the practice of writing the word for equals in full was most commonly used. From that date to the time of Newton very many symbols were used. It was not until seventeen hundred, that the present sign of equality was generally adopted.

The symbol \( \frac{\delta}{\pi} \) was used by Oughtred (1647) and Barrow (1664) to denote the ratio of the diameter to the circumference. The first person to use \( \pi \) definitely to represent the ratio of the circumference to the diameter was William Jones in 1706. Euler adopted the symbol in 1737 and since that time it has been generally used to express that ratio. (2)

Brief histories of the development of other symbols might be given, but these few will serve to show that the general adoption of a completely symbolized algebra was impossible.

(1) Smith "History of Math"—vol. 2 p406.
(2) Smith "History"—vol.2 p312.
in the Middle Ages. Even the Hindu Arabic numerals were not generally used in Europe until the sixteenth century. Hence the great advances made in mathematics since 1600 may be directly attributed to the development of an algebraic symbolism which made possible a general symbolized formula. Before the introduction of a symbolic notation, such formulae as Taylor's theorem would be unintelligible to us. We have only to glance at this formula:

\[ f(x + h) = f(x) + f'(x)h + f''(x)h^2/2! + \cdots + f^n(x)h^n/n! + \cdots \]

to realize the immense contribution of a symbolic notation to the development of a comprehensive, concise formula. The development of a comprehensive symbolic formula is not yet in a state of completion, although rapid progress has been made in this direction. We have yet to hope for a universal adoption of these symbols and formulas which will make possible a closer understanding between mathematicians of various countries.

In this discussion, all generalized principles used by early mathematicians will be translated into a modern symbolic formula. Even in the Ahmes papyrus, where we find no evidence of a definite statement of rules, the solutions of various problems give us the keynote to the
general principles which were used and which may be interpreted in our modern notation.

If we employ the chronological order in the discussion of various formulas and their errors, perhaps the first formula that deserves mention is the one which gives us the area of a triangle. In the Ahmes papyrus we find a problem which asks for the acreage of a triangular piece of land of ten khet in height and four khets in its base. (1) The answer is given as twenty. There is no doubt but that the answer was determined by taking one half of the product of the base and height of the triangle. The question at once arises as to what the word height would represent. Is it synonymous with the altitude of a triangle? If so, the formula was, of course, correct. In the case of a scalene triangle, there is no other possibility, for otherwise two legs would be given. The Egyptians were certainly not measuring a right triangle, for they considered a right triangle as one half of a rectangle and hence used this symbol for it. If the triangle under consideration was an isosceles triangle, then the height might represent either the altitude or one of

(1) No. 51 Rhind papyrus.
the equal sides. However, we have other evidence which leads us to believe definitely that the ancients did not use the altitude of the triangle in order to compute the area of a triangle. In the inscription of the temple of Edfu, built by Ptolemy, mention is made of a large number of fields owned by the priesthood and their areas are given. In each case, the area of the quadrilateral is given by the formula:

\[ A = \frac{a+c}{2} \cdot \frac{b+d}{2} \]

where \( a, b, c, \) and \( d \) represent the four linear dimensions of the field. (1) In those days, triangles were not regarded separately, but were considered as quadrilaterals where the length of one side equalled zero. (2) Hence in the case of an isosceles triangle the area would be calculated by the formula:

\[ A = \frac{(a+0)}{2} \cdot \frac{(b+b)}{2} = \frac{a}{2} \cdot b \]

Moritz Cantor in his "Geschichte der Mathematik" confirms this statement when he asserts that the Egyptians used the formula \( A = \frac{1}{2}ab \) to represent the area of a triangle whose sides

(1) Cajori "History of Mathematics" p10
(2) Heath "History of Greek Geometry" p124.
null
are $a$, $a$, and $b$. (1) Although this formula is considered false to-day, it does give a crude approximation to the area of certain triangles. If we calculate the area of a triangle by this formula, we find that as the base is decreased relative to $a$, the area approximates the area given by the correct formula: \[ A = \frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}} \]

For example, the degree of error in an isosceles triangle of base 2 and leg 3 is 17.7%, whereas the percentage of error in a similar triangle of base 1 and slant height 4 is only 1.73%.

