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Continuous-time trading with multiple insiders and price impact

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CONTINUOUS-TIME TRADING WITH MULTIPLE INSIDERS AND PRICE IMPACT

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ABSTRACT

We study imperfect competition among risk-averse insiders in a continuous time Kyle-Back model framework with price impact, allowing informed traders with heterogeneous CARA risk aversion to enter the market sequentially. We show that, unlike in the risk-neutral case with perfectly correlated signals, a linear equilibrium exists even when multiple insiders observe the same signal. Our main application considers a market in which one insider observes the asset value at time 0, while a second insider receives the same signal at a later date. We characterize the equilibrium explicitly and compare it with the single-insider benchmark. Before the second insider enters, the first insider trades more aggressively in anticipation of future competition, which accelerates early information revelation and lowers market depth. After entry, however, the first insider's trading intensity declines because of the risk-sharing effect. As a result, price informativeness is always higher before entry than in the single-insider benchmark, whereas after entry it may be either higher or lower, depending on the timing of the second insider's arrival. Market depth displays a similar pattern: it is lower before entry, while after entry it may remain lower the benchmark or cross it, depending on the timing of entry. We also show that the first insider's welfare

is lower than in the single-insider benchmark, but it increases as the second insider enters the market later. More generally, we characterize how differences in insiders' risk aversion shape their equilibrium trading behavior.

Contents

List of Abbreviations

HJB	....	Hamilton–Jacobi–Bellman equation
ODE	....	Ordinary differential equation

Chapter 1

Introduction

How privately informed insiders trade in equilibrium in the presence of market impact has been a central question in the finance literature. Kyle (Kyle, 1985) studies this problem in a one-period linear equilibrium with a single insider, in which a risk-neutral informed trader trades on private information, noise traders submit random orders, and competitive market makers set prices based on aggregate order flow. The framework was later extended to settings with multiple informed traders, giving rise to imperfect competition among insiders. For risk-averse agents, this setting was first studied by Subrahmanyam (Subrahmanyam, 1991) in a one-period model. It was later extended to multiperiod settings with risk-neutral agents by Holden and Subrahmanyam (Holden and Subrahmanyam, 1992), who consider insiders observing the asset's fundamental value, and by Foster and Viswanathan (Foster and Viswanathan, 1996), who assume that informed traders observe correlated signals and study how the degree of correlation affects the equilibrium. These models were subsequently extended to a continuous-time setting by Back, Cao, and Willard (Back et al., 2000).

This paper studies imperfect competition among risk-averse insiders in continuous time, focusing on how the arrival of additional insiders affects trading intensity[1], mar-

[1] Trading intensity (We denote this quantity by $A(t)$ in the proof of the equilibrium construction and by $\beta(t)$ later in the equilibrium property analysis.) measures how aggressively the insider trades on her informational advantage. Later we show that $\Pi(t) = \Sigma^{-1} + 1/\sigma^2 \int_0^t \beta(u)^2 du$. This implies that higher trading intensity over a period of time increases price informativeness. Information advantage is the difference between asset's value and price, capturing the profit of each trade from the insider.

ket depth[2], and the price informativeness[3]. We also allow insiders to enter the market sequentially at different times and characterize the corresponding equilibrium. Our main question is how the arrival of a second insider changes the first insider's trading strategy and, in turn, affects market depth, price informativeness, and welfare[4]. In addition, we study how differences in insiders' risk aversion shape each other's optimal trading strategies. Our analysis builds on the work of Back (Back, 1992) and Cho (Cho, 2003), who study continuous-time trading with market impact under risk-neutral and risk-averse preferences respectively, by extending their frameworks to allow for multiple insiders trading in the market.

We first study the special case in which insiders receive their signals and enter the market simultaneously at time 0. Our finding is that a linear equilibrium exists in continuous time when insiders are risk-averse. This differs from the risk-neutral case studied in (Holden and Subrahmanyam, 1992) and (Back et al., 2000), where no continuous-time equilibrium exists when multiple insiders observe perfectly correlated signals about the asset's fundamental value. Under risk neutrality, insiders compete to trade before others move the price toward the fundamental value v, which leads to very aggressive trading. Under risk aversion trading also reflects a risk-sharing

[2]Market depth is defined as the size of an innovation in order flow required to move prices by a given amount, following (Kyle, 1985). In our model, market depth is measured by $1/\lambda(t)$, where $\lambda(t)$ is the market impact coefficient and captures how strongly the market maker adjusts prices in response to order flow at time t. It is also understood as the cost of trading of insiders and learning parameter for market maker. In equilibrium, $\lambda(t) = \beta(t)\Sigma(t)/\sigma^2$. A larger $\beta(t)$ means that insiders trade more aggressively on their informational advantage, so order flow is more informative. A larger $\Sigma(t)$ means that insiders have a greater informational advantage over the market maker. A smaller σ^2 means that noise trading provides less camouflage for informed trading. All three forces increase $\lambda(t)$, and therefore reduce market depth. Literature often use this quantity to measure the market liquidity.

[3]Price informativeness at time t, denoted by $\Pi(t)$, is measured by $1/\Sigma(t)$, where $\Sigma(t)$ denotes the posterior variance of the asset's value given the aggregate order flow. A larger $\Sigma(t)$ means that the market maker's posterior estimate of the asset's value is less precise, so prices contain less information about the asset's fundamental value. Equivalently, a larger $\Sigma(t)$ implies that insiders retain a greater informational advantage over the market maker, since more private information remains to be incorporated into prices.

[4]Welfare is the expected utility of insider given her private information at time 0.

motive, since each insider's optimal trading strategy is inversely related to her degree of risk aversion. As more insiders enter the market, the informed sector becomes more risk tolerant to future price risk[5] overall. This makes insiders less eager to trade aggressively, weakens competition, and helps sustain equilibrium. At the same time, price informativeness declines as the number of insiders increases. The reason is that greater aggregate risk tolerance leads insiders to trade less aggressively at each time point, so less private information is incorporated into prices early in the trading process and the cumulative amount of information revealed during trading is smaller.

We then analyze the model that allows insiders to enter the market sequentially at different times and explicitly characterize the equilibrium. Our main application considers a setting in which one insider receives a private signal at time 0, while a second insider receives the same signal later, at time t_1, and begins trading as soon as she receives it. Relative to the single-insider benchmark, the first insider's trading intensity increases before the second insider arrives. The intuition is that she anticipates future competition and therefore wants to profit from more of her informational advantage before the second insider enters. After the second insider enters, however, the first insider trades less aggressively because the insiders as a group become more risk tolerant. This risk-sharing effect reduces the first insider's trading intensity. Compared with the single-insider benchmark, price informativeness is always higher before the second insider's arrival, because the first insider's more aggressive trading reveals more information into the price. After the second insider enters the market, however, price informativeness may be either higher or lower, depending on the timing

[5]Future price risk refers to the uncertainty of the prices at which the insider's remaining trades will be executed. Since noise trading is random, it may move prices either favorably or unfavorably from the insider's perspective. For a risk-averse insider, this creates risk. Because the terminal liquidation value of the position is known to be v, the only relevant uncertainty is the sequence of transaction prices at which the insider executes her trades. As shown in Proposition 6.1.2, a higher degree of risk aversion leads the insider to trade more aggressively at every point in time, because she places greater value on exploiting her informational advantage immediately rather than leaving any portion of her order to be executed at uncertain future prices.

of entry. Since the arrival of the second insider can reduce insiders' trading intensity and thereby slow the incorporation of information into prices, price informativeness may become lower than in the single-insider benchmark after time t_1 when the second insider enters sufficiently early. Market depth has a jump at t_1, this is because risk sharing lowers trading intensity after t_1, which increases market depth. Relative to the single-insider benchmark, market depth is lower before entry because the first insider trades more aggressively prior to the second insider's arrival. After entry, market depth again depends on the timing of the second insider's arrival. In particular, market depth is lower after entry when the second insider enters sufficiently late, otherwise, the market-depth paths may cross. These results enrich the discussion of price efficiency in the Kyle model and have important regulatory implications. Kyle (Kyle, 1985) and Back (Back, 1992) show that, in a model with a single risk-neutral insider, information is incorporated into prices in a slow and linear way.[6] This suggests that insider trading may harm price efficiency. Holden and Subrahmanyam (Holden and Subrahmanyam, 1992) show that competition in a risk-neutral setting may lead the market to become fully price efficient very quickly, which supports the view that competition among insiders improves price efficiency and therefore makes insider trading less problematic. A related conclusion appears in Holden and Subrahmanyam (Holden and Subrahmanyam, 1994), who show that when a single insider is risk averse, information is incorporated into prices more quickly than in the risk-neutral case. Our results show that, when multiple risk-averse insiders trade in the market, even in the presence of competition, the risk-sharing effect can actually harm price efficiency, so insider trading may still be problematic. However, when two insiders enter the market sequentially and the second insider enters sufficiently late, the risk-sharing effect is weak enough that competition improves price efficiency relative to the single risk-averse insider benchmark, in line with the conclusions in (Holden

[6] That is, $\Sigma(t)$ decreases linearly over time.

and Subrahmanyam, 1992) and (Holden and Subrahmanyam, 1994). From a regulatory perspective, our results suggest that competition can improve price efficiency when the risk-sharing effect is weak; in that case, insider trading is not problematic.

We also examine how the optimal trading strategy depends on insiders' risk aversion. As the risk aversion of the first insider increases, her trading intensity before the second insider's arrival also increases. The intuition is that lower risk tolerance induces the first insider to trade more aggressively in order to reduce future price risk . Her trading intensity after the second insider's arrival, however, may either increase or decrease, depending on the parameter values, including the second insider's degree of risk aversion, the arrival time, the volatility of noise trading, and the variance of the asset value. We discuss these cases in detail. The second insider's trading intensity always increases when the first insider becomes more risk averse. When the second insider becomes more risk averse, the first insider's trading intensity decreases before the entry time and increases afterward. The second insider's trading intensity may either increase or decrease, depending on her degree of risk aversion, the arrival time, the volatility of noise trading, and the variance of the asset value. The economic intuition for these results is discussed in detail in the paper.

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 describes the model setup. Sections 4.1, 4.2, and 4.3 study the insiders' optimization problems and derive necessary conditions on market liquidity dynamics and the pricing rule for an equilibrium to exist. Sections 5.1, 5.2, and 5.3 present the main results of the paper, namely the characterization of the linear equilibrium with two insiders, multiple insiders, and a flow of insiders. Sections 6.1 and 6.2 discuss the properties of equilibrium.

Chapter 2

Literature Review

Our work relates to the literature on strategic competition among multiple insiders in markets with price impact, which studies how asymmetric information affects insiders' trading strategies, market equilibrium price of the asset and market depth. This line of research begins with Kyle (Kyle, 1985), who studies a single insider's optimal trading strategy when she anticipates the immediate price impact of her own orders. The model includes a risk neutral insider, noise traders who submit random orders, and a competitive risk neutral market maker who sets the asset price equal to its conditional expectation given aggregate order flow. The paper shows that there is a unique linear equilibrium pricing rule in which price is given by a constant plus a market impact coefficient, known as Kyle's λ, multiplied by total order flow. In equilibrium, the insider's trading strategy is linear in her private information. The market price is partially reveal the information of the asset fundamental value to the market maker. The model is later extended to the multi-period setting, where the insider splits the orders each period to gradually reveal information to the market maker to reduce the price impact. The market impact parameter λ remains constant within each period, indicating that market depth does not improve as more informed trading occurs within that period.

Subrahmanyam (Subrahmanyam, 1991) extends Kyle's one-period model by introducing multiple risk-averse informed traders who observe noisy signals about the asset's fundamental value. He studies how the presence of additional risk-averse traders affects market liquidity and price efficiency. He finds that market liquidity is nonmonotonic in the number of informed traders, while price efficiency declines. He then incorporates endogenous information acquisition into the model and shows that market liquidity is nonmonotonic in the variance of liquidity trading. In later work, he considers a risk-averse market maker and shows that market liquidity then becomes monotonically increasing in the variance of liquidity trading.

Holden and Subrahmanyam (Holden and Subrahmanyam, 1992) extend Kyle's framework to a multi-period setting with multiple insiders who observe the fundamental value of the asset. They show that there exists a unique linear equilibrium in which insiders trade increasingly aggressively as the length of each trading period tends to zero and the number of trading periods tends to infinity. They conjecture that, in the continuous time limit, competition pressure among insiders will drive insiders to trade infinitely aggressively, causing the linear equilibrium to break down. In that case, the insiders' private information is released to the market immediately, inducing the full price efficiency.

Back (Back, 1992) extends Kyle's model to continuous-time and proves the existence and uniqueness of the equilibrium. The paper allows the terminal fundamental value of the asset to follow a general distribution. In equilibrium, the market impact coefficient λ is a constant and price depends linearly on total order flow, which evolves as Brownian Bridge. The price process is a martingale under market maker's information set. When the terminal fundamental value follows a lognormal distribution, the equilibrium price follows a geometric Brownian motion. The paper also shows that it is suboptimal to submit discrete orders, confirming the optimality of absolutely continuous strategies.

Holden and Subrahmanyam (Holden and Subrahmanyam, 1994) extend Kyle's model to a multi-period setting with risk averse insiders. They numerically compute the equilibirum. In equilibrium, the market impact coefficient λ is initially high and decreases over time and the private information is incorporated into price more quickly. The intuition is that risk averse insider concern about the uncertainty of the price at which future trades will occur. She thus trades more aggressively early in the trading horizon to reveal more information to the market maker to reducing uncertainty about future price.

Foster and Viswanathan (Foster and Viswanathan, 1994) studies the competition among informed traders with heterogeneous information. They extend the Kyle model to a multi-period setting with two types of traders. One is a better informed trader who observes both the asset's fundamental value and a noisy signal about it. The other one is less informed trader who observes only the same noisy signal. They solve the equilibrium numerically and show that in equilibrium, the better informed trader trades more aggressively on the common signal and less aggressively on her own private information at the early stage of the trading. Later on, she trades more aggressively on her private information and less aggressively on the common signal. The intuition is that, early in trading, the better informed trader trades on the common signal to disguise her private information and prevent the less informed trader from learning too much from her orders. As the trading deadline approaches, the less informed trader's ability to learn from order flow declines because there is little time left to infer information from others' trades. So the better informed trader trades more aggressively on her own private information.

Forster and Viswanathan (Foster and Viswanathan, 1996) extend Kyle's insider trading model to a multi-period setting with multiple risk neutral informed traders who receive correlated noisy signals about asset's fundamental value. Each informed trader uses her own signal to forecast the asset's fundamental value and also infers other traders' forecasts from observed order flow. The equilibrium is numerically solved. They show that correlation of signals plays an important role in determining equilibrium properties. When the correlation is moderate, informed traders compete less aggressively than in (Holden and Subrahmanyam, 1992), because each trader observes a different signal rather than the fundamental value itself. As trading goes on, the conditional correlation among signals can become negative. In this case, each informed trader thinks that others hold the wrong view of the asset's value and hence delays trading, expecting others to move the price in her favorable direction. Therefore, informed traders tend to trade more slowly and postpone more of their trading to later periods, which leads market depth to follow a U-shaped pattern.

Back, Cao, and Willard (Back et al., 2000) study imperfect competition among informed traders. They extend the model of Foster and Viswanathan (Foster and Viswanathan, 1996) to a continuous time setting. They characterize the unique linear equilibrium and derive its properties explicitly. They also confirm the conjecture in (Holden and Subrahmanyam, 1992) that, when traders' signals are perfectly correlated, competition becomes so aggressive that no equilibrium exists. In addition, they confirm the finding in (Foster and Viswanathan, 1996) that the initial correlation of signals plays an important role in determining equilibrium properties.

Baruch (Baruch, 2002) extends Holden and Subrahmanyam (Holden and Subrahmanyam, 1994) to a continuous-time setting. He allows the noise trading volatility to be time varying. The equilibrium is characterized implicitly. He shows that, in equilibrium, the market impact coefficient λ is deterministic and strictly decreasing over time. The paper confirms the numerical findings in (Holden and Subrahmanyam, 1994) that information is revealed to the market more rapidly than in the case of a single risk-neutral insider, and that market liquidity is higher than in the risk-neutral benchmark.

Cho (Cho, 2003) studies the continuous-time Kyle model in a mathematically rigorous framework, considering both a risk neutral insider and a risk averse insider. He explicitly characterizes the equilibrium and derives closed form solutions for the equilibrium quantities. The paper also provides an insightful methodology for establishing equilibrium in continuous-time Kyle-Back model. To solve the model, he first conjectures a pricing rule for the market maker as a functional of a state process ξ that captures aggregate market impact. He then solves the insider's utility maximization problem, which becomes a standard stochastic control problem with state variable ξ. The resulting Hamilton–Jacobi–Bellman (HJB) equation does not determine the optimal control directly, instead, it identifies the pricing rule induced by the optimal control. The optimal strategy is characterized by the requirement that the price converges to the asset's fundamental value at the end of the trading period. Since the market maker sets price as the conditional expectation of the asset's fundamental value given aggregate order flow, standard Kalman filtering can be used to derive the pricing dynamics, which depend on the insider's trading intensity. Equating these pricing dynamics with the pricing rule implied by the HJB equation yields a system of ordinary differential equations. Solving this system establishes the optimal trading strategy and the equilibrium pricing rule. Our derivation closely follows this methodology, and our model can be viewed as a multi-period extension of Cho's model.

The studies discussed above are the ones most closely related to our paper, as they study how competition and risk preferences among informed traders shape trading strategies, price efficiency, and the strength of market impact. There is also a broader literature that extends Kyle's framework along other dimensions, such as the behavior of noise traders, the arrival of information, and the way market makers form beliefs. We next briefly review several of these extensions, which provide useful complementary perspectives for our setting.

Rochet and Vila (Rochet and Vila, 1994) study an extension of Kyle's model that does not impose normality on either the asset payoff or noise trading. They also assume that the insider observes the entire net order flow in the market, whereas in the standard Kyle framework she observes the asset's fundamental value but not the noise trader's orders. They show that, under these more general distributional assumptions, the equilibrium exists and is unique. Wang (Albert Wang, 1998) extends Kyle's model to a multi-period setting in which traders hold different prior beliefs about the precision of the asset's value. The model helps explain several common intraday patterns, including higher trading volume near the market open and close, as well as a positive relation between trading volume and price volatility. Caldentey and Stacchetti (Caldentey and Stacchetti, 2010) study a multi-period Kyle model in which the insider observes a flow of noisy signals about the asset's value, which is publicly disclosed at a random time. They show that this random disclosure leads to an equilibrium with two regimes, separated by an endogenously determined switching time T. Before T, the insider trades gradually, allowing information to enter prices over time. After T, trading volume increases sharply and prices become almost fully revealing. Umut Çetin (Çetin, 2018) extends the Kyle–Back model by allowing the time at which the asset's fundamental value is publicly revealed to be random. He explicitly characterizes the equilibrium price process and the insider's optimal trading strategy in this setting. Collin-Dufresne and Fos (Collin-Dufresne and Fos, 2016) extend the continuous-time Kyle–Back model by allowing the noise trading volatility to be stochastic and time varying. They show that, in equilibrium, both the volatility of the price process and the market-impact coefficient λ are stochastic. The informed trader's optimal strategy is to trade more aggressively when current state of liquidity is high, where current state of liquidity is measured by the ratio of current noise trading variance to expected future noise trading variance. Banerjee and Breon-Drish

(Banerjee and Breon-Drish, 2020) extend (Collin-Dufresne and Fos, 2016) by introducing endogenous information acquisition. They study two forms of information acquisition. Under smooth information acquisition, the informed trader pays a flow cost to choose the precision of her signal over time. Under lumpy information acquisition, she pays a fixed cost to acquire a signal. They show that, in the smooth acquisition case, the informed trader chooses higher signal precision when noise trading volatility is high, whereas no equilibrium exists in the lumpy acquisition case. Viswanathan and Xing (Viswanathan and Xing, 2026) study a static information acquisition problem in which the asset's fundamental value follows a general distribution. Insiders is allowed to acquire signals with arbitrary distributions, subject to an entropy-based information cost. They show that continuous signals are optimal regardless of the prior distribution of the asset's value, and that all continuous information acquisition choices give the same ex-ante profit to informed traders. They also discuss the factors that shape the insider's information acquisition decision. Ekren, Mostowski, and Žitković (Ekren et al., 2022) extend (Collin-Dufresne and Fos, 2016) by allowing the asset's fundamental value to follow a general distribution and by introducing correlation between the noise trading process and the stochastic liquidity process. Bose and Ekren (Bose and Ekren, 2023) extend the continuous-time Kyle–Back model to a setting with a risk averse insider and a non-Gaussian distribution of the fundamental value. They establish equilibrium by characterizing prices and trading strategies through a coupled system linked by an optimal-transport map at maturity. Chhaibi, Ekren, and Noh (Chhaibi et al., 2025) extend the continuous-time Kyle Back model to a setting with a risk-averse informed trader who observes a noisy signal about the asset's fundamental value. They characterize the equilibrium using optimal transport theory. Kacperczyk and Pagnotta (Kacperczyk and Pagnotta, 2024), Çetin (Çetin, 2025), and Ma, Xia, and Zhang (Ma et al., 2025) extend the Kyle model to study equilibrium when insiders are subject to legal penalties.

Chapter 3

Model Setup

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which supports a standard Brownian motion B as well as a random variable v which is independent of B and normally distributed with mean μ and variance Σ. In the market there is a single risky asset which pays off v at the terminal time $T = 1$. We denote by $p(t)$ the price of this asset for $t \leq 1$ where p will be determined in equilibrium. There is also a money market account which pays an exogenous constant interest rate, which we set to 0.

There are three agents in the model: two insiders and a noise trader. For $i = 1, 2$ and a time $t \leq 1$ we denote by $X^i(t)$ the shares of the risky asset held by the i^{th} insider, and $Z(t) = \sigma B(t)$ the shares of the risky asset held by the noise trader where $\sigma > 0$ is constant. Therefore, the combined order flow of all agents is

$$Y(t) = X^1(t) + X^2(t) + Z(t); \qquad t \leq 1. \tag{3.0.1}$$

We denote by $\mathbb{F}^Y$ the filtration generated by Y.[1] The combined order process Y is sent to a risk neutral market maker who then quotes the price

$$p(t) = \mathbb{E}\left[v \middle| \mathcal{F}_t^Y\right]; \qquad t \leq 1.$$

The two agents have constant absolute risk aversion (CARA) preferences and we denote $\gamma_i, i = 1, 2$ the respective risk aversions. Each insider is able to identify the

[1] More precisely, for a given process χ we write $\mathbb{F}^\chi$ as the right-continuous enlargement of the $\mathbb{P}$-augmentation of χ's natural filtration.

terminal payoff v prior to the public identification at time 1. However, the two insiders do not see v at the same time. The first insider sees v at time 0, while the second insider does not see v until an exogenously given time t_1 in $(0, 1)$.[2] Additionally, while the second insider may view the public price prior to t_1, he does not trade until receiving the signal.[3]

Denoting by $\mathbb{F}^I = \mathbb{F}^p \vee \sigma(v)$ the filtration generated by initially enlarging the price filtration by the payoff v, the first insider uses $\mathbb{F}^I$ over $[0, 1]$ while the second uses $\mathbb{F}^I$ over $[t_1, 1]$. Lastly, as we will show below, in equilibrium $\mathbb{F}^I = \mathbb{F}^Y \vee \sigma(v)$ as the combined order flow Y will be identifiable given the price process p.

The first insider's share policy must be adapted to $\mathbb{F}^I$, and as is standard in the literature (see (Back, 1992))[4] we assume over $[0, 1]$ her strategy takes the form $X^1(t) = \int_0^t \theta^1(u)du$ for an $\mathbb{F}^I$ adapted process θ^1.[5] Using integration by parts, the first insider has gains over $[0, 1]$ is

$$\mathcal{W}^1_{0,1}(p) = \int_0^1 X^1(u)dp(u) + X^1(1)(v - p(1)) = \int_0^1 (v - p(u))\theta^1(u)du, \tag{3.0.2}$$

where we stress the dependence on the price p. The second insider's policy must all be adapted to $\mathbb{F}^I$ over $[t_1, 1]$ and we assume his share policy X^2 takes the form $X^2(t) = \int_{t_1}^t \theta^2(u)du$ where θ^2 is $\mathbb{F}^I$ adapted over $[t_1, 1]$. Therefore, his gains from trading over $[t_1, 1]$ is

$$\mathcal{W}^2_{t_1,1}(p) = \int_{t_1}^1 X^2(u)dp(u) + X^2(1)(v - p(1)) = \int_{t_1}^1 (v - p(u))\theta^2(u)du.$$

With this notation, we define an equilibrium as follows.

[2] Alternatively, t_1 could be random time in $(0, 1)$ independent of all other quantities.

[3] Allowing the insider to trade earlier as an uninformed agent from time 0 to t_1 is an important extension of the work left to future research.

[4] The suboptimality of strategies with discrete orders is shown in the Appendix.

[5] We will precisely define the admissible class of trading strategies in Definition 3.0.4.

Definition 3.0.1. *The triple* $(\widehat{\theta}^1, \widehat{\theta}^2, \widehat{p})$ *forms an equilibrium if*

(i) $\widehat{p}(t) = \mathbb{E}\left[v \middle| \mathcal{F}_t^{\widehat{Y}}\right]$ *where* $\widehat{Y}(t) = Z(t) + \widehat{X}^1(t) + 1_{t \geq t_1}\widehat{X}^2(t)$ *for* $0 \leq t \leq 1$.

(ii) $\widehat{\theta}^1$ *solves insider 1's CARA optimal investment problem*

$$\inf_{\theta^1 \in \mathcal{A}_0} \mathbb{E}\left[e^{-\gamma_1 \widehat{\mathcal{W}}^1_{0,1}(\widehat{p})} \middle| v\right].$$

(iii) $\widehat{\theta}^2$ *solves insider 2's CARA optimal investment problem*

$$\inf_{\theta^2 \in \mathcal{A}_{t_1}} \mathbb{E}\left[e^{-\gamma_2 \widehat{\mathcal{W}}^2_{t_1,1}(\widehat{p})} \middle| \mathcal{F}^I_{t_1}\right].$$

We end this section showing how one can reduce the equilibrium problem with two insiders to that of one insider whose risk aversion changes at the second entry time t_1. To do so, consider the first insider's optimal control problem. Due to CARA preferences we may work backwards in time, beginning with the interval $[t_1, 1]$ where both agents trade. By iterative conditioning, we obtain

$$\mathbb{E}\left[e^{-\gamma_1 \mathcal{W}^1_{0,1}(p)} \middle| v\right] = \mathbb{E}\left[e^{-\gamma_1 \int_0^{t_1}(v-p(u))\theta^1(u)du} \times \mathbb{E}\left[e^{-\gamma_1 \int_{t_1}^1 (v-p(u))\theta^1(u)du} \middle| \mathcal{F}^I_{t_1}\right] \middle| v\right], \quad (3.0.3)$$

which allows the first insider to first solve the problem

$$\inf_{\theta \in \mathcal{A}_{t_1}} \mathbb{E}\left[e^{-\gamma_1 \int_{t_1}^1 (v-p(u))\theta(u)du} \middle| \mathcal{F}^I_{t_1}\right],$$

where $\mathcal{A}_{t_1}$ is the admissible trading strategy class given in Definition 3.0.4 below. As the second insider has identical information set, he uses the same admissibility class, and hence solves

$$\inf_{\theta \in \mathcal{A}_{t_1}} \mathbb{E}\left[e^{-\gamma_2 \int_{t_1}^1 (v-p(u))\theta(u)du} \middle| \mathcal{F}^I_{t_1}\right].$$

Through a simple transformation, these are in fact the same problem. Indeed, define the representative risk aversion $\overline{\gamma}_2$ through

$$\frac{1}{\overline{\gamma}_2} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}, \tag{3.0.4}$$

Inspecting the above optimization problems, it is clear by symmetry the optimizers satisfy

$$\widehat{\theta}^1 = \frac{\overline{\gamma}_2}{\gamma_1}\widehat{\theta}, \qquad \widehat{\theta}^2 = \frac{\overline{\gamma}_2}{\gamma_2}\widehat{\theta}, \tag{3.0.5}$$

where $\widehat{\theta}$ solves

$$\inf_{\theta \in \mathcal{A}_{t_1}} \mathbb{E}\left[e^{-\overline{\gamma}_2 \int_{t_1}^1 (v-p(u))\theta(u)du} \middle| \mathcal{F}^I_{t_1} \right]. \tag{3.0.6}$$

Indeed, symmetry implies $\gamma_i \widehat{\theta^i}$ must be common to both agents, and hence using the above scaling

$$Y(t) = Z(t) + X^1(t_1) + \int_{t_1}^{t} \theta^1(u)du + \int_{t_1}^{t} \theta^2(u)du = Z(t) + X^1(t_1) + \int_{t_1}^{t} \theta(u)du.$$

In words, we can solve the two-insider problem over $[t_1, 1]$ by solving a single insider problem with representative risk aversion $\overline{\gamma}_2$. Moreover, if we set $\theta = \theta^1$ over $[0, t_1]$, we have

$$Y(t) = Z(t) + X^1(t_1) + \int_{t_1}^{t} \theta(u)du = Z(t) + \int_{0}^{t} \theta(u)du.$$

Therefore, we reduce the two-insider problem to a single-insider problem with time-dependent risk aversion, where the effective risk aversion is γ_1 over $[0, t_1)$ and $\overline{\gamma}_2 < \gamma_1$ over $[t_1, 1]$. Denoting $\overline{\gamma}_1 = \gamma_1$ and building on the reduction, we re-characterize the equilibrium definition to

Definition 3.0.2. *The pair $(\widehat{\theta}, \widehat{p})$ forms an equilibrium if*

(i) $\widehat{p}(t) = \mathbb{E}\left[v \middle| \mathcal{F}^{\widehat{Y}}_t \right]$ *where* $\widehat{Y}(t) = Z(t) + \int_0^t \widehat{\theta}(u)du$ *for* $0 \le t \le 1$.

