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Dynamical invariants and parameter space structures for rational maps

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Dissertation

**DYNAMICAL INVARIANTS AND PARAMETER SPACE
STRUCTURES FOR RATIONAL MAPS**

by

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Chapter 1

Introduction

In recent years there has been a great deal of work done on the dynamics of the McMullen family of rational maps on the Riemann sphere. These maps are given by

$$F_\lambda(z) = z^n + \lambda/z^d$$

with n and d usually greater than 1, and can be viewed as singular perturbations of the polynomial z^n . In the work that follows, we will always take n and d to be equal unless otherwise noted, though we shall explicitly consider the possibility of n and d different in chapter 4. When λ is non-zero, the superattracting fixed point at the origin for the map z^n is replaced with a pole, and the degree of the map jumps from n to $n + d$. There is still always a fixed (superattracting) critical point at infinity, and a critical point at the pole at 0.

Any rational map of degree k has $2k - 2$ (up to multiplicity) critical points, so here 0 and infinity account for $n + d - 2$ of the $2n + 2d - 2$ critical points for F_λ and their behavior is independent of λ for λ nonzero: 0 maps to ∞ locally d -to-1, which maps to itself locally n -to-1. The immediate basin of infinity is denoted B_λ , and the connected component of its first preimage which contains 0 is denoted T_λ . A priori these two sets may coincide. The $n + d$ other critical points that exist for λ nonzero are “free,” in that their dynamical behavior varies widely with λ , and it

is this variety that gives rise to the rich and complicated behavior of these maps. There is rigidity in their behavior, however. Symmetries in the family cause a critical orbit relation, so that for any parameter, either all critical points escape to infinity or none do, all tend to some attracting cycle or none do, etc. Hence F_λ is in a sense unicritical. For fixed n and d , F_λ thus forms a natural one-parameter family.

Recall that it is the behavior of the critical points of a rational map that controls the global dynamical behavior. Most interesting dynamical features, such as basins of attracting or parabolic cycles, must be associated to a critical point, so we can search for such phenomena by tracking the critical points. Recall also that the behavior of critical points determines the topology of the *Julia set*, which is a primary object of study in complex dynamics. This is the locus in the complex plane on which a rational map behaves chaotically, and is defined as the set of points on which the set of iterates $\{F_\lambda^k\}$ is not a normal family in the sense of Montel.

For our family F_λ , as well as for other uncritical families such as the quadratic family $Q_c(z) = z^2 + c$, we can color the parameter plane according to the behavior of this free critical orbit. As for Q_c , the superattracting fixed point at infinity gives a natural partition of parameter space (the λ - and c -plane for F_λ and Q_c respectively) into a compact region for which the critical orbit does not tend to infinity (which for Q_c corresponds to the well-known Mandelbrot set), and a domain for which the critical orbit does tend to infinity. For the quadratic family, this yields the well-known Fundamental Dichotomy: The Julia set is connected if and only if the critical point 0 tends to infinity. Unlike for the quadratic family, however, there is more to the story for F_λ . In the escaping case, the dynamics of F_λ (and hence the topology of the Julia set) depends on the number of iterates that the critical orbit takes to enter the immediate basin of infinity, and we have the following Escape Trichotomy,

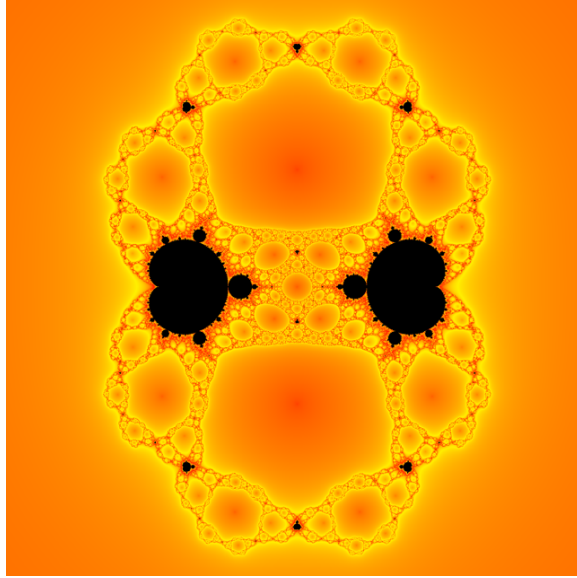


Figure 1.1: The parameter space for $n = d = 3$

proved by Devaney, Look and Uminsky in [11]:

Theorem (*The Escape Trichotomy*). Let $F_\lambda(z) = z^n + \lambda/z^d$ and consider the orbit of one of the critical values, v_λ .

1. If v_λ lies in B_λ , then $J(F_\lambda)$ is a Cantor set;
2. If v_λ lies in T_λ , then $J(F_\lambda)$ is a Cantor set of simple closed curves, each of which surrounds the origin;
3. If $F_\lambda^k(v_\lambda)$ lies in T_λ where $k \geq 1$, then $J(F_\lambda)$ is a Sierpiński curve.

Finally, if v_λ does not lie in either B_λ or T_λ , then $J(F_\lambda)$ is a connected set.

Case 1 above occurs for all large enough $|\lambda|$, and the set of all such λ is known as the *shift locus*. This is the orange region in the outer portion of figure 1.1. Not only is the Julia set homeomorphic to a Cantor set, but the dynamics on the Julia set are conjugate to the shift map on the set of infinite sequences in n symbols, which cuts off the first entry of a sequence, sending $a_0a_1a_2a_3 \dots$ to $a_1a_2a_3 \dots$.

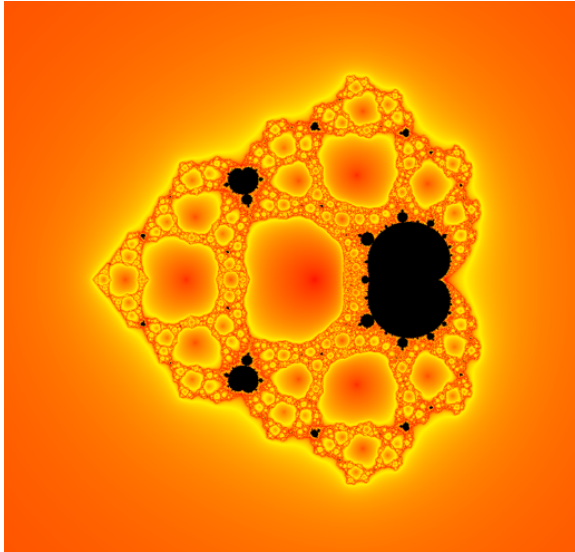


Figure 1.2: The parameter space for $n = d = 2$

Case 2 of the Escape Trichotomy result was proved by McMullen in [16], and we shall refer to the set of parameters for which this case occurs as the *McMullen domain*. This is an open topological disk containing the origin for $n \geq 3$, but is empty if $n = 2$. This is one of the reasons we restrict attention in this paper to the case $n \geq 3$ for much of this paper, with the exception of chapter 2. In figure 1.1, the McMullen domain is the small orange domain in the center of the image. Note that in figure 1.2 no McMullen domain exists. The origin is at the “tip of the tail” of the large Mandelbrot-like set straddling the real axis, and infinitely many distinct conjugacy classes exist in any neighborhood of the origin (see [6]).

The Sierpiński curve referenced in case 3 is defined as a planar set that is homeomorphic to the Sierpiński carpet fractal, which is formed by dividing a square into 9 equal sub-squares, removing the middle one, and repeating ad infinitum to each square remaining (see figure 1.3). In [22], a Sierpiński curve is characterized topologically as a planar set that is compact, connected, locally connected, nowhere dense, and with the property that its complementary domains are bounded by pairwise

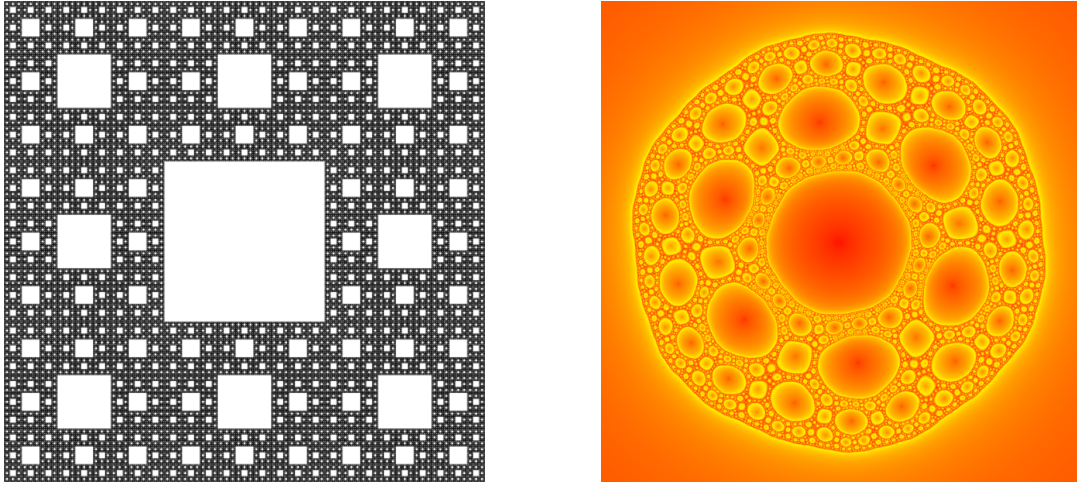


Figure 1.3: The Sierpiński carpet, and a Sierpiński curve Julia set

disjoint simple closed curves. A hyperbolic component of the parameter space for F_λ which corresponds to case 3 will be called a *Sierpiński hole*, and the parameter corresponding to the post-critically finite map for which the free critical orbit eventually lands exactly on infinity is called the *center* of the Sierpiński hole. In figure 1.1, each open orange domain other than the shift locus and the McMullen domain is a Sierpiński hole. Such maps are sometimes called *Escape Time Sierpiński* (ETS) maps, and are thoroughly examined in [18].