The formula for determining the area of any quadrilateral, namely:

\[ A = \frac{a + c}{2} \cdot \frac{b + d}{2} \]

is very inaccurate. It may be applied correctly only in the calculation of the area of a rectangle. If the quadrilateral is not rectangular, this formula yields a result considerably greater than that given by the correct formula of finding the area. Since almost all of the mensuration of the Egyptians was for the purpose of forming a basis of taxation, we see that in either case the tenant would be the loser. (2)

(1) Cantor "Geschichte" P 49
(2) Rhind Papyrus.
We find evidence of the fact that these formulas were also used by *Brahmagupta*, *Mahayira*, *Boethius*, and *Beda*.(1) Heron also knew these approximations, and it is probable that from him the Roman surveyors obtained their knowledge of them. In addition to other formulas for the area of a triangle Heron gives the formula:

\[ A = \frac{a \cdot b}{2} \]

which bears a close likeness to the formula for the area of a quadrilateral which appears on the Edfu temple. This formula, which is completely false, gives a result which is always considerably greater than the correct value. Hence all the formulas used by the ancients for discovering the area of an isosceles triangle are false or depend on false principles.

We also find other formulas which approximate rather crudely the exact area of an equilateral triangle. Among these is the formula:

\[ A = \frac{1}{8}(a^2 + a) \]

This formula yields a very inaccurate approximation to the exact area which is determined by the formula \( A = (1/4)a^2\sqrt{3} \). For example,

(1) Kaye "Indian Math." p71.
(2) Cajori "History of Elementary Mathematics" p90.
if we take a triangle one of whose sides is 4 and apply the formula \( A = \frac{1}{2}(a + b) \) we find the value of the area to be 10. The correct area is 6.928. If we take 10 as the length of one side, the area given by this approximation formula is 55 whereas the true area is only 42.5. This means that the formula \( A = \frac{1}{2}(a + b) \) gives an error of 23% when 10 is taken as the length of one side. Moreover, if we take a triangle of side 100, we find that there is only a 4% error between the area given by this inaccurate and the true one. Hence, although the formula is most inaccurate, it probably proved useful in approximating the area of large tracts of land.

It seems strange that Roman surveyors still continued to use these very inaccurate formulas after the true area could have been obtained from the formula \( A = \sqrt{s(s-a)(s-b)(s-c)} \).

The reason for this was probably due to the fact that "mathematics was studied with a view to intelligent penetration, and not practical results." (1) Hence, although the formula \( A = \sqrt{s(s-a)(s-b)(s-c)} \) had been derived, its practical application had

(1) "Fundamental Facts in Hist. of Math."

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not yet been realized.

The Romans, in addition to the formulas already mentioned, made use of the approximation formula:

\[ A = \left( \frac{13}{30} \right) a^2 \] (1)

for determining the area of an equilateral triangle of side \( a \). This formula gives us 6.933 as the approximate value of the area of a triangle of side 4, whereas the true value is 6.928. This formula, then, gives us a closer approximation to the real area of an equilateral triangle than any of the other ancient rules for determining this area.

Another formula which the Roman surveyors used extensively was the formula:

\[ A = \left( \frac{1}{2} \right) a^2 \] (2)

This was without doubt derived from the Egyptian formula \( A = \frac{a+b}{2} \cdot \frac{c+d}{2} \) which was transformed to a formula for an equilateral triangle as follows:

\[ A = \frac{a+0}{2} \cdot \frac{a+a}{2} = \frac{a^2}{2} \]. This formula gives a result which is between the very inaccurate result obtained by the formula \( A = \frac{1}{2}(a^2+a) \) and the more correct area determined by the formula \( A = \frac{13}{30} a^2 \).