(ii) $\widehat{\theta}$ *solves*

$$\inf_{\theta\in\mathcal{A}_0} \mathbb{E}\left[e^{-\overline{\gamma}_1\int_0^{t_1}(v-\widehat{p}(u))\theta(u)du-\overline{\gamma}_2\int_{t_1}^1(v-\widehat{p}(u))\theta(u)du}\middle| v\right].$$

To construct the equilibrium, we first solve the insider's optimal control problem (ii) under a conjectured functional form for the price process p over $[0,1]$. We will then identify a particular form which enforces condition (i). As a first step, we postulate that the equilibrium price takes the form

Conjecture 3.0.3. *The price process p over $[0,1]$ takes the form $p(t) = H(t,\xi(t))$ where for a trading strategy θ the state process ξ has $[0,1]$ dynamics*

$$\xi(t) = \int_0^t \lambda(s)dY(s) = \int_0^t \lambda(s)\left(\theta(s)ds + \sigma dB(s)\right), \qquad 0 \le t \le 1. \tag{3.0.7}$$

H and λ are to-be-determined functions. We require $H \in \mathcal{C}^{1,2}([0,1]\times\mathbb{R})$ and λ to be piece-wise $\mathcal{C}^1$ on $[0,t_1)$ and $[t_1,1]$ respectively, with a possible discontinuity at t_1.

As H,λ determine the map $\theta \to p$ we say that p is a "pricing rule". Having conjectured the pricing rule $p(t) = H(t,\xi(t))$, we define the class of admissible strategies, suitably modifying the class in (Cho, 2003). Fix $t,T \in [0,1]$ with $t < T$, and let $\xi(t)$ be an $\mathcal{F}_t^Y$-measurable random variable representing the state at time t. Let θ be a candidate $\mathbb{F}^I$-adapted strategy on $[t,T]$. For θ to be admissible, the state equation (3.0.7) must admit a non-explosive solution on $[t,T]$ starting from $\xi(t)$, and we additionally impose a martingale condition. This is the natural extension of the "no doubling strategies" condition (Back, 1992, Equation (7)) to the case of CARA preferences. To align the admissible class with the reduced control problem, we introduce the effective risk aversion process

$$\overline{\gamma}(t) := \overline{\gamma}_1 1_{t<t_1}(t) + \overline{\gamma}_2 1_{t\ge t_1}(t), \qquad 0 \le t \le 1. \tag{3.0.8}$$

Note that $\overline{\gamma}(t) = \overline{\gamma}_1 = \gamma_1$ on $[0,t_1)$ and $\overline{\gamma}(t) = \overline{\gamma}_2$ on $[t_1,1]$.

Definition 3.0.4. *Fix $t, T \in [0,1]$ with $t < T$, and let $\xi(t)$ be an $\mathcal{F}_t^Y$-measurable random variable. We say that a strategy θ is admissible on $[t,T]$ given $\xi(t)$ if:*

(i) θ is $\mathbb{F}^I$-adapted on $[t,T]$;

(ii) there exists a non-explosive $\mathbb{F}^Y$-adapted process ξ on $[t,T]$ satisfying, for all $s \in [t,T]$,

$$\xi(s) = \xi(t) + \int_t^s \lambda(u)\big(\theta(u)\,du + \sigma\,dB(u)\big);$$

(iii) the stochastic exponential

$$M(s) := \mathcal{E}\left(\int_t^s \overline{\gamma}(u)\,(v - H(u,\xi(u)))\,\sigma\,dB(u)\right), \qquad t \le s \le T,$$

is an $\mathbb{F}^I$-martingale on $[t,T]$.

We write $\mathcal{A}_{t,T}(\xi(t))$ for the class of admissible strategies and denote $\mathcal{A}_t(\xi(t)) := \mathcal{A}_{t,1}(\xi(t))$. When $t = 0$ and $\xi_0 = 0$, we write $\mathcal{A}_{0,T} := \mathcal{A}_{0,T}(0)$ and $\mathcal{A}_0 := \mathcal{A}_{0,1}(0)$.

Remark 3.0.5. *There are several ways to check the martingale condition in Definition 3.0.4. For example, one could use Novikov's condition*

$$\mathbb{E}\left[e^{\frac{1}{2}\sigma^2\int_t^T \overline{\gamma}(u)^2(v-H(u,\xi(u)))^2du}\right] < \infty,$$

or its relaxation in (Karatzas and Shreve, 1991, Chapter 3). Alternatively, for Markovian controls $\theta(t) = \theta(t,\xi(t))$ it is well known (see (Pinsky, 1995; Cheridito et al., 2005)) that the martingale condition will hold provided there is a non-explosive weak solution $(\xi, \widetilde{B})$ over $[t,T]$ to the stochastic differential equation

$$d\xi(t) = \lambda(t)\Big(\big(\theta(t,\xi(t)) + \overline{\gamma}(t)\sigma^2(v - H(t,\xi(t)))\big)dt + \sigma d\widetilde{B}(t)\Big).$$

When showing the candidate optimal policy is admissible, we use this latter approach. In particular, if $[t,T] \subseteq [t_1,1]$, then $\overline{\gamma}(u) = \overline{\gamma}_2$ on $[t,T]$, so the above condition reduces to the second-period admissibility criterion.

Having defined the admissible class and conjectured the pricing rule

$$p(t) = H(t,\xi(t)),$$

we consider the insider's control problem

$$\inf_{\theta\in\mathcal{A}_0} \mathbb{E}\left[e^{-\overline{\gamma}_1\int_0^{t_1}(v-p(u))\theta(u)\,du-\overline{\gamma}_2\int_{t_1}^1(v-p(u))\theta(u)\,du}\middle|v\right]. \tag{3.0.9}$$

By iterative conditioning and the tower property, we solve the nested problem

$$\inf_{\theta\in\mathcal{A}_{0,t_1}} \mathbb{E}\left[e^{-\overline{\gamma}_1\int_0^{t_1}(v-p(u))\theta(u)\,du}\left(\inf_{\vartheta\in\mathcal{A}_{t_1}(\xi(t_1))}\mathbb{E}\left[e^{-\overline{\gamma}_2\int_{t_1}^1(v-p(u))\vartheta(u)\,du}\middle|\mathcal{F}^I_{t_1}\right]\right)\middle|v\right], \tag{3.0.10}$$

where the inner infimum is taken over admissible strategies on $[t_1,1]$ with initial condition given by the realized state $\xi(t_1)$ induced by the outer control on $[0,t_1)$.

That is, we first solve the second-period control problem on $[t_1,1]$ conditional on $\mathcal{F}^I_{t_1}$, obtaining an optimal second-period control $\widehat{\theta}_{t_1}$. We then solve the first-period control problem on $[0,t_1)$, obtaining an optimal control $\widehat{\theta}_0$. Finally, we define the concatenated control

$$\widehat{\theta}(t) = \widehat{\theta}_0(t)\,\mathbf{1}_{[0,t_1)}(t) + \widehat{\theta}_{t_1}(t)\,\mathbf{1}_{[t_1,1]}(t),$$

and verify that $\widehat{\theta}\in\mathcal{A}_0$ and that it solves (3.0.9).

Having obtained the optimal control $\widehat{\theta}$, we then construct an equilibrium by imposing the pricing consistency condition $H(t,\widehat{\xi}(t)) = \mathbb{E}[v|\mathcal{F}^{\widehat{Y}}_t]$, where $\widehat{\xi}$ is the state process induced by $\widehat{\theta}$ via (3.0.7).

Chapter 4

Optimal Control Problem

4.1 Optimal Control over $[t_1, 1]$

With the admissible trading strategies well defined, and in view of (3.0.10), we now consider the second-period control problem. Fix $t \in [t_1, 1]$ and let $\xi(t)$ be the $\mathcal{F}^Y_t$-measurable state at time t. The representative insider value function over $[t, 1]$ is

$$\mathcal{J}^{\xi(t)}(t) := \operatorname*{ess\,inf}_{\theta \in \mathcal{A}_t(\xi(t))} \left\{ \mathbb{E}\left[e^{-\overline{\gamma}_2 \int_t^1 (v - H(u,\xi(u)))\theta(u)du} \middle| \mathcal{F}^I_t \right] \right\}; \qquad t \in [t_1, 1]. \tag{4.1.1}$$

where, for each admissible θ, the corresponding (non-explosive) state process ξ on $[t, 1]$ is determined by (3.0.7) with initial condition $\xi(t)$. As with the admissibility class, we are primarily interested in $\mathcal{J}^{\xi(t_1)}_{t_1}$. Below, for notational convenience, we henceforth suppress the dependence on the initial condition at time t and write $\mathcal{J} := \mathcal{J}(t)^{\xi(t)}$, where $\xi(t)$ denotes the realized state at time t induced by the control before t.

To identify $\mathcal{J}(t)$, we start by discussing the market maker and insider filtrations. First, based upon our conjecture on H, once the pricing rule $p(s) = H(s, \xi(s)), s \in [t, 1]$ is fixed, the process ξ over $[t, 1]$ can be inferred by inverting H. Therefore, as we also conjecture $\lambda > 0$, we conclude that provided $\mathcal{F}^p_t = \mathcal{F}^Y_t$, $\mathcal{F}^p_s = \mathcal{F}^Y_s$ on $[t, 1]$ as well. Next, as $dY(s) = \theta(s)ds + \sigma dB(s)$ and the (representative) insider controls θ, the insider is able to deduce

$$\overline{B}(s) := B(s) - B(t) = (1/\sigma)(Y(s) - Y(t) - \int_t^s \theta(u)du), \qquad s \in [t, 1],$$

by observing the market price. By the independence of B and v, the process $\overline{B}$ is a $\mathbb{F}^I$ Brownian motion on $[t,1]$.

We expect both the optimal control and value function to be Markovian in that $\widehat{\theta}(s) = \theta(s, \widehat{\xi}(s), v)$ and $\mathcal{J}(s) = J(s, \widehat{\xi}(s), v), s \in [t,1]$ where $\widehat{\xi}$ is obtained using $\widehat{\theta}$ with $\widehat{\xi}(t) = \xi(t)$, that is observed at t given $\mathcal{F}^I_t$. Given this, standard arguments (c.f. (Pham, 2009, Chapter 3)) can be used to obtain the HJB equation

$$\begin{aligned}\inf_{\theta}\Bigg\{ \left[\lambda(t)\frac{\partial J}{\partial \xi}(t,\xi) - \overline{\gamma}_2\big(v - H(t,\xi)\big)J(t,\xi)\right]\theta \\ + \frac{\partial J}{\partial t}(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2\frac{\partial^2 J}{\partial \xi^2}(t,\xi)\Bigg\} = 0.\end{aligned} \tag{4.1.2}$$

for $t \in [t_1, 1]$, $\xi \in \mathbb{R}$, with the random terminal condition

$$J(1, \widehat{\xi}(1)) = 1, \quad \text{a.s.}, \tag{4.1.3}$$

which is found by enforcing the condition $H(1, \widehat{\xi}(1)) = v$ a.s.

We now identify solution triples (H, λ, J) to (4.1.2). First, as the Hamiltonian is linear in the control θ, we are lead to the following systems of equations on $[t_1, 1] \times \mathbb{R}$

$$\begin{aligned}0 &= \lambda(t)\frac{\partial}{\partial \xi}J(t,\xi) - \overline{\gamma}_2(v - H(t,\xi))J(t,\xi), \\ 0 &= \frac{\partial}{\partial t}J(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2\frac{\partial^2}{\partial \xi^2}J(t,\xi).\end{aligned} \tag{4.1.4}$$

Next, following a similar approach as in (Cho, 2003, pp. 60-61) we find that if there is a solution to (4.1.4) then necessarily we must also have a solution on $[t_1, 1] \times \mathbb{R}$ to

$$\begin{aligned}0 = H_t(t,\xi) &+ \frac{1}{2}\sigma^2\lambda(t)^2 H_{\xi\xi}(t,\xi) \\ &- \lambda(t)\big(v - H(t,\xi)\big)\left[\frac{\partial}{\partial t}\left(\frac{1}{\lambda(t)}\right) - \overline{\gamma}_2\sigma^2 H_\xi(t,\xi)\right].\end{aligned} \tag{4.1.5}$$

That (4.1.4)-(4.1.5) must hold for all v, and H cannot depend upon v leads to the following lemma.

Lemma 4.1.1.

(1) Let (H, λ, J) solve (4.1.3)-(4.1.5). Then necessarily, the pricing rule (H, λ) verifies

$$\begin{aligned} C1: &\quad \frac{\partial}{\partial t}\left(\frac{1}{\lambda(t)}\right) = \overline{\gamma}_2 \sigma^2 \frac{\partial}{\partial \xi} H(t,\xi), \\ C2: &\quad \frac{\partial}{\partial t} H(t,\xi) + \frac{1}{2}\sigma^2 \lambda(t)^2 \frac{\partial^2}{\partial \xi^2} H(t,\xi) = 0, t_1 \leq t \leq 1, \xi \in \mathbb{R}. \end{aligned}$$

As such, H takes the form

$$H(t,\xi) = h_c(t_1) + h_s(t_1)\xi \tag{4.1.6}$$

for to-be-determined constants $h_c(t_1), h_s(t_1)$. λ takes the form

$$\lambda(t) = \frac{c(t_1)}{1 - \overline{\gamma}_2 \sigma^2 c(t_1) h_s(t_1)(1-t)}, \tag{4.1.7}$$

where $c(t_1)$ is a to-be-determined constant.

(2) Conversely, if a pricing rule (H, λ) verifies $C1$ and $C2$ and thus is of form of (4.1.6) and (4.1.7), then (H, λ, J), with J defined

$$\begin{aligned} J(t,\xi) = {} & e^{R(t_1)} \sqrt{1 - \overline{\gamma}_2 \sigma^2 h_s(t_1) c(t_1)(1-t)} \\ & \times \exp\left\{ -\frac{\overline{\gamma}_2 \left[1 - \overline{\gamma}_2 \sigma^2 h_s(t_1) c(t_1)(1-t)\right]}{2c(t_1) h_s(t_1)} \Big(v - \big(h_c(t_1) + h_s(t_1)\xi\big) \Big)^2 \right\}. \end{aligned} \tag{4.1.8}$$

solves (4.1.4)-(4.1.5). Here, $R(t_1)$ is a to-be-determined constant

Proof of Lemma 4.1.1. For (1) suppose (H, λ, J) solve (4.1.3)-(4.1.4). Then necessarily, (4.1.5) is satisfied. As the market maker does not observe the terminal asset value v prior to time 1, H cannot depend on v and $C1$ must be satisfied. $C1$ and (4.1.5) then imply that $C2$ is satisfied. Taking derivative on both sides of $C1$ leads to $(\partial^2/\partial \xi^2) H(t,\xi) = 0$ and therefore (4.1.5) implies H takes the form in (4.1.6). Lastly, (4.1.7) follows directly from $C1$ and (4.1.6).

As for (2), condition $C1$ and $C2$ imply $(\partial/\partial t) H(t,\xi) = 0$, and hence $H(\cdot,\xi)$ must take the form (4.1.6). Using $C1$ again gives that λ takes the form in (4.1.7). Given this, direct computation using then shows that setting J as in (4.1.8) leads to a solution of (4.1.4)-(4.1.5). $\square$

Remark 4.1.2. *Notice that the expressions for H, λ, and J in (4.1.6), (4.1.7), and (4.1.8) depend on the (as yet unspecified) constant $h_c(t_1)$, $h_s(t_1)$, and $c(t_1)$. These quantities will be identified in subsequent lemmas. However, observe from (3.0.7) and (4.1.6) –(4.1.8) that by considering the state process $t \to h_s(t_1)\xi(t)$ with dynamics $d(h_s(t_1)\xi(t)) = h_s(t_1)\lambda(t)dY(t)$ we see the $c(t_1), h_s(t_1)$ are only identified up to their product. For this reason without loss of generality, we may fix the normalization $h_s(t_1) = 1$. Additionally, note that the terminal condition (4.1.3) cannot hold unless $R(t_1) = 0$. With these two results we have*

$$H(t, \xi) = h_c(t_1) + \xi, \tag{4.1.9}$$

$$\lambda(t) = \frac{c(t_1)}{1 - \overline{\gamma}_2\sigma^2 c(t_1)(1-t)}, \tag{4.1.10}$$

and

$$J(t, \xi) = \sqrt{1 - \overline{\gamma}_2\sigma^2 c(t_1)(1-t)} \times e^{-\frac{\overline{\gamma}_2(1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t))}{2c(t_1)}\left(v-(h_c(t_1)+\xi)\right)^2}, \tag{4.1.11}$$

where $c(t_1)$ will be determined in equilibrium. Note lastly that for λ to be well-defined and positive on $[t_1, 1]$ we require $0 < c(t_1) < (\overline{\gamma}_2\sigma^2(1-t_1))^{-1}$. This will be assumed implicitly throughout.

The above Lemma shows the generic solution (H, λ, J) to (4.1.3)-(4.1.5) is obtained using (4.1.9), (4.1.10) and (4.1.11). In the lemma below we provide facts about J at time 1. To stress the dependence on random quantities we bring back the $J(t, \xi, v)$ notation.

Lemma 4.1.3 (Terminal Property)**.** *Let (H, λ, J) be a solution to (4.1.3)–(4.1.5). Then*

$$\sup_{\xi \in \mathbb{R}} J(1, \xi, v) \leq 1 \qquad a.s..$$

Moreover, for any admissible control θ and the corresponding process ξ defined by (3.0.7), we have $J\big(1, \xi(1), v\big) = 1$ a.s. if and only if $h_c(t_1) + \xi(1) = v$ a.s..

Proof of Lemma 4.1.3. This is immediate given (4.1.11). $\square$

We are now prepared to give the main verification result.

Proposition 4.1.4. *(Optimality condition) Fix $t \in [t_1, 1]$ and an $\mathcal{F}_t^Y$-measurable random variable $\xi(t)$, which represents the state variable at time t. Let (H, λ) satisfy Conditions C1–C2 in Lemma 4.1.1. The strategy $\widehat{\theta} \in \mathcal{A}_t(\xi(t))$ is optimal for the problem in (4.1.1) provided $H(1, \widehat{\xi}(1)) = v$ almost surely, for $\widehat{\xi}$ in Conjecture 3.0.3 using $\widehat{\theta}$. In particular, at $t = t_1$ if the above are satisfied then*

$$\mathcal{J}(t_1) = J(t_1, \xi(t_1), v) = \sqrt{1 - \overline{\gamma}_2 \sigma^2 c(t_1)(1 - t_1)} \times e^{-\frac{\overline{\gamma}_2 \left(1 - \overline{\gamma}_2 \sigma^2 c(t_1)(1-t_1)\right)}{2c(t_1)} \left(v - h_c(t_1) - \xi(t_1)\right)^2}. \tag{4.1.12}$$

Proof of Proposition 4.1.4. Assume (H, λ) satisfies condition $C1$, $C2$ in Lemma 4.1.1. Therefore, by Lemma 4.1.1 (H, λ, J) solves (4.1.3)-(4.1.4) for H from (4.1.9), λ from (4.1.7) and J from (4.1.11). Next, for any $\theta \in \mathcal{A}_t(\xi(t))$ let ξ be the state variable process over $[t, 1]$ driven by θ. Applying Itô to $\log J(t, \xi(t))$, and using (4.1.4), (4.1.5), we have for $s \in [t, 1]$

$$\begin{aligned} d \log\left(J(s, \xi(s))\right) = \overline{\gamma}_2 (v - H(s, \xi(s)))\theta(s) ds + \overline{\gamma}_2 (v - H(s, \xi(s)))\sigma dB(s) \\ - \frac{1}{2} \overline{\gamma}_2^2 (v - H(s, \xi(s)))^2 \sigma^2 ds. \end{aligned}$$

Next, for $s \in [t, 1]$ set

$$\psi(s) := \mathcal{E}\left(\overline{\gamma}_2 \int_t^{\cdot} (v - H(u, \xi(u)))\sigma dB(u)\right)(s); \qquad \psi(s, 1) := \frac{\psi(1)}{\psi(s)}.$$

We then have

$$-\overline{\gamma}_2 \int_t^1 (v - H(s, \xi(s)))\theta(s) ds = \log\left(J(t, \xi(t))\right) - \log\left(J(1, \xi(1))\right) + \log\left(\psi(t, 1)\right).$$

Hence,

$$\mathbb{E}\left[e^{-\overline{\gamma}_2 \int_t^1 (v - H(s, \xi(s)))\theta(s) ds} \middle| \mathcal{F}_t^I\right] = J(t, \xi(t)) \mathbb{E}\left[J(1, \xi(1))^{-1} \psi(t, 1) \middle| \mathcal{F}_t^I\right]. \tag{4.1.13}$$

As θ was arbitrary, using Lemma 4.1.3, the martingale property for ψ (c.f. Definition 3.0.4) and (4.1.1), we deduce

$$\mathcal{J}(t) \geq J(t, \xi(t)).$$

Next, (4.1.9), (4.1.11) and Lemma 4.1.3 we deduce $J(1, \widehat{\xi}(1)) = 1$ with probability one, and hence

$$\mathcal{J}(t) \leq \mathbb{E}\left[e^{-\overline{\gamma}_2 \int_t^1 (v - H(s,\widehat{\xi}(s)))\widehat{\theta}(s)ds} \middle| \mathcal{F}_t^I \right] = J(t, \xi(t)).$$

Hence, $\mathcal{J}(t) = J(t, \xi(t))$ and $\widehat{\theta}$ is the optimal strategy. The formula (4.1.12) is immediate as $\xi(t_1)$ is observable given $\mathcal{F}_{t_1}^I$.

$\square$

From Proposition 4.1.4 we obtain sufficient conditions for optimality, which allow us to verify that a given candidate strategy is optimal. Guided by these conditions, we conjecture (similarly to (Cho, 2003, pp 58)) that the equilibrium strategy on the sub-interval $[t_1, 1)$ is of affine form, and then characterize such strategies.

Conjecture 4.1.5 (Affine form of the strategy)**.** *For each staring time $t \in [t_1, 1)$ on the interval $[t, 1)$ the optimal strategy $\widehat{\theta}$ for (4.1.1) is of the affine form*

$$\widehat{\theta}(s) = A(s)(v - H(s, \widehat{\xi}(s))), \qquad s \in [t, 1),$$

where for each $T \in [t, 1)$ the map A satisfy the local bounds

$$|A(s)| \leq L(T) \quad \textit{for all } t \leq s \leq T.$$

Proposition 4.1.6. *For $t \in [t_1, 1)$, define for $s \in [t, 1)$*

$$\begin{aligned} H(s, \xi) &= h_c(t_1) + \xi, & \lambda(s) &= \frac{c(t)}{1 - \overline{\gamma}_2 \sigma^2 c(t)(1 - s)}, \\ \widehat{\theta}(s) &= A(s)\big(v - H(s, \widehat{\xi}(s))\big), & p(s) &= H(s, \widehat{\xi}(s)). \end{aligned}$$

where

$$\widehat{\xi}(s) = \xi(t) + \int_t^s \lambda(u) d\widehat{Y}(u) = \xi(t) + \int_t^s \lambda(u) \left(\widehat{\theta}(u)du + \sigma dB(u) \right).$$

If A satisfies condition in Conjecture 4.1.5 and is such that

$$r(s) := \lambda(s)\left(A(s) + \overline{\gamma}_2 \sigma^2\right) - \frac{1}{1 - s}, \tag{4.1.14}$$

is integrable over $[t,1]$ then $\widehat{\theta}(s)$ is the optimal (representative) insider policy for problem 4.1.1 at t. The value function at t is

$$\begin{aligned}\mathbb{E}&\left[e^{-\overline{\gamma}_2\int_t^1\left(v-H(u,\widehat{\xi}(u))\right)\widehat{\theta}(u)\,du}\middle|\mathcal{F}_t^I\right]\\&=\sqrt{1-\overline{\gamma}_2\sigma^2c(t_1)(1-t)}e^{-\frac{\overline{\gamma}_2\left[1-\overline{\gamma}_2\sigma^2c(t_1)(1-t)\right]}{2c(t_1)}\left(v-h_c(t_1)-\xi(t)\right)^2}.\end{aligned}$$

Proof of Proposition 4.1.6. It is clear that (H,λ) satisfies conditions $C1$ and $C2$ in Lemma 4.1.1. Thus, it remains to verify that $H(1,\widehat{\xi}(1))=v$ and that $\theta\in\mathcal{A}_t(\xi(t))$. Below, we drop the $\widehat{\cdot}$ notation for ease of notation.

Plugging θ into the evolution equation (3.0.7), we obtain

$$\begin{aligned}d\xi(s)&=\lambda(s)\left(\theta(s)\,ds+\sigma\,dB(s)\right)\\&=\lambda(s)A(s)\left(v-h_c(t_1)-\xi(s)\right)\,ds+\lambda(s)\sigma\,dB(s).\end{aligned}$$

Solving this linear SDE gives

$$\begin{aligned}\xi(s)=\xi(t)e^{-\int_t^s\lambda(u)A(u)\,du}&+\left(v-h_c(t_1)\right)\left(1-e^{-\int_t^s\lambda(u)A(u)\,du}\right)\\&+\sigma\int_t^s e^{-\int_r^s\lambda(u)A(u)\,du}\lambda(r)\,dB(r).\end{aligned}$$

Let

$$K(s):=\int_t^s\lambda(u)A(u)\,du.$$

Then

$$\xi(s)=\xi(t)e^{-K(s)}+\left(v-h_c(t_1)\right)\left(1-e^{-K(s)}\right)+\sigma e^{-K(s)}\int_t^s e^{K(r)}\lambda(r)\,dB(r).$$

Set

$$I(s):=\int_t^s e^{K(r)}\lambda(r)\,dB(r).$$

Then

$$\langle I\rangle(s)=\int_t^s e^{2K(r)}\lambda(r)^2\,dr,$$

and hence

$$\xi(s) = \xi(t)e^{-K(s)} + (v - h_c(t_1))\left(1 - e^{-K(s)}\right) + \sigma e^{-K(s)}\langle I\rangle(s)\frac{I(s)}{\langle I\rangle(s)}.$$

To show that ξ is well-defined on $[t,1]$ and that $H(1,\xi(1)) = v$ a.s., it is enough to prove that

1. $\lim\limits_{s\to 1} e^{-K(s)} = 0$;
2. $\lim\limits_{s\to 1} \int\limits_t^s e^{2K(r)}\lambda(r)^2\,dr = \infty$;
3. $\limsup\limits_{s\to 1} e^{-K(s)} \int\limits_t^s e^{2K(r)}\lambda(r)^2\,dr < \infty$.

Indeed, condition (2) implies that $\langle I\rangle(s) \to \infty$ a.s., and therefore

$$\frac{I(s)}{\langle I\rangle(s)} \to 0 \qquad \text{a.s.}$$

Together with conditions (1) and (3), this gives the desired result.

To prove (1), recall (4.1.14) and let

$$R(s) := \int\limits_t^s r(u)\,du, \qquad R := R(1).$$

Then

$$e^{-K(s)} = \frac{\lambda(t)(1-s)}{\lambda(s)(1-t)}e^{-R(s)} \to 0 \qquad \text{as } s \to 1.$$

To prove (2), the previous identity implies

$$\int\limits_t^s e^{2K(r)}\lambda(r)^2\,dr = \frac{(1-t)^2}{\lambda(t)^2}\int\limits_t^s \frac{e^{2R(r)}\lambda(r)^4}{(1-r)^2}\,dr.$$

Since λ and R are uniformly bounded on $[t,1]$, the right-hand side diverges as $s \to 1$.

To prove (3), we have

$$e^{-K(s)} \int_t^s e^{2K(r)} \lambda(r)^2 \, dr = \frac{(1-s)(1-t)}{e^{R(s)} \lambda(t) \lambda(s)} \int_t^s \frac{e^{2R(r)} \lambda(r)^4}{(1-r)^2} \, dr.$$

Since λ and R are uniformly bounded on $[t, 1]$, the above quantity remains finite as $s \to 1$.