Maps for which the free critical orbit does not escape are colored black in figure 1.1. In many cases, these parameters are known to lie in a homeomorphic copy of the Mandelbrot set, via the theory of polynomial-like maps, such as the $n - 1$ large *principal* Mandelbrot sets that touch both the McMullen domain and the boundary of the shift locus when n is greater than 2 (see [1]), or smaller *baby* Mandelbrot sets (see [4]). Maps in the main cardioids of these Mandelbrot sets will have free critical orbits that tend to some finite attracting cycle. Those lying at the center of the main cardioid will correspond to post-critically finite maps, and we call such parameters *superstable*.

With regards to the family F_λ , we have two main goals here, which reflect two of the primary goals for the general study of families of complex dynamical systems as a whole. One goal regards the question of topological conjugacy. For two parameters λ and μ , when do the maps F_λ and F_μ have conjugate dynamics? Moreover, is there a natural way to describe the existence or lack of conjugacy? In chapter 4 we use a dynamical invariant which assigns combinatorial data to maps whose parameters are drawn from main cardioids of certain Mandelbrot sets that appear in the parameter plane. These are constructed so that only maps with conjugate dynamics share the same data, which allows us to count conjugacy classes as well as the number of post-critically finite maps in each class.

The next goal is to describe the structure of the parameter space. The organization of the Sierpiński holes and baby Mandelbrot sets in the λ plane is only partially understood. One portion of this structure which has been studied is that of so-called *Mandelpinski necklaces* which surround the McMullen domain when n is greater than 2. These have been studied in [12] and [4], and consist of an infinite collection of simple closed curves converging down to the boundary of \mathcal{M} which pass alternately through post-critically finite maps of two types: ETS parameters at the centers of Sierpiński holes, and superstable parameters at the centers of main cardioids of baby Mandelbrot sets. The k th such necklace passes through $\tau_k^n = (n-2)n^k + 1$ parameters of each type.

In chapters 2 and 3 we generalize this structure in two ways. First we give a new proof of the existence of the primary necklaces which holds for n equal to 2 in addition to n greater than 2. In this case the formula for τ_k^2 reduces to 1 for all k , so that the necklaces in this case are simple closed curves converging down to the origin, each of which pass through a single center of a Sierpiński hole and a single

superstable parameter. Secondly, we show that for n greater than 2, there is a fractal *subnecklace* structure, in which each Sierpiński hole on each Mandelpinski necklace is itself surrounded by infinitely many similar necklaces, each of which passes through centers of Sierpiński holes, each of which is again surrounded by infinitely many necklaces ad infinitum.

1.1 Preliminaries

From now on, unless otherwise stated, we shall assume λ is nonzero. $F'_\lambda(z) = nz^{n-1} - n\lambda/z^{n+1}$, so the finite critical points of F_λ are at 0 and the $2n^{\text{th}}$ roots of λ . As discussed above, 0 always maps locally n -to-1 onto the superattracting fixed point at infinity, which maps locally n -to-1 onto itself. The $2n$ other critical points lie on the *critical circle* of radius $|\lambda|^{1/2n}$. Denote these by $c_j^\lambda = c_j$, where $j\pi/n \leq \arg(c_j) < (j+1)\pi/n$. The critical points map to two critical values, $v_\pm^\lambda = \pm 2\sqrt{\lambda}$, where we declare $0 \leq \text{Arg } v_+^\lambda < \pi$. We shall often write v^λ for one of the two critical values in cases when it does not matter to which one we are referring.

F_λ is an even or odd map depending on the parity of n , and so these critical values will either map to the same point after one iterate (when n is even), or map to opposite values with symmetric orbits (when n is odd). Hence for each n , there is a single free critical orbit for the family. F_λ has one pole at 0, so the pre-poles which map onto infinity after exactly two iterates lie at the zeroes of F_λ , at the $2n^{\text{th}}$ roots of $-\lambda$. Hence the pre-poles also lie on the critical circle, one between each pair of critical points. It is this property that will give us the Mandelpinski necklace structure: parameters for which v^λ eventually maps onto a pre-pole will be centers of Sierpiński holes, and parameters for which v^λ eventually maps onto a critical point will be superstable, i.e. will be post-critically finite with a finite superattracting

cycle.

Basic mapping properties of F_λ are proven in e.g. [12], but we shall summarize some of them that will be useful to us here. Let l denote the line passing through the origin and both critical values, i.e. $l = \{tv_+^\lambda \mid t \in \mathbb{R}\}$. Define the following subsets of l :

$$l_0 = \{tv_+^\lambda \mid -1 \leq t \leq 1\}$$

$$l_+ = \{tv_+^\lambda \mid t \geq 1\}$$

$$l_- = \{tv_+^\lambda \mid t \leq -1\}$$

The critical circle maps onto l_0 , $2n$ -to-1, except at the critical values, where the map is n -to-1. All other circles centered at the origin are mapped onto ellipses with foci at the critical values, since an elementary computation shows that

$$\left|F_\lambda(re^{i\theta}) - 2\sqrt{\lambda}\right| + \left|F_\lambda(re^{i\theta}) + 2\sqrt{\lambda}\right| = 2\left(r^n + \frac{|\lambda|}{r^n}\right)$$

and the right-hand side is independent of θ .

The straight lines through the origin and each critical point map onto one of the two rays l_+ or l_- . The straight lines through the origin and each pre-pole are mapped onto the entire line l^\perp which is perpendicular to the critical line l . These are the degenerate cases of a more general property: Lines through the origin are mapped onto hyperbolas with foci at the critical values, with each ray from the origin to infinity mapping onto one of the two branches of such a hyperbola. This follows from a similar straightforward computation as above, which gives

$$\left|F_\lambda(re^{i\theta}) - 2\sqrt{\lambda}\right| - \left|F_\lambda(re^{i\theta}) + 2\sqrt{\lambda}\right| = 2\sqrt{|\lambda|} \cos\left(\frac{\arg \lambda}{2} - n\theta\right)$$

where the right hand side is independent of r .

Several key symmetries exist in both the dynamical plane and the parameter plane. Let $\nu = \exp \pi i/n$. Then $F_\lambda(\nu^j z) = (-1)^j F_\lambda(z)$, so the Julia set of F_λ has $2n$ -fold rotational symmetry.

Let $H_\lambda = \lambda^{1/n}/z$, taking the principal branch of $\lambda^{1/n}$. Then $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so the Julia set is also symmetric under $z \mapsto H_\lambda(z)$.

Finally, $\overline{F_\lambda(z)} = F_{\overline{\lambda}}(\overline{z})$, so that F_λ and $F_{\overline{\lambda}}$ are antiholomorphically conjugate, and hence the parameter plane is also symmetric via reflection across the real axis.

Chapter 2

Simple Necklaces for $n=2$

For singularly perturbed maps of the form $F_\lambda(z) = z^n + \lambda/z^n$ where $n \geq 2$, it turns out that the case $n = 2$ is very different from the case $n > 2$. One reason for this is that, when $n = 2$, as $\lambda \rightarrow 0$, the Julia sets of F_λ converge to the closed unit disk (the filled Julia set of z^2), but this does not occur when $n > 2$ [9]. A second crucial difference concerns the McMullen domain in the parameter plane. Recall that this is the punctured open disk surrounding the origin when $n \geq 3$ that consists of parameters for which the Julia sets of the corresponding maps are Cantor sets of simple closed curves, all of which are dynamically and topologically the same [16]. There is no such region when $n = 2$. Rather, in any neighborhood of 0 in the parameter plane, there are infinitely many different topological types of Julia sets [6]. And a third difference is that, when $n > 2$, the second images of the free critical points all tend to ∞ as $\lambda \rightarrow 0$ (this is what generates the McMullen domain), whereas when $n = 2$, the second images of the free critical points tend to $1/4$ as $\lambda \rightarrow 0$ (and $1/4$ is not in the basin of ∞ when λ is small).

In the case $n > 2$, there is an interesting structure that surrounds the McMullen domain. In [3], [5], and [12] it is shown that this domain in the parameter plane is surrounded by infinitely many disjoint simple closed curves \mathcal{S}_k for $k = 0, 1, 2, \dots$ called Mandelpinski necklaces. Each \mathcal{S}_k passes alternately through the centers of

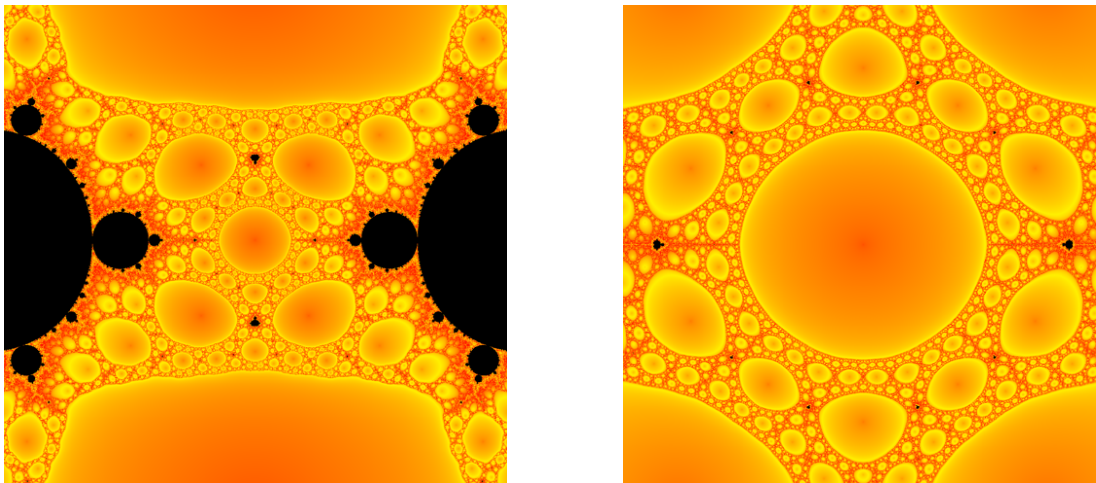


Figure 2.1: Magnifications of the parameter plane for the family $z^3 + \lambda/z^3$ around the McMullen domain (the central disk).

