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Schubert's Harmonic Language and Fourier Phase Space

—Jason Yust

The idea of harmonic space is a powerful one, not primarily because it makes visualization of harmonic objects possible, but more fundamentally because it gives us access to a range of metaphors commonly used to explain and interpret harmony: distance, direction, position, paths, boundaries, regions, shape, and so on. From a mathematical perspective, these metaphors are all inherently topological.

The most prominent current theoretical approaches involving harmonic spaces are neo-Riemannian theory and voice-leading geometry. Recent neo-Riemannian theory has shown that the *Tonnetz* is useful for explaining a number of features of the chromatic tonality of the nineteenth century (e.g., Cohn 2012), especially the common-tone principles of Schubert's most harmonically adventurous progressions and tonal plans (Clark 2011a–b). One of the drawbacks of the *Tonnetz*, however, is its limited range of objects, which includes only the twenty-four members of one set class.

The voice-leading geometries of Callender, Quinn, and Tymoczko 2008 and Tymoczko 2011 also take chords as objects. But because the range of chords is much wider—all chords of a given cardinality, including multisets and not restricted to equal temperament—the mathematical structure of voice-leading geometries is much richer, a continuous geometry as opposed to a discrete network. Yet voice-leading geometries also differ fundamentally from the *Tonnetz* in what it means for two chords to be close together: in the *Tonnetz*, nearness is about having a large number of common tones, not the size of the voice leading *per se*.

In the harmonic space described in this paper, Fourier phase space, the conception of distance is similar to that of the *Tonnetz*. But it also has the richer mathematical structure and wider range of objects that one associates with voice-leading geometries. The first part of the paper describes the particular virtues of the *Tonnetz*'s common-tone based conception of distance for analysis of Schubert, and also how, on the other hand, its highly circumscribed range of musical objects poses severe limitations on its application. The second section describes a Fourier phase space, based on the discrete Fourier transform (DFT) on pitch-class sets described in Quinn (2006), and shows how it retains the music-analytic virtues of the *Tonnetz* while expanding its range of objects and embedding it in more mathematically robust space.

One of the primary advantages of Fourier phase space over both the standard *Tonnetz* and voice-leading spaces is that it relates harmonic objects of any cardinality, including pitch-class multisets and even fractional-cardinality sets. Parts three and four exploit this aspect of the space in analyses of Schubert's tonal plans and common-tone modulations. Motivated by some of this analysis, parts five and six extend the theory by relating Fourier phase components to voice-leading properties and probing the meaning of direction in the space, and part seven applies some of these ideas in an analysis of the Trio from Schubert's String Quintet. Part eight addresses another spatial concept important to Schubert analysis: that of boundaries and regions, using the space to put a new perspective on important work in the Schubert analysis literature.

(1) Virtues and Limitations of the *Tonnetz*

What the *Tonnetz* does best is to explain the special place of mediant relationships in nineteenth-century chromatic tonality, something especially characteristic of Schubert's music. An example like the Menuetto from Schubert's String Quartet no. 13 (Figure 1) should therefore be an analytical gold mine for the neo-Riemannian. In the contrasting middle section of the piece Schubert makes a surprising move to a chromatic submediant of the relative major, A♭ major. Even more shockingly, despite the preparation of the home key (A minor) at the end of the contrasting middle, Schubert recomposes the recapitulation in a distantly related chromatic mediant key of C♯ minor.[1] Schubert exploits the common tones between the tonic triads of all these mediant-related keys: the note C is the focal point of the melody all the way from measure 12–28, resolving the B of the functionally reinterpreted diminished seventh in the C major context in mm. 16 and 20, and in an A♭ major context in m. 24. Most impressively, the recapitulation begins with the same E isolated in the cello, but instead of being $\hat{5}$, as in the exposition, in the recapitulation it turns out to be $\hat{3}$ of C♯ minor. David Kopp (2002) has shown how such common-tone links are a pervasive feature of the nineteenth-century usage of mediant relationships, and Suzannah Clark (2011a–b) has extensively demonstrated their importance for Schubert in particular. The layout of these harmonic landmarks on the *Tonnetz* in Figure 2 shows that these prominent common-tone links constitute the central axis of what Cohn (2000a, 2012) calls a "Weitzmann region," a group of six triads that all share two tones with the same augmented triad. All of the important stations of the harmonic plan (A minor, C major, A♭ major, E major, C♯ minor) belong to this region except the parallel of the contrasting key (A♭ minor).

[Figure 1]

[Figure 2]

However, Schubert's Menuetto has another exceptional feature that proves to be a stumbling block for any neo-Riemannian analysis: in the exposition, the home key is represented by its *dominant* rather than its tonic. Should this tonal area therefore be represented in the analysis by the tonic triad, which only appears in an unstable $^{6}_{4}$ position, or by an E major triad, ignoring its tonal context, which is obviously of crucial importance here?

The *Tonnetz*'s lack of analytical flexibility in this regard arises from the sparseness of its domain of musical objects, which includes only major and minor triads. A tonal area therefore must be represented by its tonic triad, a conflation of conceptually distinct objects that breaks down most spectacularly when faced with the kind of typically nineteenth-century gesture that opens this extraordinary Menuetto. On the other hand, a more literal usage whereby a triad in the *Tonnetz* always represents an explicit chord in the music (perhaps selectively divested of its sevenths or added sixths) cannot be sensitive to the often important implications of tonal context.

[1] The exceptional tonal features of the piece are also discussed by Sobaskie (2003).

Despite such limitations, this example does highlight the potential virtues of a *Tonnetz*-based approach in revealing how Schubert can take advantage of the special properties of mediant relationships in remarkable ways. In part (3), we will revisit this analysis with an analytical technique in hand that embeds the *Tonnetz*'s network of common-tone relationships in a topologically-enriched context. The expanded domain will offer multiple ways to make the more nuanced musical distinctions that are necessary to appreciate the function of harmonic subtleties in Schubert's music.

As a network, the *Tonnetz* is not only restricted in analytical usage, but also has deeper theoretical limitations, which are also related to its limited range of musical objects. Tymoczko (2010; 2011, 412–417) has pointed out that larger distances in the network do not accurately reflect voice-leading distances. A facile rejoinder to this criticism might be to assert that the *Tonnetz* provides a *different* concept of voice-leading distance, one that prioritizes common tones and therefore may have its own unique virtues. For instance, R-related triads (e.g., C major–A minor) are closer in the *Tonnetz* than S- (Slide-) related ones (e.g., C major–C♯ minor; see Lewin 2007, 178), while in voice-leading geometries, S-related triads are closer than (or at best, on the city-block metric, the same distance as) R-related ones. Yet, Tymoczko also shows that this idea of a common-tone based concept of distance fails to pan out for larger distances, using the example illustrated in Figure 3. F minor may *look* closer to C major than E♭ major, but according to the topology of the network, they are the same distance, even though F minor shares a common tone with C major while E♭ major does not.

[Figure 3]

Tymoczko's example demonstrates that the restricted domain of the *Tonnetz* is in fact a *topological* limitation. There is a relatively large distance between C major and F minor because in order to get from one to the other, one must construct a path that involves only major and minor triads. As Tymoczko points out, and implicit also in Cohn's (2000a, 2012) idea of Weitzmann regions, all it takes to fix this shortcoming is to make the augmented triad available as an object of the network, so that a shorter path, C major → A♭ augmented → F minor, is possible. Tymoczko (2012, 20–21) applies this solution in a reformulation of the *Tonnetz* as a note-based voicing-leading graph.

Reinterpreting the *Tonnetz* as a kind of voice-leading graph, however, surrenders the common-tone logic that we counted as one of its unique virtues. (The resulting network, equivalent to Douthett and Steinbach's [1998] "Cube Dance," demotes the R relation to a status equivalent to other two-semitone voice leadings like S.) Yet, there is a different way of topologically enriching the *Tonnetz* that preserves the musical insights about Schubert's use of mediant relationships and leads to a concept of harmonic distance. Note that if the pitch class C were itself an object in the space, linked to C major and F minor triads by virtue of its shared pitch-class content with them, it too would provide a shortcut that brings these triads closer together. Such mixing of different-cardinality sets is not possible in voice-leading spaces without forfeiting their basic geometric properties. It is, however, an intrinsic feature of the Fourier phase spaces described below. And, perhaps surprisingly, it leads to a different way of modeling the common-sense musical notion of voice leading.

(2) Fourier Phase Space

We have seen that there is an analytical and theoretical need for a harmonic space that preserves the common-tone logic of the *Tonnetz* while expanding its range of musical objects beyond major and minor triads. This section shows how the *discrete Fourier transform (DFT) on pcsets* introduced first in Lewin 1959 and 2001 and developed at length by Quinn (2006), Callender (2007), and Amiot and Sethares (2011), fulfills that need. This section will briefly explain how the DFT on pcsets works and define a space, first explored by Amiot (2013), using the phases of two DFT components. This geometry embeds the *Tonnetz* but is continuous and can be used to plot any pitch class set or multiset.

Most musicians will be familiar with the Fourier transform as a way of deriving frequency information from a waveform. Lewin and Quinn use the DFT instead as a set-theoretic tool by analogy: imagine that your "signal" is a pitch-class set where time is replaced by pitch class as the x-axis. For each pitch class in the set, there is a spike of height 1, and the value everywhere else is zero. The C major triad corresponds to the signal shown in solid lines in Figure 4, with spikes at x= 0, 4, 7, (12, 16, 19, . . .). Because pitch class is cyclic, the signal is infinitely periodic. The DFT decomposes this signal into twelve sinusoids, called the *Fourier components*, which are denoted by the symbols $\widehat{f_0}, \widehat{f_1}, \widehat{f_2}, \ldots, \widehat{f_{11}}$. The index of the component indicates the number of cycles per octave. Figure 4 shows three of the components for the C major triad, $\widehat{f_3}$, $\widehat{f_4}$, and $\widehat{f_5}$. Notice that the sinusoids have 3, 4, and 5 peaks respectively. These twelve sinusoids, when added back together, reproduce the pcset—i.e., spikes of height 1 at 0, 4, 7, etc. The idea of a Fourier transform is that it expresses the same pcset by a different set of parameters. The original parameters, values of 0 or 1 for each pc, are *transformed* into the twelve Fourier components.

[Figure 4]

Each component is a sinusoid, which must be specified by two parameters, a *magnitude* and a *phase*. The magnitude is the height of the sinusoid. The phase slides it to the left or the right, determining where the peaks fall along the pc-circle. The magnitude of $\widehat{f_n}$ is denoted $|\widehat{f_n}|$ and its phase as φ_n. It might seem, then, that the DFT changes 12 parameters (the values of the 12 pcs) into 24 (a magnitude and phase for each component). But this is not so: $\widehat{f_7}$–$\widehat{f_{11}}$ have the same magnitudes and opposite phases as their complimentary components $\widehat{f_1}$–$\widehat{f_5}$, and so can be ignored. Also, the zeroeth component simply reflects the cardinality of the set. Therefore the non-trivial data produced by the DFT consists of 12 quantities, the magnitudes and phases of components $\widehat{f_1}$–$\widehat{f_6}$.

The purpose of the DFT is to convert less useful information—what specific pcs are present—into more useful intervallic information, in a similar spirit to the blunter classic pcset-theory conversions into set classes and interval vectors. The magnitude of each component measures, roughly speaking, how strongly clustered the set is on the basis of a single interval type, ic1 for $\widehat{f_1}$, ic6 for $\widehat{f_2}$, ic4 for $\widehat{f_3}$, etc. Phases indicate where on the pc-circle the center of that cluster is. Magnitudes are independent of transposition and inversion, so they isolate information related specifically to the set class. This is the basic idea behind Lewin's (1959, 2001) and Quinn's (2006) use of the DFT. The phase spaces described below do roughly the opposite, using the transposition-dependent phase information and discarding (for the most part) the magnitudes.

Each Fourier component can be considered an attempt to approximate the pcset with a sinusoid. The meaning of a Fourier component is therefore derived from these two features of sinusoidal curves:

(a) The peaks and troughs of the sinusoid are evenly distributed in the octave. Therefore $\widehat{f_n}$ represents *the nearest perfectly-even superset* of size *n* or 12 – *n*. The C major triad in Figure 4, for instance, is very close to an augmented triad, so $\widehat{f_3}$ is large. This augmented triad does not need to be integer-valued, however, so that the closest augmented triad to C major, as indicated by the peaks of the sinusoid in Figure 4(a), is not {C, E, G♯}, but one slightly flat of {C, E, G♯}. This brings the note G a little bit closer to one of the peaks, improving the fit. Similarly the closest diminished seventh to the C major triad is $^1/_3$-semitone flat of {C♯, E, G, B♭}, pulling it away from E and G a little to get it closer to C. Note that the triad is only close to three of the peaks of $\widehat{f_4}$; this is what "nearest *superset*" means.

The perfectly even set of size 5 (or 7) does not occur in 12-tone equal-temperament at all. Its peaks are a slightly narrow perfect fourth apart, as one can see in Figure 4(c), which means that as one moves around the circle of fifths, the pcs gradually get closer to or further from peaks of $\widehat{f_5}$. For the C major triad, the notes G and D are closest to peaks of $\widehat{f_5}$. Other peaks are close to the next two notes in either direction on the circle of fifths, C and A, and another peak is halfway between E and F. Because it does not evenly divide 12, the peaks of $\widehat{f_5}$ can never be aligned exactly on any equal-tempered collection (other than a single pc), but they will center on collections that are compact on the circle of fifths, such as diatonic scales. For the C major triad, the peaks of $\widehat{f_5}$ center around the Guidonian hexachord CDEFGA and the troughs around the complementary hexachord F♯G♯A♯BC♯D♯.

(b) Sinusoids are continuous curves rather than a series of spikes. This is important because it means that if the component cannot align its peaks directly on members of a pcset, it will still try to get them close, and to avoid the troughs halfway between the peaks. The DFT therefore incorporates an element of what Tymoczko (2011) calls *voice-leading distance*. That is, when we say that the C major triad is close to an augmented triad, "close" means specifically that we only need to move the notes a short distance along the pc-circle to get to the augmented triad. This explains, on a very general level, Tymoczko's (2008b) result relating Fourier magnitudes to voice-leading distance from the nearest subset of a perfectly-even chord (see also Callender 2007). Voice-leading distance also plays a fundamental role in Cohn's (2012) theories of chromatic harmony, specifically the small voice-leading distance from major and minor triads to perfectly even three-note chords, which is evident in the large $|\widehat{f_3}|$ for major and minor triads.

Recent music theory has taken a great deal of interest in both of these concepts, evenness and voice-leading distance. (See, for example, Plotkin and Douthett 2013 and Tymoczko 2013.) The difference of the DFT from other theories is in the nature of its objects, which are not constrained by cardinality. Every pcset has an $\widehat{f_3}$ component regardless of whether it is a three-note chord, or whether it is triad-like. For instance, a consonant dyad, like CE, has a φ_3 value directly between the two triads that contain it (e.g., C major and A minor).

In fact, the restriction that pcs take only values of 0 and 1 is essentially arbitrary. Allowing integer values larger than 1 extends the technique to pc-multisets, but even non-integer values are allowable. Therefore the objects of the DFT are really *pitch-class distributions*. Although uncommon in music analysis, statistical distributions have been widely adopted in music cognition research; they are the basis, for example, of standard key-finding algorithms (see, e.g., Krumhansl 1990, Temperley 2007, Temperley and Marvin 2008). While we will mostly consider only pcsets and occasionally multisets in this paper, thinking of these as special cases of pc-distributions is important to understanding the theoretical basis of the DFT. For instance, when considering the Fourier components of a C major triad, among the first questions to come to mind should be ones like, what would happen if we added a little more C to this chord? or a little less C and a little more G?

The task I assigned to the DFT at the beginning of this section was to represent a *Tonnetz*-like yet continuous concept of distance—i.e., a measure of distance based on shared pc content. Distance in the *Tonnetz* is measured by counting discrete maximal common-tone moves. But how can maximal common-tone moves be made continuous? This is precisely the function of embedding pcsets in the domain of pc-distributions. Two pc-distributions can differ infinitesimally in their pc-content. The difference between this domain and that of voice-leading spaces resembles the difference between harmony and chord or key and scale. The concept of harmony, for instance, might include emphasizing a particular note as the root (e.g., to distinguish "V^{sub6}" from iii^6) or allowing other notes to occur that are de-emphasized as auxiliary tones. In other words, the concept of harmony allows for weightings of pcs and is cardinality-flexible, whereas a chord involves a specific set of voices, and therefore is fixed and integer-valued in cardinality.

Such a cardinality-flexible conception of harmony discards the basic structuring assumption of voice-leading geometries, and therefore must replace it to lead to any kind of coherent theory. The evenness criteria of the DFT fulfill this role, and do so in a way that resonates with many common ways of thinking of pitch-class sets in tonal contexts. For example, given an arbitrary pcset, such as DEG, what are its possible significations in a tonal context? It might be explained as (a) a C major triad with one note, D, displaced by step, (b) an incomplete seventh chord, or (c) a scalar fragment. Each of these convert the pcset into common tonal collections of various cardinalities, and all of these common collections are relatively even. The sense (a) is reflected by the difference in φ_3 between DEG and the C major triad, sense (b) by its φ_4-proximity to E^{m7}, and sense (c) in its φ_5-proximity to C diatonic.

So far, I have illustrated the Fourier components with sinusoids, which are conceptually useful but difficult to work with directly. Fortunately, Quinn (2006) has devised a more intuitive way to derive Fourier components with his "Fourier balances." Components can often be calculated, or at least estimated, by eye using this representation. Figure 5 shows the $\widehat{f}_3$, $\widehat{f}_4$, and $\widehat{f}_5$ balances. The $\widehat{f}_3$ balance has a "pan" for each of the four augmented triads, arranged symmetrically around a circle; the $\widehat{f}_4$ balance does the same for diminished sevenths, and the $\widehat{f}_5$ balance is the circle of fifths. Each pitch class is a vector of length one corresponding to a unit weight placed in one of the pans. Each Fourier component is the vector sum of its individual pcs on the appropriate balance. The direction of the resulting vector (the direction that the balance tips) is the phase of the component, and the length of the vector (the total force tipping in this direction) is its magnitude.

[Figure 5]

Figure 6 shows the calculation of the third, fourth, and fifth components for a C major triad through vector sums. The phase values are given in radians, where the total circumference of the circle is 2π, and increments of $\frac{1}{6}\pi$ divide the circle into 12 parts, making this a convenient unit. Notice that the sums can be at least estimated if not precisely determined by hand. Figure 7 shows the same calculation for the A minor triad. To get the precise values using vector addition often requires a little trigonometry—see the appendix for more details. Precision, however, is not especially important for present purposes; an estimate of where the value falls relative to individual pcs and other pcsets will suffice.

[Figure 6]

[Figure 7]

The concept of distance explored in this paper is one based on *differences of phase* between the Fourier components, illustrated by plotting the phases of interest in coordinate spaces, which I will call Fourier phase spaces. The strategy is similar to Quinn's (2006) quality space, except that Quinn's coordinates are Fourier magnitudes rather than phases. The Fourier phase space used throughout this paper is the one discussed in Amiot 2013, based on the third and fifth components, which I will refer to as $\varphi_{3/5}$-space. These two components are of special interest because they reflect aspects of triadic and scalar voice leading, and are the largest components for consonant triads. An advantage of limiting consideration to two dimensions is that such a space is easy to visualize. Figure 8 plots C major and A minor in $\varphi_{3/5}$-space, with individual pcs included for orientation. The grid imposed on the space is in increments of $\frac{1}{6}\pi$, which is the equivalent of a semitone on the pc-circle. The space is cyclic in both dimensions, wrapping around from top to bottom and left to right (that is, it is topologically a torus), because the phases themselves are cyclic.

[Figure 8]

One thing that is immediately apparent from Figure 8 is that the triads fall in between their constituent pcs. In fact, we can think of a pcset's position in the space as the "average" of all the positions of its individual pcs, but with a cautionary note: because the space is a torus, as one pc gets further removed from the others in a particular dimension, its contribution to the "average" is attenuated. For instance, the notes C and G have a stronger influence on φ_5 of the C major triad because they are closer together, and similarly the notes C and E determine φ_3 more strongly. Therefore the position of C major in Figure 8 is not in the exact center of the triangle made by the individual pcs C, E, and G, but leaning towards the lower left side of it.

Bartlette (2007) has proposed a space (his "h-d (harmonic distance) map") based on actual arithmetic means, with pcs are arranged in a *Tonnetz*-like pattern similar to the one that occurs in $\varphi_{3/5}$-space.[2] When limited to relatively consonant pcsets (those whose individual pcs are closely packed), Bartlette's space and $\varphi_{3/5}$-space look very similar. However, the averages can only be calculated in a tangent space—that is, a flattened two-dimensional unfolding of the torus, extending infinitely in all directions. Each pc occurs in a potentially infinite number of places in such a tangent space, and the resulting average depends on which positions are selected. This leads to "wormholes" in Bartlette's geometry, where the position of a chord jumps after a different selection from the tangent space is made. Figure 9 illustrates how arithmetic means differ from DFT phases when a pc occupies an extreme position in one dimension of the torus. Starting from a C major triad, we hold C and E constant while moving the third point continuously from G through D to A around the back side of the torus. This can be done by gradually reducing the weight of G to zero while increasing the weight of D to one, then reducing the weight of D while increasing the weight of A. The dotted line in Figure 9 shows the path of the moving point representing the weighted combination of G and D or D and A. The dashed line shows a weighted average of the pcs. The path of the weighted average is discontinuous where the moving point disappears from the right and reappears on the left. The solid line shows the change of the DFT phases. This line actually goes in the *opposite* direction from the moving point, because the influence of the moving point over φ_3 decreases as it approaches the opposite side of the torus.[3]

[Figure 9]

Amiot (2013) discovered that the major and minor triads in $\varphi_{3/5}$-space take on a *Tonnetz*-like arrangement. Figure 10 extends this observation by plotting all consonant dyads and triads and connecting individual pcs with dashed lines. The triads fall near the center of their *Tonnetz* triangles, while the dyads are the midpoints the *Tonnetz* edges between the two triads that contain them. One could also draw a "dual *Tonnetz*" in the space by connecting each triad to its nearest neighbor. The pcs would be at the center of a hexagon with all of the consonant triads containing that pitch class as its vertices and all the dyads containing it at the midpoint of its edges.

[Figure 10]

The arrangement of triads in $\varphi_{3/5}$-space is strikingly similar to the experimentally derived space of key relationships presented in Krumhansl and Kessler 1982 (see also Krumhansl 1990). The resemblance to Krumhansl and Kessler's space is especially interesting in that both spaces use the same kind of objects, pc-distributions.

It is worth keeping in mind that a great deal of information from the DFT is missing from this two-dimensional toroidal space. The magnitudes of $\widehat{f_3}$ and $\widehat{f_5}$ and phases of other components are discarded for purely heuristic reasons, though, so they remain theoretically

[2] Thanks to an anonymous JMT reviewer for bringing this work to my attention.

[3] There are actually two factors at work here: In the first half of its journey from G to D, the moving point also decreases in magnitude of $\widehat{f_3}$ from 1 to 0.7, because it involves the combination of two notes of different phases. It recovers magnitude approaching D, but by then its decreasing influence over the combined φ_3 has accelerated.

available when the situation demands. For instance, many pcsets fall on top of one another in $\varphi_{3/5}$-space. In such cases, the difference between those pcsets is captured by other components. Among the more significant of co-located set classes are:

(a) Diminished triads and their chordal thirds as pcs (e.g., BDF and D).

(b) Dominant sevenths, half-diminished sevenths, and their shared perfect fifth.

(c) Diatonic scales, pentatonic scales, and the major third that is $\hat{1}$–$\hat{3}$ of their major modes.

(d) Harmonic minor scales and their tonic minor triad (also harmonic major scales and their tonic major triads)

(e) Major and Minor seventh chords and their third-fifth dyads (a minor third for major sevenths, or a major third for minor sevenths).

These examples can all be explained either by the fact that tritones cancel out in the space (a, b, and d) or by inversional symmetries (a, c, e).[4] All of them can be disambiguated by taking φ_2 or φ_4 into account.[5]

Another feature of Fourier phase spaces is that the phases can be undefined for certain pcsets—specifically when the magnitude of the component is zero. This occurs especially in symmetrical collections such as tritones and diminished seventh chords which are undefined for both φ_3 and φ_5, or augmented triads which have a φ_3 but no φ_5. It is not limited to transpositionally symmetric collections, however: the major second, for example, has an undefined φ_3. To understand this situation, it is important to keep the magnitudes of components in mind, and the continuous nature of pc-distributions.