It remains to verify that $\theta \in \mathcal{A}_t(\xi(t))$. Since (3.0.7) has a non-explosive solution and $\widehat{\theta}$ is adapted to $\mathbb{F}^I$, we only need to show that there exists a non-explosive weak solution $(\xi, \widetilde{B})$ on $[t, 1]$ to

$$\begin{aligned} d\xi(s) &= \lambda(s) \left[\left(\theta(s, \xi(s)) + \overline{\gamma}_2 \sigma^2 (v - H(s, \xi(s))) \right) ds + \sigma \, d\widetilde{B}(s) \right] \\ &= \lambda(s) \left(A(s) + \overline{\gamma}_2 \sigma^2 \right) \left(v - h_c(t_1) - \xi(s) \right) \, ds + \lambda(s) \sigma \, d\widetilde{B}(s). \end{aligned}$$

Similarly, define

$$K(s) := \int_t^s \lambda(u) \left(A(u) + \overline{\gamma}_2 \sigma^2 \right) \, du = \log \left(\frac{1-t}{1-s} \right) + R(s).$$

For $t < s < 1$, the SDE for ξ has solution

$$\begin{aligned} \xi(s) &= \xi(t) e^{-K(s)} + (v - h_c(t_1)) \left(1 - e^{-K(s)} \right) + \sigma e^{-K(s)} \int_t^s e^{K(r)} \lambda(r) \, d\widetilde{B}(r) \\ &= \xi(t) \frac{1-s}{1-t} e^{-R(s)} + (v - h_c(t_1)) \left(1 - \frac{1-s}{1-t} e^{-R(s)} \right) \\ &\quad + \sigma (1-s) e^{-R(s)} \int_t^s \frac{e^{R(r)} \lambda(r)}{1-r} \, d\widetilde{B}(r). \end{aligned}$$

Non-explosion follows once we show that

$$\xi(s) \to v - h_c(t_1) \qquad \text{as } s \to 1.$$

The proof follows the same steps as above, so we omit the details. $\square$

4.2 Optimal Control over $[0, t_1]$

Having solved the optimal control problem over $[t_1, 1]$, we next turn to the interval $[0, t_1)$. Given the optimal control $\widehat{\theta}$ over $[t_1, 1]$, let θ denote the control on $[0, t_1)$, and let ξ denote the state process associated with θ. By Proposition (4.1.6), we have

$$\begin{aligned}&\mathbb{E}\left[e^{-\overline{\gamma}_2 \int_{t_1}^1 (v-p(u))\widehat{\theta}(u)\,du}\middle|\mathcal{F}_{t_1}^I\right]\\ &\quad= \sqrt{1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t_1)}\,e^{-\frac{\overline{\gamma}_2\left[1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t_1)\right]}{2c(t_1)}\left(v-h_c(t_1)-\xi(t_1)\right)^2}.\end{aligned}$$

As λ is a right continuous function, we have that

$$\lambda(t_1) = \frac{c(t_1)}{1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t_1)}. \tag{4.2.1}$$

The optimal control problem 3.0.10 becomes

$$\inf_{\theta\in\mathcal{A}_{0,t_1}} \mathbb{E}\left[e^{-\overline{\gamma}_1 \int_0^{t_1}(v-p(u))\theta(u)du}\sqrt{\frac{c(t_1)}{\lambda(t_1)}}\,e^{-\frac{\overline{\gamma}_2}{2\lambda(t_1)}\left(v-h_c(t_1)-\xi(t_1)\right)^2}\middle|\mathcal{F}_0^I\right].$$

Similar to the period $[t_1, 1]$, as$H$ is invertible and $\lambda > 0$ we conclude that $\mathcal{F}_t^p = \mathcal{F}_t^Y$ on $[0, t_1)$ and the process $t \to B(t)$ is a $\mathbb{F}^I$- Brownian motion. Next, we define the value function over $[0, t_1)$

$$\mathcal{J}^{\xi(t)}(t) := \operatorname*{ess\,inf}_{\theta\in\mathcal{A}_{t,t_1}(\xi(t))} \mathbb{E}\left[\begin{array}{c} e^{-\overline{\gamma}_1\int_t^{t_1}\left(v-H(u,\xi(u))\right)\theta(u)\,du}\sqrt{\dfrac{c(t_1)}{\lambda(t_1)}} \\ \times\, e^{-\frac{\overline{\gamma}_2}{2\lambda(t_1)}\left(v-h_c(t_1)-\xi(t_1)\right)^2}\end{array}\middle|\mathcal{F}_t^I\right]. \tag{4.2.2}$$

If we consider Markovian controls $\theta(t) = \theta(t, \xi(t), v)$ with ξ from (3.0.7), we expect the functional form $\mathcal{J}(t) = J(t, \widehat{\xi}(t), v)$ for a to-be-determined function J and where

$\widehat{\xi}(t)$ is obtained using the optimal control $\widehat{\theta}$. The corresponding HJB equation is

$$\inf_{\theta}\left\{ \left[\lambda(t)J_\xi(t,\xi) - \overline{\gamma}_1\big(v - H(t,\xi)\big)J(t,\xi)\right]\theta \right. \\ \left. + J_t(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2 J_{\xi\xi}(t,\xi)\right\} = 0.$$

with terminal condition

$$J(t_1,\xi,v) = \sqrt{\frac{c(t_1)}{\lambda(t_1)}}e^{-\frac{\overline{\gamma}_2}{2\lambda(t_1)}\left(v-h_c(t_1)-\xi\right)^2}, \quad \xi \in \mathbb{R}. \tag{4.2.3}$$

As the Hamiltonian is the same as over $[t_1, 1]$ (with γ_1 replacing $\overline{\gamma}$) we are led to the system of equations for $0 \le t < t_1$ and $\xi \in \mathbb{R}$.

$$\begin{aligned} 0 &= \lambda(t)J_\xi(t,\xi) - \overline{\gamma}_1\big(v - H(t,\xi)\big)J(t,\xi), \\ 0 &= J_t(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2 J_{\xi\xi}(t,\xi), \\ 0 &= H_t(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2 H_{\xi\xi}(t,\xi) \\ &\quad - \lambda(t)\big(v - H(t,\xi)\big)\left[\frac{\partial}{\partial t}\left(\frac{1}{\lambda(t)}\right) - \overline{\gamma}_1\sigma^2 H_\xi(t,\xi)\right]. \end{aligned} \tag{4.2.4}$$

The facts that (i) (4.2.4) must hold for all v and (ii) H cannot depend explicitly on v lead to the following lemma. The proof is analogous to that of Lemma 4.1.1.

Lemma 4.2.1.

(1) Let (H,λ,J) solve (4.2.4). Then necessarily, the pricing rule (H,λ) verifies on $0 \le t < t_1, \xi \in \mathbb{R}$

$$\begin{aligned} C1: &\quad \frac{\partial}{\partial_t}\left(\frac{1}{\lambda(t)}\right) = \overline{\gamma}_1\sigma^2\frac{\partial}{\partial_\xi}H(t,\xi), \\ C2: &\quad \frac{\partial}{\partial t}H(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2\frac{\partial^2}{\partial\xi^2}H(t,\xi) = 0. \end{aligned}$$

The pricing rule is of the form

$$H(t,\xi) = h_c(0) + h_s(0)\xi, \tag{4.2.5}$$

for to-be-determined constants $h_c(0), h_s(0)$. *The function* λ *takes the form*

$$\lambda(t) = \frac{c(0)}{1 - \overline{\gamma}_1 \sigma^2 c(0) h_s(0)(t_1 - t)}, \tag{4.2.6}$$

where $c(0)$ *is a to-be-determined constant.*

(2) *Conversely, if a pricing rule* (H, λ) *verifies* $C1$ *and* $C2$ *and thus is of the form* (4.2.9), (4.2.8), *then* (H, λ, J), *with* J *defined*

$$J(t,\xi) = e^{R(0)} \sqrt{1 - \overline{\gamma}_1 \sigma^2 c(0) h_s(0)(t_1 - t)} \times e^{-\frac{\overline{\gamma}_1 \left(1 - \overline{\gamma}_1 \sigma^2 c(0) h_s(0)(t_1 - t)\right)}{2c(0) h_s(0)} \left(v - h_c(0) - h_s(0)\xi\right)^2} \tag{4.2.7}$$

solves (4.2.4). *Here,* $R(0)$ *is a to-be determined constant.*

Remark 4.2.2. *Analogously to what was discussed in Remark 4.1.2, without loss of generality we can take* $h_s(0) = 1$. *Moreover, by uniqueness of pricing rule, we set* $h_c(0) = h_c(t_1)$. *Additionally, condition (4.2.3) cannot be satisfied unless* $c(0) = \lambda(t_1)\overline{\gamma}_1/\overline{\gamma}_2$ *and* $R(0) = (1/2)\log(c(t_1)/\lambda(t_1))$. *With this result, we have that*

$$\lambda(t) = \frac{\lambda(t_1)\overline{\gamma}_1/\overline{\gamma}_2}{1 - \overline{\gamma}_1 \sigma^2 \lambda(t_1)\overline{\gamma}_1/\overline{\gamma}_2\,(t_1 - t)}, \tag{4.2.8}$$

$$H(t,\xi) = h_c(t_1) + \xi, \tag{4.2.9}$$

and

$$J(t,\xi) = \sqrt{\frac{c(t_1)\,\overline{\gamma}_1}{\overline{\gamma}_2\,\lambda(t)}}\, e^{-\frac{\overline{\gamma}_1}{2\lambda(t)}\left(v - h_c(t_1) - \xi\right)^2}, \qquad 0 \le t < t_1, \tag{4.2.10}$$

The above lemmas show that 4.2.8, 4.2.9 and 4.2.10 solves (4.2.3) - (4.2.4).

Proposition 4.2.3. *(Optimality condition) Fix* $t \in [0, t_1)$ *and an* $\mathcal{F}_t^Y$*-measurable random variable* $\xi(t)$*, which represents the state variable at time* t*. Let* (H, λ) *satisfy condition* $C1$ *and* $C2$ *in Lemma 4.2.1 , then any strategy* $\theta \in \mathcal{A}_{t,t_1}(\xi(t))$ *is optimal for the problem in* (4.2.2).

Proof of Proposition 4.2.3. Assume that (H, λ) satisfies conditions $(C1)$ and $(C2)$ in Lemma 4.2.1. Then, by Lemma 4.2.1, the function J given in (4.2.10) solves (4.2.4); here we recall that $h_s(0) = 1$.

Next, fix any $\theta \in \mathcal{A}_{t,t_1}(\xi(t))$, and let ξ be the process defined by (3.0.7) on $[t, t_1)$, driven by θ with initial state $\xi(t)$. Applying Itô's formula to $s \mapsto \log J(s, \xi(s))$ and using (4.2.4), we obtain

$$\begin{aligned} d\log J(s,\xi(s)) = \overline{\gamma}_1\big(v - H(s,\xi(s))\big)\theta(s)\,ds \\ + \overline{\gamma}_1\big(v - H(s,\xi(s))\big)\sigma\,dB(s) \\ - \frac{1}{2}\overline{\gamma}_1^2\big(v - H(s,\xi(s))\big)^2\sigma^2\,ds. \end{aligned}$$

For $t \le s \le t_1$, define

$$\begin{aligned} \psi(s) &:= \mathcal{E}\left(\overline{\gamma}_1 \int_t^{\cdot} \big(v - H(u,\xi(u))\big)\sigma\,dB(u)\right)(s), \\ \psi(s,t_1) &:= \frac{\psi(t_1)}{\psi(s)}. \end{aligned}$$

Then

$$\begin{aligned} -\overline{\gamma}_1 \int_t^{t_1} \big(v - H(s,\xi(s))\big)\theta(s)\,ds = \log\left(J(t,\xi(t))\right) - \log\left(J(t_1,\xi(t_1))\right) \\ + \log\left(\psi(t,t_1)\right). \end{aligned}$$

Define

$$\widetilde{J}(t_1,\xi) = \sqrt{\frac{c(t_1)}{\lambda(t_1)}}\, e^{-\frac{\overline{\gamma}_2}{2\lambda(t_1)}\left(v - h_c(t_1) - \xi\right)^2}.$$

Hence,

$$\begin{aligned} &\mathbb{E}^v\left[e^{-\overline{\gamma}_1\int_t^{t_1}\left(v - H(s,\xi(s))\right)\theta(s)\,ds}\widetilde{J}(t_1,\xi(t_1))\,\Big|\,\mathcal{F}_t^I\right] \\ &\quad = J(t,\xi(t))\mathbb{E}^v\left[J(t_1,\xi(t_1))^{-1}\widetilde{J}(t_1,\xi(t_1))\psi(t,t_1)\,\Big|\,\mathcal{F}_t^I\right]. \end{aligned} \tag{4.2.11}$$

By Remark 4.2.2, the constants in (4.2.10) are chosen so that

$$J(t_1,\xi) = \widetilde{J}(t_1,\xi), \qquad \xi \in \mathbb{R}.$$

Since ψ is a martingale by admissibility, it follows that

$$\mathbb{E}^v\left[e^{-\overline{\gamma}_1\int_t^{t_1}\left(v-H(s,\xi(s))\right)\theta(s)\,ds}\widetilde{J}(t_1,\xi(t_1))\,\Big|\,\mathcal{F}_t^I\right]$$
$$= J(t,\xi(t)).$$

Because $\theta \in \mathcal{A}_{t,t_1}(\xi(t))$ was arbitrary, every admissible strategy attains the same value. Therefore,

$$\mathcal{J}(t) = J(t,\xi(t)),$$

and every $\theta \in \mathcal{A}_{t,t_1}(\xi(t))$ is optimal. □

We have shown that any admissible strategy is optimal for the optimization problem on $[0,t_1)$. We therefore restrict attention to affine strategies, which will help us establish equilibrium in the next section.

Conjecture 4.2.4 (Affine form of the strategy)**.** *For each starting time $t \in [0,t_1)$ on the interval $[t,t_1)$ the optimal strategy $\widehat{\theta}$ for (4.2.2) is of the affine form*

$$\widehat{\theta}(s) = A(s)(v - H(s,\widehat{\xi}(s))), \qquad s \in [t,t_1),$$

where for each $s \in [t,t_1)$ the map A satisfy the bounds

$$|A(s)| \le L.$$

Proposition 4.2.5. *For $t \in [0,t_1)$ define for $s \in [t,t_1)$*

$$\begin{aligned} H(s,\xi) &= h_c(t_1) + \xi, \\ \lambda(s) &= \frac{(\overline{\gamma}_1/\overline{\gamma}_2)\lambda(t_1)}{1-(\overline{\gamma}_1^2/\overline{\gamma}_2)\sigma^2\lambda(t_1)(t_1-s)}, \\ \widehat{\theta}(s) &= A(s)\big(v - H(s,\widehat{\xi}(s))\big), \end{aligned}$$

where (with initial condition $\widehat{\xi}(t) = \xi$)

$$\widehat{\xi}(s) = \xi + \int_t^s \lambda(u)d\widehat{Y}(u) = \xi + \int_t^s \lambda(u)\left(\widehat{\theta}(u)du + \sigma dB(u)\right).$$

If A satisfies the condition in Conjecture 4.2.4 and $\lambda(s)$ is uniformly bounded over $[t, t_1]$, then $\widehat{\theta}$ is optimal control for the problem in (4.2.2) at time t.

Proof of Proposition 4.2.5. By Proposition 4.2.3, it suffices to show that $\widehat{\theta} \in \mathcal{A}_{t,t_1}(\xi)$.

First, $\widehat{\theta}$ is $\mathbb{F}^I$-adapted. Moreover, the original state equation

$$d\widehat{\xi}(s) = \lambda(s)\Big(A(s)(v - h_c(t_1) - \widehat{\xi}(s))\,ds + \sigma\,dB(s)\Big), \qquad \widehat{\xi}(t) = \xi,$$

is a linear SDE with bounded coefficients on $[t, t_1]$, and therefore admits a unique continuous non-explosive strong solution.

It remains to verify the martingale condition. By Remark 3.0.5, it is enough to show that there exists a non-explosive weak solution $(\xi, \widetilde{B})$ on $[t, t_1]$ to

$$d\xi(s) = \lambda(s)\Big((A(s) + \overline{\gamma}_1\sigma^2)(v - h_c(t_1) - \xi(s))\,ds + \sigma\,d\widetilde{B}(s)\Big), \qquad \xi(t) = \xi.$$

Define

$$\alpha(s) := \lambda(s)(A(s) + \overline{\gamma}_1\sigma^2), \qquad \Phi(s) := \int_t^s \alpha(u)\,du.$$

Then the above SDE has the explicit solution

$$\xi(s) = (v - h_c(t_1))(1 - e^{-\Phi(s)}) + \xi\,e^{-\Phi(s)} + \sigma e^{-\Phi(s)} \int_t^s e^{\Phi(r)}\lambda(r)\,d\widetilde{B}_r.$$

Since A and λ are bounded on $[t, t_1]$, both α and Φ are bounded on $[t, t_1]$, and

$$\int_t^{t_1} e^{2\Phi(r)}\lambda(r)^2\,dr < \infty.$$

Hence $\int_t^s e^{\Phi(r)}\lambda(r)\,d\widetilde{B}_r$ is a continuous square-integrable martingale on $[t, t_1]$. It follows that ξ has continuous finite-valued paths on the compact interval $[t, t_1]$, and therefore

$$\sup_{t \le s \le t_1} |\xi(s)| < \infty \qquad \text{a.s.}$$

Thus the weak solution is non-explosive.

Therefore, $\widehat{\theta} \in \mathcal{A}_{t,t_1}(\xi)$, and Proposition 4.2.3 implies that $\widehat{\theta}$ is optimal. $\square$

4.3 Optimal Control over $[0,1]$

Having solved optimal control problem (3.0.10), we now show the optimal control to (3.0.10) is the optimal control problem to (3.0.9). We define the value function over $[0,1]$,

$$\mathcal{J}(t) := \operatorname*{ess\,inf}_{\theta\in\mathcal{A}_t(\xi(t))} \left\{ \mathbb{E}\left[e^{-\overline{\gamma}_1\left(\int_t^{t_1}(v-H(u,\xi(u))\theta(u)du\right)-\overline{\gamma}_2\left(\int_{t_1}^{1}(v-H(u,\xi(u))\theta(u)du\right)} \middle| \mathcal{F}_t^I \right] \right\}. \quad (4.3.1)$$

We have that following proposition,

Proposition 4.3.1. *Fix $t_1 \in (0,1)$ and $t \in [0,t_1)$. Assume the conditions of Proposition 4.2.5 hold on $[t,t_1)$ and the conditions of Proposition 4.1.6 hold on $[t_1,1]$. Then the piecewise-defined strategy $\widehat{\theta}$ obtained by using the optimal control on $[t,t_1)$ and the optimal second-period control on $[t_1,1]$ is optimal for the insider's control problem on $[t,1]$. Moreover, the value function at time t is*

$$\begin{aligned} &\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du-\overline{\gamma}_2\int_{t_1}^{1}(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du} \middle| \mathcal{F}_t^I \right] \\ &= \sqrt{\frac{c(t_1)\,\overline{\gamma}_1}{\overline{\gamma}_2\,\lambda(t)}}\, e^{-\frac{\overline{\gamma}_1}{2\lambda(t)}\left(v-h_c(t_1)-\xi(t)\right)^2}. \end{aligned} \quad (4.3.2)$$

Proof of Proposition 4.3.1. Define

$$\widetilde{J}(t_1,\xi) := \sqrt{\frac{c(t_1)}{\lambda(t_1)}}\, e^{-\frac{\overline{\gamma}_2}{2\lambda(t_1)}\left(v-h_c(t_1)-\xi\right)^2}, \qquad \xi\in\mathbb{R}.$$

Pick any $\theta \in \mathcal{A}_t\big(\xi(t)\big)$. By restriction, θ is admissible on $[t,t_1)$, hence $\theta \in \mathcal{A}_{t,t_1}\big(\xi(t)\big)$. Define the second-period restriction of θ by $\theta_{t_1}(u) := \theta(u)$ for $u \in [t_1,1]$, and let $\xi(\cdot)$ denote the state process associated with θ. Then $\xi(t_1)$ is $\mathcal{F}_{t_1}^Y$-measurable, and the restriction of ξ to $[t_1,1]$ solves the state equation on $[t_1,1]$ with initial condition $\xi(t_1)$. Moreover, the martingale condition in Definition 3.0.4 is preserved under restriction to a subinterval. Therefore, $\theta_{t_1} \in \mathcal{A}_{t_1}\big(\xi(t_1)\big)$.

Hence, by the tower property,

$$\begin{aligned}
&\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\xi(u)))\theta(u)\,du-\overline{\gamma}_2\int_{t_1}^1(v-H(u,\xi(u)))\theta(u)\,du}\middle|\mathcal{F}_t^I\right]\\
&=\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\xi(u)))\theta(u)\,du}\ \mathbb{E}\left[e^{-\overline{\gamma}_2\int_{t_1}^1(v-H(u,\xi(u)))\theta(u)\,du}\middle|\mathcal{F}_{t_1}^I\right]\middle|\mathcal{F}_t^I\right]\\
&\geq\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\xi(u)))\theta(u)\,du}\ \inf_{\vartheta\in\mathcal{A}_{t_1}(\xi(t_1))}\mathbb{E}\left[e^{-\overline{\gamma}_2\int_{t_1}^1\left(v-H(u,\xi^{\vartheta}(u))\right)\vartheta(u)\,du}\middle|\mathcal{F}_{t_1}^I\right]\middle|\mathcal{F}_t^I\right]\\
&=\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\xi(u)))\theta(u)\,du}\ \widetilde{J}\big(t_1,\xi(t_1)\big)\middle|\mathcal{F}_t^I\right].
\end{aligned}$$

Here ξ^{ϑ} denotes the state process associated with ϑ on $[t_1,1]$ and initial condition $\xi^{\vartheta}(t_1)=\xi(t_1)$.

By optimality of $\widehat{\theta}$ on $[t,t_1)$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\xi(u)))\theta(u)\,du}\ \widetilde{J}\big(t_1,\xi(t_1)\big)\middle|\mathcal{F}_t^I\right]\\
&\geq\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du}\ \widetilde{J}\big(t_1,\widehat{\xi}(t_1)\big)\middle|\mathcal{F}_t^I\right].
\end{aligned}$$

Hence,

$$\mathcal{J}(t)\geq\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du}\ \widetilde{J}\big(t_1,\widehat{\xi}(t_1)\big)\middle|\mathcal{F}_t^I\right].$$

It remains to note that the concatenated strategy $\widehat{\theta}$ belongs to $\mathcal{A}_t(\xi(t))$. The state process is non-explosive on $[t,t_1)$ and on $[t_1,1]$, hence also on $[t,1]$ by concatenation. Moreover, the stochastic exponential in Definition 3.0.4 factors as the product of the first-period exponential and the continuation exponential started at t_1. Since the former is an $\mathcal{F}^I$-martingale on $[t,t_1]$ and the latter is an $\mathcal{F}^I$-martingale on $[t_1,1]$, then the concatenated exponential is an $\mathcal{F}^I$-martingale on $[t,1]$. Therefore $\widehat{\theta}\in\mathcal{A}_t(\xi(t))$.

Plugging $\widehat{\theta}$ into the definition of $\mathcal{J}(t)$ and using the tower property, we obtain

$$\begin{aligned}
\mathcal{J}(t)&\leq\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du-\overline{\gamma}_2\int_{t_1}^1(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du}\middle|\mathcal{F}_t^I\right]\\
&=\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du}\ \mathbb{E}\left[e^{-\overline{\gamma}_2\int_{t_1}^1(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du}\middle|\mathcal{F}_{t_1}^I\right]\middle|\mathcal{F}_t^I\right].
\end{aligned}$$

By optimality of the second-period restriction of $\widehat{\theta}$, the inner conditional expectation is equal to $\widetilde{J}\big(t_1,\widehat{\xi}(t_1)\big)$. Therefore,

$$\mathcal{J}(t)\leq\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\,du}\ \widetilde{J}\big(t_1,\widehat{\xi}(t_1)\big)\middle|\mathcal{F}_t^I\right].$$

Therefore,

$$\mathcal{J}(t) = \mathbb{E}\left[e^{-\overline{\gamma}_1 \int_t^{t_1} (v - H(u,\widehat{\xi}(u)))\widehat{\theta}(u)\, du} \; \widetilde{J}\big(t_1, \widehat{\xi}(t_1)\big) \Big| \mathcal{F}_t^I \right].$$

This proves the claim. The optimal value function is then obtained from (4.2.10).

□

Chapter 5

Equilibrium Construction

5.1 Equilibrium Construction for Two Insiders

In this section, given the optimal control $\widehat{\theta}$, we construct the equilibrium on $[0,1]$ by imposing the rational pricing rule $H(t,\xi(t)) = \mathbb{E}\left[v \,\middle|\, \mathcal{F}_t^Y\right], 0 \leq t \leq 1$.

We first present a lemma that is useful for constructing the equilibrium. It connects the HJB equation for the insider's optimal control problem, the associated optimal trading strategy, and the distribution of the total order flow Y. The proof can be found in (Cho, 2003, Lemma 2), and we omit it here.

Lemma 5.1.1. *For $\theta \in \mathcal{A}_0$ and (H, λ, J) such that $t \to H(t,\xi(t))$ is a $\mathbb{F}^Y$ martingale on $[0,1]$ where ξ is from (3.0.7), the following are equivalent.*

(1) For $t \in [0,1]$ and $\xi \in \mathbb{R}$, (H,λ) verify

$$\frac{\partial}{\partial t} H(t,\xi) + \frac{1}{2}\sigma^2 \lambda(t)^2 \frac{\partial^2}{\partial \xi^2} H(t,\xi) = 0.$$

(2) $\mathbb{E}\left[\theta(t) \middle| \mathcal{F}_t^Y\right] = 0$ for $t \in [0,1]$.

(3) The process $\left\{\xi_t, \mathcal{F}_t^Y, 0 \leq t \leq 1\right\}$ is a martingale with quadratic variation $\langle \xi, \xi\rangle_t = \sigma^2 \int_0^t \lambda^2(s)\, ds$. Equivalently, $\left\{Y_t/\sigma, \mathcal{F}_t^Y, 0 \leq t \leq 1\right\}$ is a Brownian motion, i.e., $\langle Y/\sigma, Y/\sigma\rangle_t = t$. In particular $Y/\sigma = B$ in law.

Proposition 5.1.2. *Given (H,λ) and the strategies defined in Propositions 4.1.6 and 4.2.5, assume the pricing rule is rational in that $H(t,\xi(t)) = \mathbb{E}\left[v \,\middle|\, \mathcal{F}_t^Y\right], 0 \leq t \leq 1$.*

Then

$$h_c(t_1) = \mu, \tag{5.1.1}$$

and

$$c(t_1) = \frac{\overline{\gamma}_2 \Sigma}{2} \left(\sqrt{1 + \frac{4}{\sigma^2 \Sigma \big(\overline{\gamma}_1^2 t_1 + \overline{\gamma}_2^2 (1 - t_1) \big)}} - 1 \right). \tag{5.1.2}$$

Moreover, in the proof, we explicitly derive the optimal trading intensity $\widehat{A}(t)$, which satisfies $\widehat{\theta}(t) = \widehat{A}(t)(v - H(t, \widehat{\xi}(t)))$.

Proof of Proposition 5.1.2. In this proof, we use the rational pricing rule to derive $\widehat{A}$ and the constants $h_c(t_1)$ and $c(t_1)$. We then verify in Theorem 5.1.3 that these quantities indeed characterize an equilibrium.

We have that

$$\theta(t) = A\big(t\big)\big(v - H(t, \xi(t))\big), \qquad t \in [0, 1).$$

Hence, we have that

$$dY(t) = \theta(t)\, dt + \sigma\, dB(t) = A(t)\big(v - H(t, \xi(t))\big) dt + \sigma\, dB(t).$$

For each fixed $T < 1$, by the filtering theorem (Theorems 12.1 and 12.2 in (Liptser and Shiryaev, 2001)), the conditional mean and variance

$$\mu(t) := \mathbb{E}[v \mid \mathcal{F}_t^Y], \qquad \Sigma(t) := \mathrm{Var}(v \mid \mathcal{F}_t^Y),$$

satisfy, for $t \in [0, T]$,

$$d\mu(t) = \frac{\Sigma(t) A(t)}{\sigma^2} \Big(dY_t - A(t)\big(\mu(t) - H(t, \xi(t))\big)\, dt \Big), \qquad d\Sigma(t) = -\frac{A(t)^2}{\sigma^2} \Sigma(t)^2 dt,$$

with initial conditions $\mu(0) = \mu$ and $\Sigma(0) = \Sigma$. By our assumption of a rational pricing rule, we have

$$H(t, \xi(t)) = \mathbb{E}[v \mid \mathcal{F}_t^Y] = \mu(t).$$

Therefore,

$$d\mu(t) = \frac{\Sigma(t) A(t)}{\sigma^2}\, dY_t.$$

By assumption, conditionally on $\mathcal{F}_0^Y$ we have $v \sim N(\mu, \Sigma)$, and

$$\mu(t) = \mu + \int\limits_0^t \frac{\Sigma(s)A(s)}{\sigma^2}\, dY_s, \qquad \Sigma(t) = \left(\Sigma^{-1} + \frac{1}{\sigma^2}\int\limits_0^t A(u)^2\, du\right)^{-1}. \tag{5.1.3}$$

We also have that

$$H(t, \xi(t)) = h_c(t_1) + \int\limits_0^t \lambda(s)\, dY_s, \qquad t \in [0, 1).$$

By matching the constant term, we have that

$$h_c(t_1) = \mu.$$

By matching the integrands we obtain

$$\lambda(s) = \frac{\Sigma(s)A(s)}{\sigma^2}, A(s) = \lambda(s)\, \Sigma(s)^{-1}\sigma^2. \tag{5.1.4}$$

Using the explicit expression for $\Sigma(s)$, this can be rewritten as

$$A(s) = \lambda(s)\left(\sigma^2\Sigma^{-1} + \int\limits_0^s A(u)^2\, du\right), \qquad s \in [0, t_1). \tag{5.1.5}$$

After determining $A(s)$ for $s \in [0, t_1)$, we get

$$\Sigma(t_1) = \left(\Sigma^{-1} + \frac{1}{\sigma^2}\int\limits_0^{t_1} A(u)^2 du\right)^{-1}.$$

Then, given $\lambda(t_1)$, for $s \in [t_1, 1)$ we have

$$A(s) = \lambda(s)\left(\sigma^2\Sigma(t_1)^{-1} + \int\limits_{t_1}^s A(u)^2\, du\right), \qquad s \in [t_1, 1). \tag{5.1.6}$$

These equations identify A over $[0, 1)$. We first determine $A(s)$ for $s \in [0, t_1)$.