$(n-2)n^k + 1$ Mandelbrot sets with base period $k+1$ (with one slight exception when $k=1$) and the same number of centers of Sierpiński holes with escape time $k+3$. See Figure 2.1. A center of a Mandelbrot set of base period ℓ is a parameter in the main cardioid for which a critical point is periodic with prime period ℓ . A Sierpiński hole with escape time ℓ is a collection of parameters for which the critical orbits all land in the immediate basin of ∞ at iteration ℓ . A center of a Sierpiński hole is a parameter for which the critical orbits actually land at ∞ .

These Mandelpinski necklaces provide a great deal of structure around the McMullen domain when $n > 2$. For example, when $n = 3$, the necklace \mathcal{S}_{16} passes through exactly 43,046,722 Mandelbrot sets and Sierpiński holes. When the parameter lies in one of these Mandelbrot sets there are infinitely many small copies of quadratic Julia sets embedded in the much larger Julia set of F_λ . And when the parameter lies in a Sierpiński hole, the Julia set of F_λ is a Sierpiński curve, i.e., a set that is homeomorphic to the Sierpiński carpet fractal.

Because of the different behaviors of the critical orbits as $\lambda \rightarrow 0$ and the lack of

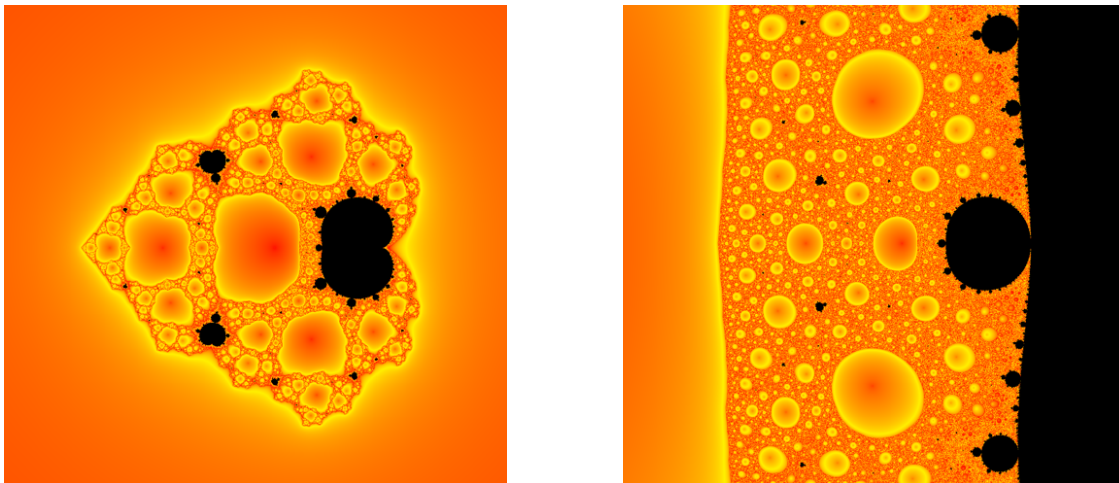


Figure 2.2: The parameter plane and a magnification around the origin for the family $z^2 + \lambda/z^2$. The large central disk is a Sierpiński hole, not the McMullen domain. The origin is located at the “tip of the tail” of the Mandelbrot set that appears to straddle the positive real axis.

a McMullen domain when $n = 2$, it was always assumed that there were no such Mandelpinski necklaces when $n = 2$. However, note that the above formula says that each necklace \mathcal{S}_k should pass through exactly $(2 - 2)2^k + 1 = 1$ Mandelbrot set and 1 Sierpiński hole when $n = 2$ for each k . In fact, as we show in this paper, this does indeed happen. So we do have some simplified Mandelpinski necklaces in this case. These necklaces no longer surround a McMullen domain; rather, they converge to the origin as $k \rightarrow \infty$. So the structure of the parameter plane around the origin when $n = 2$ is very different from the case $n > 2$. See Figure 2.2. We conjecture that the existence of these simple necklaces will allow us to begin to understand the very complicated structure of the parameter plane for $n = 2$ around the origin, just as the Mandelpinski necklaces did in the case $n > 2$.

2.1 Preliminaries

In this paper we shall concentrate on the family of complex rational maps given by

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

where $\lambda \in \mathbb{C}$. One checks easily that the point at ∞ is fixed in the Riemann sphere and $F'_\lambda(\infty) = 0$ so ∞ is a superattracting fixed point. We denote the immediate basin of attraction at ∞ by B_λ . Since 0 is a pole there is an open set about 0 that is mapped to B_λ . This set may or may not be disjoint from B_λ , but in the cases we consider in this paper, these two sets will be disjoint. We then call the preimage of B_λ surrounding 0 the *trap door* and denote this set by T_λ .

It is well known that the *Julia set* of F_λ , denoted by $J(F_\lambda)$, has several equivalent definitions [17]. One definition is that $J(F_\lambda)$ consists of all points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. A second definition is that the Julia set is the closure of the set of repelling periodic points of F_λ . And a third definition is that the Julia set is the boundary of the set of all points whose orbits tend to ∞ (not just those in B_λ). These definitions then imply that $J(F_\lambda)$ is the chaotic regime since arbitrarily close to any point in the Julia set there are points whose orbits tend to ∞ and other points whose orbits are periodic. More importantly, Montel's Theorem implies that any neighborhood of a point in $J(F_\lambda)$ is mapped over the entire Riemann sphere by the set of all iterates of maps in this family. So, on the Julia set, F_λ exhibits extreme sensitive dependence on initial conditions.

There are several symmetries in the dynamical and parameter planes for these maps. We have $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$. Therefore the orbits of z and

iz are the same after two iterations. As a consequence, the Julia set is symmetric under the map $z \mapsto iz$, i.e., $J(F_\lambda)$ has fourfold symmetry. Also, let $H_\lambda(z) = \lambda^{1/2}/z$. Then $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, so the Julia set is also symmetric under the involution H_λ . We also have that F_λ is conjugate to $F_{\bar{\lambda}}$ via the map $z \rightarrow \bar{z}$, so the parameter plane is symmetric under complex conjugation.

A straightforward computation shows that there are four free critical points for F_λ that are given by $\lambda^{1/4}$. We call these critical points “free” since there are two other critical points at ∞ and 0 , but ∞ is fixed and 0 maps directly to ∞ for each λ . However, there are only two critical values given by $\pm 2\sqrt{\lambda}$ since two of the free critical points are mapped to $+2\sqrt{\lambda}$ and the other two are mapped to $-2\sqrt{\lambda}$. In fact, just like the quadratic polynomial family $z^2 + c$, there really is only one free critical orbit as both critical values are then mapped to $4\lambda + 1/4$, so all of the critical points end up on the same orbit after two iterations.

There are also four prepoles for F_λ given by $(-\lambda)^{1/4}$. So the prepoles and critical points all lie on the circle of radius $|\lambda|^{1/4}$ centered at the origin. We call this circle the *critical circle* and denote it by C_0^λ . Another easy computation shows that F_λ maps the critical circle 4-to-1 onto the line segment connecting the two critical values $\pm 2\sqrt{\lambda}$ and passing through the origin. We call this line the *critical segment*. Any other circle centered at the origin is then mapped as a 2-to-1 covering onto an ellipse whose foci are the critical values. In particular, the region in the exterior of the critical circle is then mapped as a 2-to-1 covering onto the complement of the critical segment in the Riemann sphere and so too is the interior of the critical circle.

We shall assume for the remainder of this paper that the critical values both lie on or inside the critical circle, so the critical segment will always lie in the the disk bounded by the critical circle. It is known [2], [14] that, in this case, $J(F_\lambda)$ is

connected and that ∂B_λ is a simple closed curve lying in the exterior of C_0^λ . Since $H_\lambda(B_\lambda) = T_\lambda$, we have that ∂T_λ is also a simple closed curve that lies inside C_0^λ .

Let \mathcal{O} be the punctured disk in the parameter plane that consists of all nonzero parameters for which the critical segment lies strictly inside the critical circle. When λ lies on the boundary of \mathcal{O} , we must have $2|\sqrt{\lambda}| = |\lambda|^{1/4}$, so it follows that $|\lambda| = 1/16$. Therefore the boundary of \mathcal{O} is the circle of radius $1/16$ centered at the origin in the parameter plane. For $\lambda \in \mathcal{O}$, F_λ maps the exterior of the critical circle as a 2-to-1 covering onto the exterior of the critical segment. Thus there is a simple closed curve in the exterior of C_0^λ that is mapped 2-to-1 onto C_0^λ . Call this curve C_1^λ . Since C_0^λ contains four critical points and four prepoles, C_1^λ contains eight pre-critical points and eight pre-prepoles. Since the exterior of C_1^λ is then mapped onto the exterior of C_0^λ as a 2-to-1 covering, there is another simple closed curve C_2^λ that lies outside C_1^λ and is mapped 2-to-1 onto C_1^λ . Continuing in this fashion, we find an infinite collection of simple closed curves C_k^λ for $k > 0$ satisfying $F_\lambda(C_k^\lambda) = C_{k-1}^\lambda$ and hence $F_\lambda^k(C_k^\lambda) = C_0^\lambda$. Note that the C_k^λ are all disjoint and these curves converge outward toward ∂B_λ as $k \rightarrow \infty$. This follows since, if this were not the case, the limiting set of the C_k^λ would be a closed, invariant set, say Λ_λ . If $\Lambda_\lambda \neq \partial B_\lambda$, then the region bounded by ∂B_λ and Λ_λ would also be invariant. But this cannot happen since there would then be points in ∂B_λ that have neighborhoods on which the family of functions $\{F_\lambda^k\}$ would be normal, which cannot happen since $\partial B_\lambda \subset J(F_\lambda)$. In addition, C_k^λ contains 2^{k+2} points that are mapped by F_λ^k to critical points and the same number of points that are mapped to the prepoles on C_0^λ . The points that map to critical points and to prepoles are arranged alternately around C_k^λ .

Since the interior of the critical circle is also mapped as a 2-to-1 covering of the exterior of the critical segment, there are other simple closed curves C_{-k}^λ for

$k = 1, 2, \dots$ such that F_λ maps C_{-k}^λ as a 2-to-1 covering of C_{k-1}^λ just as above. We have $H_\lambda(C_{-k}^\lambda) = C_k^\lambda$. The C_{-k}^λ now converge down to ∂T_λ as $k \rightarrow \infty$. And, just as above, C_{-k}^λ contains exactly 2^{k+2} points that are mapped to critical points and the same number to prepoles by F_λ^k .

