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*SAMPLE PATH PROPERTIES OF
STOCHASTIC PROCESSES REPRESENTED
AS MULTIPLE STABLE INTEGRALS*

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Sample path properties of stochastic processes represented as multiple stable integrals ^{*†‡§}

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Abstract

This paper studies the sample path properties of stochastic processes represented by multiple symmetric α -stable integrals. It relates the “smoothness” of the sample paths to the “smoothness” of the (non-random) integrand. It also contains results about the tail behavior of the distribution of suprema and zero-one laws.

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1 Introduction and Preliminaries

In this paper we study stochastic processes of the form

$$X(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_t(x_1, \dots, x_n) M(dx_1) \dots M(dx_n), \quad t \in T, \quad (1.1)$$

where M is a symmetric α -stable ($S\alpha S$), $0 < \alpha < 2$, random measure on $(\mathbf{R}, \mathcal{B})$ with a Radon control measure m (i.e. m is finite on compact subsets of \mathbf{R}), and $\{f_t, t \in T\}$ is a family of real measurable functions $\mathbf{R}^n \rightarrow \mathbf{R}$ symmetric with respect to permutations of their arguments and vanishing on the diagonals. Such processes can be regarded as an extension of both $S\alpha S$ processes (to which they reduce when $n = 1$), and multiple Gaussian integrals, which corresponds to the case $\alpha = 2$.

Stochastic processes of the form (1.1) can exhibit long range dependence and high variability, and they are useful for modelling of various natural phenomena (see Taqqu [Taq87] and references therein.) It is therefore of interest to study properties of their sample path. This paper is a first step in that direction.

Multiple stable intergrals defining the stochastic process $\{X(t), t \in T\}$ have been a focus of many studies in recent years (see for example [RW86], [MT86], [KW87], [KS88b].) Samorodnitsky and Szulga [SS89] proposed a series representation for multiple stable integrals, later improved and generalized by Samorodnitsky and Taqqu [ST88a], [ST88b].

Let M be a $S\alpha S$ random measure on $(\mathbf{R}, \mathcal{B})$ with a Radon control measure m . The product random measure $M^{(n)}$ on $(\mathbf{R}^n, \mathcal{B}^n)$ is defined as the product of the marginal random measures on measurable rectangles, and it can be extended to an L^p -valued, $0 < p < \alpha$, vector measure on symmetric measurable subsets of \mathbf{R}^n which either do not include the diagonals, or include them fully (see Krakowiak and Szulga [KS88b] and Samorodnitsky and Taqqu [ST88a]). Let now \mathbf{f} be a symmetric, vanishing on the diagonals, Banach-space valued Borel function on \mathbf{R}^n . We say that \mathbf{f} is $M^{(n)}$ -integrable if there is a sequence of simple functions of the type

$$\mathbf{f}^{(m)} = \sum_{i=1}^{N_m} \mathbf{a}_i(m) \mathbf{1}_{A_i(m)}, \quad (1.2)$$

where

1. $A_1(m), \dots, A_{N_m}(m)$ are disjoint symmetric Borel subsets of \mathbf{R}^n with finite $m^{(n)} = m \times m \times \dots \times m$ measure and which do not include the diagonals;
2. the $\mathbf{a}_i(m)$'s are Banach-valued coefficients, such that $\mathbf{f}^{(m)} \rightarrow \mathbf{f}$ as $m \rightarrow \infty$ in measure $m^{(n)}$;
3. the sequence $\mathbf{I}_n(\mathbf{f}^{(m)} \mathbf{1}_C)$, $m = 1, 2, \dots$ converges in probability for any symmetric Borel subset C of \mathbf{R}^n which does not include the diagonals, where,

as usual, for simple functions,

$$\mathbf{I}_n(\mathbf{f}^{(m)} \mathbf{1}_C) = \sum_{i=1}^{N_m} \mathbf{a}_i(m) M^{(n)}(A_i^{(m)} \cap C).$$

In this case we define

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{f}(x_1, \dots, x_n) M(dx_1) \dots M(dx_n) = \text{plim}_{m \rightarrow \infty} \mathbf{I}_n(\mathbf{f}^{(m)}),$$

where plim denotes limit in probability.

We now quote two results from Samorodnitsky and Szulga [SS89] and Samorodnitsky and Taqqu [ST88b] which play a major role in the present work.

Let S be a real Banach space. S is said to be of the *Rademacher-type* (R-type) p if the random series $\sum_{j=1}^{\infty} \epsilon_j \mathbf{x}_j$ converges a.s. for every sequence $\{\mathbf{x}_j\}_{j=1}^{\infty}$ of elements of S satisfying $\sum_{j=1}^{\infty} \|\mathbf{x}_j\|^p < \infty$, where $\epsilon_1, \epsilon_2, \dots$ are i.i.d. Rademacher random variables, i.e. $P(\epsilon_j = 1) = P(\epsilon_j = -1) = \frac{1}{2}$. Recall that every Banach space has R-type at least 1; every Hilbert space is of R-type 2, as is \mathbf{R}^n equipped with the maximum norm, $n < \infty$.

Given a Banach space S of R-type p , let $\mathbf{f} : \mathbf{R}^n \rightarrow S$ be a symmetric, vanishing on the diagonals, strongly measurable function, and let m be a Radon Borel measure on \mathbf{R} . This ensures that m is σ -finite. The following notation will be used throughout the paper. We denote by ψ a measurable function: $\mathbf{R} \rightarrow (0, \infty)$ satisfying:

- $$\int_{-\infty}^{+\infty} \psi(x)^\alpha m(dx) = 1, \quad (1.3)$$
- $\epsilon_1, \epsilon_2, \dots$ is an i.i.d. Rademacher sequence,
- $\Gamma_1, \Gamma_2, \dots$ is the sequence of arrival times of a Poisson process with unit rate,
- Y_1, Y_2, \dots are i.i.d. S -valued random variables with common distribution $m_\psi(dx) = \psi(x)^\alpha m(dx)$.

All three sequences of random variables are always assumed independent. For an $x > 0$ we denote

$$\ln_+ x = \begin{cases} \ln x & \text{if } x \geq 1, \\ 0 & \text{if } x < 1. \end{cases}$$

THEOREM 1.1 (i) Let M be a S α S random measure on $(\mathbf{R}, \mathcal{B})$ with a Radon control measure m , $0 < \alpha < p$. Suppose that the function \mathbf{f} satisfies

$$\int_{\mathbf{R}^n} \|\mathbf{f}(x_1, \dots, x_n)\|^\alpha \left(\ln_+ \frac{\|\mathbf{f}(x_1, \dots, x_n)\|}{\psi(x_1) \dots \psi(x_n)} \right)^{n-1} m(dx_1) \dots m(dx_n) < \infty. \quad (1.4)$$

Then the series

$$S_n(\mathbf{f}) = C_\alpha^{n/\alpha} \sum_{j_1=1}^{\infty} \dots \sum_{j_n=1}^{\infty} \epsilon_{j_1} \dots \epsilon_{j_n} \Gamma_{j_1}^{-1/\alpha} \dots \Gamma_{j_n}^{-1/\alpha} \psi(Y_{j_1})^{-1} \dots \psi(Y_{j_n})^{-1} \mathbf{f}(Y_{j_1}, \dots, Y_{j_n}) \quad (1.5)$$

converges a.s., where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x \, dx \right)^{-1}. \quad (1.6)$$

Moreover, if the space S is separable, then the multiple integral

$$I_n(\mathbf{f}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{f}(x_1, \dots, x_n) M(dx_1) \dots M(dx_n)$$

exists and

$$I_n(\mathbf{f}) \stackrel{d}{=} S_n(\mathbf{f}).$$

(ii) If $S = \mathbf{R}$, then $I_n(f)$ exists if and only if $S_n(f)$ converges, and $I_n(f) \stackrel{d}{=} S_n(f)$.