Differentiating both sides of equation (5.1.5) with respect to s

$$A'(s) = \lambda'(s)\,\frac{A(s)}{\lambda(s)} + \lambda(s)A(s)^2.$$

From condition C1 of Lemma 4.2.1 and (4.2.9) we know $\lambda'(s)/\lambda(s) = -\overline{\gamma}_1\sigma^2\lambda(s)$. Substituting into the previous expression and collecting terms yields

$$\frac{A'(s)}{A(s)} - \frac{A'(s)}{A(s) - \overline{\gamma}_1\sigma^2} = -\overline{\gamma}_1\sigma^2\lambda(s).$$

Integrating this first-order equation from 0 to s and using the explicit form of λ in (4.2.8) gives

$$A(s) = \frac{\overline{\gamma}_1\sigma^2}{1 - K\big(1 - \sigma^2\lambda(t_1)\overline{\gamma}_1^2/\overline{\gamma}_2(t_1 - s)\big)},$$

for some constant K. From equation (4.2.8) , we have

$$A(0) = \lambda(0)\sigma^2\Sigma^{-1} = \frac{\lambda(t_1)\overline{\gamma}_1\sigma^2\Sigma^{-1}}{\overline{\gamma}_2 - \overline{\gamma}_1^2\sigma^2\lambda(t_1)t_1},$$

$$A(0) = \frac{\overline{\gamma}_1\sigma^2}{1 - K\left(1 - \frac{\overline{\gamma}_1^2}{\overline{\gamma}_2}\sigma^2\lambda(t_1)t_1\right)},$$

which yields that

$$K = \frac{\overline{\gamma}_2\left(\lambda(t_1) - \overline{\gamma}_2\Sigma + \lambda(t_1)\Sigma\overline{\gamma}_1^2\sigma^2 t_1\right)}{\lambda(t_1)\left(\overline{\gamma}_2 - \overline{\gamma}_1^2\sigma^2\lambda(t_1)t_1\right)}.$$

We now pin down $A(s)$ for $s \in [t_1, 1)$. Indeed, differentiating both sides of equation (5.1.6), we have

$$A'(s) = \lambda'(s)\,\frac{A(s)}{\lambda(s)} + \lambda(s)A(s)^2.$$

From condition C1 of Lemma 4.1.1 and (4.1.9) we know $\lambda'(s)/\lambda(s) = -\overline{\gamma}_2\sigma^2\lambda(s)$. Substituting into the previous expression and collecting terms yields

$$\frac{A'(s)}{A(s)} - \frac{A'(s)}{A(s) - \overline{\gamma}_2\sigma^2} = -\overline{\gamma}_2\sigma^2\lambda(s).$$

Integrating this first-order equation and using the explicit form of λ in (4.1.7) gives

$$A(s) = \frac{\overline{\gamma}_2\sigma^2}{1 - C\big(1 - \overline{\gamma}_2\sigma^2 c(t_1)(1-s)\big)},$$

for some constant C. To identify C note that the optimality condition in Proposition 4.1.4 requires

$$\begin{aligned}
\lim_{s\uparrow 1}\Sigma(s) = 0 \iff & \lim_{s\uparrow 1}\int_{t_1}^{s} A(u)^2\,du = \infty \\
\iff & \lim_{s\uparrow 1}\big(A(s)\lambda(s)^{-1} - \sigma^2\Sigma(t_1)^{-1}\big) = \infty \\
\iff & C = 1.
\end{aligned}$$

Thus

$$A(s) = \frac{1}{c(t_1)(1-s)}, \qquad s \in [t_1, 1).$$

This identifies θ. In the remainder of the proof we identify $c(t_1)$. Here, the condition in Proposition 4.1.4 also implies that $H(1,\xi(1))$ and v have the same distribution. First, we can write

$$v = \mu + \sqrt{\Sigma}\,Z, \qquad Z \overset{\mathcal{F}_0^Y}{\sim} N(0,1).$$

Next, by Lemma 5.1.1, we have that Y/σ is a $\mathbb{F}^Y$ Brownian motion over $[0,1]$. Therefore, from (3.0.7), we see that

$$\xi(1) \overset{\mathcal{F}_0^Y}{\sim} N\Big(0,\ \sigma^2\int_0^1 \lambda(s)^2\,ds\Big),$$

so that

$$H(1,\xi(1)) = \mu + \xi(1) \sim N\Big(\mu,\ \sigma^2\int_0^1 \lambda(s)^2\,ds\Big).$$

Matching the variances gives

$$\Sigma = \sigma^2\int_0^1 \lambda(s)^2\,ds.$$

Using the explicit form for λ in (4.1.7), we obtain

$$\frac{\Sigma}{\sigma^2} = \int_0^{t_1} \lambda(s)^2\, ds + \int_{t_1}^{1} \lambda(s)^2\, ds$$
$$= \frac{c(t_1)^2(1-t_1)}{1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t_1)}$$
$$+ \frac{\overline{\gamma}_1^2 t_1 c(t_1)^2}{\overline{\gamma}_2\left[1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t_1)\right]\left[\overline{\gamma}_2 - \sigma^2 c(t_1)\left(\overline{\gamma}_1^2 t_1 + \overline{\gamma}_2^2(1-t_1)\right)\right]}.$$

Solving for $c(t_1)$ yields (5.1.2), finishing the result.

□

Theorem 5.1.3. *For $c(t_1)$ in (5.1.2), $\lambda(t_1)$ in (4.2.1) and fix any $t \in [0, t_1)$, define for $s \in [t, 1)$*

$$H(s, \xi) = \mu + \xi.$$

For $s \in [t, t_1)$, let

$$\lambda(s) = \frac{(\overline{\gamma}_1/\overline{\gamma}_2)\lambda(t_1)}{1-(\overline{\gamma}_1^2/\overline{\gamma}_2)\sigma^2\lambda(t_1)(t_1-s)},$$
$$K = \frac{\overline{\gamma}_2\left[\lambda(t_1) - \overline{\gamma}_2\Sigma + \lambda(t_1)\Sigma\overline{\gamma}_1^2\sigma^2 t_1\right]}{\lambda(t_1)\left[\overline{\gamma}_2 - \overline{\gamma}_1^2\sigma^2\lambda(t_1)t_1\right]} = \frac{c(t_1)}{\lambda(t_1)},$$
$$\widehat{\theta}(s) = \frac{\overline{\gamma}_1\sigma^2}{1-K\left[1-(\overline{\gamma}_1^2/\overline{\gamma}_2)\sigma^2\lambda(t_1)(t_1-s)\right]}\big(v - H(s, \widehat{\xi}(s))\big)$$
$$= \frac{\overline{\gamma}_1\overline{\gamma}_2}{c(t_1)\left[\overline{\gamma}_2^2(1-t_1) + \overline{\gamma}_1^2(t_1-s)\right]}\big(v - H(s, \widehat{\xi}(s))\big).$$

For $s \in [t_1, 1)$, let

$$\lambda(s) = \frac{c(t_1)}{1-\overline{\gamma}_2\sigma^2 c(t_1)(1-s)}, \quad \widehat{\theta}(s) = \frac{v - H(s, \widehat{\xi}(s))}{c(t_1)(1-s)},$$

where (with initial condition $\xi(t)$ given)

$$\widehat{\xi}(s) = \xi(t) + \int_t^s \lambda(u) d\widehat{Y}(u) = \xi(t) + \int_t^s \lambda(u)\left(\widehat{\theta}(u)\, du + \sigma\, dB(u)\right).$$

Then $\widehat{p}(s) = H(s, \widehat{\xi}(s))$ is an equilibrium price on $[t, 1]$ and $\widehat{\theta}(s)$ is the optimal (rep-

resentative) insider policy. The value function at time t is

$$\mathbb{E}\left[e^{-\overline{\gamma}_1\int_t^{t_1}(v-\widehat{p}(s))\widehat{\theta}(s)\,ds-\overline{\gamma}_2\int_{t_1}^{1}(v-\widehat{p}(s))\widehat{\theta}(s)\,ds}\middle|\mathcal{F}_t^I\right]=\sqrt{\frac{c(t_1)\,\overline{\gamma}_1}{\overline{\gamma}_2\,\lambda(t)}}\,e^{-\frac{\overline{\gamma}_1}{2\lambda(t)}\left(v-\mu-\xi(t)\right)^2}.$$

Proof of Theorem 5.1.3. Set

$$A(s)\;:=\;\begin{cases}\dfrac{\overline{\gamma}_1\sigma^2}{1-K\big(1-\overline{\gamma}_1\sigma^2\lambda(t_1)\overline{\gamma}_1/\overline{\gamma}_2(t_1-s)\big)}, & s\in[t,t_1),\\[2ex] \dfrac{1}{c(t_1)(1-s)}, & s\in[t_1,1),\end{cases}$$

so that

$$\widehat{\theta}(s)=A(s)\big(v-H(s,\widehat{\xi}(s))\big),\qquad s\in[t,1).$$

We first verify that A satisfies the conditions in Propositions 4.1.6 and 4.2.5. For the second period, let $\tau\in[t_1,1)$ and $T\in(\tau,1)$. Then for all $u\in[\tau,T]$,

$$|A(u)|=\frac{1}{c(t_1)(1-u)}\le\frac{1}{c(t_1)(1-T)}\;:=\;L(T),$$

so the required local bound holds.

Next, for every $\tau\in[t_1,1)$,

$$\begin{aligned}\lambda(\tau)\big(A(\tau)+\sigma^2\overline{\gamma}_2\big)&=\frac{c(t_1)}{1-\overline{\gamma}_2\sigma^2c(t_1)(1-\tau)}\left(\frac{1}{c(t_1)(1-\tau)}+\sigma^2\overline{\gamma}_2\right)\\&=\frac{1}{1-\tau}+\frac{2\overline{\gamma}_2\sigma^2c(t_1)}{1-\overline{\gamma}_2\sigma^2c(t_1)(1-\tau)}.\end{aligned}$$

Therefore, the desired representation holds with

$$r(\tau)\;:=\;\frac{2\overline{\gamma}_2\sigma^2c(t_1)}{1-\overline{\gamma}_2\sigma^2c(t_1)(1-\tau)},\qquad\tau\in[t_1,1).$$

Since $1-\overline{\gamma}_2\sigma^2c(t_1)(1-t_1)>0$, it follows that for all $\tau\in[t_1,1)$,

$$1-\overline{\gamma}_2\sigma^2c(t_1)(1-\tau)\ge1-\overline{\gamma}_2\sigma^2c(t_1)(1-t_1)\;:=\;\delta>0.$$

Hence

$$|r(\tau)|\le\frac{2\overline{\gamma}_2\sigma^2c(t_1)}{\delta},$$

and consequently $r \in L^1([t_1, 1])$. Thus the conditions of Proposition 4.1.6 are satisfied on $[t_1, 1]$.

For the first period, note that the explicit formula (5.1.2) implies $0 < c(t_1) < \overline{\gamma}_2 / \big(\sigma^2(\overline{\gamma}_1^2 t_1 + \overline{\gamma}_2^2(1-t_1))\big)$. Therefore, for $s \in [t, t_1)$,

$$1 - \overline{\gamma}_1 \sigma^2 \lambda(t_1) \overline{\gamma}_1 / \overline{\gamma}_2 \, (t_1 - s) = \frac{\overline{\gamma}_2 - \sigma^2 c(t_1)\big(\overline{\gamma}_1^2(t_1 - s) + \overline{\gamma}_2^2(1-t_1)\big)}{\overline{\gamma}_2\big(1 - \overline{\gamma}_2 \sigma^2 c(t_1)(1-t_1)\big)} > 0.$$

By explicit calculation, we have that $c\,(t_1) < (\overline{\gamma}_2 \sigma^2 \,(1-t_1))^{-1}$, hence the value function is well defined for $t \in [t_1, 1)$. Hence λ is continuous and bounded on $[t, t_1]$. Moreover, the first-period coefficient A is continuous on $[t, t_1]$, and therefore bounded there. Thus the conditions of Proposition 4.2.5 are satisfied on $[t, t_1)$.

Next, by Proposition 5.1.2, the pricing rule is rational, i.e. $\widehat{p}(s) = H(s, \widehat{\xi}(s)) = \mathbb{E}\left[v \,\middle|\, \mathcal{F}_s^{\widehat{Y}} \right], s \in [t, 1]$.

Since the conditions of Propositions 4.2.5 and 4.1.6 have been verified, the restriction of $\widehat{\theta}$ to $[t, t_1)$ is optimal on $[t, t_1)$, and the restriction of $\widehat{\theta}$ to $[t_1, 1]$ is optimal on $[t_1, 1]$. Therefore, by Proposition 4.3.1, the concatenated strategy $\widehat{\theta}$ is optimal for the insider's control problem on $[t, 1]$, and the value function at time t is

$$\mathbb{E}\left[e^{-\overline{\gamma}_1 \int_t^{t_1} (v - \widehat{p}(s))\widehat{\theta}(s)\,ds - \overline{\gamma}_2 \int_{t_1}^1 (v-\widehat{p}(s))\widehat{\theta}(s)\,ds} \,\middle|\, \mathcal{F}_t^I \right] = \sqrt{\frac{c(t_1)\,\overline{\gamma}_1}{\overline{\gamma}_2\,\lambda(t)}}\, e^{-\frac{\overline{\gamma}_1}{2\lambda(t)}\left(v - \mu - \xi(t)\right)^2}.$$

Thus $\widehat{p}(s) = H(s, \widehat{\xi}(s))$ is an equilibrium price on $[t, 1]$ and $\widehat{\theta}(s)$ is the optimal (representative) insider policy. In particular, taking $t = 0$ yields the equilibrium on $[0, 1]$.

We next show that show that $K = c(t_1)/\lambda(t_1)$. Using the definition of K and substituting for $\lambda(t_1)$ gives

$$K = \frac{\overline{\gamma}_2\big(\lambda(t_1) - \overline{\gamma}_2 \Sigma + \lambda(t_1)\Sigma \overline{\gamma}_1^2 \sigma^2 t_1\big)}{\lambda(t_1)\big(\overline{\gamma}_2 - \overline{\gamma}_1^2 \sigma^2 \lambda(t_1) t_1\big)} = \frac{\overline{\gamma}_2 d\big(c(t_1) - \overline{\gamma}_2 \Sigma d + c(t_1)\Sigma \overline{\gamma}_1^2 \sigma^2 t_1\big)}{c(t_1)\big(\overline{\gamma}_2 d - \overline{\gamma}_1^2 \sigma^2 c(t_1) t_1\big)},$$

where $d = 1 - \overline{\gamma}_2 \sigma^2 c(t_1)(1-t_1) = c(t_1)/\lambda(t_1)$.

Let $q = \overline{\gamma}_1^2 t_1 + \overline{\gamma}_2^2(1-t_1)$, we have that $\overline{\gamma}_2 d - \overline{\gamma}_1^2 \sigma^2 c(t_1) t_1 = \overline{\gamma}_2 - \sigma^2 c(t_1) q$, and $c(t_1) - \overline{\gamma}_2 \Sigma d + c(t_1)\Sigma \overline{\gamma}_1^2 \sigma^2 t_1 = c(t_1) - \overline{\gamma}_2 \Sigma + \sigma^2 c(t_1)\Sigma q$. Since $c(t_1)$ satisfies $\sigma^2 q\, c(t_1)^2 + \sigma^2 q\, \overline{\gamma}_2 \Sigma\, c(t_1) - \overline{\gamma}_2^2 \Sigma = 0$, we have $c(t_1) - \overline{\gamma}_2 \Sigma + \sigma^2 c(t_1)\Sigma q = \frac{c(t_1)}{\overline{\gamma}_2}\big(\overline{\gamma}_2 - \sigma^2 c(t_1) q\big)$. Substituting this into the formula for K yields $K = d = c(t_1)/\lambda(t_1)$.

Substitute $K = c(t_1)/\lambda(t_1)$ into $A(s)$ and use $d = c(t_1)/\lambda(t_1)$ gives that

$$A(s) = \frac{\overline{\gamma}_1 \overline{\gamma}_2}{c(t_1)\big(\overline{\gamma}_2^2(1-t_1) + \overline{\gamma}_1^2(t_1 - s)\big)}$$

for $s \in [0, t_1)$. $\square$

5.2 Extension to Multiple Insiders

In this section, we extend the model by allowing multiple agents with heterogeneous risk-aversion parameters to enter the market sequentially at different times.

Theorem 5.1.3 yields three useful insights. First, all equilibrium objects are pinned down by a single constant $c(t_1)$, which is determined by imposing the pricing rule is rational. Second, the equilibrium depends on a weighted average of the effective risk-aversion parameters, where the weights are the lengths of the intervals between arrival dates, namely $\overline{\gamma}_1^2 t_1 + \overline{\gamma}_2^2(1-t_1)$. Third, the market impact parameter λ and the insider's value function share the same functional form, with parameters determined by the relevant effective risk aversion and time horizon. These observations motivate the extension to n sequentially arriving agents, and further to a continuum of agents arriving continuously over time.

We start with the case of n agents with risk-aversion parameters γ_i who enter the market sequentially at times $t_0 = 0 < t_1 < \cdots < t_{n-1}$, and we set $t_n = 1$. Let

$$\overline{\gamma}_i := \left(\sum_{j=1}^{i} \frac{1}{\gamma_j} \right)^{-1}$$

denote the representative agent's risk-aversion parameter. To define an equilibrium in this setting, we modify condition (ii) in Definition 3.0.2 as follows:

$$\inf_{\theta \in \mathcal{A}_0} \mathbb{E}\left[e^{-\sum_{i=1}^{n} \overline{\gamma}_i \int_{t_{i-1}}^{t_i} \left(v - \widehat{p}(u)\right) \theta(u)\, du} \middle| v \right]. \tag{5.2.1}$$

We have the following proposition.

Proposition 5.2.1. *Define the constant*

$$c(t_{n-1}) = c_{n-1} = \frac{\Sigma \overline{\gamma}_n}{2} \left(\sqrt{1 + \frac{4}{\sigma^2 \Sigma \sum_{k=1}^{n} \overline{\gamma}_k^2 \left(t_k - t_{k-1}\right)}} - 1 \right). \tag{5.2.2}$$

Define the pricing rule by

$$H(t,\xi) = \mu + \xi, \qquad t \in [0,1].$$

For each interval $[t_i, t_{i+1})$, $0 \le i \le n-1$, *define*

$$\lambda(t) = \frac{c(t_i)}{1 - \overline{\gamma}_{i+1}\sigma^2\, c(t_i)\,(t_{i+1} - t)}, \qquad t_i \le t < t_{i+1}. \tag{5.2.3}$$

The coefficients $\{c(t_i)\}_{i=0}^{n-2}$ *satisfy the recursion*

$$c(t_i) = \lambda(t_{i+1})\,\frac{\overline{\gamma}_{i+1}}{\overline{\gamma}_{i+2}}, \qquad 0 \le i \le n-2. \tag{5.2.4}$$

The corresponding optimal strategy is

$$\widehat{\theta}(t) = A(t)\big(v - H(t, \widehat{\xi}(t))\big), \tag{5.2.5}$$

where, for $t \in [t_i, t_{i+1})$ *and* $0 \le i \le n-2$,

$$A(t) = \frac{\overline{\gamma}_{i+1}\sigma^2}{1 - B_i\left(1 - \overline{\gamma}_{i+1}\sigma^2\, c(t_i)\,(t_{i+1} - t)\right)}, \tag{5.2.6}$$

with

$$B_i = \frac{1 - \frac{\overline{\gamma}_{i+1}\Sigma(t_i)}{\lambda(t_i)}}{1 - \overline{\gamma}_{i+1}\sigma^2\, c(t_i)\,(t_{i+1} - t_i)}, \qquad 0 \le i \le n-2, \tag{5.2.7}$$

and, on the last interval,

$$A(t) = \frac{1}{c(t_{n-1})(1-t)}, \qquad t \in [t_{n-1}, 1).$$

Moreover, for $1 \le i \le n-1$, $\Sigma(t_i)$ *evolves according to*

$$\Sigma\left(t_i\right) = \Sigma\left(t_{i-1}\right) - \frac{\sigma^2 c\left(t_{i-1}\right)^2\left(t_i - t_{i-1}\right)}{1 - \overline{\gamma}_i\sigma^2 c\left(t_{i-1}\right)\left(t_i - t_{i-1}\right)} \tag{5.2.8}$$

with initial condition $\Sigma(t_0) = \Sigma$.

Here

$$\widehat{\xi}(t) = \int_0^t \lambda(u)\, d\widehat{Y}(u) = \int_0^t \lambda(u) \left(\widehat{\theta}(u)\, du + \sigma\, dB(u)\right).$$

Then $\widehat{p}(t) = H(t, \widehat{\xi}(t))$ *is an equilibrium price, and* $\widehat{\theta}(t)$ *is the optimal (representative) insider policy.*

Finally, the value function is given by

$$J(t,\xi) = \sqrt{\frac{c(t_{n-1})\overline{\gamma}_{i+1}}{\overline{\gamma}_n\, \lambda(t)}}\, e^{-\frac{\overline{\gamma}_{i+1}}{2\lambda(t)}(v-\mu-\xi)^2}, \qquad t_i \le t < t_{i+1}, \quad 0 \le i \le n-1. \tag{5.2.9}$$

Proof. **Step 1.** Assume a Markovian control θ and derive the HJB equation. Let J denote the value function on each interval. For $t \in [t_i, t_{i+1})$, this leads to the following system of equations:

$$\begin{aligned}
0 &= \lambda(t) J_\xi(t,\xi) - \overline{\gamma}_{i+1}\big(v - H(t,\xi)\big) J(t,\xi), \\
0 &= J_t(t,\xi) + \frac{1}{2}\sigma^2 \lambda(t)^2 J_{\xi\xi}(t,\xi), \\
0 &= H_t(t,\xi) + \frac{1}{2}\sigma^2 \lambda(t)^2 H_{\xi\xi}(t,\xi) \\
&\quad - \lambda(t)\big(v - H(t,\xi)\big) \left[\frac{\partial}{\partial t}\left(\frac{1}{\lambda(t)}\right) - \overline{\gamma}_{i+1}\sigma^2 H_\xi(t,\xi)\right].
\end{aligned} \tag{5.2.10}$$

Following a similar analysis as in Section 2, we obtain

$$\frac{\partial}{\partial t}\Big(\frac{1}{\lambda(t)}\Big) - \overline{\gamma}_{i+1}\sigma^2 \frac{\partial}{\partial \xi} H(t,\xi) = 0.$$

Solving the resulting ODE yields (5.2.3). Solving the system of PDEs and imposing continuity of J at t_{i+1} then give (5.2.4) and (5.2.9).

Moreover, (5.2.4) implies that, for $0 \le i \le n-1$,

$$c\,(t_i) = \frac{\overline{\gamma}_{i+1} c_{n-1}}{\overline{\gamma}_n - \sigma^2 c_{n-1} \sum_{k=i+2}^{n} \overline{\gamma}_k^2\,(t_k - t_{k-1})},$$

where the sum is understood to be 0 when $i = n-1$. Finally, Since

$$H(1, \xi(1)) = \mu + \xi(1) \sim N\left(\mu, \sigma^2 \int_0^1 \lambda(s)^2\, ds\right).$$

Matching variances gives

$$\Sigma = \sigma^2 \int_0^1 \lambda(s)^2 \, ds.$$

Using the identity

$$\frac{\Sigma}{\sigma^2} = \int_0^1 \lambda(t)^2 \, dt,$$

we obtain

$$\frac{\Sigma}{\sigma^2} = \frac{c_{n-1}^2 \left(\sum_{k=1}^n \overline{\gamma}_k^2 (t_k - t_{k-1})\right)}{\overline{\gamma}_n \left[\overline{\gamma}_n - \sigma^2 c_{n-1} \left(\sum_{k=1}^n \overline{\gamma}_k^2 (t_k - t_{k-1})\right)\right]}.$$

Solving this equation for c_{n-1} yields (5.2.2).

Step 2. For each period $[t_i, t_{i+1})$, following (5.1.5)–(5.1.6) and given $\lambda(t_i)$, we have

$$A(s) = \lambda(s) \left(\sigma^2 \Sigma(t_i)^{-1} + \int_{t_i}^{s} A(u)^2 \, du \right), \qquad s \in [t_i, t_{i+1}). \tag{5.2.11}$$

Here

$$\begin{aligned} \Sigma(t_i) &= \left(\Sigma^{-1} + \frac{1}{\sigma^2} \int_0^{t_i} A(u)^2 \, du \right)^{-1} \\ &= \left(\Sigma(t_{i-1})^{-1} + \frac{1}{\sigma^2} \int_{t_{i-1}}^{t_i} A(u)^2 \, du \right)^{-1}. \end{aligned} \tag{5.2.12}$$

Differentiating (5.2.11) with respect to s yields

$$A'(s) = \lambda'(s) \frac{A(s)}{\lambda(s)} + \lambda(s) A(s)^2.$$

Since $\lambda'(s)/\lambda(s) = -\overline{\gamma}_{i+1} \sigma^2 \lambda(s)$ on $[t_i, t_{i+1})$, substituting into the previous display and collecting terms give

$$\frac{A'(s)}{A(s)} - \frac{A'(s)}{A(s) - \overline{\gamma}_{i+1} \sigma^2} = -\overline{\gamma}_{i+1} \sigma^2 \lambda(s).$$

Solving this ODE yields (5.2.6). Evaluating (5.2.11) at $s = t_i$ gives

$$A(t_i) = \lambda(t_i)\sigma^2\Sigma(t_i)^{-1},$$

which in turn implies (5.2.7). Finally, applying (5.2.6) over $[t_i, t_{i+1})$ in (5.2.12) yields (5.2.8).

Step 3. We verify that the quantities established above indeed construct an equilibrium. It therefore suffices to show that the function A on the last period satisfies the condition in Proposition 4.1.6, while A on the earlier period satisfies the condition in Proposition 4.2.5. Since the argument is very similar to the proof of Theorem 5.1.3, we omit it here. □

5.3 Extension to a Flow of Insiders

We now generalize the model by allowing a continuum (flow) of informed traders to arrive in the market at each instant t. We model this flow as a time-varying (representative) risk-aversion parameter $\overline{\gamma}(t)$, assumed to be continuous and positive on $[0, 1]$ and differentiable on $(0, 1)$. In this setting, condition (ii) in the definition of equilibrium becomes

$$\inf_{\theta \in \mathcal{A}_0} \mathbb{E}\left[e^{-\int_0^1 \overline{\gamma}(u)\left(v - \widehat{p}(u)\right)\theta(u)\,du} \middle| v \right]. \tag{5.3.1}$$

We have the following proposition.

Proposition 5.3.1. *Let $\overline{\gamma} \in \mathcal{C}^1[0, 1]$. Define the constant*

$$c(1) = \frac{\Sigma\overline{\gamma}(1)}{2}\left(-1 + \sqrt{1 + \frac{4}{\Sigma\sigma^2 \int_0^1 \overline{\gamma}(u)^2\,du}} \right). \tag{5.3.2}$$

Let

$$H(t, \xi) = \mu + \xi, \qquad t \in [0, 1].$$

Define

$$\lambda(t) = \frac{\overline{\gamma}(t)c(1)}{\overline{\gamma}(1) - \sigma^2 c(1) \int_t^1 \overline{\gamma}(u)^2\, du}, \qquad t \in [0,1). \tag{5.3.3}$$

Set

$$\widehat{\theta}(t) = \frac{\overline{\gamma}(t)\overline{\gamma}(1)}{c(1)\int_t^1 \overline{\gamma}(u)^2 du}\big(v - H(t, \widehat{\xi}(t))\big), \qquad t \in [0,1), \tag{5.3.4}$$

where

$$\widehat{\xi}(t) = \int_0^t \lambda(u)\, d\widehat{Y}(u) = \int_0^t \lambda(u)\big(\widehat{\theta}(u)\, du + \sigma\, dB(u)\big).$$

Then $\widehat{p}(t) = H(t, \widehat{\xi}(t))$ *is an equilibrium price, and* $\widehat{\theta}(t)$ *is the optimal (representative) insider policy. Moreover, the value function is given by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^1 \overline{\gamma}(u)(v-\widehat{p}(u))\widehat{\theta}(u)\, du}\,\middle|\,\mathcal{F}_t^I\right] \\
&= \sqrt{\frac{c(1)\overline{\gamma}(t)}{\overline{\gamma}(1)\lambda(t)}}\, e^{-\frac{\overline{\gamma}(t)}{2\lambda(t)}\left(v-\mu-\widehat{\xi}(t)\right)^2} \\
&= \sqrt{\frac{\overline{\gamma}(1) - \sigma^2 c(1)\int_t^1 \overline{\gamma}(u)^2\, du}{\overline{\gamma}(1)}}\, e^{-\frac{\overline{\gamma}(1)-\sigma^2 c(1)\int_t^1 \overline{\gamma}(u)^2\, du}{2c(1)}\left(v-\mu-\widehat{\xi}(t)\right)^2}.
\end{aligned} \tag{5.3.5}$$

Proof. **Step 1.** Assume a Markovian control θ and derive the HJB equation. Let J denote the value function. For $t \in [0,1)$, we obtain the system

$$\begin{aligned}
0 &= \lambda(t)J_\xi(t,\xi) - \overline{\gamma}(t)\big(v - H(t,\xi)\big)J(t,\xi), \\
0 &= J_t(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2 J_{\xi\xi}(t,\xi), \\
0 &= H_t(t,\xi) + \frac{1}{2}\sigma^2\lambda(t)^2 H_{\xi\xi}(t,\xi) \\
&\quad + \big(v - H(t,\xi)\big)\left[\sigma^2\lambda(t)\overline{\gamma}(t)H_\xi(t,\xi) - \left(\frac{\overline{\gamma}'(t)}{\overline{\gamma}(t)} + \frac{\left(\frac{1}{\lambda(t)}\right)'}{\frac{1}{\lambda(t)}}\right)\right].
\end{aligned} \tag{5.3.6}$$

Let $H(t,\xi) = \mu + \xi$, following the similar analysis we have that

$$\left(\frac{\overline{\gamma}(t)}{\lambda(t)}\right)' = \sigma^2 \overline{\gamma}(t)^2, \tag{5.3.7}$$

which is equivalent to

$$\sigma^2 \lambda(t)\overline{\gamma}(t) - \left(\frac{\overline{\gamma}'(t)}{\overline{\gamma}(t)} + \frac{\left(\frac{1}{\lambda(t)}\right)'}{\frac{1}{\lambda(t)}}\right) = 0.$$

Integrating (5.3.7) from t to 1 and using $\lambda(1^-) = c(1)$, we obtain $\overline{\gamma}(t)/\lambda(t) = \overline{\gamma}(1)/c(1) - \sigma^2 \int_t^1 \overline{\gamma}(u)^2\,du$, which yields $\lambda(t) = \overline{\gamma}(t)c(1)/(\overline{\gamma}(1) - \sigma^2 c(1)\int_t^1 \overline{\gamma}(u)^2\,du)$. This proves (5.3.3).

