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Describing pre-service teachers' developing understanding of elementary number of theory topics

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BOSTON UNIVERSITY
SCHOOL OF EDUCATION

Dissertation

**DESCRIBING PRE-SERVICE TEACHERS' DEVELOPING UNDERSTANDING
OF ELEMENTARY NUMBER THEORY TOPICS**

by

ZIV FELDMAN

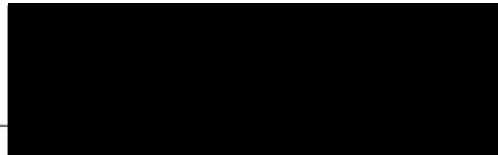
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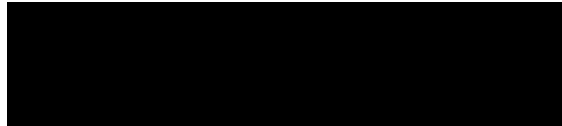
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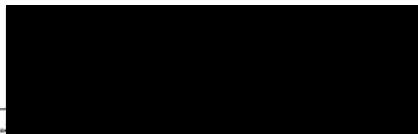
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**DESCRIBING PRE-SERVICE TEACHERS' DEVELOPING UNDERSTANDING
OF ELEMENTARY NUMBER THEORY TOPICS**

(Order No.)

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ABSTRACT

Although elementary number theory topics are closely linked to foundational topics in number and operations and are prevalent in elementary and middle grades mathematics curricula, little is currently known about how students and teachers make sense of them. This study investigated pre-service elementary teachers' developing understanding of elementary number theory topics, including factors, divisibility, greatest common factor, and least common multiple. Fifty-nine participants in a college mathematics course for pre-service elementary teachers participated in a three-week unit of instruction on number theory. All participants completed the Number Theory Knowledge Test (NTKT) before and after instruction. Additionally, individual clinical interviews were conducted with six participants before and after instruction. Each interview was recorded and transcribed.

In order to describe how participants' understanding of number theory developed during instruction, analysis of the interview data was done using Dubinsky's (1991) APOS theory. Results showed that the most notable changes were in participants' attention to prime factorization, their notions of factor, and their abilities to coordinate

multiple processes. Prior to instruction, participants' understanding was characterized by a need to convert prime factored numbers into decimal form, the notion of factor as a number that is visible within the prime factorization of another number, and an inability to coordinate multiple processes. Following instruction, participants attended to a number's prime factorization in order to solve problems, identified factors as combinations of prime numbers within the prime factorization of another number, and coordinated multiple processes. Statistical analysis of the NTKT scores supported the interview data, showing that participants significantly improved in their performance on questions requiring both procedural and conceptual knowledge.

Contrary to prior research, this study provides evidence that pre-service elementary teachers can develop a deep and connected understanding of number theory topics. Future research should continue to investigate pre-service teachers' understanding of number theory by examining their work during periods of classroom instruction. Additionally, mathematical tasks that attend to prime factorization as a foundational tool should be investigated for their effects on pre-service teachers' understanding of number theory.

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CHAPTER I
THE RESEARCH PROBLEM

Section I: Introduction

The mathematics education that U.S. elementary and middle school students receive continues to be a major source of concern for educators, researchers, and policymakers. Although national assessment data show improving scores for U.S. fourth and eighth graders (National Center for Education Statistics, 2011), the achievement gap between white and minority students is not shrinking. Additionally, international comparisons on standardized measures in the last ten years have shown that the United States continues to lag behind many other nations in mathematics achievement (Gonzales et al., 2008; Organization for Economic Co-operation and Development, 2010). Such news has sparked national debate over how best to improve mathematics education: should teaching focus on developing students' conceptual understanding or procedural knowledge? Should mathematics standards emphasize a narrow set of key concepts or a long list of specific skills?

While such debates continue unresolved, one source of agreement among educators, researchers, and now policymakers is that teachers make a difference in student achievement and the mathematical knowledge that teachers bring with them to the classroom has a profound effect on instruction (National Commission on Teaching and America's Future, 1996; Ball, 2003; National Mathematics Advisory Panel, 2008; National Council on Teacher Quality, 2008). In the standards document, *Principles and Standards for School Mathematics* (NCTM, 2000), NCTM states that "students learn

mathematics through the experiences that teachers provide...students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school" (p. 17). The National Mathematics Advisory Panel (2008) supported the council's statement when, after analyzing numerous high-quality research studies, they concluded that variability in student achievement gains (12% to 14% over a one year period) was due to differences in the effectiveness of the students' mathematics teachers.

In order to be effective, teachers must know the mathematics that they teach their students and be able to make meaningful connections between key mathematics concepts across grade levels (Ball, Thames, & Phelps, 2008; Ma, 1999; Ball, 1990b; Shulman, 1986). According to Ma (1999), teachers who develop a *profound understanding of fundamental mathematics* possess a depth and a breadth of understanding of mathematics topics. They recognize the connectedness of both central and peripheral mathematical topics and use such connections to clarify mathematical concepts in meaningful ways for their students. Ball and colleagues (2008) developed a theoretical framework for teacher knowledge called *mathematical knowledge for teaching* that emphasizes the importance of subject-matter knowledge that is specific to the task of teaching. To be effective, teachers must know the content they teach, be able to explain it to others in multiple ways, and anticipate and react productively to students' misconceptions. Underlying Ma's and Ball's models is the importance of a flexible and connected understanding of school mathematics.

Research results indicate, however, that pre-service and in-service elementary

teachers do not have a deep, connected understanding of the mathematics that they teach (van Es & Conroy, 2009; Philipp et. al., 2007; Kilpatrick, Swafford, & Findell, 2001; Conference Board of the Mathematical Sciences, 2001; Ma, 1999; Ball, 1990). One reason is that teacher education programs do not offer mathematics coursework that focuses on the mathematics of the elementary curriculum. Many preparation programs do not require pre-service teachers to take any mathematics coursework; other programs require them to take mathematics courses geared toward STEM careers, such as calculus (National Mathematics Advisory Panel, 2008; National Council on Teacher Quality, 2008).

In light of these findings, there have been calls for changes at the state and federal levels to strengthen teacher preparation programs. It is recommended that mathematics teachers show evidence of relevant mathematics knowledge in order to obtain a license to teach (No Child Left Behind, 2001). The implication of this recommendation is that the coursework for elementary education majors should focus on topics relevant to the K-8 curriculum rather than advanced mathematics. In Massachusetts, the Department of Education has made these recommendations a matter of policy; elementary teachers must now take 9-12 credits of mathematics coursework focused on elementary grade-level topics, successfully complete a separate mathematics subject test, and demonstrate “that they possess both fundamental computation skills and comprehensive, in-depth understanding of K-8 mathematics” in order to receive a teaching license (MA Department of Education, 2007, p. 1). It is no longer sufficient for elementary teachers to only know how to execute procedures accurately; they must also know why those

procedures make sense.

One branch of mathematics that is extremely important for all teachers to understand is number theory. Number theory is the study of the properties of integers, and commonly includes topics such as divisibility, prime factorization, least common multiples (LCM), and greatest common factors (GCF). Campbell and Zazkis (2002) make a distinction between *elementary* number theory and *advanced* number theory: the former restricts focus to the domain of integers; the latter expands its focus to non-integral domains. Elementary number theory is typically taught throughout the K-12 level, but it is rarely explored as its own subject area. Number theory topics such as whole number patterns and sequences, multiplication, multiples, factors, primes, relatively prime numbers, division, and divisibility rules are explored within well-known arithmetic and algebra units (Campbell & Zazkis, 2002). Advanced number theory topics such as Diophantine equations, linear and polynomial congruence, and quadratic residues are often taught in number theory courses as part of an undergraduate mathematics program. Such courses take a more formal approach to the study of integers, and often involve rigorous proof of fundamental concepts seen at the K-12 level.

The study of elementary number theory is an important part of mathematics education at all levels. In the elementary grades, understanding the properties of the set of whole numbers allows students to make better sense of arithmetic operations (Kilpatrick, Swafford, & Findell, 2001). In later grades, students' ability to classify number systems in different ways (e.g., as primes and composites, odds and evens) can generate richer schemas of number properties and operations (Zazkis, 1998). In the

middle grades, understanding the prime factorization of natural numbers helps students develop more abstract divisibility concepts and identify common factors and multiples. And in secondary school, number theory concepts serve as a foundation for students' understanding of algebra (National Council of Teachers of Mathematics, 2000).

The existing research on pre-service teachers' knowledge of number theory is informative but insufficient. In order to improve pre-service teachers' ability to teach elementary number theory, teacher educators must first know how pre-service teachers come to make sense of these topics. While prior research investigated pre-service elementary teachers' understanding of various number theory topics (Zazkis & Campbell, 1996a, 1996b; Zazkis & Gadowsky, 2001; Zazkis, 1999; Campbell, 2002), none did so over the course of instruction. Most studies analyzed participants' conceptions at a single point in time and focused primarily on identifying pre-service teachers' misconceptions following instruction. Since none had provided any detail about how pre-service teachers' understanding developed, it is difficult to make any inferences about how their knowledge was generated. Therefore, additional research is needed to inform researchers and teacher educators about how pre-service teachers' tend to develop an understanding of number theory.

This study contributes to the knowledge base by generating detailed descriptions of pre-service elementary teachers' developing understanding of number theory topics. It extends prior research on pre-service teachers' knowledge of number theory by identifying their levels of understanding before and after instruction. As such, this study helps identify the cognitive processes that pre-service teachers undergo to make sense of

number theory topics. This study can also inform teacher educators in the development of more focused curriculum materials and instructional strategies. Since many current teacher education programs do not require or even offer mathematics content courses that are relevant to teaching (National Council on Teacher Quality, 2008), there is a critical need for developing coursework that promotes understanding of the mathematics that pre-service teachers will teach their future students.

Section II: Statement of the Problem

The purpose of this study was to investigate undergraduates' conceptions of elementary number theory topics. In particular, this study sought to identify the levels of understanding of number theory topics such as factors, prime factors, prime factorization, divisibility, greatest common factor, and least common multiple by pre-service elementary teachers and to describe changes in their levels of understanding after relevant instruction in a teacher education content course. The following questions were addressed:

1. Do participants exhibit identifiable levels of understanding of divisibility, greatest common factor, and least common multiple prior to instruction, and if so, what are they?
2. Do participants' levels of understanding of divisibility, greatest common factor, and least common multiple change after the course of instruction of a unit on number theory, and if so, how?

Section III: Definitions of Terms

Elementary Number Theory Topics

Elementary number theory is defined as the study of the properties of the set of integers (Campbell & Zazkis, 2002); although in this study it is restricted to the set of non-zero whole numbers (i.e., natural numbers). The elementary number theory topics discussed in this study include factors, multiples, prime factors, prime factorization, divisibility, greatest common factor, and least common multiple. This study defines these terms according to the standard mathematical definitions most often presented to pre-service elementary teachers (Billstein, Libeskind, & Lott, 2010; Bennett, Burton, & Nelson, 2010).

Factors

A factor of a number can be defined in two equivalent ways: using multiplication and division (Zazkis & Campbell, 1996a). Using multiplication, if a and b are any whole numbers where $b \neq 0$, then b is a factor of a if, and only if, there exists a unique whole number q such that $a = bq$. Using division, any whole number b , where $b \neq 0$, is a factor of any whole number a if, and only if, $a \div b = q$, where q is also a whole number. This second definition implies that when a number is divided by its factor, the remainder is zero. The term *divisor* is a synonym for factor (Chapin & Johnson, 2006).

Every whole number has a finite number of factors or divisors. A whole number's greatest factor must be the whole number itself, while its least factor must be 1. For example, the factors, or divisors, of 12 are 1, 2, 3, 4, 6, and 12. Each factor can be

multiplied by some whole number to equal 12 (e.g., $3 \times 4 = 12$). Equivalently, if 12 is divided by one of its divisors, the quotient is some whole number (e.g., $12 \div 3 = 4$).

Multiples

If a and b are any whole numbers, then a is a multiple of b if, and only if, there exists a unique whole number q such that $a = bq$. Equivalently, a is a multiple of b if, and only if, there exists some whole number q such that $a \div b = q$. Any whole number, a , has an infinite number of multiples, including $a, 2a, 3a, 4a, 5a$, etc. Note that while zero is a multiple of a ($0 = 0 \times a$), this study does not consider zero to be a multiple because of the common restriction to the set of natural numbers. For example, the multiples of 8 are 8, 16, 24, 32, etc. Each multiple can be expressed as a product of 8 and some whole number (e.g., $24 = 8 \times 3$); alternatively, each multiple results in a zero remainder when divided by 8.

There is a direct relationship between factors and multiples. If a is a multiple of b , then b is a factor or divisor of a . For example, since $56 = 7 \times 8$, 56 is a multiple of 7 and 8, and 7 and 8 are both factors of 56. Also, a whole number is simultaneously its own factor and multiple. For example, 36 is a factor and a multiple of 36, since $36 = 36 \times 1$.

Prime and Composite Numbers

All non-zero whole numbers except 1 can be classified as either prime or composite. A *prime number* is defined as any non-zero whole number with exactly two

distinct, positive factors (or divisors). For example, 3 is a prime number because it has exactly two distinct factors, 1 and 3. A *composite number* is defined as any non-zero whole number that has more than two distinct, positive factors. For example, 15 is a composite number because it has four distinct factors: 1, 3, 5, and 15. The number 1 is neither prime nor composite because it has only one distinct factor.

Prime Factors

A prime factor is any factor that is, itself, a prime number. For example, while the factors of 12 are 1, 2, 3, 4, 6, and 12, the prime factors of 12 are 2 and 3.

Any composite number can be represented as a product of its prime factors. This notion is critical to the field of elementary number theory and is stated as the *fundamental theorem of arithmetic*: each composite number can be written as a unique product of primes. In other words, every composite number has exactly one *prime factorization*, or a unique representation of the number as a product of its prime factors. For example, the prime factorization of 12 is $2^2 \times 3$, and the product of all of the prime factors of 12 must be 12.

Divisibility

The concept of divisibility refers to the multiplicative relationship between whole numbers. Given whole numbers a and b , where $b \neq 0$, a is divisible by b if, and only if, a is a multiple of b . For example, 18 is divisible by 1, 2, 3, 6, 9, and 18 since 18 is a multiple of each of these numbers. Alternatively, given whole numbers a and b , where b

$\neq 0$, b divides a if, and only if, b is a factor of a . For example, 2 divides 18 because 2 is a factor of 18. As a result, a number is always divisible by each of its factors and the factors of a number always divide that number.

Divisibility rules refer to simple tests that can be performed quickly, in place of dividing, to determine if any natural number is divisible by some natural number. There exist common divisibility rules for 2, 3, 4, 5, 7, 8, 9, 10, and 11, although the rules for 7 and 11 are often considered too tedious to implement. For example, the divisibility rule for 3 states that a number is divisible by 3 if, and only if, the sum of its digits is divisible by 3.

Greatest Common Factor (GCF)

The greatest common factor of two natural numbers a and b is defined as the greatest natural number that divides both a and b . For example, the greatest common factor of 12 and 18 is 6 because 6 divides both 12 and 18, and it is the greatest such natural number.

Least Common Multiple (LCM)

The least common multiple of two natural numbers a and b is defined as the least natural number that is a multiple of both a and b . For example, the least common multiple of 12 and 18 is 36 because 36 is a multiple of both 12 and 18, and it is the least such common multiple.

Schema

A schema is a conceptual structure that consists of connections between individual concepts (Skemp, 1987). Such connections are initially constructed from the individual's experiences with the outside world. As the individual makes connections between objects in the external environment, he begins to form connections between internal, mental representations of those objects (Hiebert & Carpenter, 1992). As the number and strength of the connections between internal representations grow, the schema becomes more robust and better able to integrate new and existing information.

Understanding

Although multiple definitions exist (Piaget, 1970; Skemp, 1987; Sfard, 1991; Dubinsky, 1991; Sierpiska, 1994; Pirie & Kieren, 1994; Carpenter & Lehrer, 1999), this study defines understanding as a continuous process of organizing one's conceptual structures (von Glasersfeld, 1987). Organizing such structures involves two processes: assimilation and accommodation. Assimilation is the individual's attempt to fit experiences into pre-existing conceptual structures, while accommodation is the modification of one's cognitive structure. Understanding develops as the processes of assimilation and accommodation near equilibrium (Piaget 1970, 1971).

To develop understanding of a concept, an individual attempts to explain a relevant experience by aligning it with her prior knowledge. If her prior knowledge cannot explain the experience, assimilation is unsuccessful and the individual undergoes an accommodation whereby her prior knowledge is modified, reorganized, or replaced by

information gathered from the new experience. New connections are formed, either between previously constructed concepts or between old and new information (Hiebert & Carpenter, 1992).

Levels of Understanding

An individual's achieved level of understanding of a concept refers to the degree to which the individual has organized conceptual structures around pieces of information. One reaches a higher level of understanding by making more and stronger connections between different pieces of information (Hiebert & Carpenter, 1992). Since it is not possible for researchers to directly access an individual's conceptual structure, the individual's levels of understanding must be determined by analyzing his written work or verbal explanations.

This study identifies levels of understanding using Dubinsky's (1991) APOS model as its theoretical framework. The APOS model identifies four levels of conception: *action*, *process*, *object*, and *schema*. An individual's understanding develops from more concrete (e.g., actions) to more abstract (e.g., schema) mental representations of a concept. The model also describes five mental constructions that individuals might make as their understanding of a concept develops through the four conception levels: *interiorization*, *coordination*, *encapsulation*, *generalization*, and *reversal*. These will be described more fully in Chapter II.

Section IV: Justification for the Study

Number theory's resurgence in national and state mathematics standards documents (Common Core State Standards, 2010; MA Department of Education, 2007) makes it critical to prepare pre-service elementary teachers to teach such topics with understanding. The Common Core State Standards for Mathematics (2010) identifies elementary number theory topics as stand-alone standards in the elementary and middle grades instead of as secondary topics as had been previously recorded in prior standards documents. Additionally, recent state guidelines identify number theory as a critical component of elementary school teacher preparation (MA Department of Education, 2007, p. 30).

In addition to its prevalence in mathematics education policy, number theory topics are closely connected to the study of number and operations, the most heavily emphasized topic at the elementary level (NCTM, 2000). The close link suggests that knowledge of elementary number theory can help pre-service teachers strengthen their understanding of other key mathematics topics. For instance, the multiplicative structure of number is central to the study of number theory and leads to an understanding of key number properties. Defined as "conceptual attributes and relations pertaining to and implied by the decomposition of natural numbers as unique products of prime factors" (Zazkis & Campbell, 1996a, p. 541), multiplicative structure highlights the importance of the commutative and associative properties. These properties form the basis for understanding and operating in our number system. Knowledge of multiplicative structures can also help develop an understanding of rational numbers and functions

(Vergnaud, 1988). As such, developing an understanding of number theory can strengthen pre-service teachers' understanding of other key concepts that they will need to teach their future students.

Prior studies acknowledge that more research is needed to determine precisely how pre-service teachers come to make sense of number theory topics (Zazkis & Campbell, 1996a, 1996b). Specifically, it is unclear what levels of understanding of number theory topics they possess before and after relevant instruction in teacher education programs. Vergnaud's work (1988, 1994) around the *multiplicative conceptual field* – the set of all situations, schemes, and concepts that require the use of multiplication, division, or a combination of both – illustrated the challenging nature of learning multiplicative concepts by showing that multiple concepts underlie any single multiplicative situation and a single concept refers to several multiplicative situations. Noting similar levels of complexity, Campbell and Zazkis (2002) identified a *number theory conceptual field* – the set of abstract conceptual and procedural understandings of the arithmetic of integers – as a “newly emerging area of research in mathematics education” (p. 8). Our understanding of how pre-service teachers make sense of number theory topics appears to be in its nascent phase.

Prior research on pre-service teachers' knowledge of number theory serves as a starting point for describing their understanding and for explicating the number theory conceptual field. It provided examples of individuals' conceptions and misconceptions of divisibility and prime factorization by analyzing their responses during individual interviews and on written questionnaires (Zazkis & Campbell, 1996a, 1996b; Zazkis &

Gadowsky, 2001). However, it did not provide detailed descriptions of how individuals' understanding of number theory changed, if at all, during instruction. It also focused primarily on divisibility concepts, leaving greatest common factor and least common multiple topics unexamined. Detailed descriptions of how pre-service teachers' understanding of divisibility, greatest common factor, and least common multiple develops over the course of instruction can aid teacher educators in the development of more focused number theory curricula and instruction. Given the need to improve the mathematics education of pre-service teachers (Ball, 1990; Ma, 1999), more research is needed to determine precisely how pre-service teachers' knowledge develops over time with regards to a variety of number theory topics.

This chapter presented the problem and provided a justification for addressing this problem. Given the need to improve the quality of teacher knowledge and training and the lack of detail around pre-service teachers' knowledge of elementary number theory, this study identified and described pre-service teachers' levels of understanding of elementary number theory topics before and after instruction in an elementary mathematics education content course. Chapter II provides a review of the relevant literature around teacher knowledge, elementary number theory, and theories of understanding. Chapter III describes the study's research design, including sampling, data collection and procedures for data analysis. Chapter IV presents the results of analysis of the testing instrument, while Chapter V presents the results of analysis of the sub-sample of six interview participant. Chapter VI discusses conclusions from both analyses, identifies study limitations, and makes recommendations for future research.

CHAPTER II

REVIEW OF THE LITERATURE

The purpose of this study was to describe pre-service elementary teachers' developing understanding of elementary number theory topics. Specifically, the study used the APOS theoretical framework to identify participants' levels of understanding as indicated by their written and verbal responses in solving number theory problems during interviews and classroom activities. In order to gain insight into the various aspects of this study, this chapter presents a review of the relevant mathematics education research. In Section I, research on the undergraduate education of pre-service elementary teachers is presented. In Section II, research on elementary number theory is described. In Section III, the literature on analyzing pre-service teachers' knowledge is discussed and the theoretical framework for the study is described.

Section I: Undergraduate Education of Pre-Service Elementary Teachers

Teachers often begin their teaching careers lacking the necessary mathematical knowledge needed to help their students learn (Conference Board of the Mathematical Sciences, 2001). Recent research has begun identifying components of the mathematical knowledge needed for teaching and calling for pre-service and in-service teachers to develop such knowledge (Shulman, 1986; Ball, Thames, & Phelps, 2008). Unimpressive standardized test scores of U.S. elementary and middle school students on *international* assessments (Gonzales et al., 2008; OECD, 2010) have also contributed to growing

national pressure to improve mathematics teacher education. Although it is not yet clear exactly what effective teacher preparation in mathematics should include, the preparation of pre-service teachers begins predominately in university-level teacher education programs (Elliott, 2009). Therefore, it is at these institutions where one must examine the mathematical knowledge development of pre-service elementary teachers.

Unfortunately, teacher education programs do not consistently foster the kinds of mathematical knowledge that pre-service teachers need to help their future students learn effectively. One reason for this problem is the large variability in mathematics course requirements for pre-service elementary teachers across institutions (National Council on Teacher Quality, 2008). Many teacher education programs only require their students to take mathematics content courses such as calculus or statistics, while others require enrollment in mathematics content or methods courses specifically designed for future teachers. In some cases, teacher education programs do not require pre-service teachers to take any mathematics coursework. What mathematics should pre-service elementary teachers learn? Without nationwide agreement on the answer, it is clear that teacher education programs have yet to establish clear guidelines for how best to prepare pre-service elementary teachers for teaching.

The need for more focused research on teacher knowledge originated from a need to fill in research gaps dating back to the 1960's and 1970's. Two widespread modes of research at that time, *process-product research* and *educational production function research*, attempted to identify the knowledge that teachers needed in order to be effective teachers. Both attempted to identify the teaching behaviors and teacher

characteristics that support effective instruction, studied teachers in multiple subject areas, compared novice to expert teachers, and made broad generalizations about effective instruction.

Process-product research consisted primarily of correlation studies. These studies entailed extensive classroom observations aimed at identifying any relationships between teaching behaviors in the classroom and student achievement (Hill, Rowan, & Ball, 2005). Researchers would count the frequency of teaching behaviors such as praising students, setting clear expectations, actively teaching, and questioning students. Once counted, these behaviors would be correlated with student achievement on paper-and-pencil assessments in the content area studied. Results showed that actively teaching content, providing students with opportunities to learn, organizing content in a structured format, pacing students briskly but in small steps through the curriculum, and circulating around the classroom to supervise student seatwork are all correlated with student achievement gains (Brophy, 1986, 1988, 1999; Grouws & Cebulla, 2000).

Despite its prevalence, process-product research was methodologically and theoretically flawed. Its reliance on correlation analysis and use of non-random samples meant that causality could never be concluded and results could rarely be generalized to larger populations (Brophy, 1986; Hiebert & Grouws, 2007; Hill et al., 2005). Most importantly, since process-product research did not take into account a teacher's subject-matter knowledge when correlating teaching behavior to student achievement, results could only identify generic teaching behaviors as effective (Shulman, 1986; Hill et al., 2005). With such a limited emphasis on content-specific issues, the process-product

research was inadequate in identifying appropriate teacher knowledge of mathematics.

The educational production function literature, on the other hand, focused on the relationship between educational resources – years of teaching experience, certification level, number of college-level mathematics courses taken, performance on tests of subject-matter knowledge – and student outcomes in order to identify the knowledge teachers needed to effect positive student achievement. Results showed that teachers with bachelor's or master's degrees in mathematics had a statistically significant positive relationship with high school students' mathematics achievement, but not with elementary student achievement (Bolyard & Moyer-Packenham, 2008). Other studies showed a positive relationship between teacher certification in mathematics and student achievement (Darling-Hammond, 2000, as cited by Bolyard & Moyer-Packenham, 2008). Many studies, however, showed mixed results when attempting to find a significant relationship between a teacher's educational background and student achievement.

The production function literature represented only a moderate improvement over process-product research in attempting to identify teachers' mathematical knowledge. While the focus on teachers' subject-matter knowledge instead of general pedagogy was a marked improvement, the production function literature continued to use imprecise proxy measures of teacher knowledge of mathematics (Hill et al., 2005). The number of college-level mathematics courses taken, the type of certification earned, and the years spent teaching do not directly determine the mathematics that teachers know or how they know it. For example, a veteran teacher may have only a superficial understanding of mathematics, while a novice teacher may have a deeper understanding of fundamental

concepts. Furthermore, since advanced coursework in mathematics does not typically address content issues related to teaching, production function research was unable to identify mathematics knowledge *relevant* to the teaching profession.

Given the limitations of process-product and educational production function research, new theoretical models that directly address teacher knowledge of mathematics were developed. The basis for all of these models is the premise that teachers must know the mathematics that they teach in order to teach effectively (Shulman, 1986; Ball, 1990b; Fennema & Franke, 1992; Bolyard & Moyer-Packenham, 2008). While this idea is not disputed by researchers, what is uncertain is the type of mathematical knowledge that teachers must know in order to teach students well (Hill et al., 2005).

Lee Shulman's (1986) seminal work presented a new model of teacher knowledge comprised of three categories: (a) *subject-matter content knowledge*, (b) *pedagogical content knowledge*, and (c) *curricular knowledge*. He defined subject-matter content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (Shulman, 1986, p. 9). This type of knowledge encompasses knowledge of facts and concepts, the structure of the subject, why certain concepts are worth knowing, and how one establishes and justifies claims within the field of study. Pedagogical content knowledge, on the other hand, involves knowing "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Teachers who develop pedagogical content knowledge not only know how to do mathematics, but they also

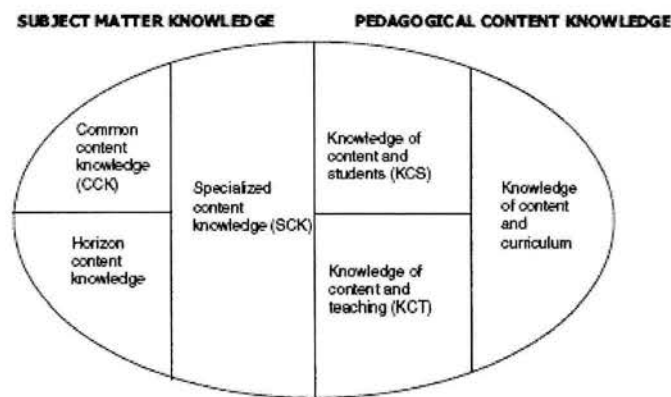
know how to present mathematics to their students. They understand how their students come to understand and misunderstand concepts, and they know specific methods to correct misconceptions. Shulman's third category, curricular knowledge, includes an understanding and awareness of curriculum materials. Teachers should know the types of curricular tools available to them for teaching specific content, as well as decide which resources are and are not useful for instruction. Teachers should also know the trajectory of their subject area from the early grades to the upper grades, so they can anticipate the kind of prior knowledge their students bring to class as well as the knowledge their students must develop in order to be prepared for future coursework.

Shulman's model (1986) broadened the way the field thought about teacher knowledge because it provided detailed descriptions of the types of knowledge that teachers must possess in order to teach well. The impetus for the model came from what Shulman called the *missing paradigm problem* – research and policy's over-emphasis on teachers' pedagogical knowledge at the expense of content knowledge. This phenomenon, he noted, all but eliminated content knowledge from teacher certification examinations and policy recommendations (Shulman, 1986). With Shulman's model, researchers now had a framework with which they could examine the knowledge that teachers possessed in specific subject areas. Cited in more than 1,200 refereed journal articles and 125 journals, the model represented a redefining period in the research on teacher knowledge (Ball et al., 2008).

Building on Shulman's model, a new theoretical framework called *mathematical knowledge for teaching* (MKT) has been developed by Deborah Ball and colleagues

(2008). It attempts to specify the types of knowledge mathematics teachers should have. The framework explains that knowing mathematics in the way most educated adults do is not sufficient for effective mathematics instruction. Mathematics teachers must also be able to correctly explain content, provide mathematical justifications of key ideas, use multiple representations, evaluate multiple solution strategies, and anticipate and remedy common student errors (Ball et al., 2008).

Figure 2.1.1: Domains of Mathematical Knowledge for Teaching (Ball et al., 2008)



As shown in Figure 2.1.1 above, MKT is divided into two major categories: *subject-matter knowledge* and *pedagogical content knowledge*. Ball and her colleagues divided Shulman's subject-matter knowledge into three forms of knowledge: *common content knowledge (CCK)*, *specialized content knowledge (SCK)*, and *horizon content knowledge*. The model also divided Shulman's pedagogical content knowledge into three forms: *knowledge of content and student (KCS)*, *knowledge of content and teaching (KCT)*, and *knowledge of content and curriculum*. In order to fully understand this new theoretical framework for teacher knowledge of mathematics, one must examine each of the subcategories in detail.

Within the subject-matter knowledge domain, *common content knowledge* is defined as the knowledge of mathematical concepts and procedures that any educated adult holds (Ball et al., 2008). For example, knowing how to find the factors of 12 is knowledge that many adults outside of teaching possess. Since simply knowing the procedure needed to solve this problem is not inherently specific to teaching, it is considered common content knowledge. On the other hand, *specialized content knowledge* is mathematical knowledge that is specific to teaching. This type of knowledge “involves an uncanny kind of unpacking of mathematics that is not needed – or even desirable – in settings other than teaching” (Ball et al., 2008, p. 400). Choosing appropriate representations, evaluating the validity of students’ mathematical responses, appreciating the differences between various strategies, and justifying concepts all require teachers to have specialized knowledge of mathematics. For example, the ability to justify the divisibility rule for 3 is a form of specialized content knowledge because such knowledge is relevant to the work of teachers, but not to the work of adults outside of teaching. Ball and colleagues clarify that specialized content knowledge is distinct from knowledge of pedagogy. For instance, the ability to justify the divisibility rule for 3 is independent of a teacher’s knowledge of student thinking or instructional strategies.

The third category of subject-matter knowledge is *horizon content knowledge*. This type of teacher knowledge involves knowing how specific topics develop over time and how topics are related to one another. For example, an elementary school teacher should know how the topic of even and odd numbers in the early grades relates to notions of divisibility in upper elementary and middle grades, and how such concepts are used in

middle school and high school (Zazkis, 1998). Such knowledge helps a mathematics teacher decide which topics to discuss and in what detail.

The pedagogical content knowledge domain is a hybrid of content and pedagogical knowledge. Within this domain, *knowledge of content and student* represents knowledge of how students learn mathematical content, including which examples students will find most helpful, the misconceptions students bring with them to the classroom, and the topics students will find most interesting. An example of this type of teacher knowledge is teachers' understanding that students often erroneously believe that prime numbers are inherently small and that every large composite number must be divisible by some small prime number (Zazkis & Liljedahl, 2004). *Knowledge of content and teaching* incorporates knowledge of mathematics and teaching methods. Teachers with such knowledge can evaluate the usefulness of various representations and examples and can properly sequence their lessons. As Ball and colleagues (2008) note, "a teacher must decide when to pause for more clarification, when to use a student's remark to make a mathematical point, and when to ask a new question or pose a new task to further students' learning" (p. 401). *Knowledge of content and curriculum* reflects Shulman's curricular knowledge, and includes familiarity with a variety of curricular programs, how they are typically used, and their benefits and disadvantages. Teachers with this type of knowledge are also aware of a curriculum's philosophy of teaching and learning (e.g., constructivist, teacher-centered) and the particular tools that such a curriculum requires.

A recent large-scale study by Hill and colleagues (2005) called the Study of Instructional Improvement, measured the effects of elementary teachers' mathematical

knowledge for teaching (MKT) on student achievement. Nearly 3,000 first and third grade students at 115 elementary schools, of which 89 randomly-selected elementary schools participated in a school reform program, were tested using standardized assessments. Teacher data were collected via questionnaires and daily classroom logs. Teachers answered questions testing their common and specialized content knowledge, including their ability to provide mathematically correct explanations and representations and to work with alternative solution methods. The focus of the questions was on number, operations, and pre-algebra. Results of the longitudinal, four-year, study showed that specialized content knowledge was the strongest teacher predictor of student test score gains. In fact, specialized content knowledge was nearly as strong a predictor of student achievement as the student's socioeconomic status. Hill's results support conclusions from a variety of educational production function research studies by finding no significant correlation between MKT and the number of advanced mathematics courses a teacher has taken, years of experience in teaching, and certification status.

In a separate study, Hill and her colleagues (2008) examined ten elementary school teachers, each of whom taught in a school district noted for its socially and economically diverse student body. Each participant completed an MKT assessment at the beginning and end of the study, was videotaped during regular instruction, and participated in post-observation debriefings and general and clinical interviews. Based on close examination of their instruction, the researchers generated a mathematical quality of instruction (MQI) scoring rubric. The MQI rubric scored each participant's quality of instruction by examining items such as participants' abilities to connect

classroom activities to mathematics, respond to students appropriately, use appropriate mathematical language, make errors, and develop rich mathematics (Hill et al., 2008). Using a correlation analysis, the study showed that MQI and MKT scores were significantly correlated along specific domains of instruction. For example, responding to students appropriately is significantly correlated to higher MKT scores, while making mathematical and language errors during instruction are significantly associated with lower MKT scores. The authors concluded that there is a “powerful relationship between what a teacher knows, how she knows it, and what she can do in the context of instruction” (Hill et al., 2008, p. 496).

The current theoretical and empirical research on teachers’ mathematical knowledge states that the quality of mathematics instruction depends largely on the teacher’s knowledge of mathematics *for* teaching (Ball, Hill, & Bass, 2005). The implication for teacher education programs is significant – pre-service elementary teachers should take mathematics content courses that foster deep, conceptual understanding of the mathematics they will teach their future students (National Council on Teacher Quality, 2008). What remains unclear is the nature of this knowledge within specific mathematical domains. What *does* pre-service teachers’ content knowledge currently look like within specific mathematical domains? Additionally, what *should* pre-service teachers’ relevant mathematical knowledge look like within such domains? In the following section, these questions will be addressed for the domain of elementary number theory.

Section II: Elementary Number Theory

Elementary number theory is the study of the properties of integers. Within typical K-12 curricula, elementary number theory emphasizes positive integers and includes the study of factors, divisors, multiples, prime factorization, prime and composite numbers, divisibility rules, and divisibility (Campbell & Zazkis, 2002). Number theory is typically emphasized in upper elementary and middle grades curricula, and teachers are expected to help students develop robust knowledge of these topics. In order for elementary students to make sense of elementary number theory topics, however, pre-service teachers must first develop a deep understanding of this domain.

Despite its importance as a foundational set of topics in elementary mathematics, number theory has received inconsistent attention in policy and standards documents over the past twenty years. For a short time, number theory held a prominent place in curricular recommendations such as the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), in which number systems and number theory were explicitly identified as one of thirteen middle school standards. This standard's focus was on the underlying structure of mathematics, understanding of which "promotes students' efficiency in investigating the arithmetic of fractions, decimals, integers, and rationals through the unity of common ideas" (p. 91). The standard also emphasized how topics in number theory might be used to provide students with opportunities for enjoyable and relevant explorations of key mathematical concepts – divisibility, prime, composite, square, even, and odd numbers. Understanding of number theory concepts might also help students see mathematics as a coherent and sensible system instead of as

a collection of arbitrary rules.

During the first several years of the twenty-first century, however, the importance of understanding number theory topics was deemphasized in policy documents. In *Principles and Standards for School Mathematics* (NCTM, 2000), the study of the structure of number systems was subsumed under the *number and operations* standard for grades 3-5; the only explicit mention of number theory topics involved identifying classes of numbers by the nature of their factors (p. 148). In grades 6-8, number theory topics were again incorporated into the *number and operations* standard and described briefly as providing students opportunities to problem solve and reason as they work with prime numbers, factors, multiples, and divisibility (p. 217). Similarly, the NCTM's *Curriculum Focal Points* (2006) explicitly identified number theory topics only twice in its K-8 content guidelines: exploration of prime and composite numbers and factors and multiples in grade 5, and prime factor decomposition in grade 7. Both were labeled as secondary topics that can support the learning of primary focal points at each grade level.

Despite inconsistent attention to number theory topics over the past twenty years, the most recent policy documents have placed a renewed emphasis on the importance of learning number theory topics (*Common Core State Standards*, 2010; MA Department of Education, 2007). Adopted by forty-five states to be the new set of mathematics content standards, the *Common Core State Standards for Mathematics* (2010) identifies factors and multiples, including finding factor pairs and prime and composite numbers, as a grade 4 standard. It also includes a separate grade 6 standard for finding the greatest common factor and least common multiple of two whole numbers. And, in the recent

Massachusetts guidelines for elementary and middle school teacher preparation, number theory is listed as a critical concept that is often misunderstood by teachers (MA Department of Education, 2007, p. 30). With new math standards placing greater emphasis on number theory topics, researchers must begin to carefully examine the concepts and procedures underlying the field.

The domain of elementary number theory is related to Vergnaud's (1988, 1994) *multiplicative conceptual field*, which he defines as the set of all situations that require multiplication, division, or some combination of the two operations, as well as the schemes (i.e., organized cognitive structures), concepts, theorems, and symbolic representations necessary to deal with these situations. The concepts include linear and nonlinear functions, fraction, ratio, rate, rational number, dimensional analysis, and vector spaces (Vergnaud, 1994). Multiplicative situations are classified as simple or double proportion problems, or as comparisons of rates and ratios. Analyzing multiplicative situations requires an understanding of the concepts listed above. As Vergnaud (1988) illustrates, even an easy multiplication problem such as finding the total cost of purchasing four plastic cars at a cost of \$5 each requires an understanding of scalar (4×5) and functional (5×4) relationships in order to properly analyze students' reasoning. The driving force behind describing mathematics in terms of conceptual fields is that concepts do not develop in isolation: "A concept is made meaningful through a variety of situations...at the same time, a situation cannot be analyzed with the help of just one concept; at least several concepts are necessary" (Vergnaud, 1994, p. 46). By studying conceptual fields instead of single concepts, researchers may better understand

and represent individuals' cognitive development.

In an effort to articulate the number theory conceptual field, similar to Vergnaud's multiplicative conceptual field, Campbell and Zazkis (2002) offer an initial description of a *number theory conceptual field* (NTCF) as "primarily concerned with the development of abstract procedural and conceptual understandings of the arithmetic and algebra of whole numbers and integers" (p. 8). They claim that the part of Vergnaud's multiplicative conceptual field that deals with integers may be incorporated into a definition for the NTCF. However, little is currently known about the types of situations that can help make sense of number theory concepts (Campbell & Zazkis, 2002). In order to build toward a more detailed description of the field, it is first necessary to examine prior number theory research in order to identify the concepts and procedures fundamental to its development.

The foundation for the study of elementary number theory is the *multiplicative structure of number*. Zazkis and Campbell (1996a) define the multiplicative structure of number as "conceptual attributes and relations pertaining to and implied by the decomposition of natural numbers as unique products of prime factors" (p. 541). Knowledge of multiplicative structure requires an understanding of the properties of our number system (i.e., distributive, commutative, and associative), arithmetic operations and the inverse relationship between multiplication and division, and the unique factorization of any number into a product of primes (Brown, Thomas, & Tolia, 2002). Multiplicative structure is not only directly related to prime factor, divisibility, and multiples concepts, but it is also related to number theory procedures, such as finding the

number of factors of a number, the greatest common factor (GCF) of two numbers, and the least common multiple (LCM) of two numbers. For example, adequate attention to the multiplicative structure of a number requires the ability to construct the number's prime factorization and recognize the divisibility properties inherent in its structure (Zazkis & Gadowsky, 2001). In fact, developing an understanding for divisibility appears to begin with an understanding of multiplicative structure (Brown, Thomas, & Tolias, 2002).

Surprisingly, there is limited research on teachers' knowledge of multiplicative structure and other number theory topics (Campbell & Zazkis, 2002). One reason might be that number theory is not typically designated as a stand-alone unit in K-12 curricula; it is often intertwined with arithmetic, algebra, and geometry units. The research literature that does exist regarding number theory focuses on pre-service elementary teachers' understanding of specific number theory topics, such as divisibility and prime factorization.

Zazkis and colleagues (1996a, 1996b, 1998, 1999) conducted a three-year study of pre-service elementary teachers' understanding of number theory concepts. The participants were all enrolled in a required course called "Foundations of Mathematics for Teachers." Questionnaires and interviews were administered following a number theory unit that covered topics including factorization, LCM, GCF, prime and composite numbers, and divisibility. During one-hour long individual interviews, the authors probed participants' understanding by asking them to explain and justify their responses. Participants' responses to the questionnaire and interview questions were used to conduct

multiple analyses of participants' understanding of the various number theory topics.

The following are summaries of five of these analyses.

In one analysis of the interviews, Zazkis and Campbell (1996a) examined the responses of twenty-one participants to clarify their understanding of divisibility concepts and procedures. Results showed that most pre-service teachers had to execute procedures in order to solve divisibility problems. Even those who displayed some conceptual understanding still exhibited what the authors referred to as “procedural attachments,” meaning that they reverted back to procedural calculations in order to verify their solutions. For example, when examining the number, $M = 3^3 \times 5^2 \times 7$, only six out of the twenty-one participants were able to use the prime factorization of M to determine if M was divisible by specific prime and composite numbers, while the remaining fifteen participants had to, at some point, perform long division. Furthermore, when determining which numbers between 12,358 and 12,368 were divisible by 7, many participants expressed the need to divide each number between the two by 7 to make sure that one of them was divisible. Additional findings from the data included: inferring divisibility by composite numbers was much more difficult for participants than inferring divisibility by prime numbers; inferring indivisibility was more difficult for participants than inferring divisibility; and participants had a tendency to incorrectly apply known divisibility rules to inappropriate situations (e.g., 391 is prime because the sum of its digits is a prime number). The authors suggested that helping participants make connections between divisibility and prime factorization would greatly contribute to a conceptual understanding of divisibility, and ultimately, the multiplicative structure of number.

Additional analysis of the same data (Zazkis & Campbell, 1996b) focused on pre-service elementary teachers' understanding of the *Fundamental Theorem of Arithmetic*, a foundational concept in number theory that states that “decomposition of a composite number into its prime factors exists and is unique except for the order in which the prime factors appear in the product” (p. 207). The authors concluded that participants did not have a strong grasp of the Fundamental Theorem of Arithmetic – while they knew how to decompose a number into a product of prime factors, they had difficulty applying the concept to different problems. They also hypothesized that this may help explain why participants' understanding of divisibility appeared to develop before their understanding of indivisibility, as discovered in the previous analysis. While more than half of the fifty-four participants who took a written questionnaire could explain why $M = 3^3 \times 5^2 \times 7$ was divisible by 7, 5, and 3, many of them could not provide justifications for indivisibility by 2 and 11. The authors suggested that these participants did not have a well-formed notion of the *uniqueness* feature of the Fundamental Theorem of Arithmetic; their responses showed an implicit belief in the possibility of alternative prime factorizations for M that could show divisibility by 2 and 11. Similarly, when asked to decide whether $K = 16,199 = 97 \times 167$, where 97 and 167 are each prime, was divisible by other prime numbers, only one-fourth of the participants could correctly explain that no primes other than 97 and 167 can be divisors of K since they are not in K 's prime factorization. This was further evidence of participants' inadequate understanding of the Fundamental Theorem of Arithmetic. As a result, the authors suggested that educators place a greater focus on teaching prime and composite numbers, noting that “if the concepts of prime and

composite numbers have not been adequately constructed, this will likely inhibit any meaningful conceptualization of prime decomposition” (p. 217).

Upon further examination of the data from the three-year study discussed above, Zazkis (1998) observed that pre-service teachers’ understanding of even and odd numbers may also be disconnected from divisibility concepts. To investigate this observation, she administered a written questionnaire and conducted individual interviews with the last cohort of seventy-three pre-service elementary teachers enrolled in the “Foundations of Mathematics for Teachers” course. One of the questions asked participants to decide if numbers such as 1234567, 3^{100} , 3^{99} , $2^{100} + 3$, and $7^{50} \times 3^{40}$ are even or odd, and to briefly explain their decisions. Its purpose was to identify participants’ conceptions of even and odd numbers, and the extent to which they could use the multiplicative structure of number to determine a number’s parity. Results of the study showed that many participants perceived parity to be a function of the last digit of the number, not a function of the number’s divisibility by 2. For example, all participants identified 1234567 as odd because its last digit is 7 and used the argument that the product of odd numbers is odd when examining numbers such as 3^{99} . Although some participants did use the existence of 2 as a factor to identify evenness, few used the absence of 2 as a factor to identify a number as odd. For example, while at least twenty-five participants claimed that 1234567×2^{40} is even for reasons such as “it is multiplied by 2” or “it has a factor of 2”, only six participants mentioned a lack of 2 as a factor to explain why $7^{50} \times 3^{40}$ is odd. The authors contended that pre-service teachers do not perceive evenness to be equivalent to divisibility by 2, and that such equivalence is

necessary in order to help foster deep understanding of divisibility. In fact, they argue that instruction should focus on divisibility by 2 as a special case of divisibility by a prime number in order to bridge the gap between evenness and divisibility by other numbers. To do so, they claim, requires postponing the teaching of the last digit rule for evenness “in order to achieve even/odd classification by substance and not by form” (p. 86).