(1) Cajori "History of Mathematics" P66
(2) Cajori "History of Mathematics" P66
Let us now glance at some of the other approximation formulas used by the ancient mathematicians. Great difficulty was encountered in the development of a formula for the area of a trapezoid. Before taking up the formula, it is first necessary to note the confusion which has arisen throughout the centuries between the words trapezoid and trapezium. Euclid defined a trapezium as any quadrilateral except a square, oblong, or rhomboid. Later Greek geometers applied the term trapezium to denote any quadrilateral with one pair of parallel sides. A figure with no two sides parallel was called a trapezoid. In European countries, this distinction still holds. However, the words became interchanged by English writers on mensuration, so that the term trapezoid is used with us to mean a quadrilateral with two sides parallel, while a trapezium denotes a figure with no sides parallel. In this discussion the terms will be considered in this latter interpretation.

In the Ahmes papyrus we find evidence of the fact that the formula: 
\[ A = \frac{b_1 + b_2}{2} \cdot a \]  
\[ (1) \]  
(1) Cantor "Geschichte" p86.
where $b_1$ and $b_2$ represent the two parallel sides and $a$ is the length of one of the equal sides, was used to find the area of an isosceles trapezoid. This is but another case in which the length of one side is used instead of the altitude of the figure. Fink tells us that the Egyptians not only used these approximate formulas throughout all their work, but they were evidently considered perfectly correct.(1) Although this formula of Ahmes gave inaccurate approximations to the area of an isosceles trapezoid, we find the case of a much more inaccurate formula in use among the Romans. Among these peoples, the inaccurate formula of Ahmes is applied to any trapezium whatsoever! Hence we must agree with Cajori when he says "Thus the formulas of more than 2000 B.C. yield closer approximations than those written two centuries after Euclid."(2)

The rule for the area of any quadrilateral, is stated by Brahmagupta is as follows: "Half the sum of the sides set down four times, and severally lessened by the sides, being multiplied together, the square of the product is

(1) Fink "History of Math" p192.
(2) Cajori "History of Elementary Math" p45.
the exact area." (1) How much more simply this rule may be expressed in terms of one modern symbolic formula:

\[ A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \]

where \( s = \frac{1}{2}(a+b+c+d) \). This formula does not hold true unless the figure is inscribed. Nevertheless this formula marked an important step in the development of algebra, for it gave us the correct formula \( A = \sqrt{s(s-a)(s-b)(s-c)} \) where one of the sides is equal to zero.

Mahavira makes the same mistake in regard to the formula for the trapezium in that he also does not limit it to an inscribed figure. This same error is again responsible for the errors in these two formulas for the length of the diagonal of a quadrilateral:

\[ m = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}} \]

\[ n = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \]

(2)

Of much greater importance geometrically are certain calculations upon the area and

(1) Smith—vol. 1 p158.
(2) Smith—vol. 2 p287.
volumes of pyramids. In the papyrus of Ahmes we find that the lateral area of a regular pyramid is given as:

\[ A = \frac{1}{2} Bh. \]  

(B = perimeter of base.)

In this case where the slant height of the figure should be used, curiously enough, we find the ancient mathematicians going out of their way in order to use the altitude of the figure. In calculating the volume of a pyramid, Aryabhata used this formula:

\[ V = \frac{1}{2} Bh \] (1)

rather than \[ V = \frac{1}{3} Bh. \] This formula of course is very inaccurate and the absolute error necessarily increases as the figure becomes larger. In the Ahmes papyrus, we find the solution of an example which evidently makes use of the formula:

\[ V = \frac{x}{3} (a^2 + ab + b^2) \] (2)

to express the volume of the frustum of a pyramid. Confusion arises in the minds of our historians as to whether the Egyptians really used the altitude of the frustum or the slant height which they had employed in other mensuration examples. If \( x \) denotes the altitude and the bases are squares, the formula is correct.

(1) Kaye "Indian Math" p13
(2) P93 Rhind Papyrus.
A great portion of the work in approximation formulas deals with the areas of circles, spheres, and other figures which involve the ratio of the circumference of a circle to its diameter. This constant ratio, which we now call $\pi$, has had a most interesting development. At times the value used for $\pi$ has been a very close approximation to the true value and then this has curiously been abandoned in favor of a much more inaccurate one. There are probably hundreds of values of $\pi$ which have been advanced by one mathematician and another during the four thousand years of our mathematical history.

Many mathematicians have used two values of $\pi$, termed an exact and an approximate value. For a great many years $\pi = \frac{22}{7}$ was the approximate one, while $\pi = \frac{355}{113}$ was the "exact" value. But to include even a few of the more important values of $\pi$ would necessitate too much time and space.