Remarks

- Bold letters denote vectors. We write $I_n(f)$ if f is real, and $I_n(\mathbf{f})$ if \mathbf{f} is a vector.
- Theorem 1.1 can be applied to $X(t) = I_n(f_t)$, $t \in T$, where f_t is a real-valued function for each $t \in T$. If we view $\mathbf{f} = \{f_t, t \in T\}$ as a vector in a Banach space S , we need to ensure that S is separable in order to apply Theorem 1.1 (i).

To shorten the notation we will write

$$\begin{aligned} \mathbf{N}_n &= \{1, 2, \dots\}^n, \\ [a_j] &= a_{j_1} a_{j_2} \dots a_{j_n}, \\ \mathbf{f}(\mathbf{Y}_j) &= \mathbf{f}(Y_{j_1}, \dots, Y_{j_n}) \end{aligned}$$

for $\mathbf{j} = (j_1, \dots, j_n)$. Also, $I_n(\mathbf{f}) = \int_{\mathbf{R}^n} \mathbf{f} dM^{(n)}$ and

$$S_n(\mathbf{f}) = C_\alpha^{n/\alpha} \sum_{\mathbf{j} \in \mathbf{N}^n} [\epsilon_j] [\Gamma_j]^{-1/\alpha} [\psi(Y_j)]^{-1} \mathbf{f}(\mathbf{Y}_j).$$

Thus, with a real-valued stochastic process $\{X(t), t \in T\}$ as in (1.1), we conclude immediately that

$$\{X(t), t \in T\} \stackrel{d}{=} \{S_n(f_t), t \in T\}. \quad (1.7)$$

We call $\{S_n(f_t), t \in T\}$ the series representation of the stochastic process $\{X(t), t \in T\}$. The series representation is very important in our study of the sample path properties of the process $\{X(t), t \in T\}$ for it shows that the properties of the integrands $\{f_t(x_1, \dots, x_n), t \in T\}$, $x_1, \dots, x_n \in R$, as functions

on T , have impact on the sample path properties of $\{X(t), t \in T\}$. This phenomenon has been observed and studied by Rosinski [Ros86], [Ros87] in the case of stable and infinitely divisible processes; some of the ideas used in the present paper originate from the papers of Rosinski.

In Section 2 we find conditions for a stochastic process of the form (1.1) to have “smooth” sample paths, more generally, for the sample paths of the process to belong to a vector space. The case of bounded sample paths is handled in Section 3. The tail behavior of the distribution of $\sup_{t \in T} |X(t)|$ is studied in Section 4 for bounded stochastic processes of the form (1.1). Finally, Section 5 states zero-one laws for stochastic processes of the form (1.1) with $n = 2$. These zero-one laws complement the results of Section 2.

2 Processes with sample paths in a vector space

Let $\{X(t), t \in T\}$ be a stochastic process of the form (1.1) and let V be a vector space of real-valued functions on T . We study, in this section, whether $\{X(t), t \in T\}$ has a version with all sample paths belonging to V . This question is of interest because path properties can be typically formulated in terms of vector subspaces V of \mathbf{R}^T . Much is known in the case of Gaussian and stable processes. Our results shed some light in the case of multiple stable integrals (1.1).

In order to make our discussion meaningful and to avoid obvious measurability problems, we introduce some assumptions.

From now on, the parameter space T is assumed to be a separable metric space. We extend the notion of *separable representation*, introduced by Rosinski [Ros87] to multiple stochastic integrals. Let $m^{(n)} = m \times \dots \times m$. An integral representation (1.1) of $\{X(t), t \in T\}$ is said to be separable if there is a countable subset $T_0 \subset T$ and a Borel measurable symmetric set $N_0 \subset \mathbf{R}^n$ which does not include the diagonals, such that $m^{(n)}(N_0) = 0$, and for every $(x_1, \dots, x_n) \notin N_0$ and for every $t \in T$, there is a sequence $\{t_j\}_{j=1}^\infty \subset T_0$ such that $f_t(x_1, \dots, x_n) = \lim_{j \rightarrow \infty} f_{t_j}(x_1, \dots, x_n)$. Separability of the integral representation (1.1), as pointed out in [Ros87] for the case $n = 1$, is an assumption which can always be made. Indeed, $m^{(n)}$, if restricted to the Borel subsets of the lower-triangular space $L^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_1 < x_2 < \dots < x_n\}$ is a σ -finite measure, so letting $\hat{m}^{(n)}$ be a probability measure on the Borel subsets of L^n , equivalent to the measure $m^{(n)}$, we may regard $\{f_t, t \in T\}$ as a stochastic process indexed by T with $\hat{m}^{(n)}$ as underlying probability measure. By Doob’s theorem (see [Doo53], Theorem 2.4, Chapter II), there is a countable subset $T_0 \subset T$, a measurable set $M_0 \subset L^n$ with $\hat{m}^{(n)}(M_0) = 0$ (and, therefore, with $m^{(n)}(M_0) = 0$), and a family of measurable functions $\{g_t, t \in T\}$, $g_t : L^n \rightarrow \overline{\mathbf{R}}$ (where $\overline{\mathbf{R}}$ is the

two-point compactification of \mathbf{R}) such that for every $t \in T$,

$$m^{(n)}\{(x_1, \dots, x_n) \in L_n : f_t(x_1, \dots, x_n) \neq g_t(x_1, \dots, x_n)\} = 0, \quad (2.1)$$

and for every $t \in T$ and every $(x_1, \dots, x_n) \notin M_0$ there is a sequence $\{t_j\}_{j=1}^\infty \subset T_0$ such that

$$g_t(x_1, \dots, x_n) = \lim_{j \rightarrow \infty} g_{t_j}(x_1, \dots, x_n).$$

g_t , at this point, is defined only on L^n . We further extend g_t to the whole of \mathbf{R}^n by setting $g_t(x_1, \dots, x_n) = g_t(x_{(1)}, \dots, x_{(n)})$, where $x_{(1)}, \dots, x_{(n)}$ is an increasing rearrangement of x_1, \dots, x_n if the numbers x_1, \dots, x_n are all different, otherwise we define $g_t(x_1, \dots, x_n) = 0$. Then (2.1) extends to \mathbf{R}^n , so that we obtain

$$X(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g_t(x_1, \dots, x_n) M(dx_1) \dots M(dx_n), \quad t \in T. \quad (2.2)$$

The integral representation (2.2) is separable by construction, with

$$N_0 = \{(x_1, \dots, x_n) \in \mathbf{R}^n : x_i \neq x_j \text{ if } i \neq j, (x_{(1)}, \dots, x_{(n)}) \in M_0\}.$$

This completes the argument.