Zazkis (1999) further investigated pre-service elementary teachers’ intuitive notions of factors, specifically their belief that *the greater a number the more factors it must have*. Zazkis conducted individual interviews with fifteen pre-service teachers enrolled in the same required university course as in the prior analyses. During these interviews, more than half of the participants claimed that $A = 3^2 \times 7$ has fewer factors than $B = 3^2 \times 17$ (Note: A and B have the same number of factors). In fact, several participants had to list the factors of A and B in order to acknowledge their assertion was incorrect. In a second phase of the study, Zazkis (1999) administered a written questionnaire to a different cohort of fifty-eight pre-service elementary teachers. The questionnaire asked participants to agree or disagree with the conjecture that the greater the magnitude of a number, the greater the number of factors it must have. Participants responded to this conjecture for both natural numbers and composite natural numbers.

Most participants correctly claimed that greater natural numbers do not necessarily have more factors. Forty-two out of forty-eight participants gave prime numbers as counter-examples for their justifications. Likewise, most participants correctly claimed that greater composite natural numbers do not necessarily have more

factors either. However, the number of participants who correctly made this second claim was fewer than the number who made the same claim for natural numbers. The majority of participants' justifications for composite natural numbers involved squares of primes, since all squares of primes have just three factors. Zazkis (1999) suggested that such results are misleading – although it appears pre-service teachers do not hold the intuitive belief that the size of a number correlates positively with its number of factors, their investigation revealed that participants did, in fact, hold this belief but viewed prime numbers as the sole exception to the rule. Zazkis claimed that knowledge of students' intuitive beliefs is critical for developing curricula and planning instruction. By understanding the nature of these intuitive beliefs, teacher educators can anticipate misconceptions and modify instruction to help pre-service teachers make connections between multiplicative structure and the number of factors of a number. Zazkis (1999) suggests constructing activities that examine a variety of special cases such as cubes of primes or products of primes.

Pre-service teachers' understanding of the connections between division and divisibility (Campbell, 2002) was also investigated in this set of studies. Prior analyses by Zazkis and Campbell (1996a) suggested that pre-service teachers' difficulty with divisibility and multiplicative structure was related to their conceptions of division. As such, the researcher conducted individual interviews with pre-service elementary teachers around their notions of quotient and remainder. One question asked participants to find the remainder and quotient if $A = 6 \times 147 + 1$ is divided by 6 or by 2. Another asked them to find the remainder and quotient when $B = 6 \times 147 + 2$ is divided by 2. Results

showed that many participants had difficulty applying their knowledge of division to novel problems. Most were forced to first compute the value of A and B since they were not accustomed to their non-traditional representation. Participants also exhibited inadequate understanding of the relationship between whole number and rational number division. When asked to use a calculator to divide 10561 by 24 so that the answer was expressed as a quotient with a remainder, most could not use the rational quotient, $440.04\overline{16}$, to find the whole number remainder. Campbell (2002) argued that poor understanding of the two types of divisions is rooted in participants' confusion between quotients and remainders, since "whole number division, in contrast with rational number division, yields an answer with not one, but two components (the quotient and the remainder)" (p. 34). These results provided additional evidence of pre-service teachers' confusion with divisibility and division. Furthermore, procedural attachments permeated participants' work with division problems, and the use of abstract terminology, such as *divisor*, *dividend*, *quotient*, and *remainder* exacerbated their confusion (Zazkis & Campbell, 1996a; Campbell, 2002).

In a separate study, pre-service elementary teachers' understanding of prime numbers was found to be further influenced by the representation used. Zazkis and Liljedahl (2004) administered a written questionnaire to a sample of 116 participants enrolled in a university mathematics education course, and conducted individual interviews with a subgroup of eighteen volunteers. These data were collected after participants completed a 3-4 week unit on number theory topics. The written questionnaire asked participants to determine if $F = 151 \times 157$ is prime, and to explain

their decision. Following the written questionnaire, participants were also asked two interview questions: 1) describe prime and composite numbers and their interrelationship, and 2) determine if a number in the form of $m(2k+1)$, where m and k are whole numbers, is prime. Participant responses showed that their understanding of primes was “incomplete, inconsistent, fragile, algorithm-driven, and significantly influenced by particular examples” (p. 180). While most participants were able to articulate definitions of prime and composite numbers, less than half were able to apply those definitions to justify that $F = 151 \times 157$ is composite. Most remaining participants either reverted to using a known algorithm for determining if a number is prime, or incorrectly claimed that F is prime because the product of two primes is a prime number. The authors also identified three ways in which participants understand the concept of prime number: as the result of a factoring process that leads to trivial factors; by generalizing from examples (e.g., prime numbers are small); or by what prime numbers are not (i.e. primes cannot be factored by any number other than 1 and itself). Regardless of the way in which participants came to understand prime number, the authors argued that the lack of transparent representations of prime number was an obstacle in developing participant understanding.

The notion of *transparent representations* as it pertains to number theory comes from Zazkis and Gadowsky’s work (2001) with elementary students and pre-service teachers on using different representations of a number to identify its properties. Based on prior research (Lesh, Behr, & Post, 1987, as cited by Zazkis & Gadowsky, 2001), Zazkis and Gadowsky defined a *transparent representation* of a number as a

representation specifically designed to illustrate a particular property of the number (e.g., the representation 6^2 is transparent in showing that 36 is a perfect square). On the other hand, an *opaque representation* with respect to a particular property is one that de-emphasizes that property (e.g., while 6^2 makes it clear that 36 is a perfect square, it is not apparent from this representation that 9 is a factor of 36). As part of the study, participants were asked if $A = 15 \times 5623 + 60$ could be represented as a multiple of 15. To answer this question, most participants first calculated the value of A and then divided it by 15 instead of using distributive property to represent A as $15 \times (5623 + 4)$. The researchers concluded from their study that students do not attend to *transparent* features of number representations, and instead, typically revert to the number's conventional decimal form (e.g., represent 6^2 as 36). Such tendencies, the authors argued, are evidence of students' need to verify number properties computationally and of their narrow conception of the representation of number as a single numeral. Zazkis and Liljedahl (2004) defended the need for analysis of transparent and opaque representations by claiming that "representations are often considered as a means to form conceptual understanding" (p. 167).

Brown, Thomas, and Tolia (2002) investigated pre-service elementary teachers' understanding of divisibility using a series of case studies of participants enrolled in a university course focused on the foundations of arithmetic. Each participant was interviewed at the end of the course regarding divisibility vocabulary, making divisibility decisions based on a number's prime factorization, and least common multiple (LCM). Participants showed strong tendencies to associate divisibility with the act of dividing

instead of attending to the number's multiplicative structure. The authors noted that such a dependency on procedure can inhibit participants from developing stronger schemas of multiplication and division concepts. To remedy such habits, the authors recommended that pre-service teachers be exposed to operations on numbers in prime factorization form so that they can begin to perceive this representation of a number as a single value and not as a signal to compute.

Brown and colleagues (2002) also uncovered three prevailing approaches that participants used to find the least common multiple of a pair of numbers: *set intersection*, *create a multiple and divide*, and *prime factorization*. The researchers noted that while the prime factorization method is the most commonly taught method in textbooks, most participants could not explain why the method worked even if they were able to execute it well. They recommended linking different representations of the LCM. For example, the authors suggested that linking the *create a multiple and divide* method to the *prime factorization method* could help students develop a deeper understanding for the “higher exponent” rule for finding the LCM of two numbers presented in prime factored form.

Research on pre-service teachers' knowledge of multiplicative relationships supports the number theory research that suggests pre-service teachers struggle to make sense of the multiplicative relationships that underlie many number theory topics. In her analysis of interviews with pre-service elementary and secondary teachers, Deborah Ball (1990a) showed that the majority of her subjects held a narrow, rule-based, understanding of division. Research on pre-service teachers' misconceptions regarding multiplication and division showed that many continue to be influenced by primitive multiplicative

models that promote incorrect notions, such as “multiplication always makes bigger” and “division always makes smaller” (Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1989; Simon, 1993). In a study of fifteen pre-service teachers who had finished their mathematics content course requirements, it was found that participants struggled to make sense of the rectangular array model of multiplication and the distributive and commutative properties of multiplication despite having a well-developed understanding of multiplication as repeated addition (Chinnappan, 2005). Thus, the researcher suggested that participants’ use of repeated addition might be taken as evidence of their superficial understanding of multiplication.

Unlike the previous studies, a more recent study examined the effects of classroom technology on pre-service elementary teachers’ understanding of factors, multiples, and prime numbers (Sinclair, Zazkis, & Liljedahl, 2004; Liljedahl, Sinclair, & Zazkis, 2006). The purpose of the study was to see if a computer-based program, or *micro-world*, could promote a more connected understanding of elementary number theory. The micro-world is an interactive number grid that shows square numbers, multiples, primes, and factors up to and including a specified value. The width of the grid can also be altered or reset. Approximately forty-five pre-service teachers volunteered to participate in the study, which took place after all participants completed a unit on elementary number theory. Participants were observed completing several tasks using the micro-world software, for which they answered several written questions. Individual clinical interviews were then conducted with a smaller sample of seventeen volunteers from the course.

Results from the micro-world study found that the visual display of the micro-world helped shape participants' images of multiples and factors. Participants developed an image of multiples as having "constant spread", and an image of factors as "non-patterned, sporadic, or asymmetrical". Unlike traditional instruction, the micro-world allowed participants to make clear distinctions between factors and multiples (Sinclair et al., 2004). Given that pre-service teachers often confuse factors and multiples (Zazkis, 2000a, as cited in Sinclair et al., 2004), one important outcome from the study was that the micro-world clarified the finite nature of factors and the infinite nature of multiples (Liljedahl et al., 2006). The researchers also found that some common student misconceptions, such as the notion that the greater a number the more factors it has, were absent from participants' work. The authors attributed participants' success to the unique visual feature of the micro-world, which they claim provided an unfamiliar, challenging way of examining number theory concepts (Sinclair et al., 2004).

The prevailing research on pre-service elementary teachers' understanding of number theory topics has focused on their displayed subject matter knowledge – specifically the common and specialized content knowledge (Ball et al., 2008) that they exhibit on written assessments and during individual clinical interviews following elementary number theory instruction at the university level. Such research has shed considerable light on the knowledge that pre-service elementary teachers do not have: confusion with mathematical terminology such as factors, divisors, and multiples; tendency to conflate divisibility concepts with long division; lack of attention to multiplicative structure; and a score of intuition-based, erroneous beliefs about the nature

of prime numbers and factors (Zazkis & Campbell, 1996a; Zazkis, 1999; Zazkis, 2000a; Brown et al., 2002).

Section III: Analyzing Pre-Service Teachers' Knowledge

Although prior research on number theory has identified pre-service elementary teachers' key misconceptions, it has not begun to identify the ways in which pre-service teachers develop their understanding of this content. What understanding do they initially exhibit and how does their understanding progress over the course of instruction? By generating rich descriptions of the thought processes they take in order to make sense of specific concepts, research will be better able to develop, test, and refine mathematics instruction that promotes the mathematical knowledge for teaching elementary number theory that is critically needed in teacher education programs.

This study is founded on a constructivist notion of understanding. Its theoretical framework is Dubinsky's (1991) APOS theory, a constructivist model for the development of individual understanding. APOS theory arose from Piaget's *reflective abstraction* and *equilibration*, which is described in detail below. Additionally, Sfard's (1991) three stages of concept development are also presented as a model of understanding that anticipates APOS theory and its usefulness in analyzing pre-service teachers' understanding of number theory concepts.

In order to describe pre-service teachers' developing understanding of elementary number theory, one must first define understanding. In the past thirty years, several theories of understanding have emerged to describe how individuals come to understand

mathematics (von Glasersfeld, 1995; Sierpiska, 1994; Sfard, 1991, 1994; Skemp, 1987; Piaget, 1970). Despite making significant improvements over prior notions of understanding, these theories do not provide researchers with a clear consensus of the meaning of understanding (Meel, 2003). To clarify what is meant by understanding, it is important to describe some of these theories and their commonalities. Since many of these theories are grounded in the constructivist notion of understanding, it is appropriate to begin there.

Constructivism

In the fifth century BC, the Greek philosopher Protagoras once said, “Man is the measure of all things.” Though his words should not be taken literally, what he most likely meant was that notions such as truth and knowledge are derived from human experience (Goldman, 2006). Protagoras opposed the popular notion that there exists an objective reality that one strives to uncover, and instead favored the idea that what one calls reality is based on experience. For example, there was a time when we had no concept of *integer*, yet a need eventually arose when we had to consider solving problems such as 4-6. Our current number classification system – *counting, whole, integer, rational, irrational, real, and complex* – developed from such human needs to solve real-life problems.

Protagoras’s view represents an early version of a theory of knowing called *constructivism*. The eminent Swiss psychologist and philosopher, Jean Piaget, is widely credited with the development of the constructivist theory of knowing (von Glasersfeld,

1995; Ernest, 1994). The main tenet of this theory is that knowledge arises from individuals' actions on their environment (Piaget, 1970). Individuals act on their experiences through physical or mental activity, and it is through such activity that knowledge develops. In other words, knowledge is a process by which humans construct more organized mental structures to help them make sense of their own experiences (Piaget, 1971). Piaget summarizes this assertion concisely: "The mind organizes the world by organizing itself" (Piaget, 1937, as cited by von Glasersfeld, 1995, p. 57).

Constructivist theory opposes conventional epistemology, which states that there exists an objective reality that is independent of human thought (von Glasersfeld, 1987). For example, conventional wisdom states that the notion of *prime number* exists independently of how humans think about primes – it is only a matter of time before humans discover its true meaning. In order to understand prime number concepts, then, one must compare what one knows about a prime number (i.e., an individual's notion of prime) with its true meaning (i.e., an absolute truth). However, according to Piaget (1970), such a comparison implies that one must already have a model for the true meaning of primes. With such circular logic, it is impossible to know for certain if one's knowledge of prime number is the true meaning of prime number. Constructivist theory uses this critique of an objective reality to explain that one's view of reality is constantly shaped by one's experiences. It refers to this reality as an "experiential reality" that is governed by the individual's interpretation of his experiences (von Glasersfeld, 1995).

Piaget's constructivist theory of knowing, which he referred to as *genetic epistemology*, attempts to create a model for human cognition that explains what

knowledge consists of and how it develops from states of lesser to greater knowledge (Campbell, 2001). Underlying this model is the notion that “knowledge is necessarily determined to a large extent, if not altogether, by the knower’s ways of perceiving and conceiving” (Glaserfeld, 1995, p. 55). An individual’s knowledge consists of cognitive structures, or mental schemes, that can develop as the individual acts on his experiences. Von Glaserfeld (1987) refers to the way in which the individual organizes such structures as understanding.

Organizing cognitive structures involves two fundamental processes: *assimilation* and *accommodation* (Piaget, 1970, 1971, 1985). Assimilation is defined as “the incorporation of an external element, for example, an object or event, into a sensorimotor or conceptual scheme of the subject” (Piaget, 1985, p. 5). In other words, assimilation occurs when one fits an experience into one’s pre-existing cognitive structure. Piaget (1985) notes that the assimilating process can take on multiple forms: interactions with the external environment versus interactions between mental schemes. When assimilating an experience into a known structure or scheme, the individual disregards, or is unaware of, elements of the experience that do not fit into his scheme. In fact, cognitive assimilation results in the individual perceiving only that which can fit into his pre-existing structure (von Glaserfeld, 1995). In this sense, the development of understanding is adaptive to the constraints established by one’s experiences. Once a situation is recognized as fitting into a known scheme, the situation is assimilated, resulting in the individual taking a specific action. For example, when asked if 30 is divisible by 5, a student who associates divisibility with the act of division might

assimilate the situation by performing long division.

Accommodation is the second process of organizing cognitive structures. It refers to a modification of one's cognitive structure when an individual does not successfully assimilate an experience into his mental scheme (e.g., the experience induces cognitive conflict). In such a case, he will accommodate the scheme to elements of the conflicting experience by reorganizing or replacing his scheme. In some cases, the new experience may be ignored to preserve the old scheme. For example, if a student is asked whether 5^{20} is divisible by 5, his divisibility scheme may not be able to support such a calculation. Instead, he may use a calculator to convert 5^{20} to decimal form and then use long division to determine divisibility. Upon performing this action several times for other division situations, the individual may notice an emerging new pattern that informs his divisibility scheme (e.g., a number is divisible by 5 if 5 is one of its factors). Such recognition leads to an accommodation of the old divisibility scheme, so that new problems are more easily assimilated. When a scheme is accommodated, new connections are formed between previously constructed concepts or between old and new information (Hiebert & Carpenter, 1992).

Understanding develops as the processes of assimilation and accommodation near a balance, or *equilibration* (Piaget, 1985). Piaget claims that an equilibrium between assimilation and accommodation "forms the point of departure of all knowledge and is presented at the very outset in the form of a complex relation between the subject and the objects" (Piaget, 1971, p. 108). Von Glasersfeld (1995) refers to this equilibrium as the "elimination of perturbations" (p. 67). In other words, understanding develops when

individuals are able to accommodate, or eliminate, a greater range of perturbations. Accommodation naturally follows assimilation and both processes occur continuously as individuals constantly attempt to make sense of their environment. This helps to explain why understanding is said to always be developing (Carpenter & Lehrer, 1999). It also supports the constructivist notion that knowledge derives from human experience and not an objective reality (von Glasersfeld, 1987).

While the notion of equilibration can help to explain the development of understanding, both as interactions with the external environment and between mental schemes, it does not explain the nature of such development (Campbell, 2001). In order to understand the processes that lead to higher levels of understanding, one must first make sense of the role of *abstraction* in Piaget's genetic epistemology theory.

An abstraction is a type of lasting mental change resulting in the development of a concept (Skemp, 1987). A concept has been abstracted, to some degree, when similarities among its instantiations are recognized. An individual who recognizes similarities between several objects or actions can begin to identify new objects as possessing those similarities, allowing for their classification. Higher-order abstractions result in the ability of the individual to visualize the concept without seeing it in context, reflect on its structure, and operate on it mentally (Battista, 2007).

Piaget (1970) identifies three different types of abstraction. The lowest level, *empirical abstraction*, refers to knowledge generation from experience with an external object, such as when a young child holds several apples and recognizes that bigger apples tend to weigh more than smaller apples (Piaget, 1970). In this case, the individual acts on

an object and derives knowledge exclusively from the properties of the object itself. The key characteristic of empirical abstraction is that knowledge is constructed from the individual's perceptions of an external object, and not by the individual's actions on that object. Such abstraction is most often associated with sensory-motor or physical actions and is only indirectly associated with the development of cognitive structures (Piaget, 1971, 1985).

Reflective abstraction is Piaget's highest level of abstraction. It refers to concept formation derived exclusively from reflecting on one's actions on objects (Piaget, 1970). When one acts on an object, one applies new characteristics to the object without changing its original properties. The subject then reflects on his or her own coordinated actions and makes connections between the new characteristics (Piaget, 1970, 1971). An example of reflective abstraction is an individual's coordination of counting, reordering, and recounting actions to abstract the notion of commutativity (Dubinsky, 1991). An individual might count two or more sets of chips, note their sum, reorder the chips, and then note their sum. Knowledge of the commutative property is not derived from the characteristics of the chips themselves; instead, it is generated by reflecting on the reordering action taken on the chips.

Piaget (2001) classified *pseudo-empirical abstraction* as an intermediate level of abstraction. It refers to knowledge building that results from the individual acting on a physical object (e.g., ordering a group of objects) in a way that introduces new properties to the object (e.g., the order of each object in the group). So while the new properties are identified from the objects themselves, these properties are introduced not from the

objects but from the actions of the individual. For example, in order to understand multiplication, a child first observes, via empirical abstraction, that a fixed number of chips are added each time. Then, as the child continues to add a fixed number of chips, he begins to keep track of how many times he is adding this fixed amount (Campbell, 2001). This recognition is conscious on the part of the child, and requires the child to reflect on his actions, even though the properties are perceptible from the chips themselves. Piaget sees pseudo-empirical abstraction as an intermediate form of abstraction because it relies on both the individual's perceptions of the elements of an external object (i.e., empirical abstraction) and on reflections of the individual's actions on that object (i.e., reflective abstraction). However, he does note that "it is really a special case of reflective abstraction and not in any way a derivative of empirical abstraction" (Piaget, 2001, p. 303). In line with other researchers in the field (Campbell, 2001; Simon et al., 2004), this study will not distinguish between pseudo-empirical abstraction and reflective abstraction.

Reflective abstraction is a two-phase process of *projection* followed by *reflection* (Campbell, 2001; Piaget, 1985). The initial projection phase involves moving from a lower to a higher level of thinking by becoming conscious of certain concepts that were previously used without awareness. For example, Piaget (2001) conducted an experiment in which young children were asked to make equal-length towers of different colored chips taken two at a time (A chips) and three at a time (B chips). Subjects who did not exhibit reflective abstraction focused entirely on the chips and not on their actions. They could not anticipate the result and were not aware that the number of times they added

each type of chip mattered. Projection, and thus the beginnings of reflective abstraction, was evident when some subjects became aware of the number of times that they had used each type of chip. Since the A chips were taken two at a time while the B chips were taken three at a time, they began to recognize that a greater number of groups of A chips must be taken in order to create equal amounts of A and B chips.

The second phase of reflective abstraction, *reflection*, involves using the new information that was projected to help reorganize one's cognitive structure at the higher level (Campbell, 2001). At this phase, the individual begins to generalize her thinking by constructing relationships between what was projected and what already existed at the higher level of thought. In the equal-length towers example above, reflection occurs when the subject articulates the multiplicative operation " n times x ", where n represents the number of times the A or B chips were added. In this scenario, n is no longer seen as the number of groups of chips, but rather as the number of times each pair or triplet of chips is added. At this phase, the subject's notion of multiplication develops from "addition of additions" to a generalized operation.

Piaget's constructivist theory of knowing, with its emphasis on equilibration and reflective abstraction, forms the basis for the current study's operationalization of understanding. The study views its participants as undergoing a variety of number theory tasks that they either assimilate into their prior mental schemes or that require them to modify their schemes to make sense of those tasks. As participants attempt to accommodate their schemes to the new information, they may become aware of new properties or concepts and begin to reflect on and reorganize their prior knowledge.

Piaget's theory is prevalent in the literature on concept formation, and has been used in a variety of studies on students' understanding of mathematical content (Cappetta, 2007; Simon, Tzur, Heinz, & Kinzel, 2004; Cooley, 2002; Goodson-Espy, 1998). However, research does reveal some variation on the meaning of understanding (Meel, 2003). In the thirty years since Piaget's work, several theories of understanding have been developed in the hopes of clarifying the notion. Two of these theories are described briefly, while a third that pertains directly to APOS theory is described in greater detail.

Skemp's (1976) influential work views understanding as either *relational* or *instrumental*. Relational understanding represents knowing what to do and why it is done, while instrumental understanding refers to knowing rules without the reasoning behind them. Although Skemp (1987) identifies relational and instrumental as two types of understanding, he clarifies that only relational understanding represents conceptual understanding, or in Piaget's language, "understanding as assimilation into an appropriate schema" (p. 29). Skemp's view of understanding centers around the notion of *schema*, or a collection of higher-order conceptual structures that organize knowledge into a coherent whole. A schema is useful because "it integrates knowledge, it acts as a tool for future learning and it makes possible understanding" (p. 24). An individual who has relational understanding of a concept assimilates that concept into a coherent and organized mental system. As a result, she is able to adapt to new problems, remember rules more easily, and generate new understanding. An individual with instrumental understanding, on the other hand, can apply a remembered rule to a problem accurately, but does not know why that rule makes sense (Skemp, 1987). Such understanding is associated with fast recall

and efficient computational skills, but it does not lead to new understanding because one's schema is not connected and well-organized. If the individual makes a mistake while solving a problem, for example, he will have difficulty troubleshooting his error because he does not have a viable schema on which to rely.

Sierpinska (1990) defines understanding as both an act and a process that is based on one's prior beliefs, preconceptions, convictions, and mental schemas. An *act of understanding* refers to a mental experience in which the individual is able to make sense of a concept or solve a difficult problem; it is often characterized as a flash of recognition. The act typically occurs at a brief moment in time following a prolonged period of mental effort. A *process of understanding* refers to mental activity that occurs over a longer period of time, within which acts of understanding serve as critical steps (Sierpinska, 1994). Understanding develops as one makes a conjecture about an object of interest, leading to an act of understanding. The act of understanding induces the individual to reason in a way that helps validate or reject the conjecture and leads to a second act of understanding. Sierpinska (1994) proposes that understanding can also develop as one faces *cognitive obstacles* and attempts to overcome them. Overcoming such obstacles leads to processes of understanding, whereby the learner must revisit her prior knowledge, reassess her preconceptions, and reorganize her mental networks. The result is a new way of understanding.

Both Skemp's and Sierpinska's definitions of understanding share features of Piaget's constructivist theory. Primarily, they subscribe to the notion that a learner develops understanding of an object by constructing increasingly stable and connected

mental representations of the object (Meel, 2003). Skemp (1976, 1987) uses the notion of schema and its cognitive structures to describe connected mental representations. Similarly, Sierpiska (1994) uses acts and processes of understanding, as well as cognitive obstacles, to explain how the learner revisits old information in order to reorganize cognitive structures and create better mental representations.

With a better understanding of Piaget's theory of understanding and its influence on other definitions, one must now turn to Sfard's (1991) work on concept development in order to better understand this study's theoretical framework. A *concept* refers to the official, accepted form of a mathematical idea, while a *conception* refers to an individual's internal representations evoked by a concept. As the individual gains more experience with a concept, his or her conception is susceptible to change. According to Sfard (1991), there are two types of conceptions: *operational* and *structural*. The former views mathematical concepts as outcomes of processes, algorithms, or actions, while the latter treats them as abstract, static objects. For example, an operational conception of *factor* might involve seeing a factor, A , of another number, B , as a number that generates a zero remainder when B is divided by A . A structural conception of factor, however, might define A as an element (or a composition of elements) of B 's prime factorization. The structural conception of factor above is static because it reflects the multiplicative structure of the number, not the result of some operation or transformation.

Sfard (1991) claims that operational and structural conceptions are complementary elements of the same mathematical concept, as the ability to see an idea as both a process and an object is needed for a deep understanding of mathematics.

Nevertheless, a structural conception represents a more advanced stage of concept development and is typically preceded by an operational conception (Sfard, 1991). As learners develop more abstract conceptions their mental representations shift from operational to structural. Once abstracted, the concept can then be acted on operationally to begin an entirely new cycle of concept development: “Such hierarchy emerges in a long sequence of reifications, each one of them starting where the former ends, each one of them adding a new layer to the complex system of abstract notions” (p. 16).

Sfard (1991) describes three stages of concept development: *interiorization*, *condensation*, and *reification*. During the interiorization stage, a learner becomes familiar with mathematical processes that will eventually lead to a new concept. A process is interiorized when it can be executed mentally. For example, a learner striving to understand divisibility will begin by using long division to determine if one number is divisible by another. Once this process is interiorized, the learner becomes more skilled at determining divisibility and can often perform such calculations mentally, often by invoking specific divisibility rules. At the condensation stage, the learner begins to look at the process as a whole without attending to all of its details. In the divisibility example, the learner begins to determine divisibility by using a number’s prime factorization instead of performing computations, or she might combine divisibility rules to determine divisibility by composite numbers (e.g., if a number is divisible by 3 and by 2, then it is also divisible by 6). Reification, the final stage of concept development, occurs when the mathematical idea can be seen as a single object or entity completely independent of its underlying process. For example, the concept of divisibility is reified

when the learner sees divisibility as an intrinsic property of number (i.e., as an element of its prime factorization) and not as the outcome of performing division.

Sfard's three stages of concept development directly lead to Dubinsky's (1991) *action-process-object-schema (APOS)* theory, which is this study's theoretical framework. APOS theory is based on the constructivist philosophy of learning, according to which individuals construct their own understanding by interacting with the environment around them. It is primarily an elaboration of Piaget's notion of *reflective abstraction*. The theory was originally used to describe how undergraduate mathematics students develop understanding of advanced mathematical concepts, including calculus, functions, and group theory (Dubinsky & McDonald, 2001). However, researchers have also successfully applied APOS theory to elementary mathematics topics (Zazkis & Campbell, 1996a, 1996b).

According to APOS theory, understanding a mathematical concept involves constructing mental representations (e.g., processes and objects) and organizing them in ways that help make sense of a concept or problem. Similar to Sfard's concept development from operational to structural, understanding under APOS theory develops as mental representations become more abstract and connected. Dubinsky and his colleagues explain the development process as follows:

Understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas. (Asiala et al., 1996, p. 9)

The mental constructions involved in the development of concepts are *interiorization*, *coordination*, *encapsulation*, *generalization*, and *reversal* (Dubinsky, 1991). As the individual makes these constructions, he reaches increasingly sophisticated levels of understanding: *action*, *process*, *object*, and *schema*.

An *action* is any transformation of a previously known object (physical or mental) where the transformation is seen as external to the individual performing the transformation (Asiala et al., 1996). Actions are typically repeatable, algorithmic procedures for altering the object in some fashion. Since the transformation is external to the individual, he who possesses an action conception of some mathematical concept requires step-by-step instructions on how to perform the action. For example, Zazkis and Campbell (1996a) describe students who need to perform long division by 5 in order to see whether a given number is divisible by 5 as having an action conception of divisibility.

When an individual repeats an action and begins to reflect on its steps, he can construct an internal representation for that action. At this point, the individual no longer requires external cues to perform the action, and can often execute it mentally without actually performing the steps. If this occurs, we say that the action has been interiorized into a *process* (Dubinsky & McDonald, 2001). This interiorization is identical to Sfard's (1991) interiorization of procedures into mental processes. In a study of college mathematics students, Breidenbach, Dubinsky, Hawks, and Nichols (1992) examined students' conceptions of function in the context of a computer learning environment. They found that students with a process conception can imagine a transformation taking

place (i.e., an input being operated on and resulting in an output) while an action conception requires the existence of an explicit formula or directions for transforming an input. An appropriate example would be the sine function, $\sin(x)$, since in order to transform an input value the individual must be able to imagine a process of associating an input value with its output value (Asiala et al., 1996).

In the context of elementary number theory, interiorizing an action to form a process conception of division means that the learner can imagine the process of long division without having to perform the algorithm itself. For example, when asked if there is a number between 12,358 and 12,368 that is divisible by 7, one participant in Zazkis and Campbell's (1996a) study computed $12,362 \div 7$, saw a remainder of 4, and counted up 3 units to show that 12,365 is divisible by 7. The researchers concluded that this participant was developing a process conception for divisibility because she was able to reenact the division procedure in her imagination (i.e., *intended* division) with some anticipation of the outcome. Those at the action level performed *explicit* division – divided each number between 12,358 and 12,368 by 7 – in order to answer this question.

In addition to interiorizing an action, an individual can construct new processes by combining two or more processes or reversing a process (Dubinsky, 1991; Breidenbach et al., 1992). APOS theorists refer to these two mental constructions as *coordination* and *reversal*, respectively. Coordination is quite similar to Sfard's (1991) condensation, where lengthy processes are compressed into manageable chunks, such as when an individual might view the GCF of two numbers as the “overlap” or the LCM as “all of the numbers multiplied together” in a Venn diagram. According to APOS theory,

coordination occurs when a student can combine his interiorized conceptions of divisibility by 5 and by 7 to infer divisibility by 35. Coordination can also take place with seemingly unrelated processes, such as combining the processes of finding the least common multiple (LCM) of two numbers and the prime factorization of a number in order to create a new, more efficient, process for finding the LCM. Reversal takes place when, for example, the student can undo the steps of his coordinated process for inferring divisibility by 35 in order to identify divisibility by 5 and by 7 if a number is said to be divisible by 35. In all three constructions of process development, the individual reflects on the process and anticipates its results in advance of executing the procedure.

An individual encapsulates a process into an *object* when he transforms the process via a set of actions, reflects on those actions, and begins to see the process as a single entity (Asiala et al., 1996). Zazkis and Campbell (1996a) define *encapsulation* as the development of conceptual understanding. This advanced stage of concept development is comparable to Sfard's (1991) reification, in which something is suddenly seen as consisting of a static structure existing independent of its associated process. The key component of encapsulation is that the learner can now operate on the process itself, transforming it into an object of thought. Encapsulating the divisibility concept, for example, results in seeing divisibility as an intrinsic property of whole numbers – one number is or is not divisible by some other number. An individual with an object conception of divisibility can infer divisibility by attending to the multiplicative structure of a number without the need to perform or imagine the division procedure. For example, the inference “if X is divisible by Y and Y is divisible by Z, then X must be

divisible by Z " is evidence of an object conception of divisibility (Zazkis & Campbell, 1996a).

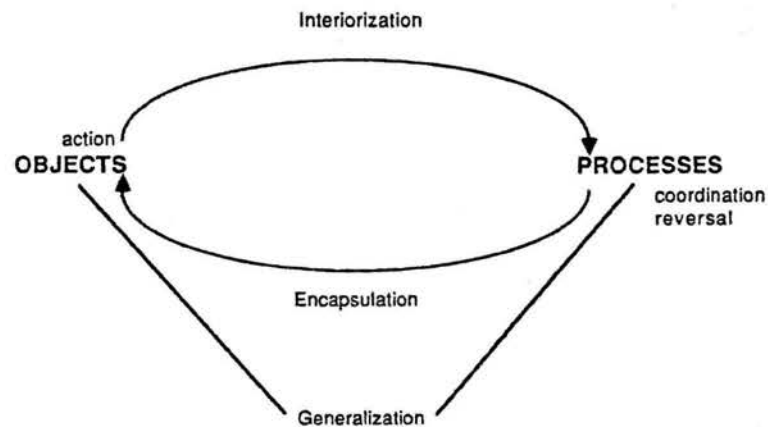
When processes and objects become connected, a *schema* is formed. An individual's schema represents a structured mental network that allows the learner to make sense of observed information as it relates to the concept at hand. More powerful, connected schemas allow the individual to assimilate new information more readily. For example, a schema for divisibility might result when the learner connects notions of divisibility with factor concepts and prime factorization to infer divisibility by composite numbers (Zazkis & Campbell, 1996a). According to Asiala and colleagues (1996), a schema can be *thematized* (i.e., the process of becoming conscious of something) to an object when the schema is treated as an object itself and assimilated to higher-level schemas.

Applying one's schema to other situations occurs by *generalization*, a mental construction in which a pre-existing schema is applied to a wider array of problems. For example, the schema for the factorization of positive integers can be generalized to the factorization of polynomials. Generalizing widens the range of applicability of a schema without altering the structure of the schema itself. This can happen in two ways: the individual simply becomes aware of new applications of the schema, or a process is encapsulated into an object so that a pre-existing schema can be applied to it (Dubinsky, 1991). Regardless of the source of generalization, the pre-existing schema remains unchanged.

The APOS theory constructions and the conception levels that they achieve are

represented in Figure 2.3.1 below. Starting on the left, known objects are transformed via procedural activity to form action conceptions. Such conceptions are then interiorized into processes so that the action can exist entirely in the individual's mind. Processes can often be coordinated or reversed to create new processes. When a process is viewed as a singular whole that can be operated on, it is said to be encapsulated into an object. Objects and processes form the components of a schema, which when applied to a broader range of situations, is said to be generalized.

Figure 2.3.1: Constructions of Processes and Objects in APOS Theory (Asiala et al., 1996)



APOS theory has been used primarily the undergraduate level to describe mathematics students' concept development for topics including sequences, the derivative, the chain rule, group theory, function, and mathematical induction. Weller and colleagues (2000) examined student performance data across thirteen of these studies. Their analysis suggests that APOS theory is an effective tool to help undergraduate students learn mathematics with conceptual understanding. However, in a study examining the nature of students' development of a schema for the chain rule,

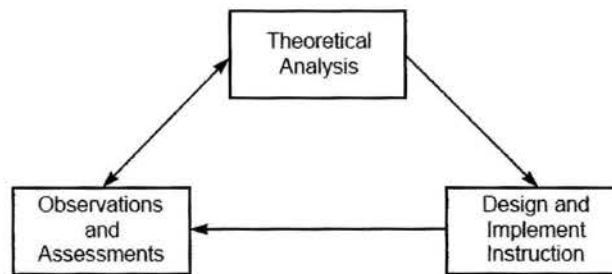
Clark and colleagues (1997) found students referred to different instances of the chain rule without regard to the relationships between these instances or to the general chain rule itself. As such, they claimed that APOS theory does not sufficiently explain schema development since the concept under investigation does not result from encapsulation of a single process.

In order to better identify schema development, Clark and colleagues (1997) introduced the triad model as a framework for understanding how students progress from constructing objects (i.e., encapsulation) to structuring schemas. The triad model has three stages of concept development: *Intra*, *Inter*, and *Trans*. At the *Intra* stage, the individual is incapable of recognizing relationships between the actions, processes, and objects that make up the schema. At the *Inter* stage, the individual begins to identify some of these relationships, but he does not yet understand how the components all fit together within one schema. This stage can be referred to as a *pre-schema* (Clark et al., 1997). At the *Trans* stage, the individual can construct a coherent structure that makes the relationships between schema components clear. Only when such a structure is developed can the individual make full sense of the concept and the relationships between its components.

APOS theory uses a three-part framework – theoretical analysis, instruction, and data analysis – to identify how students’ understanding of mathematical concepts develops (Dubinsky & McDonald, 2001). The framework is shown in Figure 2.3.2. Initially, researchers conduct a theoretical analysis of the mathematical content area by developing a genetic decomposition of a concept. A genetic decomposition is a proposed

set of mental constructions, with descriptions, that one might take in order to make sense of a concept. It is constructed using the researcher's knowledge of APOS theory and the mathematical concept itself. For example, the descriptions of the mental constructions leading to the various conception levels (action, process, object, and schema) described above for Zazkis and Campbell's (1996a) study represent a possible genetic decomposition for divisibility.

Figure 2.3.2: APOS Framework (Asiala et al., 1996)



Once a genetic decomposition is constructed, the researcher designs and implements instruction designed to elicit those mental constructions (Dubinsky & McDonald, 2001). Instruction is characterized by a *holistic spray*, during which students are placed “into an intentionally disequilibrating environment which contains as much as possible about the material being studied” (Asiala et al., 1996, p. 14). The entirety of the concept is presented to them at once (as opposed to in sequential order) in order to promote the disequilibrium. The entirety of the concept is then re-presented in a different way, providing additional opportunities for students to piece together an understanding of the concept bit by bit. Such instruction is often marked by students' frustration as they try to make sense of the concept under investigation; the instructor's role is only to help

them manage their frustration, not to eliminate it. Much of the students' work is done in cooperative groups where they are more likely to reflect on their procedures and communicate their understanding to others (Meel, 2003).

Data are collected about students' mental constructions using predominately detailed classroom observations, student interviews, and quantitative assessments. The data are then analyzed to determine if the students developed the understanding that was anticipated based on the initial genetic decomposition. Upon comparison, the researcher may need to modify the genetic decomposition to more accurately reflect the mental constructions that the students undergo. Ideally, the modified genetic decomposition is then used as the initial stage in a new cycle of theoretical analysis, instruction, and data analysis. APOS researchers continue to conduct research using this model until the data they generate from instruction results in a stable genetic decomposition.

According to Meel (2003), the APOS model satisfies Schoenfeld's (2000) criteria for classification as a theory, as it has descriptive, explanatory, and predictive power in identifying participants' mental constructions. It has also been shown to effectively describe students' mental constructions in a variety of research studies across multiple content domains, including elementary number theory, calculus, limits, group theory, and functions (Cappetta, 2007; Brown, Thomas, & Tolia, 2002; Clark et al., 1997; Zazkis & Campbell, 1996a; Breidenbach et al., 1992; Dubinsky, 1991).

While prior research shows that pre-service teachers' understanding of number theory topics is superficial, fragmented, and plagued by procedural attachments, it does not describe how pre-service teachers' understanding of number theory topics changes

over time. In this study, pre-service elementary teachers' developing understanding of elementary number theory was described by identifying their levels of understanding using Dubinsky's APOS model. The following chapter describes the study's research design, including the methods of collecting and analyzing data.

CHAPTER III

METHODS AND PROCEDURES

This study investigated pre-service elementary teachers' developing understanding of elementary number theory concepts before and after relevant classroom instruction. It used the APOS research-based theoretical framework (Dubinsky & McDonald, 2001) to identify participants' levels of understanding of a variety of number theory topics. The design of the study included both quantitative and qualitative data, specifically, the Number Theory Knowledge Test (NTKT) and individual clinical interviews. In Section I of this chapter, a description of the instruments used in the study is presented. Section II describes the sample and procedures for data collection, and Section III presents the methods of data analysis.

Section I: Instruments

Number Theory Knowledge Test

The Number Theory Knowledge Test (NTKT) is a written assessment consisting of sixteen questions that assess participants' conceptual and procedural knowledge of number theory topics. Two versions of the assessment were created: the NTKT pre-test and post-test. All questions were identical on both assessments, except that numerical values on all but one question were changed. Topics assessed included factors and prime factors, divisibility, greatest common factors, and least common multiples. The NTKT assessment was written by the researcher and content validity was established in two

ways: first, by adapting questions from the work of Zazkis and Campbell (1996a, 1996b) on the conceptions and misconceptions of pre-service elementary teachers, and second, by consultation with a mathematics education professor.

Two types of questions were included on the NTKT: *abstract* and *concrete*.

Abstract questions cannot be solved easily by executing a step-by-step procedure.

Instead, participants must use solution strategies that address, explicitly or implicitly, the inherent relationships among number theory concepts. As such, abstract questions require a measure of conceptual knowledge, which Hiebert and Lefevre (1986) define as knowledge that is rich in relationships. In terms of APOS theory, success on abstract questions requires, at minimum, internalizing the underlying procedure into a process and coordinating or reversing processes in order to find a solution (Dubinsky, 1991).

Concrete questions, on the other hand, can be solved by a straight-forward application of known procedures or algorithms to find solutions. There is typically a clear solution method and minimal need to attend to relationships. As such, concrete questions require procedural knowledge, which Hiebert and Lefevre (1986) define as knowledge of the rules, algorithms, and procedures that are used to solve mathematical problems.

Nevertheless, certain questions requiring concrete skills can be solved more efficiently using abstract skills, as the results of this study will demonstrate.

Table 3.1.1 summarizes the types of questions included on the NTKT. The researcher classified each NTKT question as either abstract or concrete based on Hiebert and Lefevre's (1986) definitions of conceptual and procedural knowledge and Zazkis and Campbell's (1996a, 1996b) research describing pre-service elementary teachers'

conceptions needed to solve similar types of number theory problems. If a question could be solved by executing a step-by-step procedure, regardless of how inefficient that procedure may be, then it was classified as concrete. If a question required the use or understanding of the relationship between two or more concepts, then it was classified as abstract.

Table 3.1.1: Describing the NTKT by Question and Type

Test Question	Description	Question Type
1a	Given a number in prime factored form: Identify divisibility by a prime	Abstract
1b	Identify indivisibility by a prime	Abstract
1c	Identify divisibility by a composite	Abstract
1d	Identify divisibility by a composite	Abstract
1e	Identify indivisibility by a composite	Abstract
2a	Find number of factors of a number in decimal form	Concrete
2b	Find number of factors of a number in prime factored form	Concrete
2c	Compare the number of factors between two numbers in factored form	Concrete
3a	Define term <i>prime number</i>	Concrete
3b	Determine if a given number is prime	Concrete
4a	Ascertain validity of a divisibility rule claim	Concrete
4b	Show divisibility rule claim is invalid	Concrete
5	Find the LCM of the first ten counting numbers	Abstract
6	Find all possible pairs of numbers with known GCF and LCM values	Abstract
7a	Identify the GCF or LCM concept in a context	Abstract
7b	Identify the GCF or LCM concept in a context	Abstract

The purpose of the concrete questions was to assess participants' abilities to execute familiar number theory procedures, such as finding the factors of a number or determining if a number is prime. For example, Questions 2a and 2b require participants to identify the number of factors of two numbers, one represented in decimal form and the other in prime factored form. Both questions are classified as concrete questions because they *can* be solved by familiar procedures: constructing factor trees or executing long division. However, Question 2b can also be solved more efficiently by relating the concept of prime factorization to divisibility. The purpose of this question, therefore, is to distinguish between participants who continue to execute long division versus those who begin to attend to structural relationships. Thus, classification as concrete versus abstract is determined by the minimal skill set required to successfully solve a problem.

The purpose of the abstract questions is to assess participants' developing conceptions of number theory topics. The instrument has numerous abstract questions, although they vary in the level of difficulty. Question 1, for example, provides participants with a number in prime factored form and asks them to identify divisibility by several prime and composite numbers. Pre-service teachers typically solve such questions by computing the value of the given number and dividing by each potential divisor, denoting procedural tendencies (Zazkis & Campbell, 1996b). Since the question prohibits calculating the value of the given number, Question 1 requires the abstract skill of connection-making between divisibility and prime factorization concepts. Questions 5 and 6 are also abstract questions because they require knowledge of the relationships between prime factorization, LCM, GCF, and divisibility concepts. Without such

understanding, participants cannot execute an effective solution strategy.

Test-retest reliability of the NTKT was established during a pilot study that the researcher conducted in June 2011. The purpose of the pilot study was to assess the appropriateness of the research methodology, including instrument reliability and methods of analysis, used during the study. Nineteen participants, all graduate students in an elementary mathematics education content course, were administered a pilot version of the NTKT on two separate occasions (i.e., test-retest analysis) three days apart. Results showed a high correlation ($r = 0.96$) between the pilot NTKT test and retest scores, indicating that the NTKT instrument. Based on pilot data results, some modifications were made to the NTKT. Some questions were omitted because they did not assess participant understanding as anticipated, while others were omitted because they were extraneous. These modifications were not significant because the questions assessing the key number theory concepts and on which the current study's analysis focused most heavily, remained unchanged.

The NTKT pre-test and post-test was scored by the researcher for correctness using a pre-determined scoring scheme. Each test contained a total of 50 possible points, with 29 points distributed to abstract questions and 21 points to concrete questions. The point values for each question varied depending on the depth of knowledge or number of procedural steps required to reach a correct solution. Scoring included the possibility of partial credit to account for intermediate levels of understanding. The scoring scheme reflected the typical scoring of written assessments in the mathematics education content course.