2.2 Rings in the Parameter Plane

In this section, we prove that the origin in the parameter plane is surrounded by infinitely many disjoint simple closed curves \mathcal{S}_k with the \mathcal{S}_k converging to 0 as $k \rightarrow \infty$. The curve \mathcal{S}_k will consist of parameters for which the critical orbit lands on the critical circle after exactly $k + 1$ iterations in a manner specified below. We shall show that \mathcal{S}_k contains exactly one parameter for which one of the critical points is periodic with period $k + 1$. Results in [7] shows that this parameter is a center of the main cardioid of a Mandelbrot set in the parameter plane (with two exceptions noted at the end of this section). And we shall show that there is one other parameter in \mathcal{S}_k for which the critical orbits all land on ∞ at iteration $k + 3$. It is known [19] that this parameter is then the center of a Sierpiński hole with escape time $k + 3$. The parameters that are centers of a main cardioid of a Mandelbrot set will lie in \mathbb{R}^+ while the parameters that are centers of a Sierpiński hole will lie in \mathbb{R}^- .

We first describe the ring \mathcal{S}_0 in the parameter plane. This curve consists of λ -values for which the critical values lie on the critical circle C_0^λ in the dynamical plane. So, on this set, we must have $|\lambda|^{1/4} = 2|\sqrt{\lambda}|$. Solving this equation shows that \mathcal{S}_0 is the circle of radius $1/16$ centered at the origin in the parameter plane, i.e., the boundary of \mathcal{O} . When $\lambda \in \mathcal{S}_0$, the critical circle C_0^λ is the circle of radius $1/2$ centered at the origin. Note that, as λ rotates around \mathcal{S}_0 , the critical points and prepoles each rotate around C_0^λ by a quarter of a turn while the critical values

rotate by half a turn. It then follows that there is exactly one parameter in \mathcal{S}_0 for which the critical values land on a critical point, namely $\lambda = 1/16$, and one other parameter for which they land on a prepole, namely $\lambda = -1/16$. So, for $\lambda = 1/16$, F_λ has a superattracting fixed point while, for $\lambda = -1/16$, the critical orbit escapes at iteration 3. This gives the result for \mathcal{S}_0 .

For $\lambda \in \mathcal{O}$ with $0 \leq \text{Arg } \lambda < 2\pi$, let $c_0^\lambda = \lambda^{1/4}$ denote the critical point satisfying $0 \leq \text{Arg } c_0^\lambda < \pi/2$ and let c_j^λ , $j = 1, 2, 3$ denote the other three critical points where the c_j^λ are arranged in the counterclockwise direction around the origin. Let I_0^λ denote the closed sector in \mathbb{C} bounded by the two critical point rays that are given by tc_0^λ and tc_3^λ with $t \geq 0$. Let I_j^λ denote the similar sector bounded by tc_{j-1}^λ and tc_j^λ . Note that the interior of each I_j^λ is mapped one-to-one onto \mathbb{C} minus the two critical value rays given by $\pm tv_\lambda$, $t \geq 1$. One of the critical point rays that bounds each I_j^λ is mapped onto one of these critical value rays while the other critical point ray is mapped to the other critical value ray. Note also that, when $\lambda \in \mathbb{R}^+$, the critical value rays lie in $I_0^\lambda \cap I_1^\lambda = \mathbb{R}^+$ and $I_2^\lambda \cap I_3^\lambda = \mathbb{R}^-$. For all other λ -values, one of the critical value rays lies in the interior of I_1^λ while the other lies in the interior of I_3^λ . See Figure 2.3.

Since C_0^λ is an actual circle, we may define a natural parametrization $C_0^\lambda(\theta)$ of this curve by setting $C_0^\lambda(0) = c_0^\lambda$. We choose this parameterization so that $C_0^\lambda(\theta)$ rotates in the clockwise direction as θ increases. Here we again assume that $0 \leq \text{Arg } \lambda < 2\pi$. Let η_0^λ be the portion of C_0^λ that lies inside I_0^λ , i.e., $\eta_0^\lambda(\theta) = C_0^\lambda(\theta)$ where $0 \leq \theta \leq \pi/2$. Now the sector I_0^λ is mapped over itself univalently (except when $\lambda \in \mathbb{R}^+$ in which case one boundary curve is mapped 2-to-1 to a portion of \mathbb{R}^+). In all cases there is then a smooth curve η_1^λ lying in $C_1^\lambda \cap I_0^\lambda$ that is mapped univalently onto η_0^λ . We define $\eta_1^\lambda(\theta)$ to be the point on this portion of C_1^λ that is mapped to $\eta_0^\lambda(\theta)$. Inductively, we

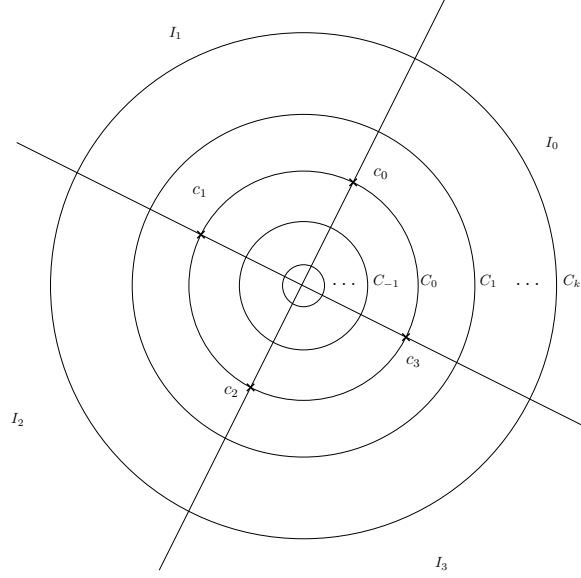


Figure 2.3: The critical circle and its preimages together with the sectors I_j .

then define $\eta_k^\lambda(\theta)$ to be the point in $C_k^\lambda \cap I_0^\lambda$ for which $F_\lambda(\eta_k^\lambda(\theta)) = \eta_{k-1}^\lambda(\theta)$ for each $k \geq 1$. Then we let $\eta_{-k}^\lambda(\theta) = H_\lambda(\eta_k^\lambda(\theta))$ where H_λ is the involution $z \rightarrow \lambda^{1/2}/z$ that fixes the critical points $\pm c_0^\lambda$. So $F_\lambda(\eta_{-k}^\lambda(\theta)) = \eta_{k-1}^\lambda(\theta)$. One checks easily that H_λ interchanges I_0^λ and I_1^λ , so $\eta_{-k}^\lambda(\theta)$ lies in $I_1^\lambda \cap C_{-k}^\lambda$ for each $k > 0$.

Lemma. *Given $k > 0$, there exists $\lambda^* > 0$ such that, if $|\lambda| \leq \lambda^*$, then both critical values of F_λ lie strictly inside the curve C_{-k}^λ .*

Proof: For $|\lambda|$ sufficiently small, the critical circle C_0^λ has magnitude that is very small. Since $F_\lambda \approx z^2$ away from the origin when $|\lambda|$ is small, we may choose λ_1 so that, if $|\lambda| < \lambda_1$, then the closed curve C_{k-1}^λ lies strictly inside the circle of radius $1/8$ surrounding the origin. Also, since $F_\lambda(v_\lambda) = 1/4 + 4\lambda$, we may choose λ_2 so that, if $|\lambda| < \lambda_2$, then $|F_\lambda(v_\lambda)| > 1/8$. Let $\lambda^* = \min(\lambda_1, \lambda_2)$. Then we have that, for each λ inside the circle of radius λ^* , the image of the critical value lies outside the circle of radius $1/8$ and hence outside C_{k-1}^λ . Therefore $\pm v_\lambda$ lies strictly inside the closed curve C_{-k}^λ .

□

We now define the rings \mathcal{S}_k for $k \geq 1$ in the parameter plane. Recall that \mathcal{O} is the set of nonzero parameters for which v_λ lies inside the critical circle.

Proposition. *Suppose $\lambda \in \mathcal{O}$. Fix $k \geq 1$ and θ in the interval $[0, \pi/2]$. Then there is a unique parameter $\lambda = \lambda_\theta^k$ in \mathcal{O} for which a critical value lies at the point $\eta_{-k}^\lambda(\theta)$. Moreover, λ_θ^k varies continuously with θ and $\lambda_0^k = \lambda_{\pi/2}^k$ is a parameter in \mathbb{R}^+ .*

Proof: Since \mathcal{O} is the open disk of radius $1/16$ with the origin removed, we have the universal covering half-plane $\tilde{\mathcal{O}}$ given by $\operatorname{Re} z < \log(1/16)$. We then have two maps defined on $\tilde{\mathcal{O}}$.

The first is a map that we shall denote by $\tilde{V}(\lambda)$. To define this map, let v_λ be the critical value that lies in the upper half plane when $0 < \operatorname{Arg} \lambda < 2\pi$. Clearly, the map $\lambda \mapsto v_\lambda$ is not well-defined on \mathcal{O} since v_λ moves to $-v_\lambda$ as $\operatorname{Arg} \lambda$ rotates from 0 to 2π . However, we can lift this map to a new map $V : \tilde{\mathcal{O}} \rightarrow \mathcal{X}$ where \mathcal{X} is the annulus $0 < |z| < 1/2$ so that V agrees with the map $\lambda \mapsto v_\lambda$ when $0 < \operatorname{Arg} \lambda < 2\pi$. Let $\tilde{\mathcal{X}}$ be the universal covering of \mathcal{X} . Then we can lift V to a map $\tilde{V} : \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{X}}$. Note that \tilde{V} is an analytic, invertible map on $\tilde{\mathcal{O}}$, and, as the argument of λ increases by 2π in \mathcal{O} , the imaginary part of \tilde{V} in $\tilde{\mathcal{X}}$ increases by exactly π .