Hence an approximation of $\pi$ will be mentioned only in connection with its value in the solution of some of the formulas.

Again glancing at the Ahmes papyrus, we find an example which gives a solution for the area of a circular piece of land.(1) This ex-

(1) No. 50 Rhind papyrus—p90.
ample asks the reader to reckon the acreage of a circular piece of land of diameter 9 khet. The ancient mathematician then tells us to solve the example in the following manner.

You are to subtract $1/9$ of the diameter

$9 \text{khet} = 8$

You are to multiply 8, eight times $= 64$

This is its area in land. The process involved would be expressed today by the formula:

$$A = (d - 1/9d)^2 = \left(\frac{8}{9}d\right)^2$$

We may easily determine the value of $\pi$ employed in the solution of this example from the relation:

$$\frac{1}{4} \pi d^2 = \left(\frac{8}{9}d\right)^2$$. The ratio $\frac{c}{d}$ or $\pi$ will be expressed as $(\frac{16}{9})^2 = 3.1604$ which is a very good approximation for such an early mathematical work. This gives us an error of only .59% from the value of $\pi$ with which we are familiar today.

The early Hebrews made use of a cruder approximation to the value of $\pi$. In the first book of Kings, the seventh chapter and twenty-third verse we find the following: "And he made a molten sea ten cubits from one brim to the other; it was round all about and his height was five cubits; and a line of thirty cubits did compass it about". This shows us that the ancients considered $\frac{30}{10}$ or 3 as the ratio of the circumfer-
null
ence to the diameter. Hence the very crude value of \( \pi = 3 \) was evidently used by the ancient Hebrews.

In Aryabhata's mathematical treatise written in the sixth century, the following rule for finding the circumference of a circle is given: "Add 4 to 100. Multiply by 8 and add 62,000. The result is the value of the circumference of the diameter is 20,000." This rule gives a very accurate value of \( \pi \)--namely 3.1416.

As we trace through the centuries various formulas which give us the areas of circles, we find that the Hindus attempted to construct a circle equivalent to a given square. Here the diameter is made equal to four-fifths of the diagonal of the square. Although this construction is now known to be impossible, much mathematical work was accomplished in attempting to solve the problem. The value of \( \pi \) resulting from the above assumption would be 3 1/8, and we find that this value of \( \pi \) was used generally among the Romans.

Heron of Alexandria gives as his value for the diameter of a circle

\[
d = \sqrt{\frac{154s + 841}{11}} - 29
\]

In this formula "s" represents the sum of the diameter, circumference and area. This formula is interesting because it sheds some light on the kind of mathematical work accomplished by Heron. In order to derive this formula he must have solved a quadratic equation. If we express the area of a circle as \( \frac{\pi d^2}{4} \) and the circumference as \( \pi \) and take \( \frac{\pi}{7} = \frac{22}{7} \) we have the equation:

\[
s = d + \frac{\pi d^2}{4} + \pi d \text{ or } \frac{11d^2}{14} + \frac{29}{7} d = s
\]

Clearing of fractions and completing the square we have:

\[
121d^2 + 638d + 841 = 154s + 841
\]

whence

\[
d = \sqrt{\frac{154s + 841}{11}} - 29
\]

Hence Heron was an algebraist, although he lacked the symbolism for expressing himself.

In addition, Heron most certainly used \( \pi \) as the approximation \( \frac{22}{7} \).

After discovering the area of the circle the Egyptians next devised a method of finding the volumes of cylinders, spheres, and hemispheres. Again referring to our oldest mathematical treatise, the Ahmes papyrus, we note this example involv-

ing the volume of a cylinder. The example asks for the volume of a circular container of diameter 9 and height 10.(1) From the solution given, we know that the principle used would be expressed today by the formula

\[ V = \left( \frac{16r^2}{9} \right)^2. \]

The value of \( \pi \) would evidently be \( \frac{213}{61} \). There are other examples involving the calculation of volumes which are mentioned in the Ahmes papyrus, but it is impossible to ascertain the formulas from the examples because we are not acquainted with the shape of the buildings or monuments whose volumes have been computed. However in the Kahun papyrus there is an example given which calculates the amount of corn contained in a hemisphere.(2) Heath represents the principle employed by them in the measurement of this hemisphere in modern symbols as:

\[ V = \frac{8}{30} \cdot d^3 \]

The true formula is \( V = \frac{1}{12} \cdot \pi \cdot d^3 \). Hence \( \frac{\pi}{12} = \frac{8}{30} \) and the value of \( \pi \) used by the Hindus was 3.20.