Since

$$H(1,\xi(1)) = \mu + \xi(1) \sim N\left(\mu, \sigma^2 \int_0^1 \lambda(s)^2 ds\right),$$

matching the variances gives $\Sigma = \sigma^2 \int_0^1 \lambda(s)^2 ds$. Let $Q(t) := \overline{\gamma}(1) - \sigma^2 c(1)\int_t^1 \overline{\gamma}(u)^2\,du$. Then $Q'(t) = \sigma^2 c(1)\overline{\gamma}(t)^2$, and hence

$$\begin{aligned}\int_0^1 \lambda(t)^2\,dt &= c(1)^2 \int_0^1 \frac{\overline{\gamma}(t)^2}{Q(t)^2}\,dt = \frac{c(1)}{\sigma^2}\int_0^1 \frac{Q'(t)}{Q(t)^2}\,dt \\ &= \frac{c(1)}{\sigma^2}\left(\frac{1}{Q(0)} - \frac{1}{Q(1)}\right) = \frac{c(1)^2 \int_0^1 \overline{\gamma}(u)^2\,du}{\overline{\gamma}(1)\left(\overline{\gamma}(1) - \sigma^2 c(1)\int_0^1 \overline{\gamma}(u)^2\,du\right)}.\end{aligned}$$

Equating this with Σ/σ^2 and solving the resulting quadratic equation for $c(1)$ gives (5.3.2).

Solving the systems of PDE gives (5.3.5).

Step 2. Following (5.1.5), and given λ, we have

$$A(s) = \lambda(s)\left(\sigma^2\Sigma^{-1} + \int_0^s A(u)^2\,du\right), \qquad s \in [0,1). \tag{5.3.8}$$

Define

$$\Sigma(t) = \left(\Sigma^{-1} + \frac{1}{\sigma^2} \int_0^t A(u)^2 \, du \right)^{-1}. \tag{5.3.9}$$

Differentiating (5.3.8) with respect to s yields

$$A'(s) = \lambda'(s) \frac{A(s)}{\lambda(s)} + \lambda(s) A(s)^2.$$

From (5.3.7), we have $\lambda'(t)/\lambda(t) = \overline{\gamma}'(t)/\overline{\gamma}(t) - \sigma^2 \lambda(t) \overline{\gamma}(t)$. Substituting this into the previous equation gives

$$A'(t) = \frac{\overline{\gamma}'(t)}{\overline{\gamma}(t)} A(t) - \sigma^2 \lambda(t) \overline{\gamma}(t) A(t) + \lambda(t) A(t)^2.$$

Now define $\widetilde{A}(t) = A(t)/\overline{\gamma}(t)$. Then

$$\widetilde{A}'(t) = \frac{A'(t)\overline{\gamma}(t) - A(t)\overline{\gamma}'(t)}{\overline{\gamma}(t)^2} = \lambda(t)\overline{\gamma}(t) \, \widetilde{A}(t) \big(\widetilde{A}(t) - \sigma^2 \big).$$

Solving this ODE yields

$$\widetilde{A}(t) = \frac{\sigma^2}{1 - C e^{\sigma^2 \int_0^t \lambda(s) \overline{\gamma}(s) \, ds}}$$

for some constant C. Since

$$A(0) = \lambda(0) \sigma^2 \Sigma^{-1},$$

we have

$$\widetilde{A}(0) = \frac{A(0)}{\overline{\gamma}(0)} = \frac{\lambda(0)\sigma^2}{\Sigma \overline{\gamma}(0)},$$

which implies

$$C = 1 - \frac{\Sigma \overline{\gamma}(0)}{\lambda(0)}.$$

Therefore

$$A(t) = \frac{\sigma^2 \overline{\gamma}(t)}{1 - \left(1 - \frac{\Sigma \overline{\gamma}(0)}{\lambda(0)} \right) e^{\sigma^2 \int_0^t \lambda(s) \overline{\gamma}(s) ds}}.$$

We have that

$$\int_0^t \lambda(u)\bar{\gamma}(u)du = \frac{1}{\sigma^2}\log\frac{\bar{\gamma}(1)-\sigma^2 c(1)\int_t^1 \bar{\gamma}(u)^2 du}{\bar{\gamma}(1)-\sigma^2 c(1)\int_0^1 \bar{\gamma}(u)^2 du},$$

which yields

$$A(t) = \frac{\overline{\gamma}(t)\overline{\gamma}(1)}{c(1)\int_t^1 \overline{\gamma}(u)^2 du}.$$

Hence

$$\widehat{\theta}(t) = A(t)\big(v - H(t,\widehat{\xi}(t))\big)$$

is given by (5.3.4).

Step 3. We verify that the quantities established above indeed construct an equilibrium. It therefore suffices to show that the function A on the last period satisfies the condition in Proposition 4.1.6. Since the argument is very similar to the proof of Theorem 5.1.3, we omit it here.

$\square$

Chapter 6

Properties of Equilibrium

6.1 Properties of Equilibrium: $t_1 = 0$

We first consider a special case where there are N insiders entering the market at time 0. This setup is consistent with the discrete-time model in (Holden and Subrahmanyam, 1994), where there are N insiders trade strategically. In our case, we allow each insider to have different risk aversions. Our main purpose in studying this special case is twofold. First, we compare our setting with the results in (Holden and Subrahmanyam, 1994) and (Back et al., 2000), showing that an equilibrium exists when multiple risk-averse insiders are present in the market. Second, we examine how the arrival of additional insiders affects market depth and price informativeness, allowing us to compare our findings with some of the results in (Subrahmanyam, 1991).

From the observation in equation (3.0.6), the problem with N agents with different risk aversion parameters can be treated as an equivalent single-representative-agent problem. Therefore, the results in Theorem 5.1.3 can be applied directly to our analysis here. Let each agent with risk aversion parameter $\gamma_i, i = 1, .., N$ and let $\overline{\gamma} = 1/\left(\sum_{i=1}^{N} 1/\gamma_i\right)$ be the risk aversion parameter for the representative agent.

Definition 6.1.1 (Definition of Auxiliary quantities)**.** *We begin by defining the variables and functions that will be used consistently throughout this section, both in the statements of the propositions and in their proofs.*

1. *Recall that*
$$\overline{\gamma}_1 = \gamma_1, \qquad \overline{\gamma}_2 = \frac{1}{1/\overline{\gamma}_1 + 1/\gamma_2}.$$
Then $\overline{\gamma}_1 > \overline{\gamma}_2$.

2. *Let*
$$\kappa = \frac{4}{\sigma^2\Sigma}, \qquad M = \frac{(v-\mu)^2}{\Sigma}.$$

3. *Let*
$$q(t_1) = \overline{\gamma}_1^2 t_1 + \overline{\gamma}_2^2(1-t_1) = \overline{\gamma}_1^2 - (\overline{\gamma}_1^2 - \overline{\gamma}_2^2)(1-t_1).$$

4. *Define the function* $s : (0,\infty) \to \mathbb{R}$ *by*
$$s(x) = \sqrt{1 + \frac{\kappa}{x}}.$$
Then s *is strictly decreasing in* x. *Moreover, the function*
$$x \mapsto x\big(s(x) - 1\big)$$
is strictly increasing in x.

5. *Define the function* $r : (1,\infty) \to \mathbb{R}$ *by*
$$r(x) = \sqrt{\frac{x-1}{x+1}}\, e^{-M/(x+1)}.$$
Then r *is strictly increasing in* x.

The following result is an immediate consequence of Proposition 4.1.6 and Theorem 5.1.3 in the special case $t_1 = 0$, we hence omit the proof.

Proposition 6.1.2. *Let*
$$c = \frac{\Sigma\overline{\gamma}}{2}\left(\sqrt{1 + \frac{4}{\sigma^2\Sigma\overline{\gamma}^2}} - 1\right).$$
Define for each $t \in [0,1)$
$$H(t,\xi) = \mu + \xi, \qquad \lambda(t) = \frac{c}{1 - \overline{\gamma}\sigma^2 c(1-t)}, \qquad \widehat{\theta}(t) = \frac{v - H(t, \widehat{\xi}(t))}{c(1-t)},$$

where

$$\widehat{\xi}(t) = \int_0^t \lambda(u) d\widehat{Y}(u) = \int_0^t \lambda(u)\big(\widehat{\theta}(u)du + \sigma B(u)\big).$$

Then $\widehat{p}(t) = H(t, \widehat{\xi}(t))$ *is an equilibrium price on* $[0,1]$ *and* $\widehat{\theta}(t)$ *is the optimal (representative) insider policy. The* i*th insider's optimal trading intensity is*

$$\widehat{\theta^i}(t) = \frac{\overline{\gamma}}{\gamma_i}\widehat{\theta}(t), \qquad 0 \le t < 1.$$

Moreover, each insider's conditional expected utility at time 0 *(We also call it welfare) is*

$$\mathcal{J}(0) = -\sqrt{1-\overline{\gamma}\sigma^2 c}\, e^{-\frac{\overline{\gamma}}{2\lambda(0)}(v-\mu)^2},$$

and the posterior variance is

$$\Sigma(t) = \left(\Sigma^{-1} + \frac{1}{\sigma^2 c^2}\left(\frac{1}{1-t} - 1\right)\right)^{-1} = \frac{\Sigma\,\sigma^2 c^2(1-t)}{\sigma^2 c^2(1-t) + \Sigma t}, \qquad 0 \le t < 1.$$

Proposition 6.1.3. *To stress the dependence upon the risk aversion, we write* $c(\overline{\gamma}) = c$*. Following the equilibrium quantities defined in Proposition 6.1.2, we have the following properties of the equilibrium:*

1. *The welfare* $\mathcal{J}(0;\overline{\gamma})$ *is strictly increasing in* $\overline{\gamma}$*.*

2. *Define the market impact coefficient at time* 0 *to be*

$$\lambda(0;\overline{\gamma}) = \frac{c(\overline{\gamma})}{(1-\overline{\gamma}\sigma^2 c(\overline{\gamma})}, \tag{6.1.1}$$

   then $\lambda(0)$ *is strictly increasing in* $\overline{\gamma}$*.*

3. *The market impact coefficient at time* 1*,* $\lambda(1;\overline{\gamma}) = c(\overline{\gamma})$*, is strictly decreasing* $\overline{\gamma}$*.*

4. *For each fixed* $t \in (0,1)$*, the posterior variance* $\Sigma(t;\overline{\gamma})$ *is strictly decreasing in* $\overline{\gamma}$*. Equivalently, the price informativeness* $\Pi(t;\overline{\gamma}) = \Sigma(t;\overline{\gamma})^{-1}$ *is strictly increasing in* $\overline{\gamma}$*.*

5. *The trading intensity of the representative agent,* $\beta(t;\overline{\gamma}) = 1/(c(\overline{\gamma})(1-t))$*, is strictly increasing in* $\overline{\gamma}$*. The trading intensity of each individual,* $\beta_i(t;\overline{\gamma}) = \overline{\gamma}/\gamma_i\beta(t;\overline{\gamma})$*, is strictly increasing in* $\overline{\gamma}$ *for any fixed* $t \in [0,1)$ *and* $\gamma_i > 0$*.*

Proof of Proposition 6.1.3. Recall the function and variable definitions in Definition 6.1.1.

First notice that
$$c(\overline{\gamma}) = \frac{\Sigma\overline{\gamma}}{2}\big(s(\overline{\gamma}^2) - 1\big).$$
Then
$$\overline{\gamma}\sigma^2 c(\overline{\gamma}) = \frac{\sigma^2\Sigma\overline{\gamma}^2}{2}\big(s(\overline{\gamma}^2) - 1\big) = \frac{2}{s(\overline{\gamma}^2)+1}.$$
Hence
$$1 - \overline{\gamma}\sigma^2 c(\overline{\gamma}) = \frac{s(\overline{\gamma}^2)-1}{s(\overline{\gamma}^2)+1}.$$

Therefore,
$$\lambda(0;\overline{\gamma}) = \frac{c(\overline{\gamma})}{1-\overline{\gamma}\sigma^2 c(\overline{\gamma})} = \frac{\Sigma\overline{\gamma}}{2}\big(s(\overline{\gamma}^2)+1\big),$$
and
$$\mathcal{J}(0;\overline{\gamma}) = -\sqrt{1-\overline{\gamma}\sigma^2 c(\overline{\gamma})}\, e^{-\frac{\overline{\gamma}}{2\lambda(0;\overline{\gamma})}(v-\mu)^2} = -r\big(s(\overline{\gamma}^2)\big).$$

(1) Since $s(x)$ is strictly decreasing in x, the map $\overline{\gamma} \mapsto s(\overline{\gamma}^2)$ is strictly decreasing in $\overline{\gamma}$. Moreover, $r(x)$ is strictly increasing in x. Therefore $r(s(\overline{\gamma}^2))$ is strictly decreasing in $\overline{\gamma}$, and hence
$$\mathcal{J}(0;\overline{\gamma}) = -r(s(\overline{\gamma}^2))$$
is strictly increasing in $\overline{\gamma}$.

(2) Taking the derivative with respect to $\overline{\gamma}$, we obtain
$$\begin{aligned}
\partial_{\overline{\gamma}}\lambda(0;\overline{\gamma}) &= \frac{\Sigma}{2}\big((s(\overline{\gamma}^2)+1) + \overline{\gamma}\partial_{\overline{\gamma}} s(\overline{\gamma}^2)\big)\\
&= \frac{\Sigma}{2}\left((s(\overline{\gamma}^2)+1) - \frac{s(\overline{\gamma}^2)^2-1}{s(\overline{\gamma}^2)}\right)\\
&= \frac{\Sigma}{2}\left(1 + \frac{1}{s(\overline{\gamma}^2)}\right) > 0.
\end{aligned}$$
Hence $\lambda(0;\overline{\gamma})$ is strictly increasing in $\overline{\gamma}$.

(3) Observe that
$$c(\overline{\gamma}) = \sqrt{\frac{\Sigma^2\overline{\gamma}^2}{4} + \frac{\Sigma}{\sigma^2}} - \frac{\Sigma\overline{\gamma}}{2}.$$
Hence $c(\overline{\gamma})$ is strictly decreasing in $\overline{\gamma}$. Therefore $\lambda(1;\overline{\gamma}) = c(\overline{\gamma})$ is strictly decreasing in $\overline{\gamma}$.

(4)
$$\Sigma(t;\overline{\gamma}) = \left(\Sigma^{-1} + \frac{1}{\sigma^2 c(\overline{\gamma})^2}\left(\frac{1}{1-t} - 1\right)\right)^{-1}.$$
Since $c(\overline{\gamma})$ is strictly decreasing in $\overline{\gamma}$, the term $1/c(\overline{\gamma})^2$ is strictly increasing in $\overline{\gamma}$. Therefore the quantity inside the parentheses is strictly increasing in $\overline{\gamma}$, and hence $\Sigma(t;\overline{\gamma})$ is strictly decreasing in $\overline{\gamma}$.

(5) The result follows from the fact that $c(\overline{\gamma})$ is strictly decreasing in $\overline{\gamma}$. □

It should be noted in proposition 6.1.2 that in our model there is an equilibrium even if multiple informed traders observe the same (perfect) signal. This is in contrast to the case in (Back et al., 2000), where under risk-neutral conditions and with perfect information, due to competition between insiders for the same information, no equilibrium exists.

In the risk-neutral model, insiders choose their trading strategies to maximize expected profit by balancing trading intensity against price impact. More aggressive trading allows an insider to profit more from her informational advantage.[1] However, aggressive trading also increases price impact, causing the insider to trade at less favorable prices. When there are multiple risk-neutral insiders, each has an incentive to trade more aggressively to exploit informational advantage before the others push the price toward v. This competition leads insiders to concentrate their trading at the beginning of the trading period. In continuous time, the result is that insiders would like to trade infinitely aggressively at the beginning, which causes equilibrium to fail to exist.

When insiders are risk averse, the tradeoff is different. In addition to balancing trading intensity against price impact, they also care about future price risk. This gives each insider an incentive to trade more aggressively at every point in time rather than postpone trading to a later date, when the remaining order would be exposed

[1] The informational advantage is captured by the mispricing $v - p(t)$. Because the insider observes v, this difference measures the profit opportunity associated with each trade she executes.

to future price risk.[2] When multiple risk-averse insiders are present, they will trade in a way to reduce the future price risk together. As a result, the informed traders as a whole behaves like a single representative insider with higher aggregate risk tolerance, or equivalently lower effective risk aversion. In equilibrium, each insider's position is inversely related to her degree of risk aversion, reflecting the risk-sharing effect.[3] This reduces the need for aggressive trading as shown in Proposition 6.1.3 (5), thereby supporting the existence of equilibrium.

Proposition 6.1.3 (2) shows that $\lambda(0)$ decreases as more insiders enter the market. Since $\lambda(t) = \Sigma(t)\beta(t)/\sigma^2, t > 0$ by equation (5.1.4), this decline at $t = 0$ is driven by the fall in $\beta(0)$, by the fact that insiders trade less aggressively at the beginning of the trading period. As a result, less information is revealed early on, which leads to a higher $\Sigma(t)$ for $t > 0$. This is supported by Proposition 6.1.3 (4), which shows that $\Sigma(t)$ is higher later in the trading period when more insiders are present. Consequently, $\lambda(1)$ increases even though $\beta(1)$ decreases with the number of insiders. Therefore, market depth initially rises and then falls over the trading period. The behavior of λ in our model differs from that in (Subrahmanyam, 1991). In the one-period model studied there, λ is not monotonic in the number of insiders: it may first increase and then decrease as more insiders enter the market. By contrast, in our continuous-time setting, for each fixed time $t, \lambda(t)$ changes monotonically with the number of insiders. This difference arises because trading behavior is fundamentally different in one-period and continuous-time models. In a one-period model, insiders must concentrate all of their trading in a single round, whereas in a continuous-time model they can spread their trades over time. As a result, the effect of additional

[2] This is reflected in Proposition 6.1.3(5), which shows that trading intensity of representative agent is strictly increasing in $\bar{\gamma}$.

[3] Risk sharing here means that insiders bear risk jointly, with each insider bearing a share of risk that is inversely proportional to her risk-aversion parameter. At the aggregate level, they trade as if there were a single representative agent in the market. We call the trading intensity of this representative agent the aggregate trading intensity.

insiders on price impact is qualitatively different across the two settings. Proposition 6.1.3 (1) shows that each insider's welfare decreases as more insiders enter the market, since stronger competition [4] reduces the informational advantage of any given insider and therefore lowers the profit she can get from her private information.

[4] Competition here refers to the situation in which insiders profit from the same source of information. When insiders are risk averse, the risk-sharing effect implies that each insider takes a share of aggregate informed trading that is inversely proportional to her risk-aversion parameter. Since they share the profits generated by the same information, this gives rise to competition.

6.2 Properties of Equilibrium: $t_1 > 0$

In this section, we present the main result of our paper. We allow that two insiders to enter the market at different times: the first agent arrives at time 0, and the second agent arrives at time $t_1 > 0$. We then discuss the equilibrium properties: Trading intensity, Price informativeness and market depth.

6.2.1 Trading Intensity $\beta(t)$

Proposition 6.2.1. *For each $t_1 \in (0,1)$, in the single-insider economy with risk aversion $\overline{\gamma}_1$, the trading intensity is*

$$\beta^s(t) = \frac{1}{c^s(1-t)}, \ \text{where } c^s = \frac{\overline{\gamma}_1 \Sigma}{2}\left(\sqrt{1+\frac{4}{\sigma^2\Sigma\overline{\gamma}_1^2}}-1\right), \quad t \in [0,1).$$

In the delayed-entry economy, let $c(t_1), K$ be given in Theorem 5.1.3. The first insider's trading intensity $\beta_1(t)$ is

$$\beta_1(t) = \begin{cases} \frac{\overline{\gamma}_1\overline{\gamma}_2}{c(t_1)\left(\overline{\gamma}_2^2(1-t_1)+\overline{\gamma}_1^2(t_1-t)\right)}, & t \in [0,t_1), \\ \dfrac{\overline{\gamma}_2}{\overline{\gamma}_1}\dfrac{1}{c(t_1)(1-t)}, & t \in [t_1,1). \end{cases}$$

The second insider's trading intensity $\beta_2(t)$ is

$$\frac{\overline{\gamma}_2}{\gamma_2}\frac{1}{c(t_1)(1-t)}, \quad t \in [t_1,1).$$

Then $\beta_1(t) > \beta^s(t)$ for all $t \in [0,t_1)$, while $\beta_1(t) < \beta^s(t)$ for all $t \in [t_1,1)$. Moreover, for each fixed time $t \in [0,1)$, the map $t_1 \mapsto \beta_1(t)$ is strictly decreasing on $(t,1)$ and strictly increasing on $(0,t)$. Equivalently, $\partial_{t_1}\beta_1(t) < 0$ whenever $t < t_1 < 1$, and $\partial_{t_1}\beta_1(t) > 0$ whenever $0 < t_1 < t$.

Proof of Proposition 6.2.1. We have that

$$c(t_1) = \frac{\Sigma\overline{\gamma}_2}{2}\big(s(q(t_1))-1\big).$$

Since $x \to s(x)$ is strictly decreasing in x and $q(t_1) < \overline{\gamma}_1^2$, we have

$$\frac{\overline{\gamma}_1}{\overline{\gamma}_2}c(t_1) = \frac{\overline{\gamma}_1\Sigma}{2}\big(s(q(t_1)) - 1\big) > \frac{\overline{\gamma}_1\Sigma}{2}\big(s(\overline{\gamma}_1^2) - 1\big) = c^s.$$

Moreover,

$$\frac{q(t_1)c(t_1)}{\overline{\gamma}_1\overline{\gamma}_2} = \frac{\Sigma}{2\overline{\gamma}_1}\, q(t_1)\big(s(q(t_1)) - 1\big).$$

Since the function $x \mapsto x(s(x) - 1)$ is strictly increasing and $q(t_1) < \overline{\gamma}_1^2$, it follows that

$$\frac{q(t_1)c(t_1)}{\overline{\gamma}_1\overline{\gamma}_2} < \frac{\Sigma}{2\overline{\gamma}_1}\, \overline{\gamma}_1^2\big(s(\overline{\gamma}_1^2) - 1\big) = \frac{\overline{\gamma}_1\Sigma}{2}\big(s(\overline{\gamma}_1^2) - 1\big) = c^s.$$

Step 1: Comparison on $[0, t_1)$.

For $t \in [0, t_1)$, we have that

$$\frac{1}{\beta_1(t)} - \frac{1}{\beta^s(t)} = \left(\frac{q(t_1)\, c(t_1)}{\overline{\gamma}_1\overline{\gamma}_2} - c^s\right) + t\left(c^s - \frac{\overline{\gamma}_1}{\overline{\gamma}_2}c(t_1)\right).$$

By the two inequalities above, both coefficients on the right-hand side are strictly negative. Hence $1/\beta_1(t) - 1/\beta^s(t) < 0$ for all $t \in [0, t_1)$, and therefore $\beta_1(t) > \beta^s(t)$ for all $t \in [0, t_1)$.

Step 2: Comparison on $[t_1, 1)$.

For $t \in [t_1, 1)$, we have that

$$\frac{1}{\beta_1(t)} - \frac{1}{\beta^s(t)} = \big(\frac{\overline{\gamma}_1}{\overline{\gamma}_2}c(t_1) - c^s\big)(1 - t).$$

$1/\beta_1(t) = \overline{\gamma}_1/\overline{\gamma}_2\, c(t_1)(1 - t), 1/\beta^s(t) = c^s(1 - t)$. Since $\overline{\gamma}_1/\overline{\gamma}_2 c(t_1) > c^s$, it follows that $1/\beta_1(t) > 1/\beta^s(t)$, and hence $\beta_1(t) < \beta^s(t)$ for all $t \in [t_1, 1)$.

Step 3: Monotonicity in t_1 when $t < t_1$.

Fix $t \in [0, 1)$ and suppose $t < t_1 < 1$, we have that

$$\ln\beta_1(t) = \ln\overline{\gamma}_1 + \ln\overline{\gamma}_2 - \ln c(t_1) - \ln\Big(q(t_1) - \overline{\gamma}_1^2 t\Big). \tag{6.2.1}$$

$$\partial_{t_1}q(t_1) = \overline{\gamma}_1^2 - \overline{\gamma}_2^2, \partial_{t_1}c(t_1) = -\frac{\Sigma\bar{\gamma}_2\left(\bar{\gamma}_1^2 - \bar{\gamma}_2^2\right)}{4q(t_1)s(q(t_1))}\left(s\left(q\left(t_1\right)\right)^2 - 1\right)$$

$$\frac{\partial_{t_1}c(t_1)}{c(t_1)} = -\frac{\left(\bar{\gamma}_1^2 - \bar{\gamma}_2^2\right)\left(s\left(q\left(t_1\right)\right) + 1\right)}{2q\left(t_1\right)s\left(q\left(t_1\right)\right)}.$$

By (6.2.1), we have

$$\partial_{t_1} \ln \beta_1(t) = -\frac{\partial_{t_1} c(t_1)}{c(t_1)} - \frac{\partial_{t_1} q(t_1)}{q(t_1) - \overline{\gamma}_1^2 t} = -\frac{\partial_{t_1} c(t_1)}{c(t_1)} - \frac{\partial_{t_1} q(t_1)}{q(t_1) - \overline{\gamma}_1^2 t}$$
$$= \partial_{t_1} q(t_1)\Big(\frac{s(q(t_1)) + 1}{2s(q(t_1))}\frac{1}{q(t_1)} - \frac{1}{q(t_1) - \overline{\gamma}_1^2 t}\Big).$$

Observe that
$$\frac{s(q(t_1)) + 1}{2s(t_1)}\frac{1}{q(t_1)} < \frac{1}{q(t_1)} \leq \frac{1}{q(t_1) - \overline{\gamma}_1^2 t}.$$
gives us $\partial_{t_1} \ln \beta_1(t) < 0$, and since $\beta_1(t) > 0$, we conclude that $\partial_{t_1}\beta_1(t) < 0$ whenever $t < t_1 < 1$.

Step 4: Monotonicity in t_1 when $t_1 < t$.

Fix $t \in [0, 1)$ and suppose $0 < t_1 < t$. The result follows from the fact that $\partial_{t_1} c(t_1) < 0$. □

Proposition 6.2.2. *Define $\beta(t) = \beta_1(t)$ for $t \in [0, t_1)$ while $\beta(t) = 1/(c(t_1)(1 - t))$ for $t \in [t_1, 1)$. $\beta(t)$ is the representative informed trading intensity. Then fix $t \in [t_1, 1)$,*
$$\beta(t) \geq \beta^s(t) \iff c(t_1) \leq c^s.$$
Moreover, fix $\gamma_1 > 0$, $\gamma_2 > 0$, $\Sigma > 0$, and $\sigma^2 > 0$. Then there exists a unique $t_1^* \in (0, 1)$ such that
$$c(t^*) = c^s,$$
and
$$c(t_1) \leq c^s \iff t_1 \geq t_1^*.$$
Equivalently,
$$\beta(t) \geq \beta^s(t) \quad \textit{for } t \in [t_1, 1) \iff t_1 \geq t_1^*.$$

The cutoff is given by
$$t_1^* = \frac{q^* - \overline{\gamma}_2^2}{\overline{\gamma}_1^2 - \overline{\gamma}_2^2}, \qquad q^* = \frac{\frac{4}{\sigma^2\Sigma}\overline{\gamma}_2^2}{\overline{\gamma}_1\Big(\sqrt{1 + \frac{4}{\sigma^2\Sigma\overline{\gamma}_1^2}} - 1\Big)\Big(\overline{\gamma}_1\Big(\sqrt{1 + \frac{4}{\sigma^2\Sigma\overline{\gamma}_1^2}} - 1\Big) + 2\overline{\gamma}_2\Big)}.$$

Proof of Proposition 6.2.2. It is trivial that $\beta(t) \geq \beta^s(t) \iff c(t_1) \leq c^s$. We prove that there exists a unique $t_1^* \in (0, 1)$ such that $c(t_1^*) = c^s$ and $c(t_1) \leq c^s \iff t_1 \geq t_1^*$.

At $t_1 = 1$,
$$c(1) = \frac{\Sigma\overline{\gamma}_2}{2}\big(s(\overline{\gamma}_1^2) - 1\big) = \frac{\overline{\gamma}_2}{\overline{\gamma}_1}c^s < c^s.$$

At $t_1 = 0$,

$$c(0) = \frac{\Sigma\overline{\gamma}_2}{2}\big(s(\overline{\gamma}_2^2) - 1\big).$$

Since $\overline{\gamma}_2 < \overline{\gamma}_1$, we obtain

$$\overline{\gamma}_2\big(s(\overline{\gamma}_2^2) - 1\big) > \overline{\gamma}_1\big(s(\overline{\gamma}_1^2) - 1\big),$$

and therefore

$$c(0) > c^s.$$

Since c is continuous and strictly decreasing, with $c(0) > c^s > c(1)$, there exists a unique $t_1^* \in (0,1)$ such that

$$c(t_1^*) = c^s.$$

Moreover, because c is strictly decreasing,

$$c(t_1) \leq c^s \iff t_1 \geq t_1^*.$$

It remains to compute t_1^*. The equation $c(t_1) = c^s$ is equivalent to

$$\overline{\gamma}_2\big(s(q(t_1)) - 1\big) = \overline{\gamma}_1\big(s(\overline{\gamma}_1^2) - 1\big),$$

that is,

$$s(q(t_1)) = 1 + \frac{\overline{\gamma}_1}{\overline{\gamma}_2}\big(s(\overline{\gamma}_1^2) - 1\big).$$

it becomes

$$\sqrt{1 + \frac{\kappa}{q(t_1)}} = 1 + \frac{\overline{\gamma}_1}{\overline{\gamma}_2}\big(s(\overline{\gamma}_1^2) - 1\big).$$

Hence

$$1 + \frac{\kappa}{q(t_1)} = \left(1 + \frac{\overline{\gamma}_1}{\overline{\gamma}_2}\big(s(\overline{\gamma}_1^2) - 1\big)\right)^2,$$

so

$$q(t_1) = \frac{\kappa}{\left(1 + \frac{\overline{\gamma}_1}{\overline{\gamma}_2}\big(s(\overline{\gamma}_1^2) - 1\big)\right)^2 - 1}.$$

Therefore

$$q^* = \frac{\kappa}{\left(1 + \frac{\overline{\gamma}_1}{\overline{\gamma}_2}\big(s(\overline{\gamma}_1^2) - 1\big)\right)^2 - 1} = \frac{\kappa\,\overline{\gamma}_2^2}{\overline{\gamma}_1\big(s(\overline{\gamma}_1^2) - 1\big)\Big(\overline{\gamma}_1\big(s(\overline{\gamma}_1^2) - 1\big) + 2\overline{\gamma}_2\Big)}.$$

by $q(t^*) = q^*$, we have

$$t_1^* = \frac{q^* - \overline{\gamma}_2^2}{\overline{\gamma}_1^2 - \overline{\gamma}_2^2}.$$

$\square$

The first part of Proposition 6.2.1 shows that, relative to the single-insider benchmark, the first insider's trading intensity in the delayed-entry economy is higher before t_1, the arrival time of the second insider, and lower after t_1. The intuition is as follows. In the single-insider benchmark, the first insider is the only informed trader throughout $[0, 1]$, and therefore the only trader who can exploit the informational advantage. In the delayed-entry economy, by contrast, she remains the sole informed trader only until t_1. Anticipating that a second informed trader will enter at t_1 and compete the remaining mispricing opportunities, the first insider has an incentive to trade more aggressively before t_1 in order to extract more profit while she is still alone.