For fixed values of $k \geq 1$ and $\theta \in [0, \pi/2]$, we also have the map $\lambda \mapsto \eta_{-k}^\lambda(\theta)$ defined when $0 \leq \operatorname{Arg} \lambda < 2\pi$. Again this map is not well-defined on \mathcal{O} , but we can lift it to a new map $L : \tilde{\mathcal{O}} \rightarrow \mathbb{C}$ as above. By construction, $L(\lambda)$ is strictly contained inside the annulus \mathcal{X} . So we may lift L to a map $\tilde{L} : \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{X}}$.

As we have shown, the point $\eta_{-k}^\lambda(\theta)$ is contained in the sector I_1^λ as long as $\theta \in [0, \pi/2]$. And this sector rotates by exactly $\pi/2$ radians as $\operatorname{Arg} \lambda$ increases from 0 to 2π . Moreover, the argument of $\eta_{-k}^\lambda(\theta)$ never increases by π as λ rotates around the origin, since this would imply that this point visited both the positive and negative

real axis enroute. Hence the argument of $\tilde{L}(\lambda)$ increases by an amount strictly less than π as the argument of λ increases by 2π .

Now, since \tilde{V} is invertible, we may consider the composition $\Phi = \tilde{V}^{-1} \circ \tilde{L} : \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{O}}$. We claim that there is a unique fixed point for Φ in $\tilde{\mathcal{O}}$. To see this, first note that we may extend both \tilde{V} and \tilde{L} to the boundary $\text{Re } z = 1/16$ of $\tilde{\mathcal{O}}$ and Φ maps this boundary strictly inside $\tilde{\mathcal{O}}$. This follows since η_{-k}^λ always lies strictly inside the critical circle for F_λ , which then lies inside the critical circle for F_μ when μ lies on the boundary of \mathcal{O} . But this is the circle $r = 1/2$. Hence there are no fixed points on the boundary of $\tilde{\mathcal{O}}$. Next note that there are no fixed points in the far left half-plane in $\tilde{\mathcal{O}}$. This follows immediately from the previous Lemma. Finally, since the argument of \tilde{V} increases by more than the argument of \tilde{L} as λ rotates around the origin, it follows that Φ must have a fixed point in $\tilde{\mathcal{O}}$ and, by the Schwarz Lemma, this fixed point must be unique. Then the projection of this point into \mathcal{O} is a parameter λ for which a critical value lands on the point $\eta_{-k}^\lambda(\theta)$. This is the parameter λ_θ^k . Since all of the above varies continuously with θ , it follows that $\theta \mapsto \lambda_\theta^k$ traces out a continuous curve in \mathcal{O} .

Now when $\lambda \in \mathbb{R}^+$, elementary arguments using real dynamics shows that there is a superstable parameter value for which

$$v_\lambda < c_0^\lambda = F_\lambda^{k+1}(c_0^\lambda) < F_\lambda^k(c_0^\lambda) < \dots < F_\lambda^2(c_0^\lambda).$$

This is then the parameter λ_0^k . Similarly, when $\text{Arg } \lambda = 2\pi$, the above result shows that we have a similar unique parameter for which v_λ lies in \mathbb{R}^- and then maps onto the same superattracting cycle. This is now the parameter $\lambda_{\pi/2}^k$. But this then implies that $\lambda_0^k = \lambda_{\pi/2}^k$. Note that these are the only two θ -values for which two “different” λ_θ^k 's coincide. Therefore the parameters λ_θ^k lie along a simple closed curve

surrounding the origin in the parameter plane.

□

Thus we may define the ring \mathcal{S}_k to be the simple closed curve parametrized by $\theta \mapsto \lambda_\theta^k$.

Corollary. *There is a unique parameter in \mathcal{S}_k for which a critical point lies on a superattracting cycle of period $k + 1$ and another unique parameter for which the critical orbits escape at iteration $k + 3$.*

Proof: As shown above, the parameter $\lambda_0^k = \lambda_{\pi/2}^k$ is the unique parameter in \mathcal{S}_k for which $F_\lambda^k(v_\lambda)$ lands on a critical point in the curve η_0^λ . There is also a unique parameter for which $F_\lambda^k(v_\lambda)$ lands on the prepole in η_0^λ and hence the critical orbit escapes at iteration $k+3$. The graph of the real function F_λ shows that this parameter lies in \mathbb{R}^- .

□

Remarks:

1. Note that the parameter λ_0^0 is the parameter for which c_0^λ is a superattracting fixed point and this parameter appears to lie at the center of the main cardioid of a Mandelbrot set that straddles the positive real axis. However, this is not quite a “full” Mandelbrot set, as the tip of the tail (i.e., the parameter corresponding to $c = -2$ for the quadratic Mandelbrot set) lies at the origin, so the dynamics associated to this parameter do not correspond to those for the parameter $c = -2$. We conjecture that this is the only portion of the Mandelbrot set that is missing.
2. Also, the parameter value λ_0^1 does not lie at the center of a main cardioid of a baby Mandelbrot set; rather, this parameter lies at the center of the period 2 bulb of the above Mandelbrot set.
3. All other parameters λ_0^k do lie at the center of a baby Mandelbrot set that lies

inside the main Mandelbrot set on the real axis. This follows from a polynomial-like map construction. See [7] for details.

Chapter 3

Subnecklaces Around the McMullen Domain

In [12] and [4], a structure in the parameter space near the McMullen domain is described. There is an annular region surrounding the McMullen domain containing a countably infinite collection of simple closed curves $\{S_k\}$, each containing the next in its interior, and converging to $\partial\mathcal{M}$ as k tends to infinity, with the property that S_k passes through the centers of $(n-2)n^k + 1$ Sierpiński holes, and the same number of centers of main cardioids of baby Mandelbrot sets. For this reason, these curves were named *Mandelpinski necklaces*, and in this discussion, we shall refer to them as *primary necklaces*. In this chapter, we prove that this structure persists at deeper and deeper levels, i.e. that each Sierpiński hole on each primary necklace necklace is itself surrounded by infinitely many sub-necklaces, and each Sierpiński hole on each of those is itself surrounded by infinitely many sub-necklaces, ad infinitum.

3.1 The Dynamical Plane

As discussed in [18], consider the $2n$ rays $\{\rho_{(j)} | j = 0, 1, \dots, 2n-1\}$, where ρ_j extends from the origin through the critical point c_j . Define the open sector $R_{(j)}$ to be bounded by $\rho_{(j)}$ and $\rho_{(j+1)}$, together with a circle lying entirely within the trap door

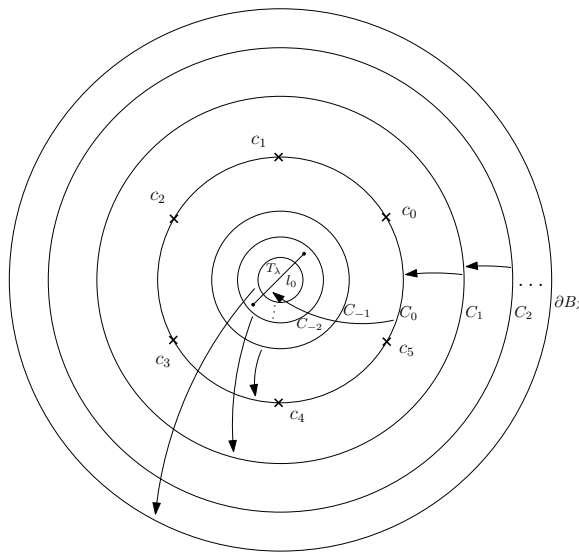


Figure 3.1: Structure in the dynamical plane

and another circle lying entirely in the basin of infinity. Suppose $\text{Arg } \lambda \neq 2q\pi/(n-1)$ for any integer q , so that the critical values do not lie on any of these rays. In each opposite pair of sectors which do not contain critical values, there exists an invariant cantor necklace, and by taking first preimages of this necklace, we partition the complex plane into $2n$ open “dynamical sectors.” This partition depends on a choice of a pair of R_i sectors, and it will be convenient here to use R_0 and R_n , i.e. the partition bounded by preimages of the invariant cantor necklace which lies in $R_0 \cup R_n$ (in fact, the critical values never lie in this union, so we can always do this).

With respect to this partition, denote the dynamical sector containing the critical point c_j by I_j .

Now we characterize certain pre-images of the critical circle in the dynamical plane. Much of the following discussion appears in [12] but we recall the construction here with slightly modified notation.

Denote the critical circle of radius $|\lambda|^{1/2n}$ by C_0 . Suppose that the critical values v_\pm^λ lie in the interior of C_0 . Since C_0 maps $2n$ -to-1 onto the line segment l_0 between

the critical values, the critical circle then maps entirely inside itself. Both the interior O^+ and the exterior O^- of the critical circle map n -to-1 onto the complement of this line segment. Since the critical circle itself lies in this complement, O^+ and O^- each contain a preimage of C_0 . Denote the preimage in O^+ by C_1 and the preimage in O^- by C_{-1} . These curves contain no critical points, since they are disjoint from C_0 , 0 and infinity, and hence each is a simple closed curve covering C_0 n -to-1. Moreover, each must wrap around the origin, since the critical point 0 lies inside C_0 .

Now consider the curve C_1 in O^+ . This curve also lies in the complement of l_0 . Hence like C_0 , C_1 has a preimage in O^+ and a preimage in O^- . As before, these preimages are simple closed curves surrounding the origin, and we denote them by C_2 and C_{-2} respectively. Clearly C_2 must lie in the exterior of C_1 and C_{-2} in the interior of C_{-1} . Continuing in this fashion, there exist infinitely many curves C_k , where the curve C_k in O^+ has a preimage $C_{-(k+1)}$ surrounding the origin in O^- and a preimage C_{k+1} surrounding the origin in O^+ . This yields a collection C_k , $k = 0, \pm 1, \pm 2, \dots$, of concentric simple closed curves surrounding the origin, with the property that C_k maps n -to-1 onto $C_{|k|-1}$. Moreover, as $k \rightarrow \infty$, the curves accumulate on ∂B_λ (and likewise for ∂T_λ as $k \rightarrow -\infty$, by symmetry) (see [12]).