A second scorer recruited from among mathematics education doctoral students scored a random subset of 17% of the NTKT pre- and post-test using the same scoring scheme used by the researcher in order to assess inter-rater reliability. The subset included an equal amount of pre-tests and post-tests to make sure that the reliability measure accounted for possible changes in scoring from pre- to post-test. The second scorer matched the researcher's scoring on this subset of tests 83% of the time. Any discrepancies in scoring were subsequently resolved by back-and-forth discussion. A copy of the NTKT pre-test and post-test can be found in Appendix A, while their solutions and scoring rubrics can be found in Appendix B.

Individual Clinical Interviews

Individual clinical interviews consisted of thirteen questions. Questions were all of a mathematical nature and, similar to the NTKT, assessed participants' understanding of elementary number theory topics before and after instruction. Two versions of the interview were created: the pre-interview and post-interview. All questions were identical across both versions, with two exceptions: Question 2 was changed on the post-interview, and some of the numbers of other questions were modified. Most of the interview questions were taken from the NTKT assessment, yet three discrepancies remained. Question 4 did not appear on the NTKT assessments; Question 3b was an expansion on Question 2c of the NTKT assessments; and Questions 7 and 8 required participants to find solutions to the real-life scenarios presented in addition to identify the relevant concept. Since the solutions to these real-life scenarios may not be truly realistic

(e.g., it is unlikely that a school would purchase 8 feet by 8 feet tiles to tile a cafeteria floor), participants were told not to be concerned if their solutions appeared absurd. Question 3 on the NTKT was not included as an interview question because participants' responses on the NTKT were deemed sufficient evidence of their understanding of the concept. Table 3.1.2 displays the interview questions and their brief descriptions:

Table 3.1.2: Describing the Individual Clinical Interview by Question, Type, and Correspondence to NTKT Questions

Interview Question	Description	Question Type	Corresponding Question on NTKT
1a 1b 1c 1d 1e	Given a number in prime factored form: Identify divisibility by a prime Identify indivisibility by a prime Identify divisibility by a composite Identify divisibility by a composite Identify indivisibility by a composite	Abstract Abstract Abstract Abstract Abstract	1a-e
2	Explain the validity of a proposed divisibility rule (pre-interview) Construct a divisibility rule given two divisors (post- interview)	Concrete Abstract	4 None
3a 3b	Compare the number of factors between two numbers in factored form Find the factors of the two numbers	Concrete Concrete	2c None
4	Find a number with a specific number of factors	Concrete	None
5	Find the LCM of the first ten counting numbers	Abstract	5
6	Find all possible pairs of numbers with known GCF and LCM values	Abstract	6
7	Solve a real-world problem involving the GCF concept	Abstract	7a
8	Solve a real-world problem involving the LCM concept	Abstract	7b

The purpose of the interviews was to identify participants' developing understanding of number theory concepts in action. Questions 1-5 primarily addressed divisibility and factor concepts, while Questions 6-8 attended to GCF and LCM concepts.

However, several questions required knowledge of multiple concepts, such as Questions 3, 5, and 6. Participants solved number theory problems and described their reasoning out loud as they solved each problem. During this *think-aloud protocol* (Patton, 2002), the interviewer remained quiet except to prompt participants to continue articulating their thinking or to ask them to clarify specific statements. Such a design has been used and recommended in numerous studies as an effective method for ascertaining participants' authentic cognitive processes (Wilson, 2000; Snook, 1997; Zazkis & Campbell, 1996a, 1996b; Asiala et. al., 1996). Pre-interview and post-interview questions can be found in Appendix C. Solutions to interview questions can be found in Appendix D.

Section II: Sample and Procedures for Data Collection

Setting and Sample

The setting for this study was a semester-long mathematics education content course at a major private institution in the Northeastern U.S. The course was the first in a sequence of two required mathematics education content courses for pre-service teachers majoring in elementary education and special education. The course comprised three separate sections taught by full-time mathematics education professors and graduate student instructors, including the researcher of this study. Each section met three times per week for a total of three and a half hours. The goal of the course was to foster pre-service teachers' conceptual understanding of topics in number and operations, number theory, and proportionality. The course used discourse-based instruction as its primary pedagogical strategy because such instruction has been recommended as a way to foster

teachers' conceptual understanding of key mathematical concepts (Hufferd-Ackles, Fuson, & Sherin, 2004; NCTM, 2000; Chapin, O'Connor, & Anderson, 2009; Cohen, 1994).

The sample for this study consisted of fifty-nine undergraduate students enrolled in the first semester mathematics education content course described above. All participants were sophomores or juniors, and most were seeking teaching licensure in elementary, special, or early childhood education. 90% of the sample was female and 10% were male. Approximately 70% of the study participants majored in elementary, special education, or both, while the remaining 30% majored in early childhood education. Most have studied at least three years of high school mathematics through at least Pre-Calculus and tended to exhibit a moderate level of procedural familiarity with elementary mathematics topics. As seen in prior research, relatively few participants exhibited an adequate level of conceptual understanding of the mathematics needed to teach at the elementary school level (Ball, 1990).

All participants and instructors signed informed consent forms agreeing to participate in the study. The informed consent forms asked participants if they would agree to being videotaped during class sessions, if they would be willing to take part in two individual interviews outside of class hours, and if they would allow the researcher to include their math SAT[®] or ACT[®] scores in the study's analysis. All participants agreed to be videotaped during class sessions. Approximately 51% of the sample agreed to be interviewed, and roughly 88% of the sample agreed to release their SAT[®] or ACT[®] scores for the purpose of data analysis. A copy of the informed consent forms is

available in Appendix E.

Of the 51% of the sample who agreed to be interviewed, a sub-sample of six female participants was selected by stratified random sampling to participate in one hour-long individual clinical pre- and post-interviews. The six participants represented a full range of mathematics achievement levels (low, average, and high), as determined by instructor recommendations and NTKT pre-test scores. Three participants were selected from each of two sections not taught by the researcher. No interview participants were selected from the section taught by the researcher in order to avoid any conflict that could have arisen from the researcher interviewing his own students.

Based on statistical results from the pilot study, the sample size for the study was more than sufficient for the purpose of conducting statistical analysis. Power analysis was conducted prior to the study using a standard 80% desired level of statistical power, a 0.95 effect size computed from the pilot study, and a two-tailed α -level of .05. Results showed a minimum total sample size of thirty-eight needed to conduct the study, which is well below the study's sample size.

Implementation of Instruction

Instructors in each of the three sections of the course taught a three-week number theory unit. The unit was composed of five mathematical tasks specifically designed to elicit participants' verbal and written generalizations and justifications regarding factor, prime factorization, divisibility, greatest common factor, and least common multiple concepts. All tasks were written by a team of mathematics education professors and

graduate students, including the researcher, as part of the NSF-funded Elementary Pre-Service Teachers Mathematics Project (EMP, 2011, Grant No: NSF DUE-0837349). Early versions of the tasks were piloted and field tested in multiple university settings. As such, the tasks have been repeatedly revised to support the development of pre-service teachers' conceptual understanding of relevant number theory content. Copies of all five mathematical tasks are included in Appendix F. Solutions to all five tasks are included in Appendix G.

Each task contained subsets of content-focused questions followed by *Group Discussion Questions*, or overarching questions meant to focus participants' attention on the key mathematical ideas embedded within the subset. Each subset of questions typically included between two and four related questions, and was followed by approximately two group discussion questions. At the end of each task, *Summarize and Connect* questions asked participants to reflect on and articulate the main ideas of the task and their connectedness. Participants were instructed to complete these questions for homework; their written responses were collected, but they were not analyzed for the purposes of this study. Each task was designed to be completed in approximately eighty minutes.

The first task, called *Factors – Part I*, served as an initial exploration of the factors and prime factors of numbers. The purpose of this task was to introduce participants to the concept of *factor*. It began with a discussion of mathematical vocabulary that pre-service teachers often confuse, such as *factor*, *divisor*, and *multiple* (Zazkis, 2002). Participants then explored factor characteristics by playing the *Factor*

Game (Lappan et al., 2006). This activity allowed them to develop some intuition regarding which types of numbers tend to have a lot or few factors. Participants then concluded the task by examining and reasoning through differences between numbers with an odd or an even number of factors.

The second task, *Factors – Part II*, was a follow-up to *Factors – Part I*. Its purpose was to help participants make explicit connections between a number's factors and its prime factorization. Participants began the task by generating examples of numbers that have exactly two, three, four, or five distinct factors, and using their data to make conjectures about which types of numbers have two, three, four, or five distinct factors (Teppo, 2002). Participants then used their data to define *prime* and *composite* numbers. The task then introduced participants to the use of factor trees and prime factorization as tools for representing numbers by their prime factors, leading to a discussion of the *Fundamental Theorem of Arithmetic*, which states that any composite number can be decomposed into a unique product of prime numbers. Following this discussion, participants revisited the first activity of the task and re-presented their generated examples using prime factorization, leading to confirmation or rejection of their initial conjectures. The task ended with participants finding all of the factors of a much greater number represented in prime factored form.

The third task, *Divisibility*, examined the relationship between prime factorization and divisibility concepts. The task began with participants determining if two given numbers, in prime factored form, are divisible by other given numbers. They are not allowed to compute the decimal values or use long division to solve these problems. This

activity led to a discussion of what it means for one number to be divisible by another. Following the discussion, the task presented participants with a list of common divisibility rules, and asked them to explain divisibility rules for 6 and 10 by listing each of their multiples in prime factored form. The final part of the task involved participants checking the validity of proposed divisibility rules for greater composite numbers, such as 20 and 24, and developing generalized guidelines for creating divisibility rules for any composite number.

The purpose of the fourth task, *Greatest Common Factor*, was to help participants make sense of different methods for finding the greatest common factor (GCF) of two numbers. Venn diagrams were introduced as a tool for representing numbers and, coupled with prime factorization, were used to find the GCF of different pairs of numbers representing different cases (no shared factors, some shared factors, and all shared factors). The prime factorization method was introduced, and participants were required to justify it and compare it to the Venn diagram method. The last part of the task had participants using the GCF to solve contextualized word problems and creating their own word problems.

The fifth and final task, *Least Common Multiple*, was intended to develop participants' understanding of the least common multiple (LCM) concept, and to relate this concept to prime factorization. As such, participants were initially presented with a typical LCM strategy of listing out consecutive multiples of two numbers. They had to determine if the strategy was valid and explain why or why not. They then used prime factorization to make connections between the prime factorizations of two numbers and

the prime factorization of their LCM. The second part of the task presented participants with erroneous and valid ways of using prime factorization to find the LCM of two numbers. Participants were required to explain if each method was valid. The last part of the task had participants solving contextualized word problems and creating their own word problems for which the LCM was necessary.

Measures were taken to ensure that implementation of the five tasks was consistent in each of the three course sections. The researcher, who was one of the three instructors, met with the other two instructors prior to task implementation to determine the order, pacing, teacher questioning, and content focus that all instructors were to follow when implementing each task. Implementation of the five number theory tasks was characterized by the use of small-group and whole-class discussion formats, as well as focused questioning by the instructors to elicit participants' reasoning. After a brief introduction to each task by the instructor, participants worked in small groups to solve a subset of the task's questions, during which time the instructors moved from group to group to monitor each group's progress. After most groups made sufficient progress on the subset of questions, the instructor facilitated a whole-class discussion around the subset's group discussion questions. This cycle repeated for each subset of questions in a task.

Each course section was divided into four to five small groups of approximately four participants each. Group assignments were determined by stratified random sampling based on participants' NTKT pre-test scores and the course instructors' informal evaluations of participants' mathematics achievement to date. As such,

participants were rated as high, average, and low achieving. The researcher then randomly selected one high achieving student, two average achieving students, and one low achieving student to constitute each small group. Participants remained in their groups for the duration of the three-week unit.

During small-group and whole-class discussion formats, instructors asked questions focused on eliciting participants' reasoning instead of finding correct solutions. Research recommends providing students opportunities to articulate their reasoning to the rest of the class as a way to help participants make sense of the mathematics for themselves (Smith & Stein, 2011; Hiebert et al., 1997). During small group work, instructors facilitated each group by quietly observing how participants talked about the mathematics. Instructors asked guiding questions only if they observed that a group was not making any progress on the task. Instructors were able to use their observations of participant thinking to inform subsequent whole-class discussions. Whole-class discussions provided opportunities for all participants to share their thinking with others with the goal of achieving a shared understanding of the key mathematical concepts. To this end, instructors used specific *talk moves* such as revoicing, repeating, and wait time, to prompt participants to listen to others' ideas and articulate their own thinking (Chapin, O'Connor, & Anderson, 2009).

Each course section was videotaped and audiotaped throughout the three-week unit in order to analyze their mathematical thinking during classroom instruction. One focus group was randomly selected in each course section to be videotaped and audiotaped during small group work. The researcher made sure to place all six interview

participants into focus groups so as to compare their mathematical thinking during interviews and classroom instruction. Unfortunately, due to lack of resources and time constraints, this study did not include an analysis of the classroom videotapes. Future research will examine these videotapes in the hopes of constructing more in-depth analyses of participants' developing understanding.

Administration of Instruments

The Number Theory Knowledge Test (NTKT) was a paper-and-pencil assessment administered to all study participants during regular course hours before and after the three-week number theory unit. The NTKT pre-test was administered approximately one week prior to instruction, while the NTKT post-test was administered one week following instruction. The NTKT pre-test included a Background Information Form (Appendix H) that participants completed immediately prior to completing the pre-test. The NTKT post-test was used as a course assessment grade in all three sections.

Participants had sixty minutes to complete each assessment, and no calculators were allowed. The instructors of each section administered the assessments to their own section. They were allowed to clarify instructions, but could not assist participants in solving any of the questions. Participants were instructed to show all of their work on the test paper itself, and to try to complete as many questions as possible.

Since added complexity derives from the fact that some participants may have already possessed some conceptual or procedural knowledge of number theory from prior instruction, individual interviews were conducted during and after the number theory unit

with a sub-sample of six participants. The pre-interview was conducted with each participant within one week following the administration of the NTKT pre-test, but prior to commencement of the number theory unit. The post-interview was conducted within two weeks following the administration of the NTKT post-test. All interviews were conducted by the researcher outside of regular course hours. Each interview lasted for approximately one hour, and calculators were allowed except when a question explicitly prohibited its use. For their time, each interview participants was given a gift card to a vendor of their choice.

The researcher followed a strict interview protocol during each interview. Interview participants were given access to a pad of paper, pen or pencil, and a TI-83 Plus calculator. With participants' permission, each interview was audio recorded. The researcher began each interview by stating the purpose of the interview and providing verbal directions on how the interview would be conducted. Interview guidelines are provided in Appendix C. Once these directions were read to each participant, the researcher and interview participant engaged in a non-mathematical exercise in order to give the participant practice thinking out loud.

Following the non-mathematical exercise, each participant was asked to read the first interview question out loud and to proceed to try to solve it on the paper provided by articulating her thinking out loud. The role of the researcher was to listen to each participant's thinking and to remind the participant to continue thinking out loud. When a participant completed a question, it was the participant's responsibility to determine if she had provided sufficient reasoning in order to proceed to the next question. If a

participant asked for help or appeared confused at any time, the researcher asked her to describe her confusion and to try to identify why she was having difficulty solving the problem. The researcher did not assist the participant in solving the problem. If the participant could not make any progress, the researcher asked if she wanted to move on to the next question with the possibility of returning to the current question at a later time.

All data collected from classroom activities, testing, and interviews were stored in a secure location. Participants recorded their names on the cover sheet of each NTKT assessment. To ensure confidentiality, each test was assigned a non-identifiable code number and the cover sheet and background information form was removed. Participants' interview transcripts and written work were also assigned corresponding code numbers. Participant names were only used to identify prospective interview participants and, as part of on-going course assessments, to assign individual quiz grades based on their performance on the NTKT post-tests.

Section III: Methods of Data Analysis

Each research question was answered using analysis of both quantitative and qualitative data collected from the Number Theory Knowledge Test and individual clinical interviews.

Question 1

Do participants exhibit identifiable levels of understanding of divisibility, greatest common factor, and least common multiple prior to instruction, and if so, what are they?

Question 1 was investigated using study participants' scores on the Number Theory Knowledge Test (NTKT) pre-test, written responses on Question 2b and Question 6 from the NTKT pre-test, and responses during the pre-interviews. Each NTKT pre-test was scored by the researcher using the NTKT scoring rubric (Appendix B); reliability of the rubric was verified by a second scorer. Subset scores for abstract and concrete question types were also determined. Descriptive statistics for the NTKT pre-test were calculated for the entire sample and by course section. These statistics included sample means and standard deviations for the test as a whole and for each question type.

A paired-sample *t*-test was conducted in order to compare participants' performance on abstract versus concrete questions. Since the total number of points possible for each question type was unequal (twenty-nine points for abstract questions versus twenty-one points for concrete questions), percent scores were computed by dividing the number of points earned on all questions of one subset by the total points possible for that subset. These percent scores were used to create a standardized scale of the two subset scores for each participant.

All participants' written responses on two questions from the NTKT pre-test (#2b and 6) were analyzed using the APOS model (Dubinsky & McDonald, 2001). Each conception level of the model (*action, process, object, and schema*) was used to code

participants' written responses. Frequencies of each code were determined across the entire sample and by each course section. These results provided more detailed descriptions of participants' understandings prior to instruction. Question 2b and Question 6 were chosen as focus questions for analysis because they represented two primary areas of interest within the field of number theory: divisibility and the relationship between greatest common factor (GCF) and least common multiple (LCM). Also, pilot test results showed that these questions provided opportunities for participants to demonstrate a variety of conceptions. Table 3.3.1 shows both questions, as they appear on the NTKT pre-test:

Table 3.3.1: NTKT Pre-Test Focus Questions

Question 2b	How many factors does $5^2 \times 7$ have? How do you know?
Question 6	The prime factorization of the greatest common factor of two numbers, A and B , is $2^3 \times 5$. The prime factorization of the least common multiple of the same two numbers is $2^5 \times 5^2$. Give all possible values for A and B . Show your work.

Question 2b can be answered by computing the decimal value of the number and using long division to find its factors, or by using its prime factorization to infer divisibility. Similarly, Question 6 can be answered by computing the decimal values of the GCF and LCM first or by using their prime factorizations to find solutions to the question. On both questions, the degree to which participants attend to prime factorization in their solution methods can signify varying levels of understanding of divisibility and of the GCF-LCM relationship, respectively.

Pre-interview data in the form of audiotapes and transcripts were also analyzed

using the APOS conception levels as codes. Each transcript was read twice to determine participants' success at solving each problem and to identify examples of participant thinking at specific APOS conception levels. A *transcript coding form* was created for each interview transcript to provide a dynamic format for recording examples and determining specific APOS codes. Table 3.3.2 is an example of a portion of a transcript coding form for one participant:

Table 3.3.2: Coding of Transcript by Question, Explanation, and APOS Conception

Question/Text	Explanation	APOS conception
<p>Q2: Do you agree or disagree with the following statement? Explain your response. <i>A number is divisible by 18 if it is divisible by 3 and by 6.</i></p> <p>“I’m just trying to think of another example right now that I could disprove divisibility by 3, 6, and 18...I’m just going to do 4-48.”</p>	<p>Guesses and checks numbers that might disprove the statement. Stops when finds a counter-example.</p>	<p>Action</p>

Each transcript coding form was analyzed twice – by participant and by question – in order to refine the codes created for each participant. A more detailed description of this analysis process is provided in Chapter V.

Question 2

Do participants' levels of understanding of divisibility, greatest common factor, and least common multiple change after the course of a unit on number theory, and if so, how?

Question 2 was investigated using study participants' responses and scores on the Number Theory Knowledge Test (NTKT) pre-test and post-test, analyses of written responses on Question 2b and Question 6 from the NTKT pre-test and post-test, and responses during the pre-interviews and post-interviews. In order to compare NTKT pre-test scores to post-test scores, each NTKT post-test was first scored by the researcher using the NTKT scoring rubric (Appendix B); reliability of the rubric was verified by a second scorer. Subset scores for abstract and concrete question types were also determined. Descriptive statistics for the NTKT post-test were calculated for the entire sample and by course section. These statistics included sample means and standard deviations for the test as a whole and for each problem type.

Three paired-sample *t*-tests were performed to identify significant differences between pre-test and post-test with regards to total scores, abstract subset scores, and concrete subset scores. Effect sizes (*Cohen's d*) were also computed to determine the size of any significant differences found when performing the three *t*-tests. Additionally, correlations were computed between pre-test and posttest total scores, abstract subset scores, and concrete subset scores to determine if pre-test scores were associated with post-test scores for each subset.

A one-way ANOVA was also conducted to see if there were significant

differences in performance by course section, from pre-test to post-test, with regards to total scores, abstract subset scores, and concrete subset scores. Unstandardized residual values for each score type (total, abstract, concrete) were obtained by running three linear regression models for the three score types, where participants' pre-test scores and average SAT[®] and ACT[®] percentile scores were used as covariates. This analysis was necessary in order to determine if differences in participants' understanding were due to differences in instruction among the three course sections.

In order to determine if there was a significant difference between participants' performance on abstract versus concrete NTKT questions from pre-test to post-test, a paired-sample *t*-test was performed using unstandardized residual values of participants' percent scores on abstract versus concrete questions. First, post-test abstract and concrete percent scores were computed for each participant in the same way they were computed for each participant's pre-test. The resultant percent scores created a standardized scale of each subset score for each participant. Second, if permission was granted, participants' average percentile score on the mathematics portion of the SAT[®] and ACT[®] was computed to serve as a measure of their prior mathematics achievement. If a participant took only one of the two standardized tests, then that test's percentile score served as the average. Percentile scores for each test were computed by finding the difference between a participant's raw score and the nationwide mean score, dividing that difference by the nationwide standard deviation, and converting the resulting standardized value into a percentile using a standard normal probabilities table (Moore, 1995). A mean percentile score was then computed. Given the age of the participants,

national SAT[®] and ACT[®] mean scores and standard deviations from the year 2010 were used (<http://www.collegeboard.com>; <http://nces.ed.gov>). Since research has shown that one of the most influential determinants of mathematics achievement is prior achievement (Hill et al., 2005), analysis controlled for participants' prior mathematics achievement by using average SAT[®] and ACT[®] percentile scores and pretest percent scores as covariates. If participants did not grant permission to use their SAT[®] and ACT[®] data, then their NTKT results were not included in this part of the analysis. Third, the pre-test abstract percent scores and the average of each participant's SAT[®] and ACT[®] scores were linearly regressed onto the post-test abstract percent scores in order to extract the set of residuals indicating the growth from pre-test abstract percent scores to post-test abstract percent scores while controlling for participants' prior mathematics achievement. A similar procedure was followed for the participants' concrete percent scores. Lastly, the paired-sample *t*-test was performed comparing those two sets of residual values to determine if, after controlling for prior mathematics achievement, there was a significant difference between the sizes of the changes in participants' performance on abstract versus concrete questions.

All participants' written responses on Questions 2b and 6 from the NTKT post-test were analyzed using the APOS model (Dubinsky & McDonald, 2001). Each conception level of the model (*action, process, object, and schema*) was used to code participants' written responses. Frequencies of each code were determined across the entire sample and by each course section, and compared to frequencies from participants' work on the NTKT pre-test. Results provided the researcher with more detailed

descriptions of participants' developing understanding of the divisibility, GCF, and LCM concepts inherent in the two questions. Both questions are identical to Questions 2b and 6 on the NTKT pre-test, except that the numerical values within each question were changed.

Post-interview data in the form of audiotapes and transcripts were also analyzed using the APOS conception levels. Similar to analysis of the pre-interview, each transcript was initially read twice to identify examples of participant thinking at specific APOS conception levels. Transcript coding forms were created and analyzed by participant and by question in order to refine codes. Following these preliminary analyses, the researcher compared each participant's response for individual interview questions across pre-test and post-test. Such analysis not only further refined the coding scheme, but also helped to determine changes in participants' conception levels following instruction. A more detailed description of this analysis process is provided in Chapter V.

To assess the reliability of the coding scheme in evaluating participants' conception levels during each interview, a second rater with knowledge of elementary number theory and the APOS model analyzed approximately 15% of the interview transcripts, including both pre-interview and post-interview transcripts. Specifically, he was given portions of each interview where the identity of a specific conception level was unclear. For example, in several cases, a participant's thinking exhibited qualities of *process* and *object* level conceptions, so it was difficult to decide on a single code. The second rater's coding of these transcripts showed 85% agreement with initial coding,

where agreement is defined as either coming up with the same code or, in cases where initial coding identified two adjacent levels as possible codes, coming up with either of the two levels. For the remaining 15%, the first and second coder eventually reached full agreement by back-and-forth negotiation.

This chapter presented the methods and procedures used to conduct the study. Descriptions of all study instruments, number theory tasks, and procedures for collecting participant data were provided. Additionally, the methods used to analyze the data were presented. In Chapter IV, the results of data analysis of the NTKT instrument are presented. In Chapter V, the results of data analysis of the individual interviews are presented.

CHAPTER IV

ANALYSIS OF TESTING DATA

This study analyzed pre-service elementary teachers' understanding of number theory topics and developed detailed descriptions of how their understanding of these topics develops. In November 2011, the Number Theory Knowledge Test (NTKT) assessments were administered to fifty-nine pre-service elementary teachers at a large private university in the northeast before and after a three-week unit of instruction on number theory. The results of these assessments were analyzed quantitatively and qualitatively. Section I of this chapter presents the results of the quantitative analysis in the context of the study's research questions. Section II provides a more detailed, qualitative analysis of two key questions within the NTKT assessments.

Section I: Quantitative Analysis

Question 1

Do participants exhibit identifiable levels of understanding of divisibility, greatest common factor, and least common multiple prior to instruction, and if so, what are they?

Data from the NTKT pre-test were analyzed to answer Question 1. The test was administered one week prior to a three-week instructional unit on number theory.

Participants were given one-hour to complete the pre-test; calculators were not allowed. A total of fifty points were possible on the pre-test; twenty-nine points were allocated to

abstract questions and twenty-one points were allocated to concrete questions. Abstract questions typically required participants to solve complex non-algorithmic problems or to make use of inherent relationships between various concepts. Concrete questions, however, could be solved using known, straightforward procedures. They did not require knowledge or use of concept relationships. A brief summary of the scoring data for all participants is presented by question type in Table 4.1.1. Participants' mean scores were low across all question types, but their mean score on concrete questions was greater than their mean score on abstract questions (37.6% versus 24.6%). Mean scores are also presented as percentages of the total possible points for each question type and for the entire pre-test.

Table 4.1.1: NTKT Pre-Test Performance by Question Type (n = 59)

Question Type	Mean Scores	Standard Deviation	Minimum Score	Maximum Score
Abstract	7.1 (24.6%)	3.6	0.0 (0.0%)	15.5 (53.4%)
Concrete	7.9 (37.6%)	3.0	1.0 (4.8%)	14.0 (66.7%)
Total	15.0 (30.1%)	5.4	5.0 (10.0%)	28.5 (57.0%)

A paired-sample *t*-test was conducted to compare participants' performance on abstract versus concrete questions on the NTKT pre-test. Percentage scores were used instead of raw scores because NTKT subtotals for abstract and concrete question types are not equal (twenty-nine points versus twenty-one points, respectively). Results of the *t*-test indicated a significant difference between mean abstract and concrete scores, $t(58) = 6.225, p < .05$. This suggests that prior to instruction participants were less successful at answering abstract questions requiring knowledge of concepts and their relationships than they were at answering concrete questions that only required knowledge of how to

execute step-by-step procedures. This result supports the contention that participants' level of understanding prior to instruction was at the *action* or *process without coordination* levels, since those levels are associated with procedural attachments rather than recognition of the meanings and interconnectedness of concepts (Zazkis & Campbell, 1996a).

In summary, prior to instruction participants exhibited low levels of understanding across all tested number theory topics. They scored significantly lower on abstract questions than on concrete questions, suggesting *action* and *process without coordination* levels of understanding. Participants did a better job solving problems that required executing procedures than problems that required a deeper understanding of concepts.

Question 2

Do participants' levels of understanding of divisibility, greatest common factor, and least common multiple change after the course of a unit on number theory, and if so, how?

Data from the NTKT pre- and post-tests were analyzed to answer Question 2. The post-test was administered one week following the three-week instructional unit on number theory. Participants were given one-hour to complete the post-test; calculators were not allowed. The post-test was composed of the same questions as the pre-test, except that the numerical values were changed on most questions. A brief summary of the scoring data for all participants is presented by question type in Table 4.1.2. Mean

scores are also presented as percentages of the total possible points for each type of question and for the entire post-test.

Table 4.1.2: NTKT Post-Test Performance by Question Type ($n = 59$)

Question Type	Mean Scores	Standard Deviation	Minimum Score	Maximum Score
Abstract	19.8 (68.3%)	4.0	10.0	28.0
Concrete	17.0 (81.0%)	2.7	8.0	21.0
Total	36.8 (73.6%)	5.5	21.0	47.0

In order to analyze participants' score gains from pre-test to post-test, three paired-sample t -tests were conducted to identify any significant differences by question type, controlling for participants' prior mathematics achievement. Results indicated significant differences between pre- and post-test total scores ($t(58) = 27.845, p < .05, d = 4.06$), abstract scores ($t(58) = 22.851, p < .05, d = 3.49$), and concrete scores ($t(58) = 18.083, p < .05, d = 3.01$). Effect sizes were also computed using Cohen's d in order to quantify the size of the improvements. Participants' performance on abstract and concrete questions improved following instruction, suggesting that their conceptual and procedural understanding of number theory topics became more robust. Additionally, the effect sizes indicate a large effect in all cases when compared to standard effect size benchmarks: $d = .20$ represents a small effect, $d = .50$ represents a medium effect; and $d = .80$ represents a large effect (Cohen, 1988). In other words, an effect size of 4.06 means that approximately 97.7% of the post-test total scores exceed 97.7% of the pre-test total scores. An effect size of 3.49 means that approximately 96% of the post-test abstract scores exceed 96% of the pre-test abstract scores, and an effect size of 3.01 means that 93.3% of post-test concrete scores exceed 93.3% of pre-test concrete scores

(Cohen, 1988, p. 22).

Correlations were also computed by question type in order to assess the strength of association between pre- and post-test scores. Table 4.1.3 displays results of the correlation analysis:

Table 4.1.3: Correlations by Question Type

	Correlation (r)	P-value
Pre-test vs. Post-test Total Scores	.388	0.002
Pre-test vs. Post-test Abstract Scores	.382	0.003
Pre-test vs. Post-test Concrete Scores	.104	0.434

Results indicate significant but weak positive associations between pre- and post-test total scores and between pre- and post-test abstract scores, but nearly no association between pre- and post-test concrete scores. In other words, it appeared that performance on the pre-test did not have a strong effect on performance on the post-test.

A paired-sample *t*-test was also performed to compare participants' gains on abstract versus concrete questions from pre- to post-test, while controlling for participants' prior mathematics achievement. Unstandardized residuals for abstract and concrete scores were first derived by conducting two linear regression models using abstract and concrete percentage scores instead of raw scores. Percentage scores were used instead of raw scores to standardize the unequal scales for each question type. Each regression model included its respective pre-test score type and participants' average SAT[®] and ACT[®] math percentile scores as covariates in order to control for participants' pre-test scores and prior mathematics achievement. Results of the *t*-test showed no significant difference between changes in participants' performance on abstract versus

concrete questions from pre- to post-test, $t(51) = 0.000, p > .05$, indicating that participants' improvements on abstract and concrete questions were not significantly different following number theory instruction.

Participants' performance on the post-test was also analyzed by course section in order to determine if differences in how instruction was implemented across course sections contributed to differences in participants' performance on the NTKT. Table 4.1.4 shows the means and standard deviations of participants' scores on the NTKT post-test by question type and by section:

Table 4.1.4: NTKT Post-Test Means and Standard Deviations by Question Type and Course Section

Question Type	Section 1 (n = 20)		Section 2 (n = 18)		Section 3 (n = 21)	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Abstract	19.2 (66.3%)	4.1	19.0 (65.4%)	4.1	21.1 (72.7%)	3.7
Concrete	16.0 (76.0%)	3.3	18.1 (86.2%)	2.3	17.1 (81.4%)	2.2
Total	35.2 (70.4%)	6.5	37.1 (74.2%)	5.4	38.2 (76.3%)	4.2

A one-way ANOVA was also conducted in order to identify if there were any significant differences in participants' score improvements between the three course sections. Three linear regression models were constructed used total, abstract, and concrete raw scores in order to obtain unstandardized residuals for each score type. Each regression model included its respective pre-test score type and participants' average SAT[®] and ACT[®] math percentile scores as covariates in order to control for participants' pre-test scores and prior mathematics achievement. Once residuals were obtained for all

score types, they were used to perform the ANOVA.

Results of the ANOVA indicate that there were no significant differences between any of the course sections with regards to changes in participants' performance on total scores ($F(51) = 0.206, p > .05$), abstract scores ($F(51) = 1.893, p > .05$), and concrete scores ($F(51) = 1.509, p > .05$). Participants in all three course sections improved similarly across all question types following instruction, suggesting that any differences between participants' understanding of number theory was not impacted by differences in how each course instructor implemented the number theory unit.

A paired-sample *t*-test was then conducted to determine the nature of participants' understanding of number theory topics following instruction. Post-test mean abstract percentage scores were compared to post-test mean concrete percentage scores in order to standardize the unequal question type scales. Results showed a significant difference between participants' post-test scores for abstract and concrete questions, $t(58) = 6.140, p < .05$, indicating that participants did perform better on concrete questions than on abstract questions following instruction. This result supports the claim that although all participants understanding of number theory grew, their ability to solve procedurally-focused problems remained stronger than their ability to solve more conceptually-oriented problems.

In summary, participants displayed significant improvement on abstract and concrete question types following number theory instruction. Their understanding of number theory topics improved to more conceptual levels; they were better able to answer questions requiring knowledge of concepts and their meanings and

interrelationships. This result is in stark contrast to prior research, which described pre-service elementary teachers' understanding of number theory after instruction as superficial and focused primarily on executing procedures (Zazkis & Campbell, 1996a, 1996b). Despite improving significantly on conceptual questions, participants still showed greater success solving problems requiring only procedural knowledge. This suggests that their developing understanding of number concepts may have strengthened their ability to execute and apply known algorithms. Additionally, since the entire sample showed similar improvements in NTKT scores, it is reasonable to assume that changes in participants' levels of understanding were comparable across the sample.

Section II: Qualitative Analysis

Analysis of Questions 2b and 6 on the Number Theory Knowledge Test (NTKT) assessments provided added detail of participants' number theory conceptions. The APOS levels of understanding (Dubinsky & McDonald, 2001) were used to assess each participant's level of understanding of divisibility, GCF, and LCM concepts when solving the two questions. When comparing pre-test to post-test results, improved understanding was evident for concepts related to both questions. Specifically, participants' understanding of divisibility and prime factorization concepts became more connected in Question 2b, while their ability to coordinate their schemes for greatest common factor (GCF) and least common multiple (LCM) became more pronounced in Question 6.

Question 2b was presented as follows on the NTKT pre-test and post-test,

respectively: a) *How many factors does $5^2 \times 7$ have? How do you know?*; b) *How many factors does $3^2 \times 5^2$ have? How do you know?* Each question can be solved by computing the decimal value of the given number and then using long division to compute individual factors. However, a more efficient method involves using the prime factorization of each number to directly identify factors. For example, the factors of $5^2 \times 7$ are 1, 5, 7, 5^2 , 5×7 , and $5^2 \times 7$ because each number is the product of a combination of the prime factors visible in the prime factorization of the number. Computation is completely unnecessary because the prime factorization provides all the needed information directly. Not only is this method more efficient, but it shows a deeper understanding of the role of multiplicative structure in identifying number properties (Zazkis & Campbell, 1996b). Analysis showed that participants' use of prime factorization varied from identifying only the prime factors as factors to identifying all combinations of prime factors as factors. Table 4.2.1, shown below, presents sample-wide frequencies of each level of understanding as exhibited by participants when answering Question 2b on both pre-test and post-test.

Table 4.2.1: NTKT Question 2b Total Frequencies and Percentages for each Level of Understanding on APOS

Code	Pre-Test Frequencies	Post-Test Frequencies
Action	14 (23.7%)	0 (0.0%)
Process without coordination	35 (59.3%)	1 (1.7%)
Process with coordination	8 (13.6%)	3 (5.1%)
Object	2 (3.4%)	27 (45.8%)
Schema	0 (0.0%)	25 (42.4%)
Total	59 (100.0%)	56* (94.9%)

*Three responses on the post-test were not coded because they were unclear

Prior to instruction, 83% of all participants exhibited lower levels of understanding (*action* or *process without coordination*), characterized by a need to compute decimal values first, or an inability to identify composite factors when attending to prime factorization. At the action level, participants had to compute the value of $5^2 \times 7 = 175$ and then use long division to find its factors. These participants had not yet internalized an image for divisibility, and so had to use long division. At the process without coordination level, participants were able to identify 1, 5, 25, 7, and 175 as factors without computing, but had difficulty coordinating 5 and 7 to identify 35 as another factor. Even though a majority of the sample (76.3%) had internalized an image for divisibility prior to instruction, 77.7% of this majority held an image that was only applicable to prime factors and could not yet be extended to identify many composite factors.

Following instruction, 88.2% of participants exhibited *object* and *schema* level

understandings of divisibility and prime factorization on Question 2b. Such understanding was marked by an ability to identify all of the number’s factors without needing to compute its decimal value first, and by an ability to explain the relationship between prime factorization and divisibility – *the factors of the number are products of various combinations of the prime factors in its prime factorization*. The ability to articulate this relationship represented the highest level of understanding exhibited by participants in this study.

Results of participants’ work on Question 2b were also analyzed by section. By converting each level to a numeric value (1 to 5), the researcher computed a mean level of understanding exhibited by the entire sample and by each section on this particular question during the pre-test and post-test. Table 4.2.2 below presents these mean values.

Table 4.2.2: NTKT Question 2b Mean Levels of Understanding by Course Section

Code	Pre-Test Mean	Post-Test Mean
Section 1	1.9	4.4
Section 2	1.9	4.4
Section 3	2.1	4.3
All Sections	2.0	4.4

Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Prior to instruction, the sample-wide mean was 2.0, indicating a process without coordination level of understanding of divisibility. Following instruction, this mean increased to 4.4, indicating object level understanding of divisibility. Participants maintained these levels of understanding across all three course sections. Although a separate analysis of classroom instruction is necessary to explain this result, it does provide support for the claim that instruction in all three sections was similar with respect

to divisibility and prime factorization concepts.

Question 6 was presented as follows on the NTKT pre-test and post-test, respectively: a) *The prime factorization of the greatest common factor of two numbers, A and B, is $2^3 \times 5$. The prime factorization of the least common multiple of the same two numbers is $2^5 \times 5^2$. Give all possible values for A and B. Show your work;* b) *The prime factorization of the greatest common factor of two numbers, A and B, is $3^2 \times 7^2$. The prime factorization of the least common multiple of the same two numbers is $3^4 \times 7^3$. Give all possible values for A and B. Show your work.* Question 6 was a complex question that could not be solved by traditional methods. For example, if $2^3 \times 5$ is the GCF of A and B, then A and B must be multiples of $2^3 \times 5$. Since the LCM of A and B is $2^5 \times 5^2$, then A and B must also be factors of $2^5 \times 5^2$. Thus, $A = (2^3 \times 5) \times f$ and $B = (2^3 \times 5) \times g$, where f and g are factors of $(2^5 \times 5^2) \div (2^3 \times 5)$ or $2^2 \times 5$. The factors of $2^2 \times 5$ are 1, 2, 2^2 , 5, 2×5 , and $2^2 \times 5$. However, if $f=2$, then $g=2 \times 5$ resulting in a change to the GCF of A and B. As such, the only possible values of f and g are 1, 2^2 , 5, and $2^2 \times 5$. Thus, there are four possible values of A and B: $2^3 \times 5$, $2^5 \times 5$, $2^3 \times 5^2$, and $2^5 \times 5^2$. Since the GCF and LCM were already provided, it was necessary to reverse the normal process of finding these values in order to find the values of A and B. However, reversal is a difficult mental construction to make because it requires the individual to already have internalized the process that needs reversing (Dubinsky, 1991). Additionally, Question 6 also requires an understanding of the relationship between GCF and LCM.

Results showed that participants struggled to solve this question during both the pre- and post-test. Table 4.2.3 presents sample-wide frequencies of each level of

understanding as exhibited by participants when answering Question 6 on both the pre and post-test.

Table 4.2.3: NTKT Question 6 Total Frequencies and Percentages for each Level of Understanding on APOS

Code	Pre-Test Frequencies	Post-Test Frequencies
Action	55 (93.2%)	4 (6.8%)
Process without coordination	1 (1.7%)	3 (5.1%)
Process with coordination	2 (3.4%)	37 (62.7%)
Object	1 (1.7%)	9 (15.3%)
Schema	0 (0.0%)	6 (10.2%)
Total	59 (100.0%)	59 (100.0%)

Prior to instruction, an overwhelming 93.2% of all participants exhibited an *action* level of understanding. They either could not initiate any solution strategy, or they computed the decimal values of the GCF and LCM. Participants at this level of understanding could not attend to either the GCF or LCM concept because they had not yet internalized a mental image for either concept. As such, all they could do was revert to needless computation.

Following instruction, the majority of participants (62.7%) showed an understanding of the GCF and LCM concepts at the *process with coordination* level. Such understanding was marked by an ability to attend to both GCF and LCM concepts at the same time. Participants recognized that the solution values must be factors of the LCM and multiples of the GCF, and they were able to find the factors of $(LCM \div GCF)$

needed to solve the problem. However, they could not distinguish between the factors that maintained or changed the GCF of A and B. As a result, many either could not find all of the possible solutions or found additional incorrect solutions. While they were able to use the relationship between the GCF and LCM to find some solution values, they did not yet possess an explicit understanding of that relationship.

Approximately 25% of the sample operated at the two highest levels of understanding (*object* and *schema*) for Question 6 during the post-test. At the object level, they were able to find all possible solution values, but they still could not identify why certain cases were incorrect. Typically, this was not an explicit error, but more likely a result of not recognizing the possibility of other cases. Participants at the schema level, however, did recognize the possibility of erroneous cases and immediately dismissed them. They were able to provide clear reasons for the dismissal, showing they possessed an explicit understanding of the relationship between the GCF and LCM.

Results of participants' work on Question 6 were also analyzed by section. Table 4.2.4 below shows mean values for each section's displayed level of understanding on the pre- and post-tests.

Table 4.2.4: NTKT Question 6 Mean Levels of Understanding by Course Section

Code	Pre-Test Mean	Post-Test Mean
Section 1	1.0	3.2
Section 2	1.2	3.1
Section 3	1.2	3.2
All Sections	1.1	3.2

Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Participants' displayed levels of understanding were nearly identical in all three course

sections. Although a separate analysis of classroom instruction is necessary to explain this result, it does provide support for the claim that instruction in all three sections was similar with respect to GCF and LCM concepts.

This chapter presented analysis of the Number Theory Knowledge Test (NTKT) instrument. Results were analyzed by overall scores, subset scores, and course section. Additionally, a more in-depth, qualitative analysis of two NTKT questions was provided in order to better describe participants' initial and developing understanding of divisibility, prime factorization, GCF, and LCM concepts. The next chapter presents the results of a qualitative analysis of the interview data from a sub-sample of the study.

CHAPTER V

ANALYSIS OF INTERVIEW DATA

This study analyzed participants' developing understanding of elementary number theory topics by conducting two one-hour long clinical interviews with six participants from the sample. The six participants were selected by random sampling, stratified by prior achievement levels based on their NTKT pre-test scores and their course instructor's recommendations. Each participant was interviewed by the researcher approximately one week before the start of a three-week unit on number theory and then again one week following the completion of the unit and the administration of the NTKT post-test. Each interview consisted of thirteen questions on divisibility, factors, greatest common factor (GCF), and least common multiple (LCM).

Participants were asked to describe their thinking out loud as they solved each question in order to obtain a genuine account of their thinking (Patton, 2002). The interviewer did not provide any assistance, and instead, only prompted participants to articulate their thinking out-loud. Such an approach increased the likelihood that the participants' own conceptions came to the forefront. In some instances, the interviewer asked participants to repeat their previous comments in order to more accurately record their thinking.

In Section I of this chapter, a detailed description of the analysis of the interview transcripts is presented. Section II provides definitions of each code that emerged from analysis of the interview data. Section III presents profiles of each of the six interview

participants, and Section IV discusses overall trends in the interview data.

Section I: Interview Transcript Analysis

Analysis of the six interviews was completed using the APOS model of concept development (Dubinsky & McDonald, 2001). The APOS model was used to describe individuals' mental constructions of mathematical concepts. Analysis of the six interview participants' transcripts resulted in the development and refinement of the APOS codes as they pertain to elementary number theory topics. The APOS codes emerged from a three-phase process of analyzing pre-interviews, post-interviews, and across interviews. Within each phase, transcripts were analyzed across two perspectives: by participant and by question.

In order to identify participants' initial understanding of number theory topics, each participant's pre-interview transcript was read twice, once to identify solution strategies and a second time to identify specific evidence of understanding. Initial codes emerged as a result of the two readings of each transcript, and *transcript coding forms* were created for each participant's pre-interview. For each interview question, the form included direct quotations, the researcher's explanation of the quotation, and a preliminary APOS code. The pre-interview transcripts were then re-analyzed by participant and by question. Each participant's pre-interview transcript was re-examined in order to confirm or reject the initial codes. If evidence was found to reject the initial codes, the transcript coding form for that particular participant was edited to account for the change.

Once the initial coding by participant was completed, pre-interview transcripts were re-analyzed by question (i.e., Question 1 was analyzed across all participants, then Question 2, etc.). This approach gave the researcher a more detailed view of initial conception levels for specific questions. It was during this second re-analysis that the researcher began constructing meanings for each code pertaining to specific questions. Again, transcript coding forms were further refined if necessary.

Participants' understanding of number theory topics following instruction as determined by analyzing participants' post-interviews used the same approach as the pre-interviews. Each post-interview transcript was read twice to identify solution strategies and find evidence of participants' conceptions. Transcript coding forms were made for each participant to record this information and initial codes for each question. Re-analysis of the post-interview transcripts was executed by participant and then by question. Any modifications to initial codes were recorded on participants' transcript coding forms. Analysis by participant provided detail regarding each participant's conception levels for various number theory topics. Analysis by question contributed to the researcher's developing understanding of the meaning of each code for each question.