The impossibility of locating of certain pcsets might seem like a handicap if we want the Fourier phases spaces to behave like voice-leading geometries, or the voice-leading based networks of Douthett and Steinbach 1998, where many of these set classes play a central role. From another perspective, though, it provides a window into how the meaning of positions in Fourier space differ from those in voice-leading spaces. It also reflects certain real features of tonality: symmetrical sets such as tritones or diminished seventh chords do not, *qua* pcsets, have a specific harmonic function. Instead they are equally balanced between multiple opposing functions. In real musical situations, such ambiguities are resolved by surrounding context. Addition of context and/or weighting of pcs can similarly resolve an undefined Fourier phase.

[4] The explanation from inversional symmetries is essentially statistical. Any phase component must correspond to one of the two inversional centers, so if two symmetrical pcsets have the same inversional centers, there is a 1/4 likelihood they will coincide in any two-dimensional phase space. Or, to put it differently, there are only four distinct locations (after factoring out the twelve transpositions) for inversionally symmetric sets in $\varphi_{3/5}$-space. The pcs and the three consonant dyads occupy each of these positions.

[5] φ_2 works for all but (d) and φ_4 works for all but (a) and (e).

One way to think of undefined phases is that in the vicinity of these ambiguous pcsets, position in phase space becomes highly sensitive. A scrap of contextual evidence might have a large effect on the phase. This accurately reflects how these pcsets often behave in tonal music. For instance, consider the remarkable opening progression to Schubert's song "Daß sie hier gewesen" in Figure 11(a). The only conventionally functional harmony present in these two measures is a diminished seventh chord, which in principle is ambiguous between four possible functions. In reality however, the function of the chord is not so ambiguous (that is, it is not enharmonically ambiguous—the key of the piece is not yet clear), because it is clarified by the appoggiaturas. In $\varphi_{3/5}$-space, adding these auxiliary tones places the harmony in the vicinity of D minor, even if they are very weakly weighted relative to the chord tones. Schubert's function-defining appoggiaturas are an inspired representation of the fragility of love. Appoggiaturas do not usually have this much influence in determining harmonic function. Figure 11(b) shows four other similar appoggiaturas from the song. Because they resolve to a dominant seventh, these do not have the same kind of harmony-defining power. They add modal coloration to the chord, but do not change its function. The C♯→C♮ substitution imparts a distinctly Schubertian interpretation of the text: by transferring love into the beautiful object, the artwork, it gains substance and resilience.

[Figure 11]

(3) Schubert's mediants in $\varphi_{3/5}$-space

Now we may return to the *Tonnetz* analysis of Schubert's Menuetto (Fig. 1), reformulated in Fourier phase space, and explore the expanded domain of objects as analytical resource.

One of the problems of the *Tonnetz* analysis noted above is its conflation of chord and key, and its reduction of seventh chords to triads. Figure 12 offers three interpretations in Fourier phase space that redress these problems in different ways. Figure 12(a) treats the tonal plan as a pathway between chords, giving a picture similar to the *Tonnetz* analysis. The main difference is that now the E dominant seventh chord can be distinguished from the E major triad. This analysis shows that most of the action of the piece is centered on a hexatonic cycle beginning with the V^7 of the exposition and ending with the V^7 just before the recapitulation. The recapitulation, despite its unusual non-tonic beginning, does achieve "resolution" in the sense that it shifts into the hexatonic cycle of the home-key tonic (if one, following Cohn [1999], attributes functional identity to hexatonic cycles.) The *Tonnetz* in the phase space is rotated slightly from its traditional depiction (as in Figure 1) so that the hexatonic strips (see Cohn 2012, ch. 2) are precisely vertical. This reflects the fact that hexatonic progressions give maximally balanced voice leading between triads and that they produce the most efficient enharmonic cycles.

[Figure 12]

One might complain, however, that this chordal pathway still neglects basic facts about how tonal contexts operate in the piece—specifically that the piece begins in A minor and returns to this key at the end of the recapitulation, whereas the chord-based analysis shows it ending in a very different place than it began. One way to represent such tonal contexts might be to plot their characteristic scales in the space, as in Figure 12(b). This pathway consists mainly of harmonic minor scales, which are in the same place as their tonic triads, and major scales, which are in the same place as their tonic major third. It is similar to the chordal trajectory except that it begins at A minor and gradually works its way into the hexatonic region of the dominant and relative major before returning at the recapitulation.

The scalar positions are not ideal representations of the tonal plan either, however, because they do not distinguish the dominant-weighted A minor beginning from a more conventional tonic-weighted A minor. An even more sensitive analysis is possible by plotting a pathway on multisets that combine scalar context and chord (in other words, scales with chord tones doubled), as shown in Figure 12(c). Such combinations average the positions of their subsets, but weighted according to the magnitudes of each component individually. For example, "V^7 of A minor" is between the A harmonic minor scale (equal to the A minor triad) and the E dominant seventh, but somewhat weighted toward A minor. "I of C major" is very close to the C diatonic scale (located at CE) in the vertical dimension because the scale has a strong fifth component, but somewhat closer to the C major triad in the horizontal direction.

This pathway on chord/scale combinations begins on the boundary between the two hexatonic regions, very close to the pitch class E (the first note), moves gradually into the dominant region, then back to the tonic region at the recapitulation. This analysis reveals another interesting feature to the tonal plan by showing a consistent motion in a direction downward and slightly to the right. The trajectory of the exposition (V^7 of A minor to I of C major) is thus very similar to the overall trajectory of the contrasting middle (I of C major to I of A♭ minor), and the first part of the recapitulation (i of C♯ minor to V^7 of A minor). This interesting feature of the tonal plan might be described as a mediant-based generalization of the subdominant recapitulation principle. Schubert is well known for occasionally recapitulating main themes in the subdominant key in his sonata forms (see Webster 1978–9; Clark 2011b, ch. 4). Such subdominant recapitulations can be explained as a way to retrace the trajectory (modulation up a fifth) of the exposition in the recapitulation. Something similar happens in this Menuetto, where a non-tonic beginning allows the recapitulation to return to the initial harmonic state (V of A minor) via the same kind of path that departed from that state (because i of C♯ minor → V^7 of A minor goes in a similar direction as V^7 of A minor → I of C major). This feature is reinforced by the fact that the contrasting middle also goes in a similar overall direction, from I of C major to I of A♭ minor.

One thing worth noting about all of the analyses in Figure 12 is that they are all quite similar, showing an enharmonic tour centered around a particular vertical axis associated with the prominent common-tone links in the piece. That we could track a tonal plan like this as a progression of very different kinds of objects (e.g., chords vs. scales) and get the same basic result illustrates the robustness of the space with respect to the omission and addition of pc-content, particularly when such additions and omissions reflect conventional logic about relatedness of chords, keys, and pitch classes. Scales, for instance, are very close to the chords that typically function as their tonic harmonies.

One could, of course, continually refine such plots by giving more nuanced weightings to recognize the tonic status of a note or its prominence in the music, but the influence on the resulting path diminishes rapidly the more subtle such refinements become, and the basic features of the analysis are generally resilient to such variation in interpretative method.

(4) Modulation by Scalar Common Tone

When Schubert wants to emphasize the sense of distance between two tonal areas, a favorite method is to isolate a common tone between the tonic chord of one key and the characteristic scale of the other. When encountering this technique, it is essential for an analyst to have a consistent way of relating objects of different cardinality: chords, scales, and single pitch classes. Fourier phase spaces are particularly effective in this respect.

In the slow movement of the String Quintet, Schubert modulates from E major to F minor for the interior theme in the passage shown in Figure 13(a). He isolates the tonic note of the home key of E major which is then reinterpreted as leading tone of the new key of F minor. The half-step trill on E erases the previous tonal context a moment before the assertion of the new key. Kinderman (1997) notes ramifications of this trill elsewhere in the movement, interpreting it a symbol of the "anguished, antithetical forces that . . . lurk behind the sublime surface of this music" (214).

[Figure 13]

The most efficient way to get from E major to F minor is via the slide relation, shown by the dashed line in Figure 13(b). Because the common tone of the slide relation is the third of the major triad (G♯) the interior theme would be in a respelled E♯ minor if this were the correct key relationship. However, Schubert's purpose here is clearly to send the music plummeting deep into the dark, impossibly flatward world of F minor. The enharmonic distinction corresponds to possible paths between E major and F minor on the torus, upward or downward (Fig. 14). The space also shows what harmonic intermediaries will effectively secure the flatward enharmonic interpretation: those that lie between E major and F minor in the downward direction: A minor, C major, etc. Schubert's solution is breathtakingly efficient: by isolating the single tone, E, the key of F minor comes closer in the downward direction. He thereby achieves a stark juxtaposition of keys whose remoteness is otherworldly, more than half a turn around the φ_5-cycle, further than the tritone-related key or the hexatonic pole (both of which are a precise half turn around the cycle).

A more understated application of the scalar common-tone method occurs in the modulation from the Scherzo to the Trio of the same work, shown in Figure 14(a). In this beguiling little Trio, a quiet respite from the hyperbolically extrovert Scherzo, the first eight measures serve both as main theme and as transition into a distant tonality (♭II). The theme is a completely scalewise melody played *unisono* by the viola and second cello; beginning from the tonic of the Scherzo (C) it descends, tentatively exploring F minor before taking a headlong descent into D♭ major. A plot of the process in $\varphi_{3/5}$-space (Fig. 14b) shows something noteworthy about it: the two primary transitional elements that Schubert uses to make a seamless passage between the keys, the common tone C and the F minor triad, lie almost directly in-between the two tonic chords, C major and D♭ major. The first chord of the new tonality, however, is not the tonic but the even more remote subdominant, G♭ major. The way that the theme approaches the G♭ major chord involves another interesting collinearity: halfway between the F minor and G♭ major triads is the major third D♭F, which is also the position of the D♭ diatonic scale, the basis of the scalewise passage that connects these two chords in the theme.

[Figure 14]

Such relationships may seem at first uncanny—suggestive but mysterious. They are in fact part of a larger compositional design involving the play of stepwise intervals between and within scales, pulled in different directions by the gravitational forces of tonality. To explain this design, however, we need more refined concept of interval and triadic voice-leading positions. We will revisit Schubert's Trio in §7 after deriving such concepts from the Fourier coefficients in the next two sections.

(5) Fourier Phases as Voice Leading

As is probably apparent already, the voice-leading geometries of Callender, Quinn, and Tymoczko (2008) have inspired and driven the present study in a number of ways. One immediately intriguing feature of the Fourier phase spaces is that they mimic voice-leading distances when restricted to the relatively even sets of a given cardinality—specifically a cardinality equal to one of the Fourier component indices or its complement. In fact, a voice-leading based space proposed in Yust 2013b has the same topology as $\varphi_{3/5}$-space and lays out the major, minor, diminished, and augmented triads in the same basic pattern. This is surprising given the incompatibility of initial premises of the two spaces. But there is also an underlying conceptual kinship between them: the technique of iterated quantization used to construct the diatonic-triad space of Yust 2013b functions as an evenness criterion (see Tymoczko 2013, Yust 2013a, 2015). The Fourier components, as perfectly-even approximations of pcsets, do much the same.

The relationship of $\varphi_{3/5}$-space to voice-leading geometries can be summarized by the following two facts:

(1) For relatively even three-note chords (augmented, major, minor, diminished, and "sus4"), changes of φ_3 are approximately equal to overall voice-leading ascent and descent. A decrease in φ_3 by ~$^1/_6\pi$ corresponds to an overall voice-leading ascent of one semitone, and an increase of ~$^1/_6\pi$ in φ_3 to a one-semitone descent. Chords related by balanced voice leading (like C major – E major or C♯ diminished – C augmented) have the same φ_3, so φ_3 differences are not the same as voice-leading distances. Instead, φ_3 is roughly equivalent to the projection of these chords onto the center axis of three-note chord space. Or, we could say the φ_3 sorts the chords into twelve ordered categories equivalent to Cohn's (2012) voice-leading zones.

(2) The same is true of φ_5 for relatively even seven-note scales. For diatonic and acoustic scales, a voice-leading ascent of one semitone corresponds exactly to $+^1/_6\pi$ in φ_5. This concept of voice leading between scales has been applied extensively by, e.g., Tymoczko 2004 and Hook 2008. The positions of φ_5 (in increments of $^1/_6\pi$) correlate precisely with

Hook's (2011) accidental indices for these scales as spelled heptachords, up to enharmonic equivalence.

Similar points could be made for all of the Fourier coefficients, such as φ_4 on four-note chords, and from the discussion of §2 the reasons should be apparent: for relatively even *n*-note chords, $\widehat{f_n}$ is a good approximation, representing a nearby perfectly-even chord in *n*-note chord space. The perfectly even chords define the center axis of chord space.

Of course, such comparisons are limited to chords of a fixed cardinality, which disqualifies the feature of Fourier spaces identified above as a principle hermeneutic asset, their flexibility with respect to cardinality. These observations then motivate the development of a cardinality-independent concept of voice leading for the interpretation of Fourier phases.

Up to now I have used the term voice leading in the part-writing sense formalized by Tymoczko (2008a) as *bijective voice leadings*, which implies a fixed number of voices. Yet many important uses of the term in music analysis assume more flexibility in the number and independence of voices. Callender's (1998) influential and analytically useful idea of "split" and "fuse" operations, for example, are explicitly presented as parsimonious but non-bijective voice leadings. Figure 15 reproduces one of Callender's examples where allowing for a fluidity of voice membership allows for a smooth voice leading that would be impossible if the voices were fixed and independent. Tymoczko (2004) uses split and fuse voice leadings to relate scalar collections of different cardinality. Callender, Quinn, and Tymoczko's (2008) inclusion of cardinality equivalence in their "OPTIC" symmetries attests to the significance of such inter-cardinality operations. This recognition is especially noteworthy since it is essentially honorary: there are no true voice-leading geometries that assume C-equivalence.[6]

[Figure 15]

The DFT helps capture this kind of cardinality-flexible sense of voice leading in a way that is geometrically tractable. Consider $\widehat{f_3}$, which, as shown in Figure 16, divides the pc-circle into three regions. The boundaries between each region (the points of the triangle) are where the troughs of the sinusoidal curve bottom out, while the centers of each region (dashed lines) are the peaks. I will call these regions *triadic orbits*. For a simple triad they consist of all the notes that act as neighbors or substitutes for each note of the triad. Ordinary voice leading between triads consists of motions restricted to specific orbits. Such restricted voice-leading motions, as the example in Figure 16(a) shows, turn the triadic orbit boundaries in the same direction as the voice leading. This is one way to explain the correlation of φ_3 with voice-leading direction for relatively even three-note chords. However, the reasoning is not restricted to three-note chords, because multiple notes could potentially be used to represent an orbit, or an orbit might be unoccupied altogether, without effecting the result. For instance, Figure 16(b) gives a voice leading between dyads, but through the lens of φ_3, these dyads are actually being viewed as incomplete triads. The behavior of φ_3 approximates that

[6] The reason for this is that cardinality equivalence does not preserve basic voice-leading metrics, as explained on pp. 6–7 of Callender, Quinn, and Tymoczko's supporting online materials.

of related triadic progressions like C major – F major, or A minor – F major, or C major – Csus4, etc. Figure 16(c) shows that the seventh of a chord, viewed through φ_3, is a substitute tone for the root—that is, the seventh and root occupy the same triadic orbit. Figure 16(d) shows that the correlation of φ_3 with voice-leading direction holds also for seventh chords when the moving notes stay in the central zone of their triadic orbits.

[Figure 16]

So what happens when the voice leading crosses the triadic orbit boundaries? Figure 16(e) appears to be a descending voice leading between three-note chords. But $\widehat{f_3}$ does not see it this way: the unevenness of the GB♭C chord means that B♭ and C occupy the same triadic orbit, as in a C minor seventh chord, so the φ_3 value approximates that of a C minor triad. Therefore, the effect on φ_3 is that of an *ascending* voice leading. In essence, φ_3 idealizes the voice leading on a specifically triadic model, so it treats the progression in Figure 16(e) as a variant on a comparable triadic progression like C minor – F minor.

The value of φ_4, on the other hand, *would* indicate a descending voice leading for the progression of Figure 16(e). This might seem like a contradiction, but it is not, because φ_3 and φ_4 are mathematically independent. Thus, what seem like different theories of voice leading, one based on f_4, in which the sevenths are true chord tones, and one based on f_3, where the seventh is a displacement of the root (a domesticated passing dissonance, perhaps), are actually two facets of the same voice-leading reality.

The sequence from Schubert's song "Gruppe aus dem Tartarus," shown in Figure 17, illustrates how such a cardinality-flexible concept of voice leading may be useful in a situation that is challenging to accurately represent with fixed-cardinality bijective voice leadings. Taken very literally, the sequence consists entirely of three-note chords, two per measure. The voice leading suggested by the music is shown with arrows in Figure 18a: the suspensions must be prepared, but there is also a clear stepwise motion in the vocal part that creates the suspensions. This means that approaching each suspension one voice must split and one of the lower voices must simply disappear. A strictly three-voice pattern would have to either approach the suspension by upward leap, or approach the note in the vocal line by downward leap, both highly implausible analyses.

[Figure 17]

[Figure 18]

In $\varphi_{3/5}$-space the three-chord pattern gives the path shown in Figure 18b. The overall motion follows an ascending semitonal axis shown with the dashed line. There is some backtracking at the i–V progressions in each key, while the suspension chords and incomplete sevenths mediate the progression into each new key. The suspension chords, like A♭E♭F♭, are close to the preceding tonics (like A♭ minor) because they share two common tones and imply similar scalar contexts.

Figure 19 shows the triadic orbits for one stage of the sequence. When the E♭ major triad goes to the incomplete seventh chord F♭C♭E♭ (= EBD♯) the E♭ splits within its triadic orbit, and the triadic orbit previously occupied by B♭ is left empty. Two voices move up within their orbits, the E♭→F♭ of the split, and G→A♭, so the triadic orbits turn clockwise. In the next two stages, the resolving suspension E♭→D followed by the resolving seventh D→C crosses a orbit boundary. The triadic orbits therefore respond by turning the opposite way, in the direction associated with ascending voice leading. (This is also reinforced by the ascent G♯→A.) The boundary-crossing restores the missing triadic orbit, necessary to beginning the next stage of the sequence.

[Figure 19]

Usual analytical convention would identify the suspension chords as non-harmonic embellishments of a dominant seventh, implicitly eliminating the suspension and adding the missing fifth to the seventh chord, giving the progression shown in Figure 20(b). The resulting path in $\varphi_{3/5}$-space is very similar to the path for the "raw" progression in (a). The pattern could also be plausibly interpreted, somewhat more abstractly, as a pattern on triads, as in Figure 20(c). Here again, the basic contours of the path traced in $\varphi_{3/5}$-space remain intact.

[Figure 20]

The situation is quite different in voice-leading geometries. The triadic voice leading of Figure 20(c) consists of relatively small 2–3 semitone voice leadings confined to the center of three-note chord space. The more literal progression in (a), however, charts a radically different course: thrusting out towards the periphery with a large seven-semitone voice leading where the suspension chord occurs, then easing back toward the center of the space with the two smaller voice leadings that follow before cycling back though the pattern sequentially.

In other words, voice-leading geometries are highly sensitive to standard operations of basic harmonic theory that involve changes of cardinality: omission, addition, or doubling of voices. Using them effectively usually requires the analyst to first idealize the voice leading by hand. One advantage of Fourier phase spaces is that they are robust with respect to these operations: the raw progression will typically already approximate the idealized one. Since conventional theories tend to model harmony on relatively even collections, the DFT simulates much of what we do automatically when interpreting the harmony of a passage. (See also Callender [2007], who makes a related argument comparing voice-leading distance to changes in DFT magnitudes.)

(6) Intervallic Axes

In §4, a plot of the transition into the Trio of Schubert's String Quintet (Fig. 14) suggested that there might be musical significance to *direction* in phase space as well as proximity. The F minor tonicization is an intermediary between C and D♭ not only in the sense of distance but also of direction, meaning that C major → F minor is a similar kind of motion as F minor → D♭ major.

In this section we will define *intervallic axes* as a way to associate directions with intervals. Two mathematical features of the space, related to transposition and inversion, make this possible. First, a given transposition always has the same vector, regardless of the set classes involved (assuming that the relevant phases are defined on that set class). For instance, in the Trio, the C major → D♭ major vector is a transposition by semitone. The transposition between pcs C→D♭, or minor thirds EG→FA♭, or in fact any transposition by semitone has the same vector.

However, a given vector may be represented by multiple paths. This is true even if one considers only straight paths, because of the topology of the torus.[7] For instance, between C and D♭, the shortest path goes down and to the left, as shown in Figure 21, but there is also path up and to the left that circles the torus in the opposite direction. Since φ_5 generally reflects circle-of-fifths distances, the difference between the upward and downward paths can be captured by spelling. The shorter downward path is a *diatonic semitone* (flatward) while the upward path is a *chromatic semitone* (sharpward). In principle, we could take even less efficient paths involving multiple cycles around the torus, such as a doubly-diminished third. All these paths involve different directions in the space, which means that to associate intervals with directions, we have to specify spelled intervals to capture the different available paths.

[Figure 21]

This is not enough, though: we must also distinguish between different ways to connect the same two points in the φ_3 direction. Consider the two minor-third paths shown in Figure 22. Both are minor thirds (not augmented seconds, etc.) because both move the same short vertical distance. The difference between them is therefore not captured by spelling, but can be understood through the idea of triadic orbits. Recall from §5 that φ_3 motion to the left corresponds to ascending triadic voice leadings. The most efficient minor-third motion, however, has a small *rightward* φ_3. This is because a smaller triadic voice leading result from the assumption that the minor third crosses a triadic orbit boundary. If we force the minor third to occur within one triadic orbit, then we drag φ_3 the long way around (to the left). We can refer to the first interval as a *chordal* minor third (one that changes triadic orbits) and the second as a *voice-leading* minor third (occurring within a single triadic orbit).[8]

[7] Meaning paths that are straight in a tangent space. The mathematical term for these is *geodesics*.

[8] This terminology, though sufficient for present purposes, only allows for a distinction between two triadic interval varieties. In principle there are an infinite number (analogous to doubly-diminished thirds, etc.). For a more systematic terminology, one can indicate changes in triadic orbit with a signed numerical prefix, such as a "+1-chordal minor third" (which would be an ordinary minor third, moving one triadic orbit clockwise) versus a "+2-chordal minor third" (a minor third moving two triadic orbits clockwise, which would imply a very large voice leading descent). The implicit notion of equivalence here is called *homotopy*. Two paths are homotopic if one can be continuously deformed into the other. On a torus, homotopic paths are those that cycle each dimension the same number of times. For more discussion of homotopy as a voice-leading concept, see Yust 2013b.

[Figure 22]

Viewing intervals through their most efficient φ_3 paths thus does, automatically, what, e.g., a Schenkerian analyst might do by hand: when a melody leaps by third, an analysis might tend to say that the underlying voices have not moved, that instead the melody has moved from one voice to another. Triadic orbits are not equivalent to such notions of conceptual voice, but they do capture this typical feature of them.

The interval of a whole tone is actually ambiguous in the φ_3 dimension: it is the same distance to the left (a chordal whole tone) or to the right (a voice-leading whole step). These are shown in Figure 23, along with less efficient upward paths representing augmented sixths (with the usual augmented sixth going to the right). The difference between the chordal and voice-leading whole tones depends on context. The voice-leading whole step from B♭ to A♭ goes past the notes F and D♭, so a context involving these notes will tend to imply this interval. For instance, in a progression from B♭ minor to D♭ major, B♭ and A♭ belong to a single triadic orbit. A context featuring E♭, on the contrary, would tend to imply that the B♭–A♭ interval is chordal, as in an E♭sus4 chord.