Each of these curves inherits a natural parametrization. Since C_0 is an actual circle, let

$$C_0(\theta) = |\lambda|^{1/2n} \exp \left(i \left(-\theta + \frac{\text{Arg}(\lambda)}{2n} \right) \right)$$

so that $\theta = 0$ corresponds to the critical point c_0^λ , and $C_0(\theta)$ is periodic of period 2π , with θ increasing in the atypical clockwise direction.

Coordinates on C_k are defined as follows: Let $C_k(0)$ for $k > 0$ be the unique k th

preimage of $C_0(0)$ which lies in the sector

$$\frac{\text{Arg } \lambda - \pi}{2n} < \text{Arg } z < \frac{\text{Arg } \lambda + \pi}{2n}$$

(i.e., the sector of points within one $2n$ th of a turn from c_0^λ). Then define $C_k(\theta)$ to be continuous in θ , with $F_\lambda(C_k(\theta)) = C_{k-1}(\theta)$. For $k < 0$, let $C_k(\theta) = H_\lambda(C_{-k}(\theta))$. Note that for $k < 0$, θ now increases in the counter-clockwise direction. Note also that $C_k(\theta)$ is $2n|k|\pi$ -periodic.

We shall denote by A_k the open annulus bounded by C_{k-1} and C_k . Note that A_k maps as an n -to-1 covering of A_{k-1} for $k \geq 2$, while A_{-k} maps as an n -to-1 covering of A_k for $k \geq 1$. A_0 and A_1 each map onto $O^- \setminus l_0$.

Now suppose moreover that the critical values v^λ lie inside C_{-1} as defined above, but outside the boundary of the trap door ∂T_λ . This is equivalent to taking λ to lie in the annulus Λ bounded by S_1 and the boundary of the McMullen domain, where S_1 is the simple closed curve around the origin which consists of parameters for which the critical values lie on C_{-1} , and which was proved to exist in [12]. Unless otherwise noted, λ shall lie in Λ for the remainder of the discussion.

Recall that for a domain D and $z_0 \in D$, we say that D is a *star-shaped domain with respect to z_0* if, for all $z \in D$, the line segment from z_0 to z lies entirely within D .

Lemma 3.1.1. *If $\lambda \in \Lambda$, then the interior of C_{-1} is a star-shaped domain with respect to the origin. In particular, the entire critical segment l_0 lies in the interior of C_{-1} .*

Proof. Recall that each ray from the origin maps to a single branch of a hyperbola with foci at the critical values. Any such branch intersects the critical circle C_0 in exactly two points, and intersects the critical segment l_0 in between, at its vertex.

Pulling back under F_λ , this gives each ray from the origin intersecting the full preimage of C_0 in exactly two points, one on C_{-1} and one on C_1 , since the preimage of the vertex of the hyperbola branch must lie on C_0 . Hence no ray from the origin intersects C_{-1} in more than one point. For any z in the interior of C_{-1} , the segment from the origin to z then lies entirely inside the interior of C_{-1} , since otherwise the full ray would intersect C_{-1} at least twice. \square

By the above lemma, when λ lies in Λ , C_0 maps strictly inside C_{-1} , and C_{-1} maps onto C_0 , so that A_0 maps over itself. There is thus a preimage of C_{-1} in A_0 , and the annulus between this preimage and which maps *strictly* over itself after two iterates. A standard quasiconformal surgery argument then gives the existence of a forward-invariant simple closed curve lying in this annulus, and hence inside A_0 , which we will denote by γ_0 . Moreover, $F_\lambda|_{\gamma_0}$ is quasiconformally conjugate to $z \mapsto z^{-n}$ via a homeomorphism ϕ . γ_0 therefore contains $n + 1$ of the fixed points for F_λ , and ϕ is unique up to rotation by $2\pi k/(n + 1)$.

γ_0 is contained in the open annulus A_0 , and thus cannot intersect l_0 . As in the discussion of the C -curves above, γ_0 then has preimages that are simple closed curves surrounding the origin in both O^- and O^+ , but here note that the preimage in O^- is just γ_0 itself. Denote the preimage in O^+ by γ_1 , and note that this curve lies in A_1 . γ_1 then has preimages in both A_2 and A_{-1} , which we denote γ_2 and γ_{-1} respectively. Continuing in this fashion, we get another doubly infinite sequence γ_k of concentric simple closed curves surrounding the origin, where γ_k covers γ_{k-1} n -to-1 for $k \geq 1$ and γ_{-k} covers γ_k n -to-1 for $k \geq 0$. Denote by B_k the open annulus between γ_k and γ_{k+1} . Note that B_0 maps $2n$ -to-1 onto the interior of the curve γ_0 , a domain we denote by Ω . F_λ maps B_k onto $B_{|k|-1}$ as an n -fold covering map for $k \neq 0$.

Let $\gamma_0(0)$ be the fixed point on γ_0 closest to the real axis, and use ϕ to induce

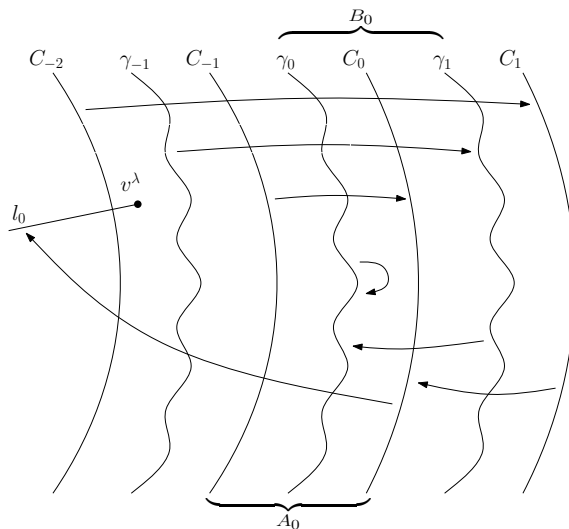


Figure 3.2: Dynamics of the preimages of the critical circle and of the invariant curve

a natural parametrization $\gamma_0(\theta)$ on γ_0 , with $0 \leq \theta < 2\pi$. As with the C_k above, we can pull back this parametrization to all γ_k via F_λ and the involution H_λ .

Previously we constructed the preimages of C_k and γ_k for $k \geq 0$. Now we turn our attention to the preimages of the inner curves. Recall that the annulus B_0 maps $2n$ -to-1 onto Ω , which contains C_k and γ_k for $k < 0$. Thus B_0 contains preimages of all of these curves. Their topology, however, depends on the location of v^λ . Suppose the critical values lie in the annulus B_{k^*} . Then the full preimage of C_k for $k < k^*$, γ_k for $k \leq k^*$, and ∂T_λ itself, consist of $2n$ disjoint simple closed curves surrounding the pre-poles, each mapped 1-to-1 by F_λ . For $k^* < k < 0$, the preimage of C_k and γ_k is a pair of simple closed curves, one in $B_0 \cap A_0$ and one in $B_0 \cap A_1$, mapped as an n -to-1 covering by F_λ . The preimage of C_{k^*} itself may be one of the above two cases, or a pair of curves in B_0 “pinched” together at each critical point, depending on whether v^λ lies inside, outside, or exactly on C_{k^*} .

For now we’ll focus our attention on the first case, i.e. the preimages of those

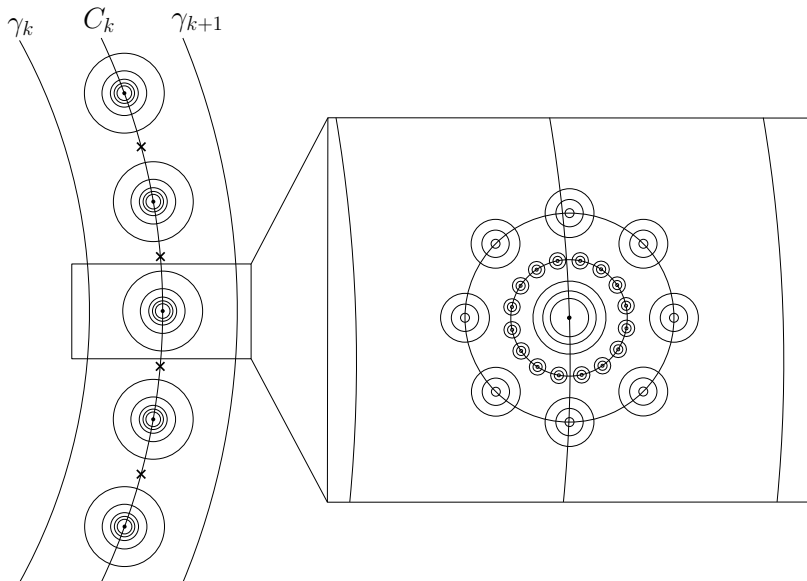


Figure 3.3: Structure in the annulus B_k . Note there are really infinitely many small simple closed curves around each preimage of 0 (marked with dots).

curves which do not surround the critical values. For $k < k^*$, denote by $C_{0(i),k}$ the preimage of C_k which surrounds the pre-pole $C_0\left(\frac{2\pi(2i+1)}{4n}\right)$, where $0 \leq i < 2n$. Each such preimage inherits natural coordinates from its image, so that $F_\lambda(C_{0(i),k}(\theta)) = C_k(\theta)$. Note we can define and parametrize preimages of γ_k for $k \leq k^*$ analogously.

The picture is therefore this: B_0 contains C_0 , which in turn contains the $2n$ critical points and the $2n$ pre-poles. Each pre-pole is surrounded by a preimage of T_λ , which is in turn surrounded by infinitely many simple closed curves, preimages alternately of C_k and γ_k .

Now recall that for $k \neq 0$, F_λ maps B_k onto $B_{|k|-1}$ as an n -fold covering and so B_k covers B_0 $n^{|k|}$ -to-1 under $F_\lambda^{|k|}$. Thus B_k has $2n^{|k|+1}$ $|k|+1$ st preimages of the trap door, each containing a preimage of a pre-pole, and each surrounded by $|k|+1$ st preimages of C_k for $k < k^*$ and of γ_k for $k \leq k^*$. We will now assume $k < 0$, since we are interested in the regions that can potentially contain the critical values.