The third phase of transcript analysis involved examining levels of understanding across interviews by participant and by question. First, each participant's pre-interview and post-interview transcripts were compared in order to identify changes in understanding following instruction. Then, each question was analyzed from pre-interview to post-interview across all participants. The first approach provided details about individual participant's developing understanding of number theory topics across

all questions. The second approach provided details about how individual questions were understood across all participants. It was found that the dual perspective provided greater insight into determining participants' levels of understanding and the meanings of the APOS codes than either perspective alone.

Although participants' responses during the interviews corresponded well to the APOS levels, the researcher discovered some variability among participant responses at the APOS *process* level. The process level is characterized by an interiorization of physical or mental actions so that the individual no longer needs to run through each step of the action; the action can instead be executed more efficiently with some anticipation of what the result should be. Some participants who displayed a process level of understanding did not implicitly attend to relationships between key number theory concepts, while others did. For example, when asked to determine if $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$ was divisible by 5 and by 51, all participants classified at the process level could identify divisibility by 5 without computation, but some of these participants could not recognize that N was divisible by 51. These participants could not coordinate some of N 's prime factors (3×17) in order to identify composite factors (51). As such, it was necessary to sub-divide the process code into two sub-codes: *process without coordination* and *process with coordination*. The former reflects a process level of understanding in which the individual is unable to combine, explicitly or implicitly, two or more concepts to make sense of the problem situation; the latter reflects one's ability to make such combinations. At neither level, however, is the individual able to clearly articulate a rationale for the process or combination.

Section II: APOS Code Descriptions

Each interview participant's responses to every interview question prior to and following instruction were coded according to APOS theory. Analysis resulted in deep descriptions of their understanding of a variety of number theory topics. It also clarified the fact that any specific APOS level of understanding is characterized differently for different questions. For example, an *object* level of understanding for a divisibility question does not mean the same exact thing as an *object* level of understanding for a GCF question. As such, each APOS level of understanding must be clearly characterized for each question.

The result of the interview analysis is the following set of code descriptions for each interview question. Each pair of corresponding pre- and post-interview questions is presented, followed by schema-level explanations of both questions' solutions. Then, each of the five APOS levels of understanding is described for that particular pair of questions as they were observed in interview participants' work. Note that Question 2 differs substantively from pre- to post-interview, Question 5 is identical on both interviews, and all remaining questions are the same except their numbers are changed from pre- to post-interview.

Question 1 (pre-interview)

In number theory, “divisible” means divided by with no remainder. Consider the number $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$. Without calculating the value of N, answer each question by circling one of the three possible options. Explain each choice.

- | | | | |
|---------------------------|-----|----|---------------|
| a) Is N divisible by 5? | Yes | No | I am not sure |
| b) Is N divisible by 7? | Yes | No | I am not sure |
| c) Is N divisible by 4? | Yes | No | I am not sure |
| d) Is N divisible by 51? | Yes | No | I am not sure |
| e) Is N divisible by 105? | Yes | No | I am not sure |

Question 1 (pre-interview) Solution

- a) Yes, since 5 is a factor in the prime factorization of N.
- b) No, since 7 is not in the prime factorization of N.
- c) Yes, since $4 = 2^2$ is a factor of 2^3 , which is a factor in the prime factorization of N.
- d) Yes, since $51 = 3 \times 17$ and 3×17 is a factor in the prime factorization of N.
- e) No, since $105 = 3 \times 5 \times 7$ and 7 is not a prime factor of N.

Question 1 (post-interview)

In number theory, “divisible” means divided by with no remainder. Consider the number $N = 3^2 \times 5^4 \times 11 \times 17^3$. Without calculating the value of N, answer each question by circling one of the three possible options. Explain each choice.

- | | | | |
|-------------------------|-----|----|---------------|
| a) Is N divisible by 3? | Yes | No | I am not sure |
|-------------------------|-----|----|---------------|

b) Is N divisible by 35?	Yes	No	I am not sure
c) Is N divisible by 19?	Yes	No	I am not sure
d) Is N divisible by 75?	Yes	No	I am not sure
e) Is N divisible by 51?	Yes	No	I am not sure

Question 1 (post-interview) Solution

- a) Yes, since 3 is a factor in the prime factorization of N.
- b) No, since $35 = 5 \times 7$ and 7 is not a prime factor of N.
- c) No, since 19 is not in the prime factorization of N.
- d) Yes, since $75 = 3 \times 5^2$ and 3×5^2 is a factor in the prime factorization of N.
- e) Yes, since $51 = 3 \times 17$ and 3×17 is a factor in the prime factorization of N.

Question 1 Levels of Understanding

Action Level: Participants at this level compute the value of N first and then perform long division in order to answer this question. Such participants have tremendous difficulty answering this question because the question explicitly does not allow computing the value of N.

Process Without Coordination Level: Participants at this level are able to identify divisors of N if the divisors are visible within N's prime factorization (e.g., 5) or if the divisor is a factor of one of the prime powers in N's prime factorization (e.g., 4). Such participants cannot identify divisors of N that are products of distinct prime factors of N (e.g., 51).

Process With Coordination Level: Participants at this level are able to identify divisors in the same way as those at the *process without coordination* level, plus they can begin to identify composite divisors, but typically only those that are the product of two distinct prime factors of N . Such participants struggle to identify composite divisors that are composed of three or more distinct prime factors of N . Additionally, they are unable to explain why certain composite numbers (e.g., 105) are not divisors of N using prime factorization.

Object Level: Participants at this level can identify all prime and composite divisors of N , as well as use prime factorization to explain why certain composite numbers cannot be divisors of N . Such participants are not yet able to explicitly relate notions of prime factorization to divisibility, and so cannot yet provide generalized explanations for determining divisors and non-divisors of any number in prime factored form.

Schema Level: In addition to being able to identify all divisors and non-divisors of N , participants at this level are able to connect prime factorization and divisibility concepts in order to articulate generalized statements regarding divisibility. Such participants often provide explanations that go beyond the specific values of N , such as “a divisor’s prime factors must be in the prime factorization of the number.”

Question 2 (pre-interview)

Do you agree or disagree with the following statement? Explain your response.

A number is divisible by 18 if it is divisible by 3 and by 6.

Question 2 (pre-interview) Solution

Disagree. A number can be divisible by 3 and by 6 but not be divisible by 18 (e.g., 12). Since 3 and 6 share a factor of 3, divisibility by 6 already implies divisibility by 3. As such, if a number is divisible by 6, then no new information is gained from checking divisibility by 3. Thus, if a number is divisible by 3 and by 6, the greatest number for which divisibility can be inferred is 6.

Question 2 (post-interview)

A certain mystery number, N , is divisible by 18 and by 15. What is the greatest number that N must be divisible by? How do you know?

Question 2 (post-interview) Solution

The greatest number that N must be divisible by is $2 \times 3^2 \times 5$, or 90. If N is divisible by 18, then its prime factorization must include the prime factorization of 18, or 2×3^2 . If N is divisible by 15, then its prime factorization must also include the prime factorization of 15, or 3×5 . The 3 from 15's prime factorization is already accounted for by one of the 3's in 18's prime factorization. The solution, written in prime factored form, includes both 18's and 15's prime factorizations with no extraneous prime factors of any kind.

Question 2 Levels of Understanding

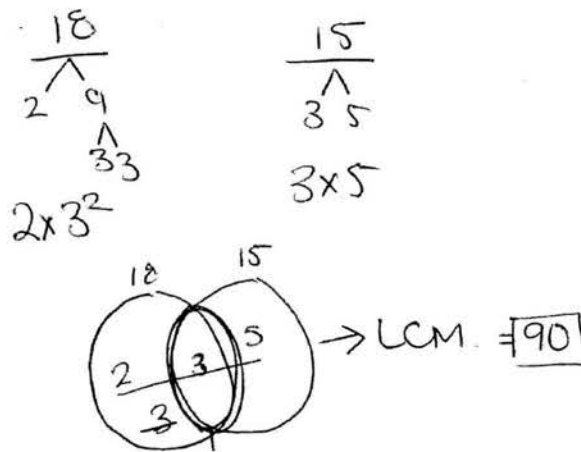
Action Level: Participants at this level cannot execute an efficient solution strategy, so they either use a guess-and-check strategy to find a solution or they provide unreasonable answers. Correct responses to the pre-interview question are possible at this level because participants can guess and check random numbers and find a counter-example. Unreasonable responses have no relation to the underlying concepts (i.e., divisibility, least common multiple, and prime factorization) and occur because participants do not know what else to do. On the post-interview question, they include “the greatest number that N must be divisible by is 3 because 3 is the greatest common factor of 18 and 15” or “18 is the answer because it is bigger than 15.”

Process Without Coordination Level: Participants at this level begin to use a more efficient solution strategy, such as a modified guess-and-check procedure in which they begin to anticipate characteristics of the solution. Nevertheless, they cannot combine characteristics of both numbers (3 and 6, or 18 and 15) to find a solution. For example, participants at this level solve the pre-interview question by using their number sense to recognize that they only need to check numbers less than 18 for divisibility by 3 and 6. Participants begin to internalize notions of divisibility, but they cannot yet coordinate multiple aspects of the problem to efficiently solve it or provide a rationale.

Process With Coordination Level: Participants at this level can combine characteristics of the two given numbers (3 and 6, or 18 and 15) in order to find a solution. For

example, Figure 5.2.1 shows how one participant used the prime factorizations of 18 and 15 to construct a Venn diagram and identify the least common multiple (LCM) as the solution to the post-interview question. At this level, participants' conception of divisibility is internalized to the extent that they can construct new processes (e.g., finding the LCM of two numbers). Participants are unable to explain why their solution is correct.

Figure 5.2.1: Process With Coordination Level Response to Question 2



Object Level: Participants at this level are now able to explain why the underlying concept (e.g., LCM) is necessary. The ability to provide sound reasoning for a solution is evidence of encapsulation of the LCM procedure (Brown, Thomas, & Tolia, 2002). However, they are not yet able to explicitly articulate the inherent relationships between divisibility, LCM, and prime factor concepts needed to solidify their schema. For example, one participant's response to the post-interview question included an explanation such as, "the answer is the LCM of 18 and 15 because since the number must

be divisible by 18 and 15, it must be a multiple of 18 and 15.”

Schema Level: Participants at this level can solve each problem efficiently and are now able to articulate the key relationships between the LCM and the prime factors of the two given numbers in each problem. One participant at the schema level explained this relationship:

In order to be divisible by 18 and 15, this number must include all of the prime factors of 18 and 15, but the reason it doesn't have to include an extra 3 is because 18 and 15 share that factor...you don't have to list it for both times.

It is the realization and articulation of the relationship between concepts that marks participants' progress to the schema level. Zazkis and Campbell (1996a) refer to the construction of relationships between concepts as the *thematization* of an object to a schema.

Question 3 (pre-interview)

Consider two numbers, $N = 3 \times 6 \times 7$ and $R = 11 \times 13 \times 17$.

- a) Do you think N has more factors than R , less factors than R , or the same number of factors as R ? Explain your choice.
- b) Find all of the factors of N and R .

Question 3 (pre-interview) Solution

- a) N has more factors than R because its prime factorization has a greater number of prime factor combinations. Since one of N 's factors shown above can be further

decomposed ($6 = 2 \times 3$), N can be represented as a product of 4 prime numbers ($N = 2 \times 3^2 \times 7$), resulting in 12 factor combinations. R is composed of only 3 prime numbers, resulting in 8 factor combinations. In general, a number whose prime factorization is in the form $P^A \times Q^B$ has $(A + 1) \times (B + 1)$ factors.

- b) N 's twelve factors are made of combinations of zero, one, two, three, or four prime factors. They are: 1, 2, 3, 7, 2×3 , 2×7 , 3^2 , 3×7 , 2×3^2 , $2 \times 3 \times 7$, $3^2 \times 7$, and $2 \times 3^2 \times 7$.

R 's eight factors are made of combinations of zero, one, two, or three prime factors. They are: 1, 11, 13, 17, 11×13 , 11×17 , 13×17 , and $11 \times 13 \times 17$.

Question 3 (post-interview)

Consider two numbers, $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$.

- a) Do you think N has more factors than R , less factors than R , or the same number of factors as R ? Explain your choice.
- b) Find all of the factors of N and R .

Question 3 (post-interview) Solution

- a) N has more factors than R because its prime factorization has a greater number of prime factor combinations. Since one of N 's factors shown above can be further decomposed ($4 = 2 \times 2$), N can be represented as a product of 4 prime numbers ($N = 2^2 \times 3 \times 5$), resulting in 12 factor combinations. R is composed of only 3 prime numbers, resulting in 8 factor combinations. In general, a number whose prime

factorization is in the form $P^A \times Q^B$ has $(A + 1) \times (B + 1)$ factors.

b) N's twelve factors are made of combinations of zero, one, two, three, or four prime factors. They are: 1, 2, 3, 5, 2^2 , 2×3 , 2×5 , 3×5 , $2^2 \times 3$, $2^2 \times 5$, $2 \times 3 \times 5$, and $2^2 \times 3 \times 5$.

R's eight factors are made of combinations of zero, one, two, or three prime factors. They are: 1, 13, 17, 19, 13×17 , 13×19 , 17×19 , and $13 \times 17 \times 19$.

Question 3 Levels of Understanding

Action Level: Participants at this level cannot answer either question without first computing the values of N and R and then executing long division to find which number has more factors. At this level, participants' conceptions of factor and divisibility are extremely algorithmic and dependent on seeing numbers in a familiar (non-factored) form. Figure 5.2.2 shows an example of a study participant exhibiting an action level of understanding when solving Question 3.

Figure 5.2.2: Action Level Response to Question 3

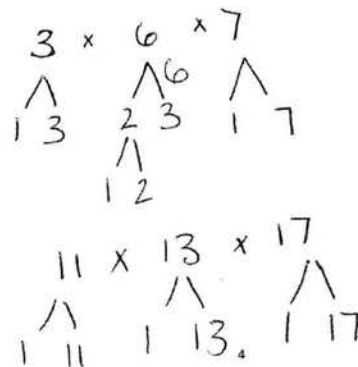
$N = 3 \times 6 \times 7$
 $\begin{array}{r} 18 \\ \times 7 \\ \hline 126 \end{array}$
 126
 2 126
 3 63

$R = 11 \times 13 \times 17$
 $\begin{array}{r} 143 \\ \times 17 \\ \hline 2431 \end{array}$
 $\begin{array}{r} 11 \\ \times 2431 \\ \hline 2431 \end{array}$

126 has more factors be
 its an even #, above 2
 + 267 2 + 13 so 126/2 = 63. But
 318 is only divisibly by 1 + 31
 so there are only 2
 factors w 2431 four
 Thus $N > R$ in factors.

Process Without Coordination Level: Participants at this level internalize the long division procedure and immediately claim that N has more factors than R because N can be further decomposed into a product of 4 prime factors, while R is already decomposed into a product of only 3 prime factors. However, they have not yet internalized the procedure for finding composite factors, so they can only identify numbers that are visible, factors of visible numbers (e.g., 2 is a factor of 6 so it is a factor of N), 1, and the number itself as factors of N or R. Figure 5.2.3 serves as a typical example of this type of thinking.

Figure 5.2.3: Process Without Coordination Level Response to Question 3



Process With Coordination Level: Participants at this level immediately identify that N has more factors than R because it can be further decomposed into a greater number of prime factors, and they continue to claim that N has only one more factor than R using the same inappropriate reasoning as above. Surprisingly, when they begin finding the factors, they are able to identify composite numbers as factors by coordinating two or more distinct prime factors (e.g., $11 \times 13 = 143$ is a factor of R). As such, they revise their original claims regarding the number of factors of N and R without explaining why

they were wrong beforehand. Participants at this level still have difficulty coordinating all combinations of prime factors, and often miss some factors. Figure 5.2.4 illustrates a process with coordination level of understanding; note that this participant had originally missed some factors of N and could find all twelve factors only after redoubling her efforts.

Figure 5.2.4: Process With Coordination Level Response to Question 3

$$\begin{array}{l}
 N = 3 \times 6 \times 7 \\
 N = 3 \times 3 \times 2 \times 7 \\
 3, 2, 7, 9, 6, 21, \\
 14, 1, (3 \times 6 \times 7) \\
 42, 18, 63, \\
 12 \text{ factors}
 \end{array}
 \qquad
 \begin{array}{l}
 R = 11 \times 13 \times 17 \\
 11, 13, 17, (11 \times 13), \\
 (13 \times 17), (11 \times 17), 1 \\
 (11 \times 13 \times 17) \\
 8 \text{ factors}
 \end{array}$$

Object Level: Participants at this level encapsulate the notion of factor, so they begin to see a factor as a product of some combination of prime factors. As such, they are able to find all factors of N and R by listing all of their prime factor combinations. Typically, they use a systematic approach to finding the factors, such as a Cartesian product (tree) diagram shown in Figure 5.2.5.

Figure 5.2.5: Object Level Response to Question 3

Schema Level: Participants at this level are not only able to find all factors of N and R , but they can also explicitly explain the relationship between the prime factorization of a number and its number of factors by stating that a number whose prime factorization is $P^A \times Q^B$ has $(A + 1) \times (B + 1)$ factors.

Question 4 (pre-interview)

Find a number that has exactly 4 factors. Explain how you know it has 4 factors.

Question 4 (pre-interview) Solution

6 has four factors because its prime factorization (2×3) contains four prime factor combinations: 1, 2, 3, and 2×3 . In fact, any number whose prime factorization is the product of exactly two distinct prime factors ($p \times q$) will have four factors (1, p , q , and $p \times q$). Additionally, any number whose prime factorization is the cube of a prime (p^3) also

has exactly four prime factor combinations (1, p , p^2 , and p^3).

Question 4 (post-interview)

Find a number that has exactly 6 factors. Explain how you know it has 6 factors.

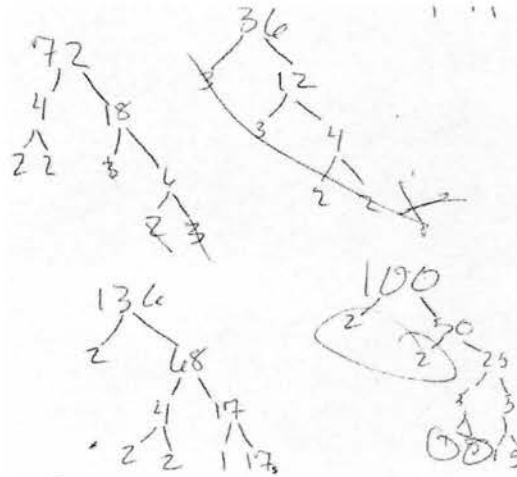
Question 4 (post-interview) Solution

12 has six factors because its prime factorization ($2^2 \times 3$) contains six prime factor combinations: 1, 2, 3, 2^2 , 2×3 , and $2^2 \times 3$. In fact, any number whose prime factorization is the product of a prime and the square of a different prime ($p^2 \times q$) will have six factors (1, p , q , $p \times q$, p^2 , and $p^2 \times q$). Additionally, any number whose prime factorization is the fifth power of a prime (p^5) also has exactly six prime factor combinations (1, p , p^2 , p^3 , p^4 , and p^5).

Question 4 Levels of Understanding

Action Level: Participants at this level randomly guess numbers and find all of their factors using tree diagrams or long division to check to see if they have four factors. In Figure 5.2.6, the participant is using factor trees to find the factors of a random selection of numbers. However, she is only considering the prime factors as factors of each number.

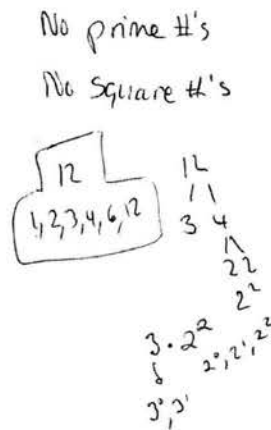
Figure 5.2.6: Action Level Response to Question 4



Process Without Coordination Level: Participants at this level use a more systematic method of guessing-and-checking that is guided by their knowledge of factor properties. For example, they avoid checking prime numbers and square numbers because they know primes only have 2 factors and square numbers have an odd number of factors. Such participants, however, are not yet able to account for combinations of prime factors, so they at times do not find all of a number's factors.

Process With Coordination Level: Participants at this level use a more systematic method of guessing-and-checking, as mentioned above. However, when checking the number of factors, they are now able to account for combinations of prime factors, so they are better able to find solutions. Figure 5.2.7 illustrates this level of understanding.

Figure 5.2.7: Process With Coordination Level Response to Question 4



Object Level: Participants at this level no longer use a guess-and-check strategy to find numbers. They are now able to reverse the procedure for finding a number's factors and construct the number via its prime factorization. For example, they may select two distinct prime numbers and claim that their product is a number with four factors (e.g., $2 \times 3 = 6$). However, they typically need to make a list of all of the number's prime factor combinations to check if their assertion is correct. Such participants have not yet thematized the prime factorization concept to their factor scheme, and so they are unable to explicitly relate the two concepts.

Schema Level: Participants at this level have thematized prime factorization to their factor schema. They construct the prime factorization of a number based on their knowledge of the relationship between the form of the prime factorization of a number and its number of factors. Additionally, and most importantly, they articulate this relationship clearly. For example, a number whose prime factorization is $P \times Q$ has four

factors (2×2); a number whose prime factorization is $P^2 \times Q$ has six factors (2×3).

Question 5 (pre- and post-interview)

What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Explain your work. Note: you may leave your answer in factored form.

Question 5 (pre- and post-interview) Solution

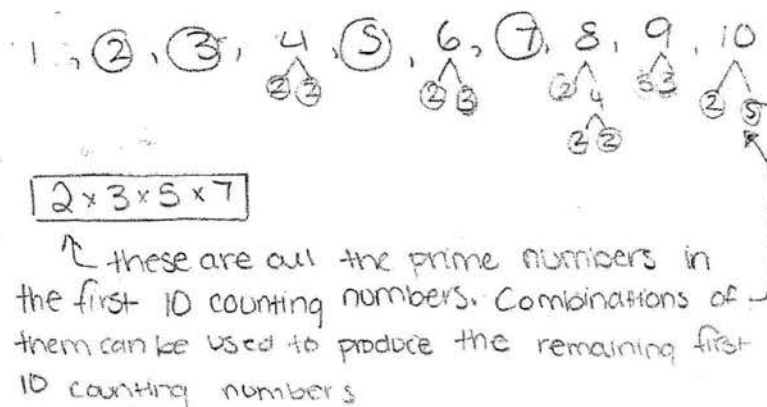
$2^3 \times 3^2 \times 5 \times 7$, or 2,520, is the smallest positive integer that is divisible by all of the first ten counting numbers. Since the number must be divisible by 1-10, we can build its prime factorization to include all of these numbers as factors. In order to be divisible by 2, 4, and 8, its prime factorization must contain 2^3 . In order to be divisible by 3 and 9, its prime factorization must contain 3^2 . To be divisible by 5 and 7, the number must contain 5 and 7 in its prime factorization. To be divisible by 6, the number must contain 2 and 3 in its prime factorization. Both are already accounted for in the 2^3 and 3^2 . To be divisible by 10, the number must contain 2 and 5 in its prime factorization. Both are already accounted for in the 2^3 and 5. As always, 1 is a factor of any whole number. Also, $2^3 \times 3^2 \times 5 \times 7$ is the smallest positive integer that satisfies the given criterion because if any of the prime factors are removed from its prime factorization, at least one of the first ten counting numbers will cease to be its factor.

Question 5 Levels of Understanding

Action Level: Participants at this level can only guess and check random numbers to see if they are divisible by the first ten counting numbers. They have not internalized the divisibility concept, so any checking must involve executing long division.

Process Without Coordination Level: Participants at this level begin to attend to prime factorization in order to solve the problem. They internalize the long division procedure, so they can identify divisibility by a number if that number is visible within the prime factorization of the potential solution. Such participants claim that the product of the first ten counting numbers might be the solution since each counting number is represented, or they might claim that the solution is $1 \times 2 \times 3 \times 5 \times 7$ (as shown in Figure 5.2.8) since this product represents all of the prime factors of the first ten counting numbers. However, these participants do not yet coordinate distinct prime factors to form composite factors, so their solution either repeats factors unnecessarily (e.g., 2 and 8) or does not account for all composite factors (e.g., 9).

Figure 5.2.8: Process Without Coordination Level Response to Question 5



Process With Coordination Level: Participants at this level begin to account for composite factors and notice that certain counting numbers already account for others (e.g., divisibility by 8 implies divisibility by 4 and 2). They are able to articulate this relationship, but they cannot accurately apply it in all cases. Figure 5.2.9 provides an example where a participant incorrectly solved the problem by inadvertently repeating certain prime factors (i.e., 2 and 3 are overused twice and once, respectively).

Figure 5.2.9: Process With Coordination Level Response to Question 5

$$\begin{array}{l}
 \cancel{1} \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5} \times 6 \times 7 \times 8 \times 9 \times 10 \\
 6 \times 7 \times 8 \times 9 \times 10 \\
 42 \times 72 \times 10 \\
 30240
 \end{array}$$

Object Level: Participants at this level are able to correctly construct the prime factorization of the solution by attending to prime and composite factors. As Figure 5.2.10 shows, they recognize that certain counting numbers are divisible by other counting numbers, and so they eliminate redundancies. Such participants attend to the least number of prime factors necessary to create a composite number divisible by the first ten counting numbers, but they are unable to clearly articulate how they know their solution represents the least such number.

Figure 5.2.10: Object Level Response to Question 5

$$\begin{array}{cccccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 \downarrow & \downarrow & & \wedge & \downarrow & \downarrow & \downarrow & \wedge & \downarrow & \downarrow \\
 1 & 2 & 3 & 2 \cdot 2 & 5 & 2 \cdot 3 & 7 & 2 \cdot 2 \cdot 2 & 3 \cdot 3 & 2 \cdot 5
 \end{array}$$

$$N = 2^3 \cdot 3^2 \cdot 5 \cdot 7$$

$$\begin{array}{l}
 2^0 = 1 \\
 2^1 = 2 \\
 2^2 = 4 \\
 2^3 = 8
 \end{array}
 \quad
 \begin{array}{l}
 3^0 = 1 \\
 3^1 = 3 \\
 3^2 = 9
 \end{array}
 \quad
 \begin{array}{l}
 5^0 = 1 \\
 5^1 = 5
 \end{array}
 \quad
 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

Schema Level: Participants at this level make explicit connections between prime factorization, divisibility, and LCM concepts by explaining why their solution represents the least positive integer divisible by the first ten counting numbers. For example, one participant, whose written work is shown below in Figure 5.2.11, articulated why attention to prime factors is all that is necessary to find the correct solution:

If you have just the basic prime numbers, then that would be the smallest number, whereas if you add in – if you multiply it also by 4 and 6 and 8 and 9 and 10, that’s just something you don’t need, and so it will make it a larger number that is divisible by all of them.

Figure 5.2.11: Schema Level Response to Question 5

$$\begin{array}{cccccccc}
 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 2^2 \quad 2 \cdot 3 \quad 2^3 \quad 3^2 \quad 5 \cdot 2
 \end{array}$$

$$1 \cdot 2^3 \cdot 3^2 \cdot 5 \cdot 7$$

Question 6 (pre-interview)

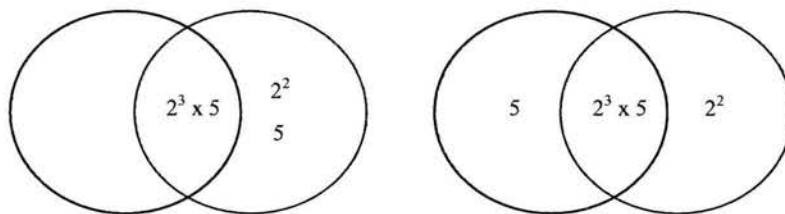
The prime factorization of the greatest common factor of two numbers, A and B, is $2^3 \times 5$.

The prime factorization of the least common multiple of the same two numbers is $2^5 \times 5^2$.

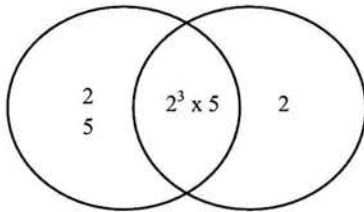
Give all possible values for A and B. Explain your work. Note: you may leave your answer in factored form.

Question 6 (pre-interview) Solution

The possible values of A and B are: $2^3 \times 5$, $2^5 \times 5^2$, $2^5 \times 5$, and $2^3 \times 5^2$ (i.e., 40, 800, 160, and 200), as shown in the Venn diagrams below. Each circle below represents a possible value of A or B:



Each solution must be divisible by the GCF of A and B and a factor of the LCM of A and B. Let $K = \text{LCM} \div \text{GCF} = (2^5 \times 5^2) \div (2^3 \times 5) = 2^2 \times 5$. Therefore, all possible solution values are in the form $\text{GCF} \times (\text{factor of } K)$. The factors of K that are not permissible are 2 and (2×5) because they would alter the value of the GCF. This erroneous case is shown in the Venn diagram below. Notice that in this case, the GCF cannot equal $2^3 \times 5$ because A and B share an additional factor of 2.

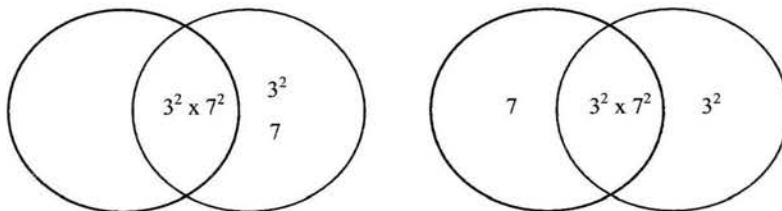


Question 6 (post-interview)

The prime factorization of the greatest common factor of two numbers, A and B , is $3^2 \times 7^2$. The prime factorization of the least common multiple of the same two numbers is $3^4 \times 7^3$. Give all possible values for A and B . Explain your work. *Note: you may leave your answer in factored form.*

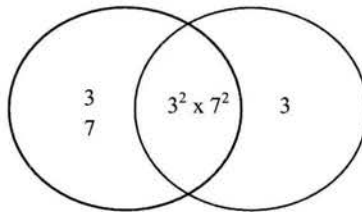
Question 6 (post-interview) Solution

The possible values of A and B are: $3^2 \times 7^2$, $3^4 \times 7^3$, $3^2 \times 7^3$, and $3^4 \times 7^2$ (i.e., 441, 27783, 3087, and 3969), as shown in the Venn diagrams below. Each circle below represents a possible value of A or B :



Each solution must be divisible by the GCF of A and B and a factor of the LCM of A and B . Let $K = \text{LCM} \div \text{GCF} = (3^4 \times 7^3) \div (3^2 \times 7^2) = 3^2 \times 7$. Therefore, all possible solution values are in the form $\text{GCF} \times (\text{factor of } K)$. The factors of K that are not permissible are

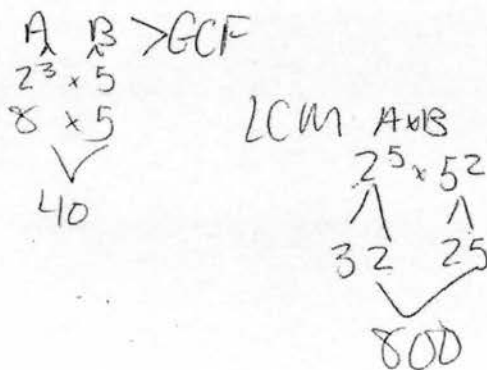
3 and (3×7) because they would alter the value of the GCF. This erroneous case is shown in the Venn diagram below. Notice that in this case, the GCF cannot equal $3^2 \times 7^2$ because A and B share an additional factor of 3.



Question 6 Levels of Understanding

Action Level: Participants at this level can only guess-and-check possible solution values, or they are unable to execute any strategy. At this conception level, participants do not have an internalized notion of the GCF or LCM procedures, so the only action they can take is to compute the values of the GCF and LCM of A and B, as shown in Figure 5.2.12.

Figure 5.2.12: Action Level Response to Question 6



Process Without Coordination Level: Participants at this level do not guess-and-check values because they understand the meaning of the GCF and LCM concepts. They find some of the solution values by attending to either the GCF or the LCM concept, but not both. For example, participants might claim that the solution values must be multiples of the GCF, but disregard the importance of the LCM.

Process With Coordination Level: Participants at this level coordinate both GCF and LCM concepts to find some of the possible solution values. At this level, participants recognize that the solutions are related to both the GCF *and* the LCM of A and B. As such, they compute $K = \text{LCM} \div \text{GCF}$ and find possible solution values by multiplying the GCF with factors of K. As Figure 5.2.13 shows, however, they are not yet able to distinguish between factors of K that preserve or alter the GCF of A and B. Also, they do not yet view solutions as multiples of the GCF or as factors of the LCM.

Figure 5.2.13: Process With Coordination Level Response to Question 6

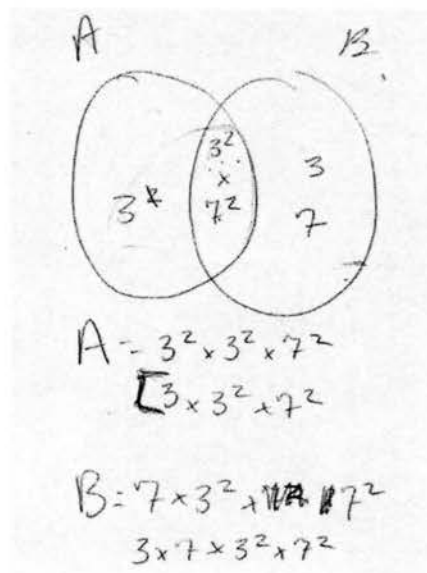
Handwritten work showing calculations and factorizations:

- $2^3 \times 5 = 40$
- $32 \times 25 = 800$
- $40 \overline{)800}$ (with 20 written above the line)
- Factor tree for 20: 20 branches to 5 and 4; 4 branches to 2 and 2.
- Handwritten list under "A or B":
 - 40
 - $40 \cdot 2 = 80$
 - $40 \cdot 4 = 160$
 - $40 \cdot 5 = 200$
 - $40 \cdot 10 = 400$
 - $40 \cdot 20 = 800$

Object Level: Participants at this level see the GCF and LCM as single entities, so they can operate on them as objects, typically by using a visual representation such as a Venn diagram. Figure 5.2.14 displays a participant's work at an object level of understanding.

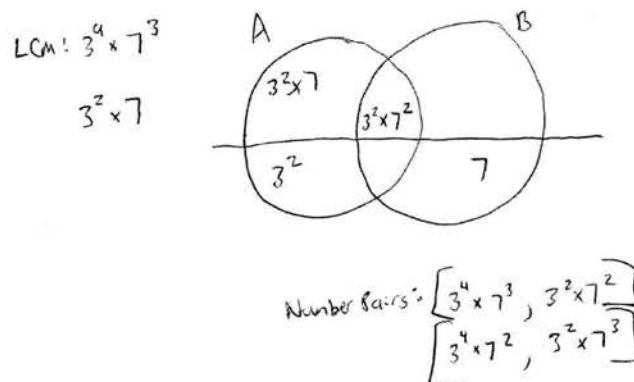
They begin to conceptualize solution values as multiples of the GCF, where the multiple is a factor of K. However, they may still not be able to distinguish between factors of K that maintain or alter the GCF of A and B.

Figure 5.2.14: Object Level Response to Question 6



Schema Level: In addition to encapsulating the concepts of GCF and LCM, participants at this level exhibit a clear understanding of the relationship between the LCM and GCF of A and B. They recognize that solutions values are specific multiples of the GCF, and they can identify and explain which multiples are valid and which are not. For example, one participant explained that her solutions, shown in Figure 5.2.15, could not possibly include other values because one “can’t put one 3 on one side, and one 3 on the other side because then their greatest common factor would include another 3.”

Figure 5.2.15: Schema Level Response to Question 6



Question 7 (pre-interview)

Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 96 feet by 56 feet. What are the dimensions of the largest square tile that the school can buy to completely tile the cafeteria floor?

Question 7 (pre-interview) Solution

The dimensions of the largest square tile are 8 feet by 8 feet. Each dimension of the square tile must be a factor of the dimensions of the cafeteria floor in order to fit along the edges of the floor perfectly. Since the tile is square, its length equals its width, so its dimensions must be a common factor of 96 and 56. To determine the largest square tile needed, then, one must identify the greatest common factor of 96 and 56. The prime factorization of 96 is $2^5 \times 3$; the prime factorization of 56 is $2^3 \times 7$. The greatest common factor of these two numbers is 2^3 , or 8, because that is the greatest product of prime factors that both prime factorizations share. Note: all interview participants were told that the solution to this problem may not accurately reflect reality. They were advised to

not let that interfere with their thinking.

Question 7 (post-interview)

Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 84 feet by 76 feet. What are the dimensions of the largest square tile that the school can buy to completely tile the cafeteria floor?

Question 7 (post-interview) Solution

The dimensions of the largest square tile are 4 feet by 4 feet. Each dimension of the square tile must be a factor of the dimensions of the cafeteria floor in order to fit along the edges of the floor perfectly. Since the tile is square, its length equals its width, so its dimensions must be a common factor of 84 and 76. To determine the largest square tile needed, then, one must identify the greatest common factor of 84 and 76. The prime factorization of 84 is $2^2 \times 3 \times 7$; the prime factorization of 76 is $2^2 \times 19$. The greatest common factor of these two numbers is 2^2 , or 4, because that is the greatest product of prime factors that both prime factorizations share. Note: all interview participants were told that the solution to this problem may not accurately reflect reality. They were advised to not let that interfere with their thinking.

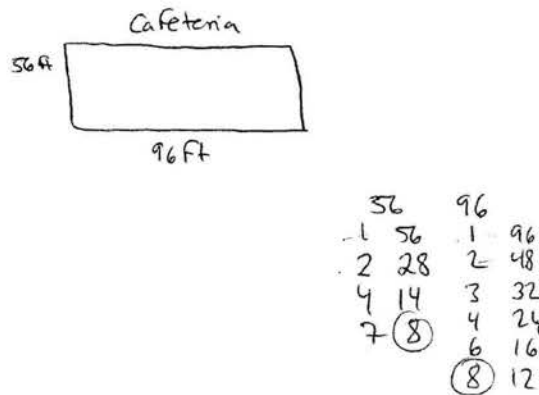
Question 7 Levels of Understanding

Action Level: Participants at this level use a guess-and-check strategy by randomly selecting a number and using long division to determine if it is a factor of the two given

values. They may or may not recognize that the solution is the GCF of the two numbers, but they do not know a systematic method for computing its value.

Process Without Coordination Level: Participants at this level anticipate the solution is the GCF of the two given numbers, and they use a more efficient method for computing the GCF. However, their method does not attend to the relationship between the GCF and the prime factorization of the two given numbers. For example, participants at this level compute all of the factor pairs of each given number separately and then identify the greatest factor that appears on both lists, as shown in Figure 5.2.16 below. Such participants cannot explain why the GCF is necessary to solve this problem.

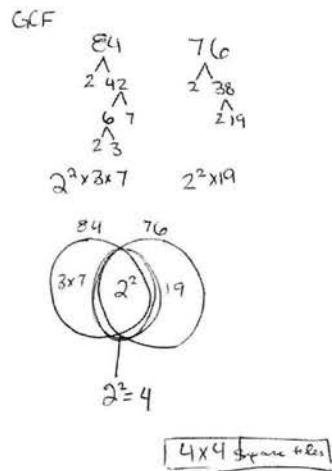
Figure 5.2.16: Process Without Coordination Level Response to Question 7



Process With Coordination: Participants at this level begin to use the relationship between the GCF and the prime factorizations of the two given numbers to more efficiently find a solution. For example, as shown in Figure 5.2.17, they find the prime factorization of each given number and use a Venn diagram to identify the GCF. There is

no need to compute factor pairs any longer because participants at this level have internalized the divisibility process and begun relating divisibility to prime factorization. As with the *process without coordination* level, however, they cannot yet explain why the GCF is necessary to solve this problem.

Figure 5.2.17: Process With Coordination Level Response to Question 7



Object Level: Participants at this level are easily able to find the GCF by using prime factorization, Venn diagrams, and relating the former to the GCF concept. Additionally, they can now explain why the GCF is needed to find the solution. For example, one participant used the context of the question (square tiles) to explain:

Since they have to be square, it means they have to have all the sides the same, so one side, like, of the tiles has to be a factor of 84, and the other side – like the other side has to be a factor of 76, so since they have to be the same, it has to be a factor that both of the tiles share, and you want to find like the GCF because it's the greatest factor that they both share.

As such, participants at this level show an explicit understanding of the meaning of the GCF concept (i.e., encapsulation).

Schema Level: Participants at this level can explain why the GCF is needed to solve the problem, and they can also explain why their method for finding the GCF must be correct. Such explicit reasoning requires an understanding of the relationship between GCF and prime factorization concepts. Participants who can provide this level of reasoning have a schema for the GCF concept that is connected to prime factorization and possibly divisibility concepts.

Question 8 (pre-interview)

Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 135 seconds. Jean runs one complete lap in 80 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

Question 8 (pre-interview) Solution

It will take both runners 2160 seconds to stop running. Rita will stop running only at multiples of 135 seconds, since it takes her 135 seconds to finish each lap. Jean will stop running only at multiples of 80 seconds, since it takes her 80 seconds to finish each lap. Therefore, to stop together, the time elapsed must be a multiple of both 135 and 80. To find the first time that both runners stop, the time elapsed must represent the least common multiple of 135 and 80. The prime factorization of 135 is $3^3 \times 5$ and the prime factorization of 80 is $2^4 \times 5$. The least common multiple of these two numbers is $2^4 \times 3^3 \times 5$, or 2160, because that is the least product of prime factors of which both prime

factorizations are factors. Note: all interview participants were told that the solution to this problem may not accurately reflect reality. They were advised to not let that interfere with their thinking.

Question 8 (post-interview)

Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 80 seconds. Jean runs one complete lap in 105 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

Question 8 (post-interview) Solution

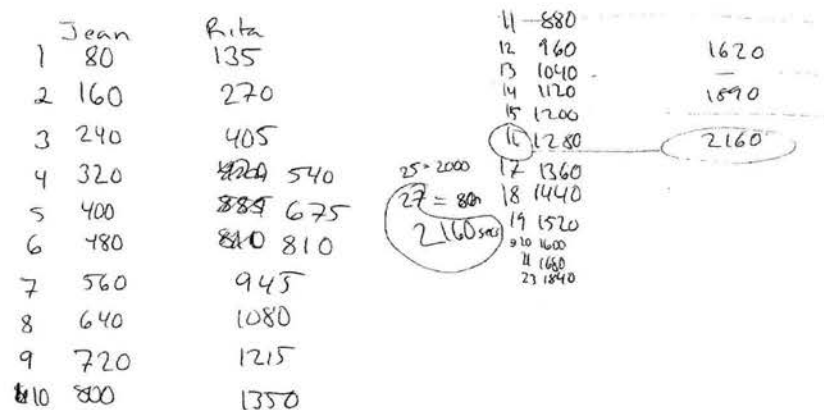
It will take both runners 1680 seconds to stop running. Rita will stop running only at multiples of 80 seconds, since it takes her 80 seconds to finish each lap. Jean will stop running only at multiples of 105 seconds, since it takes her 105 seconds to finish each lap. Therefore, to stop together, the time elapsed must be a multiple of both 80 and 105. To find the first time that both runners stop, the time elapsed must represent the least common multiple of 80 and 105. The prime factorization of 80 is $2^4 \times 5$ and the prime factorization of 105 is $3 \times 5 \times 7$. The least common multiple of these two numbers is $2^4 \times 3 \times 5 \times 7$, or 1680, because that is the least product of prime factors of which both prime factorizations are factors. Note: all interview participants were told that the solution to this problem may not accurately reflect reality. They were advised to not let that interfere with their thinking.

Question 8 Levels of Understanding

Action Level: Participants at this level use a guess-and-check strategy to find a multiple of both given numbers. They may or may not recognize that the solution is the LCM of the two given numbers, but they do not possess a systematic method for finding it.

Process Without Coordination Level: Participants at this level anticipate the solution is the LCM of the two given numbers, and they use a more efficient method for computing the LCM. However, their method does not attend to the relationship between the LCM and the two given numbers. Participants at this level typically compute consecutive multiples of each given number and then search for common multiples across both lists. Figure 5.2.18 is an example of a study participant using this strategy. Such participants cannot explain why the LCM is necessary to solve this problem.

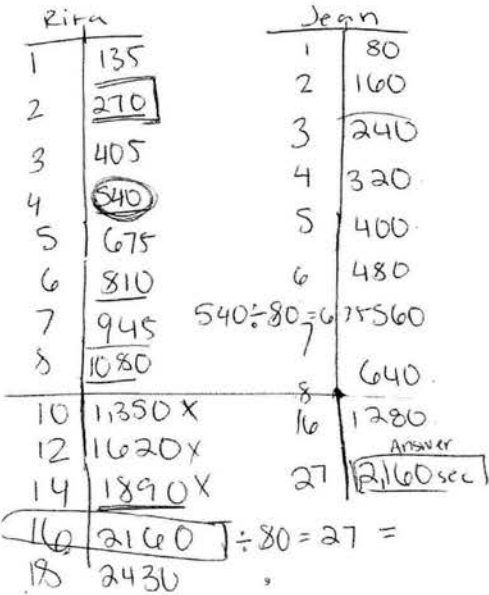
Figure 5.2.18: Process Without Coordination Level Response to Question 8



Process With Coordination Level: Participants at this level begin to use the relationship between the LCM and the two given numbers to more efficiently find a solution. For

example, they might compute the product of the two given numbers and then divide by their GCF, or they may find consecutive multiples of one given number and divide it by the other given number. As with the process without coordination level, however, they cannot yet explain why the LCM is necessary to solve this problem, nor can they explain why their methods work. Figure 5.2.19 shows an example of a participant's thinking at this level of understanding – although she begins listing consecutive multiples of both given numbers, she successfully finds the LCM by periodically dividing multiples of 135 by 80.