In example number forty-three of the Kahun papyrus, we find the solution of an example which calls for the contents of a space round

(1) Rhind papyrus--No. 41, page 50.
(2) Heath "History of Greek Geometry"--p125.
in form, nine inches in height and six in breadth. Expressed in modern symbols, the formula used by them would be represented as:

\[ V = (4/3 \cdot 8/9 \cdot k)^2 \cdot 2/3 \cdot h \]

where \( k \) represents the base and \( h \), the altitude.

We cannot account for the fact that \( 4/3 \) of the area of the circle is taken as the base. The fact that this result is multiplied by \( 2/3 \cdot h \) rather than \( h \) is probably due to confusion with another formula commonly used for the volume of a hemisphere in which \( 2/3 \cdot r \) was needed.

In Aryabhata's work, a very inaccurate formula for the volume of a sphere is given. The volume of the sphere is said to be the product of the area of a circle which has the same radius as the sphere, and the root of this area. The formula is:

\[ V = \pi r^2 \cdot \sqrt{\pi r^2} \tag{1} \]

rather than the true formula \( V = 4/3 \pi r^3 \). This gives \( 16/9 \) as the value of \( \pi \). This may possibly be an error for the more accurate value \((16/9)^2\) which is found in the Ahmes papyrus.

In Mahavira's work we find another formula which gives a very inaccurate value of \( \pi \). The volume of the sphere is given by this empirical formula:

\[ V = 9/2 \cdot (1/2d)^3 \]

(1) Smith "History of Math." vol.I. P.156
Also Kaye "Indian Math." P. 13.
Another formula which he claims is an accurate one is:

$$V = \frac{9}{10} \cdot \frac{9}{2}(1/2a)^3$$

The first of these two formulas yields 3.375 as the value of $\pi$, while the "accurate" formula gives $\pi$ as 3.03.

Ketsugiši Ō, a Japanese mathematician of the seventeenth century, expresses the surface of a sphere as:

$$A = \frac{1}{4}c^2 = \pi^2 r^2$$  \(1\)

This formula is also incorrectly stated by other Japanese writers. The error arose from considering the surface of a sphere as if it were the skin of an orange which could be removed and cut into triangular forms and fitted together again. Then the plane area was discovered and the curvature of the sphere was absolutely neglected.

In the "Arithmetic in Nine Sections" the area of a segment of a circle is given as:

$$A = \frac{1}{2} (c+a)a$$  \(2\)

where $c$ is the chord, and $a$, the altitude. The area of a circle is given as:

$$A = \frac{3}{4} d^2$$

(1) Smith and Mikarni "Japanese Mathematics" p73.
and $A = \frac{1}{12} c^2 \quad (1)$

Both of these formulas assume the value of $\pi$ to be equal to 3.

We will now leave the field of mensuration formulas and discover what other approximation or fallacious formulas have been used by various mathematicians. In the realm of trigonometry, Aryabhata worked out a crude table of sines. In order to perform this task, he devised a rule which is probably represented by the formula:

$$\sin(n+1)a - \sin na = \sin na - \sin(n-1)a - \sin na \csc a.$$ \quad (2)

in which $a$ stands for 3 3/4, and $n$ stands for the natural numbers. The correct formula is:

$$\sin(n+1)a - \sin na = \sin na - \sin(n-1)a - 4 \sin na \cdot \sin^{1/2}a.$$ 

Hence, Aryabhata used $4 \sin^2 1/2a$ as equal to cosec $a$. In other words he assumed $2 \sin a = 1 + \sin 2a$ which is not true. The results obtained, however, were fairly good approximations to the results which would have been secured by the correct formula.