We may and will assume, therefore, that the integral representation (1.1) is, to start with, separable.

As in [CR73] and [Ros87], we consider the following vector spaces V of functions on T :

- (a) space of bounded functions on T ,
- (b) space of continuous functions on T ,
- (c) space of uniformly continuous functions on T ,
- (d) space of Lipschitz continuous functions on T ,

and, if $T = \mathbf{R}$,

- (e) space of functions without oscillatory discontinuities on T ,
- (f) space of functions of locally bounded variation on T ,
- (g) space of absolutely continuous functions on T ,
- (h) space of differentiable functions on T .

This list can be continued. In fact, we consider any function space V which satisfies the following condition:

CONDITION 2.1 *There exists a linear measurable subspace \hat{V} of \mathbf{R}^∞ such that for every separable stochastic process $\{Y(t), t \in T\}$, there is an event Ω_1 with $P(\Omega_1) = 1$, such that for the countable subset $T_0 \subset T$ in the definition of separability,*

$$\{\omega : \{Y(t), t \in T\} \in V\} \Delta \{\omega : \{Y(t), t \in T_0\} \in \hat{V}\} \in \Omega_1^c.$$

For example, if $T = [0, 1]$ and $V =$ space of continuous functions on T , then we can take $T_0 =$ rationals and $\hat{V} =$ space of uniformly continuous functions on T_0 . Cambanis and Rajput [CR73] showed that the function spaces V in (a)-(h) all satisfy Condition 2.1.

We are now ready to state our first theorem. It gives necessary conditions for a stochastic process of the type (1.1) to have a version with sample paths in a function space.

THEOREM 2.1 *Let $\{X(t), t \in T\}$ be a separable stochastic process with a separable representation (1.1) and let V be one of the vector spaces (a)-(h) above (or any other vector space satisfying Condition 2.1). If there is an event Ω_0 with $P(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$,*

$$\{X(t, \omega), t \in T\} \in V,$$

then there is a Borel measurable set $S_0 \subset \mathbf{R}^n$ such that $m^{(n)}(\mathbf{R}^n \setminus S_0) = 0$ and for every $(x_1, \dots, x_n) \in S_0$,

$$\{f_t(x_1, \dots, x_n), t \in T\} \in V.$$

PROOF: It is sufficient to suppose Condition 2.1 holds. We apply first this condition to the stochastic process $\{X(t), t \in T\}$. Setting $\Omega_2 = \Omega_0 \cap \Omega_1$, we obtain $P(\Omega_2) = 1$ and for every $\omega \in \Omega_2$, $\{X(t, \omega), t \in T_0\} \in \hat{V}$. Using Theorem 1.1, we conclude that

$$Z(t) = \sum_{j \in \mathbf{N}_n} [\epsilon_j][\Gamma_j]^{-1/\alpha} [\psi(Y_j)]^{-1} f_t(\mathbf{Y}_j), \quad t \in T_0,$$

satisfies

$$P(\omega : \{Z(t, \omega), t \in T_0\} \in \hat{V}) = 1 \quad (2.3)$$

since $X \stackrel{d}{=} Z$ and T_0 is countable. Now let $\tilde{\epsilon}_1 = -\epsilon_1$, $\tilde{\epsilon}_j = \epsilon_j$, $j \geq 2$. Clearly, $\{\tilde{\epsilon}_j\}_{j=1}^\infty$ is a Rademacher sequence independent of the sequences $\Gamma_1, \Gamma_2, \dots$ and Y_1, Y_2, \dots . Therefore

$$\tilde{Z}(t) = \sum_{j \in \mathbf{N}_n} [\tilde{\epsilon}_j][\Gamma_j]^{-1/\alpha} [\psi(Y_j)]^{-1} f_t(\mathbf{Y}_j), \quad t \in T_0,$$

is a version of $\{Z(t), t \in T_0\}$, and hence

$$P(\omega : \{\tilde{Z}(t, \omega), t \in T_0\} \in \hat{V}) = 1. \quad (2.4)$$

Since \hat{V} is a linear space, we conclude by (2.3) and (2.4) that

$$P(\omega : \{Z(t, \omega) - \tilde{Z}(t, \omega), t \in T_0\} \in \hat{V}) = 1.$$

But for each $t \in T_0$,

$$Z(t) - \tilde{Z}(t) = 2\epsilon_1 \Gamma_1^{-1/\alpha} \psi(Y_1)^{-1} \sum_{\substack{j \in \mathbf{N}_{n-1} \\ j_1 \geq 2}} [\epsilon_j][\Gamma_j]^{-1/\alpha} [\psi(Y_j)]^{-1} f_t(Y_1, Y_{j_1}, \dots, Y_{j_{n-1}}).$$

Thus, with

$$Z_1(t) = \sum_{\substack{\mathbf{j} \in \mathbf{N}_{n-1} \\ j_1 \geq 2}} [\epsilon_{\mathbf{j}}][\Gamma_{\mathbf{j}}]^{-1/\alpha} [\psi(Y_{\mathbf{j}})]^{-1} f_t(Y_1, Y_{j_1}, \dots, Y_{j_{n-1}}), \quad t \in T_0,$$

we get

$$P(\omega : \{Z_1(t, \omega), t \in T_0\} \in \hat{V}) = 1. \quad (2.5)$$

Repeating the procedure which led us from (2.3) to (2.5) $n-1$ times, we conclude

$$P(\omega : \{f_t(Y_1(\omega), Y_2(\omega), \dots, Y_n(\omega)), t \in T_0\} \in \hat{V}) = 1. \quad (2.6)$$

Let

$$S_2 = \{(x_1, \dots, x_n) \in \mathbf{R}^n : \{f_t(x_1, \dots, x_n), t \in T_0\} \in \hat{V}\}.$$

Since each Y_i has distribution m_ψ , relation (2.6) implies $m_\psi^{(n)}(\mathbf{R}^n \setminus S_2) = 0$ and thus

$$m^{(n)}(\mathbf{R}^n \setminus S_2) = 0. \quad (2.7)$$

We now apply Condition 2.1 to $\{f_t(\cdot), t \in T\}$ regarded as a separable stochastic process on the probability space $(\mathbf{R}^n, \mathcal{B}^n, m_\psi^{(n)})$. We may assume without loss of generality, that the countable set T_0 here is the same as before (take the union of the two sets, if necessary), and we replace, in this case, Ω_1 by S_1 . Set $S_0 = S_1 \cap S_2$. Then $m^{(n)}(\mathbf{R}^n \setminus S_0) = 0$, and since $S_0 \subset S_2$, we conclude that for every $(x_1, \dots, x_n) \in S_0$, $\{f_t(x_1, \dots, x_n), t \in T\} \in V$. This completes the proof of the theorem. ■