The case after t_1 is more subtle. To interpret the result, we first consider the representative insider's trading intensity after t_1. Proposition 6.2.2 shows that the representative insider's trading intensity may be either higher or lower than in the single-insider benchmark, depending on the arrival time of the second insider. If the second insider enters early, then the representative insider's trading intensity is lower than in the single-insider benchmark, if she enters late, then it is higher. To understand this, note first that the constant $c(t_1) = \Sigma\overline{\gamma}_2/2\,(s(q(t_1)) - 1)$ fully determines trading intensity after t_1. Since this quantity depends on t_1, the effect of risk sharing also depends on the timing of the second insider's arrival. Risk sharing pushes trading intensity downward, when the second insider enters the market, and this effect is weaker when t_1 is larger. A stronger risk-sharing effect pushes down the post-entry trading more substantially, so post-entry trading intensity differs more from pre-entry trading intensity, in particular, the downward jump in trading intensity at t_1 is larger. By contrast, when the risk-sharing effect is weak, post-entry trading behavior is much closer to pre-entry trading behavior. As a result, when t_1 is small, the risk-sharing effect is strong, which raises the representative insider's risk tolerance and lowers post-entry trading intensity sufficiently that it falls below the single-insider benchmark. When t_1 is large, the risk-sharing effect is weak, so post-entry trading remains close to pre-entry trading. Since pre-entry trading intensity is higher than in the single-insider benchmark, post-entry trading intensity may then exceed the benchmark as well.

For each individual insider, after t_1, the risk-sharing effect implies that each insider takes a share of the representative trader's order flow that is inversely proportional to her risk-aversion parameter. As a result, the first insider's trading intensity is lower than in the single-insider benchmark, even though the representative trading intensity may be higher than in that benchmark. More generally, once both insiders are active after t_1, risk sharing reduces the first insider's trading intensity, so her post-entry trading intensity is lower than in the single-insider benchmark.

Part 2 of Proposition 6.2.1 shows that the first insider's trading intensity decreases with t_1 before entry and increases with t_1 after entry. As t_1 increases, the first insider remains the only insider for longer over $[0, t_1)$. As a result, she faces less competition to exploit her informational advantage before the second insider arrives, and therefore trades less aggressively before entry. After entry, a larger t_1 weakens the effect of risk sharing, so the first insider's post-entry trading intensity is pushed down less and hence becomes closer to her pre-entry trading intensity and therefore increases.

Proposition 6.2.3. *Fix $\gamma_2 > 0$, $\Sigma > 0$, $\sigma^2 > 0$, and $t_1 \in (0,1)$. Then:*

(i) For each fixed $t \in [0, t_1)$, $\partial_{\gamma_1}\beta_1(t) > 0$.

(ii) For each fixed $t \in (t_1, 1)$, the sign of $\partial_{\gamma_1}\beta_1(t)$ may be either positive or negative, depending on the parameter values. The detailed case analysis is provided in the proof.

(iii) For each fixed $t \in (t_1, 1)$, $\partial_{\gamma_1}\beta_2(t) > 0$.

Proof of Proposition 6.2.3. Taking derivatives with respect to γ_1, we have

$$\partial_{\gamma_1} q(t_1) = 2\gamma_1 t_1 + 2\overline{\gamma}_2 \partial_{\gamma_1} \overline{\gamma}_2 (1 - t_1) = 2\gamma_1 t_1 + \frac{2\overline{\gamma}_2^3}{\gamma_1^2}(1 - t_1).$$

It follows that

$$\begin{aligned}\gamma_1 \partial_{\gamma_1} q(t_1) - 2q(t_1) &= 2\gamma_1^2 t_1 + 2\gamma_1 \overline{\gamma}_2 \partial_{\gamma_1} \overline{\gamma}_2 (1 - t_1) - 2\gamma_1^2 t_1 - 2\overline{\gamma}_2^2 (1 - t_1) \\ &= 2(1 - t_1)\overline{\gamma}_2 \big(\gamma_1 \partial_{\gamma_1} \overline{\gamma}_2 - \overline{\gamma}_2\big) < 0.\end{aligned}$$

(i) Case $t < t_1$.
For $t \in [0, t_1)$, we have

$$\beta_1(t) = \frac{\overline{\gamma}_1 \overline{\gamma}_2}{c(t_1)\big(q(t_1) - \overline{\gamma}_1^2 t\big)}, \qquad c(t_1) = \frac{\Sigma \overline{\gamma}_2}{2}\big(s(q(t_1)) - 1\big).$$

Therefore, we have that

$$\beta_1(t) = \frac{2\overline{\gamma}_1}{\Sigma\big(s(q(t_1)) - 1\big)\big(q(t_1) - \overline{\gamma}_1^2 t\big)}.$$

Hence

$$\log \beta_1(t) = \log \gamma_1 - \log\big(s(q(t_1)) - 1\big) - \log\big(q(t_1) - \gamma_1^2 t\big) + \text{const}.$$

Differentiating with respect to γ_1, we obtain

$$\partial_{\gamma_1} \log \beta_1(t) = \frac{1}{\gamma_1} - \partial_{\gamma_1} \log\big(s(q(t_1)) - 1\big) - \frac{\partial_{\gamma_1} q(t_1) - 2\gamma_1 t}{q(t_1) - \gamma_1^2 t}.$$

We have that

$$\partial_{\gamma_1} \log\big(s(q(t_1)) - 1\big) = -\frac{s(q(t_1)) + 1}{2s(q(t_1))} \frac{\partial_{\gamma_1} q(t_1)}{q(t_1)}.$$

Substituting this into the previous expression gives

$$\partial_{\gamma_1} \log \beta_1(t) = \frac{1}{\gamma_1} + \frac{s(q(t_1)) + 1}{2s(q(t_1))} \frac{\partial_{\gamma_1} q(t_1)}{q(t_1)} - \frac{\partial_{\gamma_1} q(t_1) - 2\overline{\gamma}_1 t}{q(t_1) - \overline{\gamma}_1^2 t}.$$

Next, we have that

$$\frac{\partial_{\gamma_1} q(t_1) - 2\overline{\gamma}_1 t}{q(t_1) - \overline{\gamma}_1^2 t} - \frac{1}{\overline{\gamma}_1} - \frac{1}{2}\frac{\partial_{\gamma_1} q(t_1)}{q(t_1)} = \frac{\big(\overline{\gamma}_1 \partial_{\gamma_1} q(t_1) - 2q(t_1)\big)\big(\overline{\gamma}_1^2 t + q(t_1)\big)}{2\overline{\gamma}_1 q(t_1)\big(q(t_1) - \overline{\gamma}_1^2 t\big)}.$$

Since

$$\overline{\gamma}_1 \partial_{\gamma_1} q(t_1) - 2q(t_1) < 0,$$

we have that

$$\frac{\partial_{\gamma_1} q(t_1) - 2\overline{\gamma}_1 t}{q(t_1) - \overline{\gamma}_1^2 t} < \frac{1}{\overline{\gamma}_1} + \frac{1}{2}\frac{\partial_{\gamma_1} q(t_1)}{q(t_1)}.$$

Since $s(q(t_1)) > 1$, we have

$$\frac{s(q(t_1)) + 1}{2s(q(t_1))} > \frac{1}{2}.$$

Therefore

$$\begin{aligned}\partial_{\gamma_1} \log \beta_1(t) &= \frac{1}{\overline{\gamma}_1} + \frac{s(q(t_1)) + 1}{2s(q(t_1))}\frac{\partial_{\gamma_1} q(t_1)}{q(t_1)} - \frac{\partial_{\gamma_1} q(t_1) - 2\overline{\gamma}_1 t}{q(t_1) - \overline{\gamma}_1^2 t} \\ &> \frac{1}{\overline{\gamma}_1} + \frac{1}{2}\frac{\partial_{\gamma_1} q(t_1)}{q(t_1)} - \frac{\partial_{\gamma_1} q(t_1) - 2\overline{\gamma}_1 t}{q(t_1) - \overline{\gamma}_1^2 t} > 0.\end{aligned}$$

Hence, we have that

$$\partial_{\gamma_1} \beta_1(t) > 0,$$

(ii) Case $t > t_1$. For $t \in [t_1, 1)$, we have

$$\beta_1(t) = \frac{\overline{\gamma}_2}{\overline{\gamma}_1}\frac{1}{c(t_1)(1-t)}, c(t_1) = \frac{\Sigma\overline{\gamma}_2}{2}\big(s(q(t_1)) - 1\big),$$

it follows that

$$\beta_1(t) = \frac{2}{\Sigma(1-t)\,\overline{\gamma}_1\,\big(s(q(t_1)) - 1\big)}.$$

Hence

$$\log \beta_1(t) = -\log \overline{\gamma}_1 - \log\big(s(q(t_1)) - 1\big) + \text{const}.$$

Differentiating with respect to γ_1, we obtain

$$\partial_{\gamma_1} \log \beta_1(t) = -\frac{1}{\overline{\gamma}_1} - \partial_{\gamma_1} \log\big(s(q(t_1)) - 1\big) = -\frac{1}{\overline{\gamma}_1} - \frac{\partial_q s(q(t_1))}{s(q(t_1)) - 1}\,\partial_{\gamma_1} q(t_1).$$

we have that

$$\frac{\partial_q s(q(t_1))}{s(q(t_1))-1} = -\frac{s(q(t_1))+1}{2\,s(q(t_1))\,q(t_1)}.$$

Therefore

$$\partial_{\gamma_1} \log \beta_1(t) = -\frac{1}{\overline{\gamma}_1} + \frac{s(q(t_1))+1}{2\,s(q(t_1))} \frac{\partial_{\gamma_1} q(t_1)}{q(t_1)}.$$

Since $\beta_1(t) > 0$, it follows that

$$\begin{aligned}
&\partial_{\gamma_1}\beta_1(t) > 0 \iff \partial_{\gamma_1} \log \beta_1(t) > 0 \iff \gamma_1 \frac{\partial_{\gamma_1} q(t_1)}{q(t_1)} > \frac{2\,s(q(t_1))}{s(q(t_1))+1} \\
&\iff t_1\big(1+\frac{\gamma_1}{\gamma_2}\big)^3 + (1-t_1) > (1-t_1)\frac{\gamma_1}{\gamma_2} s(q(t_1)) \\
&\iff \frac{t_1\left(1+\frac{\gamma_1}{\gamma_2}\right)^3 + (1-t_1)}{(1-t_1)\,\frac{\gamma_1}{\gamma_2}} > s(q(t_1)) = \sqrt{1+\frac{\kappa}{q(t_1)}},
\end{aligned}$$

Square both sides, the condition becomes

$$1 + \frac{\kappa/\gamma_2^2}{(\frac{\gamma_1}{\gamma_2})^2\left(t_1 + \frac{1-t_1}{(1+\frac{\gamma_1}{\gamma_2})^2}\right)} < \frac{\left(t_1(1+\frac{\gamma_1}{\gamma_2})^3 + (1-t_1)\right)^2}{(1-t_1)^2\,(\frac{\gamma_1}{\gamma_2})^2},$$

Simplify it, we have that

$$\begin{aligned}
\partial_{\gamma_1}\beta_1(t) > 0 \iff & \frac{\left(1+2t_1\frac{\gamma_1}{\gamma_2} + t_1\left(\frac{\gamma_1}{\gamma_2}\right)^2\right)^2}{(1-t_1)^2\left(1+\frac{\gamma_1}{\gamma_2}\right)} \\
& \times \left[1 + (4t_1-1)\frac{\gamma_1}{\gamma_2} + 3t_1\left(\frac{\gamma_1}{\gamma_2}\right)^2 + t_1\left(\frac{\gamma_1}{\gamma_2}\right)^3\right] > \frac{\kappa}{\gamma_2^2}.
\end{aligned}$$

Denote the left-hand side function $f(\gamma_1)$.
So

$$\partial_{\gamma_1}\beta_1(t) > 0 \iff f(\gamma_1) > \frac{\kappa}{\gamma_2^2},$$

We have that

$$f'(\gamma_1) = \frac{2\left(t_1\gamma_1^2 + 2t_1\gamma_1\gamma_2 + \gamma_2^2\right)\left(t_1\gamma_1^3 + 3t_1\gamma_1^2\gamma_2 + 3t_1\gamma_1\gamma_2^2 + \gamma_2^3\right)}{\gamma_2^6(\gamma_1+\gamma_2)^2(1-t_1)^2} \\ \times \left(3t_1\gamma_1^2 + 6t_1\gamma_1\gamma_2 + (4t_1-1)\gamma_2^2\right).$$

The sign of $f'(\gamma_1)$ is determined by $3t_1\gamma_1^2 + 6t_1\gamma_1\gamma_2 + (4t_1-1)\gamma_2^2$.
If $0 < t_1 < 1/4$, then $3t_1\gamma_1^2 + 6t_1\gamma_1\gamma_2 + (4t_1-1)\gamma_2^2 = 0$ has one positive root

$$\gamma_1^* = \gamma_2\left(\sqrt{\frac{1-t_1}{3t_1}} - 1\right),$$

So $f'(\gamma_1) < 0$ for $0 < \gamma_1 < \gamma_1^*$, $f'(\gamma_1) > 0$ for $\gamma_1 > \gamma_1^*$. So f is first strictly decreasing and then strictly increasing. Hence the minimum value of f is

$$f(\gamma_1^*) = \frac{32}{27}\left(3\sqrt{3t_1(1-t_1)} + t_1 - 1\right),$$

If $1/4 \le t_1 < 1$, then

$$3t_1\gamma_1^2 + 6t_1\gamma_1\gamma_2 + (4t_1-1)\gamma_2^2 \ge 0,$$

so for all $\gamma_1 > 0$, so f is increasing.

We have following cases:

1. If $t_1 = 0$, then
$$f(\gamma_1) = \frac{1-\frac{\gamma_1}{\gamma_2}}{1+\frac{\gamma_1}{\gamma_2}},$$
$$\partial_{\gamma_1}\beta > 0 \iff \frac{\kappa}{\gamma_2^2} < \frac{1-\frac{\gamma_1}{\gamma_2}}{1+\frac{\gamma_1}{\gamma_2}}.$$
If $\gamma_2 \le 2/(\sigma\sqrt{\Sigma})$, then $\partial_{\gamma_1}\beta_1 < 0$ for all $\gamma_1 > 0$. If $\gamma_2 > 2/(\sigma\sqrt{\Sigma})$, then there exists cutoff
$$\gamma^* = \gamma_2\frac{1-\frac{\kappa}{\gamma_2^2}}{1+\frac{\kappa}{\gamma_2^2}}$$
such that $\partial_{\gamma_1}\beta_1 > 0$ for $0 < \gamma_1 < \gamma^*$ and $\partial_{\gamma_1}\beta_1 < 0$ if $\gamma_1 > \gamma^*$.

2. If $0 < t_1 \le 1/28$, then we have that $f(\gamma_1^*) \le 0$. Since $\kappa/\gamma_2^2 > 0$, the constant

line lie above the minimum of f. Since

$$\lim_{\gamma_1 \to 0} f(\gamma_1) = \frac{1}{(1-t_1)^2}, \lim_{\gamma_1 \to \infty} f(\gamma_1) = \infty.$$

If $\kappa/\gamma_2^2 < 1/(1-t_1)^2$, there exists $0 < \underline{\gamma_1} < \gamma_1^* < \overline{\gamma_1}$ such that

$$\partial_{\gamma_1}\beta_1 \begin{cases} > 0, & 0 < \gamma_1 < \underline{\gamma}_1, \\ < 0, & \underline{\gamma}_1 < \gamma_1 < \bar{\gamma}_1, \\ > 0, & \gamma_1 > \bar{\gamma}_1. \end{cases}$$

If $\kappa/\gamma_2^2 \geq 1/(1-t_1)^2$, we have that there exists a cut-off γ^*

$$\partial_{\gamma_1}\beta_1 \begin{cases} < 0, & 0 < \gamma_1 < \gamma^*, \\ > 0, & \gamma_1 > \gamma^*. \end{cases}$$

3. If $1/28 < t_1 < 1/4$, then we have that

   If $\kappa/\gamma_2^2 \leq 32/27(3\sqrt{3t_1(1-t_1)} + t_1 - 1)$, then the constant lie below the curve of f. Hence $\partial_{\gamma_1}\beta_1 \geq 0$ for all $\gamma_1 > 0$.

   If $32/27(3\sqrt{3t_1(1-t_1)} + t_1 - 1) < \kappa/\gamma_2^2 < 1/(1-t_1)^2$, so there exists $\underline{\gamma_1} < \gamma_1^* < \overline{\gamma_1}$, such that

$$\partial_{\gamma_1}\beta_1 \begin{cases} > 0, & 0 < \gamma_1 < \underline{\gamma}_1, \\ < 0, & \underline{\gamma}_1 < \gamma_1 < \overline{\gamma}_1, \\ > 0, & \gamma_1 > \overline{\gamma}_1. \end{cases}$$

   If $\kappa/\gamma_2^2 \geq 1/(1-t_1)^2$, then there exists a cut-off γ^*,

$$\partial_{\gamma_1}\beta_1 \begin{cases} < 0, & 0 < \gamma_1 < \gamma^*, \\ > 0, & \gamma_1 > \gamma^*. \end{cases}$$

4. If $1/4 \leq t_1 < 1$, then we have that if $\kappa/\gamma_2^2 \leq 1/(1-t_1)^2$, then $\partial_{\gamma_1}\beta_1 > 0$ for all

$\gamma_1 > 0$. If $k/\gamma_2^2 > 1/(1-t_1)^2$, then we have that

$$\partial_{\gamma_1}\beta_1 \begin{cases} < 0, & 0 < \gamma_1 < \gamma^*, \\ > 0, & \gamma_1 > \gamma^*. \end{cases}$$

(iii) For $t > t_1$, we have

$$\beta_2(t) = \frac{\overline{\gamma}_2}{\gamma_2}\frac{1}{c(t_1)(1-t)}, \quad c(t_1) = \frac{\Sigma\overline{\gamma}_2}{2}\big(s(q(t_1)) - 1\big),$$

it follows that

$$\beta_2(t) = \frac{2}{\Sigma(1-t)\gamma_2\big(s(q(t_1)) - 1\big)}.$$

Hence

$$\log\beta_2(t) = \text{const} - \log\gamma_2 - \log\big(s(q(t_1)) - 1\big).$$

Then we have,

$$\partial_{\gamma_1}\log\beta_2(t) = -\partial_{\gamma_1}\log\big(s(q(t_1)) - 1\big).$$

Now

$$\partial_{\gamma_1}\log\big(s(q(t_1)) - 1\big) = \frac{\partial_q s(q(t_1))}{s(q(t_1)) - 1}\,\partial_{\gamma_1}q(t_1).$$

we obtain

$$\frac{\partial_q s(q(t_1))}{s(q(t_1)) - 1} = -\frac{s(q(t_1)) + 1}{2\,s(q(t_1))\,q(t_1)}.$$

Therefore

$$\partial_{\gamma_1}\log\beta_2(t) = \frac{s(q(t_1)) + 1}{2\,s(q(t_1))}\frac{\partial_{\gamma_1}q(t_1)}{q(t_1)} = \frac{s(q(t_1)) + 1}{2\,s(q(t_1))}\frac{\partial_{\gamma_1}q(t_1)}{q(t_1)}$$

so

$$\partial_{\gamma_1}q(t_1) = 2\overline{\gamma}_1 t_1 + 2\overline{\gamma}_2\,\partial_{\gamma_1}\overline{\gamma}_2\,(1-t_1),$$

where

$$\partial_{\gamma_1}\overline{\gamma}_2 = \frac{\gamma_2^2}{(\gamma_1+\gamma_2)^2} > 0.$$

Hence $\partial_{\gamma_1}q(t_1) > 0$, and therefore $\partial_{\gamma_1}\log\beta_2(t) > 0$. Since $\beta_2(t) > 0$, it follows that $\partial_{\gamma_1}\beta_2(t) > 0$. $\square$

Proposition 6.2.3 (i) shows that as γ_1 increases, the first insider's trading intensity before the entry increases. This is because as γ_1 increases, the first insider becomes more risk averse to future price risk and therefore trades more aggressively before t_1 to reduce the future price risk.

Proposition 6.2.3 (ii) shows that after t_1, the effect of an increase in γ_1 on the first insider's trading intensity depends on parameter values. The intuition is that two opposing forces are at work. On the one hand, when γ_1 increases, the representative insider's trading intensity, $\beta(t) = 1/(c(t_1)(1-t))$, also increases. This is because a higher γ_1 makes the first insider more risk averse, which strengthens the incentive to trade more aggressively in order to reduce future price risk. However, risk sharing after t_1 pushes down the first insider's individual trading intensity. The first effect nominates the second effect and therefore increase the representative's trading intensity. On the other hand, the first insider's share of aggregate trading $\overline{\gamma}_2/\overline{\gamma}_1$ decreases. As a result, $\beta_1(t)$ may either increase or decrease, depending on which effect dominates.

Proposition 6.2.3 (iii) shows that the second insider's trading intensity also increases with $\overline{\gamma}_1$. This is because the second insider's share of aggregate trading also increases. Therefore, the second insider's trading intensity rises.

Proposition 6.2.4. *Fix $\gamma_1 > 0$, $\Sigma > 0$, $\sigma^2 > 0$, and $t_1 \in (0,1)$. Then:*

(i) For each fixed $t \in [0, t_1)$, $\partial_{\gamma_2}\beta_1(t) < 0$, whereas for each fixed $t \in (t_1, 1)$, $\partial_{\gamma_2}\beta_1(t) > 0$.

(ii) For each fixed $t \in (t_1, 1)$, the sign of $\partial_{\gamma_2}\beta_2(t)$ may be either positive or negative, depending on the parameter values. The detailed case analysis is provided in the proof.

Proof of Proposition 6.2.4. We first show that $\partial_{\gamma_2} q(t_1) > 0$. We have

$$\partial_{\gamma_2}\overline{\gamma}_2 = \frac{\gamma_1^2}{(\gamma_1+\gamma_2)^2} = \frac{\overline{\gamma}_2^2}{\gamma_2^2}.$$

Therefore

$$\partial_{\gamma_2} q(t_1) = 2(1-t_1)\overline{\gamma}_2\, \partial_{\gamma_2}\overline{\gamma}_2 = 2(1-t_1)\frac{\overline{\gamma}_2^3}{\gamma_2^2} > 0.$$

(i) Suppose first that $t < t_1$. We have that

$$\log \beta_1(t) = \text{const} - \log\bigl(s(q(t_1)) - 1\bigr) - \log\bigl(q(t_1) - \gamma_1^2 t\bigr).$$

Hence

$$\begin{aligned}
\partial_{\gamma_2} \log \beta_1(t) &= -\partial_{\gamma_2} \log\bigl(s(q(t_1)) - 1\bigr) - \frac{\partial_{\gamma_2} q(t_1)}{q(t_1) - \gamma_1^2 t} \\
&= \left(\frac{s(q(t_1)) + 1}{2s(q(t_1))}\frac{1}{q(t_1)} - \frac{1}{q(t_1) - \gamma_1^2 t}\right)\partial_{\gamma_2} q(t_1).
\end{aligned}$$

By

$$\frac{s(q(t_1)) + 1}{2s(q(t_1))}\frac{1}{q(t_1)} < \frac{1}{q(t_1)} \leq \frac{1}{q(t_1) - \gamma_1^2 t}, \quad \partial_{\gamma_2} q(t_1) > 0,$$

we have that $\partial_{\gamma_2} \log \beta_1(t) < 0$, and therefore $\partial_{\gamma_2} \beta_1(t) < 0$.

If $t > t_1$, then $\log \beta_1(t) = \text{const} - \log\bigl(s(q(t_1)) - 1\bigr)$.

Hence

$$\partial_{\gamma_2} \log \beta_1(t) = \left(\frac{s(q(t_1)) + 1}{2s(q(t_1))}\frac{1}{q(t_1)}\right)\partial_{\gamma_2} q(t_1) > 0.$$

Therefore $\partial_{\gamma_2}\beta_1(t) > 0$.

(ii) If $t > t_1$, then

$$\log \beta_2(t) = \text{const} - \log \gamma_2 - \log\bigl(s(q(t_1)) - 1\bigr),$$

and so

$$\partial_{\gamma_2} \log \beta_2(t) = -\frac{1}{\gamma_2} - \partial_{\gamma_2} \log\bigl(s(q(t_1)) - 1\bigr) = -\frac{1}{\gamma_2} + \frac{s(q(t_1)) + 1}{2s(q(t_1))}\frac{\partial_{\gamma_2} q(t_1)}{q(t_1)}.$$

$$\begin{aligned}
\partial_{\gamma_2} \log \beta_2(t) > 0 &\iff \gamma_2 \frac{\partial_{\gamma_2} q(t_1)}{q(t_1)} > \frac{2s(q(t_1))}{s(q(t_1))+1} \\
&\iff \frac{(1-t_1)\gamma_1\gamma_2^2}{(\gamma_1+\gamma_2)\big(t_1\gamma_1^2+2t_1\gamma_1\gamma_2+\gamma_2^2\big)} > \frac{s(q(t_1))}{s(q(t_1))+1} \\
&\iff \frac{(1-t_1)\gamma_1\gamma_2^2}{t_1(\gamma_1+\gamma_2)^3+(1-t_1)\gamma_2^3} > s(q(t_1)) \iff g(\gamma_2) > \frac{\kappa}{\gamma_1^2},
\end{aligned}$$

where

$$g(\gamma_2) = -\frac{\left(t_1\gamma_1^2+2t_1\gamma_1\gamma_2+\gamma_2^2\right)^2\left(\gamma_2^3+(4t_1-1)\,\gamma_1\gamma_2^2+3t_1\gamma_1^2\gamma_2+t_1\gamma_1^3\right)}{(\gamma_1+\gamma_2)\left(t_1\,(\gamma_1+\gamma_2)^3+(1-t_1)\,\gamma_2^3\right)^2}.$$

We have following cases:

1. If $t_1 = 0$, then
$$g(\gamma_2) = \frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2},$$
$$\partial_{\gamma_2}\beta_2 > 0 \iff \frac{\kappa}{\gamma_1^2} < \frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2} \iff \gamma_2 < \gamma_1 \frac{1-\frac{4}{\sigma^2\Sigma\gamma_1^2}}{1+\frac{4}{\sigma^2\Sigma\gamma_1^2}}.$$
If $\gamma_1 \le 2/(\sigma\sqrt{\Sigma})$, then $\partial_{\gamma_2}\beta_2 < 0$ for all $\gamma_2 > 0$. If $\gamma_1 > 2/(\sigma\sqrt{\Sigma})$, then there exists a cutoff
$$\gamma^* = \gamma_1 \frac{1-\frac{\kappa}{\gamma_1^2}}{1+\frac{\kappa}{\gamma_1^2}},$$
such that $\partial_{\gamma_2}\beta_2 > 0$ for $0 < \gamma_2 < \gamma^*$ and $\partial_{\gamma_2}\beta_2 < 0$ if $\gamma_2 > \gamma^*$.

2. If $0 < t_1 < \frac{1}{28}$, then first notice that
$$g(\gamma_2) > 0 \iff C(\gamma_2) = \gamma_2^3 + (4t_1-1)\gamma_1\gamma_2^2 + 3t_1\gamma_1^2\gamma_2 + t_1\gamma_1^3 < 0.$$
The discriminant of this cubic equation is
$$\Delta = -4\gamma_1^6 t_1(1-t_1)^2(28t_1-1).$$
If $t_1 < \frac{1}{28}$, then $\Delta > 0$, so C has two distinct positive roots. Therefore, there

exists $\gamma_{2,-}$ and $\gamma_{2,+}$ such that

$$g(\gamma_2) > 0, \text{ for } 0 < \gamma_{2,-} < \gamma_2 < \gamma_{2,+}.$$

Compute the derivative, we have that

$$g'(\gamma_2) = -\frac{P(\gamma_2)2(1-t_1)\gamma_1\gamma_2(t_1\gamma_1^2 + 2t_1\gamma_1\gamma_2 + \gamma_2^2)}{(\gamma_1+\gamma_2)^2(t_1\gamma_1^3 + 3t_1\gamma_1^2\gamma_2 + 3t_1\gamma_1\gamma_2^2 + \gamma_2^3)^3},$$

where

$$\begin{aligned} P(\gamma_2) =& \gamma_2^6 + 6t_1\gamma_1\gamma_2^5 + 3t_1\left(4t_1+1\right)\gamma_1^2\gamma_2^4 + 4t_1(6t_1-1)\gamma_1^3\gamma_2^3 \\ &+ 3t_1\left(6t_1-1\right)\gamma_1^4\gamma_2^2 + 6t_1^2\gamma_1^5\gamma_2 + t_1^2\gamma_1^6. \end{aligned}$$

As the signs of the coefficients of P change twice, Descartes' rule of signs implies that P has at most two positive roots. Next, we note that

$$\lim_{\gamma_2\to 0} g(\gamma_2) = -t_1 < 0 \qquad \text{and} \qquad \lim_{\gamma_2\to\infty} g(\gamma_2) = -1 < 0.$$

Moreover, $P(0) = t_1^2\gamma_1^6 > 0$, so for sufficiently small $\gamma_2 > 0$, we have $P(\gamma_2) > 0$. Since the sign of $g'(\gamma_2)$ is the opposite of the sign of $P(\gamma_2)$, it follows that $g'(\gamma_2) < 0$ for sufficiently small $\gamma_2 > 0$. Hence g is strictly decreasing near 0.