[Figure 23]

Interestingly, the distinction between different kinds of whole tones is precisely analogous to the intonational distinction between large and small whole steps (9/8 and 10/9) in the just major scale (9/8→chordal, 10/9→voice leading). This means that the classic syntonic comma problem can also be formulated as a voice-leading paradox associated with φ_3-cycles: for example, if $\hat{2}$ is approached as a voice-leading interval from $\hat{1}$ (for example in the progression I^6–IV–ii^6) but resolved back to $\hat{1}$ as a chordal whole tone (e.g., in a V–I progression), then there is a φ_3-cycle from the first $\hat{1}$ to the concluding $\hat{1}$. The ending note is the same (the same point in the space) yet not the same (because it has crossed a triadic orbit boundary). This is analogous to the fact if the same progression were realized with justly tuned harmonies, the concluding tonic note would be about a fifth of a semitone flat of the initial one. Žabka's (2013, 2014) work suggests similar generalizations of classic tuning-theory ideas.[9]

While context is essential to resolving ambiguities, it can also imply less efficient paths. For instance, B♭–A♭ interval is not by itself φ_5-ambiguous because the flatward path is much more efficient. But given a strong D minor context, especially one emphasizing its dominant, the interval may instead be interpreted as an augmented sixth, B♭–G♯, because the note A♭/G♯ is closer on the sharp side when reckoned from, e.g., an A major chord. Notice that the harmonic objects along the augmented-sixth path in Figure 23 include all the critical elements of contexts that imply the augmented sixth interval (such as D minor and A major triads).

[9] The intervals in Žabka's chromatic comma lattice correspond one-to-one to the non-homotopic paths from any point to itself in $\varphi_{3/5}$-space.

With this expanded concept of interval, each interval now has a specific direction in the space. An *intervallic axis* is a line that circles the space in the direction corresponding to a given interval, connecting transpositions of a pcset by that interval. There may be multiple intervallic axes for the same interval passing through different set classes, or for the same set class if the interval's semitonal size is a factor of 12. Also, the intervallic axes are the same for two intervals if one is a multiple of the other (such as perfect fifths and chordal whole tones).

Intervallic axes, by definition, pass through multiple instances of the same set class, but the motivating examples from the Trio of the String Quintet (Figure 14) involve approximate collinearity of *different* set classes, indicating that different set classes often fall on or near the same intervallic axes. For instance, F minor falls approximately along the diatonic-semitone axis for pcs, between C and D♭. Does this indicate that the path from C to F minor is somehow related to the diatonic semitone interval? The answer is yes, but to understand why requires another result about inversions in phase space.

Amiot (2013) discovered that the axes traced through $\varphi_{3/5}$-space by diatonic-semitone transpositions of major and minor triads are remarkably close together, and also remarkably close to the diatonic-semitone axes of diminished triads (which are equivalent to single pcs in the space). This seemingly remarkable fact is not coincidental. Its explanation relies upon the following proposition:

Proposition: Let A and B be inversionally symmetric pcsets with well-defined phases. Let T_xI be the inversion such that $T_xI(B) = B$. Let $T_y(A)$ be the transposition such that $T_y(A) = T_xI(A)$. Then B falls on a line halfway between A and $T_y(A)$ in Fourier phase space.

Proof: For any inversionally symmetric set, each phase value is equal to that of one its two inversional centers. For some component, let a be the phase for A and b the phase for B. Then $T_xI(b) = b$. Also, the phase of $T_xI(A)$ ($= T_y(A)$) is $T_xI(a)$. We then have $b - a = T_xI(a) - T_xI(b) = T_xI(a) - b$, which means the difference in phase from A to B is the same as from B to $T_y(A)$—i.e., the phase-space coordinate of B is halfway between those of A and $T_y(A)$ along some path (less than a full cycle of the space in any one dimension). This holds for the phase of any component, so the proposition holds for any Fourier phase space.

The proposition implies that any two inversionally symmetric pcsets lie on a common intervallic axis. For instance, let A = C and B = FA♭. FA♭ is symmetrical under T_1I, and T_1I(C) is D♭, or $T_1(A)$. Therefore, FA♭ falls exactly on the midpoint between C and D♭, on a diatonic-semitone axis. If we continue applying the proposition from the last two sets, A = FA♭ and B = D♭, we will continue to transpose by 1: T_2I(FA♭) = F♯A = T_1(FA♭) . . . . Therefore, the minor thirds and pcs lie evenly spaced along a diatonic semitone axis in the sequence . . . –C–FA♭–D♭–F♯A–D– . . . .

This axis is particularly interesting because the combination of adjacent pcs/minor thirds is always a major or minor triad. The resulting sequence of triads is a favorite pattern of Schubert's and other nineteenth-century composers (see Cohn 2012, 94–5, 100–1; Yust 2013b). Because such combinations will fall between, though not necessarily precisely halfway between, the two constituent subsets, they allow us to extend the significance of intervallic axes to non-symmetric set classes like major/minor triads. The exact location of a combination with respect to its subsets depends upon the relative magnitudes the subsets' components. For minor thirds, $|\widehat{f}_3|$ and $|\widehat{f}_5|$ are both 1.41, while for single pcs all components have magnitude 1. Because the minor third have larger components, the triads are closer to them. But because the proportional difference is the same for both components, the triads fall very close to the line between the minor thirds and singletons. Of the three ways to partition major and minor triads into dyad + singleton, the diatonic-semitone axis is the most compact and most evenly spaced, making it uniquely significant.

Schubert's use of F minor at the beginning of the Trio therefore does relate directly to an overall strategy of transitioning via a diatonic-semitone axis from C to D♭. It reflects a well-rehearsed move in ascending minor-second sequences. The intervallic axis of Fourier phase space show that this is not an accidental association; it is built into the topology of tonal relations.

Diatonic-semitone transpositions play important roles in other parts of the Trio as well. Figure 24 shows how the main theme ends, extending four measures beyond the music of Figure 14a. Here (in mm. 221–24) Schubert includes a formally superfluous repetition of the previous four measures with only one change, the G♭ major chord becomes G♭ minor. This introduces a diatonic semitone, B𝄫–A♭, in the most prominent moving voices of the progression. The replacement itself (B♭→B𝄫) involves a chromatic semitone.

[Figure 24]

The contrasting middle section of the Trio (mm. 236–243) echoes this chromatic line, B♭–B𝄫–A♭, but with an enharmonic reinterpretation of the chromatic note, B♭–A♮–A♭, in the context of the relative key, as shown in Figure 25. The enharmonic distinction between these two chromatic lines is graphed in Figure 26, in the large outer triangles. The lower triangle represents the progression B♭–B𝄫–A♭, where the chromatic note is a flatward displacement. The enharmonic reinterpretation flips this triangle over the common diatonic whole tone, B♭–A♭, by reordering the two kinds of semitone intervals, so that the chromatic note represents a sharpward displacement, B♭–A♮–A♭.

[Figure 25]

[Figure 26]

A number of features of this graph are noteworthy: first, directly in the center of each chromatic triangle lies another tone that can be used to create one of the tonal contexts that can resolve the enharmonic ambiguity of the chromatic line: The note D♭ defines the line as $\hat{6}$–♭$\hat{6}$–$\hat{5}$ in D♭ major, while the note F defines the line as $\hat{1}$–♮$\hat{7}$–$\hat{7}$ in B♭ minor. These notes are prominent in each melody respectively, as shown in Figures 24–25. When these notes are added to each chromatic line, to create a progression of dyads, the triangles shrink around the common tone, pulling the chromatic note up or down to resolve the ambiguity. The interior triangles are equivalent in shape to the exterior ones—consisting of lines in the diatonic-semitone direction, the chromatic-semitone direction, and the voice-leading whole-step direction—but the edges are half the distance because they are not complete transpositions by half step or whole step, but voice leadings involving those intervals. This illustrates a general principle that adding additional context typically shrinks a tonal structure in Fourier spaces—preserving their shapes to varying degrees—and can thereby resolve ambiguities in either dimension.

It is also interesting in Figure 26 that the important harmonies of the Trio's main theme and contrasting middle lie approximately along the lines defined by the melodic motions. Notice also that the interval D♭F lies directly at the center of the figure, at the midpoint of B♭–A♭ as well as D♭–F, and also halfway between D♭ major and B♭ minor triads. This is also the position of the D♭ diatonic scale and played an important role in Figure 14(b) (the transition into D♭ major at the beginning of the main theme).

Figure 26 also shows, with dotted lines, that multiple important harmonic progressions of the passage follow the diatonic-semitone axis, including the i–V progression in B♭ minor and the iv–I progression in D♭ major. The latter of these connects to the line from C major to D♭ major traced at the beginning of the main theme.

(7) Chordal semitones in the Trio of the String Quintet

In §4, I asserted that Schubert's harmonic and motivic design for the String Quintet Trio centered around the play of contrasting types of stepwise interval. The previous sections have expanded the concept of interval to include not only differences of spelling but also analogous distinctions between chordal and voice-leading interval types. The latter distinction has to do with orientation within triadic orbits, the sense of placement within a contoured gravitational field aligned to the triadic structure of the harmony. These distinctions, and their associations with directions in Fourier phase space, will help us understand the more striking and unusual harmonic features of Schubert's Trio as purposeful explorations of the possible triadic and enharmonic orientations of stepwise intervals and their expressive effects.

We have already seen that one of the linear relationships, the F minor triad between C and D♭, corresponds one important type of scale step, the diatonic semitone. However, we also observed that Schubert seems to exploit another linear relationship in the transition from Scherzo into Trio: as Figure 14(b) shows, the D♭ major scale appears halfway between the F minor and G♭ major triads. The intervallic axis traced by this motion is that of a more exotic φ_3-inefficient chordal chromatic semitone, as shown in Figure 27.

[Figure 27]

An ordinary major scale has three kinds of steps, because of the distinction between chordal and voice-leading whole steps, as shown in Figure 28(a). A mixture of elements of two scales will add ordinary chromatic half steps as well as chordal chromatic half steps (Fig. 28b). The slurs in the figure show which notes belong to the same triadic orbit.

[Figure 28]

The idea of exploring of such distinctions in scale-step types and their metamorphoses within shifting harmonic contexts is set forth by the unusual main theme of the Trio, an entirely scalewise melody doubled across registers that combines the function of tonal transition with thematic initiation. The reduction of Figure 29 highlights properties of the theme related to this topic. The melody gives special attention to the interval A♭–F, which is filled with the passing tone G♮ in the F minor portion of the melody. The subsequent scalar descent puts a heavy emphasis on G♭ where the first literal chord appears. This G♭ resolves to F, filling in another A♭–F interval. The G♭–F resolution is further reinforced at the cadence, where the first cello leaps suddenly into its upper register to cover the melody with a G♭ over the dominant. The A♭–G♭–F melody also appears in the cadence of the contrasting middle, also shown in the figure.

[Figure 29]

The plot in Figure 30 shows that a structure relating the principal harmonies of the passage, F minor, D♭ major, G♭ major, and the B♭ minor of the contrasting middle, mirrors on a smaller scale the structure created by the two ways of filling the A♭–F minor third. The outer edges of each parallelogram follow diatonic-semitone and chordal whole-tone axes. (The latter is the same as a perfect-fifth axis, since a chordal whole tone is equal to two perfect fifths). The long inner diagonal represents the difference between the chordal whole tone and diatonic semitone, the chordal chromatic semitone. This is one of the most distant intervals in the complex of F minor and D♭ major tonalities, and Schubert's theme highlights it by pointedly dramatizing the conversion of G into G♭. Association of the smaller parallelogram of triads with the larger melodic structure is also strongly apparent in the passage: the note G♮ belongs to the F minor tonicization, the G♭ appears first in the G♭ major chord at the dynamic climax of the theme, and the F is most prominent melodically in the contrasting middle where the B♭ minor area appears. (The remaining association is A♭ with D♭ major, which is most evident in the return to D♭ major in the contrasting middle.) The larger parallelogram is centered on FA♭, the melodic interval repeatedly outlined in the passage, while the smaller parallelogram shifts down and to the left because, in the broader context, the passage is centered on the D♭ diatonic scale (which coincides with D♭F).

[Figure 30]

Thus the harmonic plan as well as aspects of the thematic material of the first part grow out of a chromatic conversion that embodies the stark contrast of Scherzo and Trio. The fifth of C major, which drives the energetic Scherzo, undergoes a two-part metamorphosis after being converted into a passing tone at the beginning of the Trio: it is chromatically softened, and it crosses the boundary between voices, entering a triadic orbit that draws it downward instead of upward. Both facets of the metamorphosis involve large distances in Fourier space, but especially the change of voices, which traverses three quarters of a φ_3-cycle. The combined effect of chromatic softening and the re-orientation of triadic gravitational forces casts a blanket of enchanted sleep over the music.

The recapitulation of the Trio further develops these relationships. Mirroring the unusual exposition, it begins the process of retransition at the very moment that thematic recapitulation begins. The reduction in Figure 31 respells the recapitulation's sequential melody to show its proper tonal relationship to the home key. It begins with a melody based on the main theme in the key of ♭III, the relative of the parallel, then sequences this melody down a step to the key of ♭II (E𝄫 major). This is followed by a cadence in the home key that directly juxtaposes it with the key of ♭II. Schubert uses the same sequence for the true retransition beginning on ♭II (transposed down a whole step) so that it ends in "♭I," enharmonically C major, the key of the Scherzo.

[Figure 31]

The graph of the sequence in Figure 32 shows that it circles the space in the φ_3 dimension. This kind of "voice-leading tour" is analogous to an enharmonic tour, and involves a similar kind of paradox. The sequence creates a continual descent by voice-leading whole steps, which is particularly evident in the upper voice whose chromatic descent divides them into alternating diatonic and chromatic semitones. However, the overall descent outlines chordal intervals in the tonic harmony of D♭ major. In enharmonic tours, a respelling is required at the global level of the entire cycle, but the local placement of the respelling is essentially arbitrary. Similarly in voice-leading tours, the global progression requires a change of triadic orbits to appear somewhere in the process, but none of the local progressions themselves imply such a shift. The voice-leading paradox is not unique to geometrical theories of harmony: the same paradox arises conspicuously in Schenker's theory, where the endpoints of a linear progression or unfolding represent a vertical interval, belonging to separate voices at some middleground level, but are "horizontalized" into a single voice at the next level.[10] Some Schenkerians have pointed out how enharmonicism creates paradoxes within the theory,[11] but the voice-leading paradoxes—which are actually more significant because of their universality—are not as commonly acknowledged. The reproduction of these endemic music theoretic paradoxes as features of a topology actually helps to more clearly rationalize them.[12]

[Figure 32]

As the slurring of Figure 31 shows, the $\hat{6}$–$\hat{5}$ resolutions and $\hat{3}$–$\hat{4}$–$\hat{5}$ motions are prominent in the sequential melody. Figure 33 isolates these and uses slurs to show orbits as in Figure 28, and beams to show changes of orbit. The φ_3 relationships between notes in different tonal areas are shown with downward-pointing beams and slurs. The tonal areas are in precisely

[10] See Schenker 1979, part II, particularly the sections on "Unfolding," "Linear Progression," "Reaching Over," "Motion from an Inner Voice," and "Initial Ascent."
[11] Proctor (1978, 131–143), Damscroder (2006, 261–4)
[12] See also the analysis of "Nacht und Traume" in Yust 2013b.

the same relationship as melodic figures within the sequential melody: from D♭ major to F♭ major is a minor third filled in with a diatonic semitone plus chordal whole step, D♭ major → E𝄫 major → F♭ major. Taking a large-scale view, the analysis orients these tonal areas with respect to the home key, which is why the two sequential repetitions relate by a chordal whole tone (rather than the voice-leading whole step suggested by the local relationships).

[Figure 33]

The dotted triangles in Figure 34 extend the A♭–G–F and A♭–G♭–F structures of Figure 30 (associated with F minor and D♭ major), to a larger series of triangles including the melodic $\hat{3}$–$\hat{4}$–$\hat{5}$ cells of the three keys from the sequential retransition (enharmonically E major, D major, and C major). At the center of the figure, shown with dashed lines, the diatonic scales associated with three of these keys (D♭ major–E major–D major) duplicate the shape of the individual melodic $\hat{3}$–$\hat{4}$–$\hat{5}$ cells. This super-structure for the recapitulation surrounds G♭ minor and D♭, indicating a kind of governing presence of G♭ minor as iv. G♭ minor does not literally appear as a harmony in the recapitulation, but its intangible presence might be evidence of the influence of the striking iv–I progression that concludes the first part (see Fig. 24). A closer examination of the music confirms this connection: the return to D♭ major approaching the parallel cadence ending the second part involves a respelled B𝄫–A♭ resolution with a held D♭ in the upper voice, exactly the essential contrapuntal components of the iv–I progression. The relationship of E𝄫 major to D♭ major reinforces a more subtle link in that G♭ is the common tone between their respective $\hat{3}$–$\hat{4}$–$\hat{5}$ melodic cells.

[Figure 34]

What could be the musical purpose of Schubert's expanding these motivic and harmonic features of the first part in the recapitulation, using the scalar motive at the foreground and as a modulatory plan, in a harmonic context that recalls the mode mixture on the subdominant? The triangle created by the $\hat{3}$–$\hat{4}$–$\hat{5}$ motive has a wide φ_3 spread, which is doubled at the local level by this procedure of "motivic multiplication." The fish-eye effect opens up the stage for the conversion necessary to restore the tonal state of the Scherzo and burn off the Trio's somnolent fog.

Figure 35 shows the status of G♮/G♭ in each significant tonal area of the piece, starting and ending from the Scherzo. The Trio begins by destabilizing this tonal pillar of the Scherzo: its geographic position does not change at first (in F minor), but it becomes a non-tonic passing note, inhabiting the outskirts of its triadic orbit. Placing the note in this peripheral status is a necessary preparation for what happens next: with its chromatic mutation to G♭ in D♭ major, it crosses into the next orbit below, still peripheral but now in the upper reaches of its new orbit. The key of E𝄫 major confers a fleeting status of tonal stability, a place at the center of a triadic orbit, to this G♭. It also reintroduces G (= A𝄫), and in the same orbit as G♭. The restoration of the Scherzo tonality then requires just the reversal of the φ_3 orientation of the

G♭–G♮ path to return to G♮ into an upward-pulling gravitational orbit (see the arrows in Fig. 34). Schubert has already composed the necessary traversal of φ_3 space into the sequence of the recapitulation, so all he has to do to complete the retransition is to repeat the sequence starting from E𝄫 (= D) major. Figure 35 summarizes the overall process, falling into the Trio by the chordal chromatic semitone from G to G♭, then rising out with the chordal diatonic semitone back to G.

[Figure 35]

(8) Boundaries and Regions

This paper began with an observation about the ubiquity and abundance of types of spatial metaphor used to explain music. We have investigated concepts of place, path, and direction within Fourier phase space. But among the most common spatial metaphors in music are those of region and boundary, used especially to describe tonality and key areas. As spatial concepts, these are especially rich with meaning and implications.

The *Tonnetz* is drawn in Figure 10 as a set of three intervallic axes, corresponding to the three consonant intervals and oriented to pass through the dyads and singletons. Unlike the axes investigated in the previous section, however, these are not primarily used as pathways from one object to another, but to create boundaries between them. The basic neo-Riemannian operations cross over them, passing through the dyad representing the common tones between triads. The dyadic axes are boundaries in the sense that one can approach them by gradually removing the note that distinguishes one triad from another.

Thinking of these axes as boundaries implies that the *Tonnetz* is a partition of the Fourier space into *regions*. As a scheme of regions, the *Tonnetz* exhibits a number of special properties:

(1) It has twenty-four regions with translational and rotational symmetry (equating to transposition and inversion respectively), such that the regions can be associated one-to-one with the members of a set class—e.g., consonant triads. The major and minor triads fall near the centers of regions, and edges and vertices corresponds neatly to subset dyads and singletons.

(2) The regions are *convex*. This means that it is impossible to take a straight path crossing a boundary out of a region and re-enter that region on the same path, except possibly by circling the entire torus. The convexity property is important because it means there is no ambiguity about whether a given straight path crosses a particular border in some neighborhood of the space. To satisfy this property, the boundaries must consist of straight lines between vertices (points shared by three or more regions).

(3) The boundaries are meaningful intervallic axes. In the *Tonnetz*, the boundary axes represent the constituent dyads of the major and minor triads. The triad at the center of each region can be considered the combination of a point at the center of any one boundary (a dyad) with the vertex at the intersection of the other two axes (the other pc).

(4) A network of relationships between triads can be derived from the regions sharing borders, which, for the *Tonnetz* regions, is the dual *Tonnetz* or "chicken-wire Torus" (Douthett and Steinbach 1998)—see Figure 3.

Property (4) is not a constraint on possible schemes, since such a network can always be constructed. Yet it may be a consideration, since the resulting network may help in interpreting the regions, and the links of this network may correspond to meaningful intervallic axes. Borrowing Lewin's (2007) familiar term, we can call these *transformational axes*, to distinguish them from the *boundary axes*. In the *Tonnetz* regions the interval for each transformational axis is the voice-leading interval that connects the corresponding triads, as shown in Figure 36. It is also the difference between intervals for two of the boundary axes (P4–M3, P4–m3, M3–m3).

[Figure 36]

The *Tonnetz* might therefore, in recognition of all these special properties, be offered as a theoretical model of the concept of tonal region. But, if we mean by tonal region something like the conventional notion of major and minor keys, the *Tonnetz* regions are not a very good fit. In the harmonic syntax of common practice tonality, a tonal area is typically secured by the combination of its dominant seventh and tonic triad. Dominant sevenths, however, do not fall in the same *Tonnetz* region as the triads they tonicize. As Figure 37 shows, they are on the boundary of two regions that are not even adjacent to the regions associated with their tonics.

[Figure 37]

Conventional keys, then, if they can be represented by regions, are not like the purely triad-based regions of the *Tonnetz*. The arrows in Figure 38 show the shortest paths for all V^7–I/i progressions. Since none of them cross, a scheme of regions containing them is possible, but is constrained by the fact that the parallel modes share a dominant. This means that the dominant sevenths must fall on a boundary that splits parallel modes, and the only available axis is the perfect-fifth axis shown with a dashed line in the figure. This is also the parallel-mode boundary in the *Tonnetz* regions; approaching this boundary involves removing or negating mode-defining elements. The axis interval defines neutrality for the boundary: in this case, the open perfect fifth.

[Figure 38]

This perfect-fifth axis does not separate closely-related major and minor keys or fifth-related keys of either mode. Note that there is no way to draw a straight line between the dashed lines of Figure 38 without crossing one of the arrows. This means that it is not possible to draw *Tonnetz*-like triangular regions that contain the V^7–I/i progressions. However, the closely-related major/minor regions can be separated by another perfect-fifth axis as shown by the dotted line in Figure 38. This axis is halfway between adjacent portions of the parallel-mode axis and goes through the major and minor thirds (. . . –FA–AC–CE–EG– . . .). The points on this boundary either lack fifths (major or minor thirds) or are balanced between fifths on either side (major and minor seventh chords), making them ambivalent with respect to possible roots a third apart.[13] By separating the major and minor keys into parallel strands, the fifth-related region on each strand can be separated with different kinds of boundaries, making it possible to avoid cutting across the V^7–I/i progressions.

There are many possible ways to draw boundaries separating fifth-related regions that satisfy properties (2) and (3) above (convexity and intervallic axes). By using different kinds of boundaries for major keys versus minor keys, however, we are already forfeiting one aspect of property (1): although the regions will all be parallelograms of the same area and the scheme will have translational symmetry, it will not have rotational symmetry. This property is actually inappropriate for keys, though: the 180º rotational symmetries of the *Tonnetz* correspond to inversions of the pcsets represented by each region. But conventionally understood as keys (strict dualist theories aside), major and minor are not inversionally related, as manifest in the differently oriented arrows of Figure 38.

One way to draw appropriate boundaries is to first find appropriate transformational axes. Because the tonal regions will be parallelograms, the transformational axes will run parallel to the border axes, rather than bisecting them as in the *Tonnetz*. One of these will be a perfect-fifth axis between the two perfect-fifth boundaries. The others should run parallel to the borders between fifth-related regions. We can determine the border axes, then, by finding a transformational axis through a central characteristic pcset for the key, then transposing this axis by half of a fifth in either direction.

The *Tonnetz* regions reflect the total subset structure of major and minor triads: each region contains the three pcs and three dyads belonging to its triad. This principle can be generalized to other trichords.[14] However, it does not readily generalize to larger sets. A region containing all pcs and dyads of a major scale would occupy a quarter of the entire torus, six times too large for a scheme of 24 regions. The principle deduced in the last section, that averaging over larger objects leads to smaller areas on the torus, advises us to consider larger subsets. Indeed, it would be possible to contain all major and minor triads of

[13] Such parallel intervallic axes can be associated with set-class multiplication (or division). Starting from the perfect-fifth axis through the pcs and fifth dyads (. . . – F – FC – C – CG – G – . . .) multiply by major third (= . . . – FA – F^{M7} – CE – C^{M7} – GB – . . .) or by minor third / major sixth (= . . . – DF – D^{m7} – AC – A^{m7} – EG – . . .). One can show that this works for any such multiplication using the convolution theorem.