Remarks

- Since \mathbf{f} is symmetric and vanishes on diagonals, the measurable set S_0 in Theorem 2.1 can always be chosen to be symmetric and to include all the diagonals.
- In the case $n = 2$ the statement of Theorem 2.1 remains true if we replace the assumption $P(\Omega_0) = 1$ by $P(\Omega_0) > 0$, since our process satisfies an appropriate zero-one law. See Section 5 for more details.
- Note the relation between the two notions of separability appearing in Theorem 2.1. We should understand it as follows: $\{X(t), t \in T\}$ is separable, $\{X(t), t \in T\} \stackrel{d}{=} \{I_n(f_t), t \in T\}$, where the equality is in terms of finite-dimensional distribution, and the integral representation defined by the functions $\{f_t, t \in T\}$ is separable.

Theorem 2.1 provides a *necessary* condition for a stochastic process of the type (1.1) to have almost all sample paths in a vector space V . We now focus on *sufficient* conditions and assume that the space V satisfies the following:

CONDITION 2.2 *V is a normed space of real-valued functions on T such that all evaluations $\pi_t : V \rightarrow \mathbf{R}$ defined by $\pi_t(\mathbf{x}) = (\mathbf{x})_t$ are continuous.*

The following result ensures that the multiple integral $\mathbf{I}_n(\mathbf{f})$ of a function \mathbf{f} taking values in V may be regarded as the vector of the multiple integrals of the evaluations $I_n(f_t)$ of this function.

PROPOSITION 2.1 *Let \mathbf{f} be a symmetric, vanishing on the diagonals, measurable function from \mathbf{R}^n to a vector space V satisfying Condition 2.2 and suppose that the multiple integral $\mathbf{I}_n(\mathbf{f})$ exists. Then for each $t \in T$ the evaluation $f_t := \pi_t(\mathbf{f})$ is $M^{(n)}$ -integrable, and for each $t \in T$,*

$$I_n(f_t) = (\mathbf{I}_n(\mathbf{f}))_t, \quad \text{a.s.}, \quad (2.8)$$

where $(\mathbf{I}_n(\mathbf{f}))_t = \pi_t(\mathbf{I}_n(\mathbf{f}))$ for each $t \in T$.

PROOF: Suppose first that \mathbf{f} is a simple function, i.e.

$$\mathbf{f}(x_1, \dots, x_n) = \sum_{i=1}^N \mathbf{a}(i) \mathbf{1}((x_1, \dots, x_n) \in A_i) \quad (2.9)$$

where $\mathbf{a}(1), \dots, \mathbf{a}(N) \in V$, and A_1, \dots, A_N are disjoint symmetric Borel sets in \mathbf{R}^n with finite $m^{(n)}$ measure and which do not include the diagonals. Then

$$I_n(f_t) = \sum_{i=1}^N (\mathbf{a}(i))_t M^{(n)}(A_i) = \left(\sum_{i=1}^N \mathbf{a}(i) M^{(n)}(A_i) \right)_t = (\mathbf{I}_n(\mathbf{f}))_t.$$

Thus (2.8) holds for simple functions \mathbf{f} .

Now let \mathbf{f} be $M^{(n)}$ -integrable. Then, by definition, there is a sequence of simple functions $\{\mathbf{f}^{(k)}\}_{k=1}^\infty$ as in (2.9) converging to \mathbf{f} in measure $m^{(n)}$, such that $\{\mathbf{I}_n(\mathbf{f}^{(k)} \mathbf{1}_C), k = 1, 2, \dots\}$ converges to $\mathbf{I}_n(\mathbf{f} \mathbf{1}_C)$ in probability for each symmetric Borel set C of \mathbf{R}^n .

Now, for every $t \in T$, $m^{(n)}\{\mathbf{x} : \|\mathbf{f}^{(k)}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\|_V > \epsilon_1\} \rightarrow 0, \forall \epsilon_1$, implies $m^{(n)}\{\mathbf{x} : |f_t^{(k)}(\mathbf{x}) - f_t(\mathbf{x})| > \epsilon_2\} \rightarrow 0, \forall \epsilon_2$, by Condition 2.2, and hence the sequence $\{f_t^{(k)}\}_{k=1}^\infty$ converges to f_t in measure $m^{(n)}$. Similarly, the sequence $\{(\mathbf{I}_n(\mathbf{f}^{(k)} \mathbf{1}_C))_t, k = 1, 2, \dots\}$ converges in probability to $(\mathbf{I}_n(\mathbf{f} \mathbf{1}_C))_t$.

Since $\mathbf{f}^{(k)} \mathbf{1}_C$ is a simple function, (2.8) holds for each $\mathbf{f}^{(k)} \mathbf{1}_C$ and $t \in T$. Letting $k \rightarrow \infty$, we infer that for each $t \in T$, f_t is $M^{(n)}$ -integrable and $I_n(f_t) = (\mathbf{I}_n(\mathbf{f}))_t$ a.s.. ■

The following result gives *sufficient* conditions for a stochastic process of the type (1.1) to have almost all its sample paths in a Banach space with special properties.

THEOREM 2.2 *Let V be a separable Banach space of the R -type p satisfying Condition 2.2, and let $\{X(t), t \in T\}$ be a stochastic process given in the form of a multiple $S\alpha S$ integral with a separable representation (1.1), $0 < \alpha < p$. Suppose that there is a Borel measurable set $S_0 \subset \mathbf{R}^n$ such that $m^{(n)}(\mathbf{R}^n \setminus S_0) = 0$, and for*

every $(x_1, \dots, x_n) \in S_0$, $\{f_t(x_1, \dots, x_n), t \in T\} \in V$, and that there is a function ψ as in (1.3) such that

$$\int_{\mathbf{R}^n} \|\mathbf{f}(x_1, \dots, x_n)\|_V^\alpha \left(\ln_+ \frac{\|\mathbf{f}(x_1, \dots, x_n)\|_V}{\psi(x_1) \dots \psi(x_n)} \right)^{n-1} m(dx_1) \dots m(dx_n) < \infty, \quad (2.10)$$

where

$$\mathbf{f}(x_1, \dots, x_n) = \begin{cases} \{f_t(x_1, \dots, x_n), t \in T\} & \text{if } (x_1, \dots, x_n) \in S_0, \\ \mathbf{0} & \text{otherwise,} \end{cases} \quad (2.11)$$

and $\|\cdot\|_V$ is the V -induced norm. Then there is a version of $\{X(t), t \in T\}$ with all sample paths in V .