On the other hand, $g(\gamma_2) > 0$ on some nonempty interval, while $g(\gamma_2) < 0$ for γ_2 sufficiently small and for γ_2 sufficiently large. Therefore g must increase somewhere in between, and so g' must be positive at some point. Since g' is negative near 0 and again negative for large γ_2, continuity implies that g' has at least two positive zeros. Because P has at most two positive roots, g' also has at most two positive zeros. We conclude that g has exactly two positive critical points, denoted by

$$0 < \gamma_{2,-}^* < \gamma_{2,+}^*.$$

Thus g is strictly decreasing on $(0, \gamma_{2,-}^*)$, strictly increasing on $(\gamma_{2,-}^*, \gamma_{2,+}^*)$, and strictly decreasing on $(\gamma_{2,+}^*, \infty)$. In particular, $\gamma_{2,+}^*$ is the unique global maximizer of g on $(0, \infty)$.

Therefore, the maximum value is

$$M := \max_{\gamma_2 > 0} g(\gamma_2) = g(\gamma_{2,+}^*).$$

If $\kappa/\gamma_1^2 \geq M$, then $\partial_{\gamma_2}\beta_2 \leq 0$ for all $\gamma_2 > 0$.

If $\kappa/\gamma_1^2 < M$, then there exists $\underline{\gamma_2} < \overline{\gamma_2}$, such that

$$\partial_{\gamma_2}\beta_2 \begin{cases} < 0, & 0 < \gamma_2 < \underline{\gamma}_2, \\ > 0, & \underline{\gamma}_2 < \gamma_2 < \overline{\gamma}_2, \\ < 0, & \gamma_2 > \overline{\gamma}_2. \end{cases}$$

3. If $1/28 \leq t_1 < 1$, then we have that $\Delta \leq 0$, so the cubic function C only has one real root. Since $C(0) = t_1\gamma_1^3 > 0$ and $\lim_{\gamma_2 \to \infty} C(\gamma_2) = \infty$, hence the real root is negative. So $C(\gamma_2) > 0$ for all $\gamma_2 > 0$. Therefore $g(\gamma_2) \leq 0$ and $g(\gamma_2) > \kappa/\gamma_1^2$ is impossible. So $\partial_{\gamma_2}\beta_2 < 0$ for all $\gamma_2 > 0$.

□

Proposition 6.2.4 (i) shows that, as γ_2 increases, the first insider's trading intensity before entry decreases. This is because, as γ_2 increases, the second insider becomes more risk averse and is less willing to bear future risk. As a result, due to the risk-sharing effect after t_1, her share of aggregate informed trading becomes smaller and the first insider's share becomes larger. This implies that the first insider's profit from informed trading is greater than that of the second insider, and therefore from the first insider's perspective the competition becomes weaker. Anticipating this, the first insider's trading intensity before entry decreases. This intuition is also justified by Proposition 6.2.11 that the increase of γ_2 will increase the first insider's welfare.

To explain the other results, we first consider the representative insider's trading intensity. As γ_2 increases, the first insider's trading intensity before entry decreases. The risk-sharing effect then further lowers trading intensity at t_1. This effect becomes weaker as γ_2 increases. If the weakening of the risk sharing is only slight, the risk sharing will push the post-entry trading intensity lower than before the increase in γ_2. If the risk-sharing effect becomes sufficiently weak, however, then post-entry trading intensity might lower slightly and remain closer to the pre-entry intensity and therefore become higher than it was before the increase in γ_2.

Proposition 6.2.4 (i) also shows that, after t_1, an increase in γ_2 raises the first insider's trading intensity. This is because the first insider's share of aggregate informed trading increases with γ_2. This effect is strong enough that, even though aggregate informed trading decreases, the first insider's own trading intensity still rises.

Proposition 6.2.4 (ii) shows that the second insider's trading intensity may either increase or decrease with γ_2. This is because the second insider's share of aggregate informed trading decreases as γ_2 increases, while aggregate informed trading itself may either increase or decrease. These two opposing effects determine whether the second insider's trading intensity rises or falls, depending on the parameter values.

6.2.2 Price Informativeness $\Pi = \Sigma^{-1}$

Proposition 6.2.5. *The price informativeness for two-insider economy and single-insider economy is* $\Pi(t) = \Sigma^{-1} + 1/\sigma^2 \int_0^t \beta(u)^2\, du$ *and* $\Pi^s(t) = \Sigma^{-1} + 1/\sigma^2 \int_0^t \beta^s(u)^2\, du$. *Then, for all* $t \in [0, 1)$,

$$\Pi^s(t) = \Sigma^{-1} + \frac{1}{\sigma^2 (c^s)^2} \left(\frac{1}{1-t} - 1 \right),$$

and

$$\Pi(t) = \Sigma^{-1} + \frac{1}{\sigma^2 c(t_1)^2} \begin{cases} \frac{\overline{\gamma}_1^2 \overline{\gamma}_2^2 t}{q(t_1)(q(t_1) - \overline{\gamma}_1^2 t)}, & 0 \le t < t_1, \\ \left(\frac{1}{1-t} - \frac{\overline{\gamma}_2^2}{q(t_1)} \right), & t_1 \le t < 1. \end{cases}$$

Proof of Proposition 6.2.5. The formulas follow by direct integration. We have

$$\int_0^t \left(\beta^s(u)\right)^2 du = \frac{1}{(c^s)^2} \int_0^t (1-u)^{-2}\, du = \frac{1}{(c^s)^2} \left(\frac{1}{1-t} - 1 \right).$$

Hence
$$\Pi^s(t) = \Sigma^{-1} + \frac{1}{\sigma^2 (c^s)^2}\left(\frac{1}{1-t} - 1\right).$$

If $0 \le t < t_1$, then $\beta(u) = \beta_1(u)$ on $[0,t]$, so

$$\int_0^t \beta(u)^2\,du = \int_0^t \beta_1(u)^2\,du = \frac{\overline{\gamma}_2}{\overline{\gamma}_1 c(t_1)}\big(\beta_1(t) - \beta_1(0)\big).$$

Therefore,

$$\Pi(t) = \Sigma^{-1} + \frac{1}{\sigma^2}\frac{\overline{\gamma}_2}{\overline{\gamma}_1 c(t_1)}\big(\beta_1(t) - \beta_1(0)\big), \qquad 0 \le t < t_1.$$

Substitute
$$\beta_1(t) - \beta_1(0) = \frac{\overline{\gamma}_1\overline{\gamma}_2}{c(t_1)}\frac{\overline{\gamma}_1^2 t}{q(t_1)(q(t_1) - \overline{\gamma}_1^2 t)}$$
into the equation yields the result.

If $t_1 \le t < 1$, then we split the integral at t_1:

$$\int_0^t \beta(u)^2\,du = \int_0^{t_1} \beta_1(u)^2\,du + \int_{t_1}^t \frac{1}{c(t_1)^2(1-u)^2}\,du.$$

The first term is
$$\int_0^{t_1} \beta_1(u)^2\,du = \frac{\overline{\gamma}_2}{\overline{\gamma}_1 c(t_1)}\big(\beta_1(t_1^-) - \beta_1(0)\big),$$
and the second term is

$$\int_{t_1}^t \frac{1}{c(t_1)^2(1-u)^2}\,du = \frac{1}{c(t_1)^2}\left(\frac{1}{1-t} - \frac{1}{1-t_1}\right).$$

Thus

$$\Pi(t) = \Sigma^{-1} + \frac{1}{\sigma^2}\Bigg[\frac{\overline{\gamma}_2}{\overline{\gamma}_1 c(t_1)}\big(\beta_1(t_1^-) - \beta_1(0)\big) + \frac{1}{c(t_1)^2}\left(\frac{1}{1-t} - \frac{1}{1-t_1}\right)\Bigg], \qquad t_1 \le t < 1.$$

Substitute

$$\beta_1({t_1}^-) - \beta_1(0) = \frac{\overline{\gamma}_1^3 t_1}{\overline{\gamma}_2 c(t_1)(1-t_1)q(t_1)}$$

into the equation yields the result. □

Proposition 6.2.6. *(i) For every $t \in (0, t_1]$, $\Pi(t) > \Pi^s(t)$ (in particular, $\Pi(t_1) > \Pi^s(t_1)$).*

(ii) For every $t \in [t_1, 1)$,

$$\Pi(t) - \Pi^s(t) = \big(\Pi(t_1) - \Pi^s(t_1)\big) + \frac{1}{\sigma^2}\left(\frac{1}{c(t_1)^2} - \frac{1}{(c^s)^2}\right)\left(\frac{1}{1-t} - \frac{1}{1-t_1}\right).$$

Consequently, if $c(t_1) \le c^s$, then $\Pi(t) > \Pi^s(t)$ for all $t \in [t_1, 1)$. Proposition 6.2.2 shows that $c(t_1) \le c^s \iff t_1 \ge t_1^*$ for some t_1^*. If $t_1 < t_1^*$, then there exists a unique $t^* \in (t_1, 1)$ such that $\Pi(t) > \Pi^s(t)$ for $t < t^*$ and $\Pi(t) < \Pi^s(t)$ for $t > t^*$, where t^* is characterized by

$$\frac{1}{1-t^*} = \frac{1}{1-t_1} + \frac{\sigma^2\big(\Pi(t_1) - \Pi^s(t_1)\big)}{\frac{1}{(c^s)^2} - \frac{1}{c(t_1)^2}}.$$

Proof of Proposition 6.2.6. For $t \in (0, t_1]$, we have

$$\Pi(t) - \Pi^s(t) = \frac{1}{\sigma^2}\int_0^t \Big(\beta_1(u)^2 - (\beta^s(u))^2\Big)\,du.$$

By the previous comparison result, $\beta_1(u) > \beta^s(u)$ for every $u \in (0, t_1]$. Hence the integrand is strictly positive on $(0, t]$, and therefore

$$\Pi(t) > \Pi^s(t), \qquad t \in (0, t_1].$$

This proves part (i).

Now let $t \in [t_1, 1)$. Using the formulas established in the previous proposition, we have

$$\Pi(t) - \Pi^s(t) = \big(\Pi(t_1) - \Pi^s(t_1)\big) + \frac{1}{\sigma^2}\left(\frac{1}{c(t_1)^2} - \frac{1}{(c^s)^2}\right)\left(\frac{1}{1-t} - \frac{1}{1-t_1}\right),$$

which proves the identity in part (ii).

Then it is obvious that $\Pi(t) > \Pi^s(t)$ if $c(t_1) \leq c^s$. If $c(t_1) > c^s$, notice that

$$\frac{1}{1-t} - \frac{1}{1-t_1}$$

is continuous and strictly increasing on $[t_1, 1)$, with value 0 at $t = t_1$ and tending to $+\infty$ as $t \uparrow 1$, it follows that $(\Pi(t) - \Pi^s(t))$ is continuous and strictly decreasing on $[t_1, 1)$, with

$$\Pi(t_1) - \Pi^s(t_1) > 0 \qquad \text{and} \qquad \lim_{t\uparrow 1}(\Pi(t) - \Pi^s(t)) = -\infty.$$

Therefore there exists a unique $t^* \in (t_1, 1)$ such that $\Pi(t^*) - \Pi^s(t^*) = 0$. Equivalently,

$$\big(\Pi(t_1) - \Pi^s(t_1)\big) + \frac{1}{\sigma^2}\left(\frac{1}{c(t_1)^2} - \frac{1}{(c^s)^2}\right)\left(\frac{1}{1-t^*} - \frac{1}{1-t_1}\right) = 0,$$

that is,

$$\frac{1}{1-t^*} = \frac{1}{1-t_1} + \frac{\sigma^2\big(\Pi(t_1) - \Pi^s(t_1)\big)}{\frac{1}{(c^s)^2} - \frac{1}{c(t_1)^2}},$$

and thus we have

$$\Pi(t) > \Pi^s(t) \quad \text{for } t < t^*, \qquad \Pi(t) < \Pi^s(t) \quad \text{for } t > t^*.$$

□

Proposition 6.2.5 provides an explicit characterization of price informativeness. Proposition 6.2.6 then compares price informativeness in the single-insider economy and in the delayed-entry economy. The result shows that the entry of a second insider improves price informativeness relative to the single-insider benchmark before t_1. The mechanism is a competition effect: anticipating the second insider's arrival in the second period, the first insider trades more aggressively before t_1, so more information is incorporated into prices by time t_1. After t_1, however, price informativeness remains higher than in the benchmark only if the second insider enters sufficiently late. The reason is that, once two insiders are active, risk sharing can reduce insiders' trading intensity at t_1. As shown in Proposition 6.2.2, the representative insider's trading intensity exceeds that in the single-insider benchmark if the second insider enters sufficiently late, that is, when $t_1 \geq t^*$. When entry occurs late, the additional information revealed before t_1 dominates, and information is incorporated into prices more quickly after t_1, so price informativeness remains above the benchmark for the rest of the horizon. When entry occurs early, the risk-sharing effect lowers trading intensity below that in the single-insider benchmark, so although informativeness is still higher at t_1, it may eventually fall below the benchmark after the cutoff time t^*.

Proposition 6.2.7. *(i) Fix $t \in (0,1)$. If $0 < t < t_1$, then*

$$\partial_{t_1}\Pi(t) = \frac{\overline{\gamma}_2^2\overline{\gamma}_1^2\, t\,(\overline{\gamma}_1^2 - \overline{\gamma}_2^2)}{\sigma^2 c(t_1)^2\, q(t_1)^2 \big(q(t_1) - \overline{\gamma}_1^2 t\big)^2}\left(\frac{q(t_1) - \overline{\gamma}_1^2 t}{s(q(t_1))} - q(t_1)\right) < 0.$$

Hence $t_1 \mapsto \Pi(t)$ is strictly decreasing on $(t,1)$.

(ii) If $t_1 < t < 1$, then

$$\partial_{t_1}\Pi(t) = \frac{\overline{\gamma}_1^2 - \overline{\gamma}_2^2}{\sigma^2 c(t_1)^2\, q(t_1)^2\, s(q(t_1))}\left(\frac{(s(q(t_1)) + 1)q(t_1)}{1-t} - \overline{\gamma}_2^2\right) > 0.$$

Hence $t_1 \mapsto \Pi(t)$ is strictly increasing on $(0,t)$.

Proof of Proposition 6.2.7. For (i), let $t \in (0, t_1)$. Differentiating $\Pi(t)$ with respect to t_1 gives

$$\begin{aligned}
\partial_{t_1}\Pi(t) &= \frac{\overline{\gamma}_2^2\overline{\gamma}_1^2 t}{\sigma^2}\partial_{t_1}\left(\frac{1}{c_1^2 q_1(q_1 - \overline{\gamma}_1^2 t)}\right)\\
&= \frac{\overline{\gamma}_2^2\overline{\gamma}_1^2 t}{\sigma^2}\left[\partial_{t_1}\left(\frac{1}{c_1^2}\right)\frac{1}{q_1(q_1 - \overline{\gamma}_1^2 t)} + \frac{1}{c_1^2}\partial_{t_1}\left(\frac{1}{q_1(q_1 - \overline{\gamma}_1^2 t)}\right)\right]\\
&= \frac{\overline{\gamma}_2^2\overline{\gamma}_1^2 t}{\sigma^2}\left[\frac{(\overline{\gamma}_1^2 - \overline{\gamma}_2^2)(s_1+1)}{c_1^2 q_1 s_1}\frac{1}{q_1(q_1 - \overline{\gamma}_1^2 t)} - \frac{\overline{\gamma}_1^2 - \overline{\gamma}_2^2}{c_1^2}\frac{2q_1 - \overline{\gamma}_1^2 t}{q_1^2(q_1 - \overline{\gamma}_1^2 t)^2}\right]\\
&= \frac{\overline{\gamma}_2^2\overline{\gamma}_1^2 t(\overline{\gamma}_1^2 - \overline{\gamma}_2^2)}{\sigma^2 c_1^2 q_1^2 (q_1 - \overline{\gamma}_1^2 t)^2}\left(\frac{q_1 - \overline{\gamma}_1^2 t}{s_1} - q_1\right).
\end{aligned}$$

Moreover, $s(q(t_1)) > 1$ and $q(t_1) - \overline{\gamma}_1^2 t > 0$, so

$$\frac{q(t_1) - \overline{\gamma}_1^2 t}{s(q(t_1))} < q(t_1) - \overline{\gamma}_1^2 t < q(t_1).$$

Hence

$$\frac{q(t_1) - \overline{\gamma}_1^2 t}{s(q(t_1))} - q(t_1) < 0,$$

and therefore $\partial_{t_1}\Pi(t) < 0$. Thus $t_1 \mapsto \Pi(t)$ is strictly decreasing on $(t, 1)$.

For (ii), let $t \in (t_1, 1)$. Differentiating $\Pi(t)$ with respect to t_1 yields

$$\begin{aligned}
\partial_{t_1}\Pi(t) &= \frac{1}{\sigma^2}\left[\partial_{t_1}\left(\frac{1}{c(t_1)^2}\right)\left(\frac{1}{1-t} - \frac{\overline{\gamma}_2^2}{q(t_1)}\right) + \frac{1}{c(t_1)^2}\partial_{t_1}\left(-\frac{\overline{\gamma}_2^2}{q(t_1)}\right)\right] \\
&= \frac{\overline{\gamma}_1^2 - \overline{\gamma}_2^2}{\sigma^2 c(t_1)^2}\left[\frac{s(q(t_1)) + 1}{q(t_1)s(q(t_1))}\left(\frac{1}{1-t} - \frac{\overline{\gamma}_2^2}{q(t_1)}\right) + \frac{\overline{\gamma}_2^2}{q(t_1)^2}\right] \\
&= \frac{\overline{\gamma}_1^2 - \overline{\gamma}_2^2}{\sigma^2 c(t_1)^2\, q(t_1)^2\, s(q(t_1))}\left(\frac{(s(q(t_1)) + 1)q(t_1)}{1-t} - \overline{\gamma}_2^2\right).
\end{aligned}$$

We then have

$$s(q(t_1)) + 1 > 2, \qquad \frac{1}{1-t} > 1, \qquad q(t_1) \geq \overline{\gamma}_2^2,$$

and therefore

$$\frac{(s(q(t_1)) + 1)q(t_1)}{1-t} > 2q(t_1) \geq 2\overline{\gamma}_2^2 > \overline{\gamma}_2^2.$$

Thus

$$\frac{(s(q(t_1)) + 1)q(t_1)}{1-t} - \overline{\gamma}_2^2 > 0,$$

which implies $\partial_{t_1}\Pi(t) > 0$. Hence $t_1 \mapsto \Pi(t)$ is strictly increasing on $(0, t)$. □

Proposition 6.2.7 shows that, for any fixed time $t < t_1$, price informativeness decreases as t_1 increases before the second insider enters the market. This is because, when the second insider enters later, the first insider faces weaker competition over $[0, t_1)$ and therefore has less incentive to trade aggressively and reveal information before entry. As a result, less information is incorporated into prices before t_1, and price informativeness is lower.

By contrast, for any fixed time $t > t_1$, price informativeness increases as t_1 increases. The reason is that a later entry weakens the effect of risk sharing, so trading intensity after t_1 increases, as shown in Proposition 6.2.1. Consequently, information is incorporated into prices more quickly after entry, and price informativeness is higher.

6.2.3 Market Impact Parameter λ and Market Depth $1/\lambda$

Proposition 6.2.8. *$\lambda(t)$ decreases with time t. There is a jump in $\lambda(t)$ at time t_1; in particular, $\lambda(t_1) < \lim_{t\uparrow t_1} \lambda(t)$.*

$\lambda(t)$ decreases over time because progressively more information is incorporated into prices. As a result, order flow becomes less informative, so the market impact of a given trade declines. $\lambda(t)$ jumps down at t_1, when the second insider enters the market. Intuitively, the nature of order flow changes abruptly at t_1. Once the second insider becomes active, risk sharing reduces the insiders' aggregate trading intensity, which lowers the market impact parameter $\lambda(t)$, market depth jumps up at t_1 equivalently.

Proposition 6.2.9. *Fix $t_1 \in (0,1)$, Let λ^s denote the single-insider price impact with risk aversion $\overline{\gamma}_1$,*

$$\lambda^s(t) = \frac{c^s}{1 - \overline{\gamma}_1 \sigma^2 c^s (1-t)}, \qquad c^s = \frac{\Sigma \overline{\gamma}_1}{2}\left(\sqrt{1 + \frac{4}{\sigma^2 \Sigma \overline{\gamma}_1^2}} - 1\right).$$

Let λ denote the equilibrium price impact in Theorem 5.1.3. Define $t^* \in (0,1)$ by

$$t_1^* = \frac{q^* - \overline{\gamma}_2^2}{\overline{\gamma}_1^2 - \overline{\gamma}_2^2}, \qquad q^* = \frac{\frac{4}{\sigma^2 \Sigma}\, \overline{\gamma}_2^2}{\overline{\gamma}_1\left(\sqrt{1 + \frac{4}{\sigma^2 \Sigma \overline{\gamma}_1^2}} - 1\right)\left(\overline{\gamma}_1\left(\sqrt{1 + \frac{4}{\sigma^2 \Sigma \overline{\gamma}_1^2}} - 1\right) + 2\overline{\gamma}_2\right)}.$$

which is the same definition in Proposition 6.2.2. Then:

(i) For all $t \in [0, t_1)$, $\lambda(t) > \lambda^s(t)$.

(ii) For $t \in [t_1, 1)$: if $t_1 \geq t_1^*$, then $\lambda(t) < \lambda^s(t)$ for all $t \in [t_1, 1)$; if $t_1 < t_1^*$, then λ^s crosses λ exactly once on $(t_1, 1)$: there exists a unique $t^* \in (t_1, 1)$ such that $\lambda(t) < \lambda^s(t)$ for $t \in [t_1, t^*)$ and $\lambda(t) > \lambda^s(t)$ for $t \in (t^*, 1)$, where

$$t^* = 1 - \frac{\frac{1}{c^s} - \frac{1}{c(t_1)}}{(\overline{\gamma}_1 - \overline{\gamma}_2)\sigma^2}.$$

Proof of Proposition 6.2.9. (1) Let $t < t_1$. Since

$$\frac{1}{\lambda(t)} = \frac{1}{\lambda(t_1^-)} + \overline{\gamma}_1\sigma^2(t-t_1), \qquad \frac{1}{\lambda^s(t)} = \frac{1}{\lambda^s(t_1)} + \overline{\gamma}_1\sigma^2(t-t_1),$$

we have

$$\lambda(t) > \lambda^s(t) \quad \text{for all } t < t_1 \iff \lambda(t_1^-) > \lambda^s(t_1).$$

Now

$$\lambda(t_1^-) = \frac{\overline{\gamma}_1}{\overline{\gamma}_2}\lambda(t_1), \qquad \lambda(t_1) = \frac{c(t_1)}{1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t_1)}.$$

Hence

$$\frac{1}{\lambda(t_1^-)} = \frac{\overline{\gamma}_2}{\overline{\gamma}_1}\left(\frac{1}{c(t_1)} - \overline{\gamma}_2\sigma^2(1-t_1)\right).$$

Using

$$\frac{1}{\sigma^2 c(t_1)} = \frac{q(t_1)\big(s(q(t_1))+1\big)}{2\overline{\gamma}_2},$$

we obtain

$$\frac{1}{\lambda(t_1^-)} = \sigma^2\left(\frac{q(t_1)\big(s(q(t_1))-1\big)}{2\overline{\gamma}_1} + \overline{\gamma}_1 t_1\right).$$

Similarly,

$$\frac{1}{\lambda^s(t_1)} = \frac{1}{c^s} - \overline{\gamma}_1\sigma^2(1-t_1) = \sigma^2\left(\frac{\overline{\gamma}_1\big(s(\overline{\gamma}_1^2)-1\big)}{2} + \overline{\gamma}_1 t_1\right).$$

Therefore

$$\frac{1}{\lambda(t_1^-)} < \frac{1}{\lambda^s(t_1)} \iff q(t_1)\big(s(q(t_1))-1\big) < \overline{\gamma}_1^2\big(s(\overline{\gamma}_1^2)-1\big).$$

Since the map $x \mapsto x(s(x)-1)$ is strictly increasing and $q(t_1) < \overline{\gamma}_1^2$, the last inequality holds. Thus

$$\frac{1}{\lambda(t_1^-)} < \frac{1}{\lambda^s(t_1)},$$

so $\lambda(t_1^-) > \lambda^s(t_1)$. Hence

$$\lambda(t) > \lambda^s(t), \qquad t \in [0, t_1).$$

(2) Let $t \in [t_1, 1)$. Then

$$\frac{1}{\lambda(t)} = \frac{1}{c(t_1)} - \overline{\gamma}_2\sigma^2(1-t), \qquad \frac{1}{\lambda^s(t)} = \frac{1}{c^s} - \overline{\gamma}_1\sigma^2(1-t).$$

Then we have

$$\frac{1}{\lambda(t)} - \frac{1}{\lambda^s(t)} = \left(\frac{1}{c(t_1)} - \frac{1}{c^s}\right) + (\overline{\gamma}_1 - \overline{\gamma}_2)\sigma^2(1-t).$$

Since $\overline{\gamma}_1 > \overline{\gamma}_2$, $1/\lambda(t) - 1/\lambda^s(t)$ is strictly decreasing in t.

By the characterization established earlier,

$$c(t_1) \le c^s \iff t_1 \ge t^*.$$

If $t_1 \ge t^*$, then $c(t_1) \le c^s$, so

$$\frac{1}{c(t_1)} - \frac{1}{c^s} \ge 0.$$

Since $1/\lambda(t) - 1/\lambda^s(t)$ is strictly decreasing, it follows that

$$\frac{1}{\lambda(t)} - \frac{1}{\lambda^s(t)} \ge \frac{1}{\lambda(1)} - \frac{1}{\lambda^s(1)} \ge 0, \qquad t \in [t_1, 1).$$

therefore

$$\lambda(t) \le \lambda^s(t), \qquad t \in [t_1, 1).$$

Now suppose $t_1 < t^*$. Then $c(t_1) > c^s$, so

$$\frac{1}{c(t_1)} - \frac{1}{c^s} < 0.$$

We claim that

$$\frac{1}{\lambda(t_1)} - \frac{1}{\lambda^s(t_1)} > 0.$$

Using

$$\frac{1}{\sigma^2 c(t_1)} = \frac{q(t_1)\big(s(q(t_1)) + 1\big)}{2\overline{\gamma}_2}, \qquad \frac{1}{\sigma^2 c^s} = \frac{\overline{\gamma}_1\big(s(\overline{\gamma}_1^2) + 1\big)}{2},$$

we obtain

$$\frac{1}{\sigma^2}\left(\frac{1}{\lambda(t_1)}-\frac{1}{\lambda^s(t_1)}\right)=\frac{1}{\sigma^2 c(t_1)}-\frac{1}{\sigma^2 c^s}+(\overline{\gamma}_1-\overline{\gamma}_2)(1-t_1)$$
$$=\frac{1}{2}\left(\frac{q(t_1)\big(s(q(t_1))-1\big)}{\overline{\gamma}_2}-\overline{\gamma}_1\big(s(\overline{\gamma}_1^2)-1\big)\right)+\overline{\gamma}_1 t_1\left(\frac{\overline{\gamma}_1}{\overline{\gamma}_2}-1\right).$$

Since $c(t_1) > c^s$, we have

$$\overline{\gamma}_2\big(s(q(t_1))-1\big) > \overline{\gamma}_1\big(s(\overline{\gamma}_1^2)-1\big).$$

Moreover,

$$q(t_1) \geq \overline{\gamma}_2^2,$$

and therefore

$$\frac{q(t_1)\big(s(q(t_1))-1\big)}{\overline{\gamma}_2}=\frac{q(t_1)}{\overline{\gamma}_2^2}\overline{\gamma}_2\big(s(q(t_1))-1\big)\geq\overline{\gamma}_2\big(s(q(t_1))-1\big)>\overline{\gamma}_1\big(s(\overline{\gamma}_1^2)-1\big).$$

Hence the first term is strictly positive. The second term is also nonnegative, and therefore

$$\frac{1}{\lambda(t_1)}-\frac{1}{\lambda^s(t_1)}>0.$$

Since $\frac{1}{\lambda(t)}-\frac{1}{\lambda^s(t)}$ is continuous and strictly decreasing in t,

$$\frac{1}{\lambda\left(t_1\right)}-\frac{1}{\lambda^s\left(t_1\right)}>0,\quad \lim_{t\uparrow 1}\left(\frac{1}{\lambda(t)}-\frac{1}{\lambda^s(t)}\right)=\frac{1}{c\left(t_1\right)}-\frac{1}{c^s}<0,$$

there exists a unique $t^* \in (t_1, 1)$ such that

$$\frac{1}{\lambda\left(t^*\right)}-\frac{1}{\lambda^s\left(t^*\right)}=0.$$

Hence

$$\lambda(t)<\lambda^s(t) \text{ for } t\in\left[t_1,t^*\right),\quad \lambda(t)>\lambda^s(t) \text{ for } t\in\left(t^*,1\right).$$

□

Proposition 6.2.9 shows that, relative to the single-insider benchmark, price impact is higher before entry in the delayed-entry economy, that is, $\lambda(t) > \lambda^s(t)$ for all $t < t_1$.