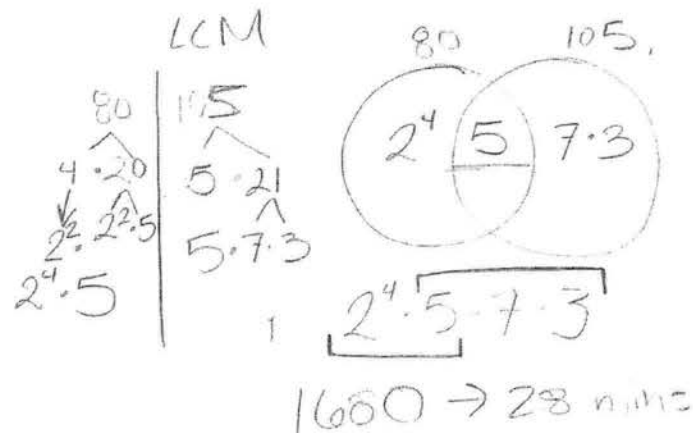
Figure 5.2.19: Process With Coordination Level Response to Question 8



Object Level: Participants at this level are easily able to find the LCM by using prime factorization and, typically, a Venn diagram, as shown in Figure 5.2.20. Additionally, they show an explicit understanding of the LCM concept by explaining why the LCM concept is necessary to solve the problem, or less frequently, by explaining why their

solution method is correct. At this level, participants do not give both explanations because understanding of both reasons requires consciously relating the LCM and prime factorization concepts.

Figure 5.2.20: Object Level Response to Question 8



Schema Level: Participants at this level can explain why the LCM is needed to solve the problem and why their method for finding the LCM must be correct. Such explicit reasoning requires an understanding of the relationship between LCM and prime factorization concepts. Participants who can provide this level of reasoning have a schema for the LCM concept that assimilates prime factorization and divisibility concepts.

The APOS code descriptions provided in this section were a result of analysis of all interview participants' responses during pre- and post-interview sessions. These descriptions were used to code participants' responses and draw conclusions about

participants' developing understanding of number theory topics. Regardless of the question or topic, it became clear that *action* and *process without coordination* levels of understanding were associated with participants' tendencies to compute decimal values and execute procedures, while *object* and *schema* levels were associated with participants viewing the concepts as static entities instead of as results of a procedure. The *process with coordination* level was a critical level of understanding during which participants began transitioning from procedural to more conceptual views of number theory topics.

Section III: Individual Interview Participant Profiles

This section will examine each of the six interview participants' conception levels as identified during their pre- and post-interviews. Each participant profile will include detailed descriptions of their understanding of divisibility, prime factorization, GCF, and LCM concepts as it develops from the pre- to post-interview. Following the six participant profiles, a summary of general patterns of participant understanding that emerged from the analysis of the interviews will be provided.

Ellen

During this study, Ellen was a sophomore undergraduate student majoring in elementary education and special education. As a result of her concentrations, this course was the first of two required mathematics education content courses. Compared to her peers in the sample, Ellen was considered a high achieving mathematics student based on her course instructor's recommendation and her performance on the Number Theory Knowledge Test (NTKT) pre-test (57% vs. sample mean of 30%). In fact, her NTKT pre-test score was the highest score in the entire sample. Her prior high school mathematics coursework included Algebra I, Algebra II, Geometry, Pre-calculus, and Statistics. Although she did not take any advanced placement examinations in mathematics, she scored in the 96th percentile nationwide on the mathematics portion of the SAT[®] examination.

Prior to Number Theory Instruction

Based on her pre-interview and NTKT pre-test, Ellen's understanding of number theory topics prior to instruction was at a higher overall level than most of the other interviewees. During her pre-interview, Ellen most often exemplified an understanding at the *process with coordination* level because she was able to internalize certain procedures, use more efficient solution strategies than guess-and-check, and make implicit connections among concepts. Her knowledge of some number theory topics, however, appeared more robust than her knowledge of other topics.

Ellen displayed limited understanding with regard to divisibility and factor

concepts because she was unable to coordinate prime factors when identifying divisibility. Instead, she only identified prime factors, prime non-factors, and composite factors that were powers of a single prime factor (e.g., 4). When asked to identify if $N=2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$ was divisible by 51, Ellen could not coordinate the fact that 3 times 17 equaled 51:

Ellen: Is N divisible by 51? Um, this one I'm not sure about right away, but I'm looking at each of the smaller numbers that multiply to get the big number. And I know that 2^3 is 8, and that's not a factor of 51. 3^2 is 9, and that's not a factor of 51. 5^2 is 25, which is not a factor of 51. And 17^3 I'm pretty sure is not a factor of 51 either...And 31 or 31^5 are not factors of 51. So, I don't think that N is divisible by 51.

In another part of the same question, Ellen was asked if N is divisible by 105. She showed increased understanding verging toward the *process with coordination* level:

Ellen: I'm thinking that it's possible that some combination of the numbers could produce 105, but I'm not sure what it is. Because, well, 3×31 would be 93, which is kind of close to 105, but it wouldn't make that factor.

Interviewer: What were you saying about the possibility of some combinations?

Ellen: ...So, because of the commutative property of multiplication, I'm not sure that if somehow different factors of the big number ended up next to each other, and you multiplied them in a different order – because of the associative property – that maybe it could make 105, but I'm not seeing how that would happen right now.

In the excerpt above, Ellen began to consider the possibility that a factor might be the product of distinct prime factors of N, but was unable to identify the necessary prime factors of 105 to make an informed decision. By Question 3, however, when asked to identify all of the factors of $N=3 \times 6 \times 7$ and $R=11 \times 13 \times 17$, Ellen was able to fully coordinate distinct prime factors:

Ellen: For actually finding the factors, I first started with N, which was $3 \times 6 \times 7$, and I broke up the 6 into 3×2 , and then I used each of the individual factors, and then used a combination of two at a time, and then I used a combination of three at a

time. And then it always has a factor of itself.

While Ellen's level of understanding of divisibility reached the *process with coordination* level, there was no evidence that she was able to coordinate prime factors systematically across situations. In fact, on the NTKT pre-test, she was unable to coordinate all prime factors when presented with $5^2 \times 7$ (she omitted 5×7). And, on questions asking Ellen to comment on the validity of divisibility rules (e.g., a number is divisible by 18 if it is divisible by 3 and by 6) or to find a number that has exactly four factors, Ellen resorted to guess-and-check methods instead of prime factorization.

With regards to greatest common factor (GCF) and least common multiple (LCM) concepts, Ellen displayed a higher level of understanding than her peers during the pre-interview and on the NTKT pre-test. Her understanding was coded at the *process with coordination* level across all relevant questions, and was characterized by knowledge of more efficient solution strategies, attention to prime factorization, and coordination between the GCF and LCM.

For example, on Question 8 of the pre-interview, Ellen initially attempted to find the LCM of 135 and 80 by listing consecutive multiples of each number. Partway through her work, she recognized the tediousness of her initial strategy and attempted an alternate method:

Ellen: I think I'm going to leave my chart thing as it is for now and try to see if I can do it the other way faster, because it's getting rather tedious. So I'm gonna multiply 135×80 – which I probably should've done in the first place – and get 10,800. So then I'm gonna try to figure out the prime factorization of, first 135...and then the prime factorization of 80...I'm noticing that they both have a factor – a prime factor – of 5, so I'm gonna try to see that, if I divide this number – 10,800 – by 5, what I get. And I got 2,160. And, since I was really close to getting to that number with this annoying chart thing, I think I'm just gonna see if it checks

out...

Ellen found the LCM more efficiently by attending to prime factorization and using the inherent relationship between 80, 135, and their LCM. Part of her work is displayed in Figure 5.3.1 below. It was not clear if Ellen understood why her method worked, since she reverted back to listing consecutive multiples of each number to check her work.

Figure 5.3.1: Ellen's Solution Strategy for Question 8 of the Pre-Interview

The image shows handwritten mathematical work. On the left, the number 135 is written above a diagonal line that splits into 27 and 5. On the right, the number 80 is written above a diagonal line that splits into 10 and 8. Below 10 is a diagonal line splitting into 5 and 2. Below 8 is a diagonal line splitting into 4 and 2. Below 4 is a diagonal line splitting into 2 and 2. Below these factorizations, the calculation $135 \times 80 = 10,800$ is written. Below this, a horizontal line is drawn, and the number 2,160 is written above it. Below the horizontal line, the number 5 is written to the left of a vertical line, and 10,800 is written to the right of the vertical line, representing the division $5 \overline{) 10,800}$.

Ellen similarly attended to prime factorizations to find the GCF of 56 and 96 on Question 7 of the pre-interview. In this example, she used factor trees to find the prime factorization of each number and then identified $2 \times 2 \times 2$, or 8, as their greatest common factor. Her use of prime factorization to find the GCF was evidence that she internalized the procedure for calculating the GCF of two numbers – she no longer needed to list all of the factors of 56 and 96 separately.

Following Number Theory Instruction

Ellen's work during the post-interview and NTKT post-test showed evidence of a

structured mental network of interconnected concepts. She was able to make explicit connections among prime factorization, divisibility, factor, GCF, and LCM concepts. In fact, her responses to multiple questions centered around the systematic use of prime factorization and its relationship to other concepts. Table 5.3.1 below shows Ellen’s levels of understanding as displayed during her interviews before and after number theory instruction. It illustrates the increasing depth and connected nature of her developing understanding.

Table 5.3.1: Ellen’s Coded Levels of Understanding by Question

Question	Pre-interview Codes	Post-interview Codes
1	Process without coordination*	Schema
2	Action	Schema
3	Process with coordination	Object
4	Action	Process with coordination
5	Process with coordination	Process with coordination
6	Process with coordination	Schema
7	Process with coordination	Schema
8	Process with coordination	Object

*Evidence of movement toward process with coordination level

Ellen’s understanding of divisibility improved significantly as her responses following instruction were classified predominately at the *object* and *schema* levels. Prior to instruction, she was only beginning to coordinate prime factors; following instruction she was able to fully coordinate prime factors and explicitly relate prime factorization to divisibility. For example, in Question 1 of the post-interview, Ellen generalized her strategy of using prime factorization to check divisibility:

Ellen: If the number can be broken down again into prime factors, I would do that first and then look to see if all of the prime factors needed for the number it’s asking for were listed in the number N, because if the prime factors of – all of the prime factors of the number they’re looking for are inside N, then it must be divisible by that number.

Ellen's attention to the link between prime factorization and divisibility was prevalent throughout the NTKT post-test as well. When asked to comment on the validity of a proposed divisibility rule (i.e., any number is divisible by 18 if it is divisible by 3 and by 6), she found the prime factorizations of 18, 6, and 3 and wrote the following:

Since 6 contains a prime factor of 3, checking for 3 does not guarantee both 3s that are needed to make up 18. If prime factors are repeated, you must be sure repeats are accounted for in the divisibility rule. A correct divisibility rule could be a number is divisible by 18 if it is divisible by 9 and by 2. This way, all prime factors of 18 are accounted for.

Ellen's reasoning directly addressed the role of prime factorization in determining divisibility – A is divisible by B if and only if all of B's prime factors are accounted for in A's prime factorization. Instead of finding a counter-example (as she had done on the pre-test), Ellen attended to the underlying relationship between divisibility and prime factors.

Ellen's understanding of the GCF and LCM concepts also improved from *process* levels to *object* and *schema* levels. Such improvement was characterized by a shift toward more conceptually-based reasoning, attention to prime factorization, and connection-making. For example, Question 6 on both the post-interview and NTKT post-test required Ellen to find all possible values for A and B when given the prime factorizations of their GCF ($3^2 \times 7^2$) and their LCM ($3^4 \times 7^3$). Ellen used the relationship between the GCF and LCM to explain why certain solutions were valid and why others were not valid. Her thinking was made clear during the post-interview:

Ellen: So, right away I know that I can't put one 3 on one side [of the Venn diagram], and one 3 on the other side because then their greatest common

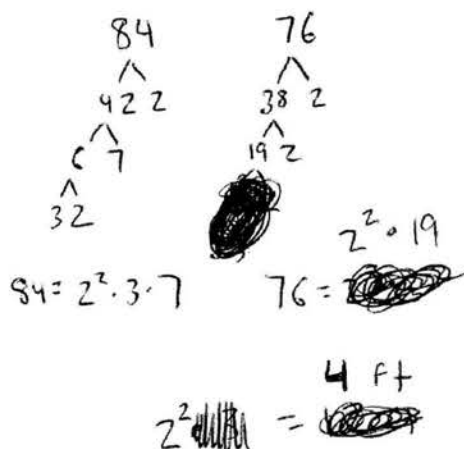
Interviewer: factor would include another 3. So the 3^2 has stay together...
 Okay. I just want to make sure I have everything down, Ellen. Can you please explain your reasoning briefly again, how you went about solving this?

Ellen: ...So I decided that either one side, A or B, would have to contain all of the extra factors and then the other one would contain none, or one type would contain 3^2 and the other would contain 7. And then to get the actual numbers themselves, I just multiplied the extra factors by the greatest common factor for each side, and then I came up with two pairs.

By explaining why 3^2 must stay together, Ellen displayed an understanding of the relationship between GCF and prime factorization. Her use of “extra factors” is evidence that she understood that the product of the GCF and the remaining extra factors of both numbers A and B must be the LCM.

Similarly, her ability to compute the GCF and the LCM of two numbers also revealed a more connected level of understanding. On Question 7 of the post-interview, Ellen was given a problem that required finding the GCF of 84 and 76. Figure 5.3.2 illustrates her work on this question.

Figure 5.3.2: Ellen’s Solution Strategy for Question 7 of the Post-Interview



Unlike her work during the pre-interview, Ellen was now able to provide sound reasoning for why the GCF calculation was correct by connecting the GCF concept to prime factorization:

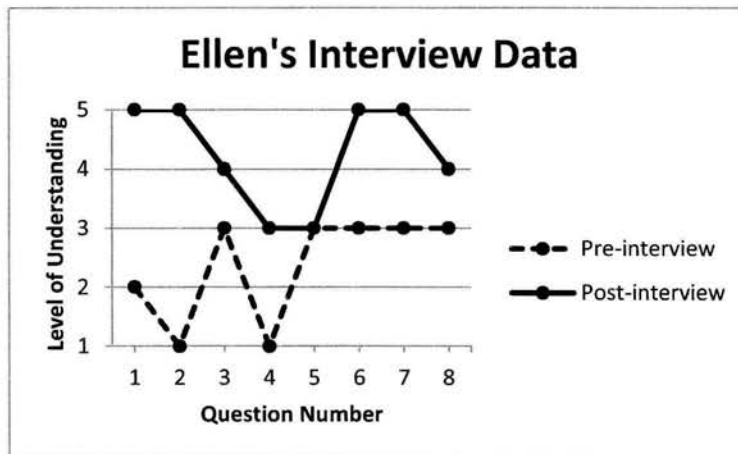
Ellen: ...as long as my factorizations are correct, it [4] has to be the only answer if I include all of the factors... 19 is extra from the 76, but I can't use it because it wouldn't fit into 84. And then the same thing goes for the 3×7 within 84, because there's no 3×7 in 76, so I can't use it.

Once again, Ellen used the term "extra factors," this time to explain why the GCF of 84 and 76 must be 4. The extra factors refer to the prime factors of 84 and 76 that are not in common. As such, it seems that Ellen was able to see the prime factors as two types: common and non-common. This compartmentalization of prime factors is evidence of Ellen's ability to view the GCF as a single object, operate on it, and connect it to prime factorization concepts. Such connection-making is the hallmark of the *schema* level of understanding.

Reflections on Ellen's Developing Understanding

Ellen's understanding of number theory topics became more conceptual and interconnected, and less procedural and superficial following relevant instruction. Figure 5.3.3 below shows a summary of Ellen's levels of understanding as displayed during the pre- and post-interviews. While her pre-interview work showed a tendency to work with procedures (*action* and *process* levels), her post-interview work revealed a greater ability to make connections across topics (*schema* level) and view concepts flexibly as objects that could be operated on (*object* level).

Figure 5.3.3: Ellen’s Displayed Levels of Understanding



Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Her work on the NTKT post-test also improved, increasing from a score of 57% to 92%. Table 5.3.2 shows the percentage of possible points earned on questions requiring concrete skills (e.g., computation, executing known procedures) and abstract skills (e.g., attention to underlying concepts and relationships, executing unfamiliar solution strategies):

Table 5.3.2: Ellen’s Percentage of Possible Points Earned by NTKT Question Type

	<i>Abstract Skills</i>	<i>Concrete Skills</i>
<i>Pre-Test</i>	53.4%	61.9%
<i>Post-Test</i>	86.2%	100.0%

Following instruction, Ellen’s performance on all question types improved. Specifically, the data concerning abstract skills provides further evidence that her conceptual understanding of number theory topics improved. She was better able to make stronger connections across topics, articulate sound reasoning, and solve complex problems.

Margaret

At the time of this study, Margaret was a sophomore undergraduate student majoring in elementary education and special education. Her prior high school mathematics coursework was typical of many of her peers in the course, and included Algebra I, Algebra II, Geometry, and Pre-calculus. While she did not take any advanced placement examinations in mathematics, she did take both the SAT[®] and ACT[®] examinations, scoring, on average, in the 73rd percentile nationwide. This course was the first of two required mathematics education content courses for Margaret at the university. Based on her course instructor's recommendation and on her initial score on the Number Theory Knowledge Test (NTKT) pre-test (42% vs. sample mean of 30%), Margaret was considered a high achieving mathematics student relative to her peers in the sample.

Prior to Number Theory Instruction

Based on her work on the clinical pre-interview and NTKT pre-test, Margaret's understanding of elementary number theory topics was coded primarily at the *process without coordination* level. Such understanding is characterized by an internalization of specific actions, such as long division or finding the greatest common factor of two numbers. By internalizing certain computational procedures, Margaret could construct mental images of those procedures to help her either execute procedures mentally or develop more sophisticated solution strategies. However, Margaret's mental images of these procedures were limited and only displayed in non-complex situations. Problems

that required a more abstract understanding of underlying concepts, such as coordinating distinct prime factors, caused Margaret a great deal of difficulty. In such situations, she reverted to long or unnecessary computations in order to find solutions.

Throughout the pre-interview, Margaret's success solving problems involving factors and divisibility was limited by her inability to successfully coordinate prime factors. Coordination of prime factors refers to an ability to recognize composite numbers as the product of distinct prime factors. When asked to determine which number, $N=3 \times 6 \times 7$ or $R=11 \times 13 \times 17$, has more factors, Margaret's response was:

Margaret: So, N has the factors 1, which appears in 3, 6, and 7. It has 2, which is from 2×3 for 6. It has 3, which is from 1×3 and from 2×3 to get 6. And it has 6, from 1×6 , and it has 7 from 1×7 . So, N has one, two, three, four, five factors. And R has 1, which appears in all three of the numbers...and then it has 11, from 1×11 . It has 13 from 1×13 . And, it has 17 from 1×17 . So, after looking at all the factors that N and R have, I would say that N has more factors than R because N has...five factors while R has only four.

In this short excerpt, Margaret was only able to identify a factor of N or R if it was clearly visible in the factorizations or if it was a factor of clearly visible numbers (e.g., 6) in the factorizations. Her ability to identify 6 as a factor of $N = 3 \times 6 \times 7$ without resorting to long division is evidence that she had internalized the long division procedure and no longer needed to execute it to determine divisibility. However, she did not internalize the long division procedure for identifying divisibility by other composite factors, such as 21 (21 is a factor of N since 3×7 is part of N's factorization).

Throughout the pre-interview and the NTKT pre-test, Margaret showed a lack of understanding that combinations of prime factors are, themselves, factors.

Based on the clinical pre-interview and NTKT pre-test, Margaret's understanding

of concepts related to the greatest common factor (GCF) and least common multiple (LCM) was coded primarily at the *process without coordination* level. Although she did not need to rely on inefficient guess-and-check strategies, her methods did not attend to relationships between the GCF, LCM, and prime factorization. On the pre-interview, for example, she computed the GCF of two numbers by listing out each number's factor pairs and circling the greatest number that appeared on both lists. When required to find the LCM of two numbers, Margaret made two long lists of each number's multiples until she found a match. She repeatedly tried to think of a more efficient method, but could not:

Margaret: I'm trying to think if there's an easier way to – a quicker way to find this...I can't think of one at the moment, so I guess I'll just keep going like I'm doing it. Um, okay, so I'm just gonna find some more multiples for Jean.

After a few minutes of finding more multiples of Jean's time:

Interviewer: What are you thinking about right now?

Margaret: I'm, again, just trying to think if I can do it faster, but I can't think of a way.

Margaret's attachment to procedures was also evident on the NTKT pre-test, where her articulation of the meaning of the LCM was more a recitation of how to compute the LCM. With regards to GCF and LCM concepts, it was clear that Margaret had difficulty recognizing efficient, conceptually-based methods. Her understanding tended to consist of numerical procedures.

Following Number Theory Instruction

Following the three-week unit on number theory, Margaret's understanding of

elementary number theory topics changed, albeit inconsistently. As evidenced by her work on the clinical post-interview and NTKT post-test, she showed an increasing, but tenuous ability to coordinate prime factors so as to identify a number’s divisibility. Margaret’s understanding of GCF and LCM concepts also improved, as she was able to find more efficient solution methods and explain her reasoning more accurately. Table 5.3.3 shows codes for Margaret’s levels of understanding on each question on the pre- and post-interviews:

Table 5.3.3: Margaret’s Coded Levels of Understanding by Question

Question	Pre-interview Codes	Post-interview Codes
1	Process without coordination*	Object
2	Process without coordination	Action
3	Process without coordination	Process without coordination
4	Action	Process with coordination
5	Process without coordination	Object
6	Action	Action
7	Process without coordination	Process with coordination
8	Process without coordination	Object

*Evidence of movement toward the next higher level of understanding

Despite showing improved understanding, the table illustrates that Margaret’s understanding was highly inconsistent during the post-interview. On the one hand, she was able to identify divisibility by composite factors when presented with a number in prime factored form, $N = 3^2 \times 5^4 \times 11 \times 17^3$, as in Question 1 of the post-interview:

Margaret: Um, yes, I think that N is divisible by 75, because N has 3^1 which is 3, and it also has 5^2 which is 25. So, if you multiply 3×25 that’s 75. So, I think N is divisible by 75. And is N divisible by 51? Yes, I think that N is divisible by 51 because N has 3^1 which is 3 and then it also has 17^1 which is 17, so if we multiply 3×17 that’s 51. So, I think that N is divisible by 51.”

For this question, her understanding of divisibility and factors was coded at the *object*

level because she was beginning to abstract the notion of factor as a single entity – a combination of the prime factors of a given number. On the other hand, when asked to identify the factors of $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$ in Question 3 of the post-interview, Margaret could not identify any composite factors that were not already visible in the factorizations. Her work is shown in Figure 5.3.4, followed by her verbal explanation:

Figure 5.3.4: Margaret’s Solution Strategy for Question 3 of the Post-Interview

$$\begin{array}{l}
 N = 3 \times 4 \times 5 \\
 3^0, 3^1 \quad 4^0, 2, 4^1 \quad 5^0, 5^1 \\
 \\
 R = 13 \times 17 \times 19 \\
 13^0, 13^1 \quad 17^0, 17^1 \quad 19^0, 19^1
 \end{array}$$

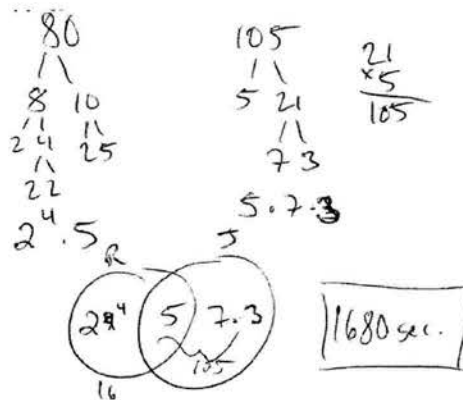
Margaret: Um, so, yes, so R has six factors while N has seven, so I think that N has more factors than R. And, um, I’ll find all the factors of N and R. So, for N, all the factors are... $3^0, 3^1$, and then for 4 it’s $4^0, 2$, and 4^1 and then for 5 it’s 5^0 and then 5^1 . And then R is, um, $13^0, 13^1, 17^0, 17^1$, and 19^0 and 19^1 . Yeah, so that’s it.

By reverting back to similar thinking from her pre-interview (also coded at the *process without coordination* level), Margaret showed evidence that her understanding of factors and divisibility, though improving, remained fragile.

Margaret’s understanding of GCF and LCM concepts improved, except when she was required to construct relationships between the GCF and the LCM of two numbers. Her understanding of the GCF concept was coded at the *process with coordination* level because she was able to find a more efficient strategy (i.e., prime factorization) that

implicitly coordinated the relationship between the GCF and prime factorization. Her understanding of the LCM concept was coded at the *object* level because Margaret began to show an explicit understanding of her method, shown in Figure 5.3.5, for finding the LCM:

Figure 5.3.5: Margaret's Solution Strategy for Question 8 of the Post-Interview



Interviewer: Can you just explain your thinking about why you multiply across the Venn diagram to get the LCM?

Margaret: Well, you multiply across because you're trying to find the smallest number that is – or the smallest multiple that's common to both 80 and 105. So, um, in this Venn diagram, you have the left section shows what's common only to Rita's prime factorization, and then the right section shows what's common to only Jean's, and then the middle there's a 5 because both of their prime factorizations have a 5. So you multiply across because that ensures that you're not repeating a 5...you use the prime factors that are in both the prime factorizations for Rita and Jean and you don't repeat any numbers, yeah.

In this excerpt, Margaret provided correct reasoning for why multiplying across the Venn diagram results in the LCM of 80 and 105. She no longer needed to list out multiples; instead, she operated on her mental image of a Venn diagram in order to develop meaning. This understanding represents a substantial improvement over her work prior

to instruction – she used more efficient methods, relied less on computation, and provided reasoning for her methods.

Margaret’s deeper understanding of GCF and LCM concepts did not translate to complex, non-algorithmic questions. Question 6, for example, asked Margaret to find all possible pairs of values (A and B) that had a GCF of $3^2 \times 7^2$ and an LCM of $3^4 \times 7^3$. In order to solve this problem, which was presented on both the NTKT post-test and post-interview, participants had to attend to the relationship between the GCF and the LCM (i.e., $A \times B = \text{GCF} \times \text{LCM}$). Margaret’s understanding was coded at the *action* level, however, because she was unable to execute a solution strategy aside from computing the values of the prime factorizations of the GCF and LCM. In fact, her comfort with computation proved to be an obstacle for her in solving this problem:

Margaret: I feel like if this was, for some reason like was not in factored form, I would have an easier time with doing it. It’s just whenever I see sometimes that kind of factored form, it throws me.

Interviewer: Can you explain why you think that throws you?

Margaret: Just because it’s – like to me it seems kind of abstract. I know if you actually do it, if I were to do it $3 \times 3 \times 7 \times 7 \dots$ if I solve that, then that would be a number, and then the same thing for $3^4 \times 7^3$, but this for some reason, whenever I see it, I just kind of think confusion...

Margaret’s confusion centers on her discomfort with numbers represented in prime factored form. Since prime factored form did not trouble Margaret so much on other post-interview and post-test questions, it appears that the complex nature of the problem – viable solution methods must attend to the relationship between GCF and LCM – coupled with her fragile understanding of prime factorization may have led to her inability to make progress on this question.

Reflections on Margaret's Developing Understanding

During instruction, Margaret's understanding of number theory topics developed inconsistently. Her understanding of divisibility and the relationship between factors and prime factors reached greater levels of abstraction for certain problems, but then regressed back to concrete and superficial thinking for others. Her understanding of GCF and LCM concepts showed consistent improvement, but only when the problems were relatively straight-forward. When a problem had no obvious solution strategy, Margaret reverted back to lower levels of understanding highlighted during the pre-interview. Below, Figure 5.3.6 illustrates the inconsistent nature of Margaret's levels of understanding on questions related to divisibility and factors (Questions 1-5). Overall, Margaret's levels of understanding improved from *process without coordination* to *process with coordination*, as a result of her increasing (yet still fragile) ability to identify factors as products of combinations of prime factors.

Figure 5.3.6: Margaret's Displayed Levels of Understanding



Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Her results on the NTKT pre-test and post-test support this improved understanding, but it is interesting to note that while her pre-test score was above the sample mean (42% versus 30%), her post-test score was below than the sample mean (62% versus 73.6%). Table 5.3.4 shows the percentage of possible earned that Margaret earned on questions requiring concrete skills (e.g., computation, executing known procedures) and abstract skills (e.g., attention to underlying concepts and relationships, executing unfamiliar solution strategies):

Table 5.3.4: Margaret’s Percentage of Possible Points Earned by NTKT Question Type

	<i>Abstract Skills</i>	<i>Concrete Skills</i>
<i>Pre-Test</i>	44.8%	38.1%
<i>Post-Test</i>	51.7%	76.2%

Following instruction, Margaret showed increasing success on all question types. However, her improvement on questions requiring abstract skills was not very big, as evidenced by the fact that the entire sample’s mean scores on questions requiring abstract skills improved by a much wider margin (24.6% on the pre-test to 68.3% on the post-test). Her much larger improvement on concrete questions provides some evidence that her understanding remained relatively superficial and procedure-focused.

Amanda

Another interview participant, Amanda, was a sophomore undergraduate student majoring in elementary education and special education. As a result of her concentrations, this course was the first of two required mathematics education content courses. For the purposes of this study, Amanda was considered an average achieving mathematics student based on her course instructor's recommendation and the Number Theory Knowledge Test (NTKT) pre-test (26% vs. sample mean of 30%). Her prior high school mathematics coursework included Algebra I, Algebra II, Geometry, Pre-calculus, and Calculus. Amanda took the Calculus AB advanced placement examination and received a score of 3 out of 5 points. She also took both the SAT[®] and ACT[®] examinations, and scored on average in the 91st percentile nationwide on the mathematics portion of both examinations.

Prior to Number Theory Instruction

Prior to instruction, Amanda's understanding of number theory topics was difficult to pinpoint because of inconsistent responses to topics and individual questions. She appeared to reach the *process with coordination* level for several topics. Her ability to coordinate prime factors, for example, varied greatly, ranging from a complete lack of coordination to a coordination of three prime factors at a time. Likewise, her knowledge of greatest common factor (GCF) and least common multiple (LCM) concepts reflected, at times, dependence on computation, and at other times, an understanding of underlying meaning.

The inconsistent nature of Amanda's understanding of divisibility was evident throughout the pre-interview and NTKT pre-test. On Question 1 of both assessments, Amanda was asked to determine if $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$ was divisible by 105. On the pre-test, Amanda wrote that she did not know if N was divisible by 105, but during the pre-interview she coordinated three distinct prime factors at a time to make an accurate determination:

Amanda: ...you can break 105 down into its prime factorization, which is $7 \times 3 \times 5$. And, although prime factorization of N includes a 3 and a 5, it does not include a 7, which means that N would not be divisible by 105.

Several reasons may help explain this inconsistency. For one, Amanda deferred answering Question 1 on the pre-interview until the end of the interview. This move allowed her to work on the other interview questions first, which may have helped focus her thinking. Second, all interview participants completed the pre-test before the pre-interview, so Amanda had seen Question 1 before. Her ability to indicate what she did know about divisibility may have been enhanced through the opportunity to reflect on this question (Campbell, 2001). And third, unlike her work on the pre-test, working on the interview questions afforded Amanda the opportunity to verbalize her thinking. Prior research describes the potential benefit to mathematical understanding of articulating one's thinking (Hiebert & Carpenter, 1992; Hiebert et al., 1997; Donovan & Bransford, 2005).

Likewise, she displayed inconsistent coordination on Question 2b of the pre-test and Question 3 of the pre-interview. On the former, Amanda showed no ability to coordinate prime factors when asked to find the number of factors of $5^2 \times 7$. She

computed its value (175), identified 1, 5, 25, and 7 as its only factors, and completely ignored combinations of its distinct prime factors (5×7 and $5^2 \times 7$). On the pre-interview, however, when asked to find the factors of $N = 3 \times 6 \times 7$ and $R = 11 \times 13 \times 17$ she could coordinate two prime factors at a time:

Amanda: I'm trying to think if there are other numbers that I can multiply together to get, um, larger numbers than just the prime factorization – to get more factors. For example, 18 is also a factor of N because 3×6 is 18.

Once again, it is possible that the pre-interview afforded Amanda an opportunity to make sense of divisibility by repeatedly articulating her thinking in a way that the pre-test did not.

There was also evidence of varied understanding with regards to GCF and LCM concepts prior to instruction. For example, Question 7 of the pre-interview and Question 7a of the pre-test presented Amanda with the following real-life situation:

Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 96 feet by 56 feet. What are the dimensions of the largest square tile that the school can buy to completely tile the cafeteria floor?

On the pre-test, Amanda was unable to determine which concept was needed to solve the problem (GCF, LCM, or neither). On the pre-interview, however, she displayed an *object* level of understanding by recognizing the need to find the GCF of 96 and 56, using an efficient solution strategy based on prime factorizations, and providing a correct explanation for why the GCF was needed. Figure 5.3.7 displays her written work, and is followed by her explanation for why she needed to find the GCF:

Figure 5.3.7: Amanda's Solution Strategy for Question 7 of the Pre-Interview

$$\begin{array}{r} 56 \\ \hline 8 \cdot 7 \\ 2^3 \cdot 7 \end{array} \qquad \begin{array}{r} 96 \\ \hline 12 \cdot 8 \\ (2^2 \cdot 3) \cdot 2^3 \\ 2^5 \cdot 3 \end{array}$$

2^3 8 feet

Amanda: Um, because it asked for the largest square tile that the school could buy. So you're looking for multiple – you're gonna be buying multiple tiles, but you want them to be the largest sizes for it to fit evenly. So, if you find the greatest common factor, then you know that it'll still evenly go into 56 and 96 and be able to cover the cafeteria floor.

Amanda's understanding of the LCM concept was coded at the *process with coordination* level because she was able to coordinate an LCM relationship. For example, when she had to compute the LCM of 135 and 80, she found consecutive multiples of 135 and checked to see which was divisible by 80:

Amanda: So, what I'm gonna try to do is multiply the larger number, which is 135, by 2, and see if 80... will divide evenly into it. And then, I'll probably just keep multiplying 135 by higher numbers until 80 divides into it evenly.

Amanda appeared to understand that 135 and 80 must be factors of their LCM, and so it was not necessary for her to find multiples of both numbers. Having internalized this concept well, Amanda sought to find an even more efficient method:

Amanda: Okay, so I'm just adding 135 to my other answers, and I'm not really finding a common multiple, so I guess I will just do 135×80 which equals 10,800. However, to find the least common multiple between those two numbers, I think maybe doing the greatest common factor would help between the numbers, so I could possibly cross out some numbers that are the same between them.

Although Amanda did not successfully execute this new strategy, her attempt is evidence of some conceptual understanding of the LCM concept; she relied less on executing step-by-step procedures and more on the intrinsic relationship between the LCM and the two given numbers.

Following Number Theory Instruction

Following the three-week unit of instruction, Amanda’s understanding of number theory developed to *object* and *schema* levels across multiple topics. While her knowledge prior to instruction was mainly characterized by an erratic ability to internalize procedures, her knowledge now became more stable and focused on identifying relationships between various number theory topics. Table 5.3.5 shown below summarizes Amanda’s coded levels of understanding during her pre- and post-interviews.

Table 5.3.5: Amanda’s Coded Levels of Understanding by Question

Question	Pre-interview Codes	Post-interview Codes
1	Process with coordination	Schema
2	Action	Schema
3	Process with coordination	Object
4	Action	Process without coordination
5	Process with coordination	Object
6	Action	Schema
7	Object	Schema
8	Process with coordination	Object*

*Evidence of movement toward the next higher level of understanding

Her understanding of divisibility and factor concepts became more robust because she was now able to explicitly identify the critical relationship between divisibility and prime factorization. For example, on Question 1 of the post-interview, Amanda was not

only able to skillfully coordinate prime factors in order to identify the divisors of $N = 3^2 \times 5^4 \times 11 \times 17^3$, but she was also able to generalize the relationship between prime factors and divisibility:

Amanda: According to the fundamental theorem of arithmetic, it says that each number is composed of a unique combination of prime factors. I know that any of the numbers, if I can find their prime factorizations within N then N would be divisible by the numbers.

The excerpt above is evidence that Amanda's understanding of divisibility developed to the *schema* level – she connected her knowledge of prime factor coordination to divisibility, allowing her to generalize divisibility across all numbers.

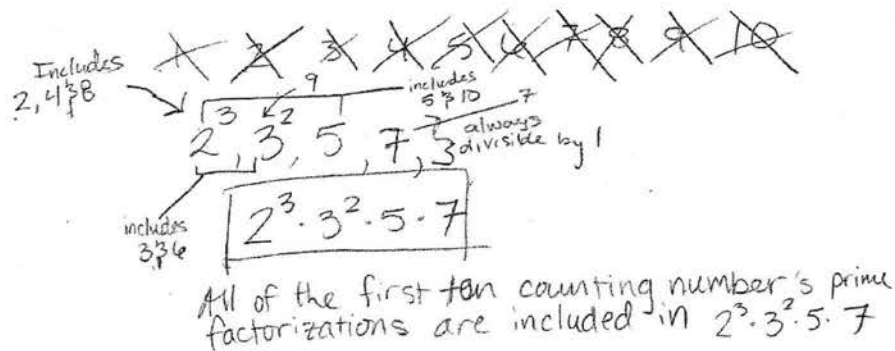
A similar example of Amanda's increasingly connected understanding of divisibility was visible in her work on Question 2 of the post-interview. In this example, Amanda was asked to find the greatest number that N must be divisible by if N is divisible by 18 and 15. She was able to answer the question correctly using the prime factorizations of 18 and 15, and she was also able to properly relate the prime factors of 18 and 15 to the answer:

Amanda: So, if N is divisible by 18 and 15, that means I would have to be able to find the prime factorization of 18 and 15 separately in my number that N must be divisible by. So, you know that you have $3 \times 3 \times 2$ which equals 18 in that number, and then you also have 3×5 which is 15. So because you see that N is divisible by 18 and 15, because it includes all the prime factors that would compose that number, you know that the greatest number that N must be divisible by is 60, because, um, $3 \times 2 \times 3 \times 5$ is 60, no 90.

In this example, her recognition that divisibility by 18 and 15 had a direct implication on the prime factorization of the solution was the salient feature of her schema-based understanding.

On the NTKT post-test, Amanda also showed evidence of a high level of understanding of divisibility concepts. Figure 5.3.8 illustrates her work on Question 5, for which she found the smallest positive integer that is divisible by the first ten counting numbers by constructing its prime factorization from scratch. By recognizing the divisibility relationships among the counting numbers (e.g., if the solution must be divisible by 8, then it must also be divisible by 2 and 4), Amanda was able to recognize the least number of prime factors necessary to construct all ten counting numbers. The power of her understanding is evident in the fact that she did not need to check that the ten counting numbers were factors of her answer – by attending to their prime factorizations Amanda was assured of their presence. This was evidence of her *object* level understanding of factors as products of distinct prime factor combinations.

Figure 5.3.8: Amanda’s Solution Strategy for Question 5 of the NTKT Post-Test



Amanda’s knowledge of GCF and LCM concepts also improved to *object* and *schema* levels. Although prior to instruction Amanda was able to explain why the GCF was needed to solve the real-life scenario presented in Question 7 of the pre-interview about the dimensions of square tiles needed to tile a cafeteria floor, she was unable to

explain why her method of finding the GCF was correct. During the post-interview, however, she provided this missing explanation:

Amanda: ...because 84 and 76 both have 4, or both have 2×2 in their prime factorization, so that means if I use a square tile that was 4 feet by 4 feet, you could tile the entire floor. And that's the largest one I could use because there are no other prime factors that both 84 and 76 have in common...

Interviewer: Can you repeat that last part you said? Do you give an explanation for why your answer is the largest square tile – can you repeat that last part, please?

Amanda: ...you know that the largest square tile that you're looking for will have to be composed of numbers or prime numbers that 76 and 84 both have in common. So once you break down their prime factorizations, you see that 2^2 is in common between both of those numbers and there's no way you could make a bigger number because the prime numbers between 84 and 76 that are left aren't in common, so the number that is common between them is 4.

Amanda's explanation above was evidence that she understood the relationship between the GCF and prime factorization – the GCF of two numbers represents a coordination of all of the prime factors that are common to both numbers. As such, her knowledge of the GCF was coded at the *schema* level.

Though slightly weaker than her understanding of the GCF, Amanda's understanding of the LCM concept was coded at the *object* level with a noticeable transition to the *schema* level. On Question 8 of the post-interview, Amanda was solving the following real-life scenario:

Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 80 seconds. Jean runs one complete lap in 105 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

Although she correctly explained why her solution strategy resulted in the LCM

(prompting the *object* code), she used a superficial reason to explain why the LCM was necessary in the first place.

Amanda: So, you know you'll be looking for the least common multiple in this problem because you'll be creating a number that's bigger than both 80 and 105...

Instead of providing reasoning based on the underlying meaning of the LCM, she used the common association between *multiple* and *bigger* (many also associated *factor* with *smaller*) to explain why the scenario warranted finding the LCM. Her transition to the *schema* level, however, was evident in her explanation of the validity of her solution strategy:

Amanda: I can still compose 80 from my new number and I can still compose 105 as well, and I know it's the least number, because, all the numbers that are in common, which was just 5, are only included once, but the prime factorizations of 80 and 105 are still – they still exist there as well.

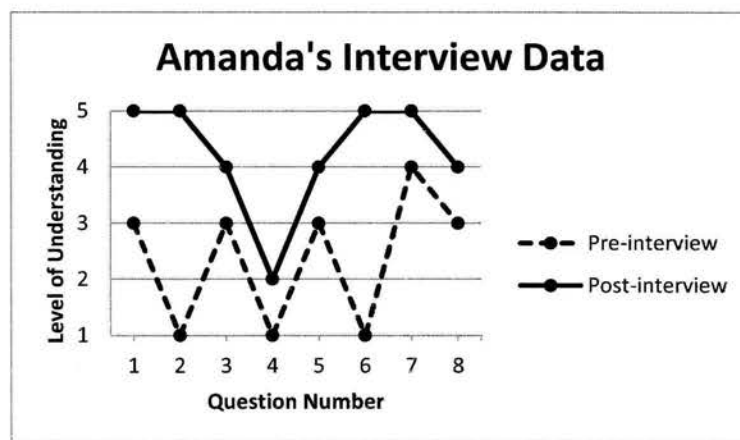
In this excerpt, Amanda explained precisely why the LCM procedure worked. Her explanation attended to prime factorization and hinted at the relationship between the LCM and the GCF of 80 and 105 (their common factors need only be included once in the LCM).

Reflections on Amanda's Developing Understanding

Below, Figure 5.3.9 shows a summary of Amanda's levels of understanding as displayed during the pre- and post-interviews. Prior to instruction, Amanda showed evidence that her understanding of number theory topics was limited and inconsistent. She was able to coordinate prime factors relatively well (*process with coordination* level), but in certain circumstances required executing tedious computation in order to

identify divisibility (*action* level). Her post-interview work, however, revealed a consistent view of number theory as a connected body of concepts (*schema* level). Her explanations began to reflect a greater awareness of the close relationships between divisibility, prime factorization, GCF, and LCM concepts.

Figure 5.3.9: Amanda's Displayed Levels of Understanding



Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Amanda's work on the NTKT post-test also improved, increasing from a score of 26% to 76%. This improvement appeared to be due to Amanda's more structured, schema-based, understanding of the multiplicative structure of number. In evaluating her written work on the post-test, it became clear that her solution strategies shifted from computation to a steadfast focus on prime factorization. Even on questions that could have easily been solved using long division, Amanda chose to attend to prime factorization.

Table 5.3.6, shown below, presents the percentage of possible points earned on questions requiring concrete skills (e.g., computation, executing known procedures) and

abstract skills (e.g., attention to underlying concepts and relationships, executing unfamiliar solution strategies):

Table 5.3.6: Amanda's Percentage of Possible Points Earned by NTKT Question Type

	<i>Abstract Skills</i>	<i>Concrete Skills</i>
<i>Pre-Test</i>	34.5%	14.3%
<i>Post-Test</i>	72.4%	81.0%

The data show that Amanda's ability to solve problems requiring both conceptual-based and procedural-based knowledge improved dramatically on the post-test. Her increased focus on prime factorization may have contributed to her improvement because most, if not all, of the post-test questions centered on prime factorization. As such, Amanda appeared to have developed a structured schema-based understanding centered around prime factorization. With her knowledge heavily structured, she could have more effectively organized her existing knowledge for the purpose of solving new problems (Skemp, 1987).

Zoe

Zoe was a sophomore undergraduate student majoring in early childhood education. As a result of her concentration, this course was her only required mathematics education content course at the university. For the purposes of this study, Zoe was considered an average achieving mathematics student based on her course instructor's recommendation and her performance on the Number Theory Knowledge Test (NTKT) pre-test (24% vs. sample mean of 30%). Her prior high school mathematics coursework included Algebra I, Algebra II, Geometry, and Pre-calculus; the curriculum used by her high school was the *SIMMS Integrated Mathematics* program, a four-year high school curriculum that uses real-world contexts and a technology-based modeling approach to teach mathematics. Although she did not take any advanced placement examinations in mathematics, she scored in the 88th percentile nationwide on the mathematics portion of the SAT[®] examination.

Prior to Number Theory Instruction

Based on pre-interview and NTKT pre-test data, Zoe's understanding prior to instruction was at the *process without coordination* level for divisibility and prime factorization concepts. Her displayed understanding was at the *action* level for the greatest common factor (GCF) concept, and at the *process without coordination* level for the least common multiple (LCM) concept. Overall, her understanding of these concepts was characterized by a limited internalization of the divisibility procedure and a procedural understanding of the GCF and LCM concepts.

During the pre-interview, Zoe appeared to have internalized an image of the divisibility procedure. When given a number in prime factored form, she was able to identify its factors without executing the long division procedure. For example, on the first question, she was able to correctly identify that 5 is a factor and 7 is not a factor of $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$, without doing any division computation:

Zoe: Well, I see that the number N equals, uh, has a 5 in it, which means you're multiplying by something that is a multiple of 5...it means you can break down into the factors, and that means that you can divide by 5. So, I would say yes. Is it divisible by 7? Well, I don't see a 7 in here. So I would, without calculating N , I would guess that it's not divisible by 7.

While Zoe could correctly identify prime factors and non-factors of N , she had difficulty identifying composite factors and non-factors:

Zoe: Is N divisible by 51? Well, I don't see anything in here that looks like 51, or could be 51, and I think – if I'd hazard a guess – I would say it's prime because I don't know much that goes into 51. So, I'm gonna say no. Is N divisible by 105? Again, I don't see anything that could be 105 in these parts that make up N . Um, so I'm just gonna say no.