By the parametrization of C_k , the preimages in B_k of the origin under F_λ^{-k+1} lie at $C_k\left(\frac{2\pi(2i+1)}{4n}\right)$, with $0 \leq i < 2n^{-k+1}$. Letting $k = k_0$, we can denote the preimage of C_{k_1} surrounding such a point on C_{k_0} by $C_{k_0(i),k_1}$, with $0 \leq i_0 < 2n^{|k|+1}$, as in B_0 . Again, natural coordinates can be pulled back to each of these curves, and to preimages of γ_0 .

We can now continue in this fashion, because the above structures in B_k for $k < 0$ all lie in Ω , and thus can be pulled back under F_λ to B_0 . For every $k < k^*$, each preimage of C_k in B_0 is a simple closed curve surrounding a pre-pole, which itself contains $2n^{-k+1}$ preimages of the origin, each of which is in turn surrounded by infinitely many simple closed curves which map to C_j for each $j < k^*$ under F_λ^{-k+2} . This structure then pulls back to each other annulus. See figure 3.3.

Since clearly we can repeat this process ad infinitum, we shall define the following notation. Inductively, $C_{k_0(i_0),k_1(i_1),\dots,k_{q-1}(i_{q-1}),k_q}$ will be a curve in B_{k_0} which maps to $C_{k_1(i_1),\dots,k_{q-1}(i_{q-1}),k_q}$ under $F_\lambda^{|k_0|+1}$, where $k_\alpha < k^*$ for each $0 < \alpha \leq q$, and $0 \leq i_\alpha < 2n^{|k_\alpha|+1}$. We call such curves *dynamical q -necklaces*. Each dynamical q -necklace is a simple-closed curve containing some number of preimages of the origin, each contained in a preimage of T_λ . These preimages of T_λ are therein surrounded by infinitely many dynamical $(q+1)$ -necklaces.

3.2 The Parameter Plane

By results in [12] and [7] we have curves in parameter space S_k and G_k , $k \leq 0$, parametrized by θ , such that $\lambda = S_k(\theta)$ if and only if $v_\lambda = C_k(\theta)$, and similarly for G_j and γ_j ([7] covers the case $n = 3$, but a completely analogous argument gives the result for $n \geq 3$). This partitions the annulus between $\partial\mathcal{M}$ and G_{-1} into infinitely many sub-annuli. In the annulus Γ_k lies the curve S_k consisting of parameters with

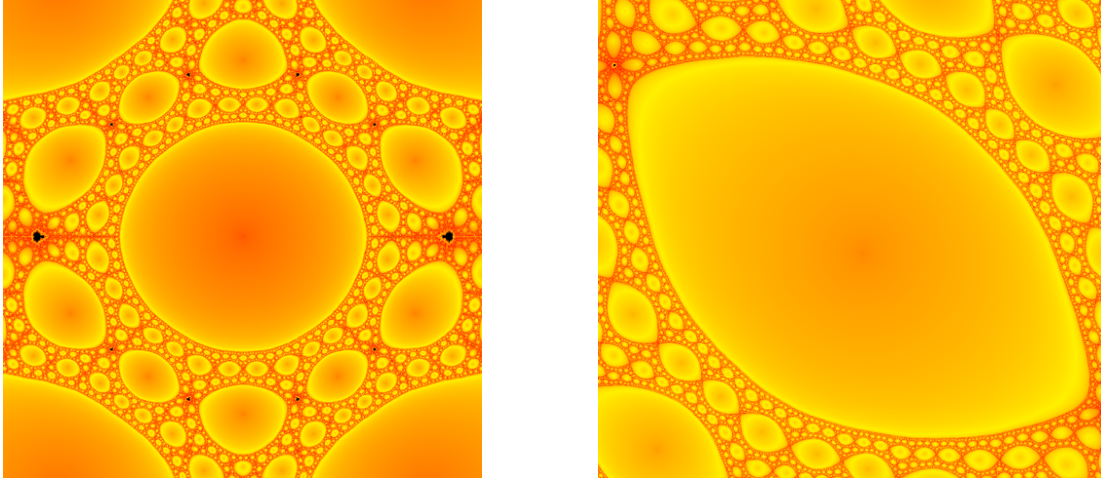


Figure 3.4: Left: Primary necklaces around the McMullen domain for $n = 3$. Right: Close-up of a Sierpiński hole on S_4 , with sub-necklaces visible.

critical values lying on C_k , where S_k contains $n^{|k|}(n-2)+1$ centers of Sierpiński holes and the same number of superstable parameters (i.e. parameters corresponding to maps with a finite attracting cycle).

Our goal here is to extend this notion to the dynamical q -necklaces described earlier. We shall prove the existence of a system of *parameter q -necklaces* in the parameter plane, corresponding to λ values for which the critical values of the map lie on a given dynamical q -necklace in the dynamical plane (see figure 3.4). We will see that each Sierpiński hole in each q -necklace is surrounded by infinitely many $(q+1)$ -necklaces, and we shall determine a count of the number of Sierpiński holes and superstable parameters on each.

For now we will fix an integer $k^* < -1$ and focus our attention on parameters lying in the closure of $\Gamma = \Gamma_{k^*}$, so that the critical values lie in $\mathcal{B}^\lambda = B_{k^*}^\lambda$. Our approach will be similar to that in chapter 2. Let $\mathbf{k} = k^*(i_0)k_1(i_1), \dots, k_{q-1}(i_{q-1}), k_q$ with $k_\alpha < k^*$ for $0 < \alpha \leq q$. By construction above, $C_{\mathbf{k}} = C_{\mathbf{k}}^\lambda$ is a simple closed curve lying in \mathcal{B}^λ , parametrized by θ so that $C_{\mathbf{k}}^\lambda(\theta)$ is an analytic function of λ .

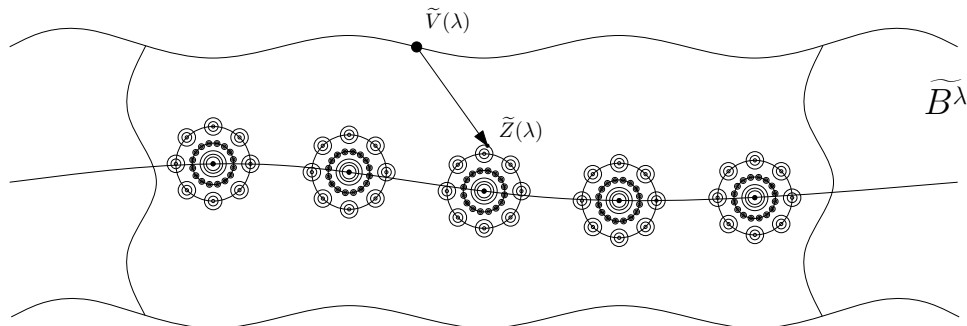


Figure 3.5: Structure in the lift $\widetilde{\mathcal{B}}^\lambda$ for λ in the boundary of $\widetilde{\Gamma}$.

Fix an itinerary \mathbf{k} as above and some θ . We then have two maps defined on Γ . Let $V(\lambda) = v_+^\lambda$, and let $Z(\lambda) = Z_{\mathbf{k},\theta}(\lambda) = C_{\mathbf{k}}(\theta)$. Note that both of these maps take values in \mathcal{B}^λ . To construct the q -necklaces in parameter space, we must find a unique λ in Γ such that $Z(\lambda) = V(\lambda)$.

An issue is that neither Z nor V are in fact well-defined on Γ . As λ runs once the origin in Γ , v_+^λ and v_-^λ interchange. Also by the results in [12], as λ runs around, $C_{\mathbf{k}}$ will move as well. We can rectify this by passing to universal covers $\widetilde{\Gamma}$ of Γ and $\widetilde{\mathcal{B}}^\lambda$ of \mathcal{B}^λ for each λ . These covers are infinite strips in the complex plane. Z and V then lift to well-defined analytic maps $\widetilde{Z}: \widetilde{\Gamma} \rightarrow \mathbb{C}$ and $\widetilde{V}: \widetilde{\Gamma} \rightarrow \mathbb{C}$. Note that for λ on the boundary of $\widetilde{\Gamma}$, $\widetilde{V}(\lambda)$ lies on the boundary of $\widetilde{\mathcal{B}}^\lambda$, but $\widetilde{Z}(\lambda)$ always lies in the interior. Now lift the portion of the real axis in Γ to a pair of arcs connecting the boundaries of $\widetilde{\Gamma}$ yielding a rectangular sector $\widetilde{\Gamma}'$, and take the image of these arcs under \widetilde{V} as well, to yield a rectangular sector $\widetilde{\mathcal{B}}^{\lambda'}$ in $\widetilde{\mathcal{B}}^\lambda$. By [12] there are $(n-2)n^{k^*} + 1$ preimages of prepoles on C_{k^*} that stay in the interior of $\widetilde{\mathcal{B}}^{\lambda'}$ for all λ in $\widetilde{\Gamma}'$. If necessary, we can enlarge these slightly to ensure that $\widetilde{Z}(\lambda)$ stays in the interior as well. See figure 3.5.

Now let $\psi(\lambda) = \widetilde{Z}(\lambda) - \widetilde{V}(\lambda)$. We have the following lemma.

Lemma 3.2.1. *Suppose $Z(\lambda) = C_k\left(\frac{2\pi(2i+1)}{4n}\right)$, i.e. one of the $|k^*|$ th preimages of the origin lying on C_{k^*} . Then ψ maps $\partial\widetilde{\Gamma}'$ around the origin with winding number*

1.

Proof. ψ is a difference of analytic functions, hence analytic, and has a unique zero, namely the center of the Sierpiński hole on S_{-k^*} for which $V(\lambda)$ coincides with $Z(\lambda)$. Call this parameter λ^* . By the argument principle, the winding number of ψ as λ moves around $\partial\tilde{\Gamma}'$ equals the number of zeroes of ψ counted with multiplicity. This winding number will be 1 then unless ψ happens to have a critical point at λ^* . But the proof of the existence of λ^* is entirely topological, and perturbing V to $V + \epsilon$ would still yield a unique solution λ_ϵ^* to the equation $Z(\lambda) = V(\lambda) + \epsilon$. This then gives a unique solution to $\psi(\lambda) = \epsilon$, and hence ψ is one-to-one in a neighborhood of λ^* . Therefore, ψ does not have a critical point at λ^* and the winding number is 1.