[14] The generalization is similar to Cohn's (1997), but with an added geometrical component. Such a generalized *Tonnetz* is possible where connecting the pcs of the trichord in the space yields a triangular region that contains no other pcs. For instance, connecting (024) trichords via major third, chordal whole tone, and voice-leading whole-tone paths creates a *Tonnetz* of 12, not 24, regions, because one edge (the chordal whole tone) passes through another pc (e.g., G between C and D). The problem with (024) is its lack of odd intervals, however, not its inversional symmetry. The (027) trichord, for instance, makes a well-formed *Tonnetz* by using a voice-leading whole tone which is the sum of an ordinary (1-chordal) fourth and a φ_3-inefficient 2-chordal fourth. The distinction between kinds of fourths makes it possible to invert a trichord onto another with the same pcs but different intervals (i.e., swapping which fourth is 1-chordal and which is 2-chordal). The simplest (012) *Tonnetz* in $\varphi_{3/5}$-space has rather imponderably narrow regions bounded by a diatonic semitone + chordal doubly diminished unison = chordal doubly diminished second.

a major scale in a region smaller by about a third. Yet this is still too large, which means that we cannot expect tonal regions to include every singleton, dyad, or triad belonging to a given key. Instead, the characteristic sets within each region should be the scales and large subsets of scales that typify specific keys.

Figure 39 shows the positions of diatonic and harmonic minor scales and some of their subset hexachords. The major regions are easy to characterize: the diatonic scales and major hexachords (i.e., Guidonian hexachords) coincide with major and minor thirds on the boundary with the closely-related minor regions. Each major hexachord is the intersection of diatonic scales on either side. One diatonic hexachord appears within the major strand, and it is an obvious candidate for the characteristic pcset of a major key, a hexachord from $\hat{7}$ to $\hat{5}$ that includes all the notes of the tonic triad and dominant seventh of the key. This hexachord falls along a voice-leading whole tone axis (which is also a transformational axis for the *Tonnetz*).[15] The axis intersects the upper boundary at the diatonic scale for the major key, which makes a natural point of transition between the major key and its relative minor. It intersects the lower boundary at the $\hat{5}$ pc. On Figure 39, this point is called "V^7+C," because the five-note set consisting of the dominant seventh plus the tonic note is also located here. This set combines all the notes common to the major key and its parallel minor, excluding the mode-determining $\hat{3}$s and $\hat{6}$s.

[Figure 39]

The boundaries for the fifth-related major keys can then be placed halfway between these transformational axes, as shown in Figure 39. These intersect the other two boundaries at the points for dominant sevenths (perfect fifths) and major hexachords. The major hexachords are subsets of the basic diatonic scales of both adjacent major keys, and include both of their tonic chords.

The minor-key strands contain more scalar sets: the harmonic minor scale itself and two hexachordal subsets occur within the strand. One of these is the minor hexachord (a diatonic hexachord consisting of the first six notes of a minor scale). The other is a harmonic-minor version of the $\hat{7}$–$\hat{5}$ hexachord, which has all the notes of V^7 and i of a minor key. These each have some associated intervallic axis, as shown in Figure 40, making the choice of minor-key boundaries less straightforward. The intervallic axis for the minor hexachord can be taken out of consideration on the grounds that it goes in the wrong direction and would lead to boundaries that cross the V^7–i progressions. The diatonic

[15] In all cases that will be described here, the criterion for associating a pcset with an axis is that it can be decomposed into two subsets adjacent on that axis and falls relatively close to it. Since tritones have no effect on the third and fifth Fourier components, an effective method of finding such an axis is to remove tritones, then decompose the set into two inversionally symmetric subsets, as balanced as possible in the magnitudes of the third and fifth components. For instance, the C major $\hat{7}$–$\hat{5}$ hexachord, with the BF tritone removed, is a combination of G and CDE. The latter coincides with the major third CE, but is better balanced between its third and fifth components. This decomposition therefore makes a more suitable axis for the set than the imbalanced CE/GD.

semitone axis and harmonic minor are more central than the tritone axis and $\hat{7}$–$\hat{5}$ hexachord in the sense that (a) boundaries evenly spaced around the tritone axis would put the minor hexachord near the edge of the region, whereas all the characteristic pcsets are well within the region defined by diatonic-semitone axes, and (b) the harmonic minor scale is between the two hexachords and represents the union of their pc-content. Using the diatonic semitone axis also simplifies the network of adjacent regions.

[Figure 40]

Figure 41 shows the resulting regions. The network of adjacent regions is slightly more complex than the *Tonnetz*, including fifth-related same-mode regions, plus three mode-change transformations that essentially reproduce the PLR relations of the *Tonnetz*. The diatonic-semitone transformational axis of the minor regions is the same as the L-axis of the *Tonnetz*, and the voice-leading whole-tone axis is the same as the R-axis. These axes collide between the parallel regions, yielding a transformation based on the intervallic difference between them, VL whole tone – diatonic semitone = chromatic semitone, which is also the interval defining the *Tonnetz*'s P-axis.

[Figure 41]

The most expressively charged of boundaries, certainly in Schubert's music, is the one that these tonal regions share with the *Tonnetz* regions, the perfect-fifth axis between parallel-mode regions. Schubert's use of mode change to communicate contrasts of emotional valence, particularly in Lieder, is too abundantly evident to require demonstration here. (Wollenberg [2011, ch. 2] discusses his use of parallel modes in instrumental music.) What a harmonic geometry can add to this already well-known aspect of Schubert's musical language is to generalize its implications via spatial metaphors. As a kind of boundary-transgression, the crossing of the fifths axis need not necessarily involve mode-change directly. Some of the most interesting examples from Schubert's music cross this boundary in the major→minor direction while pointedly avoiding the explicit articulation of a minor key through a delicate circumlocution.

The tonal digression that disturbs the unearthly calm in the main theme of the String Quintet's Adagio is one notable example. The harmonic path of the first period of the theme (mm. 1–14) is summarized in Figure 42. It begins by moving, ascending-fifth-wise, to F♯ major (mm. 1–9). It returns to E major via a tonicization of A major in mm. 12–13. The harmony that mediates between these keys is a B minor triad in m. 11. The figure tracks the tonal implications of this progression by plotting the pc-content union of adjacent harmonies. The progressions involving B minor (F♯ major – B minor / B minor – E major) are tonally ambiguous in the space, balanced on the boundary with the parallel minor keys of B minor and E minor. Schubert never explicitly crosses this boundary in the passage, but the use of B minor as a "pivot" activates it, making the listener aware of it the way one might search for a door by groping along the wall of an unfamiliar dark room at night.

[Figure 42]

When the second period of the theme begins in hushed *sotto voce* (Fig. 43a), its first harmonic move is to the B minor triad once again, this time directly from E major, approaching the parallel-key border more forcefully and impatiently. The chord may hint at another immanent tonicization of the subdominant, but Schubert instead leaps across the boundary into the remote key of G major. As Figure 43(b) shows, this progression passes through the parallel key of E minor. Schubert then returns to the home key by surfing the expressively charged parallel key boundary once again, bridging the minor strand with B^7 in m. 20, then suggesting, but not realizing, A minor with a Fr^{+6}–V^7 progression in mm. 20–21. The music returns to E major once again via the subdominant area.

[Figure 43]

The ability of this scheme of regions to accurately reflect intuitions about tonal areas has been explicitly engineered into the design of the regions. That the Fourier space can act as a mirror to these intuitions without too much distortion in the picture is impressive, but there is also more to the spatial metaphor than a correspondence between position in the space and keys. In the Adagio, a tonal area, E minor, is activated not by a progression within that region, but by a motion that passes through it. This is an effective description of the method behind Schubert's harmony in the passage. He achieves the sense of E minor as an unspoken subtext by writing a harmonic progression that invokes E minor without actually being in that key at any given moment. The sense of E minor as a necessary intermediary between E major and G major is effectively a spatial one.

We have already (in §4 above) seen what happens next in this Adagio, the tranquility of the seraphic main theme is violently upended by the intrusion of the F–E trill and the F minor interior theme that it launches. The catastrophic descent into this impossibly flatward tonality seems to be a consequence of what happens in the main theme: when it first explores, then transgresses, the charged parallel-mode boundary, it enters a forbidden region of tonal space. By opening the door, so to speak, the demons are unleashed, and the Inferno of F minor, two passes over the modal boundary from the home key, is suddenly exposed.

The idea of this kind of tonal "action at a distance" as a feature of Schubert's music was first proposed by Cone (1982) in a famous article on what he called Schubert's "promissory note," a strikingly unresolved chord in one phrase that is answered by the expected resolution appearing in another phrase. The proposal carries an air of paradox: Cone asks us to hear one chord resolving the other (in a later article he called the promissory note a "harmonic device involving an aborted and delayed resolution" (1984, 223)), even though the resolution cannot possibly be construed as structural, since both harmonic events are clearly subsidiary within their individual phrases. The paradox evaporates, however, if we reformulate the causal relationship between exceptional tonal events of different phrases in spatial terms. In Cone's examples, unresolved secondary chords activate regions of tonal space without fully committing to them. A later, more thorough realization of that region may appear to be causally related to the earlier event, fulfilling the earlier intimation that this region will play some sort of role in the tonal plan of the piece. In other words, a promissory note may be a way of defining the tonal sphere of activity for the music at the outset, before literally advancing the parts of the narrative that involve activity in certain contrasting regions.

The notion of a promissory note can also be broadened by focusing on the spatial boundary involved. Whereas the example from the Adagio of the String Quintet uses the parallel-mode boundary, Cone's examples (Moment Musical no. 6 and the first movement of the String Quintet) involve the other perfect-fifth axis that separates relative keys. The promissory note may then be invoked by other keys related to the one originally suggested. For instance, Cone (1984) cites the String Quintet first movement, where a rescinded move to E minor early in the exposition is recalled by a similar feint to the same key in the later part of the second theme (mm. 106–110) and in the closing material (mm. 139–140 / 143–144). Between these two later events, however, there is another notable tonal disruption, the repeated deceptive cadences to vii^{o7}/ii at the end of the subordinate theme (mm. 125 and 131). The same relative-key boundary is involved here, and Schubert uses B as a common-tone thread to G major as he does in all of the E minor tonicizations. In the Moment Musical that Cone discusses in the earlier (1982) article, the C major – $\text{E}\flat^{7}$ progression lives on the same relative-key boundary as the diatonic seventh chords ($\text{D}\flat^{\triangle 7}$ and $\text{B}\flat^{m7}$) that are so conspicuous in the initial thematic idea.

Schemes of regions can be seen as ways of overlaying familiar networks of chord and/or key relations onto a common underlying space. The common space can then be used to relate them and provide perspective on their differences. In particular, using the tonal regions as a stand-in for traditional harmonic theory may illuminate debates over neo-Riemannian analyses of Schubert's music. Fourier phase space is especially well-suited to this task, since it circumvents what is usually held to be a fundamental difference between the theories, that neo-Riemannian theory is based on direct relationships between triads whereas traditional tonal theories use diatonic scales as intermediaries in such harmonic relationships. While the *Tonnetz* regions may be derived from the structure of major/minor triads, and the tonal regions from scalar collections, there is nonetheless no barrier to viewing relationships between harmonic entities of any type—triads, scales, etc.—through either configuration.

One such debate was instigated by Cohn's (1999) use of hexatonic cycles in an analysis of Schubert's last piano sonata, D.960, eliciting a critique by Fisk (2000) and response from Cohn (2000b). Fisk's critique uses the metaphor of tonal region extensively to explain his dispute with Cohn's approach, claiming that Schubert's keys should be grouped by fifth-relation (B♭ major–F major / F♯ minor–C♯ minor) rather than by hexatonic cycle (B♭ major – F♯ minor / F major–C♯ minor). Fisk's "regions" group tonal regions adjacent along the circle-of-fifths strips, while the hexatonic cycles, which are vertical strips in the *Tonnetz* regions, chart a jagged, circuitous path across the tonal regions.

The recapitulation of D.960, which condenses the tonal milestones of the exposition into an enharmonic tour within the recomposed B part of the main theme, is a good place to compare the two approaches. The harmonic trajectory looks different through a *Tonnetz* framework (Figure 44a) than it does through tonal regions (Figure 44b), and the difference reflects to some extent Cohn's and Fisk's differing interpretations of the sonata. The hexatonic pole, a focal point of contention, looks simpler as a direct move in the *Tonnetz*, where it requires passing through three bordering regions in a roughly consistent vertical orientation, one of which is the G♭ major region matching the initial key of the B section of the theme. In the tonal regions, the route to a hexatonic pole is less clearly defined and more indirect. A straight path from B♭ major to G♭ major cuts indelicately across multiple regions.

The trill that intercedes between these keys, however, promotes the idea that the modulation involves a glance toward the subdominant. Such an interpretation helps to rationalize the pathway through tonal regions, making it perhaps unsurprising that Fisk emphasizes the gesture, whereas Cohn warns of overstating its importance.

[Figure 44]

In contrast, the return to the home key from A major at the beginning of the A' part is hard to rationalize in the *Tonnetz* (passing through bits of four regions). But it yields a straightforward interpretation in the tonal regions, with a single key, D minor, acting as intermediary between the distantly related keys. The tonal regions thus convey an explanation of the modulation rooted in conventional harmony: the B♭ major chord is approached via a D minor deceptive progression and immediately reinterpreted as a tonic. Marston (2000) places special emphasis on this moment as a "defamiliarization of the tonic harmony." Noting the strong implication of an unrealized key in this progression, we might read this as a reverse promissory note, recalling the extraordinary D minor return of the main theme material at the end of the development section (and also echoing a similar progression from the exposition, a true promissory note, where the resolution of an extended dominant of D minor in m. 70 is reinterpreted, by means of a textural sleight of hand, as a cadential $^{6}_{4}$ in B♭ major, evaporating the D minor tonality before it has an opportunity to fully materialize).

Conclusion

In an article comparing the Fourier transform to voice-leading geometry, Tymoczko (2008b) criticizes the Fourier transform for being a "black box," claiming that the cogs and wheels of the Fourier machine are concealed, not clearly referable to musical meaning. Indeed, the criticism is a potent one: a method whose operational premises remain mysterious is not very useful as theory. Tymoczko's recommended response to this impasse was to abandon Quinn's supposedly abstruse procedure in favor of one engineered from the ground up, voice-leading geometries. A different reaction to the black-box syndrome is to reverse-engineer the machine to better understand how to interpret it. The DFT rewards this strategy: investigating the musical meaning of Fourier phase components in this paper has led to cardinality-independent concepts of scalar and triadic voice leading, a harmonic topology that is robust with respect to cardinality-changing operations, and an embedding of the *Tonnetz* in a continuous space that expands upon the common-tone based sense of distance that has made the *Tonnetz* so useful in analysis of nineteenth-century repertoire.

Common topological metaphors such as tonal distance and tonal area become especially powerful hermeneutic tools when they are made explicit in a geometry like the Fourier space explored here. Regions and boundaries, for instance, help to generalize and make sense out of the idea of Schubert's "promissory notes," and the idea that Schubert's music activates harmonic regions by outlining or passing through them without occupying them. The cycles that constitute an intrinsic feature of this geometry help rationalize paradoxes like enharmonicism that persistently arise in music theory. And the analysis of the Trio from

Schubert's String Quintet shows that it is possible for long-range voice-leading processes to involve objects other than simple chords, such as scale segments.

Appendix: Calculating Fourier Components

Those interested in exploring Fourier phase spaces further will need a way to calculate components. The best way is simply to calculate the DFT (sometimes also referred to as FFT or "Fast Fourier transform") directly. Software packages like Mathematica have built-in Fourier analysis functions. One can also set up a spreadsheet to perform DFTs and conversion to polar coordinates; see the example available at [INSERT LINK HERE].

Another way to calculate individual components is to perform the vector addition procedure of Quinn's Fourier balances (see §2). The vector for a single pc is

$$v_{pc} = (\sin\theta, \cos\theta)$$

where θ is the angle on the given balance. For $\widehat{f_3}$, θ comes in increments of $\pi/2$, and the resulting vector will be (±1, 0) or (0, ±1). For $\widehat{f_5}$, θ takes twelve possible values in increments of $\pi/6$. To give a pc a weight *k* other than 1, multiply the vector by *k*:

$$kv_{pc} = (k\sin\theta, k\cos\theta)$$

After summing these vectors componentwise, convert back to polar coordinates:

$$\left|\widehat{f_n}\right| = \sqrt{x^2 + y^2}, \qquad \varphi_n = \arctan(x/y)$$

For example, φ_3 and φ_5 for the C major triad are given by

$$(0,1) + (0,1) + (1,0) = (1,2), \qquad \varphi_3 = \arctan(1/2) = 0.464 = 0.15\pi$$

$$(0,1) + (0.5, 0.87) + (0.87, -0.5) = (1.37, -1.37),$$

$$\varphi_5 = \arctan(-1) = 0.25\pi.$$

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Figures and Captions

Figure 1: Menuetto from Schubert's String Quartet no. 13 in A minor, mm. 1–46, with common-tone links indicated

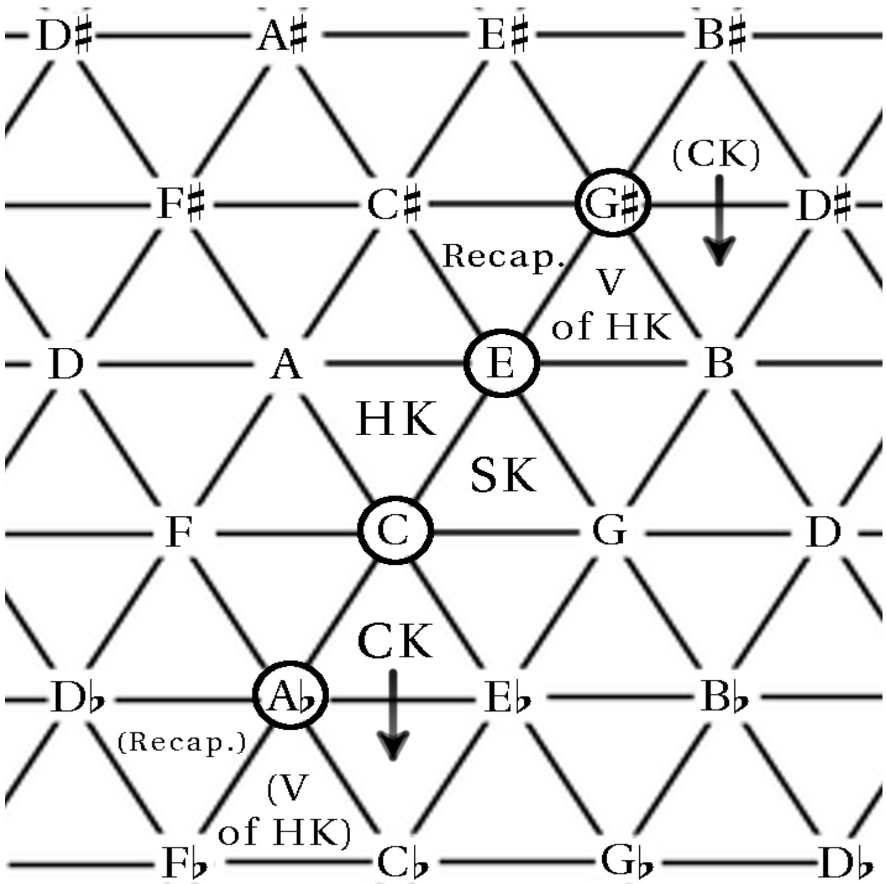


Figure 2: The harmonic stations of Schubert's Menuetto (home key, subordinate key, contrasting key, etc.) on the *Tonnetz*. Significant common tones used to link tonal areas in the music are circled.

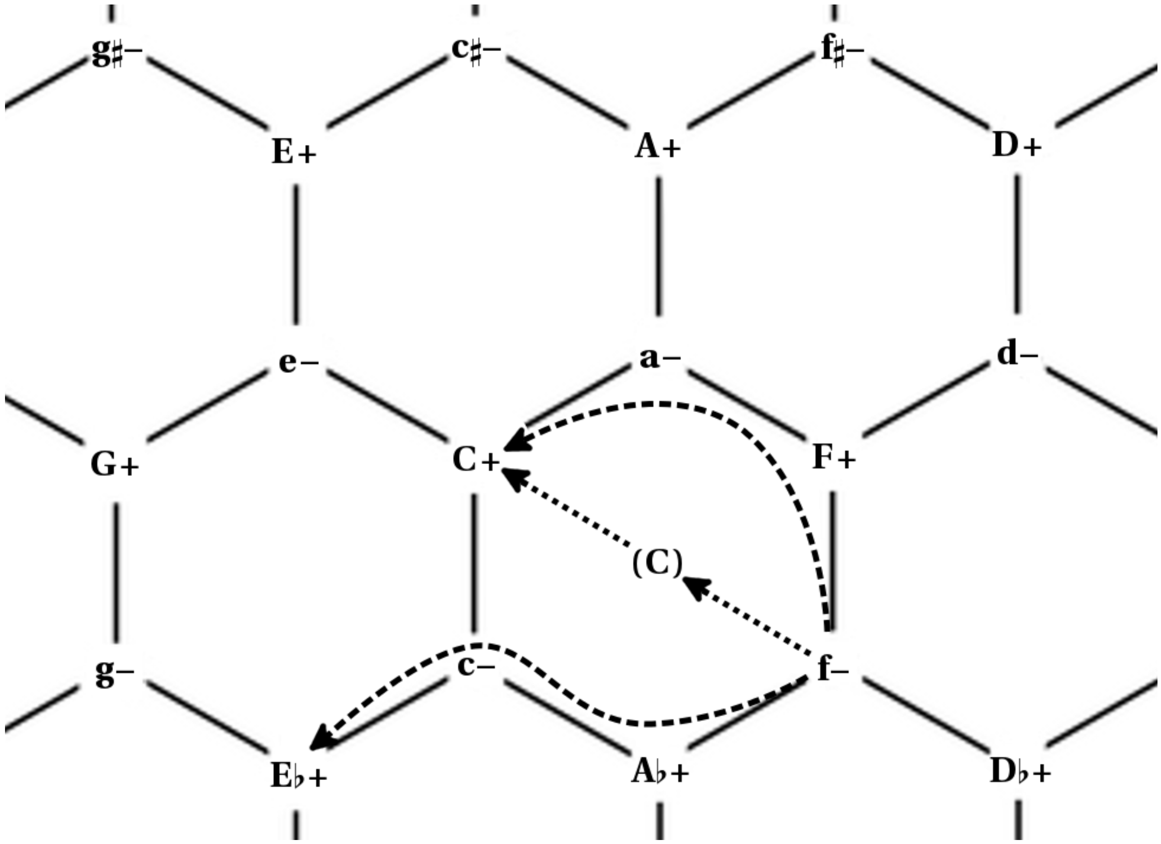


Figure 3: C major is three *Tonnetz* links away from F minor, the same as E♭ major, even though it shares one common tone while E♭ major and F minor have none.