In the excerpt above, Zoe claimed she could not *see* 51 or 105 in the prime factorization of N . This implied that her image of factor was limited to numbers that are directly visible in another number's prime factorization. Later, Zoe clarifies her limited understanding of factor:

Zoe: So, with the numbers that they give you that can't be seen in the parts [prime factorization] that they've given you, then I would say that you can't find – it's not divisible by that number.

On Question 5 of the pre-interview, Zoe had difficulty applying specific divisibility relationships and coordinating prime factors. In this example, she was asked to find the smallest positive integer divisible by all of the first ten counting numbers.

Although Zoe identified certain divisibility relationships (e.g., a number divisible by 9 is also divisible by 3), she was unable to use those relationships successfully to find a correct answer. In fact, her reasoning showed that her thinking was guided more by superficial clues than by accurate conceptual knowledge:

Zoe: Uh, anything that is divisible by 10 is also divisible by 5 because 5 goes into 10. 5 times 2 is 10. And so I can ignore the 10 because of the part of the question that says 'the smallest positive integer.' Um, if I use the 10 to find out my number, it'll be too big, so I can just go to the basic number of 10, which is 5...

Zoe's solution ($1 \times 2 \times 3 \times 5 \times 7$) does not account for several factors (4, 8, and 9) because it does not have sufficient prime factors to coordinate. Without a coordinated view of the factor concept, Zoe could not recognize that her solution must include the prime factorizations of all ten counting numbers. Her decision to exclude 10 from the prime factorization of her solution rested more on the language of the question, which asked for the *smallest* positive integer (5 is *smaller* than 10), than on the underlying divisibility relationships.

Zoe's understanding of the GCF and LCM concepts were also incomplete and limited. On Question 7 of the pre-interview, she had to find the dimensions of the largest square tile needed to perfectly tile a rectangular floor with dimensions 96 feet by 56 feet. Zoe used a guess-and-check strategy that included picking a random counting number and dividing it into 96 and 56 to see if it was a common factor:

Zoe: My reasoning was I need to find something that equally goes into – a number that goes both into 96 and 56 and I just did trial and error starting with, um, 6...Because, I don't know, but they both ended with 6...but that didn't work.

While many of her peers checked the factors of 56 and 96 immediately, Zoe checked random counting numbers for divisibility. It is unlikely that Zoe's strategy was due to

difficulty identifying factors; on Question 2a of the pre-test she was able to identify most of the factors of 45 by constructing a simple factor tree. More likely, she had not yet internalized the procedure for finding the GCF, and so did not know that only the factors of 56 and 96 needed to be checked. As such, her level of understanding of GCF was coded at the *action* level.

Zoe displayed a higher level of understanding of the LCM concept than the GCF concept. Her work on this topic was coded at the *process without coordination* level. On Question 8 of the pre-interview, she found consecutive multiples of 80 and 135 and then selected the first multiple both numbers shared as the correct answer. Unlike her work with the GCF, Zoe knew that only multiples of each number could be the LCM. This was evidence that she had internalized the concept and no longer needed to guess random values. Also, her recognition that 135×80 was an upper limit was evidence that Zoe had some understanding of the relationship between the LCM and 135 and 80:

Zoe: Um, I thought about it as a problem that it's asking for the least common multiple of those two numbers, but I'm not sure how the best way to figure that out is. I know that one way to find a multiple between two numbers is multiplying them together, but I thought that that was too big and that there had to be a number smaller. So, I just started multiplying – each time multiplying by one more counting number to see when I would find a number that worked – that both is a multiple of 135 and 80.

Following Number Theory Instruction

Zoe's understanding following instruction showed improvement across all number theory topics. Analysis of Zoe's post-interview and NTKT post-test revealed some variability in her developing understanding, as she displayed greater understanding of some concepts than of others. Specifically, her knowledge of LCM and GCF concepts

appeared more robust than her knowledge of divisibility concepts. In Table 5.3.7, Zoe's levels of understanding during her interviews before and after number theory instruction are shown.

Table 5.3.7: Zoe's Coded Levels of Understanding by Question

Question	Pre-interview Codes	Post-interview Codes
1	Process without coordination	Process without coordination*
2	Process without coordination	Object
3	Process without coordination	Process with coordination
4	Process without coordination	Process with coordination
5	Process without coordination	Schema
6	Action	Object
7	Action	Process with coordination
8	Process without coordination	Object

*Evidence of movement toward the next higher level of understanding

Zoe's understanding of divisibility and prime factorization improved primarily to the *process with coordination* level, as she was now generally able to coordinate distinct prime factors in order to identify divisibility. For example, on Question 3 of the post-interview, Zoe was able to find almost all composite factors of $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$:

Zoe: N would also have these factors, any combinations of these factors...So, N has 2×3 which is 6; 2×5 which is 10; 3×4 which is 12; 4×5 which is 20; and $3 \times 4 \times 5$ which is 12 \times 5 which is 60, because it has a factor of itself.

In this example, Zoe was able to coordinate all distinct prime factors except one combination of three primes ($2 \times 3 \times 5$). Though she recognized that factors were the result of such coordination, the non-systematic nature in which she found them suggests that she had yet to fully conceptualize this notion of factor (i.e., *object level understanding*).

The NTKT post-test provides further evidence of coordination, as well as a

stronger tendency on Zoe's part to attend to prime factorizations. On Question 2a, she re-represented 36 in its prime factored form and coordinated its prime factors to find composite factors. On Question 2b, as shown in Figure 5.3.10, she used the prime factorization given to her to find all possible prime factor combinations – her developed image of divisibility was so strong that she left her list of factors in prime factored form. This was in stark contrast to her work on a similar problem on the pre-test, in which she could only identify the numbers that were visible in the given prime factorization as factors.

Figure 5.3.10: Zoe's Solution Strategy for Question 2b of the NTKT Post-Test

b) How many factors does $3^2 \times 5^2$ have? How do you know?
 $1, 3, 3^2, 5, 5^2, 3 \cdot 5, 3^2 \cdot 5, 3 \cdot 5^2, 3^2 \cdot 5^2$
 9 factors. I know because I found all the possible combinations of the prime factors. Every factor has to be found in the prime factors, as a combination of them b/c the prime factors are the most basic factors.

Zoe's knowledge of the LCM showed the greatest improvement across all topics. Before instruction she viewed the LCM as the result of a procedure of finding consecutive multiples (*process without coordination* level). Following instruction she was able to explicitly articulate a meaning for LCM (*object* level). On one problem, she displayed a sophisticated and robust understanding of the relationship between the LCM, divisibility, and prime factorization (*schema* level).

For example, Question 8 of the post-interview required Zoe to find the time it would take two runners to stop running given that it takes the runners 80 and 105

seconds, respectively, to complete one lap and both runners would stop only when both completed a lap at the exact same time. Zoe immediately provided a rationale for why the LCM was the necessary number to compute:

Zoe: So, I know that I need to find the LCM because the question is asking to find ‘the exact time that they both stop’ and they both stop at multiples of their seconds. Uh, and, so I need to find a time when they both – the first time that they both end at the same time, which is a multiple that they both share.

Instead of describing the LCM as the result of a process (e.g., the result of listing multiples and finding a match), Zoe was able to reflect on that process until she saw it as a single object that itself can be acted upon (*object level*).

Zoe’s understanding of the GCF concept improved from an *action* level to a *process with coordination* level. On Question 7 of the post-interview, Zoe was asked to find the dimensions of the greatest square tile needed to perfectly tile an 84-foot by 76-foot rectangular floor. Although she was able to use an efficient solution strategy – find each number’s prime factorization and identify the greatest prime factor combination they share – her weak explanation of the GCF concept was not sufficient to warrant coding at the *object level*:

Zoe: So I know that this is the GCF because it has the word ‘the largest – the largest square tile’ and I know that it’s asking for something that can fit into 84 and 76, so it’s smaller than those, which means I know it’s a factor. And it’s a factor that they both have to share, because that’s how you tile floor. If it doesn’t go into 76 feet, but it goes into 84 feet, then you don’t tile the whole floor.

Although she began to identify divisibility as a reason for using the GCF, her main argument appeared to center around imprecise word associations – *largest* with *greatest* and *smaller* with *factor*. Although generally true, a factor of a number is not always smaller than the number itself, and some scenarios asking to find the largest amount or

size may not necessary call for the GCF (e.g., if two runners can run one lap in 45 and 40 seconds, respectively what is the greatest number of complete laps that both can run in 6 minutes if both start running simultaneously?) Prior research has identified pre-service elementary teachers' tendency to use imprecise, informal vocabulary as a substitute for rigor (Zazkis, 2002, 2000b).

Reflections on Zoe's Developing Understanding

Prior to instruction, Zoe's understanding of number theory topics was characterized by a limited internalization of number theory procedures. Following instruction, however, Zoe's mental representations of certain procedures strengthened, becoming more conceptually-based. Figure 5.3.11 below shows a summary of Zoe's levels of understanding as displayed during the pre- and post-interviews.

Figure 5.3.11: Zoe's Displayed Levels of Understanding



Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Zoe's progress was most apparent when solving GCF and LCM related problems.

It is not clear why she showed higher levels of understanding regarding the LCM than the GCF, though she clearly came into the study with a better understanding of LCM. One possible explanation could be that it may be easier to use informal and imprecise language when discussing the GCF, thus possibly diluting the quality of her understanding. For example, Zoe explained that the value of the GCF “fit” into the two given numbers but used the more precise term, “multiples,” to explain the LCM. Her understanding of divisibility concepts also improved as seen in Questions 1-5, yet Zoe continued to show evidence that her ability to coordinate prime factors was not fully encapsulated (e.g., Question 3). Her extremely high level of understanding on Question 5 appeared to be an isolated event.

Her work on the NTKT post-test also improved, increasing from a score of 24% to 78%. Table 5.3.8 shows the percentage of possible points earned on questions requiring concrete skills (e.g., computation, executing known procedures) and abstract skills (e.g., attention to underlying concepts and relationships, executing unfamiliar solution strategies):

Table 5.3.8: Zoe’s Percentage of Possible Points Earned by NTKT Question Type

	<i>Abstract Skills</i>	<i>Concrete Skills</i>
<i>Pre-Test</i>	20.7%	28.6%
<i>Post-Test</i>	75.9%	81.0%

Zoe showed consistent improvement on both abstract and concrete skills questions. Unlike her work on the pre-test, on both question types on the post-test she attended to prime factorizations as a way to identify factors and compute GCF and LCM values. This may have helped her solve complex problems that she was unable to solve on the pre-test.

Jane

Another interview participant in this study, Jane, was a sophomore undergraduate student majoring in elementary education. Due to her concentration, this course is the first of two required mathematics education content courses. Based on her poor performance on the Number Theory Knowledge Test (NTKT) pre-test (22% vs. sample mean of 30%) and her course instructor's recommendation, Jane was considered a low-achieving mathematics student for the purposes of this study. Her prior mathematics coursework included Algebra I, Algebra II, Geometry, Pre-calculus, and the International Baccalaureate[®] mathematics studies standard level course. The latter course is designed to be a high school student's last course in mathematics, and is specifically geared for students who do not anticipate studying mathematics in the future. Additionally, Jane took the SAT[®] examination and scored in the 82nd percentile nationwide on the mathematics portion.

Prior to Number Theory Instruction

Before relevant number theory instruction, Jane's understanding was primarily at the *action* and *process without coordination* levels. Such understanding is best characterized by procedural attachments. Despite evidence of the formation of an internalized image of several number theory procedures, Jane's limited understanding forced her to resort to inefficient methods to solve problems.

Jane's understanding of divisibility and factor concepts was coded at the *process without coordination* level. Based on her performance on the pre-interview, her

internalized image for factor was that a factor is a number that is *visible* in the prime factorization of another number. As such, her conception only helped her identify prime factors and non-factors. In Question 3 of the pre-interview, when asked to identify the factors of $N = 3 \times 6 \times 7$ and $R = 11 \times 13 \times 17$, Jane showed an inability to coordinate prime factors – she could not identify products of distinct prime factors as additional factors – limiting her ability to identify composite factors:

Jane: So, for N, if you have $3 \times 6 \times 7$, the factors of 3 are 1 and 3, the factors of 6 are 2 and 3 and 1, and the factors of – oh, and 6 is also a factor. And 7 is 1 and 7. And then R is $11 \times 13 \times 17$. It's 11 and 1, and 1 and 13, and 1 and 17. So, yes, I think that's it.

By only attending to the *visible* factors in N's and R's factorizations, Jane was unaware of additional factors of N and R (e.g., $21 = 3 \times 7$ is a factor of N). Her image of divisibility prevented her from seeing many composite factors.

Her work on the NTKT pre-test provided further evidence of her limited view of divisibility. On Question 1, when asked to identify specific factors of $2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$, Jane was unable to do so and responded “I am not sure” each time. On Question 3, she was asked if 31×17 is a prime number. Without computing its product, she wrote, “the prime numbers become other factors, so the product has more factors than 1 and itself.” Once again, we see evidence of an internalized image for factor, but one that relies exclusively on what is visible versus what can be constructed via the coordination of prime factors.

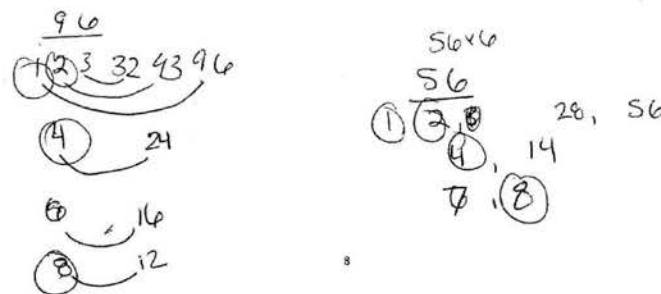
Jane's understanding of GCF and LCM concepts varied from *action* to *process* levels. Generally, Jane lacked a robust understanding of underlying concepts, and instead, focused on executing computation-heavy procedures in order to solve problems.

For example, on Question 6 of the pre-interview and pre-test, she was given the GCF ($2^3 \times 5$) and the LCM ($2^5 \times 5^2$) of two unknown numbers, A and B, and was asked to find all possible values of A and B. Jane was unable to solve this problem on either occasion and resorted to computing the values of the GCF and the LCM by multiplying their prime factors. During the pre-interview, she commented on her lack of understanding of the GCF and LCM concepts:

- Jane: ...the greatest common factor has to be this number and then the [least] common multiple has to be $2^5 \times 5^2$, but I'm confused about how they relate to each other, I guess.
- Interviewer: Why do you think you're having this confusion?
- Jane: Um. I guess maybe I don't have a very good working definition of both of these terms at the moment? Because, like, I remember learning about them in middle school, but I don't really remember what they mean.

Her procedural attachments were also evident when a problem required Jane to compute the GCF of two numbers. Her method often consisted of listing the factors of each number and identifying common values in each list.

Figure 5.3.12: Jane's Solution Strategy for Question 7 of the Pre-Interview



Such an approach was evident on Question 7 of the pre-interview, shown in Figure 5.3.12 above, where Jane tried to find the GCF of 96 and 56:

- Jane: Um, I'm trying to find factor pairs. Like 1×96 equals 96, and 2×48 equals 96. 3×32 equals 96 [she continues searching for factor pairs]...for 56, it's 1 and 56,

2 and 28 I'm gonna guess, yes. Um, 3, oops, nope, never mind. 3 isn't one [she continues searching for factor pairs]...All right, so the ones that match up are 1, and 2, and 4, and 8.

In this excerpt, Jane resorted to a guess-and-check method for finding factors without any indication that she understood the meaning of her result.

Despite her limited understanding of concepts, Jane showed evidence of higher-level thinking (*process with coordination*) with regards to the LCM concept. On Question 8 of the pre-interview, Jane was given a real-life scenario in which two women, Rita and Jean, run a complete lap in 135 seconds and 80 seconds, respectively. She was required to find the number of seconds it would take both runners to stop running, assuming both runners would stop only when both completed a lap at the exact same time:

Jane: I'm dividing the numbers that I – the times that I got for Rita based on her laps. I'm dividing them by 80 to see if Jean could've stopped at the same time... because it takes Jean 80 seconds to run a lap.

Although at first Jane found consecutive multiples of each runner's lap times, she was eventually able to find a more coordinated approach. By finding multiples of Rita's lap time and dividing them by Jean's lap time, Jane displayed an understanding of the relationship between 135, 80, and their LCM. She could not yet articulate a clear explanation of the LCM concept, but she showed movement toward a more conceptually-oriented understanding.

Following Number Theory Instruction

Jane's understanding of number theory topics improved following the three-week

unit. Based on post-interview analysis, Jane’s understanding was coded almost exclusively at the *object* level. She was now able to view many of the procedures related to divisibility, GCF, and LCM as static entities that can, themselves, be operated on. Such understanding was characterized by complete coordination of distinct prime factors, minimal reliance on lengthy computation, and the ability to articulate clear explanations of the GCF and LCM concepts. Table 5.3.9 below summarizes Jane’s levels of understanding as displayed during her pre- and post-interviews:

Table 5.3.9: Jane’s Coded Levels of Understanding by Question

Question	Pre-interview Codes	Post-interview Codes
1	Process without coordination	Object
2	Action	Process with coordination
3	Process without coordination	Object
4	Action	Process without coordination
5	Process without coordination	Object
6	Action	Object
7	Process without coordination	Object
8	Process with coordination	Object

Jane’s understanding changed from viewing factors as numbers that are *visible* in the prime factorization of a number to viewing factors as coordinated products of distinct prime factors. For example, on Question 1 of the post-interview, Jane was asked to state if $N = 3^2 \times 5^4 \times 11 \times 17^3$ is divisible by specific numbers:

Jane: Is N divisible by 35? And I say no, because you can’t – 35 is not in a prime factorization, nor can you make it with any of the numbers in the prime factorization – like there’s no 7 in there, so I can’t do 5×7 to get it.

In addition to successfully identifying composite non-factors, her ability to coordinate prime factors extended to composite factors as well:

Jane: Is N divisible by 75? And yes it is, because you can make 75 in a prime factorization by $5^2 \times 3$. And is N divisible by 51? Yes, because 51 is 3×17 , I believe, and that is part of the prime factorization [of N].

In both short excerpts above, one finds evidence that Jane began viewing factors as a coordination of distinct prime factors. She was now able to accurately identify factors and non-factors regardless of their visibility within the prime factorization of the number.

Similarly, Jane's understanding of divisibility improved as she began identifying divisibility relationships. For example, on Question 5 of the post-interview, Jane was able to successfully find the smallest positive integer that is divisible by all of the first ten counting numbers by identifying extraneous factors of factors:

Jane: So, first I'm going to list all of the counting numbers just off the top, so I can figure out what to do with them, and so, if a number is divisible by 10, it's also divisible by 5 and 2, so I can cross out 10. If a number is divisible by 9, it's also divisible by 3 and 3...which means I can cross out this 3 because 3^2 will cover that, will supersede that.

Knowledge that if A is divisible by B and B is divisible by C, then A is divisible by C is evidence of an *object* conception of divisibility because it implies that Jane was able to view divisibility as a number property (i.e., one number *is* or *is not* divisible by another number) as opposed to the result of a procedure.

On the NTKT post-test, Jane also displayed a high level of understanding with regards to factors and divisibility. For example, on the second question, when asked to find how many factors $3^2 \times 5^2$ possesses, she wrote that:

$3^2 \times 5^2$ has 9 factors because 3^2 has three factors and 5^2 has 3 factors and if you multiply the number of factors each has together (because 3 and 5 are multiplied together) the result is 9. They must be multiplied together because to find the factors of $3^2 \times 5^2$, you would find all the combinations of its prime factorization."

Unlike the pre-test, Jane now attended to prime factorizations and the coordination of

prime factors instead of computation. Her conceptual understanding was evident in general statements such as “to get the total number of factors you would need to find all the possible combinations of the factors (plus 1).”

Jane’s understanding of GCF and LCM concepts was coded at the *object* level because she showed an explicit understanding of the meaning of both concepts. Her approach to finding the GCF and LCM of two numbers improved, from listing out factors and consecutive multiples to using prime factorizations. On Question 7 of the post-interview, Jane was presented a real-life situation in which a school needs to buy square tiles to fully tile an 84-foot by 76-foot rectangular floor. In order to find the largest square dimensions that will tile the floor perfectly, she used prime factorization to find the greatest common factor of 84 and 76. Her explanation for why the GCF was the correct answer was evidence of her conceptual understanding:

Jane: So, I want to find the greatest common factor of 84 and 76 because the tiles are square, so you can’t just buy a tile that’s 84 by 76. You have to find one that’s a multiple [meant to say factor] of 84 so it can go up X number on one wall, and a factor of 76 so it can go up X number on the other wall. So, the way that I’m going to do that is I’m going to find the prime factorization of both of these numbers.

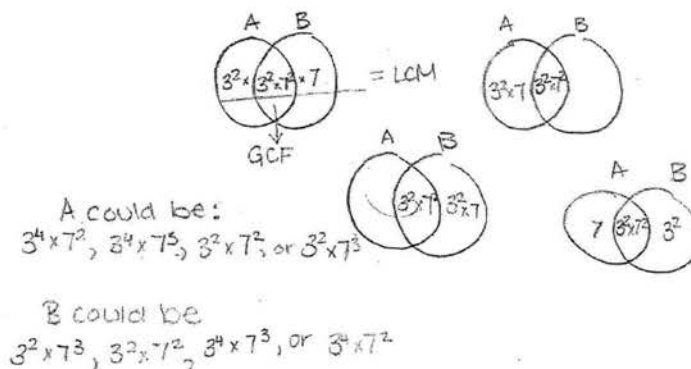
In this excerpt, Jane contextualized the meaning of factor by stating that the factor “can go up X number on one wall” and “X number on the other wall.” Later on, she provided further clarification for why this problem required finding the greatest common factor:

Jane: ...Since they have to be square, they have to be a factor of both 84 and 76...and because you want the largest square tile, you want the greatest factor that they have in common or the greatest common factor.

Jane also exhibited a deeper understanding of the relationship between the GCF and LCM concepts. For example, Question 6 on both the post-interview and post-test

required her to find all possible values for A and B such that the GCF of A and B is $3^2 \times 7^2$ and the LCM of A and B is $3^4 \times 7^3$. Prior to classroom instruction, she resorted to computing the values of the GCF and LCM. During the post-test, however, she used prime factorizations and the relationship between the GCF and LCM to find all possible solutions, as shown in Figure 5.3.13 below. She also provided reasoning for this work during the post-interview:

Figure 5.3.13: Jane's Solution Strategy for Question 6 of the NTKT Post-Test



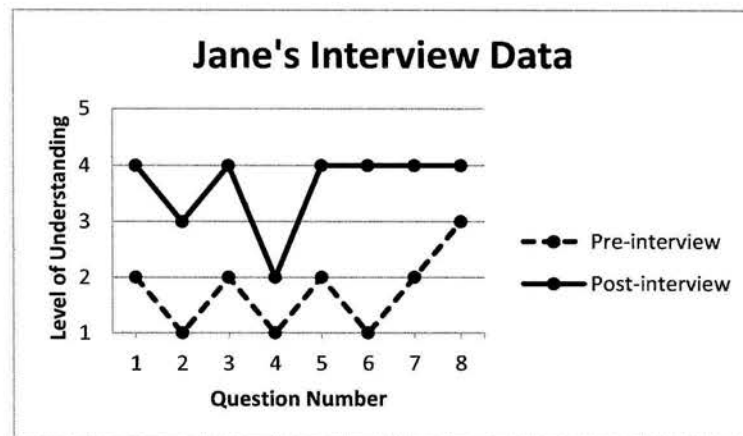
Jane: I know what's always going to be in the middle [of the Venn diagram] is going to be the GCF, and then if the LCM is – the way you find that is by multiplying all of the numbers across. So, now I just need to fill in the blanks, where this is – so I have $3^2 \times 7^2$, so the numbers that I need to add are two 3s and a 7.

The need to *add two 3s and a 7* to the Venn diagram implies that Jane understood that the product of the GCF and all remaining non-common prime factors of A and B equals the LCM. Her knowledge of the relationship between GCF and LCM helped her successfully solve this problem, whereas beforehand she was unable to use that knowledge.

Reflections on Jane's Developing Understanding

Figure 5.3.14 below shows the development of Jane's levels of understanding from pre- to post-interviews. Her understanding prior to instruction was superficial, as illustrated by her frequent reliance on tedious procedures and her inability to coordinate concepts (*action* and *process without coordination* levels). Her understanding following instruction, however, became more conceptually-oriented. She attended to prime factorizations, articulated the underlying meaning of concepts, and coordinated prime factors when identifying factors (*object* level).

Figure 5.3.14: Jane's Displayed Levels of Understanding



Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Jane's results on the NTKT post-test also showed improvement from her work on the pre-test, increasing from an overall score of 22% to 70%. Below, Table 5.3.10 shows the percentage of possible points earned on questions requiring concrete skills (e.g., computation, executing known procedures) and abstract skills (e.g., attention to underlying concepts and relationships, executing unfamiliar solution strategies):

Table 5.3.10: Jane's Percentage of Possible Points Earned by NTKT Question Type

	<i>Abstract Skills</i>	<i>Concrete Skills</i>
<i>Pre-Test</i>	10.3%	38.1%
<i>Post-Test</i>	69.0%	71.4%

The data above supports the prior contention that Jane's understanding grew to more conceptual levels. Her ability to solve problems requiring more concept-based knowledge improved from 10.3% to 69%. It should be noted that her ability to execute procedures and perform necessary computations improved as well. These results are supported by prior research that claimed that improved conceptual knowledge does not occur at the expense of procedural fluency (Fennema et al., 1996; Silver & Stein, 1996; Hiebert & Wearne, 1993).

Christina

The last interview participant, Christina, was a sophomore undergraduate student majoring in elementary education and bilingual education during this study. As a result of her elementary education concentration, this course was the first of two required mathematics education content courses. Based on her course instructor's recommendation and her performance on the Number Theory Knowledge Test (NTKT) pre-test (16% vs. sample mean of 30%), Christina was considered a low-achieving mathematics student. Her prior high school mathematics coursework included Algebra II, Geometry, Pre-calculus, and Trigonometry. Although she did not take any advanced placement examinations in mathematics, she scored, on average, in the 78th percentile nationwide on the mathematics portions of the SAT[®] and ACT[®] examinations.

Prior to Number Theory Instruction

Christina's understanding of number theory prior to instruction was the most procedurally-oriented of all of the interview participants in the study. Her pre-interview work was coded primarily at the *action* and *process without coordination* levels for divisibility topics and exclusively at the *action* level for greatest common factor (GCF) and least common multiple (LCM) topics. Her understanding was characterized by an over-reliance on computation, an inability to attend to prime factorization, and a complete lack of understanding of the meanings of GCF and LCM.

Christina displayed a strong tendency to ignore prime factorization and resort to computation (e.g., long division) when solving divisibility and factor related problems.

For example, on the second question of the NTKT pre-test, when asked which number, $N = 3 \times 6 \times 7$ or $R = 11 \times 13 \times 17$, had more factors, she computed the values of N and R first and then used long division to find several of their factors. She could not use the factorizations to identify factors (e.g., 3 is a factor of N because it appears in its factorization). Similarly, on Question 3 of the pre-test, when asked if 31×17 is a prime number, Christina concluded that 31×17 is prime “since it can’t be divided by anything else evenly.” She had to compute the product first, ($31 \times 17 = 527$) and check for factors via long division instead of identifying factors directly from its prime factorization.

During the pre-interview, however, Christina displayed a more advanced understanding of divisibility and factor concepts, one characterized by the internalization of long division for certain factor types. For example, in the first question, Christina quickly claimed that $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$ was divisible by 5 and 4, but not by 7:

Christina: I just think it’s true because it’s multiplied by 5, so then it has to be divisible by 5. That’s how I think. Um, is N divisible by 7? Going by my reasoning of 5, I would say no because there is no 7 in $2^3, 3^2$, all the factors of the problem. So, no. N divisible by 4? Um, I think N will be divisible by 4 – also going by my reasoning for “A.” Um, 2^3 is 8, so since 8 is divisible by 4, I’m just gonna assume that that’s in there.

In the excerpt above, Christina identified factors without resorting to computation. She internalized an image of factor as a number that is directly visible in the prime factorization of another number, or one that is a factor of individual prime powers of the prime factorization of another number. As such, she no longer needed to execute long division procedures to determine divisibility by prime numbers.

Her internalized image of factor did not allow her to identify divisibility by composite factors, especially when the composite factor is the product of distinct prime

numbers. This inability to coordinate prime factors was most evident on Question 4 of the pre-interview, where she was required to find a number with exactly four factors. In this example, she randomly selected numbers and tried to find how many factors they have:

Christina: I'm just going through the counting numbers to see which ones I can just knock out for having four factors...I'm just going to go for 36, a random number, and I know 36 is divisible by 3. I think we got 16 for that one. 12. Um, 12 is 3 and 4. 4 is 2 and 2. Those are all – those just count for two factors. I guess I'm having problems coming up with a number, 'cause all the numbers I'm coming up with are doubles.

Christina claimed that 36 had two factors (2 and 3). Similarly, she claimed 72 also had just two factors (2 and 3) and 100 had just three factors (2, 5, and 1). Her error was that she only counted prime factors as factors. She most likely conflated the notion of prime factor with factor because she was incapable of coordinating prime factors to generate new factors.

Christina's understanding of GCF and LCM concepts was especially weak.

During the pre-test, she did not attempt to solve any of the GCF or LCM related questions. During the pre-interview, she made unsuccessful and frustrated efforts on two of the three relevant questions. For example, on Question 6 where she was given the prime factorizations of the GCF and LCM of two unknown numbers, she could only compute the values of the GCF and LCM; she could not find the values of the two unknown numbers, as the problem had asked her to do. On Question 7, she was given the dimensions of a rectangular floor and asked to find the dimensions of the largest square tile that could tile the floor perfectly. In this example, Christina did not identify the GCF as the salient topic of the question. Her solution strategy – find the area of the floor

guess-and-check tile sizes to fill it – was unsuccessful. Due to the large amount of time it took Christina to attempt the first seven questions, she did not have time to work on the final pre-interview question regarding the LCM.

Following Number Theory Instruction

Following instruction, Christina’s displayed understanding of divisibility, prime factorization, GCF, and LCM concepts was more developed. Her mental representation of factor became more coordinated and generalizable; she showed less reliance on computation and paid more attention to prime factorization; and, she displayed an understanding of the meaning of GCF and LCM. Table 5.3.11 below shows Christina’s levels of understanding, as exhibited during the pre- and post-interviews. Overall, her level of understanding across all topics improved from the *action* level to the *process with coordination* level.

Table 5.3.11: Christina’s Coded Levels of Understanding by Question

Question	Pre-interview Codes	Post-interview Codes
1	Process without coordination	Process with coordination*
2	Action	Action
3	Process without coordination	Process with coordination
4	Action	Process without coordination
5	Process with coordination	Schema
6	Action	Object
7	Action	Process with coordination
8	Did not attempt	Object

*Evidence of movement toward the next higher level of understanding

On Question 3b of the post-interview, Christina’s understanding of factors as combinations of distinct prime factors allowed her to find almost all of the factors of $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$. The following excerpt was evidence that her internalized

image for factor from the pre-interview had become coordinated:

Christina: Um, so the factors of N are what makes up N, so N equals $3 \times 2^2 \times 5$, so I know that all those numbers are factors. So, 3, 2, 5 are factors as well as 4 for the 2^2 and 1. I know like those are just the base factors of the equation and then I also know that, any combination of those numbers will be a factor of N because they were multiplied together to get N. So, I multiplied 3×2 , 3×4 , 3×5 , 2×5 , and 4×5 to get the factors of N. And then I did the same for R and just found all the combinations of the numbers.

The only factor of N that she did not identify was $3 \times 2 \times 5 = 30$. One reason may be that Christina has some difficulty coordinating more than two factors at a time.

Question 5 of the post-interview was evidence of Christina's ability to make explicit connections between divisibility and prime factorization concepts. Her understanding of this question was coded at the *schema* level. In this example, she needed to find the smallest positive integer that was divisible by the first ten counting numbers. As shown in Figure 5.3.15, Christina successfully solved this problem by building the prime factorization of the solution and attending to the divisibility relationships among the first ten counting numbers:

Figure 5.3.15: Christina's Solution Strategy for Question 5 of the Post-Interview

LCM of 1st 10 #'s

$$1 * 2^3 * 3^2 * 5 * 7 = 2520$$

1, 2, 3, 4, 5, 6, 7
8, 9, 10

- Christina: ...I am looking for the least common multiple of the first ten counting numbers...
- Interviewer: What are you thinking about right now, Christina?
- Christina: ...So I know that any – the least common multiple must be – must have all the primes as factors. So, that would be 1, 2, 3, 5, 7. And I also know that I don't have to worry about multiplying by 10 because it is made up of 2 and 5, which if you were to combine the primes 2 and 5 you would come up with 10. I also know that 3^2 should be included in the factors, so I'm going to have the 1, 2, 3^2 , 5 and 7. And, then I also know that I need to have at least 2^2 for 4, but then I look over at 8 and I see that I need to have 2^3 . And I know if something is divisible by 8, then it can be divisible by 4. So I'm going to put 2^3 in as a factor, and I'm just going to multiply that out, so $2^3 \times 3^2 \times 5 \times 7$ and I come up with 2,520. And 2,520 will be divisible by 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.

Christina understood the divisibility relationships between counting numbers – divisibility by certain counting numbers implies divisibility by their factors. She used these relationships to simplify the prime factorization of her solution, thus providing evidence that she was making connections between divisibility and prime factorization concepts.

Christina's work on Question 5 is also an example of her developing understanding of the meaning of the LCM concept. After successfully solving this problem, she explained how she knew her answer must be the LCM of the first ten counting numbers:

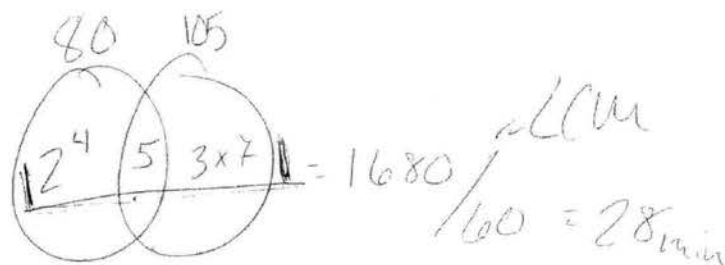
- Christina: And I know that that is the smallest number that I can come up with because there's no – like there's no doubles in the – I guess the base. And by doubles I mean I know – for example I didn't have 2^{2n} times 3^{2n} ; or to represent 4 and 8, since I knew any number divided by 8 – or divisible by 8 – would be divisible by 4, so I knew I only had to have 2^3 .

In this excerpt, Christina's mention of "no doubles in the base" reflects the need to represent each of the first ten counting numbers' prime factorizations in the most efficient way possible – removing even one prime factor would prohibit one or more of the ten

counting numbers from being a factor. During the pre-interview, though she tried to eliminate prime factors to be more efficient, she did not explicitly attend to “doubles” resulting in a solution, $6 \times 7 \times 8 \times 9 \times 10$, which was clearly not the LCM of the first ten counting numbers.

Christina’s understanding of the LCM concept reached the *object* level on Question 8 of the post-interview. In this example, she was told that two runners were running around a circular track at 80 seconds and 105 seconds per lap, respectively. Assuming the runners would stop only when both completed a full lap at the same moment, Christina had to find the time it would take both to stop running. Using prime factorizations, she was able to execute an efficient method (shown in Figure 5.3.16), attend to the relationship between LCM and the two given numbers (80 and 105), and give a valid explanation for why the LCM concept was necessary to solve the problem:

Figure 5.3.16: Christina’s Solution Strategy for Question 8 of the Post-Interview



Christina: Both runners will only stop when they both complete a lap at the exact same time. And since I knew they wouldn’t complete – since I knew from the information given that they wouldn’t complete their first lap at the same time, I knew it had to be in multiples of their first lap. And, um, I’m assuming they didn’t want to run for longer than they had to, so with the least common multiple”

In the excerpt above, Christina explained that since the runners’ lap times were different,

the time in which both would stop running would have to be a multiple of both lap times. This is evidence that she is beginning to think of the LCM as a single value, one that has a direct relationship to the two given numbers from which it is calculated.

Christina's understanding of the GCF concept also improved from pre- to post-interview. Recall that on Question 7 of the pre-interview, she could not recognize the need to compute the GCF of the dimensions of the rectangular floor – evidence of *action* level understanding. During the post-interview, however, when asked the same question with different dimensions, she not only identified the need to compute the GCF, but she also used the relationship between GCF and prime factorization to efficiently solve the problem, signs of *process with coordination* level understanding.

Analysis showed, however, that Christina's understanding of the GCF concept was not as strong as her understanding of the LCM concept. On the post-test, Christina's explanation of the LCM concept attended more to its underlying structure, while her explanations of the GCF concept rested on superficial triggers. When attempting to explain why the square tile problem necessitated finding the GCF, her explanation was based on a misconception regarding key terms rather than meaning. She wrote, "This is a GCF [problem]. I can tell because of the wording. The problem is asking me the dimensions of the 'largest square tile.' Largest tells me greatest common factor and not least common multiple." On the other hand, her explanation for why the LCM was needed to find the answer to Question 7b of the post-test (identical to Question 8 of the post-interview), attended more to meaning than to superficial indicators: "this would require the times of the runners to be multiplied to find the smallest multiple both runners

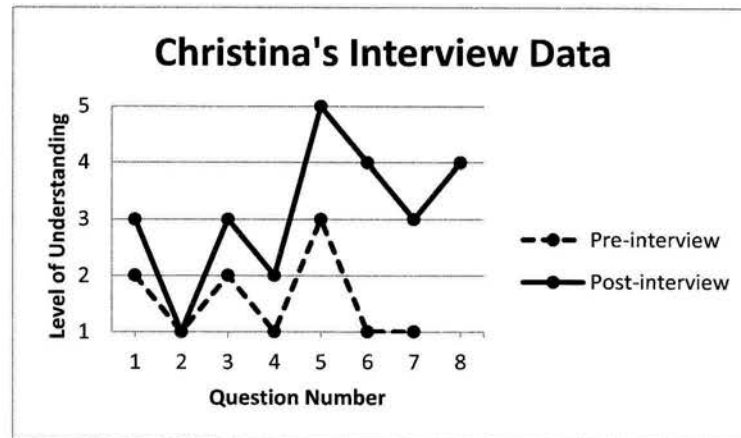
would have.” Although not completely correct, Christina is nonetheless focusing on the concept of multiples to explain the LCM.

Reflections on Christina’s Developing Understanding

Prior to instruction, Christina’s understanding of number theory topics was the weakest of all of the interview participants. It was characterized by minimal conceptual understanding and an over-reliance on computation and guess-and-check strategies. Following the number theory unit, she developed a more robust understanding of divisibility, focused more on prime factorization than on computation, and began providing valid explanations.

Figure 5.3.17 shows a summary of Christina’s levels of understanding as displayed during the pre- and post-interviews. Her progress appeared most apparent when solving GCF and LCM related problems. Her inability to compute them (*action level*) gave way to her growing ability to explain their meaning in real-life contexts (*object level*). The most significant and prevalent development with regards to divisibility and factor concepts was Christina’s newfound ability to coordinate prime factors. This development represented a major refinement of her mental image of factor (*process with coordination level*).

Figure 5.3.17: Christina's Displayed Levels of Understanding



Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Christina's work on the NTKT post-test also improved, increasing from a score of 16% to 55%. Though her post-test score was well below the sample mean post-test score (73.6%), her work on questions requiring concrete skills (e.g., computation, executing known procedures) and abstract skills (e.g., attention to underlying concepts and relationships, executing unfamiliar solution strategies) showed clear improvements. Below, Table 5.3.12 shows the percentage of possible points earned by question type on the pre-test and post-test:

Table 5.3.12: Christina's Percentage of Possible Points Earned by NTKT Question Type

	<i>Abstract Skills</i>	<i>Concrete Skills</i>
<i>Pre-Test</i>	0.0%	38.1%
<i>Post-Test</i>	53.4%	57.1%

Prior to instruction, Christina showed a complete inability to make sense of questions requiring knowledge of underlying concepts. Following instruction, however, her

developing understanding helped her achieve nearly the same percentage correctness on concept-based question as on procedure-based questions.

Section IV: Summary Analysis

Analysis of the six interview participants revealed improved understanding of number theory concepts following number theory instruction. Each code was converted into a numeric value from 1 to 5. Table 5.4.1 below shows descriptive statistics of levels of understanding for each interview participant. All participants displayed much lower than typical levels of understanding on Question 4 during pre- and post-interview sessions. While participants exhibited at least *object* level understanding on every other interview question, none reached beyond the *process with coordination* level for Question 4. In fact, Question 4 was a statistical outlier on the post-interview. As a result, the median values in Table 5.4.1 present a more accurate measure of center in this case. Mean values were also computed for each participant excluding Question 4, however, they are nearly identical to the mean values including Question 4.

Table 5.4.1: Mean and Median Levels of Understanding by Participant

	Pre-Interview Level of Understanding				Post-Interview Level of Understanding			
	Mean	Median	Min	Max	Mean	Median	Min	Max
Ellen	2.4	3.0	1.0	3.0	4.3	4.5	3.0	5.0
Margaret	1.8	2.0	1.0	2.0	2.8	3.0	1.0	4.0
Amanda	2.4	3.0	1.0	4.0	4.3	4.5	2.0	5.0
Zoe	1.8	2.0	1.0	2.0	3.5	3.5	2.0	5.0
Jane	1.8	2.0	1.0	3.0	3.6	4.0	2.0	4.0
Christina	1.6	1.0	1.0	3.0	3.1	3.0	1.0	5.0

Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

Overall, each participant's level of understanding across all interview questions improved following instruction. Ellen and Amanda exhibited the highest levels of understanding both during pre- and post-interview sessions. They began the study with coordinated images of number theory concepts that they could apply with frequent success to solving problems. They finished the study, however, with a more robust and connected understanding than their peers, one that allowed them to provide detailed explanations and justifications. For example, they often made explicit connections between concepts, such as the important relationship between divisibility and prime factorization in identifying factors.

Margaret, Zoe, and Jane began the study with uncoordinated images, so their initial success was limited (e.g., they could only identify divisibility by primes, not composites). Following instruction, however, their images for divisibility, GCF, and LCM all became more coordinated (to varying degrees) which allowed them to solve a variety of problems. Only Zoe was able to make any explicit connections between multiple concepts (*schema* level understanding), and she did this only one time. Of note was Jane's developing understanding – from *process without coordination* to *object* levels – which improved the most out of all interview participants. It is not clear why she improved more so than her peers; future analysis of classroom video should be done in order to better understand the reasons for this outcome.

Christina showed the weakest levels of understanding prior to and following instruction. Although she came into the study with an internalized image for divisibility and factors, her image was superficial and uncoordinated, and she did not possess any

internalized images for GCF and LCM concepts. Following instruction, she was able to not only internalize images for these concepts, but she began to coordinate her mental schemes to solve problems. She did, however, struggle to make explicit connections among concepts and provide any types of explanations for her work.

Analysis of the entire interview sample also revealed some broader trends in developing understandings. As Table 5.4.2 shows, on average, the six participants exhibited understanding at the *process without coordination* level (median value of 2.0) before instruction and at the *process with coordination* verging toward the *object* level (median value of 3.75) after instruction. In other words, their understanding of divisibility, prime factorization, greatest common factor (GCF), and least common multiple (LCM) concepts progressed from being limited and inconsistent to comprehensive and robust. While, on average, participants possessed internalized representations of most number theory concepts prior to instruction, those representations were limited and superficial. Following instructions, these representations became more coordinated and operational.

Table 5.4.2: Mean and Median Levels of Understanding by Question

Question	Pre-Interview Level of Understanding				Post-Interview Level of Understanding			
	Mean	Median	Min	Max	Mean	Median	Min	Max
1	2.2	2.0	2.0	3.0	3.8	4.0	2.0	5.0
2	1.3	1.0	1.0	2.0	3.2	3.5	1.0	5.0
3	2.3	2.0	2.0	3.0	3.3	3.5	2.0	4.0
4	1.2	1.0	1.0	2.0	2.5	2.5	2.0	3.0
5	2.5	2.5	2.0	3.0	4.2	4.0	3.0	5.0
6	1.3	1.0	1.0	3.0	3.8	4.0	1.0	5.0
7	2.2	2.0	1.0	4.0	3.8	3.5	3.0	5.0
8	2.6	3.0	2.0	3.0	4.0	4.0	4.0	4.0
All	2.0	2.0	1.0	4.0	3.6	3.75	1.0	5.0

Note: Level of understanding values (1, 2, 3, 4, 5) correspond to (action, process without coordination, process with coordination, object, schema) respectively.

The table also reveals that participants' understanding developed consistently across number theory topics. Questions 1-5 assessed participants' understanding of divisibility, factor, and prime factorization concepts. During the pre-interviews, participants exhibited a *process without coordination* level of understanding on most of these questions because they were able to identify divisibility without performing long division computations. However, they could not yet coordinate prime factors to determine divisibility by composite factors. As such, their internalized notions of divisibility were limited to prime factors. During post-interviews, their understanding of these topics was coded at the *process with coordination* and *object* levels because their internalizations became more developed and robust; they were able to coordinate prime factors, at times perfectly, to identify divisibility.

Their lowest levels of understanding were exhibited on Questions 2 and 4, which required participants to determine the validity of a proposed divisibility rule and to find a number that has exactly four factors, respectively. Question 2, however, could be solved using *action* level processes. Participants only needed to find a counter-example through guess and check methods, which is an action-level procedure. On the other hand, Question 4 posed a great deal of difficulty for them because it required them to construct a number with a specific property – a reversal of the process of finding a number's factors. Since reversal is a mental construction that requires a process level of understanding (Dubinsky, 1991), participants could not solve this problem without guessing and checking numbers prior to instruction. Following instruction, they were able to do so more effectively, but still not at a high level (median value of 2.5).

Participants' understanding of GCF and LCM concepts on Questions 7 and 8 was at *process* levels prior to instruction (median values of 2.0 and 3.0, respectively), and improved to *object* levels following instruction (median values of 3.5 and 4.0, respectively). In fact, results showed that participants exhibited higher levels of understanding of the LCM concept than of the GCF concept. Their initial work reflected a basic understanding of GCF and LCM definitions, but lacked coordination of the GCF concept with prime factorization (e.g., listed all factors of each number until a match was found). Some participants did exhibit coordination of the LCM concept with the two numbers needed to compute it, improving their solution strategies (e.g., listed multiples of one number and divided each one by the other number until a whole number quotient was found). During post-interviews, nearly all candidates used prime factorization to find GCF and LCM values, and were able to explain why the questions required finding the GCF and LCM. This provided evidence of object level thinking.