PROOF: Let \mathbf{f} be as in (2.11). Separability of V and Condition 2.2 imply that $\mathbf{f} = \mathbf{R}^n \rightarrow V$ is a Borel measurable function (see [Ros86], p.6). Applying Theorem 1.1, we conclude that \mathbf{f} is $M^{(n)}$ -integrable. Therefore, $\mathbf{I}_n(\mathbf{f})$ is well-defined and since \mathbf{f} is V -valued, so is $\mathbf{I}_n(\mathbf{f})$. Thus $\mathbf{I}_n(\mathbf{f})$ is the version we are looking for, since, by Proposition 2.1, $\{(\mathbf{I}_n(\mathbf{f}))_t, t \in T\}$ is a version of $\{X(t), t \in T\} \equiv \{I_n(f_t), t \in T\}$. This completes the proof. ■

Remarks

- The restriction that the space V must be of R-type p , $0 < \alpha < p$, disappears if $0 < \alpha < 1$, since every Banach space is of R-type 1.
- The results of this section do not provide a condition which is both necessary and sufficient for the sample paths of a process of type (1.1) to belong to a vector space.
- There are many unsolved questions even in the case of single stable integrals (i.e. $n = 1$ in (1.1)). Although much is known for $\alpha < 1$, general conditions for regularity are largely unknown when $\alpha \geq 1$: one has results only for specific f 's and specific path properties. The multiple integration case (i.e. $n > 1$ in (1.1)) which we are considering here, is naturally even more complicated.

3 Boundedness

Since the space of bounded functions is not separable (even on a countable set), Theorem 2.2 cannot be used to study stochastic processes of the type (1.1) with a.s. bounded sample paths. Nevertheless, one has

THEOREM 3.1 *Let $\{X(t), t \in T\}$ be a separable stochastic process represented in the form of a multiple $S\alpha S$ integral with a separable representation (1.1).*

(i) Suppose that $\{X(t), t \in T\}$ is a.s. bounded. Then

$$\int_{\mathbf{R}^n} f^*(x_1, \dots, x_n)^\alpha m(dx_1) \dots m(dx_n) < \infty, \quad (3.1)$$

where $f^*(x_1, \dots, x_n) := \sup_{t \in T} |f_t(x_1, \dots, x_n)|$.

(ii) Let $0 < \alpha < 1$, and suppose that there is a function ψ satisfying (1.3) such that

$$\int_{\mathbf{R}^n} f^*(x_1, \dots, x_n)^\alpha \left(\ln_+ \frac{f^*(x_1, \dots, x_n)}{\psi(x_1) \dots \psi(x_n)} \right)^{n-1} m(dx_1) \dots m(dx_n) < \infty. \quad (3.2)$$

Then $\{X(t), t \in T\}$ is a.s. bounded.

PROOF: By the separability assumptions, we may and will assume that the set T is countable. We identify it with the set of positive integers and denote our process $\{X(k), k = 1, 2, \dots\}$.

(i) We know that $\sup_k |X(k)| < \infty$ a.s.. But we shall, at first, make a stronger assumption, namely $\lim_{k \rightarrow \infty} X(k) = 0$ a.s.. We view then $\mathbf{X} = \{X(k), k = 1, 2, \dots\}$ as a random vector in the separable Banach space c_0 of sequences converging to zero, equipped with the supremum norm $\|\cdot\|_\infty$. Clearly, as $m \rightarrow \infty$ $\mathbf{X}^{(m)} \rightarrow \mathbf{X}$ in c_0 a.s., where

$$\mathbf{X}^{(m)} = \{X(1), \dots, X(m), 0, 0, \dots\}, \quad m = 1, 2, \dots$$

Obviously, $\mathbf{X}^{(m)} = \mathbf{I}_n(\mathbf{f}^{(m)})$, $m = 1, 2, \dots$, where

$$\mathbf{f}^{(m)} = (f_1, \dots, f_m, 0, 0, 0, \dots), \quad m = 1, 2, \dots$$

is regarded as a c_0 -valued function.

It follows from [KS88a] that the random vectors $\mathbf{X}^{(m)}$, $m = 1, 2, \dots$ belong to the same Marcinkiewicz-Paley-Zygmund class, and so for this sequence of random vectors, convergence in probability implies convergence in L^r for every $0 < r < \alpha$. We conclude that for any $0 < r < \alpha$, $\lim_{m \rightarrow \infty} E\|\mathbf{X} - \mathbf{X}^{(m)}\|_\infty^r = 0$ and $E\|\mathbf{X}\|_\infty^r < \infty$. Moreover, it follows by Proposition 5.1(ii) of [KS88b] that for any $m = 1, 2, \dots$,

$$E\|\mathbf{X}^{(m)}\|_\infty^r \geq C_{\alpha, r} \left(\int_{\mathbf{R}^n} \max_{i \leq m} |f_i(x_1, \dots, x_m)|^\alpha m(dx_1) \dots m(dx_n) \right)^{r/\alpha}, \quad (3.3)$$

where $C_{\alpha, r}$ is a positive constant depending only on α and r . Letting $m \rightarrow \infty$, we get

$$\infty > E\|\mathbf{X}\|_\infty^r \geq C_{\alpha, r} \left(\int_{\mathbf{R}^n} f^*(x_1, \dots, x_n)^\alpha m(dx_1) \dots m(dx_n) \right)^{r/\alpha}. \quad (3.4)$$

Let us now return to our original assumption $\sup_k |X(k)| < \infty$ a.s. and drop the requirement $\lim_{k \rightarrow \infty} X(k) = 0$ a.s.. Let $\{a_k, k = 1, 2, \dots\}$ belong to c_0 , and $\sup_k |a_k| \leq 1$. Then $\{a_k X(k), k = 1, 2, \dots\}$ belongs to c_0 a.s., so (3.4) gives

$$E(\sup_k |X_k|)^r \geq E(\sup_k |a_k X_k|)^r \geq C_{\alpha, r} \left(\int_{\mathbf{R}^n} \sup_k |a_k f_k(x_1, \dots, x_n)|^\alpha m(dx_1) \dots m(dx_n) \right)^{r/\alpha} \quad (3.5)$$

and we know also that

$$E(\sup_k |a_k X(k)|)^r < \infty, \quad \forall r \in (0, \alpha). \quad (3.6)$$

By taking $a_k = 1$ for $k \leq N$ and 0 otherwise, and letting $N \rightarrow \infty$, (3.5) implies

$$E(|\sup_k |X(k)||^r) \geq C_{\alpha, r} \left(\int_{\mathbf{R}^n} f^*(x_1, \dots, x_n)^\alpha m(dx_1) \dots m(dx_n) \right)^{r/\alpha}. \quad (3.7)$$

Now, (3.1) will follow if we establish $E(\sup_k |X(k)|)^r < \infty$. Assume, to the contrary, that for some $0 < r < \alpha$, $E(\sup_k |X(k)|)^r = \infty$. Choose $0 < K_1 < K_2 < \dots$ such that

$$E(\max_{k \leq K_j} |X(k)|)^r \geq j, \quad j = 1, 2, \dots$$

Choose now $a_k = j^{-1/(2r)}$ if $K_{j-1} < k \leq K_j$, $j = 1, 2, \dots$, $K_0 = 0$. Then $E(\max_{k \leq K_j} |a_k X(k)|)^r \geq j^{1/2}$ for every $j = 1, 2, \dots$, so that $E(\sup_k |a_k X(k)|)^r = \infty$, contradicting (3.6). This proves that $E(\sup_k |X(k)|)^r < \infty$ for every $0 < r < \alpha$, and thus the proof of part (i) is complete.