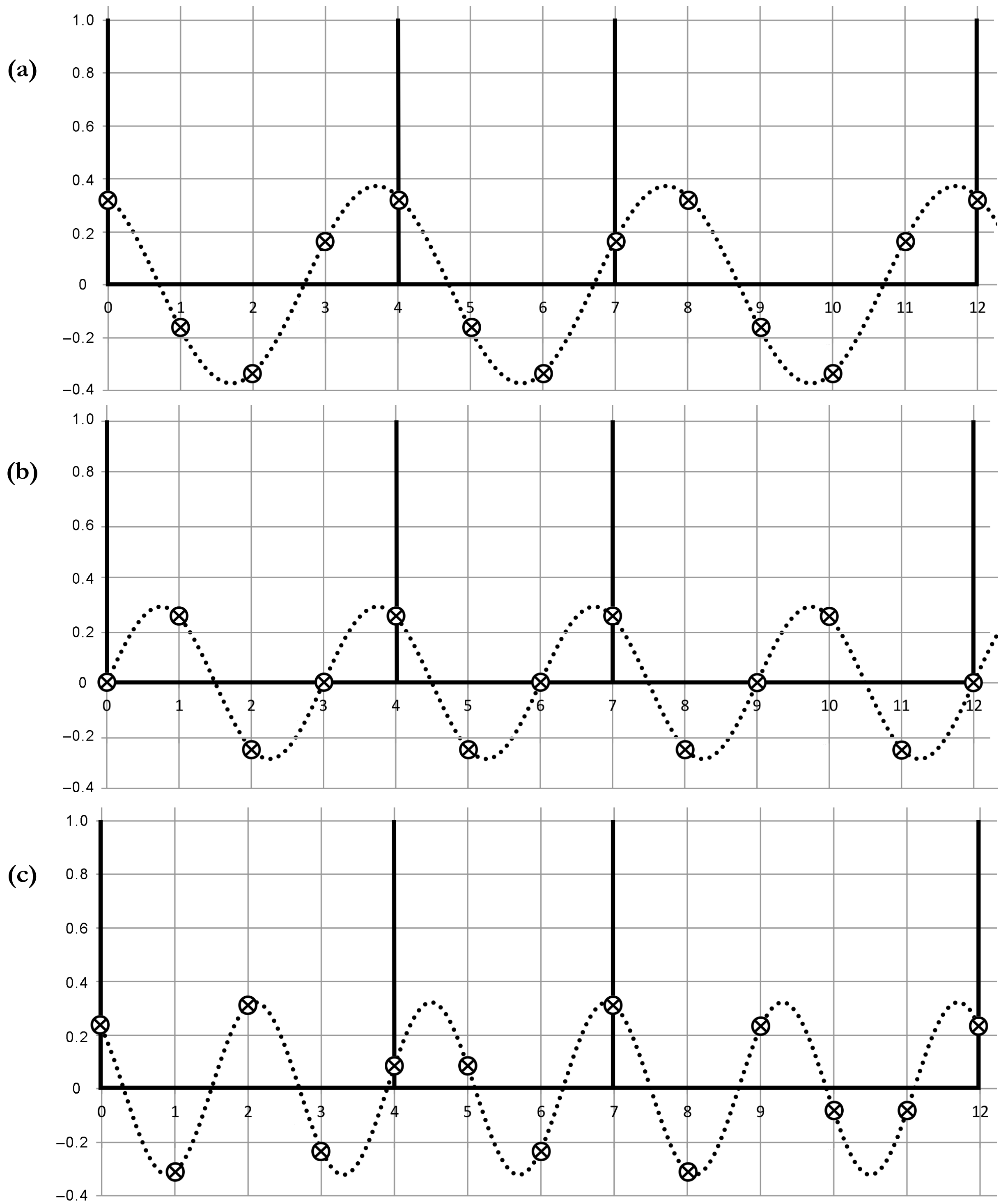


Figure 4: The three largest components of the DFT decomposition of a C major triad (dotted lines), (a) $\widehat{f}_3$, (b) $\widehat{f}_4$, and (c) $\widehat{f}_5$, superimposed on the triad itself (solid lines). The magnitudes are doubled to represent the combined effects of the given component and its complement (e.g., $\widehat{f}_9$, $\widehat{f}_8$, and $\widehat{f}_7$ respectively).

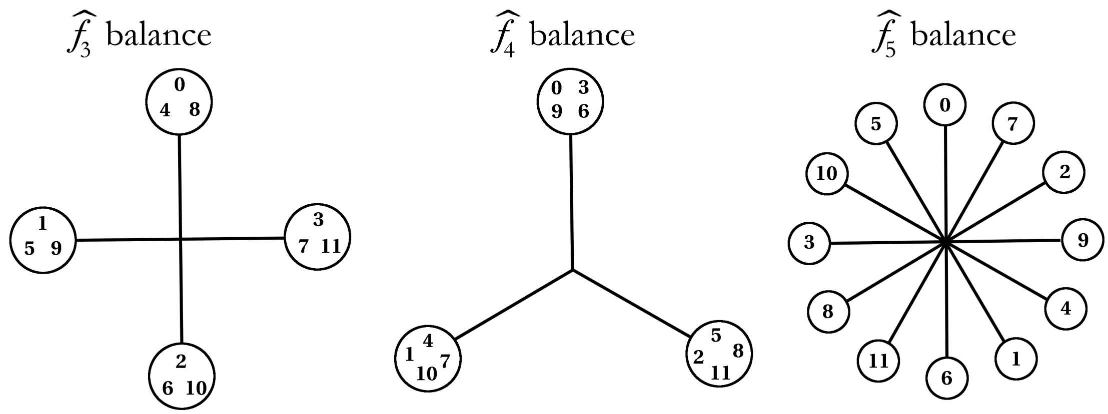
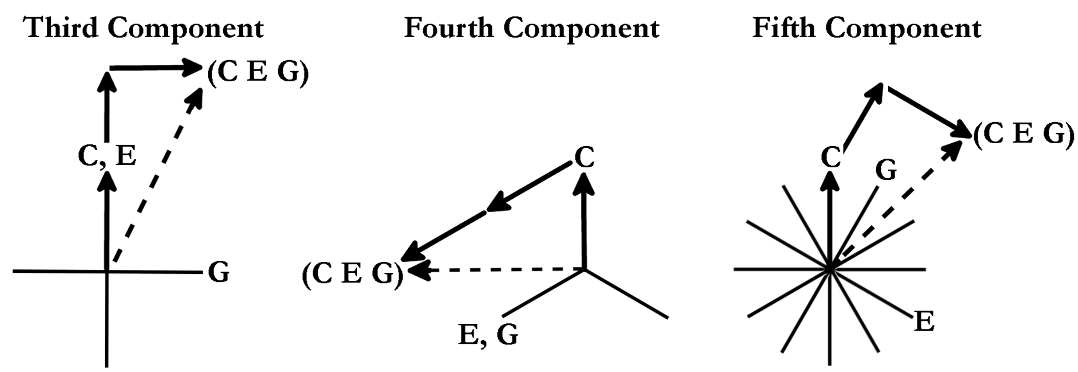
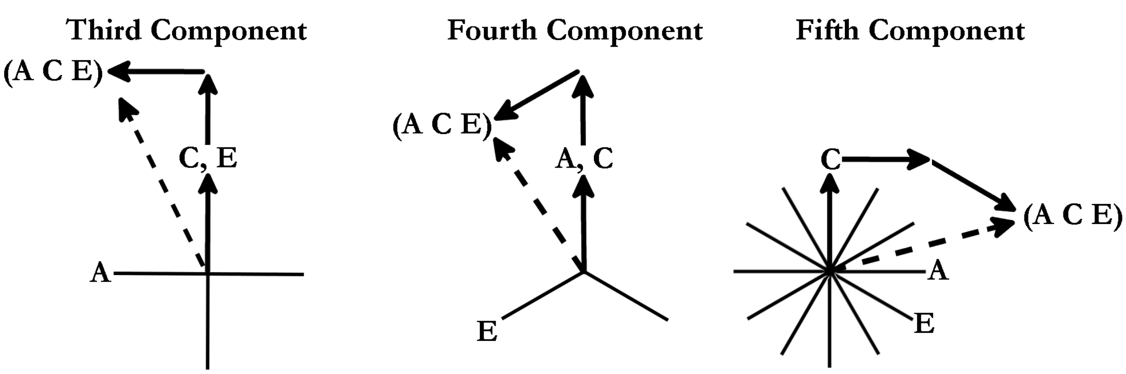


Figure 5: Quinn's Fourier balances for the third, fourth, and fifth Fourier components



$\varphi_3 = 0.89(\pi/6)$ $\varphi_4 = 9\ (\pi/6)$ $\varphi_5 = 1.5\ (\pi/6)$

Figure 6: Derivation of the third, fourth, and fifth components of the C major triad (dashed lines) by adding vectors in Quinn's Fourier balances



$\varphi_3 = 11.11(\pi/6)$ $\varphi_4 = 11\ (\pi/6)$ $\varphi_5 = 2.5\ (\pi/6)$

Figure 7: Derivation of the third, fourth, and fifth Fourier components of the A minor triad

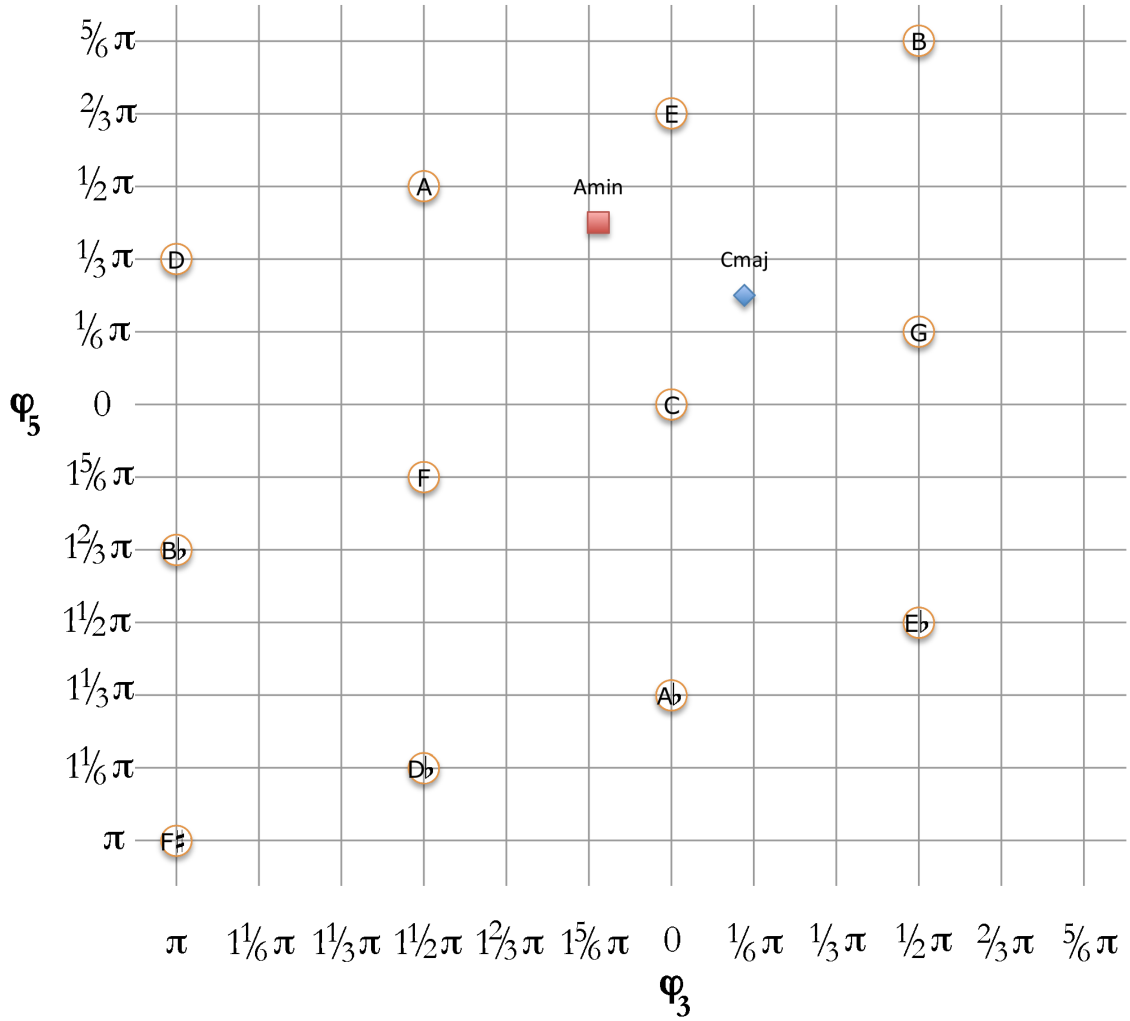


Figure 8: C major and A minor triads in $\varphi_{3/5}$-space.

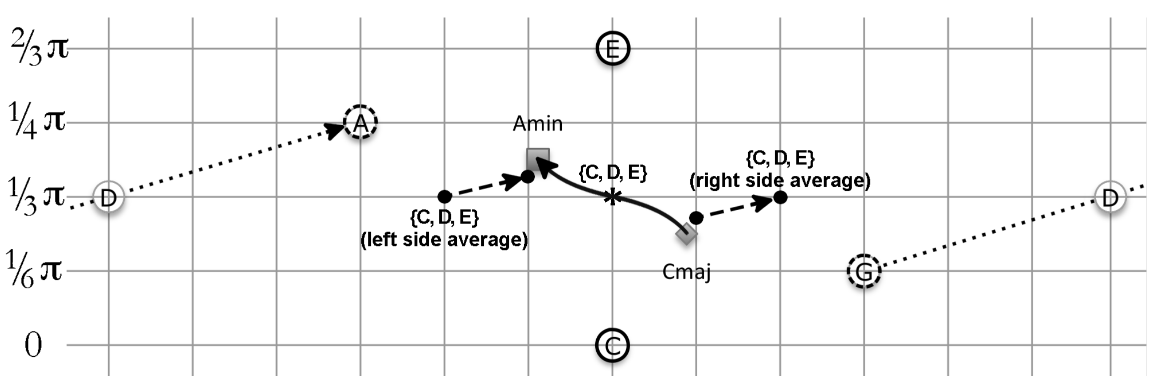


Figure 9: Continuous paths from CEG through CDE to ACE using DFT (solid line) and arithmetic mean (dashed line).

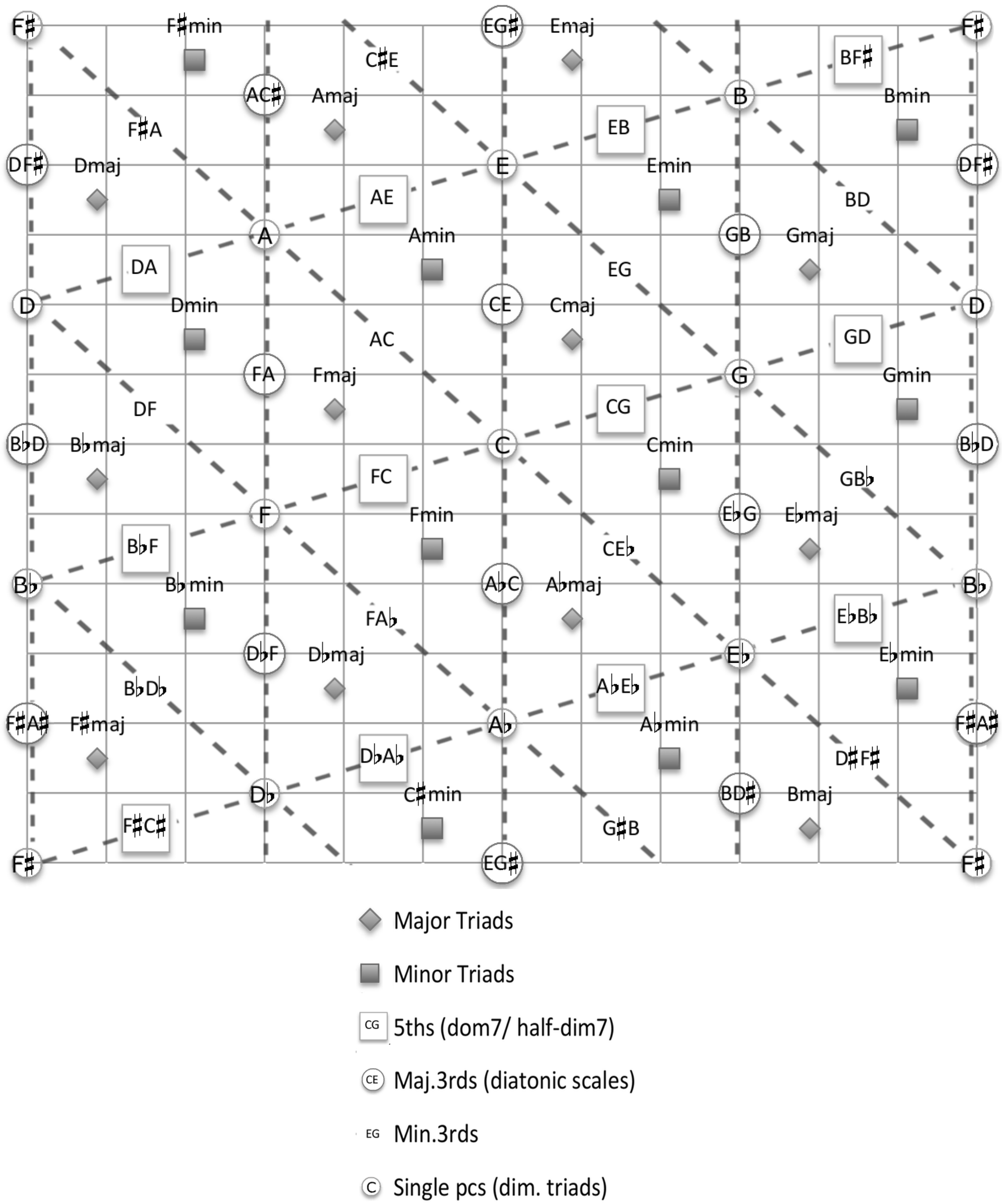


Figure 10: The *Tonnetz* in $\varphi_{3/5}$-space. Consonant triads, dyads, and single pitch-classes are plotted in the space, and the pcs are connected to their nearest neighbors in a triangular lattice.

Figure 11: Appoggiaturas from "Daß sie hier gewesen," mm. 1–2, 42–3, 48–9

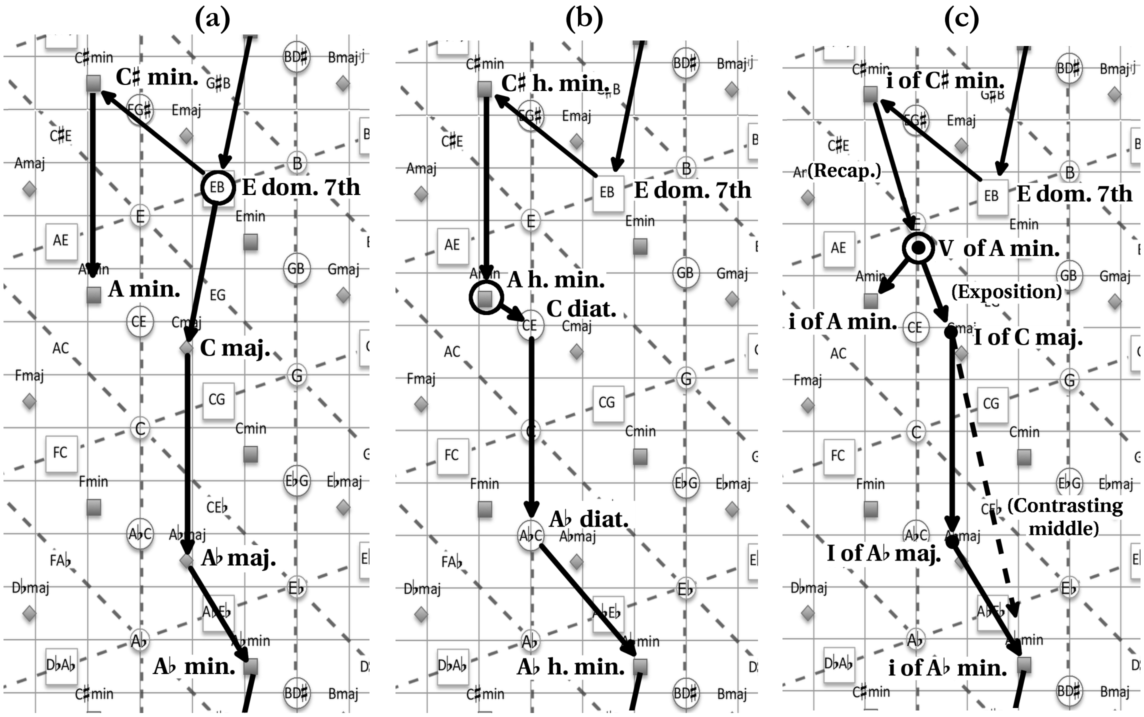


Figure 12: The tonal plan of the Menuetto as (a) a path between chords, (b) a path between scales, and (c) a path between chord + scale multisets

Figure 13: (a) The modulation into the interior theme of the Adagio movement of Schubert's String Quintet

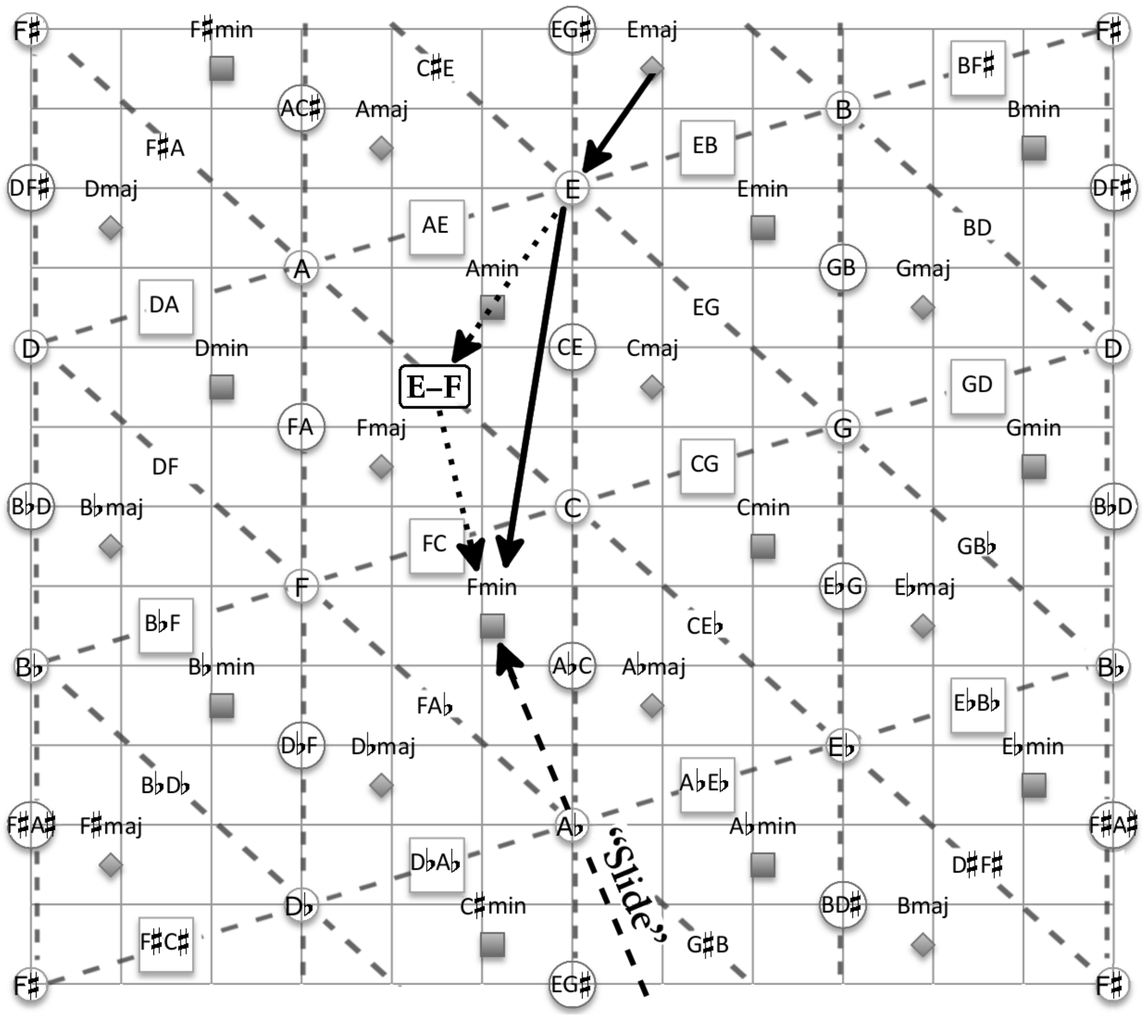


Figure 13: (b) The modulation by scalar common tone in the Adagio implies that the contrasting key is ♭ii, even though it is closer as a ♯i, by "Slide" transformation.