Question 6 was initially a difficult problem for participants because it required them to make connections between the GCF and LCM concepts. None of the participants were able to do this prior to instruction, as illustrated by the low, *action* level of understanding (median value of 1.0). Following instruction, however, most were able to coordinate prime factorization and use the relationship between the GCF and LCM of two numbers to successfully solve the problem. Using relationships between concepts requires at least a *process with coordination* level of understanding, as participants' schemes for multiple concepts (GCF and LCM in this case) need to be combined. As such, their levels of understanding for Question 6 reflected this coordination, and for

some, the growing ability to recognize the GCF as an operationalized object (median value of 4.0).

Frequencies representing instances of participant thinking at the various levels of understanding were also computed prior to and following instruction. Table 5.4.3, below, shows a clear shift from low levels of understanding to higher levels of understanding following instruction. Prior to the number theory unit, 74.4% of all instances were coded at *action* and *process without coordination* levels, while only one instance was coded at an *object* level and no instances were coded at a *schema* level. As such, their thinking tended to be quite procedural, and any internalized images they had were superficial and limited.

Table 5.4.3: Total Frequencies and Percentages for each Level of Understanding on APOS

Code	Pre-Interview Frequencies	Post-Interview Frequencies
Action	16 (34.0%)	3 (6.3%)
Process without coordination	19 (40.4%)	5 (10.4%)
Process with coordination	11 (23.4%)	11 (22.9%)
Object	1 (2.1%)	19 (39.6%)
Schema	0 (0.0%)	10 (20.8%)
Total	47*	48

*One participant did not have enough time to attempt one pre-interview question

Following instruction, however, participants' conception levels shifted heavily toward higher levels of understanding. As the table shows, 60.4% of instances were coded at the *object* and *schema* levels, while only 16.7% were coded at the *action* and

process without coordination levels. This was evidence of more developed internalized notions of divisibility, GCF, and LCM conceptions. Specifically, participants routinely coordinated prime factors, provided correct reasoning behind their strategies, and explicitly stated relationships between concepts, specifically between prime factorization and divisibility.

In this chapter, interview data for each participant was presented. Descriptions of each level of understanding for each interview question were provided and used to construct individual profiles for each participant. These profiles contained detailed descriptions of the developing understanding of number theory topics that each participant exhibited before and after instruction. Results showed that all participants improved their understanding of number theory topics following instruction. Initial levels of understanding showed that most participants already possessed internalized mental images for divisibility, GCF, and LCM concepts, but such images were uncoordinated, superficial, and narrow. Improvement was primarily characterized by more coordinated and broader mental schemes. Some participants were able to use their coordinated mental schemes to make explicit connections between concepts, but such connection-making was not the norm.

CHAPTER VI

DISCUSSION

The purpose of this study was to describe pre-service elementary teachers' developing understanding of elementary number theory. As part of the study, participants' levels of understanding of number theory topics were identified prior to and following instruction. Changes in their levels of understanding were identified and described in detail. The sample for this study consisted of fifty-nine pre-service elementary teachers enrolled in three sections of an undergraduate mathematics education content course at a four-year private university in the northeastern U.S. The instruction that was part of this study involved a three-week number theory unit on topics including factors, prime factorization, divisibility, greatest common factor (GCF), and least common multiple (LCM). Five mathematical tasks addressing these topics were used in all three course sections. In each section, all five tasks were implemented. The pedagogy used included small-group work and whole-class discussions, and focused on developing participants' abilities to explain and justify mathematical procedures and concepts.

Data were gathered using both quantitative and qualitative instruments. The Number Theory Knowledge Test (NTKT) was a paper-and-pencil test that included questions adapted from Zazkis and Campbell's (1996a, 1996b) work with pre-service elementary teachers as they solved problems related to divisibility and prime factorization. All participants were administered the NTKT before and after instruction on the number theory unit. Both versions of the NTKT were identical except that the

numerical values within several of the questions on the post-test were changed. Individual clinical interviews were also conducted with a subsample of six participants prior to and following the number theory unit. The subsample was randomly selected based on prior achievement level. During each hour-long interview, participants were asked to solve a series of number theory problems by describing their thinking as they were solving each problem. Most questions were identical to the NTKT questions. The purpose of the interviews was to reveal participants' in-the-moment thinking as they attempted to solve each problem. As such, there was minimal intervention by the interviewer as participants articulated their thinking out loud.

Data from this study were analyzed using both quantitative and qualitative methods. The NTKT was scored using scoring criteria developed by the researcher (Appendix C). Additionally, participants' success at solving questions requiring conceptual understanding versus only procedural knowledge was quantified by computing abstract and concrete sub-scores for each test. All NTKT scores were used to identify participants' levels of understanding as described by Dubinsky's APOS model (Dubinsky & McDonald, 2001). Individual clinical interview data were transcribed and coded also using the APOS levels (Dubinsky & McDonald, 2001). Analyses of both the NTKT and interview data were synthesized in order to construct profiles of each interview participant. Each profile describes the methods the participant used to solve number theory problems and changes in their understanding of the underlying concepts following relevant instruction.

Section I of this chapter presents an explication of the study's major findings.

Section II includes the limitations of this study. Lastly, Section III offers recommendations for future research.

Section I: Study Findings

Although there is research investigating pre-service elementary teachers' understanding of number theory, no research prior to this study had made direct comparisons between individuals' understanding before and after instruction. In this study, Question 1 identified participants' levels of understanding of number theory topics prior to relevant instruction using participants' performances on the NTKT pre-test and their written and oral work on individual clinical pre-interviews. Question 2 identified participants' levels of understanding following relevant instruction using participants' performances on the NTKT post-test and their written and oral work on individual clinical post-interviews. Results of both analyses were compared to help describe salient changes in participants' understanding of number theory concepts following instruction. Below, a review of the theoretical framework is presented, followed by discussions of both research questions.

Dubinsky's (1991) APOS theory is the theoretical framework within which the study's data was analyzed and interpreted. It is a theory of learning mathematics where individuals make sense of mathematical problems or concepts by constructing mental *actions, processes, objects, and schemas*. An individual develops an understanding of a mathematical concept by first using previously formed schemas (i.e., prior knowledge) to construct actions. Actions are then interiorized to form processes, which are then

encapsulated to form objects. Actions, processes, and objects can then be organized to form stronger schemas of the concept under investigation. This progression can repeat itself as the individual is faced with higher-level concepts or more complex problem situations. APOS theory was originally created to help analyze university students' work with advanced mathematical topics; however, it has also been applied to students' understanding of elementary number theory topics (Zazkis & Campbell, 1996a; Brown, 2000).

An *action* refers to a transformation of an object that is sparked by cues that are external to the individual. The transformation can only be executed if step-by-step procedures are known. For example, if an individual must perform long division to identify divisibility, then he is most likely operating at an action level of understanding of divisibility. Once this action is repeated several times, the individual may be able to perform the same action mentally without having to run through all of its steps. For example, he may be able to identify a number's factors without performing long division (e.g., by examining the number's prime factorization). When this occurs, APOS theory states that this person has interiorized the action into a *process*.

As part of APOS theory, Dubinsky (1991) describes multiple ways in which the individual can construct a process. In addition to interiorization, he identifies *coordination* and *reversal* as two mental constructions that lead to new processes. Coordination is defined as the composition of two or more processes to form a new process. For example, in order to determine if 33 is a divisor of the number $3^2 \times 5 \times 11$ without having to do long division, one must coordinate the process of identifying

divisibility by 3 with the process of identifying divisibility by 11 in order to create a third process – the process of identifying divisibility by 33. Reversal is seen as reversing a process in order to create a new process. For example, the process of identifying the factors of a number (e.g., creating a ‘factor tree’) can be reversed to find a number with a set of known factors. Likewise, the process for identifying divisibility by a composite factor can be reversed to create new processes for identifying divisibility by the factors of the composite factor.

Once a process is interiorized and the individual can then reflect on the process enough to see it as a single entity, he has encapsulated the process into an *object*. For example, object level understanding of divisibility is characterized by the ability to see a factor of a number as a product of some combination of that number’s prime factors. Zazkis and Campbell (1996a) claim that individuals who reach the object level have achieved conceptual understanding. When actions, processes, and objects are connected and the individual can explicitly identify relationships between them, he has reached a *schema* level of understanding. The ability to explain how prime factorization and divisibility are related to each other is evidence of a schema level understanding of divisibility.

The four APOS levels are based on Piaget’s notion of *equilibration* (Piaget, 1985). When an individual undergoes new experiences, he can make sense of those experiences by assimilating them to what he already knows. If his prior knowledge is not robust enough to help him make sense of his experiences, then he may either reject those experiences as anomalies or attempt to revise his prior knowledge. The process of

revision is called accommodation. Piaget claims that understanding develops as the individual progresses toward a balance, or equilibrium, between assimilation and accommodation.

APOS theory is also closely related to Sfard's (1991) theory of concept development. According to Sfard, understanding a mathematical concept requires seeing the concept from dual perspectives: *operational* and *structural*. The former refers to associating a concept with its essential algorithms and processes, while the latter sees a concept as an abstract object. Understanding develops from operational to structural conceptions along three stages: *interiorization*, *condensation*, and *reification*. The first stage corresponds directly to the mental construction needed to progress from the action to process level in APOS theory. Condensation is an intermediate stage during which the individual compresses the interiorized process into a more manageable mental representation. This is similar to a process level conception that is nearing the object level within APOS theory because it is at this point when the individual begins to view processes in more efficient ways. Finally, reification is the final stage during which the individual can see the concept as a static, whole object. This is identical to the object level of understanding of APOS theory.

Analysis of interview transcripts and participants' work on the NTKT assessments before and after instruction revealed additional levels of understanding. Participants exhibited two types of process-level understandings as they solved number theory problems: *process without coordination* and *process with coordination*. Those at the process without coordination could not, for example, identify composite factors of a

number written in prime factored form or use the inherent relationship between the GCF and LCM of two numbers, A and B (i.e., $A \times B = \text{GCF} \times \text{LCM}$), to solve complex problems. In general, these participants' conceptions were limited to the extent that they could not coordinate multiple processes in order to efficiently solve problems. Those at the process with coordination level were able to make use of multiple processes in order to create new, more efficient ones. For example, they were able to identify a number's composite factors using prime factorization and use the relationship between the GCF and LCM of two numbers to solve problems.

Progress from a *process without coordination* to a *process with coordination* level represented the most salient change in participants' understanding during the study. The change can be explained using Piaget's notion of equilibration (Piaget, 1985). When faced with a problem that required coordinating processes, participants were unable to effectively assimilate the problem to their divisibility mental schemes prior to instruction. It was only following number theory instruction that participants were able to accommodate their mental schemes to account for the need to coordinate processes. In the context of Sfard's (1991) theory, participants who achieve process with coordination levels of understanding are at the condensation stage.

Question 1

Do participants exhibit identifiable levels of understanding of divisibility, greatest common factor, and least common multiple prior to instruction, and if so, what are they?

Prior to instruction, participants' understanding of number theory topics was predominately at the *process without coordination* level. Participants exhibited signs of interiorized mental images for divisibility, factor, greatest common factor (GCF) and least common multiple (LCM). For example, most participants did not need to perform long division in order to identify 5 as a factor of $2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$; they saw 5 in the prime factorization and knew that it was a factor. This is a surprising result considering prior research's disheartening descriptions of pre-service elementary teachers' tendencies to resort to computation and procedure (Zazkis & Campbell, 1996a, 1996b; Zazkis, 1998).

However, study participants' mental images were typically not robust enough to solve problems that required coordination of multiple processes. For example, given $2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$, most could not identify 51 as one of its factors; to do so would require the ability to coordinate divisibility by 3 and by 17. Instead, they attempted to assimilate the experience to what they already knew by resorting to long division or by erroneously claiming indivisibility simply because 51 was not visible in the number's prime factorization.

The results reported in Margaret, Jane, Zoe, and Christina's profiles reflect the same level of understanding of divisibility concepts as exhibited by the majority of the

entire sample. All four interviewees displayed primarily *process without coordination* levels of understanding of divisibility and factor because they could identify prime factors but not composite factors of numbers written in prime factored form. Zazkis and Campbell's (1996a) study of pre-service elementary teachers' conceptions of multiplicative structure also showed that coordinating prime factors to identify divisibility was extremely difficult for participants. The researchers concluded that participants did not have a good understanding of what a factor means.

None of the four interviewees described above displayed an adequate understanding of what a factor means prior to instruction. Christina, for example, conflated the meanings of prime factors and factors; when constructing a factor tree for 36, she claimed that 2 and 3 are its only factors. Additionally, she, Zoe, and Jane were only able to identify factors if they were visible in the prime factorization of a number. Margaret was the only interviewee out of the four participants who briefly considered the possibility that a factor could be composed of multiple prime factors. For example, in Question 1, despite identifying that the prime factorization of the given number contains 3×17 , which is 51, she could not identify divisibility of the number by 51 because "it's not just 3 and 17, it's 3^2 and 17^3 ." This superficial emphasis on only what is visible in the prime factorization of a number is what Brown and colleagues (2002) referred to as the "action of looking." It may be due to participants' experiences working with numbers represented solely in decimal form instead of prime factored form. Analysis of their NTKT pre-test revealed a tendency to rely on arithmetic computation when it was not necessary to do so. For example, when a number was presented in decimal form, most

were easily able to find all of its factors using factor trees. However, when the number was presented in prime factored form, participants either could not find all of its factors or resorted to arithmetic to convert the number into decimal form.

Ellen and Amanda, on the other hand, showed that they could coordinate prime factors in order to identify composite factors of numbers expressed in prime factored form. As a result, both were coded as being at a *process with coordination* level understanding of divisibility. Interestingly, both were able to identify composite factors that are the product of two distinct prime factors, but had some difficulty identifying composite factors that are the product of three or more distinct prime factors. This result is consistent with the findings of Zazkis and Gadowsky (2001), who claimed that pre-service teachers struggled to identify factors of numbers written in prime factored form because they were accustomed to seeing a factor as one of two numbers multiplied together. As such, when represented as a product of many numbers, a number's composite factors were extremely difficult for them to identify. Their understanding appeared to be at Sfard's (1991) interiorization stage and nearing the condensation stage of concept development. Ellen and Amanda saw a factor as the result of multiplying two prime factors, but they did not yet see it as a product of any combination of prime factors.

With regard to GCF and LCM concepts, the six interview participants exhibited a variety of levels of understanding. For the GCF concept, participants exhibited anything from *action* level to *object* level understanding. For the LCM concept, participants displayed *process without coordination* or *process with coordination* levels of understanding.

Understanding at the *action* level was characterized by the use of guess-and-check strategies to find the GCF or LCM of two given numbers. For example, Zoe found the GCF of 96 and 56 by randomly selecting numbers and checking if they were factors of 96 and 56. In this way, she was able to find a common factor, but not the greatest common factor. Christina and Zoe were the only interview participants to exhibit this level of understanding, and only on a GCF question. Zazkis and Campbell (1996a) identified “tedious and time-consuming ‘hit and miss’ strategies” (p. 561) as the result of a dependence on carrying out procedure without any conceptual guidance. It is unclear why this level of understanding was not exhibited for the LCM question; one possibility is that finding consecutive multiples requires knowledge of repeated addition, which participants already had. However, a number’s factors cannot be found by a similar type of procedure, say, repeated subtraction.

Process without coordination level understanding involved using more systematic methods, such as listing out the factors or multiples of each number separately and stopping when a match was found. Margaret used this strategy on both GCF and LCM questions. This was a popular strategy among the interviewees, as Jane used it to find the GCF of two numbers and Zoe used it to find the LCM of two numbers. Brown and colleagues (2002) observed this strategy, which they termed the *set intersection* strategy, as one of three that participants used to find the LCM of two numbers. Several participants at this level searched for an easier way to find the LCM, but they could not find one. When trying to find the LCM of 135 and 80, Zoe claimed that the LCM must be less than the product of 135 and 80, but could not explain why and so reverted back to

listing multiples. Margaret also paused to consider if finding the GCF of 135 and 80 would help her find their LCM; she, too, returned to listing multiples. It appears that these participants had interiorized the action of finding the GCF and LCM of two numbers, but had not yet coordinated the relationship between GCF and LCM in order to find more efficient methods. Despite encountering disequilibrium every time they attempted a more efficient strategy, their recognition of the possibility of more efficient strategies was promising.

At the *process with coordination* level, participants began to make use of the relationship between prime factorization, GCF, and LCM. For example, Amanda and Ellen found the prime factorizations of 96 and 56 and used them to quickly find their GCF. To find the LCM of 135 and 80, Amanda found multiples of one of the two given numbers and periodically divided it by the other given number, while Ellen found the product of the two given numbers and then divided it by their GCF. While Ellen's approach appears to be at a high level of understanding, she was not sure if her strategy worked so she reverted back to listing out multiples of each number separately to check her result. This was evidence that Ellen had not yet encapsulated LCM as an object; she still viewed LCM as the result of a process of listing multiples, despite coordinating the relationship between LCM and GCF.

Analysis of the NTKT pre-test revealed that most participants were able to correctly identify whether a word problem required computing the GCF versus the LCM (80% and 76%, respectively). However, their ability to explain their choices was weak. Many participants chose the GCF if the word problem asked for a number that was less

than the two given numbers; they chose the LCM if the word problem asked for a number that was greater than the two given numbers. Sometimes, participants even used key words as triggers. For example, Zoe identified a word problem as requiring the GCF because the word “largest” was in the question. Zazkis (2002) found that pre-service teachers often rely on informal vocabulary to communicate mathematical ideas because formal vocabulary does not always clarify the underlying concept for them. As such, participants such as Zoe appeared to construct word associations with regard to GCF and LCM concepts because their understanding of these concepts was not yet robust enough. These results suggest that, prior to instruction, participants’ prior knowledge was sufficient to help them identify the need for GCF and LCM, but insufficient to help them explain why.

Question 2

Do participants’ levels of understanding of divisibility, greatest common factor, and least common multiple change after the course of a unit on number theory, and if so, how?

Study participants exhibited significant changes in their levels of understanding following instruction. Analysis of the NTKT post-test results showed a significant and large improvement on overall performance, as well as on questions requiring procedural knowledge and conceptual understanding. On the post-test, their mean scores on concrete questions (i.e., questions requiring only procedural knowledge) improved from 37.6% to 81.0%. Their mean scores on abstract questions (i.e., questions requiring

conceptual understanding) improved from 24.6% to 68.3%. This supports the claim that participants' understanding of number theory concepts and relationships improved.

A closer analysis of the NTKT results showed that participants' performance on divisibility-related questions improved. Mean scores on Questions 1, 2, 4, and 5 increased from a weighted mean of 34.6% to 81.2%. Participants were better able to identify prime and composite factors and non-factors of numbers represented in prime factored form. Unlike their work on the pre-test, participants did not revert to converting the prime factorization of numbers into their decimal form. Instead, they inferred divisibility by composite factors by recognizing that combinations of prime factors could be multiplied together to form composite factors. Overall, participants' ability to use prime factorization as a tool for solving and explaining divisibility problems improved.

NTKT post-results also showed that participants' performance improved on GCF and LCM questions. Mean scores on Questions 6 and 7 increased from a weighted mean of 18.1% to 58.3%. This score increase is partly due to participants' using prime factorization as the primary tool for representing GCF and LCM values; on the pre-test, participants were more likely to construct long, inefficient lists of factors or multiples to compute these values. Additionally, participants' explanations of GCF and LCM concepts on Questions 7a and 7b were better aligned with the definitions of GCF and LCM; they talked more readily about why each word problem implied the need to find a factor or a multiple, and provided some reasoning as to why the problem required finding a common factor or a common multiple. On the pre-test, these explanations were often vague and based on participants' intuition of whether the answer should be greater than

or less than the given values in each problem.

Similar results are evident upon analysis of the post-interview transcripts of the six interview participants. The most salient change in interview participants' understanding was from a *process without coordination* level to an *object* level. Analysis of post-interviews revealed that this change was primarily due to a newfound ability to coordinate multiple processes. For example, participants were able to combine divisibility processes to identify composite factors or use the relationships between prime factorization, GCF, and LCM processes to find the GCF and LCM of two numbers. Many participants also developed the ability to provide adequate reasoning for their work. Some even began to articulate generalized statements that illustrated their understanding of number theory relationships. Table 6.1.1 shows the number of pre- and post-interview questions for which each participant exhibited each APOS level of understanding. Participants frequently exhibited low levels of understanding prior to instruction – on 35 out of 47 occasions they showed no signs of coordination. During post-interviews, however, an inability to coordinate processes occurred only 8 times. Instead, participants displayed the ability to see concepts as abstract objects or as connected to other objects on 29 out of 47 occasions.

Table 6.1.1: Interview Participants' Levels of Understanding on APOS

	Action		Process without Coordination		Process with Coordination		Object		Schema	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Christina*	4	1	2	1	1	3	0	2	0	1
Margaret	2	2	6	1	0	2	0	3	0	0
Zoe	2	0	6	1	0	3	0	3	0	1
Jane	3	0	4	1	1	1	0	6	0	0
Amanda	3	0	0	1	4	0	1	3	0	4
Ellen	2	0	1	0	5	2	0	2	0	4
Total	16	3	19	5	11	11	1	19	0	10

*Christina did not attempt pre-interview Question 8

Following instruction, interview participants' understanding of divisibility no longer attended to superficial features of prime factorization. For example, on Question 3, Christina and Zoe were able to find most factors of $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$ by coordinating divisibility by individual factors. Both referred to factors as combinations of number that were already visible in the prime factorizations of N and R . This change corresponds to the condensation stage of concept development (Sfard, 1991). Instead of attending to multiple divisibility processes individually (e.g., determining if 13 and 19 are factors of R), they combined and repackaged these processes into one manageable process (e.g., determining if 13×19 is a factor of R). However, they could not find one factor that was composed of three distinct prime factors (i.e., $2 \times 3 \times 5$). It may be that their notion of factor was limited to products of two prime factors, or possibly that they did not keep a clear record of all of the combinations. The fact that both missed the exact same factor suggests the former is a more plausible reason. It is also supported by prior research of pre-service teachers' tendency to think of a factor as one of exactly two numbers multiplied together (Zazkis & Gadowsky, 2001).

Jane, Ellen, and Amanda were also able to coordinate prime factors. The difference in their work was that they explicitly recognized that a factor is any combination of a number's prime factors. For example, the factors of N and R in Question 3 can be product combinations of one, two, three, or four prime factors. Participants who recognized this idea used tree diagrams to find all combinations. Their understanding was coded at the *object* level because they had encapsulated the concept of factor into a single entity – as a combination of prime factors. This understanding made it easy for them to identify any factors using prime factorization.

Ellen and Amanda exhibited *schema* level understanding with questions that focused on divisibility. On Questions 1 and 2, they articulated the relationship between prime factorization and divisibility in order to explain their work. When a question asked them to determine if a number in prime factored form was divisible by some number, Amanda generalized, “if I can find their prime factorizations within N , then N would be divisible by the numbers.” When asked to construct a number that must be divisible by 18 and 15, Ellen provided the following explanation: “Um, because in order to be divisibly by 18 and by 15, this number must include all of the prime factors of 18 and 15.” These comments show that Ellen and Amanda understood divisibility as a relationship between divisibility and prime factors. Zazkis and Campbell (1996a) observed the importance of this relationship in helping their participants develop *schema* level understanding.

In this study, participants' understanding of divisibility appeared to be determined by their notion of factor. Seeing a factor as a combination of prime factors was a

powerful tool in helping participants identify divisibility without resorting to long division. In order to develop this notion of factor, participants first needed to coordinate divisibility processes to form new processes for determining composite factors. Even after coordination was successful, some participants like Christina and Zoe were not completely successful at identifying factors, especially factors composed of a product of three or more prime factors. Participants who were able to identify all factors of a number in prime factored form all referred to a factor as a combination of any number of prime factors. This result suggests that encapsulation of divisibility may result from thinking of a factor as *any combination of X prime factors within the prime factorization of a number*, where X is less than or equal to the number of prime factors in the prime factorization of the number. Such an interpretation of factor is typically not offered in pre-service elementary mathematics textbooks; instead, textbooks often describe factor as one of two or more integers multiplied together to form a new integer. The study results described here suggest a potential need to rethink how factor is treated in textbooks.

Interview participants' conceptions of GCF ranged from *process with coordination* to *schema* levels. On Question 7 of the post-interview, participants were asked to find the dimensions of the largest square tile that could perfectly tile a rectangular floor with dimensions 84 feet by 76 feet. All participants found the prime factorizations of 84 and 76, coordinated them using a Venn diagram, and found the correct answer. They all recognized that the overlap region of the Venn diagram represents the GCF. This method was a substantial improvement over the strategies they used prior to instruction because it allowed participants to use a simple image of an

overlapping region to represent the GCF concept. This is similar to Sfard's (1991) condensation stage, in which the individual squeezes his knowledge of a procedure into a more compact set of steps or images to help him make sense of the procedure.

However, Margaret, Zoe, and Christina were still unable to explain why the GCF was needed to solve the problem. Their reasoning was primarily based on the fact that the problem asked for the *largest* square tile. The false association between the words "largest" and "greatest" appeared to be quite strong for half of the interview participants. In fact, Zoe used the same rationale during her pre- and post-interviews. These results are evidence that an ability to find the GCF using a Venn diagram does not guarantee deep conceptual understanding of the GCF, and those participants may have simply learned this procedure without carefully considering its meaning. As a result, these three participants were coded at a *process with coordination* level of understanding of GCF. Their ability to make use of the relationship between prime factorization and GCF reflected an improvement in their understanding of the procedure; their inability to articulate a clear explanation of the concept prevented them from reaching an *object* level of understanding.

Ellen, Amanda, and Jane were able to provide an explanation for why Question 7 required finding the GCF. They explained that the tile dimension had to be a factor of 84 and 76 in order to "fit" or "cover" the floor. Jane elaborated that the tile dimension must be a factor of both 84 and 76 "so it can go up X number on one wall" and "so it can go up X number on the other." Such explanations were evidence of *object* level understanding of the GCF concept. To these three participants, the GCF was no longer seen as the

result of a computation, but rather as a single object that when repeated a certain number of times would span 84 and 76. In addition to providing explanations of the GCF concept, Ellen and Amanda also explained why their solution had to be the GCF of 84 and 76. They stated that 4 must be the GCF of 84 and 76 because the remaining prime factors of 84 and 76 are not common to each other. According to APOS theory (Asiala et al., 1996), processes and objects can be organized to form schemas; Ellen and Amanda were making connections between their notions of prime factorization and GCF to better organize their understanding of GCF. As such, they were building a schema for the GCF concept.

All six interview participants displayed *object* level understanding of the LCM concept because their explanations showed that they understood the meaning of the LCM concept. Question 8 on the post-interview asked participants to find the number of seconds it would take two runners to stop running if it took them 80 and 105 seconds, respectively, to complete a single lap and both runners would stop only when completing a lap at the exact same time. Each participant used prime factorization and a Venn diagram to find the LCM and each knew that the LCM was represented by the product of all prime factors across the three regions of a Venn diagram. However, the ability to execute procedures is not sufficient for conceptual understanding (Dubinsky, 1991; Sfard, 1991). In order to show *object* level understanding, each participant provided an explanation of one of the following: 1) why the LCM was needed to solve Question 8, or 2) why the LCM is computed by multiplying across the entire Venn diagram. Four of the six participants explained the former, while the other two explained the latter. Nobody

was able to explain both items.

It appeared that explaining the calculation of the LCM was more difficult for participants; it required an understanding of the relationship between LCM, GCF, and prime factorization. While one might expect participants who provided this explanation to have a well-developed schema for LCM, neither participant was able to explain why the LCM was the relevant concept in Question 8. For example, Amanda was able to provide a rigorous explanation of the LCM procedure, but her reasoning for why the LCM was even necessary to begin with was lacking: “you know you’ll be looking for the least common multiple in this problem because you’ll be creating a number that’s bigger than both 80 and 105.” This result was surprising, but it suggests that a strong understanding of the role of prime factorization in solving LCM related problems can potentially make up for some misconceptions that may persist. Brown and colleagues (2002) conducted interviews with pre-service elementary teachers and showed how reasoning about multiplicative structure can help one make sense of the LCM.

Interview transcript analysis also revealed that participants’ developing understanding was consistent across different number theory topics. Their knowledge improved from *process* to *object* levels regardless of the number theory topic under investigation. Participants’ median level of understanding of divisibility, GCF, and LCM related questions during pre-interviews was at approximately the *process without coordination* level. During post-interviews, their median level of understanding of both sets of concepts was primarily at the *process with coordination* and *object* levels. As such, participants’ understanding developed consistently across all topics under

investigation.

This result is not surprising considering the interrelatedness of these number theory concepts. Prime factorization and divisibility are directly connected to notions of divisor, factor, and multiple, and understanding of prime factorization has been suggested as necessary to better understand the procedures for finding greatest common factors and least common multiples (Brown et al., 2002; Campbell, 2000). As such, participants' developing understanding of divisibility and factors may be connected to their ability to make sense of GCF and LCM concepts.

Surprisingly, however, sample-wide results on the NTKT pre- and post-test showed that participants exhibited greater levels of understanding on divisibility tasks than on GCF or LCM tasks. Following instruction, participants' median level of understanding of a divisibility task (Question 2b) improved from *process without coordination* to *object* level while their understanding of a GCF and LCM task (Question 6) improved from *action* to *process with coordination* level. Scoring of the remaining NTKT questions showed similar results, suggesting that participants' understanding of GCF and LCM concepts was consistently weaker than their understanding of divisibility and factor tasks. One possible explanation for this conflicting result is that interview participants had greater opportunities to reflect on their work because of the interviewer's prompts to explain their thinking out loud. Prior research recommends providing students opportunities to reflect on and explain their mathematical thinking (Smith & Stein, 2011; Hiebert et al., 1997).

Another finding of the study was that participants' improvement on the NTKT

assessments was the same regardless of question type. In other words, changes in scores on abstract questions were similar to changes in scores on concrete questions. This suggests that participants showed similar improvements in their ability to make sense of underlying meanings and to execute straight-forward procedures. Although it is unclear which type of knowledge developed first, APOS theory describes the construction of cognitive structures that can account for both types of knowledge. Following instruction, participants' conceptual and procedural knowledge improved because their schemas became more connected. As schemas develop, procedures are interiorized and coordinated to make their execution more efficient (Asiala et al., 1996). Once procedures are interiorized, they can be encapsulated to form more abstract concepts; these concepts can further develop one's procedural knowledge as they are related to other concepts (Zazkis & Campbell, 1996a). This result is supported by prior research, which shows that improved conceptual understanding can also lead to improved procedural knowledge (Hiebert & Wearne, 1993).

The current study's results provide evidence that pre-service elementary teachers can develop a structured and connected understanding of a variety of number theory topics. These results contradict earlier research on pre-service elementary teachers' understanding of number theory that stated that the majority of its participants did not achieve higher levels of understanding (Zazkis & Campbell, 1996a, 1996b; Zazkis, 1998). This study implies that coordination of processes, encapsulation of the notion of factor, and attention to prime factorization are ways in which pre-service elementary teachers may be able to develop such understanding. The results of the interview and

NTKT analyses are quite promising in light of prior research that claimed that reaching the *object* level of understanding of mathematics is extremely difficult (Asiala et al., 1996; Sfard, 1991).

Given the significant improvements in participants' understanding, it is reasonable to conjecture that number theory instruction played a key role. Study materials consisted of mathematical tasks written at high levels of cognitive demand (Stein, Smith, Henningsen, & Silver, 2000). Additionally, great care was taken to implement these tasks in ways that preserved the high levels of cognitive demand. With a better understanding of how participants' knowledge of number theory grows, future research should investigate the role of mathematical tasks in fostering pre-service elementary teachers' understanding of number theory topics.

A common theme throughout the number theory tasks was the use of prime factorization as a tool for solving a variety of divisibility, factor, GCF, and LCM problems. Following instruction, participant explanations included more discussion around prime factorization than any other number theory topic, especially for participants exhibiting higher levels of understanding. Prime factorization may represent the underlying structure needed to make connections between number theory concepts and help participants reach *object* and *schema* level understandings. This claim is not surprising for two main reasons. First, approximately 65% of the questions across all five mathematical tasks that participants worked on during the three-week instructional period focused on or used prime factorization. Prime factorization was introduced in the second task as a tool for identifying factors. It was then used as the primary number

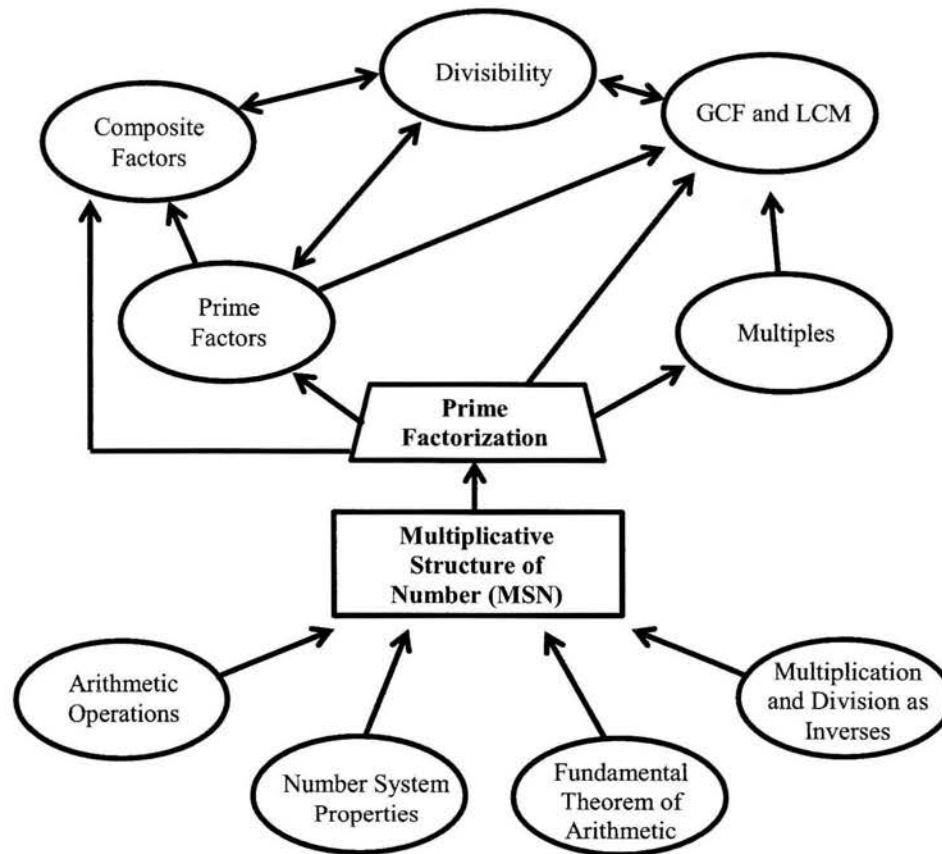
representation when discussing divisibility in the third task. In the fourth and fifth tasks, methods for finding the GCF and LCM of two numbers were tested and explained using prime factorization. And second, prior research identifies prime numbers and multiplicative structure as foundational concepts, on which many number theory topics such as divisibility, GCF, and LCM, are based (Zazkis & Liljedahl, 2004; Brown et al., 2002; Zazkis & Gadowsky, 2001).

The prevalence of prime factorization in participants' thinking about number theory topics suggests that it holds a central position in what Campbell and Zazkis (2002) describe as the number theory conceptual field. Based on the results of this study and the nature of number theory content, it is possible to construct a preliminary map of the major number theory concepts and the relationships between them. Similar to Ma's (1999) knowledge packages, this map identifies the foundational concepts of the field and shows relationships to secondary concepts and procedures. Similar to Dubinsky's (1991) genetic decomposition, this map must be revised as research on elementary number theory continues to reveal new information on concept development and student understanding. Figure 6.1.1 displays the number theory map below.

Multiplicative structure of number (MSN) is the foundational topic within the field of number theory. This study follows Zazkis and Campbell's (1996a) definition of multiplicative structure as "conceptual attributes and relations pertaining to and implied by the decomposition of natural numbers as unique products of prime factors" (p. 541). Although this study has not discussed the Fundamental Theorem of Arithmetic (FTA) in great detail, it is important to note that the FTA is inseparable from the MSN. The

uniqueness property of number under the FTA dictates that all numbers can be uniquely represented as a product of prime numbers. The importance of this property in developing one's understanding of divisibility cannot be overstated, yet past studies have shown that pre-service elementary teachers struggle to accept the FTA (Zazkis and Campbell, 1996b).

Figure 6.1.1: Elementary Number Theory Map



At the bottom of the map, the four ellipses with arrows pointing toward the MSN represent prerequisite concepts necessary in order to make sense of multiplicative structure (Brown et al., 2002). Above MSN, prime factorization represents the key

operational tool that can be applied to all other number theory topics. Prime factorization leads to an understanding of prime and composite factors, which leads to an understanding of divisibility. This study found that individuals used prime factorization to identify prime factors before composite factors, and that their ability to coordinate prime factors was a key component of their eventual understanding of divisibility. As implied by the double arrows, a developing understanding of divisibility can also strengthen one's prime and composite factor schemes. Prime factorization also leads to an understanding of GCF and LCM concepts either directly or via knowledge of prime factors and multiples. The double arrow between the divisibility and GCF and LCM ellipses implies that these two sets of topics support each other's development. Study participants who displayed high level understandings of one topic tended to display similarly high levels for the other topics.

Summary of Conclusions

1. The participants in this study exhibited low levels of understanding of divisibility and factor concepts before participating in classroom instruction. They possessed interiorized notions of divisibility, which allowed them to identify prime factors or simple composite factors (i.e., powers of prime factors). However, their inability to coordinate distinct prime factors prevented them from identifying most composite factors.
2. Study participants exhibited low levels of understanding of GCF and LCM concepts

prior to instruction. They used tedious computational strategies, and even when they considered the possibility of more efficient strategies, their lack of understanding prevented them from pursuing those strategies.

3. Prior to instruction, the form in which numbers were represented created tremendous difficulty for participants. Participants exhibited procedural fluency when numbers were presented in decimal form, but showed deep conceptual gaps when presented with numbers in prime factored form.
4. Following number theory instruction, participants exhibited high levels of understanding of divisibility and factor concepts. This change was primarily the result of their ability to coordinate multiple processes to create new processes, the development of a generalized factor scheme, and their ability to use prime factorization effectively to solve problems. These results contradict prior research that described pre-service elementary teachers' weak knowledge of divisibility and multiplicative structure even following instruction (Zazkis & Campbell, 1996a, 1996b, Zazkis, 1998).
5. Following instruction, participants displayed high levels of understanding of GCF and LCM concepts. This change was primarily due to their increasing fluency with using prime factorization. It appeared, though, that the ability to use prime factorization to solve GCF and LCM problems did not guarantee that they could provide correct

reasoning for their work. Participants who were able to provide correct reasoning appeared to also have deeper understanding of factor and divisibility concepts.

6. Prime factorization appeared to be a critical component of participants' developing understanding of number theory. Prior to instruction, they did not attend to it at all; following instruction, they used prime factorization not only to solve problems but also to explain solutions and identify relationships between concepts.
7. Participants' developing understanding of number theory topics was characterized by similar and significant improvements in both conceptual and procedural understandings. While both types of understandings improved, participants continued to exhibit relatively stronger procedural skills than abstract skills following instruction.

Section II: Limitations of the Study

The study findings must be evaluated in the context of the study design. As such, several limitations must be considered, such as a lack of generalizability of the findings to all pre-service elementary teachers or to other content areas. The following limitations are described in detail.

1. NTKT and interview questions were written to be nearly identical to each other before and after instruction. As such, participants' success on post-test and post-interview questions may have been influenced by their prior experience with these

questions on the pre-test or pre-interview.

2. Videotapes of classroom instruction were not analyzed as part of this study, so findings were based solely on participants' work prior to and following instruction. As such, these results cannot be used to infer with complete certainty any changes in their understanding that may have occurred during instruction.
3. The classification of NTKT and interview questions as abstract versus concrete may not have been the most effective classification system. Given the limited amount of time participants were given to complete each instrument, some concrete questions may have required some conceptual understanding in order to answer them efficiently. Additionally, some abstract questions required procedural knowledge. Therefore, findings representing participants' ability to solve each type of question may not have fully distinguished between participants' abilities to make connections between concepts and to execute procedures.
4. Though sufficient to establish 80% power, the study's sample size ($n = 59$) was small. Additionally, seven participants' data were not included in inferential analyses because they did not consent to release their SAT[®] or ACT[®] scores. Therefore, the study's findings cannot be generalized to other cohorts of pre-service elementary teachers.

5. Reliability was performed on an earlier version of the NTKT assessments.

Although most questions were well aligned, some questions were changed. As such, the study's findings may not be completely reliable.

Section III: Recommendations for Future Research

Based on the study's findings and limitations, the following are recommendations for future research.

1. Since the study sample was small, replicating the study with a larger sample will help confirm or elaborate on some of the findings. Specifically, the larger sample should include mathematics education content courses from multiple university teacher preparation programs. In this way, not only are multiple contexts included in the study, but the conflicting role of teacher educator and researcher is avoided (Hatch, 2002).
2. The study findings revealed significant improvements in participants' understanding of number theory topics following instruction that conflict with prior research's descriptions of pre-service elementary teachers' prevailing misconceptions. Future research could investigate the efficacy of the number theory tasks used in this study in order to determine if and in what ways the tasks contributed to the positive findings.
3. The study findings were predicated on analysis of participant work before and after instruction. In order to provide more detail about their changing

understandings, their work during classroom instruction could be analyzed. The current study collected approximately twenty-five hours of classroom video of participants working on number theory tasks. Identifying critical classroom moments, including the role of tasks, instructor questions and prompts, and student interactions could be studied in order to see if and in what ways they contributed to individual participants' developing understanding.

4. This study represented a first attempt to describe participants' developing understanding of number theory topics. A more fine-grained analysis of each APOS level of understanding can provide more detail about how understanding develops. For example, this study's analysis revealed *process* sub-levels of understanding. Are there sub-levels of understanding at the *action*, *object*, or *schema* levels? If so, what do these sub-levels reveal about pre-service elementary teachers' understandings of number theory? A refinement of the APOS model may also have implications for studies of a broad range of mathematics topics.
5. Since the ultimate purpose of this study is to improve the quality of instruction for elementary and middle school students, study participants can be followed into the elementary or middle school classroom to examine if their understanding of number theory impacts their instruction. Prior research has shown that a teacher's mathematical knowledge for teaching does correlate with her quality of

instruction (Hill et al., 2008). As such, does a teacher's APOS level of understanding have any effect on the quality of her teaching of number theory topics? Implicit in this research is the need to re-examine participants' understanding of number theory immediately prior to their teaching, as it may have changed during the two or three years between their university coursework and classroom teaching.

APPENDIX A
NUMBER THEORY KNOWLEDGE TEST

Number Theory Knowledge Test (Pre-Test)

Note: The formatting of this test was different when presented to participants

Directions: Show your work for each question. All questions refer to non-negative integers. Calculators are not allowed on this test.

1. Consider the number $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$. Without calculating the value of N, answer each question by circling one of the three possible options:

a) Is N divisible by 5?	Yes	No	I am not sure
b) Is N divisible by 7?	Yes	No	I am not sure
c) Is N divisible by 4?	Yes	No	I am not sure
d) Is N divisible by 51?	Yes	No	I am not sure
e) Is N divisible by 105?	Yes	No	I am not sure

2. a) How many factors does 45 have? Show your work.
- b) How many factors does $5^2 \times 7$ have? How do you know?
- c) Consider two numbers, $N = 3 \times 6 \times 7$ and $R = 11 \times 13 \times 17$. Do you think N has more factors than R, less factors than R, or the same number of factors as R? Explain your choice.
3. a) In your own words, what is a prime number?
- b) Is 31×17 a prime number? Why or why not?
4. Brent came up with a possible divisibility rule for 18. He claims that any number is divisible by 18 if it is divisible by 3 and by 6. For example, 36 is divisible by 3 and

36 is divisible by 6. So it must be divisible by 18.

- a) Is Brent's rule correct?
 - b) If it is correct, explain how you know. If it is incorrect, explain why not and provide a correct divisibility rule for 18.
5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Show your work. *Note: you may leave your answer in factored form.*
6. The prime factorization of the greatest common factor of two numbers, A and B , is $2^3 \times 5$. The prime factorization of the least common multiple of the same two numbers is $2^5 \times 5^2$. Give all possible values for A and B . Show your work. *Note: you may leave your answer in factored form.*
7. Without solving, decide if each situation below requires finding the greatest common factor (GCF), the least common multiple (LCM), or neither with respect to the numbers given. Briefly explain each choice.
- a) Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 96 feet by 56 feet. What are the dimensions of the largest square tile that the school can buy from Alpaca in order to completely tile the cafeteria floor?
 - b) Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 135 seconds. Jean runs one complete lap in 80 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

Number Theory Knowledge Test (Post-Test)

Note: The formatting of this test was different when presented to participants

Directions: Show your work for each question. All questions refer to non-negative integers. Calculators are not allowed on this Test.

1. Consider the number $N = 3^2 \times 5^4 \times 11 \times 17^3$. Without calculating the value of N , answer each question by circling one of the three possible options:

a) Is N divisible by 3?	Yes	No	I am not sure
b) Is N divisible by 35?	Yes	No	I am not sure
c) Is N divisible by 19?	Yes	No	I am not sure
d) Is N divisible by 75?	Yes	No	I am not sure
e) Is N divisible by 51?	Yes	No	I am not sure

2. a) How many factors does 36 have? Show your work.
b) How many factors does $3^2 \times 5^2$ have? How do you know?
c) Consider two numbers, $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$. Do you think N has more factors than R , less factors than R , or the same number of factors as R ? Explain your choice.
3. a) In your own words, what is a prime number?
b) Is 19×23 a prime number? Why or why not?
4. Brent came up with a possible divisibility rule for 18. He claims that any number is divisible by 18 if it is divisible by 3 and by 6. For example, 36 is divisible by 3 and 36 is divisible by 6. So 36 must be divisible by 18.