Before mathematics could advance very far, it became necessary to devise some means for finding the roots of numbers. Theon of Alexandria (1) Cajori "History of Mathematics." p.71

(2) Ball's "History" p.146.
is given the credit for being the first to derive an approximation for the square root of a number, and to use the method extensively in his work. Heron also used it somewhat in his work. Although the Greeks could extract the square root of a number in a manner similar to that used today, their ignorance of decimal fractions made the process difficult in case of surds. Hence this new approximation method was regarded very favorably in the eyes of the Greeks. This rule for the approximation of a square root would be expressed in modern symbols as:  
\[ \sqrt{A} = \sqrt{a + \frac{b}{2a}} + b = a + \left( \frac{b}{2a} \right) \]  
(1)

Among the Arabs this rule of Theon's was used by Albategnius (920), Abril--Wefa (950) and others. This formula gives us an approximation which is in excess of the real root. For example, if we apply this rule to find the square root of ten, we would have:

\[ \sqrt{10} = \sqrt{3^2 + 1} = 3 + \frac{1}{6} \quad \text{or} \quad 3\frac{1}{6} = 3.1666... \]

Realizing that this formula yields a root which is greater than the real root, Al-Karchi (1020) used a rule which gives a root in defect of the real root. This formula in modern symbols would be expressed as:

\[ \sqrt{A} = \sqrt{a^2 + b} = a + \frac{b}{2a + 1} \]

(1) P. 6. Nordgaard "Approximating Roots of Eq's".
\[ \sqrt{A} = \sqrt{a^2 + b} = a + \frac{b}{2a+1} \]

Hence the \( \sqrt{10} = \sqrt{3^2 + 1} = 3 + \frac{1}{6+1} = 3\frac{1}{7} \)

\[ = 3.143---- \]

The fact that \( \sqrt{10} \) is 3\( \frac{1}{7} \) according to this approximation probably accounts for the fact that \( \sqrt{10} \) was taken for \( \pi \) by so many early writers. The real value of \( \sqrt{10} \) is given today as 3.1623.....

The writers of the Middle Ages still continued to use these formulas, occasionally adding a corrective supplement. Leonardo used Theon's formula which he corrected to read:

\[ \sqrt{A} = \sqrt{a^2 + b} = a + \frac{b}{2a} - \frac{(b/2a)^2}{2(a+\frac{b}{2a})} \]

Extracting the square root of ten in this manner, we get for an answer:

\[ \sqrt{10} = \sqrt{3^2 + 3\frac{1}{6} - (1/6)^2} = 3 \frac{1}{6} - \frac{1}{228} \]

\[ \frac{2(3+1/6)}{2(3+1/6)} = 3 \frac{37}{228} = 3.1623---- \]

which is a very much closer approximation to the root.

Another formula which was used during the Middle Ages was the formula:

\[ \sqrt{A} = \frac{1}{a} \sqrt{Aa^2} \]

(1) Smith "History of Math." vol. 2 p.254.
(2) P. 10 Nordgaard "Approximations of roots."
(3) " " " " " " ""
Therefore $\sqrt[3]{10} = \frac{1}{3}\sqrt[3]{10(9)} = \frac{1}{3}(9.4868) = 3.1623$.

This approximation is very accurate for it gives the square root of 10 correct to four decimal places.

Leonardo of Pisa invented a rule for approximating irrational cube roots. Expressed in terms of a modern symbolized formula, his rule is as follows:

$$\sqrt[3]{A} = \sqrt[3]{a^3 + b} = a + \frac{b}{3a(a+1) + 1}$$  \hspace{1cm} (1)

He gives this illustration of his formula:

$$\sqrt[3]{900} = 9 + \frac{1}{171/271} = 9 \frac{171/271}{1} = 9.6309$$ \hspace{1cm} (2)

Computed by our modern method, the $\sqrt[3]{900}$ is equal to 9.6549 which is correct to four decimal places. Hence Leonardo's formula gives us an approximation which is slightly in defect of the root.