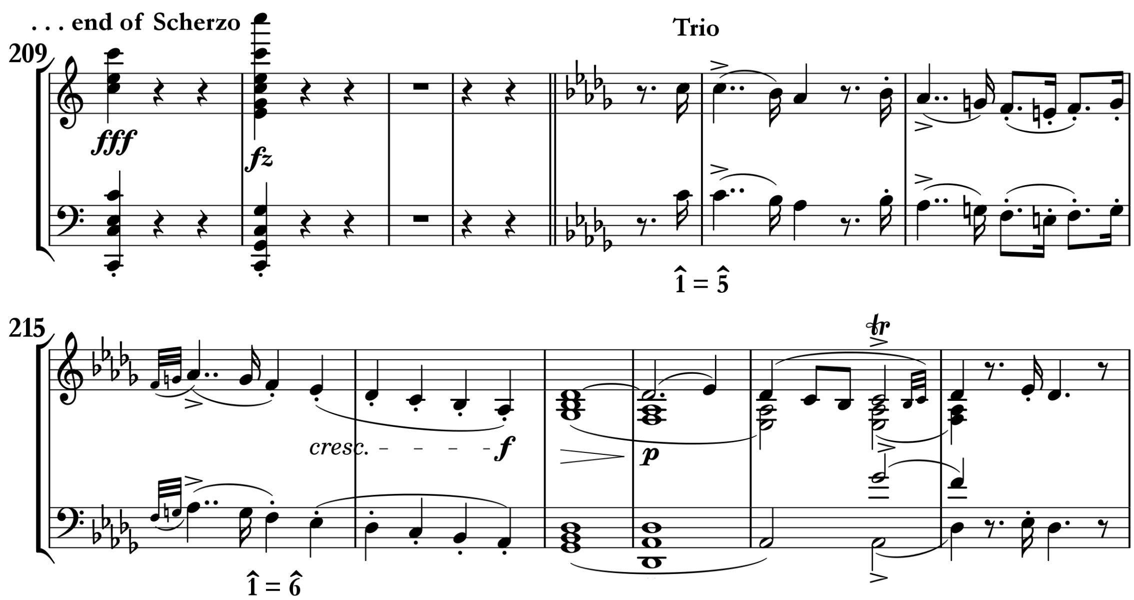


Figure 14(a): The end of the Scherzo and beginning of the Trio in the String Quintet

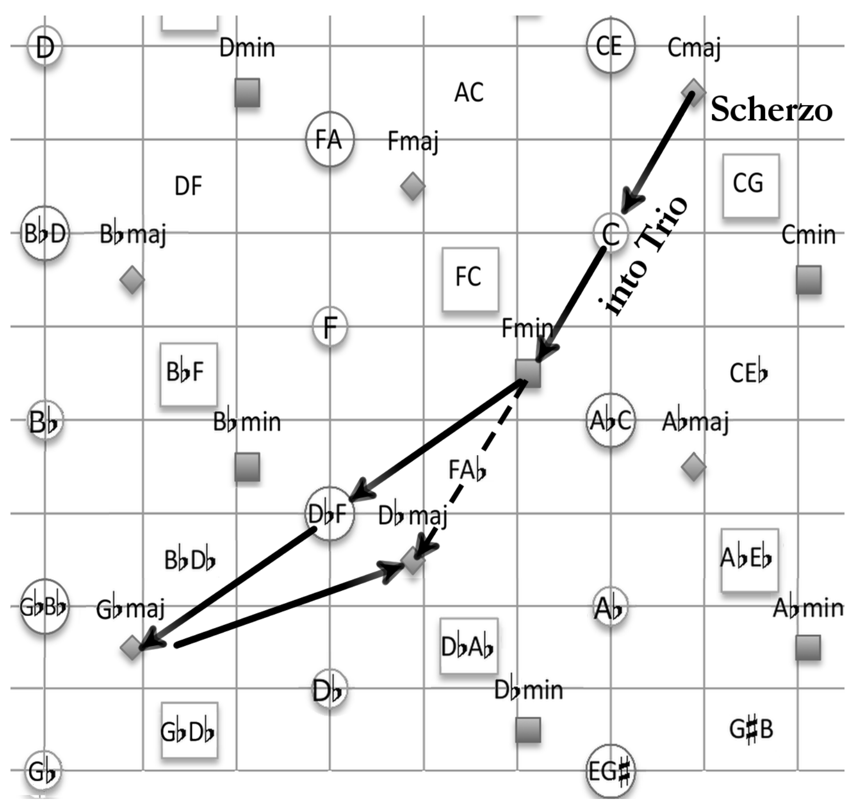


Figure 14(b): The transition into the Trio of the String Quintet in $\varphi_{3/5}$-space

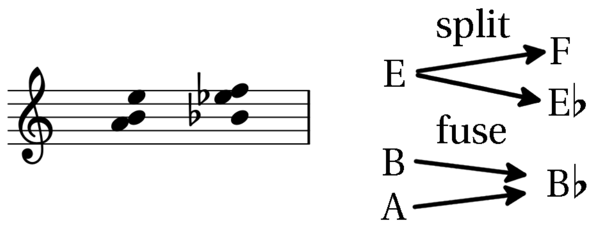


Figure 15: A split/fuse voice leading from Callender 1998

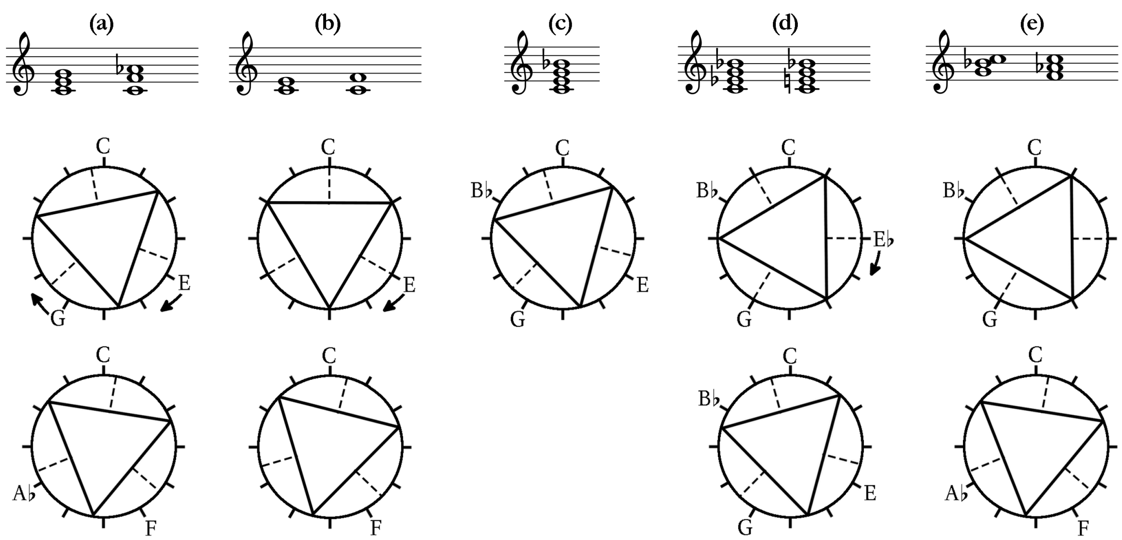


Figure 16: Changes of φ_3 interpreted as shifts of triadic orbits

Figure 17: A sequence from Schubert's song "Gruppe aus dem Tartarus"

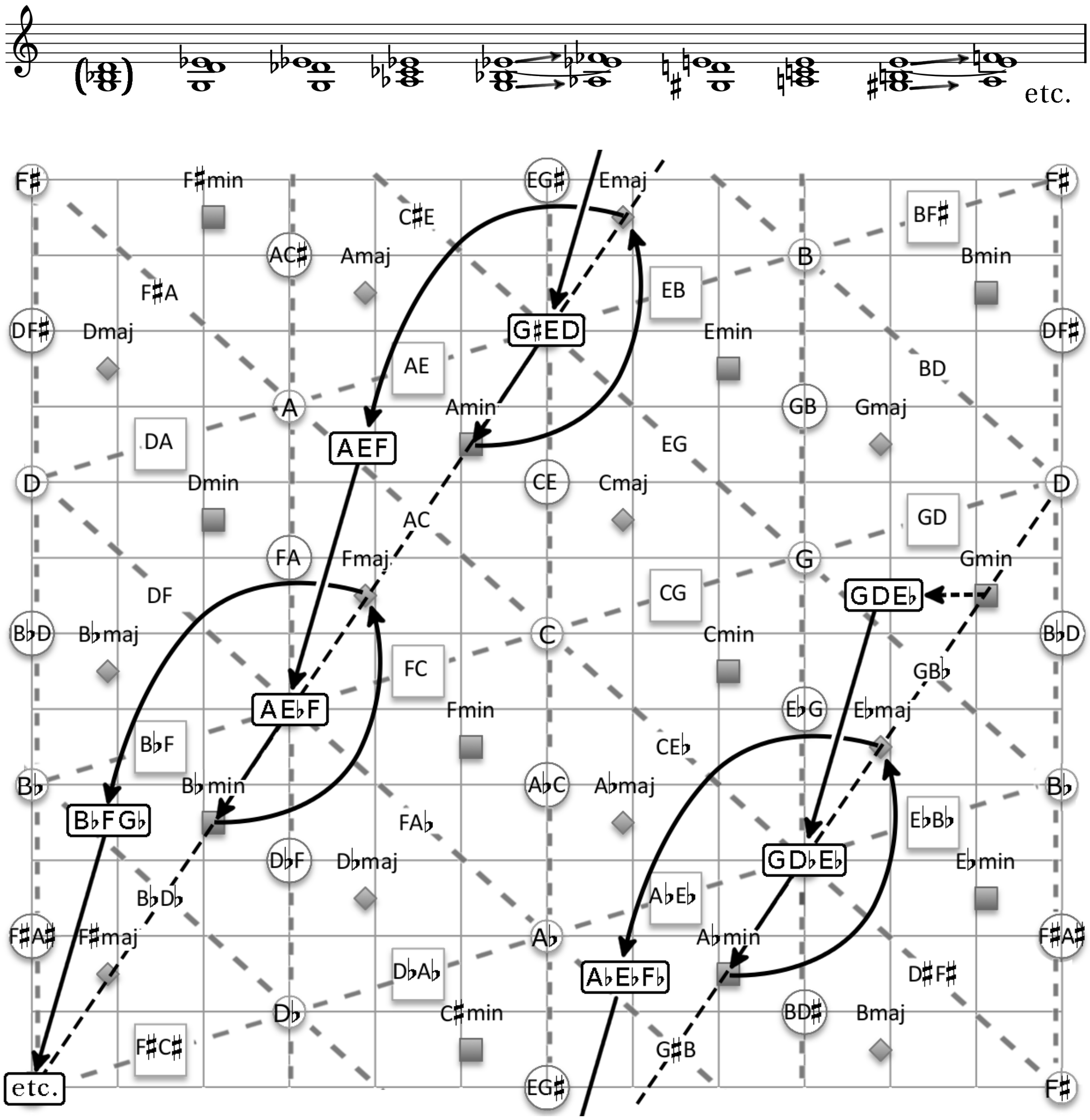


Figure 18: (a) Implied voice-leading pattern of the sequence and (b) its path in $\varphi_{3/5}$-space

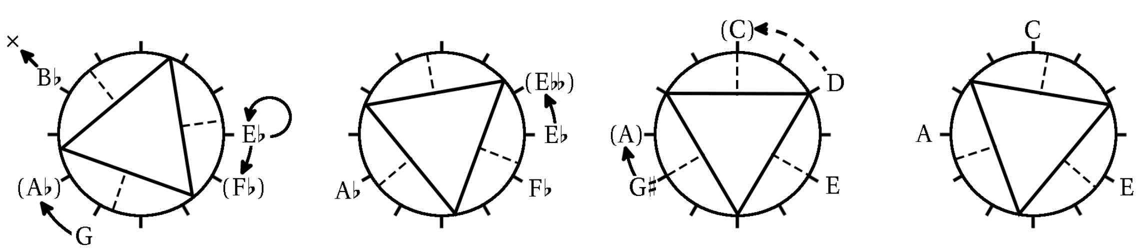


Figure 19: Triadic regions in the sequence of Fig. 18, chords 4–7

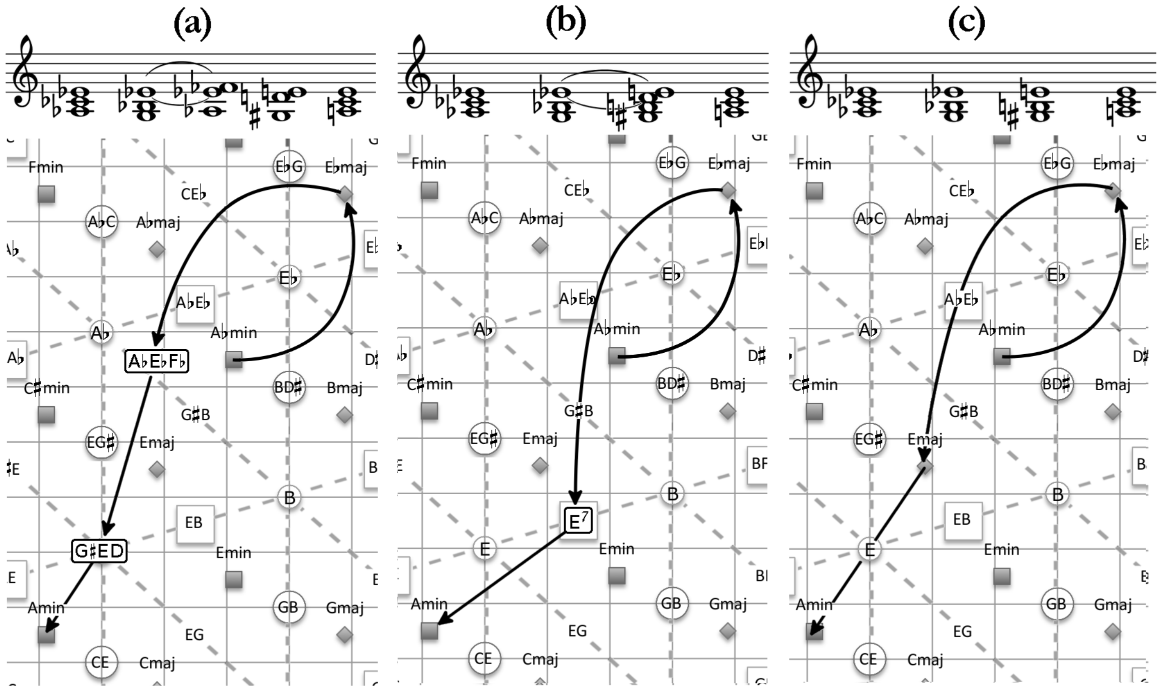


Figure 20: (a) One stage of the sequence and its path in $\varphi_{3/5}$-space, and models of the sequence using (b) triads and seventh chords and (c) only triads

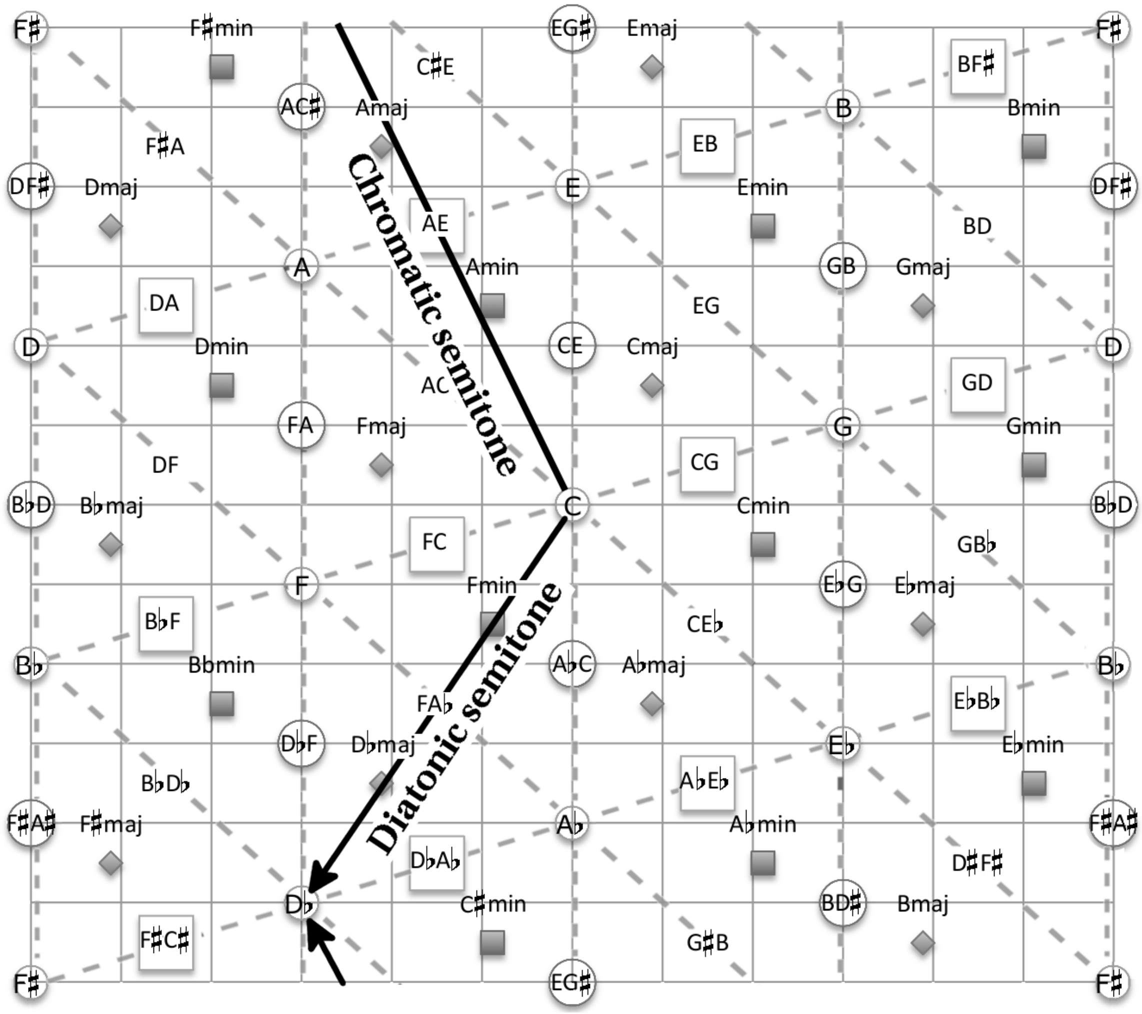


Figure 21: Chromatic and diatonic semitones are distinct paths between the same points.

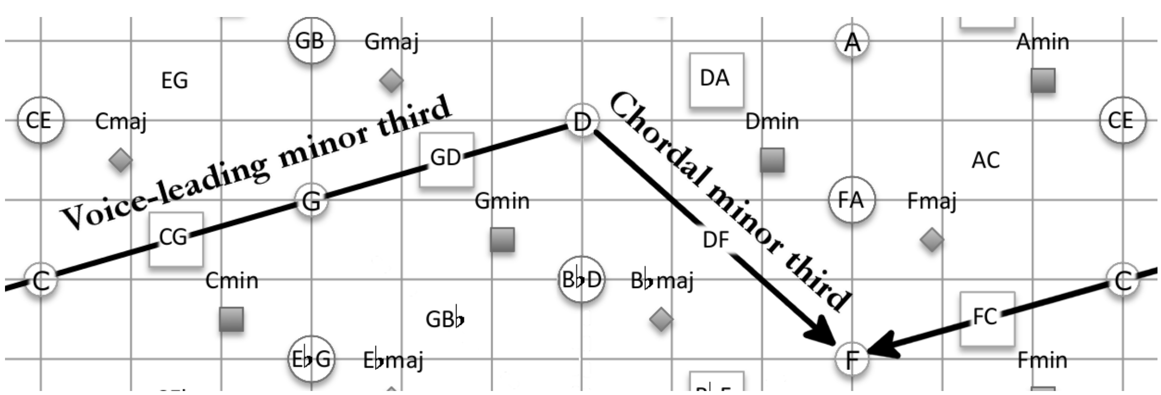


Figure 22: A voice-leading interval of a minor third is a different, and longer, path than the chordal minor third.

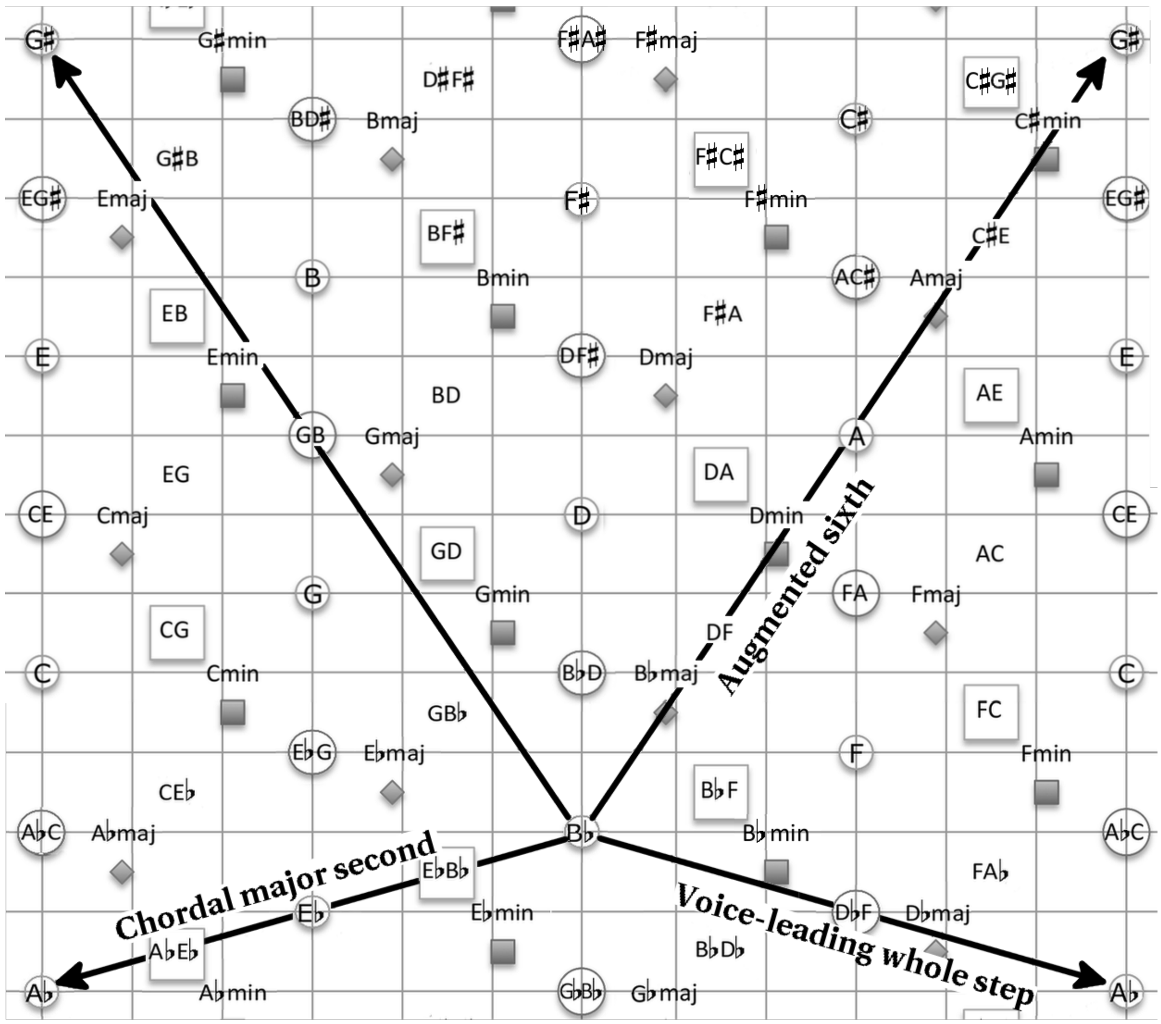


Figure 23: Four different paths between the same points, B♭→A♭, and the intervals represented by three of them

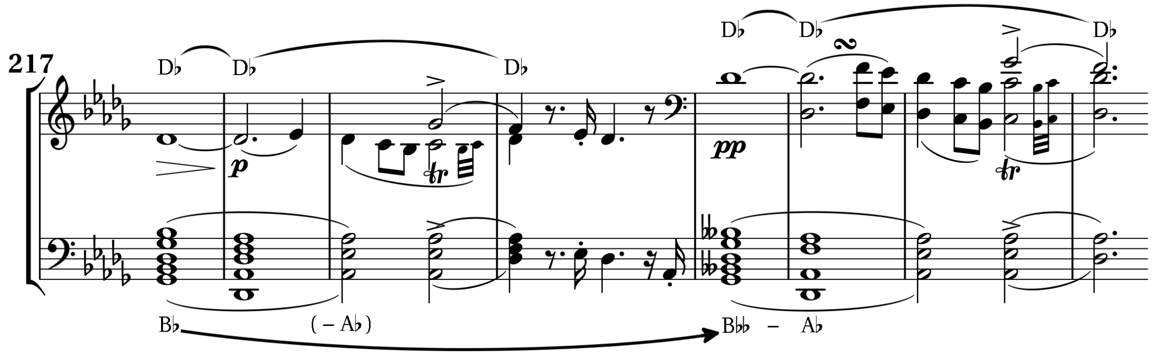


Figure 24: Schubert repeats the cadential phrase in the first part of the Trio twice, changing only the quality of the subdominant triad.