The reason is that competition from the second insider induces the first insider to trade more aggressively before t_1. This makes order flow more informative prior to entry, increases price impact, and equivalently reduces market depth on $[0, t_1)$.

At time t_1, the entry of the second insider increases aggregate risk tolerance on the informed side. The risk-sharing effect weakens informed trading intensity relative to its pre-entry level. As a result, the comparison with the single-insider benchmark after entry depends on the timing of entry. If the second insider enters late, aggregate informed trading may exceed that in the single-insider benchmark, but price informativeness at t_1 is higher than in the single-insider benchmark. Hence order flow is less informative, which lowers price impact relative to the single-insider benchmark and improves market depth. If the second insider enters sufficiently early, then the risk-sharing effect is strong, so aggregate informed trading intensity declines. Moreover, because price informativeness may eventually fall below the single-insider benchmark, $\lambda(t)$ is initially below $\lambda^s(t)$ just after t_1, but later rises relative to the benchmark, so the two curves cross exactly once on $(t_1, 1)$.

Proposition 6.2.10. *Fix $t \in (0,1)$. Then*

$$\partial_{t_1}\lambda(t) < 0 \qquad \text{for all } t_1 \in (0,t) \cup (t,1).$$

Equivalently, for each fixed t, the map $t_1 \mapsto \lambda(t)$ is strictly decreasing on $(0,t)$ and on $(t,1)$.

Proof of Proposition 6.2.10. First suppose that $t_1 < t$. Then

$$\lambda(t) = \frac{c(t_1)}{1 - \overline{\gamma}_2\sigma^2 c(t_1)(1-t)}.$$

Differentiating with respect to t_1 gives

$$\partial_{t_1}\lambda(t) = \frac{\partial_{t_1}c(t_1)}{\big(1 - \overline{\gamma}_2\sigma^2 c(t_1)(1-t)\big)^2}.$$

We have shown in proof that $\partial_{t_1}c(t_1) < 0$, it follows that $\partial_{t_1}\lambda(t) < 0$.

Now suppose that $t < t_1$. Then

$$\lambda(t) = \frac{\frac{\overline{\gamma}_1}{\overline{\gamma}_2}\lambda(t_1)}{1 - \frac{\overline{\gamma}_1^2\sigma^2}{\overline{\gamma}_2}\lambda(t_1)(t_1 - t)}.$$

Equivalently,

$$\frac{1}{\lambda(t)} = \frac{\overline{\gamma}_2}{\overline{\gamma}_1\,\lambda(t_1)} - \overline{\gamma}_1\sigma^2(t_1 - t).$$

Since

$$\lambda(t_1) = \frac{c(t_1)}{1 - \overline{\gamma}_2\sigma^2 c(t_1)(1 - t_1)},$$

We then have

$$\begin{aligned}\frac{1}{\lambda(t)} &= \frac{\overline{\gamma}_2}{\overline{\gamma}_1}\left(\frac{1}{c(t_1)} - \overline{\gamma}_2\sigma^2(1 - t_1)\right) - \overline{\gamma}_1\sigma^2(t_1 - t) \\ &= \frac{\overline{\gamma}_2}{\overline{\gamma}_1\,c(t_1)} - \frac{\sigma^2}{\overline{\gamma}_1}\big(\overline{\gamma}_2^2(1 - t_1) + \overline{\gamma}_1^2(t_1 - t)\big) \\ &= \frac{\overline{\gamma}_2}{\overline{\gamma}_1\,c(t_1)} - \frac{\sigma^2}{\overline{\gamma}_1}\big(q(t_1) - \overline{\gamma}_1^2 t\big).\end{aligned}$$

Differentiating with respect to t_1 yields

$$\partial_{t_1}\left(\frac{1}{\lambda(t)}\right) = -\frac{\overline{\gamma}_2}{\overline{\gamma}_1}\frac{\partial_{t_1}c(t_1)}{c(t_1)^2} - \frac{\sigma^2}{\overline{\gamma}_1}\partial_{t_1}q(t_1) = \frac{\partial_{t_1}q(t_1)}{\overline{\gamma}_1}\left(\frac{\overline{\gamma}_2^2}{\sigma^2\,s(q(t_1))\,q(t_1)^2\,c(t_1)^2} - \sigma^2\right).$$

It remains to simplify the term in parentheses. Since

$$c(t_1) = \frac{\Sigma\overline{\gamma}_2}{2}\big(s(q(t_1)) - 1\big),$$

we have

$$\frac{\overline{\gamma}_2^2}{\sigma^2\,s(q(t_1))\,q(t_1)^2\,c(t_1)^2} = \frac{4}{\sigma^2\Sigma^2\,s(q(t_1))\,q(t_1)^2\big(s(q(t_1)) - 1\big)^2}.$$

Using

$$q(t_1)\big(s(q(t_1))^2 - 1\big) = \kappa = \frac{4}{\sigma^2\Sigma},$$

we obtain
$$\frac{\overline{\gamma}_2^2}{\sigma^2\, s(q(t_1))\, q(t_1)^2\, c(t_1)^2} = \sigma^2\, \frac{(s(q(t_1))+1)^2}{4\, s(q(t_1))}.$$
Therefore
$$\partial_{t_1}\left(\frac{1}{\lambda(t)}\right) = \frac{\sigma^2 \partial_{t_1} q(t_1)}{\overline{\gamma}_1}\left(\frac{(s(q(t_1))+1)^2}{4\, s(q(t_1))} - 1\right) = \frac{\sigma^2 \partial_{t_1} q(t_1)}{\overline{\gamma}_1}\,\frac{(s(q(t_1))-1)^2}{4\, s(q(t_1))} > 0.$$
Since $\lambda(t) > 0$, it follows that
$$\partial_{t_1}\lambda(t) = -\lambda(t)^2\, \partial_{t_1}\left(\frac{1}{\lambda(t)}\right) < 0.$$

Thus $\partial_{t_1}\lambda(t) < 0$ for all $t_1 \in (0,t) \cup (t,1)$, as claimed. □

Proposition 6.2.10 shows that later entry by the second insider improves market depth. When the second insider arrives later, the first insider faces less competition and therefore trades less aggressively before entry. This lowers the market impact parameter and hence increases market depth. After entry, later arrival also improves price informativeness. As a result, order flow becomes less informative, which further improves market depth even when trading intensity after t_1 may be higher.

6.2.4 Insider's Welfare

Proposition 6.2.11. *Fix $\Sigma > 0, \sigma^2 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, and let $t_1 \in (0,1)$. The first insider's welfare is*
$$\mathcal{J}(0) = -\sqrt{\frac{c(t_1)\,\overline{\gamma}_1}{\overline{\gamma}_2\,\lambda(0)}}\,e^{-\frac{\overline{\gamma}_1}{2\lambda(0)}(v-\mu)^2}, \quad \text{where } \lambda(t_1) = \frac{c(t_1)}{1-\overline{\gamma}_2\sigma^2 c(t_1)(1-t_1)}.$$
Then we have $\partial_{t_1} u(0) > 0, \partial_{\gamma_2} u(0) > 0$.

Proof of proposition 6.2.11. We have that
$$u(0) = -\sqrt{\frac{s(q(t_1))-1}{s(q(t_1))+1}}\, e^{-\frac{(v-\mu)^2}{\Sigma(s(q(t_1))+1)}};$$

this can be written simply as

$$u(0) = -r\big(s(q(t_1))\big).$$

$q(t_1)$ increases with t_1 and γ_2. Therefore, $u(0)$ is strictly increasing with t_1 and γ_2. $\square$

Proposition 6.2.11 shows that the insider's welfare increases with t_1 and γ_2. The monotonicity of welfare in t_1 reflects a competition effect. The competition arises because, after entry, the second insider shares part of the profits from informed trading and therefore reduces the first insider's profit opportunities. When t_1 is larger, the first insider remains the sole informed trader for a longer period and can therefore exploit her informational advantage for longer. This increases her trading profits and, in turn, raises her welfare. When the second insider's risk aversion is higher, her share of informed trading becomes smaller, so the first insider retains a larger share of post-entry profits. In this sense, competition is weaker, which also raises the first insider's welfare.

Chapter 7

Concluding Remarks

In this paper, we study imperfect competition among risk-averse insiders in continuous time and characterize how entry, competition, and heterogeneity in risk aversion shape equilibrium trading behavior and market outcomes. Our first main result is that, unlike in the risk-neutral case, a linear equilibrium exists even when multiple insiders observe the same signal. We then analyze sequential entry and show that the timing of entry plays a central role in equilibrium dynamics. When a second insider enters the market, the first insider trades more aggressively before entry in order to exploit more of her informational advantage before the competition. After entry, however, her trading intensity declines because of the risk-sharing effect. Accordingly, price informativeness is always higher before entry than in the single-insider benchmark, whereas after entry it may be either higher or lower than in the benchmark, depending on the timing of the second insider's arrival. Market depth displays a similar pattern: it is lower before entry because insiders trade more aggressively, while after entry it is initially higher than in the benchmark and may later cross it, depending on the timing of entry. In addition, the first insider's welfare is lower than in the single-insider benchmark. Finally, we show that insiders' optimal trading strategies depend in a subtle way on their degrees of risk aversion.

We conclude by noting that, in our setting, the second insider is not allowed to trade before receiving the signal at time t_1, when she enters the market. One natural extension is to allow the second insider to trade before time t_1, without receiving the private signal and to characterize the corresponding equilibrium. Another extension is to allow the first insider to receive a noisy signal about the fundamental and to study how signal precision affects equilibrium. This is technically challenging, since even in Kyle-type models the equilibrium with risk-averse insiders and noisy signals is difficult to characterize.

Appendix A

Proof that discrete order is suboptimal

We allow insiders to submit discrete trading strategy. By allowing a possible price jump at time t_1 and time 1, the wealth process for insiders are

$$\begin{aligned}\mathcal{W}^1_{0,1}(p) &= \int_0^{t_1-} X^1(u-)\,dp(u) + X^1(t_1-)\big(p_{t_1} - p_{t_1-}\big) \\ &\quad + \int_{t_1}^{1} X^1(u-)\,dp(u) + X^1(1)\big(v - p(1)\big) \\ &= \int_0^{t_1} \big(v - p(u-)\big)\,dX^1(u) - [X^1, p](0, t_1) \\ &\quad + \int_{t_1}^{1} \big(v - p(u-)\big)\,dX^1(u) - [X^1, p](t_1, 1).\end{aligned}$$

$$\begin{aligned}\mathcal{W}^2_{t_1,1}(p) &= \int_{t_1}^{1} X^2(u-)\,dp(u) + X^2(1)\big(v - p(1)\big) \\ &= \int_{t_1}^{1} \big(v - p(u-)\big)\,dX^2(u) - [X^2, p](t_1, 1).\end{aligned}$$

We slightly modify the definition of equilibrium:

Definition A.0.1. *The triple $(\widehat{X}^1, \widehat{X}^2, \widehat{p})$ forms an equilibrium if*

(i) $\widehat{p}(t) = \mathbb{E}\left[v \middle| \mathcal{F}^{\widehat{Y}}_t\right]$ *where* $\widehat{Y}(t) = Z(t) + \widehat{X}^1(t) + 1_{t \geq t_1}\widehat{X}^2(t)$ *for* $0 \leq t \leq 1$.

(ii) $\widehat{X}^1$ *solves*

$$\inf_{X^1 \in \mathcal{A}_0} \mathbb{E}\left[e^{-\gamma_1 \widehat{\mathcal{W}}^1_{0,1}(\widehat{p})} \middle| v\right].$$

(iii) $\widehat{X}^2$ *solves*

$$\inf_{X^2 \in \mathcal{A}_{t_1}} \mathbb{E}\left[e^{-\gamma_2 \widehat{\mathcal{W}}^2_{t_1,1}(\widehat{p})} \middle| \mathcal{F}^I_{t_1}\right].$$

Following the same idea in section 3, by the property of CARA utility, one can always express the optimal control problem after t_1 as a control problem for a representative agent with risk aversion parameter $\overline{\gamma}_2$, where $\frac{1}{\overline{\gamma}_2} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$.

Denote X be the representative agent's position and define the representative agent's wealth after t_1 to be

$$\mathcal{W}_{t_1,1}(p) = \int_{t_1}^{1} (v - p(u-))dX(u) - [X, p](t_1, 1).$$

By symmetry, the optimizer satisfies

$$\widehat{X}^1 = \frac{\overline{\gamma}_2}{\gamma_1}\widehat{X}, \widehat{X}^2 = \frac{\overline{\gamma}_2}{\gamma_2}\widehat{X},$$

where $\widehat{X}$ solves

$$\inf_{X \in \mathcal{A}_{t_1}} \mathbb{E}\left[e^{-\overline{\gamma}_2 \widehat{\mathcal{W}}_{t_1,1}(\widehat{p})} \middle| \mathcal{F}^I_{t_1}\right].$$

If we set $X = X^1$ before over $[0, t_1]$, we have that

$$Y(t) = Z(t) + X(t).$$

Hence, the equilibrium could be redefined as

Definition A.0.2. $(\widehat{X}, \widehat{p})$ *forms an equilibrium if*

(i) $\widehat{p}(t) = \mathbb{E}\left[v \middle| \mathcal{F}^{\widehat{Y}}_t\right]$ *where* $\widehat{Y}(t) = Z(t) + \widehat{X}(t)$ *for* $0 \leq t \leq 1$.

(ii) $\widehat{X}$ *solves*

$$\inf_{X\in\mathcal{A}_0}\mathbb{E}\left[e^{-\gamma_1\{\int_0^{t_1}(v-p(u-))dX(u)-[X,p](0,t_1)\}-\overline{\gamma}_2\{\int_{t_1}^1(v-p(u-))dX(u)-[X,p](t_1,1)\}}\middle|v\right].$$

The conjecture of pricing rule is the same as Conjecture 3.0.3. The definition of admissible class follows the definition in 3.0.4 with (i) to be changed to be X is a $\mathbb{F}^I$-adapted on $[t,T]$.

We thus have the following proposition,

Proposition A.0.3. *Fix* $t\in[t_1,1]$ *and let* $\xi(t)$ *be an* $\mathcal{F}_t^Y$*-measurable random variable representing the state at time* t*. Suppose* (H,λ) *satisfies Conditions C1–C2 of Lemma 4.1.1. Let* J *be the utility function defined in* (4.1.11)*. If* $X\in\mathcal{A}_t(\xi(t))$ *is any admissible strategy that either*

(i) contains a discrete order (i.e. has a jump), or

(ii) has a nonzero local martingale component, or

(iii) fails to satisfy the terminal condition $H(1,\widehat{\xi}(1))=v$ *almost surely, then* X *is strictly suboptimal: its conditional expected utility at time* t *is strictly above the value* $J(t,\xi(t))$*.*

Proof of Proposition A.0.3. Given $\mathcal{F}_t^I$, define

$$\xi(s)=\xi(t)+\int_t^s\lambda(u)\,dY(u),$$

where Y is càdlàg and may jump, and λ is continuous on $[t_1, 1]$. Then

$$
\begin{aligned}
&\log J(1,\xi(1)) - \log J(t,\xi(t)) \\
&= \int_t^1 \frac{J_t}{J}\big(u,\xi(u-)\big)\,du + \int_t^1 \frac{J_\xi}{J}\big(u,\xi(u-)\big)\lambda(u)\,dY(u) \\
&\quad + \frac{1}{2}\int_t^1 \left[\frac{J_{\xi\xi}}{J} - \left(\frac{J_\xi}{J}\right)^2\right]\big(u,\xi(u-)\big)\lambda(u)^2\,d[Y^c,Y^c](u) \\
&\quad + \sum_{t<u\le 1}\Bigg[\log J\big(u,\xi(u-)+\lambda(u)\Delta Y(u)\big) - \log J\big(u,\xi(u-)\big) \\
&\qquad\qquad - \frac{J_\xi}{J}\big(u,\xi(u-)\big)\lambda(u)\Delta Y(u)\Bigg].
\end{aligned}
$$

We have

$$d[\xi^c](t) = \lambda(t)^2\,d[Y^c,Y^c](t)$$

and

$$[Y^c,Y^c](t) = [X^c,X^c](t) + 2[X^c,Z](t) + \sigma^2 t.$$

Substituting system (4.1.4) into the above yields

$$
\begin{aligned}
&\log J(1,\xi_1) - \log J(t,\xi_t) \\
&= \overline{\gamma}_2\int_t^1 \big(v - H(u,\xi(u-))\big)\,dX(u) + \overline{\gamma}_2\int_t^1 \big(v - H(u,\xi(u-))\big)\,dZ(u) \\
&\quad - \frac{1}{2}\,\overline{\gamma}_2^2\int_t^1 \big(v - H(u,\xi(u-))\big)^2\Big(d[X^c,X^c](u) + 2\,d[X^c,Z](u) + \sigma^2\,du\Big) \\
&\quad + \frac{1}{2}\int_t^1 \lambda(u)^2\frac{J_{\xi\xi}}{J}\big(u,\xi(u-)\big)\Big(d[X^c,X^c](u) + 2\,d[X^c,Z](u)\Big) \\
&\quad + \sum_{t<u\le 1}\Big[\Delta\log J(u,\xi(u)) \\
&\qquad\qquad - \overline{\gamma}_2\big(v - H(u,\xi(u-))\big)\Delta Y(u)\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&-\overline{\gamma}_2 \int_t^1 \big(v - H(u,\xi(u-))\big)\, dX(u) + \overline{\gamma}_2 [H,X](t,1) \\
&= -\log J(1,\xi_1) + \log J(t,\xi_t) + \overline{\gamma}_2 \int_t^1 \big(v - H(u,\xi(u-))\big)\, dZ(u) \\
&\quad - \frac{1}{2}\,\overline{\gamma}_2^2 \int_t^1 \big(v - H(u,\xi(u-))\big)^2 \Big(d[X^c,X^c](u) + 2\, d[X^c,Z](u) + \sigma^2\, du\Big) \\
&\quad + \frac{1}{2} \int_t^1 \lambda(u)^2 \frac{J_{\xi\xi}}{J}\big(u,\xi(u-)\big) \Big(d[X^c,X^c](u) + 2\, d[X^c,Z](u)\Big) \\
&\quad + \sum_{t<u\le 1} \Big[\Delta \log J(u,\xi(u)) - \overline{\gamma}_2\big(v - H(u,\xi(u-))\big)\Delta Y(u)\Big] + \overline{\gamma}_2 [H,X](t,1).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
[H,X](t,1) &= [H^c,X^c](t,1) + \sum_{t<u\le 1} \Delta H(u)\, \Delta X(u) \\
&= \int_t^1 H_\xi\big(u,\xi(u-)\big)\, \lambda(u)\, d[Y^c,X^c](u) + \sum_{t<u\le 1} \Delta H(u)\, \Delta X(u).
\end{aligned}
$$

Since $\Delta Y(u) = \Delta X(u)$ and $\xi(u) = \xi(u-) + \lambda(u)\Delta Y(u)$, we have

$$
\big(v - H(u,\xi(u-))\big)\Delta Y(u) + \Delta H(u)\, \Delta Y(u) = \big(v - H(u,\xi(u))\big)\Delta X(u),
$$

and therefore

$$
\begin{aligned}
&\overline{\gamma}_2 \Big[\big(v - H(u,\xi(u-))\big)\Delta Y(u) + \Delta H(u)\, \Delta Y(u)\Big] \\
&= \overline{\gamma}_2 \big(v - H(u,\xi(u))\big)\Delta X(u) = \lambda(u)\, \partial_\xi \log J\big(u,\xi(u)\big)\, \Delta X(u),
\end{aligned}
$$

where $\partial_\xi \log J\big(u,\xi(u)\big) = J_\xi\big(u,\xi(u)\big)/J\big(u,\xi(u)\big)$.

In addition,

$$
H_\xi\big(u,\xi(u-)\big) = -\frac{\lambda(u)}{\overline{\gamma}_2} \frac{J_{\xi\xi}\big(u,\xi(u-)\big)}{J\big(u,\xi(u-)\big)} + \frac{\overline{\gamma}_2}{\lambda(u)}\big(v - H(u,\xi(u-))\big)^2.
$$

Set

$$\psi(s) := \mathcal{E}\left(\overline{\gamma}_2 \int_{t_1}^{s} \big(v - H(u,\xi(u-))\big)\sigma\, dB(u)\right), \qquad \psi(s,1) := \frac{\psi(1)}{\psi(s)}.$$

Then

$$\begin{aligned}
&-\overline{\gamma}_2 \int_t^1 \big(v - H(u,\xi(u-))\big)\, dX(u) + \overline{\gamma}_2 [H,X](t,1) \\
&= -\log J(1,\xi(1)) + \log J(t,\xi(t)) + \log\big(\psi(t,1)\big) \\
&\quad + \frac{1}{2}\int_t^1 \left(\lambda(u)^2\Big(\frac{J_\xi}{J}(u,\xi(u-))\Big)^2 - \lambda(u)^2 \frac{J_{\xi\xi}}{J}\big(u,\xi(u-)\big)\right) d[X^c,X^c](u) \\
&\quad + \sum_{t<u\le 1}\Big[\Delta \log J(u,\xi(u)) - \partial_\xi \log J\big(u,\xi(u)\big)\,\Delta\xi(u)\Big], \qquad \Delta\xi(u) = \lambda(u)\Delta X(u). \\
&= -\log J(1,\xi_1) + \log J(t,\xi_t) + \log\big(\psi(t,1)\big) \\
&\quad - \frac{1}{2}\int_t^1 \lambda(u)^2 \partial_{\xi\xi} \log J(t,\xi)\, d[X^c,X^c](u) \\
&\quad + \sum_{t<u\le 1}\Big[\Delta \log J(u,\xi(u)) - \partial_\xi \log J\big(u,\xi(u)\big)\,\Delta\xi(u)\Big],
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[e^{-\overline{\gamma}_2\left\{\int_t^1 \left(v - H(u,\xi(u-))\right)\, dX(u) - [H,X](t,1)\right\}}\,\middle|\, \mathcal{F}_t^I\right] \\
&\qquad = \mathbb{E}\left[\begin{matrix} J(t,\xi(t))\, J(1,\xi(1))^{-1}\, \psi(t,1) \\ \times\, e^{-\frac{1}{2}\int_t^1 \lambda(u)^2\, \partial_{\xi\xi} \log J\left(u,\xi(u-)\right)\, d[X^c,X^c](u)} \\ \times\, e^{\sum_{t<u\le 1}\left[\Delta \log J\left(u,\xi(u)\right) - \partial_\xi \log J\left(u,\xi(u-)\right)\,\Delta\xi(u)\right]} \end{matrix}\,\middle|\, \mathcal{F}_t^I\right].
\end{aligned}$$

Since

$$\lambda(t)\partial_{\xi\xi} \log J(t,\xi) = -\overline{\gamma}_2 H_\xi(t,\xi) < 0,$$

it follows that $\log J$ is strictly concave in ξ. Hence,

$$-\frac{1}{2}\int_t^1 \lambda(u)^2 \partial_{\xi\xi} \log J(t,\xi)\, d[X^c, X^c](u) \geq 0,$$

with equality if and only if (i) does not hold, and

$$\sum_{t<u\leq 1} \Big[\Delta \log J(u,\xi(u)) - \partial_\xi \log J\big(u,\xi(u)\big)\, \Delta\xi(u)\Big] \geq 0,$$

with equality if and only if (ii) does not hold.

Therefore, if at least one of (i), (ii), or (iii) holds, then

$$\mathbb{E}\left[e^{-\overline{\gamma}_2\{\int_t^1 (v-H(u,\xi(u-)))dX(u)-[H,X](t,1)\}}\middle|\mathcal{F}_t^I\right] > J(t,\xi(t)).$$

This proves the claim. ☐

Therefore, the optimal utility at time t_1 is given by (4.1.12), and it is attained by the absolutely continuous strategy described in Section 4.1.

Over the interval $[0,t_1]$, we establish the following proposition.

Proposition A.0.4. *Fix $t \in [0,t_1]$ and let $\xi(t)$ be an $\mathcal{F}_t^Y$-measurable random variable representing the state at time t. Suppose (H,λ) satisfies Conditions C1–C2 of Lemma 4.2.1. Let J be the utility function defined in* (4.2.10)*. If $X \in \mathcal{A}_t(\xi(t))$ is any admissible strategy that either*

(i) contains a discrete order (i.e. has a jump), or

(ii) has a nonzero local martingale component, then X is strictly suboptimal: its conditional expected utility at time t is strictly above the value $J(t,\xi(t))$.

Proof of Proposition A.0.4. Applying Itô's formula to $\log J(u,\xi(u))$ on $[t,t_1]$, we obtain

$$
\begin{aligned}
&\log J\big(t_1,\xi(t_1)\big) - \log J\big(t,\xi(t)\big) \\
&= \overline{\gamma}_1 \int_t^{t_1} \frac{v - H\big(u,\xi(u-)\big)}{\lambda(u)}\, d\xi(u) \\
&\quad - \frac{1}{2}\overline{\gamma}_1^2 \int_t^{t_1} \frac{\big(v - H\big(u,\xi(u-)\big)\big)^2}{\lambda(u)^2}\, d[\xi^c](u) \\
&\quad + \frac{1}{2}\int_t^{t_1} \frac{J_{\xi\xi}}{J}\big(u,\xi(u-)\big)\Big(d[\xi^c](u) - \sigma^2\lambda(u)^2\,du\Big) \\
&\quad + \sum_{t<u\le t_1}\Bigg[\log\frac{J\big(u,\xi(u)\big)}{J\big(u,\xi(u-)\big)} \\
&\qquad\qquad - \overline{\gamma}_1 \frac{v - H\big(u,\xi(u-)\big)}{\lambda(u)}\,\Delta\xi(u)\Bigg].
\end{aligned}
$$

Proceeding as in the proof of Proposition A.0.3, we further obtain

$$
\begin{aligned}
&-\gamma_1 \int_t^{t_1} \big(v - H(u,\xi(u-))\big)\, dX(u) \;+\; \gamma_1[H,X](t,t_1) \\
&= -\log J\big(t_1,\xi(t_1)\big) + \log J\big(t,\xi(t)\big) + \log\big(\psi(t,t_1)\big) \\
&\quad + \frac{1}{2}\int_t^{t_1}\left(\lambda(u)^2\Big(\frac{J_\xi}{J}\big(u,\xi(u-)\big)\Big)^2 - \lambda(u)^2\frac{J_{\xi\xi}}{J}\big(u,\xi(u-)\big)\right) d[X^c,X^c](u) \\
&\quad + \sum_{t<u\le t_1}\Big[\Delta\log J\big(u,\xi(u)\big) - \partial_\xi \log J\big(u,\xi(u-)\big)\,\Delta\xi(u)\Big] \\
&= -\log J\big(t_1,\xi(t_1)\big) + \log J\big(t,\xi(t)\big) + \log\big(\psi(t,t_1)\big) \\
&\quad - \frac{1}{2}\int_t^{t_1} \lambda(u)^2\,\partial_{\xi\xi}\log J\big(u,\xi(u-)\big)\, d[X^c,X^c](u) \\
&\quad + \sum_{t<u\le t_1}\Big[\Delta\log J\big(u,\xi(u)\big) - \partial_\xi \log J\big(u,\xi(u-)\big)\,\Delta\xi(u)\Big].
\end{aligned}
$$

Define

$$\widetilde{J}\big(t_1,\xi(t_1)\big) = \sqrt{\frac{c(t_1)}{\lambda(t_1)}}\, e^{-\frac{\overline{\gamma}_2}{2\lambda(t_1)}\Big(v-h_c(t_1)-\xi(t_1)\Big)^2}.$$

Then

$$\mathbb{E}\left[e^{-\gamma_1\Big\{\int_t^{t_1}\big(v-H(u,\xi(u-))\big)\,dX(u)-[H,X](t,t_1)\Big\}}\,\widetilde{J}\big(t_1,\xi(t_1)\big)\Big|\mathcal{F}_t^I\right]$$
$$= \mathbb{E}\left[\begin{array}{l} J\big(t,\xi(t)\big)\, J\big(t_1,\xi(t_1)\big)^{-1}\,\psi(t,t_1)\,\widetilde{J}\big(t_1,\xi(t_1)\big) \\ \times\, e^{-\frac{1}{2}\int_t^{t_1}\lambda(u)^2\,\partial_{\xi\xi}\log J\big(u,\xi(u-)\big)\,d[X^c,X^c](u)} \\ \times\, e^{\sum_{t<u\le t_1}\Big[\Delta\log J\big(u,\xi(u)\big)-\partial_\xi\log J\big(u,\xi(u-)\big)\,\Delta\xi(u)\Big]} \end{array}\Bigg|\,\mathcal{F}_t^I\right].$$

By concavity of $\log J$, we have

$$-\frac{1}{2}\int_t^{t_1}\lambda(u)^2\,\partial_{\xi\xi}\log J\big(u,\xi(u-)\big)\,d[X^c,X^c](u)\ge 0,$$

with equality if and only if (i) fails. Moreover,

$$\sum_{t<u\le t_1}\Big[\Delta\log J\big(u,\xi(u)\big)-\partial_\xi\log J\big(u,\xi(u-)\big)\,\Delta\xi(u)\Big]\ge 0,$$

with equality if and only if (ii) fails.
Since $J\big(t_1,\xi(t_1)\big)=\widetilde{J}\big(t_1,\xi(t_1)\big)$, it follows that if at least one of (i) or (ii) holds, then

$$\mathbb{E}\left[e^{-\gamma_1\Big\{\int_t^{t_1}\big(v-H(u,\xi(u-))\big)\,dX(u)-[H,X](t,t_1)\Big\}}\,\widetilde{J}\big(t_1,\xi(t_1)\big)\Big|\mathcal{F}_t^I\right] > J\big(t,\xi(t)\big),$$

which proves the result. $\square$

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