Another formula, used in the Middle Ages gives a cube root which is slightly in excess of the real root. This is given as follows:

$$\sqrt[3]{A} = \sqrt[3]{a^3 + b} = a + \frac{b}{3a(a+1)}$$ \hspace{1cm} (3)

Another formula, which may have been used by Heron gives us the cube root of $n$ as:

(1) P. 10 Nordgaard--"Approximations of Roots"
(2) P. 11 Nordgaard
(3) P. 12 Nordgaard
where \( a \sqrt[3]{n} > b \) and \( a - b = 1 \). "By means of this rule we should find that \( \sqrt[3]{109} = 4.7785 \) instead of 4.7769. (1)

In the seventeenth century any root of a number could be found by means of the formula enunciated by Peter Hallnian:

\[
\sqrt[n]{a^n + b} = a + \frac{b}{(a + 1)^n} - b^n
\]  

(2)

Nordgaard says that this plan of finding the roots of large irrational numbers "became increasingly popular and when in 1539 a man of Cardan's prestige adopted it systematically in his arithmetic, accompanied by clear and concise rules, it became the standard method in Europe." (3)

Although it took nearly a thousand years for this approximation method to be perfected, it was an important piece of work, for it made possible the methods of Vieta and Horner.

Other methods of approximating the value of certain roots are found in the early attempts to solve an equation. Equations of the first

(2) Smith "History of Math." Vol. 2 P. 255
(3) Nordgaard "Approximation of Real Roots." P. 11.
and second degrees were classified by the Arabs into six groups:

\[ x^2 = ax \]
\[ x^2 = a \]
\[ x^2 + ax = b \]
\[ x^2 + a = bx \]
\[ ax + b = x^2 \]
\[ ax = b \]

One of the methods used by the Arabs in solving the equation \( ax+b = 0 \), is very interesting because of the fact that it was later developed as a method of approximating the roots of equations of higher degree. The method, which is now known as the method of Double False Position is substantially as follows:

Let \( g_1 \) and \( g_2 \) represent two guesses as to the possible value of \( x \), and let \( f_1 \) and \( f_2 \) represent respectively the value of the function when these guesses are substituted into the given equation. Then

\[ ag_1 + b = f_1 \]
\[ ag_2 + b = f_2 \]

Solving for \( -b/a \) we obtain

\[ -b/a = \frac{f_1 g_2 - f_2 g_1}{f_1 - f_2} \]

But since \( -b/a = x \), the formula is given by:

\[(1) \text{ Fink } "\text{History of Math.}" \text{ p. 90}\]
\[ x = \frac{f_2 g_2 - f_1 g_1}{f_1 - f_2} \] \hspace{1cm} (1)

Since the errors in the substitutions bear the same ratio to each other as the errors in the results, the approximation is accurate in the case of a straight line function. Although this rule is very awkward to use, it was employed by mathematicians for centuries, "a witness to the need for and value of a good symbolism". (2) This method was called by the Arabs the Method of Scales and was represented graphically by them. For example, if they were considering the equation \( x + 2/3x + 1 = 10 \), and the guesses as to the value of the root were 9 and 6, then the corresponding values of the function would be 61 and 1. The figures were then arranged as follows:

\[ \begin{array}{c}
    f_1 = 6 \\
    f_2 = 1 \\
    g_1 = 9 \\
    g_2 = 6
\end{array} \]

With this aid, the approximation

\[ x = \frac{6 \cdot 6 - 1 \cdot 9}{6 - 1} = \frac{27}{5} = 5 \frac{2}{5} \]

could easily be set forth.

In addition to this Rule of Double False

(2) " 439 " " " " " " "
in which two guesses were made as to the possible root, there was also another method called the Rule of Single False. Given the equation \( ax^2 + b = 0 \), and let \( g_1 \) represent the first and only guess. Then let \( f \) represent the failure of this guess, so that:

\[
ag + b = f
\]

whence \( a = \frac{f}{g - x} \)

\[
\frac{fx}{g - x} = b
\]

\[
x = \frac{gb}{b - f} = \frac{g(ab) - gf}{ag} \quad \frac{g(f - b) - gf}{f - b} \quad (1)
\]

Hence the formula:

\[
x = \frac{g(f - b) - gf}{f - b}
\]

gives the value of the root of the equation.