□

Finally, we can view the function ψ for all other choices of $Z = Z_{\mathbf{k},\theta}$ as a perturbation of the above, and hence the winding number of ψ will be equal to 1 as well. Projecting back down to Γ gives a unique parameter such that $Z(\lambda) = V(\lambda)$, which then reproduces all q -subnecklaces in the parameter plane.

Chapter 4

Conjugacy Classes of Checkerboard Julia Sets

In the paper [1], the dynamics of the family

$$F_\lambda = z^n + \frac{\lambda}{z^d}$$

are explored in cases where n and d are not necessarily equal. These maps are harder to analyze than when $n = d$ because some symmetry is lost. For example, it is no longer true that the critical points and prepoles lie on the same circle. It is also not the case that the free critical points all land on one of two opposite critical values. Some symmetries in the family do remain, however. $z \mapsto \bar{z}$ still yields a topological conjugacy between F_λ and $F_{\bar{\lambda}}$. Also, if ω is an $(n + d)$ th root of unity, then $F_\lambda(\omega z) = \omega^n F_\lambda(z)$. This gives a critical orbit relation for the $n + d$ free critical points of F_λ , meaning they all behave symmetrically under iteration. Therefore F_λ is still a natural one-parameter family when $n \neq d$.

In [1] it is shown that for given n and d , there are $n - 1$ *principal* Mandelbrot sets, i.e. homeomorphic copies of the Mandelbrot set for which the free critical points all tend to some finite attracting cycle, which touch both the boundary of the shift locus and the boundary of the McMullen domain. If λ and μ are drawn from the main

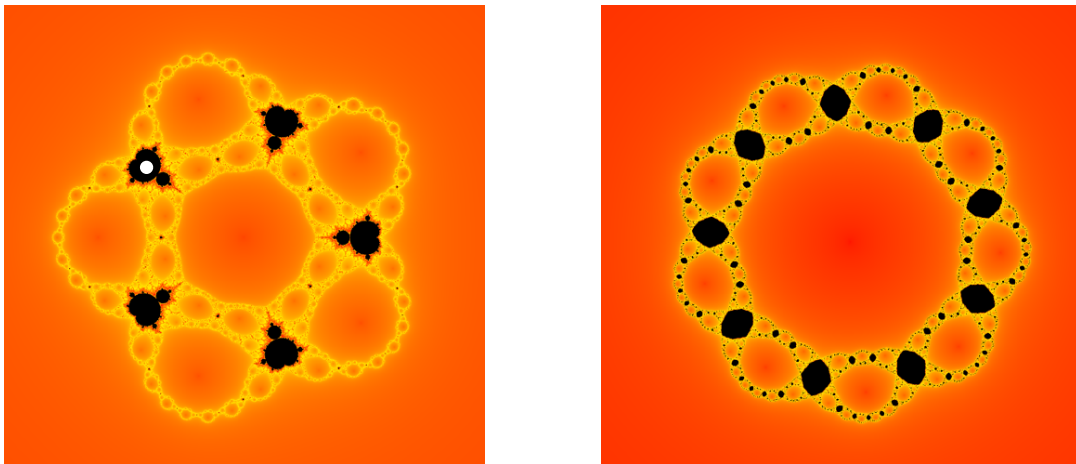


Figure 4.1: Left: the parameter space for $z^6 + \lambda/z^3$. Right: The checkerboard Julia set from the marked parameter.

cardioid of any one of these, the F_λ and F_μ will have homeomorphic *checkerboard* Julia sets (see figure 4.1). These Julia sets have two types of complementary Fatou components. Non-escaping Fatou components are colored black and contain points which tend to a finite attracting cycle, such as the free critical points. Escaping Fatou components are colored orange and consist of the full basin of infinity for the map.

Though the Julia sets are homeomorphic for any two such parameters, the dynamics are not necessarily conjugate. Let $\{\mathcal{M}_k\}$ denote the set of main cardioids of the principal Mandelbrot sets, and let λ and μ be centers of two of these. Then we have the following:

Theorem 1. F_λ is conjugate to F_μ if and only if $\mu = \nu^k \lambda$ or $\mu = \nu^k \bar{\lambda}$, where $k = j(d+1) \bmod n-1$ for some integer k .

The proof is given in [1]. Now we give the following count of the total number of conjugacy classes which are possible for the maps under consideration.

Theorem 2. Let g be the greatest common divisor of $n-1$ and $d+1$. The total

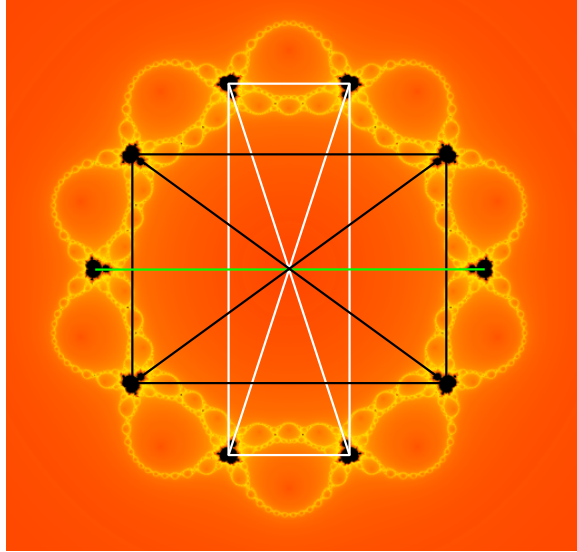


Figure 4.2: If $n = 11$ and $d = 4$, then $g = 5$, so there are three conjugacy classes. The parameters with conjugate dynamics are connected by segments of the same color.

number of conjugacy classes among the \mathcal{M}_k is $1 + g/2$ if g is even, and $(g + 1)/2$ if g is odd.

Proof. Since the conjugacies among the \mathcal{M}_k arise from reflective and rotational symmetries, we can count the number of conjugacy classes by viewing them as orbits of the action of a dihedral group on the set $\{\mathcal{M}_k\}$, viewed as the vertices of a regular $(n - 1)$ -gon.

Let $a = (n - 1)/g$. We claim that the natural group that produces these orbits is D_{2a} , the group of symmetries of a regular a -gon. Let s be the generator of D_{2a} corresponding to reflection, and r the generator corresponding to rotation, and define the action of D_{2a} on $\{\mathcal{M}_k\}$ by:

$$s\mathcal{M}_k = \mathcal{M}_{-k \bmod n-1}$$

$$r\mathcal{M}_k = \mathcal{M}_{k+g \bmod n-1}$$

This is a well-defined D_{2a} action, and since the actions on $\{\mathcal{M}_k\}$ by s and r are exactly complex conjugation and rotation by $z \mapsto \nu^g$, respectively, the orbits of this action correspond exactly to the conjugacy classes among the \mathcal{M}_k .

By Burnside's Lemma, the number of orbits is equal to the expression:

$$\frac{1}{|D_{2a}|} \sum_{x \in D_{2a}} |\text{fix}(x)|$$

where $\text{fix}(x) := \{\mathcal{M}_i \in \{\mathcal{M}_k\} : x\mathcal{M}_i = \mathcal{M}_i\}$.

D_{2a} has $2a$ elements and each can be written as r^j or sr^j with $0 \leq j < a$. The identity fixes all $n - 1$ elements of $\{\mathcal{M}_k\}$, and r^j fixes none for $0 < j < a$. Thus the number of orbits is

$$\frac{1}{2a} \left(n - 1 + \sum_{j=0}^{a-1} |\text{fix}(sr^j)| \right)$$

An element of the form sr^j rotates each \mathcal{M}_k by ν^{jg} , and then reflects it about the real axis. Equivalently, it reflects the \mathcal{M}_k through some axis of symmetry of the set viewed as a regular $(n - 1)$ -gon. Thus, if $n - 1$ is odd, every such axis passes through exactly one of the \mathcal{M}_k , and thus $|\text{fix}(sr^j)| = 1$ for all j . The formula above then shows the number of conjugacy classes to be $(n - 1 + a)/2a = (g + 1)/2$.

If $n - 1$ is even, half of the axes of symmetry pass through two of the \mathcal{M}_k , and half pass through none. Thus sr^j fixes either two or zero of the \mathcal{M}_k . There exists a j such that sr^j fixes none of the \mathcal{M}_k if and only if there is some i such that $sr^j\mathcal{M}_i = \mathcal{M}_{i+1 \bmod n-1}$ (i.e. the axis of reflection passes between \mathcal{M}_i and $\mathcal{M}_{i+1 \bmod n-1}$ for some i). For such an i , $r^j\mathcal{M}_i = s^{-1}\mathcal{M}_{i+1 \bmod n-1} = \mathcal{M}_{-i-1 \bmod n-1}$ which must equal $\mathcal{M}_{i+jg \bmod n-1}$ by the definition of the action of r . Thus $-i - 1 \equiv i + jg \bmod n - 1$ which gives $2i + jg + 1 \equiv 0 \bmod n - 1$. If either j or g is even, this is impossible (since $n - 1$ is even), and therefore sr^j must fix two of the \mathcal{M}_k . If j and g are both odd, however, any i with $i \equiv (-jg - 1)/2 \bmod n - 1$ satisfies the congruence, and

thus sr^j fixes none of the \mathcal{M}_k .

Therefore if $n - 1$ and g are even, then $|\text{fix}(sr^j)| = 2$ for all j , and the number of conjugacy classes is $(n - 1 + 2a)/2a = 1 + g/2$. If $n - 1$ is even and g is odd, $|\text{fix}(sr^j)|$ equals 2 if j is even, and 0 if j is odd. Hence there are $(n - 1 + a)/2a = (g + 1)/2$ conjugacy classes.

Finally, if $n - 1$ is odd, g must be odd, so the possible cases really depend only on the parity of g , and not of $n - 1$. Hence the number of conjugacy classes is $(g + 1)/2$ if g is odd, and $1 + g/2$ if g is even. \square

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