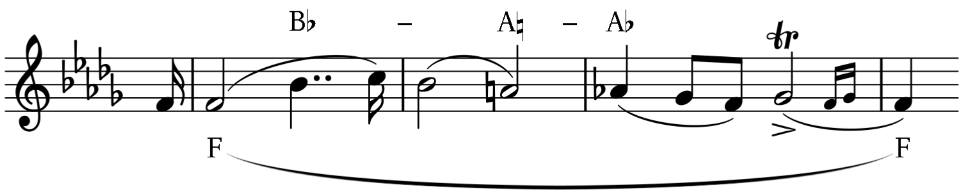


Figure 25: The melody of the contrasting middle section of the Trio creates a chromatic line by juxtaposing V of B♭ minor with the tonic of D♭ major.

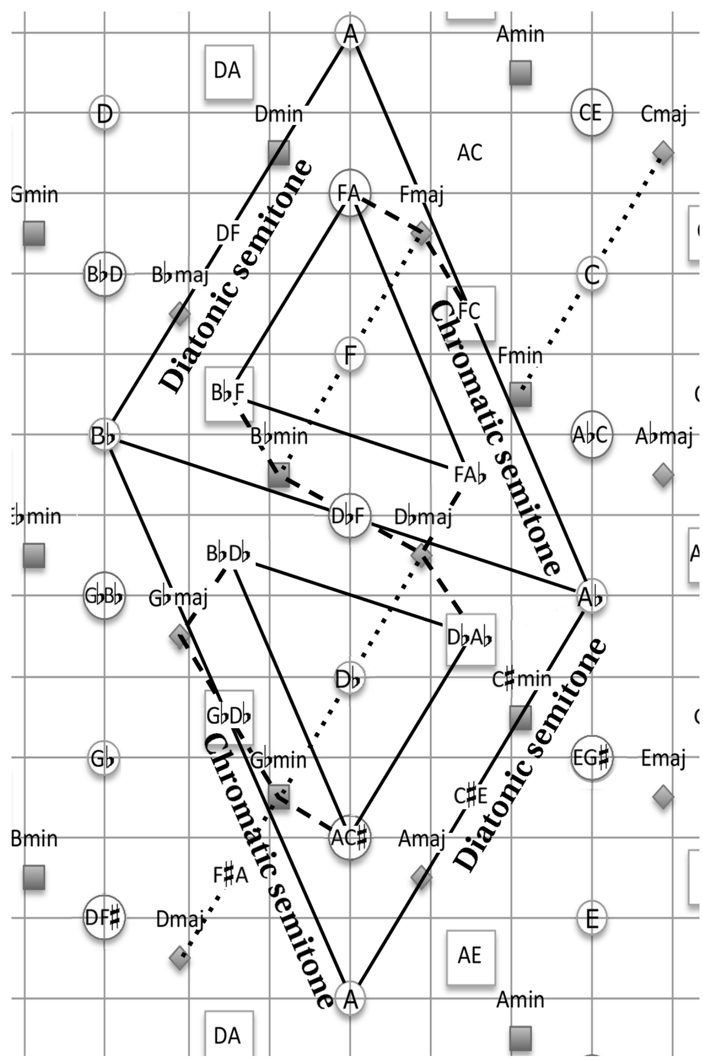


Figure 26: Two ways of chromatically filling a voice-leading whole step are inversionally related, differing only on which semitone is interpreted as diatonic and which as chromatic. The most efficient way to resolve the enharmonic ambiguity between B♭–A♮–A♭ and B♭–B𝄫–A♭ is to add the note F or D♭. Dashed lines connect triads used in the passages from Figs. 24–25 to their constituent dyads. Dotted lines show diatonic-semitone axes involving triads from the Trio A and B parts.

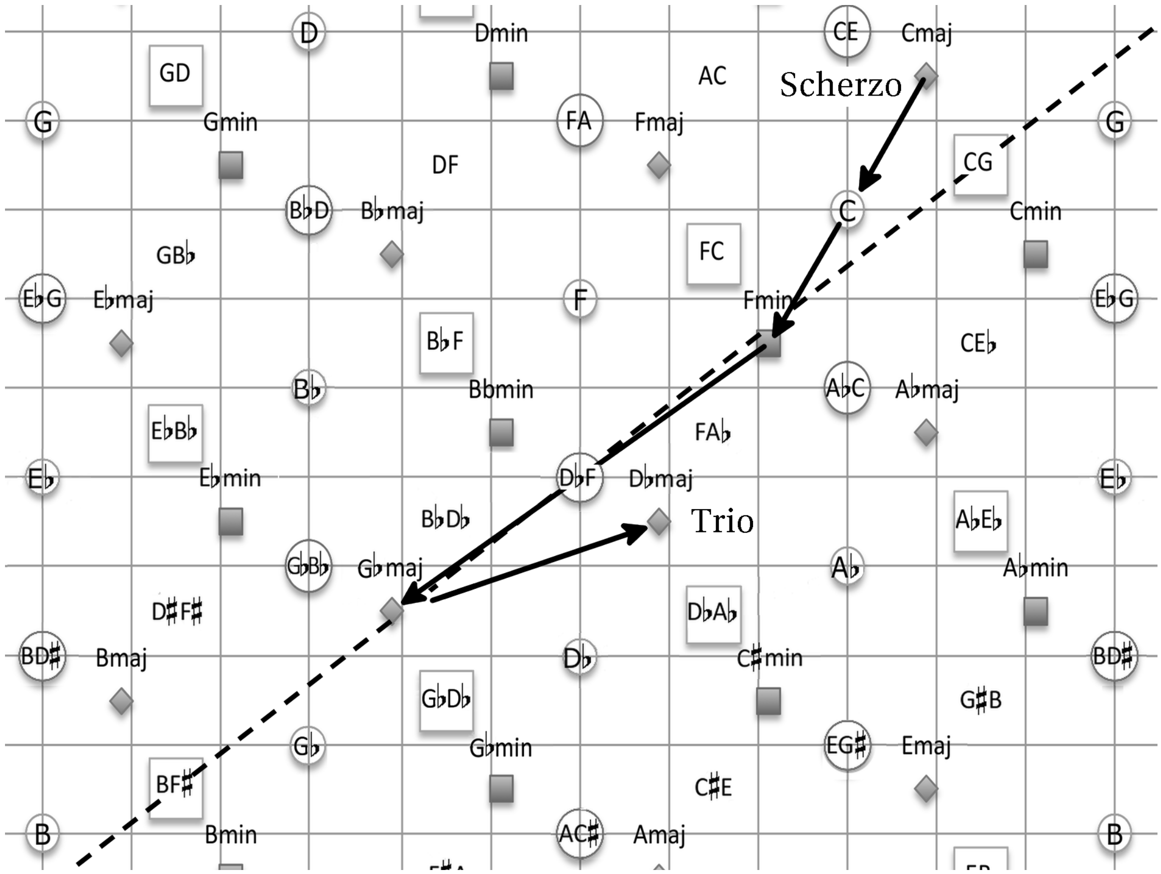


Figure 27: The path from the F minor to the G♭ major triad at the beginning of the String Quintet Trio approximately follows a chordal chromatic-semitone axis.

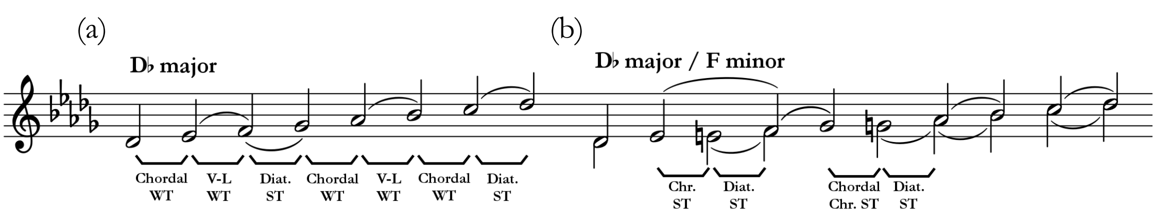


Figure 28: Steps of (a) an ordinary major scale and (b) a mixture of closely related scales. Slurs are used to group notes belonging to the same triadic orbit.

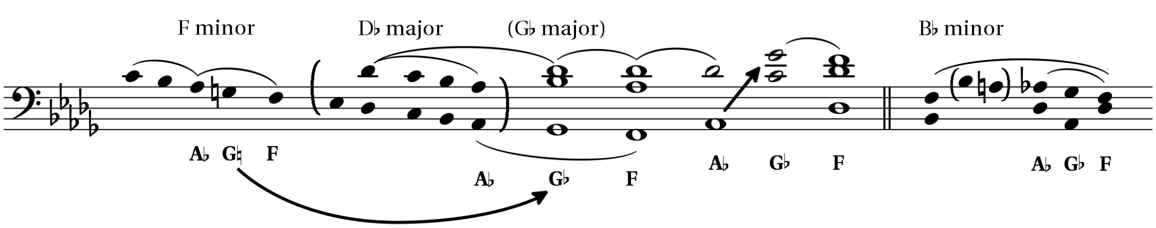


Figure 29: A reduction of the main theme and contrasting middle of the Trio, highlighting Schubert's emphasis on the idea of filling the interval A♭–F with scalar motion

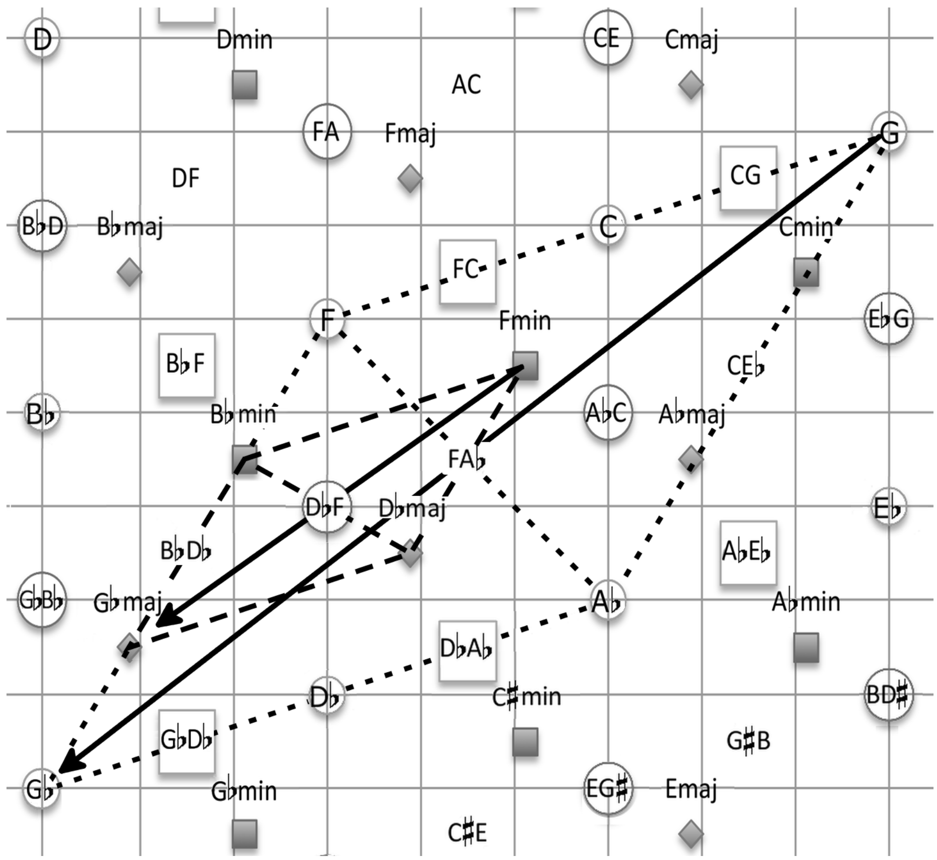


Figure 30: Two structures, chordal and melodic, from the Trio's main theme–contrasting middle outline the same shape: a parallelogram whose edges follow diatonic semitone and chordal whole-tone axes, and whose long diagonal follows a chordal chromatic-semitone axis.

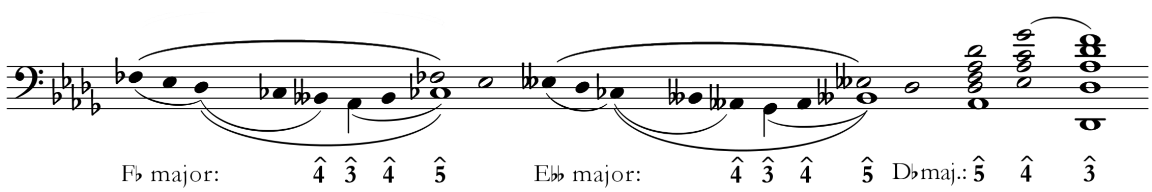


Figure 31: Reduction and respelling of the sequential recapitulation

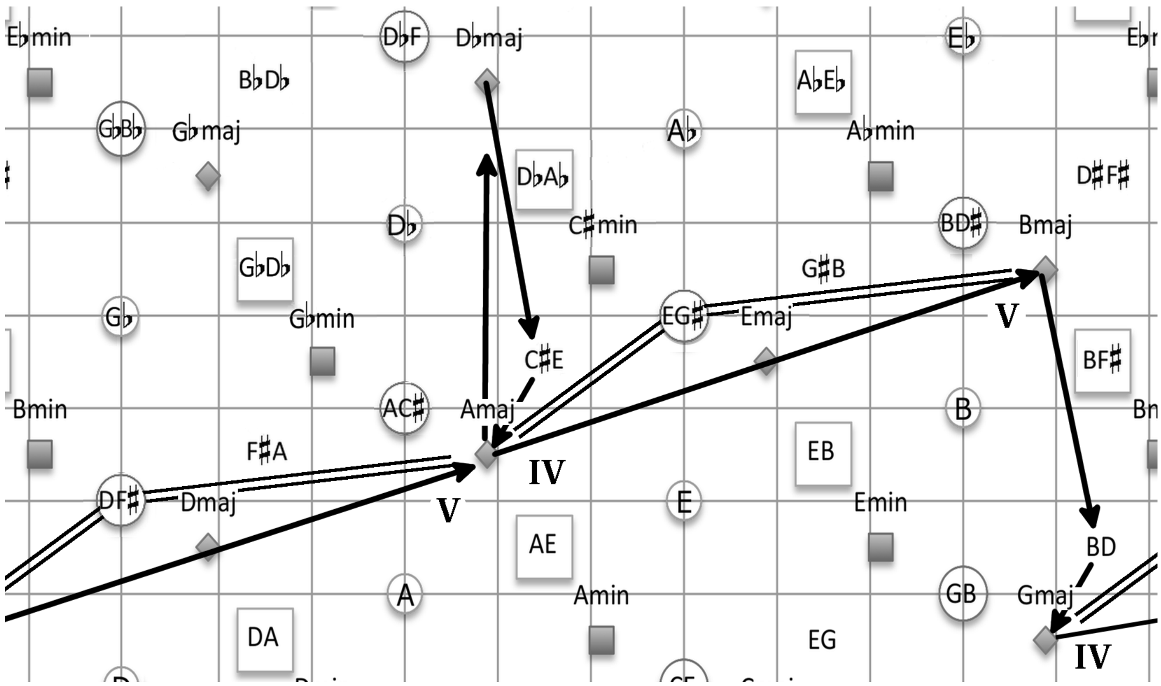


Figure 32: On a local view, the sequential recapitulation executes a voice-leading cycle.

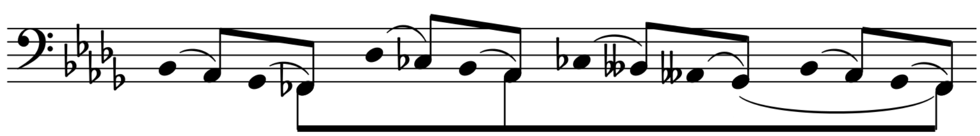
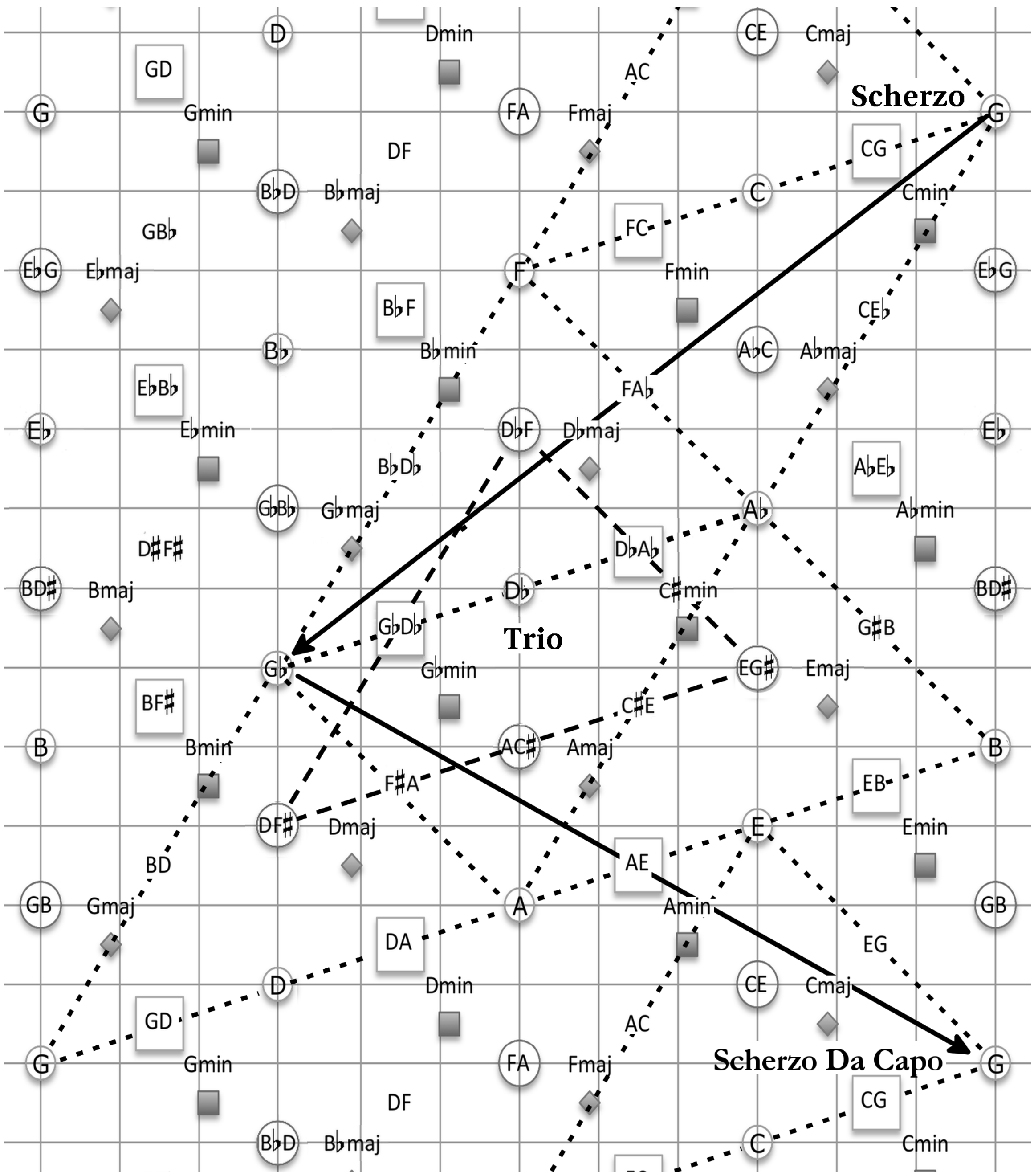


Figure 33: Tonal elements of the sequence from the recapitulation of the Trio. Slurs group notes in the same φ_3 orbit and beams show arpeggiations between adjacent voices.


Scherzo
Trio
Scherzo Da Capo
D
Dmin
CE
Cmaj
GD
AC
G
Gmin
FA
Fmaj
DF
CG
B♭D
B♭maj
C
Cmin
GB♭
FC
E♭G
E♭maj
F
Fmin
B♭F
CE♭
B♭
B♭min
A♭C
A♭maj
E♭B♭
FA♭
E♭
E♭min
D♭F
D♭maj
B♭D♭
A♭E♭
G♭B♭
G♭maj
A♭
A♭min
D♯F♯
D♭A♭
BD♯
Bmaj
D♭
C♯min
G♭D♭
G♯B
G♭
G♭min
EG♯
Emaj
BF♯
C♯E
B
Bmin
AC♯
Amaj
F♯A
EB
DF♯
Dmaj
E
Emin
BD
AE
GB
Gmaj
A
Amin
DA
EG

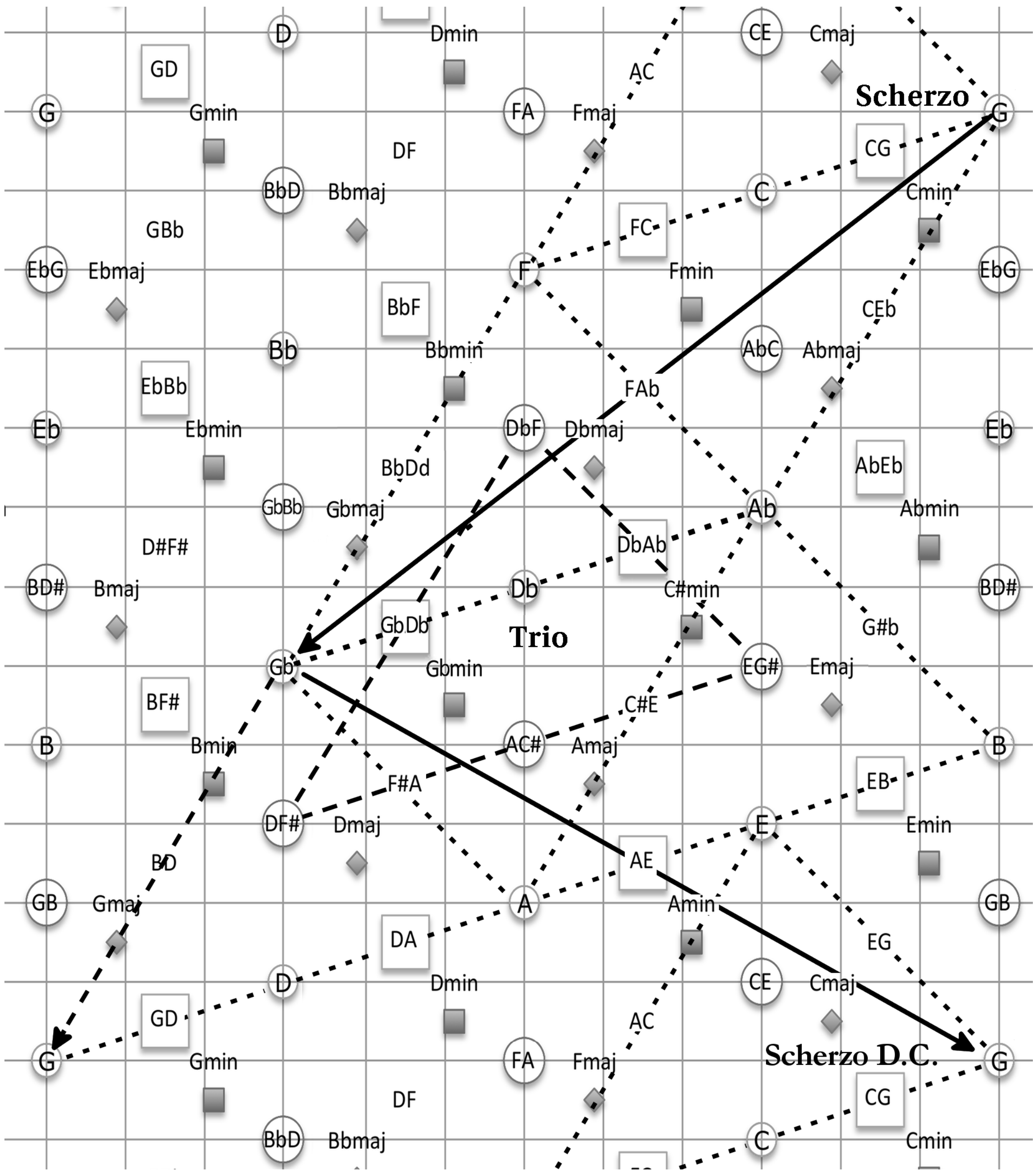


Figure 34: Dotted triangles show local melodic structures of the major $\hat{3}$–$\hat{4}$–$\hat{5}$ / minor $\hat{1}$–$\hat{2}$–$\hat{3}$ type from the main theme, recapitulation, and retransition. Three of these from the recapitulation are centered on diatonic scales that reflect the same relationships (dashed lines). Arrows show chordal semitones between G and G♭.