(ii) Let again $\{a_k, k = 1, 2, \dots\}$ belong to c_0 , and $\sup_k |a_k| \leq 1$. Set

$$\mathbf{g}(x_1, \dots, x_n) = \begin{cases} \{a_k f_k(x_1, \dots, x_n), k = 1, 2, \dots\} & \text{if } f^*(x_1, \dots, x_n) < \infty, \\ \mathbf{0} & \text{otherwise.} \end{cases} \quad (3.8)$$

Clearly \mathbf{g} is a measurable function $\mathbf{R}^n \rightarrow c_0$ and by (3.2), for every $k = 1, 2, \dots$, $\pi_k(\mathbf{g}) = a_k f_k m^{(n)}$ -almost everywhere. Since $\|\mathbf{g}(x_1, \dots, x_n)\|_\infty \leq f^*(x_1, \dots, x_n)$ for every (x_1, \dots, x_n) , we get

$$\int_{\mathbf{R}^n} \|\mathbf{g}(x_1, \dots, x_n)\|^\alpha \left(\ln_+ \frac{\|\mathbf{g}(x_1, \dots, x_n)\|}{\psi(x_1) \dots \psi(x_n)} \right)^{n-1} m(dx_1) \dots m(dx_n) < \infty.$$

Theorem 1.1 applies since $0 < \alpha < 1$ and every Banach space is of R-type $p = 1$. Hence \mathbf{g} is $M^{(n)}$ integrable and $\mathbf{I}_n(\mathbf{g})$ is a well-defined c_0 -valued random variable. By Proposition 2.1 we have $\{a_k X(k), k = 1, 2, \dots\} \stackrel{d}{=} \{\mathbf{I}_n(\mathbf{g})_k, k = 1, 2, \dots\}$, and it follows as in the proof of part (i) that $E \sup_k |a_k X(k)|^r < \infty$ for any $0 < r < \alpha$, and, thus, also $E \sup_k |X(k)|^r < \infty$. Hence $\{X(k), k = 1, 2, \dots\}$ is a.s. bounded. This completes the proof. ■

COROLLARY 3.1 *Under the conditions of Theorem 3.1, Part (i), for every $0 < r < \alpha$,*

$$(E \sup_{t \in T} |X(t)|^r)^{1/r} \geq C_{\alpha, r} \left(\int_{\mathbf{R}^n} f^*(x_1, \dots, x_n)^\alpha m(dx_1) \dots m(dx_n) \right)^{1/\alpha},$$

where $C_{\alpha, r}$ is a positive constant depending only on α and r .

PROOF: Follows immediately from the proof of part (i) of Theorem 3.1. ■

4 Tails

We now focus on the tail of the distribution of $\sup_{t \in T} |X(t)|$, where $\{X(t), t \in T\}$ is a stochastic process of the type (1.1). We obtain the best results in the case $0 < \alpha < 1$. This should not come as a surprise since even in the case of $S\alpha S$ processes ($n = 1$), no complete characterization of the tails of the distribution of extremes is known when $\alpha \geq 1$ (see [Sam88]).

The following result (lower bound) holds for any $0 < \alpha < 2$.

THEOREM 4.1 *Let $\{X(t), t \in T\}$ be a separable stochastic process represented in the form of a multiple $S\alpha S$ integral with a separable representation (1.1) satisfying the following property for every $t \in T_0$. (T_0 is the countable subset of T appearing in the definition of a separable representation): there is a function ψ satisfying (1.3) such that, for every $t \in T_0$,*

$$\int_{\mathbf{R}^n} |f_t(x_1, \dots, x_n)|^\alpha \left(\ln_+ \frac{|f_t(x_1, \dots, x_n)|}{\psi(x_1) \dots \psi(x_n)} \right)^{n-1} m(dx_1) \dots m(dx_n) < \infty, \quad (4.1)$$

if $n \geq 3$, or

$$\int_{\mathbf{R}^2} |f_t(x_1, x_2)|^\alpha \ln_+ \frac{|f_t(x_1, x_2)|}{\psi(x_1)\psi(x_2)} \ln_+ \left| \ln \frac{|f_t(x_1, x_2)|}{\psi(x_1)\psi(x_2)} \right| m(dx_1)m(dx_2) < \infty, \quad (4.2)$$

if $n = 2$. Then

$$\begin{aligned} & \liminf_{\lambda \rightarrow \infty} \frac{\lambda^\alpha}{(\ln \lambda)^{n-1}} P(\sup_{t \in T} |X(t)| > \lambda) \\ & \geq n(n!)^{\alpha-2} \alpha^{n-1} C_\alpha^n \int_{\mathbf{R}^n} f^*(x_1, \dots, x_n)^\alpha m(dx_1) \dots m(dx_n), \end{aligned} \quad (4.3)$$

where C_α is given by (1.6) and f^* is defined as $\sup_{t \in T} |f_t(x_1, \dots, x_n)|$.

PROOF: Without loss of generality we may assume that T_0 is also the separating set for the process $\{X(t), t \in T\}$. Let $\{t_1, t_2, \dots\}$ be an arbitrary enumeration of the points in T_0 . For fixed integer k ,

$$\begin{aligned} & \liminf_{\lambda \rightarrow \infty} \frac{\lambda^\alpha}{(\ln \lambda)^{n-1}} P(\sup_{t \in T} |X(t)| > \lambda) \geq \lim_{\lambda \rightarrow \infty} \frac{\lambda^\alpha}{(\ln \lambda)^{n-1}} P(\max_{i=1, \dots, k} |X(t_i)| > \lambda) \\ & = n(n!)^{\alpha-2} \alpha^{n-1} C_\alpha^n \int_{\mathbf{R}^n} \max_{i=1, \dots, k} |f_{t_i}(x_1, \dots, x_n)|^\alpha m(dx_1) \dots m(dx_n), \end{aligned}$$

by Theorem 4.1 of Samorodnitsky and Taqqu [ST88b]. Since this holds for every $k = 1, 2, \dots$, relation (4.3) follows. ■

There is asymptotic equivalence in the case $0 < \alpha < 1$.