Smith gives the following example as an illustration of the method of applying this formula:

Given the equation \( \frac{1}{5}x + \frac{1}{6}x = 20 \). "If we take \( g = 30 \), we have

\[
\left(\frac{1}{5}\right)30 + \left(\frac{1}{6}\right)30 = 11
\]

which is 9 too small, whence \( f = -9 \). Then

\[
x = \frac{30(-9+20) - 30(-9)}{-9+20} = 54 \quad \frac{6}{11}
\]

This gives an accurate value of \( x \).

The first known solution of a quadratic

(1) Smith "History of Mathematics" Vol.2 P.441
equation is found in the Berlin Papyrus. (1) The problem involves the finding of the roots of the equations \( x^2 + y^2 = 100 \) and \( y = \frac{3}{4}x \).

The solution is as follows:

"Make a square whose side is 1 and another whose side is \( \frac{3}{4} \). Square \( \frac{3}{4} \), giving \( \frac{9}{16} \). Add the squares, giving \( \frac{25}{16} \), the square root of which is \( \frac{5}{4} \). The square root of 100 is 10. Divide 10 by \( \frac{5}{4} \) giving 8 and \( \frac{3}{4} \) of 8 is 6." (2) Hence the values 8 and 6 are the values of \( x \) and \( y \) respectively. The method here implied is nothing more than a simple case of false position.

In the solution of equations, the Greeks used geometric methods almost entirely. The Hindus and Arabs, however, attempted the solution of the equations as we have seen by these other means. However, no general formula for the solution of any quadratic was given before the seventeenth century. This was due to the fact that the equation \( x^2 - px = q \) was considered an entirely different type from the equation of the form \( x^2 + px = q \) and hence a different formula was necessary for its solution. All of these

(1) Smith vol. 2 p. 441
(2) " " p. 443
formulas give accurate results, but it is interesting to note a few of the different formulas given for the solution of the quadratic. In almost every case the negative root is neglected.

Al-Khowarizmi's rule for the solution of an equation of the type \(x^2 + px = q\) is given by:

\[ x = \sqrt{1/4p^2 + q} - (1/2)p \]

while the formula \(x = 1/2p \pm \sqrt{1/4p^2 - q}\) (1) represents the root of an equation of the form \(x^2 + q = px\). Many other formulas of this same type are found in the works of Heron, Omar Khayyam, various Chinese writers, and Vieta.

After solving the quadratic, the attention of mathematicians focused on the possibilities of solving the cubic and other higher equations. The first methods of approximating the roots of these higher equations involved the method of double false position. Chuquet was the first mathematician to discover a perfectly general approximation of a root of an equation. His method is one of averaging until the mean values converge to the real root. He continually finds new limits to the root and hence by a series of computations arrives at a fairly accurate value of the root.

(1) P. 91 Fink "History of Mathematics."

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Since the time of Chuquet (1484) many other formulas have been adopted for approximating the roots of equations. Some of the more important of these are Vieta's, Newton's and Horner's Method. All of these well-known methods depend upon the approximations of earlier mathematicians. In fact, the method of double false position may still be applied today as a check upon Newton's Method.

Although the approximation formulas used today give us very accurate approximations, we must realize that this result is but the culmination of many centuries of mathematical work. In the first place, we have seen that the early mathematicians had no formulas. Later mathematicians had very crude methods of expressing the mathematical truths and principles which they discovered. It was not until the development of a fully generalized, symbolic formula that great strides in mathematical progress could be effected. In spite of the fact that they had no symbolic notation in which to express themselves, the early mathematicians made some distinct contributions to mathematics and enunciated various rules which could later be translated into a symbolized formula. Hence the formula has had a gradual evolution. Many of the formulas
used by the mathematicians of the ancient times and of the Middle Ages are fallacious, but many of them give us results which are surprisingly accurate in their approximations. We have traced the various approximative formulas used throughout the ages and have noticed their errors. It is to the contributions made to the field of mathematics by these early experimenters with mathematical formulas, that our later mathematicians have turned in order to establish some new mathematical truth. Who can say then, that this work with mathematical formulas, even though it may have been fallacious and often times crude, has not served its purpose?
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