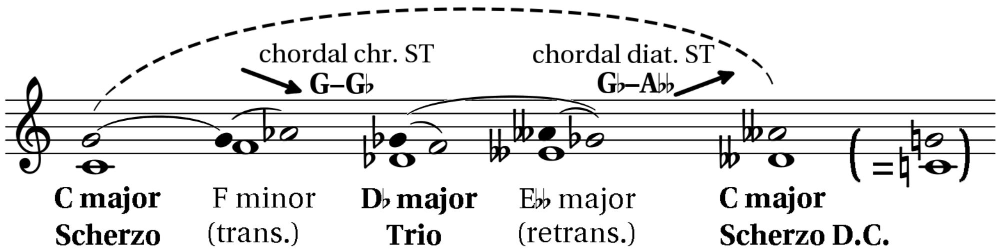


Figure 35: Transformations of G/G♭ over the course of the Trio. Slurs connect Gs or G♭s that belong to the same triadic orbit. The transitions into and out of the Trio involve moving G/G♭ into a lower orbit, then returning it to a higher orbit.

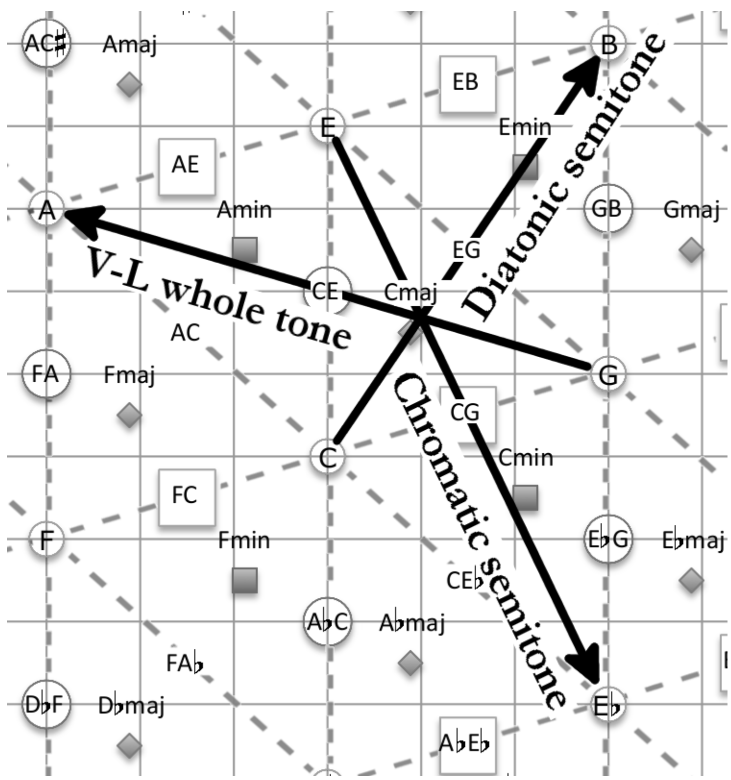


Figure 36: Transformational axes for the *Tonnetz*

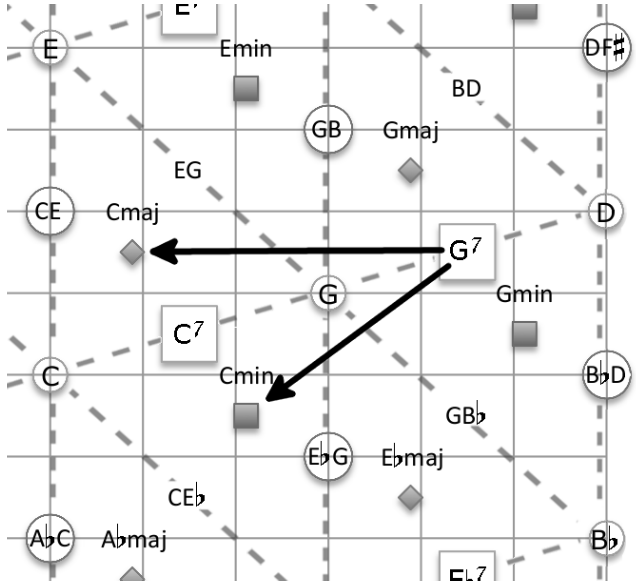


Figure 37: V^7–I/i progressions cross multiple *Tonnetz* boundaries.

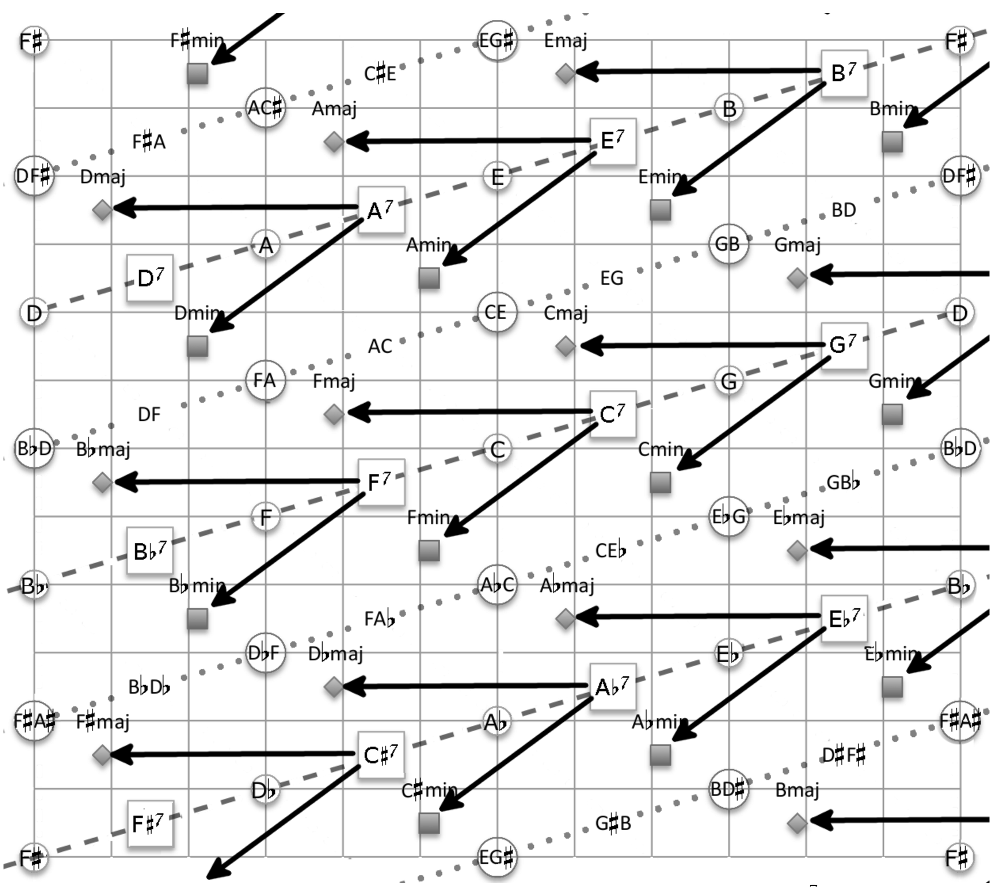


Figure 38: The perfect-fifth axis through singletons separates parallel V^7–I/i progressions in $\varphi_{3/5}$-space (dashed line). A parallel perfect-fifth axis through major and minor thirds separates relative keys (dotted line).

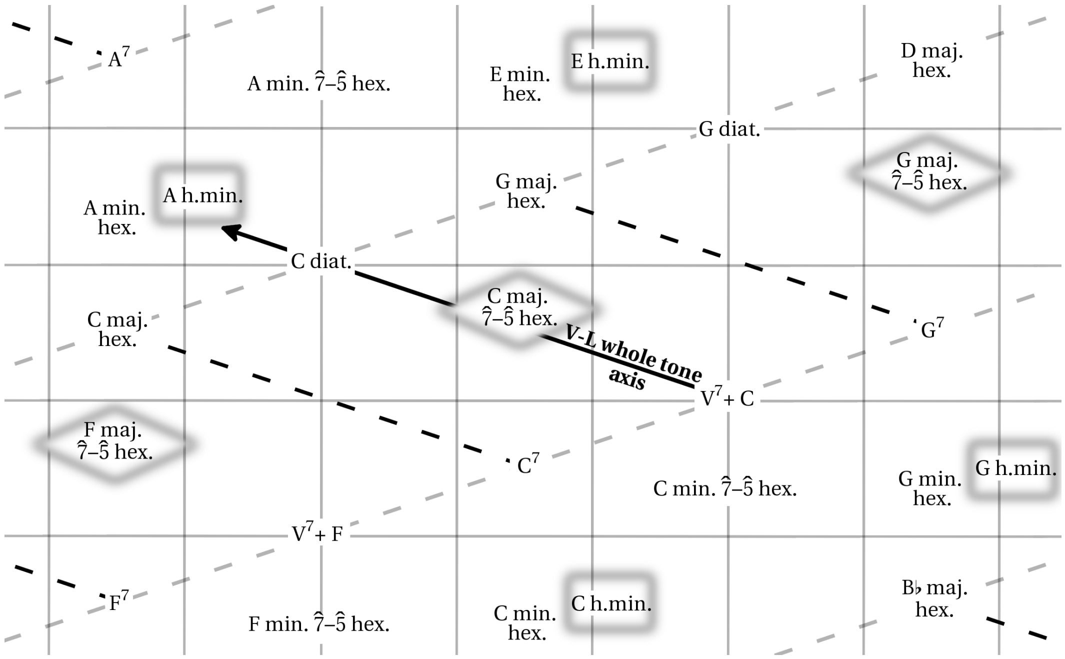


Figure 39: Tonal scales and important scalar subsets. The major-key regions have one characteristic diatonic subset in the middle of their regions, which lies along the whole-tone axis shown. The boundaries shown with dark dashed lines are whole-tone axes halfway between the transformational ones.

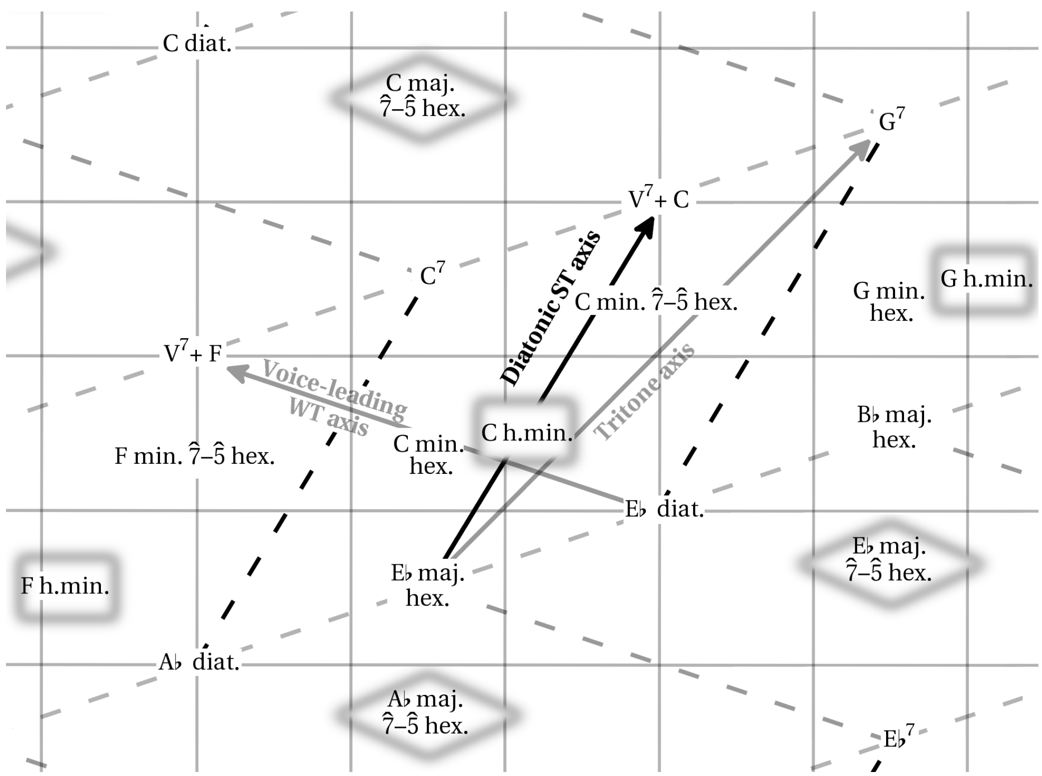


Figure 40: Three scalar sets typical of minor keys, transformational axes associated with each, and minor-key regions associated with the diatonic-semitone axis.

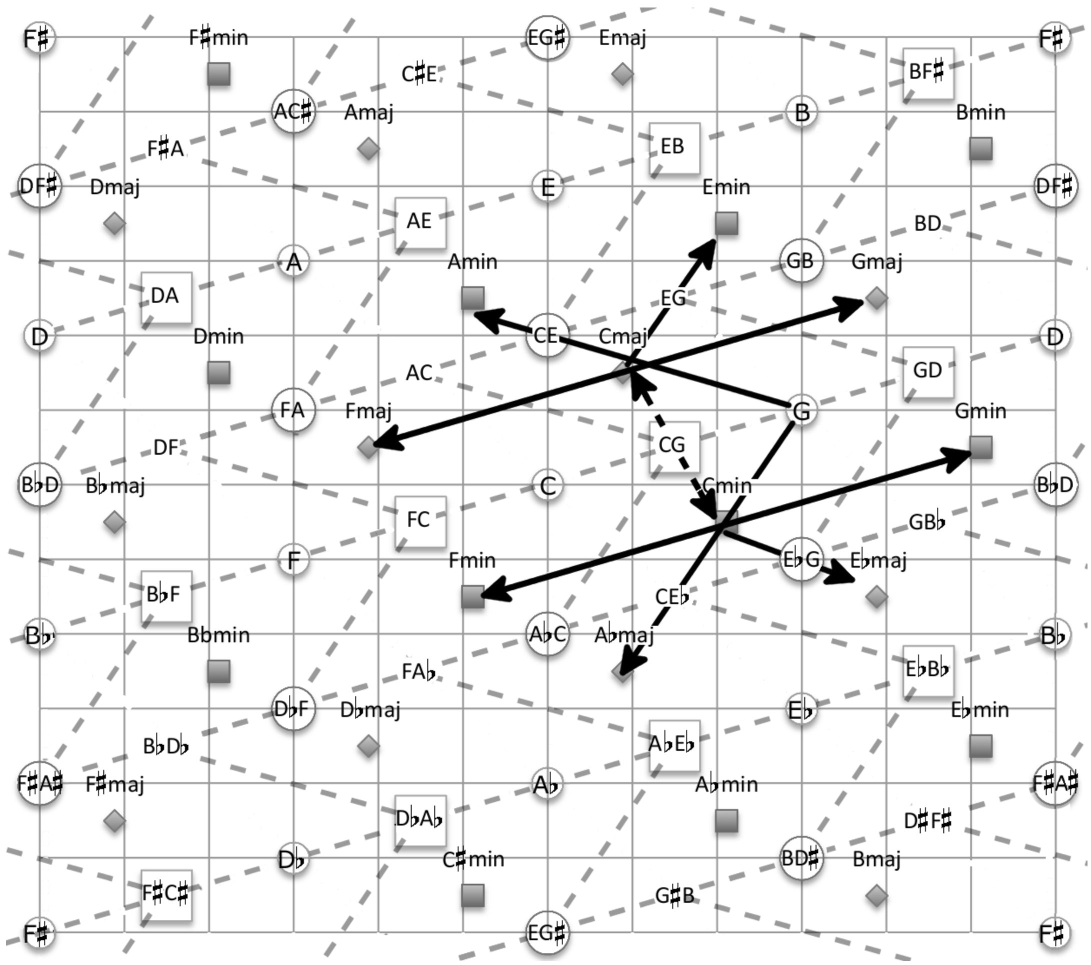


Figure 41: Tonal regions and transformational axes

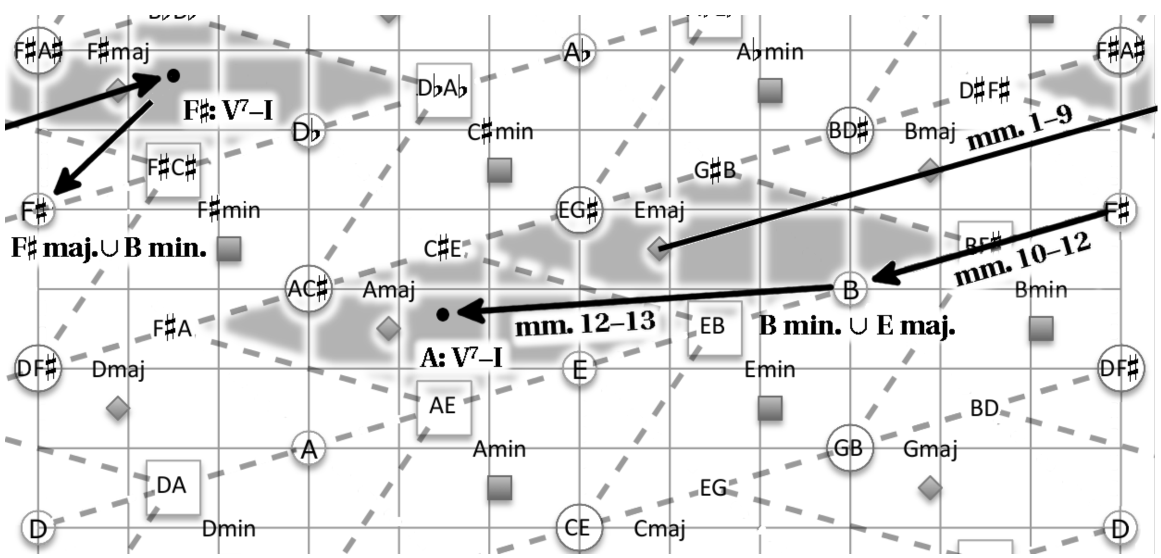


Figure 42: A harmonic summary of the first period from the main theme of the String Quintet Adagio

Figure 43(a): Second period of the main theme, mm. 15–24

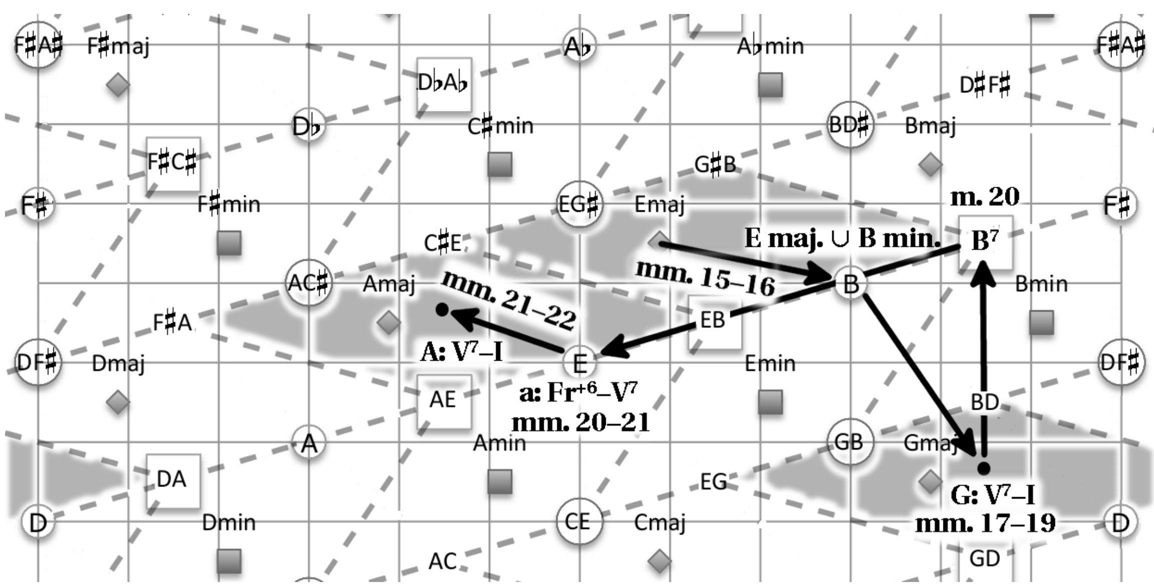


Figure 43(b): The passage crosses the modal boundary below E major.

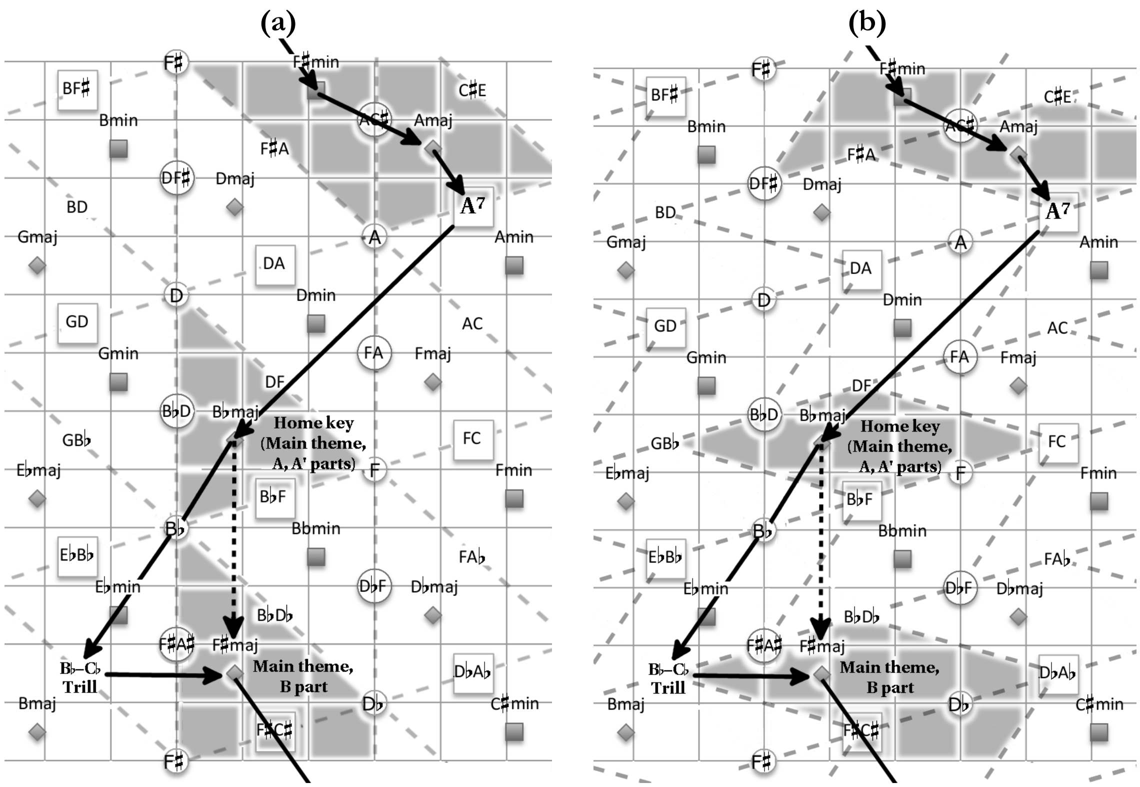


Figure 44: The progression of tonal areas in the main theme recapitulation of Schubert's B♭ major Piano Sonata, seen through (a) *Tonnetz* regions and (b) tonal regions