THEOREM 4.2 *Let $0 < \alpha < 1$. Under the assumptions of Theorem 3.1(ii), we have*

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \frac{\lambda^\alpha}{(\ln \lambda)^{n-1}} P(\sup_{t \in T} |X(t)| > \lambda) \\ &= n(n!)^{\alpha-2} \alpha^{n-1} C_\alpha^n \int_{\mathbf{R}^n} f^*(x_1, \dots, x_n)^\alpha m(dx_1) \dots m(dx_n). \end{aligned} \quad (4.4)$$

PROOF: Let T_0 be the countable subset of T appearing in the definition of separability of $\{X(t), t \in T\}$, and let $\mathbf{g}(x_1, \dots, x_n) = \{f_k(x_1, \dots, x_n), k = 1, 2, \dots\}$ if $f^*(x_1, \dots, x_n) < \infty$ and $\mathbf{0}$ otherwise. Then $\|\mathbf{g}(x_1, \dots, x_n)\|_\infty = f^*(x_1, \dots, x_n)$, $m^{(n)}$ a.e.. Moreover, $\{X(t) \stackrel{d}{=} S_n(f_t), t \in T_0\}$ by Theorem 1.1, so that $\sup_{t \in T_0} |X(t)| \stackrel{d}{=} \|\mathbf{S}_n(\mathbf{g})\|_\infty$. It is therefore enough to prove

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \frac{\lambda^\alpha}{(\ln \lambda)^{n-1}} P(\|\mathbf{S}_n(\mathbf{g})\|_\infty > \lambda) \\ &= n(n!)^{\alpha-2} \alpha^{n-1} C_\alpha^n \int_{\mathbf{R}^n} \|\mathbf{g}(x_1, \dots, x_n)\|_\infty^\alpha m(dx_1) \dots m(dx_n). \end{aligned}$$

But this follows directly from the proof of Theorem 3.2 of [ST88b]. ■

5 Zero-one laws

Sample path properties of some stochastic processes can satisfy various zero-one laws: see [Kal70] for Gaussian processes, [DK74] for stable processes and [Jan84] and [Ros89] for infinitely divisible processes. In this section we establish some zero-one laws for stochastic processes of the form (1.1) with $n = 2$. We restrict ourselves to the case $n = 2$ because we use here a result of de Acosta [DeA76] on quadratic forms in Gaussian vectors. We believe that similar zero-one laws hold for $n \geq 3$.

THEOREM 5.1 *Let $\{X(t), t \in T\}$ be a separable stochastic process with a separable representation (1.1) and with $n = 2$. Let V be a vector space of functions over T satisfying Condition 2.1. Then*

$$P\{\{X(t), t \in T\} \in V\} = 0 \text{ or } 1.$$

Before proving this theorem, we collect a number of facts which will be used in the proof.

Focus first on the $S\alpha S$ random measure M in (1.1). We assume for simplicity $m((-\infty, 0)) = 0$, and by denoting $M(t) := M([0, t])$, we may regard M as a $S\alpha S$ process with independent increments on \mathbf{R}^+ . (The case $m((-\infty, 0)) > 0$ can be treated similarly, by considering M as consisting of two independent components $\{M_1(t) = M([0, t]), t \geq 0\}$ and $\{M_2(t) = M([-t, 0]), t \geq 0\}$.) It is well-known that $\{M(t), t \geq 0\}$ has a version in the separable space $D[0, +\infty)$ equipped with

the Skorokhod topology. It follows from [FK72], as shown in [Kal73], that such a version is given in particular by the series

$$M(t) = C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \psi(Y_j)^{-1} 1_{[0,t]}(Y_j), \quad t \geq 0, \quad (5.1)$$

which converges uniformly in t on finite intervals. Here ϵ_j 's, Γ_j 's and Y_j 's are as in Section 1. It is also well-known ([LeP80]) that we may replace the Rademacher sequence $\epsilon_1, \epsilon_2, \dots$ in (5.1) by a sequence of i.i.d. standard normal random variables G_1, G_2, \dots at the expense of changing the multiplication constant in (5.1).

We will therefore assume from now on that

$$M(t) = \sum_{j=1}^{\infty} G_j \Gamma_j^{-1/\alpha} \psi(Y_j)^{-1} 1_{[0,t]}(Y_j), \quad t \geq 0. \quad (5.2)$$

Thus, $\{M(t), t \geq 0\}$ can be regarded as a random vector taking values in the space $D[0, +\infty)$ equipped with the Skorohod topology. Moreover, all finite-dimensional projections of $\{M(t), t \geq 0\}$ are $S\alpha S$. Therefore, $\{M(t), t \geq 0\}$ is a $S\alpha S$ vector in $D[0, +\infty)$, as the Skorohod Borel σ -algebra coincides with the cylindrical σ -algebra.

Assume now that the random vector $\{M(t), t \geq 0\}$ is defined on the product of two probability spaces, $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$, and let the Gaussian sequence G_1, G_2, \dots live on $(\Omega_1, \mathcal{F}_1, P_1)$, while the sequences $\Gamma_1, \Gamma_2, \dots$ and Y_1, Y_2, \dots live on $(\Omega_2, \mathcal{F}_2, P_2)$. Arguing as above, we conclude that, for a fixed $\omega_2 \in \Omega_2$, $\{M(t), t \geq 0\}$ is a zero-mean Gaussian random vector on $D[0, +\infty)$, defined on the probability space $(\Omega_1, \mathcal{F}_1, P_1)$.

We will need the following extension of the above mentioned result of deAcosta ([DeA76]).

LEMMA 5.1 *Let (E, \mathcal{B}) be a measurable vector space, and let G be a zero-mean Gaussian vector on (E, \mathcal{B}) . Let $\{A_m^{(i)}, i = 1, 2, \dots\}$, $m = 1, 2, \dots$ be sequences of Borel-measurable bilinear forms on \mathbf{R}^2 taking values in a measurable vector space (E_1, \mathcal{B}_1) . Let S be a measurable subspace of $(E_1)^\infty$. Then*

$$P((\lim_{i \rightarrow \infty} A_m^{(i)}(G, G), m = 1, 2, \dots) \in S) = 0 \text{ or } 1.$$

PROOF: Mimic the proofs of Theorems 3.1 and 3.2 in [DeA76]. ■

PROOF OF THEOREM 5.1: In view of Condition 2.1, we must prove

$$P(\{X(t), t \in T_0\} \in \hat{V}) = 0 \text{ or } 1. \quad (5.3)$$

For each $t \in T_0$, there is a sequence $\{f_t^{(i)}, i = 1, 2, \dots\}$ of simple functions as described in Section 1 such that, as $i \rightarrow \infty$, $I_2(f_t^{(i)}) \rightarrow X(t)$ in probability. Note

that any such simple function can be approximated by simple functions of the type

$$\text{sym} \left(\sum_{j=1}^K a_j 1((x_1, x_2) \in I_1^{(j)} \times I_2^{(j)}) \right),$$

where, for each $j = 1, \dots, K$ the finite intervals $I_1^{(j)}$ and $I_2^{(j)}$ are ordered (in the sense $x_1 > x_2$ if $x_k \in I_k^{(j)}$, $k = 1, 2$), and where

$$\text{sym}(f)(x_1, x_2) = f(x_1, x_2) + f(x_2, x_1)$$

for any function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$.

We may and will, therefore, assume that the simple function $f_t^{(i)}$ are themselves of the above type. Moreover, choosing, if necessary, a subsequence, we may and will assume that $I_2(f_t^{(i)}) \rightarrow X(t)$ a.s. as $i \rightarrow \infty$ for any $t \in T_0$.

Now, each $I_2(f_t^{(i)})$ is, clearly, a measurable bilinear form in $\{M(s), s \geq 0\}$. Applying Lemma 5.1 we conclude

$$P(\{X(t), t \in T_0\} \in \hat{V} | \mathcal{F}_2) = 0 \text{ or } 1 \text{ a.s.}, \quad (5.4)$$

which is a zero-one law for the (conditional) Gaussian measures.

To establish (5.3), we must remove the conditioning. Set

$$A = \{\omega_2 \in \Omega_2 : P(\{X(t), t \in T_0\} \in \hat{V} | \mathcal{F}_2) = 1\},$$

and observe that

$$P(\{X(t), t \in T_0\} \in \hat{V}) = P_2(A). \quad (5.5)$$

We want to apply the Hewitt-Savage zero-one law to the event A in order to show $P_2(A) = 0$ or 1 . Recall that $\Gamma_1, \Gamma_2, \dots$ and Y_1, Y_2, \dots live on $(\Omega_2, \mathcal{F}_2, P_2)$. In fact, $A \in \sigma((e_1, Y_1), (e_2, Y_2), \dots)$ where e_1, e_2, \dots are i.i.d. exponential random variables such that $\Gamma_j = e_1 + e_2 + \dots + e_j$. Let π be an arbitrary permutation of the numbers $\{1, \dots, k\}$ and

$$\begin{aligned} M_\pi(t) &= \sum_{j=1}^k G_j (e_{\pi(1)} + \dots + e_{\pi(j)})^{-1/\alpha} \psi(Y_{\pi(j)})^{-1} 1_{(-\infty, t]}(Y_{\pi(j)}) \\ &\quad + \sum_{j=k+1}^{\infty} G_j \Gamma_j^{-1/\alpha} \psi(Y_j)^{-1} 1_{(-\infty, t]}(Y_j) \end{aligned}$$

for $t \geq 0$. For fixed $\omega_2 \in A$, $\{M_\pi(t), t \geq 0\}$ is again a Gaussian vector in $D[0, +\infty)$, and it is easy to check that the laws of $\{M(t), t \geq 0\}$ and $\{M_\pi(t), t \geq 0\}$ are equivalent. Therefore, $P(\{\lim_{i \rightarrow \infty} I_2(f_t^{(i)}), t \in T_0\} \in \hat{V} | \mathcal{F}_2) = 1$ implies $P(\{\lim_{i \rightarrow \infty} I_2^{(\pi)}(f_t^{(i)}), t \in T_0\} \in \hat{V} | \mathcal{F}_2) = 1$, where $I_2^{(\pi)}(f_t^{(i)})$ is obtained by replacing $\{M(t), t \geq 0\}$ by $\{M_\pi(t), t \geq 0\}$ in the bilinear form $I(f_t^{(i)})$. Thus,

the event A is invariant under the permutations π of the above kind. By the Hewitt-Savage zero-one law, $P_2(A) = 0$ or 1 , and hence by (5.5),

$$P(\{X(t), t \in T_0\} \in \hat{V}) = P_2(A) = 0 \text{ or } 1.$$

This completes the proof. ■

The following proposition complements a result of Krakowiak and Szulga ([KS88a], Theorem 2.11) about the equivalence of different modes of convergence of sequences to a double $S\alpha S$ integral. Its proof uses the techniques developed in the proof of Theorem 5.1.

PROPOSITION 5.1 *Let $\mathbf{f}^{(m)}$, $m = 1, 2, \dots$ be a sequence of symmetric, vanishing on the diagonals, Banach space-valued simple functions as in (1.2), and assume that*

$$P(\mathbf{I}_2(\mathbf{f}^{(m)}), m = 1, 2, \dots \text{ converges}) > 0.$$

Then

$$P(\mathbf{I}_2(\mathbf{f}^{(m)}), m = 1, 2, \dots \text{ converges}) = 1.$$

In addition, if $\mathbf{f}^{(m)} \rightarrow \mathbf{f}$ as $m \rightarrow \infty$ in measure $m^{(2)}$ and if the Banach space satisfies the multilinear contraction principle (see [KS88b], Relation (2.1) for a definition), then \mathbf{f} is $M^{(2)}$ -integrable.

PROOF: Let $m = 1, 2, \dots$ be arbitrary. We can choose a sequence of simple functions $\mathbf{g}^{(m,k)}$, $k = 1, 2, \dots$, each one of the type $\text{sym}(\sum_{j=1}^K \mathbf{a}_j 1((x_1, x_2) \in I_1^{(j)} \times I_2^{(j)}))$, defined as in the proof of Theorem 5.1, such that $\mathbf{I}_2(\mathbf{g}^{(m,k)}) \rightarrow \mathbf{I}_2(\mathbf{f}^{(m)})$ a.s. as $k \rightarrow \infty$. Let now $\{k_m, m = 1, 2, \dots\}$ be a sequence of positive integers such that $\sum_{m=1}^{\infty} d_m < \infty$, where for $m = 1, 2, \dots$,

$$d_m = \inf\{\epsilon > 0 : P(\|\mathbf{I}_2(\mathbf{g}^{(m,k_m)}) - \mathbf{I}_2(\mathbf{f}^{(m)})\| > \epsilon) \leq \epsilon\}.$$

Then, by the Borel-Cantelli lemma,

$$P(\lim_{m \rightarrow \infty} \|\mathbf{I}_2(\mathbf{g}^{(m,k_m)}) - \mathbf{I}_2(\mathbf{f}^{(m)})\| \rightarrow 0) = 1. \quad (5.6)$$

Now $P(\mathbf{I}_2(\mathbf{f}^{(m)}), m = 1, 2, \dots \text{ converges}) > 0$ implies $P(\mathbf{I}_2(\mathbf{g}^{(m,k_m)}), m = 1, 2, \dots \text{ converges}) > 0$. But each $\mathbf{I}_2(\mathbf{g}^{(m,k_m)})$ is a measurable quadratic form in $\{M(t), t \geq 0\}$. Applying Lemma 5.1 and arguing as in the proof of Theorem 5.1, we conclude

$$P(\mathbf{I}_2(\mathbf{g}^{(m,k_m)}), m = 1, 2, \dots, \text{ converges}) = 1. \quad (5.7)$$

Now (5.6) and (5.7) imply

$$P(\mathbf{I}_2(\mathbf{f}^{(m)}), m = 1, 2, \dots, \text{ converges}) = 1,$$

establishing the first part of the theorem. The second part follows from Theorem 5.5 of Krakowiak and Szulga [KS88b].

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