- a) Is Brent's rule correct?
- b) If it is correct, explain how you know. If it is incorrect, explain why not and provide a correct divisibility rule for 18.
5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Show your work. *Note: you may leave your answer in factored form.*
6. The prime factorization of the greatest common factor of two numbers, A and B , is $3^2 \times 7^2$. The prime factorization of the least common multiple of the same two numbers is $3^4 \times 7^3$. Give all possible values for A and B . Show your work. *Note: you may leave your answer in factored form.*
7. Without solving, decide if each situation below requires finding the greatest common factor (GCF), the least common multiple (LCM), or neither with respect to the numbers given. Briefly explain each choice.
- a) Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 84 feet by 76 feet. What are the dimensions of the largest square tile that the school can buy from Alpaca in order to completely tile the cafeteria floor?
- b) Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 80 seconds. Jean runs one complete lap in 105 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

APPENDIX B
SCORING RUBRICS FOR NUMBER THEORY KNOWLEDGE TEST

**Number Theory Knowledge Test (Pre-Test)
Scoring Rubric**

Total Points = 50

1. Consider the number $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$. Without calculating the value of N , answer each question by circling one of the three possible options:

a) Is N divisible by 5?	Yes	No	I am not sure
b) Is N divisible by 7?	Yes	No	I am not sure
c) Is N divisible by 4?	Yes	No	I am not sure
d) Is N divisible by 51?	Yes	No	I am not sure
e) Is N divisible by 105?	Yes	No	I am not sure

Purpose:

Assess participants' ability to determine divisibility properties by attending to a number's prime factorization as opposed to performing long division. Increased understanding of divisibility is marked by connection-making between divisibility and prime factorization (Zazkis & Campbell, 1996a, 1996b).

Suggested Solutions:

- a) Yes
- b) No
- c) Yes
- d) Yes
- e) No

Scoring Criteria (5 points):

5 points – Earns 1 point for each correct response

- Give 0 points instead for each incorrect response.

2. a) How many factors does 45 have? Show your work.

Purpose:

This question assesses participants' knowledge of what a factor is, as well as their ability to compute factors. Participants typically use a factor tree to identify a number's factors.

Suggested Solutions:

There are 6 factors. The factors are: 1, 3, 5, 9, 15, and 45. By using a factor tree or listing out factor pairs, for example, one can find all factors of the number.

Scoring Criteria (4 points):

1 point – Correctly identifies the number of factors.

2 points – Displays accurate supporting work, such as:

- Drawing a factor tree for 45, OR
- Showing each division separately, OR
- Listing out factor pairs (e.g., 3×15), OR
- Finding the prime factorization of 45 and using it to list out all possible prime factor compositions, OR
- Using the formula *Number of factors* = *(prime factor exponent + 1) x (second prime factor exponent + 1)* with an explanation of why the formula works.

1 point – Shows or explains how supporting work leads to solution (e.g., by listing the factors).

b) How many factors does $5^2 \times 7$ have? How do you know?

Purpose:

This question is similar to part (a), but serves the added purpose of assessing participants' ability to use prime factorization as an efficient method for identifying factors. Unlike part (a), which most participants can solve easily, the prime factor representation of part (b) is challenging for them and requires some conceptual understanding of the relationship between factors and prime factors.

Suggested Solutions:

There are 6 factors. The factors are: 1, 5, 7, 5^2 , 5×7 , and $5^2 \times 7$. Using the prime

factorization of the number in question, one can find all of its factors by finding all combinations of its prime factors, since a number's factors are products of the number's prime factors.

Scoring Criteria (4 points):

1 point – Correctly identifies the number of factors.

2 points – Displays accurate supporting work, such as:

- Uses the prime factorization to determine each factor (e.g., $5^2=25$ and $5 \times 7=35$ are factors), OR
- Uses the formula *Number of factors* = *(prime factor exponent + 1) x (second prime factor exponent + 1)* with an explanation of why the formula works.
- If the prime factorization is first converted to a decimal number and then one of the following methods is used to find the factors, give 1 point instead:
 - Drawing a factor tree for 175, OR
 - Showing each division separately, OR
 - Listing out factor pairs (e.g., 25×7), OR

1 point – Shows or explains how supporting work leads to solution (e.g., by stating that the factors of a number are combinations (or products) of the number's prime factors).

- c) **Consider two numbers, $N = 3 \times 6 \times 7$ and $R = 11 \times 13 \times 17$. Do you think N has more factors than R, less factors than R, or the same number of factors as R? Explain your choice.**

Purpose:

This question assesses participants' ability to apply their knowledge of prime factorization and factors. Without the use of a calculator, this question poses a challenge to participants who are not aware of the relationship between prime factors and factors. This question distinguishes between those who feel the need to compute each number's factors and those who can identify differences in the number of factors based purely on the multiplicative structure of each number (i.e., based on their respective prime factorizations).

Suggested Solutions:

The prime factorization of N can be used to compare to the prime factorization of R. R is already presented in prime factored form with 3 distinct primes (where

each prime is raised to the first power). N, however, can be further decomposed ($6=2 \times 3$) into a product of 3 distinct primes where one of the primes is squared. Thus, N can be represented as a product of 4 prime numbers. This implies that a greater number of combinations of primes can be constructed from N's prime factorization than from R's. Thus, N has more factors than R.

Scoring Criteria (4 points):

1 point – Correctly states that N has more factors than R.

1 point – States or shows that N can be further factored because $6=2 \times 3$ while R is completely factored already.

2 points – States or shows that N has a greater number of factor combinations possible than R since it has more numbers in its prime factorization (or, because one of its prime factors is squared).

- If instead of the statement directly above the participant lists all prime factor combinations for N and R, give 1 point instead.
- If the factorizations of N and R are first converted to decimal values and then all of their factors are computed, give 0 points instead.

3. a) In your own words, what is a prime number?

Purpose:

This question assesses participants' knowledge of the precise definition of prime number. Some might provide an imprecise definition, such as "any number whose only factors are 1 and itself," which would erroneously imply that 1 is, itself, a prime number.

Suggested Solutions:

A prime number is any number that has exactly two distinct factors. Alternatively, any number (except 1) whose only factors are 1 and itself.

Scoring Criteria (2 points):

2 points – provides one of the suggested solutions above.

- If participant provides an imprecise definition, such as "any number whose only factors are 1 and itself" give 1 point instead.
- If participant provides an incorrect definition or no definition at all, give 0 points instead.

b) Is 31×17 a prime number? Why or why not?

Purpose:

This question assesses participants' ability to identify a prime number via its prime factorization. Some pre-service teachers erroneously believe that a number is prime if it is the product of two or more prime numbers.

Suggested Solutions:

31×17 is not prime because it has more than two distinct factors (1, 17, 31, and 31×17).

Scoring Criteria (2 points):

2 points – Provides suggested solution.

- Give 1 point instead if participant makes a correct claim but provides inefficient or insufficient reasoning, such as:
 - Not prime because the product of 31×17 is 527, which is not a prime number.
 - Not prime because 31 and 17 are factors of this number.
- Give 0 points instead if participant makes an incorrect claim.

4. Brent came up with a possible divisibility rule for 18. He claims that any number is divisible by 18 if it is divisible by 3 and 6. For example, 36 is divisible by 3 and 36 is divisible by 6. So 36 must be divisible by 18.

a) Is Brent's rule correct?

Purpose:

This question assesses participants' ability to determine the correctness of a divisibility statement.

Suggested Solutions:

No.

Scoring Criteria (1 point):

1 point – States that Brent's rule is not correct.

- Give 0 points instead if participant states that Brent's rule is correct.

- b) If it is correct, explain how you know. If it is incorrect, explain why not and provide a correct divisibility rule for 18.**

Purpose:

This question assesses participants' ability to justify a specific divisibility rule using counter-examples, prime factorization, and least common multiples concepts.

Suggested Solutions:

Brent's rule is incorrect. First, there are counter-examples to the rule, such as 6 and 12. 12 is divisible by 3 and by 6, but it is not divisible by 18. Another reason why the rule is incorrect is that 3 and 6 share a factor of 3; if a number is divisible by 6, then it must be divisible by 2 and 3. There is nothing gained from checking divisibility by 3 since divisibility by 6 implies divisibility by 3. Thus, the greatest number for which Brent's rule can identify divisibility is 6, not 18. A correct divisibility rule for 18 would involve checking the divisibility of two factors of 18 that are relatively prime – 2 and 9.

Scoring Criteria (4 points):

3 points – States that divisibility by 6 automatically implies divisibility by 3, so checking divisibility by 3 once divisibility by 6 is confirmed does not provide any new information.

- Give 2 points instead if participant provides a correct counter-example to Brent's rule and explains why the counter-example invalidates Brent's rule (1 pt for providing and 1 pt for explaining the counter-example)

1 point – Provides a correct divisibility rule for 18: A number is divisible by 18 if it is divisible by 2 and by 9.

- 5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Show your work.**

Purpose:

This question assesses participants' ability to extend their understanding of least common multiples to more than two numbers. It also tests their understanding of the usefulness of prime factorization to determine divisibility. This is a very challenging question because it requires participants to use prime factorization concepts to construct a number that is the least common multiple of ten other numbers. Participants need to construct a prime factorization that shows divisibility by each of

the ten numbers without any superfluous factors, otherwise the result will be a common multiple, not the *least* common multiple. This requires careful attention to avoid overlapping factors.

Suggested Solutions:

2,520 is the smallest positive integer that is divisible by all of the first ten counting numbers. Since the number must be divisible by 1-10, we can build its prime factorization to include all of these numbers as factors. The prime factorization is as follows: $2^3 \times 3^2 \times 5 \times 7$. In this prime factorization, we see that 2, 3, 5, and 7 are factors because they are prime factors of the number. Since 2^3 is part of the prime factorization of the number, 4 and 8 are also factors. Since 3^2 is represented, 9 is also a factor. Since 2 and 3 are represented as prime factors, then 6 is also a factor of the number, and since 2 and 5 are represented as prime factors of the number, 10 is also one of its factors. As always, 1 is a factor of any whole number. 2,520 is the smallest positive integer that satisfies the given criterion because if any of the prime factors are removed from its prime factorization, at least one of the first ten counting numbers will cease to be its factor.

Scoring Criteria (7 points):

2 points – Correctly identifies the solution as 2,520 OR as $2^3 \times 3^2 \times 5 \times 7$.

5 points – Correctly explains or shows how 1-10 are factors of their answer (1/2 pt for each item):

- Give 3 points instead if a participant states or shows that the solution is 3,628,800 (i.e., $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$).

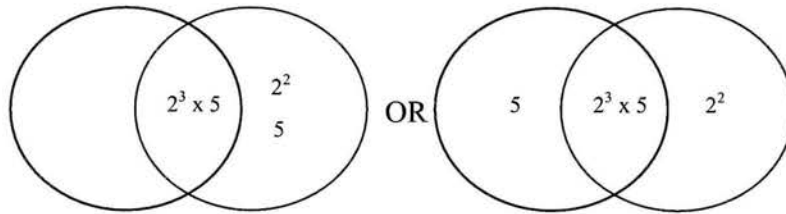
6. **The prime factorization of the greatest common factor of two numbers, A and B , is $2^3 \times 5$. The prime factorization of the least common multiple of the same two numbers is $2^5 \times 5^2$. Give all possible values for A and B . Show your work.**

Purpose:

This question assesses participants' understanding of the relationship between a number's GCF and LCM, and their ability to connect these concepts to the notion of prime factorization. As such, this is considered a challenging conceptual question.

Suggested Solutions:

The values of A and B are: $2^3 \times 5$, $2^5 \times 5^2$, $2^5 \times 5$, and $2^3 \times 5^2$ (i.e., 40, 800, 160, and 200). Each circle below can represent either the value of A or B :



Scoring Criteria (7 points):

4 points – Earns one point for each correct value.

3 points – Shows or explains one’s solutions accurately (e.g., using Venn diagrams) (1.5 points for each pair of values).

- If participant includes $2^4 \times 5$ and $2^4 \times 5^2$ as solutions in the work or explanation, give 1 point instead.

7. **Without solving, decide if each situation below requires finding the greatest common factor (GCF), the least common multiple (LCM), or neither with respect to the numbers given. Briefly explain each choice.**

- a) **Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 96 feet by 56 feet. What are the dimensions of the largest square tile that the school can buy from Alpaca in order to completely tile the cafeteria floor?**

Purpose:

This question assesses participants’ ability to identify word problems as requiring finding the GCF or LCM of two numbers.

Suggested Solutions:

Each dimension of the square tile must be a factor of the dimensions of the cafeteria floor in order to fit along the edges of the floor perfectly. Since the tile is square, its length equals its width, so its dimensions must be a common factor of 96 and 56. To determine the largest square tile needed, then, one must identify the greatest common factor of 96 and 56.

Scoring Criteria (5 points):

1 point – Correctly identifies the GCF is needed.

4 points – Correct explanation for why the GCF is needed, including:

- Each dimension of the tile must be a factor of 96 and 56 in order to fit along the edges of the cafeteria perfectly (2 points)
- The dimensions of the tile must be a common factor of 96 and 56 because the tile is square and therefore has equal dimensions (1 point).
- The dimensions of the tile must be the greatest common factor of 96 and 56 because the question asks for the largest square tile that does this (1 point).
- No points for solving the problem.

- b) Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 135 seconds. Jean runs one complete lap in 80 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?**

Purpose:

This question assesses participants' ability to identify word problems as requiring finding the GCF or LCM of two numbers.

Suggested Solutions:

Rita will stop running only at multiples of 135 seconds, since it takes her 135 seconds to finish each lap. Jean will stop running only at multiples of 80 seconds, since it takes her 80 seconds to finish each lap. Therefore, to stop together, the time elapsed must be a multiple of both 135 and 80. To find the first time that both runners stop, the time elapsed must represent the first common multiple of 135 and 80. This is the least common multiple of 135 and 80.

Scoring Criteria (5 points):

1 point – Correctly identifies the LCM.

4 points – Correct explanation for why the LCM is needed, including:

- Each runner will stop running at multiples of 135 and 80 since that is how long it takes each runner to run a complete lap, respectively (2 points).
- The answer to the question must be a common multiple of 135 and 80 because the runners are stopping at the same time (1 point).
- The answer to the question must be the least common multiple of 135 and 80 because the problem asks for the first time they stop running (1 point).
- No points for solving the problem.

**Number Theory Knowledge Test (Post-Test)
Scoring Rubric**

Total Points = 50

1. Consider the number $N = 3^2 \times 5^4 \times 11 \times 17^3$. Without calculating the value of N, answer each question by circling one of the three possible options:

a) Is N divisible by 3?	Yes	No	I am not sure
b) Is N divisible by 35?	Yes	No	I am not sure
c) Is N divisible by 19?	Yes	No	I am not sure
d) Is N divisible by 75?	Yes	No	I am not sure
e) Is N divisible by 51?	Yes	No	I am not sure

Purpose:

Assess participants' ability to determine divisibility properties by attending to a number's prime factorization as opposed to performing long division. Increased understanding of divisibility is marked by connection-making between divisibility and prime factorization (Zazkis & Campbell, 1996a, 1996b).

Suggested Solutions:

- a) Yes
- b) No
- c) No
- d) Yes
- e) Yes

Scoring Criteria (5 points):

- 5 points – Earns 1 point for each correct response
- Give 0 points instead for each incorrect response.

2. a) How many factors does 36 have? Show your work.

Purpose:

This question assesses participants' knowledge of what a factor is, as well as their ability to compute factors. Participants typically use a factor tree to identify a number's factors.

Suggested Solutions:

There are 9 factors. The factors are: 1, 2, 3, 4, 6, 9, 12, 18, and 36. One can use the prime factorization, for example, to list all prime factor combinations.

Scoring Criteria (4 points):

1 point – Correctly identifies the number of factors.

2 points – Displays accurate supporting work, such as:

- Drawing a factor tree for 36, OR
- Showing each division separately, OR
- Listing out factor pairs (e.g., 4 x 9), OR
- Finding the prime factorization of 36 and using it to list out all possible prime factor compositions, OR
- Using the formula $\text{Number of factors} = (\text{prime factor exponent} + 1) \times (\text{second prime factor exponent} + 1)$ with an explanation of why the formula works.

1 point – Shows or explains how supporting work leads to solution (e.g., by listing the factors).

b) How many factors does $3^2 \times 5^2$ have? How do you know?

Purpose:

This question is similar to part (a), but serves the added purpose of assessing participants' ability to use prime factorization as an efficient method for identifying factors. Unlike part (a), which most participants can solve easily, the prime factor representation of part (b) is challenging for them and requires some conceptual understanding of the relationship between factors and prime factors.

Suggested Solutions:

There are 9 factors. The factors are: 1, 3, 5, 3^2 , 3×5 , 5^2 , $3^2 \times 5$, 3×5^2 , and $3^2 \times 5^2$.

5^2 . Using the prime factorization of the number in question, one can find all of its factors by finding all combinations of its prime factors, since a number's factors are products of the number's prime factors.

Scoring Criteria (4 points):

1 point – Correctly identifies the number of factors.

2 points – Displays accurate supporting work, such as:

- Uses the prime factorization to determine each factor (e.g., $3^2=9$ and $3 \times 5=15$ are factors), OR
- Uses the formula *Number of factors = (prime factor exponent + 1) x (second prime factor exponent + 1)* with an explanation of why the formula works.
- If the prime factorization is first converted to a decimal number and then one of the following methods is used to find the factors, give 1 point instead:
 - Drawing a factor tree for 225, OR
 - Showing each division separately, OR
 - Listing out factor pairs (e.g., 25×9), OR

1 point – Shows or explains how supporting work leads to solution (e.g., by stating that the factors of a number are combinations (or products) of the number's prime factors).

- c) **Consider two numbers, $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$. Do you think N has more factors than R, less factors than R, or the same number of factors as R? Explain your choice.**

Purpose:

This question assesses participants' ability to apply their knowledge of prime factorization and factors. Without the use of a calculator, this question poses a challenge to participants who are not aware of the relationship between prime factors and factors. This question distinguishes between those who feel the need to compute each number's factors and those who can identify differences in the number of factors based purely on the multiplicative structure of each number (i.e., based on their respective prime factorizations).

Suggested Solutions:

The prime factorization of N can be used to compare to the prime factorization of R. R is already presented in prime factored form with 3 distinct primes (where

each prime is raised to the first power). N, however, can be further decomposed ($4 = 2 \times 2$) into a product of 3 distinct primes where one of the primes is squared. Thus, N can be represented as a product of 4 prime numbers. This implies that a greater number of combinations of primes can be constructed from N's prime factorization than from R's. Thus, N has more factors than R.

Scoring Criteria (4 points):

1 point – Correctly states that N has more factors than R.

1 point – States/shows that N can be further factored because $4 = 2 \times 2$ while R is completely factored already.

2 points – States/shows that N has a greater number of factor combinations possible than R since it has more numbers in its prime factorization (or, because one of its prime factors is squared).

- If instead of the statement directly above the participant lists all prime factor combinations for N and R, give 1 point instead.
- If the factorizations of N and R are first converted to decimal values and then all of their factors are computed, give 0 points instead.

3. a) In your own words, what is a prime number?

Purpose:

This question assesses participants' knowledge of the precise definition of prime number. Some might provide an imprecise definition, such as "any number whose only factors are 1 and itself," which would erroneously imply that 1 is, itself, a prime number.

Suggested Solutions:

A prime number is any number that has exactly two distinct factors. Alternatively, any number (except 1) whose only factors are 1 and itself.

Scoring Criteria (2 points):

2 points – provides one of the suggested solutions above.

- If participant provides an imprecise definition, such as "any number whose only factors are 1 and itself" give 1 point instead.
- If participant provides an incorrect definition or no definition at all, give 0 points instead.

b) Is 19×23 a prime number? Why or why not?

Purpose:

This question assesses participants' ability to identify a prime number via its prime factorization. Some pre-service teachers erroneously believe that a number is prime if it is the product of two or more prime numbers.

Suggested Solutions:

19×23 is not prime because it has more than two distinct factors (1, 19, 23, and 19×23).

Scoring Criteria (2 points):

2 points – Provides suggested solution.

- Give 1 point instead if participant makes a correct claim but provides inefficient or insufficient reasoning, such as:
 - Not prime because the product of 19×23 is 437, which is not a prime number.
 - Not prime because 19 and 23 are factors of this number.
- Give 0 points instead if participant makes an incorrect claim.

4. Brent came up with a possible divisibility rule for 18. He claims that any number is divisible by 18 if it is divisible by 3 and 6. For example, 36 is divisible by 3 and 36 is divisible by 6. So 36 must be divisible by 18.

a) Is Brent's rule correct?

Purpose:

This question assesses participants' ability to determine the correctness of a divisibility statement.

Suggested Solutions:

No.

Scoring Criteria (1 point):

1 point – States that Brent's rule is not correct.

- Give 0 points instead if participant states that Brent's rule is correct.

- b) If it is correct, explain how you know. If it is incorrect, explain why not and provide a correct divisibility rule for 18.**

Purpose:

This question assesses participants' ability to justify a specific divisibility rule using counter-examples, prime factorization, and least common multiples concepts.

Suggested Solutions:

Brent's rule is incorrect. First, there are counter-examples to the rule, such as 6 and 12. 12 is divisible by 3 and by 6, but it is not divisible by 18. Another reason why the rule is incorrect is that 3 and 6 share a factor of 2; if a number is divisible by 6, then it must be divisible by 2 and 3. There is nothing gained from checking divisibility by 3 since divisibility by 6 implies divisibility by 3. Thus, the greatest number for which Brent's rule can identify divisibility is 6, not 18. A correct divisibility rule for 18 would involve checking the divisibility of two factors of 18 that are relatively prime – 2 and 9.

Scoring Criteria (4 points):

3 points – States that divisibility by 6 automatically implies divisibility by 3, so checking divisibility by 3 once divisibility by 6 is confirmed does not provide any new information.

- Give 2 points instead if participant provides a correct counter-example to Brent's rule and explains why the counter-example invalidates Brent's rule (1 pt for providing and 1 pt for explaining the counter-example)

1 point – Provides a correct divisibility rule for 18: A number is divisible by 18 if it is divisible by 2 and by 9.

- 5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Show your work.**

Purpose:

This question assesses participants' ability to extend their understanding of least common multiples to more than two numbers. It also tests their understanding of the usefulness of prime factorization to determine divisibility. This is a very challenging question because it requires participants to use prime factorization concepts to construct a number that is the least common multiple of ten other numbers. Participants need to construct a prime factorization that shows divisibility by each of

the ten numbers without any superfluous factors, otherwise the result will be a common multiple, not the *least* common multiple. This requires careful attention to avoid overlapping factors.

Suggested Solutions:

2,520 is the smallest positive integer that is divisible by all of the first ten counting numbers. Since the number must be divisible by 1-10, we can build its prime factorization to include all of these numbers as factors. The prime factorization is as follows: $2^3 \times 3^2 \times 5 \times 7$. In this prime factorization, we see that 2, 3, 5, and 7 are factors because they are prime factors of the number. Since 2^3 is part of the prime factorization of the number, 4 and 8 are also factors. Since 3^2 is represented, 9 is also a factor. Since 2 and 3 are represented as prime factors, then 6 is also a factor of the number, and since 2 and 5 are represented as prime factors of the number, 10 is also one of its factors. As always, 1 is a factor of any whole number. 2,520 is the smallest positive integer that satisfies the given criterion because if any of the prime factors are removed from its prime factorization, at least one of the first ten counting numbers will cease to be its factor.

Scoring Criteria (7 points):

2 points – Correctly identifies the solution as 2,520 OR as $2^3 \times 3^2 \times 5 \times 7$.

5 points – Correctly explains or shows how 1-10 are factors of their answer (1/2 pt for each item):

- Give 3 points instead if a participant states or shows that the solution is 3,628,800 (i.e., $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$).

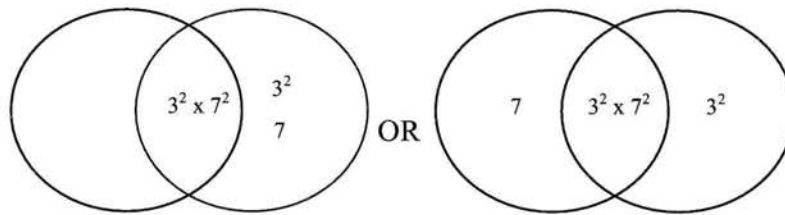
- 6. The prime factorization of the greatest common factor of two numbers, A and B , is $3^2 \times 7^2$. The prime factorization of the least common multiple of the same two numbers is $3^4 \times 7^3$. Give all possible values for A and B . Show your work.**

Purpose:

This question assesses participants' understanding of the relationship between a number's GCF and LCM, and their ability to connect these concepts to the notion of prime factorization. As such, this is considered a challenging conceptual question.

Suggested Solutions:

The values of A and B are: $3^2 \times 7^2$, $3^4 \times 7^3$, $3^2 \times 7^3$, and $3^4 \times 7^2$ (i.e., 441, 27783, 3087, and 3969). Each circle below can represent either the value of A or B :



Scoring Criteria (7 points):

4 points – Earns one point for each correct value.

3 points – Shows or explains one’s solutions accurately (e.g., using Venn diagrams) (1.5 points for each pair of values).

- If participant includes $3^3 \times 7^2$ and $3^3 \times 7^3$ as solutions in the work or explanation, give 1 point instead.

7. **Without solving, decide if each situation below requires finding the greatest common factor (GCF), the least common multiple (LCM), or neither with respect to the numbers given. Briefly explain each choice.**

- a) **Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 84 feet by 76 feet. What are the dimensions of the largest square tile that the school can buy from Alpaca in order to completely tile the cafeteria floor?**

Purpose:

This question assesses participants’ ability to identify word problems as requiring finding the GCF or LCM of two numbers.

Suggested Solutions:

Each dimension of the square tile must be a factor of the dimensions of the cafeteria floor in order to fit along the edges of the floor perfectly. Since the tile is square, its length equals its width, so its dimensions must be a common factor of 84 and 76. To determine the largest square tile needed, then, one must identify the greatest common factor of 84 and 76.

Scoring Criteria (5 points):

1 point – Correctly identifies the GCF is needed.

4 points – Correct explanation for why the GCF is needed, including:

- Each dimension of the tile must be a factor of 84 and 76 in order to fit along the edges of the cafeteria perfectly (2 points).
- The dimensions of the tile must be a common factor of 84 and 76 because the tile is square and therefore has equal dimensions (1 point).
- The dimensions of the tile must be the greatest common factor of 84 and 76 because the question asks for the largest square tile that does this (1 point).

b) Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 80 seconds. Jean runs one complete lap in 105 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

Purpose:

This question assesses participants' ability to identify word problems as requiring finding the GCF or LCM of two numbers.

Suggested Solutions:

Rita will stop running only at multiples of 80 seconds, since it takes her 80 seconds to finish each lap. Jean will stop running only at multiples of 105 seconds, since it takes her 105 seconds to finish each lap. Therefore, to stop together, the time elapsed must be a multiple of both 80 and 105. To find the first time that both runners stop, the time elapsed must represent the first common multiple of 80 and 105. This is the least common multiple of 80 and 105.

Scoring Criteria (5 points):

1 point – Correctly identifies the LCM.

4 points – Correct explanation for why the LCM is needed, including:

- Each runner will stop running at multiples of 80 and 105 since that is how long it takes each runner to run a complete lap, respectively (2 points).
- The answer to the question must be a common multiple of 80 and 105 because the runners are stopping at the same time (1 point).
- The answer to the question must be the least common multiple of 80 and 105 because the problem asks for the first time they stop running (1 point).

APPENDIX C
INDIVIDUAL CLINICAL INTERVIEWS

Individual Clinical Interview Questions (Pre-Interview)

Note: The formatting of this interview was different when presented to participants

Directions

The purpose of this interview is to find out what you know and do not know about elementary number theory topics, so that we can improve our teaching of these topics during classroom instruction. Your responses to the interview questions will in no way affect your standing in the course.

This interview consists of several math questions related to elementary number theory topics. I will ask you to solve one question at a time. If at any time you do not understand what a question is asking you to do, please ask me and I will clarify its directions. You have paper, a pen or pencil, and a calculator at your disposal. You may use your calculator any time you wish, unless a question specifically requests that you not use one. This interview will last for one hour.

For each question, it is very important that you describe to me, out loud, what you are thinking about as you try to solve the question. If at any time during the interview you stop explaining what you are doing and thinking about out loud, I will say, “please tell me what you are thinking about right now.” I may also periodically ask you to explain or your reasoning or clarify certain statements that you make if I do not understand them.

If at any point you are having difficulty solving a question, I ask that you tell me what is the difficulty you are having and why you think you are having this difficulty. Since the purpose of the interview is for me to find out what you know and do not know, I cannot give you any hints as to how to solve each question. I will only tell you to review the work you have done and try to explain what you think you should do next. If you continue to struggle with a question, that is perfectly fine. We will proceed to the next question and possibly return to any unsolved questions at the end of the interview. Do you have any questions before we begin?

1. In number theory, “divisible” means divided by with no remainder. Consider the number $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$. Without calculating the value of N , answer each question by circling one of the three possible options. Explain each choice.

a) Is N divisible by 5?	Yes	No	I am not sure
b) Is N divisible by 7?	Yes	No	I am not sure
c) Is N divisible by 4?	Yes	No	I am not sure
d) Is N divisible by 51?	Yes	No	I am not sure
e) Is N divisible by 105?	Yes	No	I am not sure

2. Do you agree or disagree with the following statement? Explain your response.

A number is divisible by 18 if it is divisible by 3 and by 6.

3. Consider two numbers, $N = 3 \times 6 \times 7$ and $R = 11 \times 13 \times 17$.
- Do you think N has more factors than R , less factors than R , or the same number of factors as R ? Explain your choice.
 - Find all of the factors of N and R .
4. Find a number that has exactly 4 factors. Explain how you know it has 4 factors.
5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Explain your work. *Note: you may leave your answer in factored form.*
6. The prime factorization of the greatest common factor of two numbers, A and B , is $2^3 \times 5$. The prime factorization of the least common multiple of the same two numbers

is $2^5 \times 5^2$. Give all possible values for A and B . Explain your work. *Note: you may leave your answer in factored form.*

Word Problems: The following two word problems represent typical word problems that students might see in mathematics textbooks. Do not worry about whether these problems represent realistic situations.

- Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 96 feet by 56 feet. What are the dimensions of the largest square tile that the school can buy to completely tile the cafeteria floor?
- Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 135 seconds. Jean runs one complete lap in 80 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

Individual Clinical Interview Questions (Post-Interview)

Note: The formatting of this interview was different when presented to participants

Directions

The purpose of this interview is to find out what you know and do not know about elementary number theory topics, so that we can improve our teaching of these topics during classroom instruction. Your responses to the interview questions will in no way affect your standing in the course.

This interview consists of several math questions related to elementary number theory topics. I will ask you to solve one question at a time. If at any time you do not understand what a question is asking you to do, please ask me and I will clarify its directions. You have paper, a pen or pencil, and a calculator at your disposal. You may use your calculator any time you wish, unless a question specifically requests that you not use one. This interview will last for one hour.

For each question, it is very important that you describe to me, out loud, what you are thinking about as you try to solve the question. If at any time during the interview you stop explaining what you are doing and thinking about out loud, I will say, “please tell me what you are thinking about right now.” I may also periodically ask you to explain or your reasoning or clarify certain statements that you make if I do not understand them.

If at any point you are having difficulty solving a question, I ask that you tell me what is the difficulty you are having and why you think you are having this difficulty. Since the purpose of the interview is for me to find out what you know and do not know, I cannot give you any hints as to how to solve each question. I will only tell you to review the work you have done and try to explain what you think you should do next. If you continue to struggle with a question, that is perfectly fine. We will proceed to the next question and possibly return to any unsolved questions at the end of the interview. Do you have any questions before we begin?

1. In number theory, “divisible” means divided by with no remainder. Consider the number $N = 3^2 \times 5^4 \times 11 \times 17^3$. Without calculating the value of N , answer each question by circling one of the three possible options. Explain each choice.

a) Is N divisible by 3?	Yes	No	I am not sure
b) Is N divisible by 35?	Yes	No	I am not sure
c) Is N divisible by 19?	Yes	No	I am not sure
d) Is N divisible by 75?	Yes	No	I am not sure
e) Is N divisible by 51?	Yes	No	I am not sure

2. A certain mystery number, N , is divisible by 18 and by 15. What is the greatest number that N must be divisible by? How do you know?
3. Consider two numbers, $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$.
- Do you think N has more factors than R , less factors than R , or the same number of factors as R ? Explain your choice.
 - Find all of the factors of N and R .
4. Find a number that has exactly 6 factors. Explain how you know it has 6 factors.
5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Explain your work. *Note: you may leave your answer in factored form.*
6. The prime factorization of the greatest common factor of two numbers, A and B , is $3^2 \times 7^2$. The prime factorization of the least common multiple of the same two numbers is $3^4 \times 7^3$. Give all possible values for A and B . Explain your work. *Note: you may*

leave your answer in factored form.

Word Problems: The following two word problems represent typical word problems that students might see in mathematics textbooks. Do not worry about whether these problems represent realistic situations.

7. Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 84 feet by 76 feet. What are the dimensions of the largest square tile that the school can buy to completely tile the cafeteria floor?

8. Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 80 seconds. Jean runs one complete lap in 105 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?

APPENDIX D
SOLUTIONS TO INDIVIDUAL CLINICAL INTERVIEWS

Individual Clinical Interview Solutions (Pre-Interview)

1. In number theory, “divisible” means divided by with no remainder. Consider the number $N = 2^3 \times 3^2 \times 5^2 \times 17^3 \times 31^5$. Without calculating the value of N, answer each question by circling one of the three possible options. Explain each choice.

a) Is N divisible by 5?	Yes	No	I am not sure
b) Is N divisible by 7?	Yes	No	I am not sure
c) Is N divisible by 4?	Yes	No	I am not sure
d) Is N divisible by 51?	Yes	No	I am not sure
e) Is N divisible by 105?	Yes	No	I am not sure

Solution

- a) Yes, since 5 is a factor in the prime factorization of N.
- b) No, since 7 is not in the prime factorization of N.
- c) Yes, since $4 = 2^2$ is a factor of 2^3 , which is a factor in the prime factorization of N.
- d) Yes, since $51 = 3 \times 17$ and 3×17 is a factor in the prime factorization of N.
- e) No, since $105 = 3 \times 5 \times 7$ and 7 is not a prime factor of N.

2. Do you agree or disagree with the following statement? Explain your response.

A number is divisible by 18 if it is divisible by 3 and by 6.

Solution

Disagree. A number can be divisible by 3 and by 6 but not be divisible by 18 (e.g., 12). Since 3 and 6 share a factor of 3, divisibility by 6 already implies divisibility by 3. As such, if a number is divisible by 6, then no new information is gained from checking divisibility by 3. Thus, if a number is divisibly by 3 and by 6, the greatest

number for which divisibility can be inferred is 6.

3. Consider two numbers, $N = 3 \times 6 \times 7$ and $R = 11 \times 13 \times 17$.

a) Do you think N has more factors than R , less factors than R , or the same number of factors as R ? Explain your choice.

b) Find all of the factors of N and R .

Solution

a) N has more factors than R because its prime factorization has a greater number of prime factor combinations. Since one of N 's factors shown above can be further decomposed ($6 = 2 \times 3$), N can be represented as a product of 4 prime numbers ($N = 2 \times 3^2 \times 7$), resulting in 12 factor combinations. R is composed of only 3 prime numbers, resulting in 8 factor combinations. In general, a number whose prime factorization is in the form $P^A \times Q^B$ has $(A + 1) \times (B + 1)$ factors.

b) N 's twelve factors are made of combinations of zero, one, two, three, or four prime factors. They are: 1, 2, 3, 7, 2×3 , 2×7 , 3^2 , 3×7 , 2×3^2 , $2 \times 3 \times 7$, $3^2 \times 7$, and $2 \times 3^2 \times 7$.

R 's eight factors are made of combinations of zero, one, two, or three prime factors. They are: 1, 11, 13, 17, 11×13 , 11×17 , 13×17 , and $11 \times 13 \times 17$.

4. Find a number that has exactly 4 factors. Explain how you know it has 4 factors.

Solution

6 has four factors because its prime factorization (2×3) contains four prime factor combinations: 1, 2, 3, and 2×3 . In fact, any number whose prime factorization is the product of exactly two distinct prime factors ($p \times q$) will have four factors (1, p , q , and $p \times q$). Additionally, any number whose prime factorization is the cube of a prime (p^3) also has exactly four prime factor combinations (1, p , p^2 , and p^3).

5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Explain your work. *Note: you may leave your answer in factored form.*

Solution

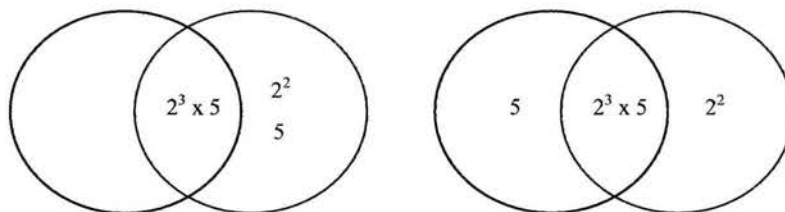
$2^3 \times 3^2 \times 5 \times 7$, or 2,520, is the smallest positive integer that is divisible by all of the first ten counting numbers. Since the number must be divisible by 1-10, we can build its prime factorization to include all of these numbers as factors. In order to be

divisible by 2, 4, and 8, its prime factorization must contain 2^3 . In order to be divisible by 3 and 9, its prime factorization must contain 3^2 . To be divisible by 5 and 7, the number must contain 5 and 7 in its prime factorization. To be divisible by 6, the number must contain 2 and 3 in its prime factorization. Both are already accounted for in the 2^3 and 3^2 . To be divisible by 10, the number must contain 2 and 5 in its prime factorization. Both are already accounted for in the 2^3 and 5. As always, 1 is a factor of any whole number. Also, $2^3 \times 3^2 \times 5 \times 7$ is the smallest positive integer that satisfies the given criterion because if any of the prime factors are removed from its prime factorization, at least one of the first ten counting numbers will cease to be its factor.

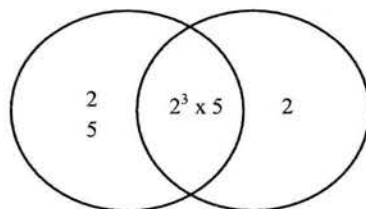
6. **The prime factorization of the greatest common factor of two numbers, A and B , is $2^3 \times 5$. The prime factorization of the least common multiple of the same two numbers is $2^5 \times 5^2$. Give all possible values for A and B . Explain your work. *Note: you may leave your answer in factored form.***

Solution

The possible values of A and B are: $2^3 \times 5$, $2^5 \times 5^2$, $2^5 \times 5$, and $2^3 \times 5^2$ (i.e., 40, 800, 160, and 200), as shown in the Venn diagrams below. Each circle below represents a possible value of A or B :



Each solution must be divisible by the GCF of A and B and a factor of the LCM of A and B . Let $K = \text{LCM} \div \text{GCF} = (2^5 \times 5^2) \div (2^3 \times 5) = 2^2 \times 5$. Therefore, all possible solution values are in the form $\text{GCF} \times (\text{factor of } K)$. The factors of K that are not permissible are 2 and (2×5) because they would alter the value of the GCF. This erroneous case is shown in the Venn diagram below. Notice that in this case, the GCF cannot equal $2^3 \times 5$ because A and B share an additional factor of 2.



Word Problems: The following two word problems represent typical word problems that students might see in mathematics textbooks. Do not worry about whether these problems represent realistic situations.

7. **Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 96 feet by 56 feet. What are the dimensions of the largest square tile that the school can buy to completely tile the cafeteria floor?**

Solution

The dimensions of the largest square tile are 8 feet by 8 feet. Each dimension of the square tile must be a factor of the dimensions of the cafeteria floor in order to fit along the edges of the floor perfectly. Since the tile is square, its length equals its width, so its dimensions must be a common factor of 96 and 56. To determine the largest square tile needed, then, one must identify the greatest common factor of 96 and 56. The prime factorization of 96 is $2^5 \times 3$; the prime factorization of 56 is $2^3 \times 7$. The greatest common factor of these two numbers is 2^3 , or 8, because that is the greatest product of prime factors that both prime factorizations share.

8. **Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 135 seconds. Jean runs one complete lap in 80 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?**

Solution

It will take both runners 2160 seconds to stop running. Rita will stop running only at multiples of 135 seconds, since it takes her 135 seconds to finish each lap. Jean will stop running only at multiples of 80 seconds, since it takes her 80 seconds to finish each lap. Therefore, to stop together, the time elapsed must be a multiple of both 135 and 80. To find the first time that both runners stop, the time elapsed must represent the least common multiple of 135 and 80. The prime factorization of 135 is $3^3 \times 5$ and the prime factorization of 80 is $2^4 \times 5$. The least common multiple of these two numbers is $2^4 \times 3^3 \times 5$, or 2160, because that is the least product of prime factors of which both prime factorizations are factors.

Individual Clinical Interview Solutions (Post-Interview)

1. In number theory, “divisible” means divided by with no remainder. Consider the number $N = 3^2 \times 5^4 \times 11 \times 17^3$. Without calculating the value of N , answer each question by circling one of the three possible options. Explain each choice.

a) Is N divisible by 3?	Yes	No	I am not sure
b) Is N divisible by 35?	Yes	No	I am not sure
c) Is N divisible by 19?	Yes	No	I am not sure
d) Is N divisible by 75?	Yes	No	I am not sure
e) Is N divisible by 51?	Yes	No	I am not sure

Solution

- a) Yes, since 3 is a factor in the prime factorization of N .
- b) No, since $35 = 5 \times 7$ and 7 is not a prime factor of N .
- c) No, since 19 is not in the prime factorization of N .
- d) Yes, since $75 = 3 \times 5^2$ and 3×5^2 is a factor in the prime factorization of N .
- e) Yes, since $51 = 3 \times 17$ and 3×17 is a factor in the prime factorization of N .
2. A certain mystery number, N , is divisible by 18 and by 15. What is the greatest number that N must be divisible by? How do you know?

Solution

The greatest number that N must be divisible by is $2 \times 3^2 \times 5$, or 90. If N is divisible by 18, then its prime factorization must include the prime factorization of 18, or 2×3^2 . If N is divisible by 15, then its prime factorization must also include the prime factorization of 15, or 3×5 . The 3 from 15's prime factorization is already accounted for by one of the 3's in 18's prime factorization. The solution, written in prime

factored form, includes both 18's and 15's prime factorizations with no extraneous prime factors of any kind.

3. Consider two numbers, $N = 3 \times 4 \times 5$ and $R = 13 \times 17 \times 19$.

- a) Do you think N has more factors than R , less factors than R , or the same number of factors as R ? Explain your choice.**
- b) Find all of the factors of N and R .**

Solution

- a) N has more factors than R because its prime factorization has a greater number of prime factor combinations. Since one of N 's factors shown above can be further decomposed ($4 = 2 \times 2$), N can be represented as a product of 4 prime numbers ($N = 2^2 \times 3 \times 5$), resulting in 12 factor combinations. R is composed of only 3 prime numbers, resulting in 8 factor combinations. In general, a number whose prime factorization is in the form $P^A \times Q^B$ has $(A + 1) \times (B + 1)$ factors.
- b) N 's twelve factors are made of combinations of zero, one, two, three, or four prime factors. They are: 1, 2, 3, 5, 2^2 , 2×3 , 2×5 , 3×5 , $2^2 \times 3$, $2^2 \times 5$, $2 \times 3 \times 5$, and $2^2 \times 3 \times 5$.
 R 's eight factors are made of combinations of zero, one, two, or three prime factors. They are: 1, 13, 17, 19, 13×17 , 13×19 , 17×19 , and $13 \times 17 \times 19$.

4. Find a number that has exactly 6 factors. Explain how you know it has 6 factors.

Solution

12 has six factors because its prime factorization ($2^2 \times 3$) contains six prime factor combinations: 1, 2, 3, 2^2 , 2×3 , and $2^2 \times 3$. In fact, any number whose prime factorization is the product of a prime and the square of a different prime ($p^2 \times q$) will have six factors (1, p , q , $p \times q$, p^2 , and $p^2 \times q$). Additionally, any number whose prime factorization is the fifth power of a prime (p^5) also has exactly six prime factor combinations (1, p , p^2 , p^3 , p^4 , and p^5).

5. What is the smallest positive integer that is divisible by all of the first ten counting numbers, 1 through 10? Explain your work. *Note: you may leave your answer in factored form.*

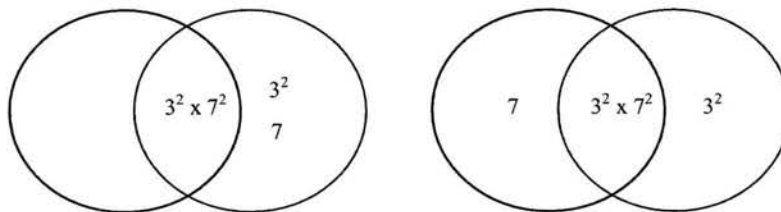
Solution

$2^3 \times 3^2 \times 5 \times 7$, or 2,520, is the smallest positive integer that is divisible by all of the first ten counting numbers. Since the number must be divisible by 1-10, we can build its prime factorization to include all of these numbers as factors. In order to be divisible by 2, 4, and 8, its prime factorization must contain 2^3 . In order to be divisible by 3 and 9, its prime factorization must contain 3^2 . To be divisible by 5 and 7, the number must contain 5 and 7 in its prime factorization. To be divisible by 6, the number must contain 2 and 3 in its prime factorization. Both are already accounted for in the 2^3 and 3^2 . To be divisible by 10, the number must contain 2 and 5 in its prime factorization. Both are already accounted for in the 2^3 and 5. As always, 1 is a factor of any whole number. Also, $2^3 \times 3^2 \times 5 \times 7$ is the smallest positive integer that satisfies the given criterion because if any of the prime factors are removed from its prime factorization, at least one of the first ten counting numbers will cease to be its factor.

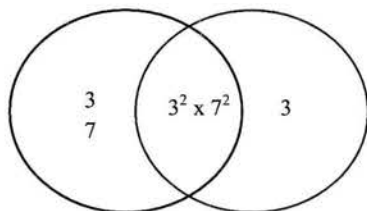
6. The prime factorization of the greatest common factor of two numbers, A and B , is $3^2 \times 7^2$. The prime factorization of the least common multiple of the same two numbers is $3^4 \times 7^3$. Give all possible values for A and B . Explain your work. *Note: you may leave your answer in factored form.*

Solution

The possible values of A and B are: $3^2 \times 7^2$, $3^4 \times 7^3$, $3^2 \times 7^3$, and $3^4 \times 7^2$ (i.e., 441, 27783, 3087, and 3969), as shown in the Venn diagrams below. Each circle below represents a possible value of A or B :



Each solution must be divisible by the GCF of A and B and a factor of the LCM of A and B . Let $K = \text{LCM} \div \text{GCF} = (3^4 \times 7^3) \div (3^2 \times 7^2) = 3^2 \times 7$. Therefore, all possible solution values are in the form $\text{GCF} \times (\text{factor of } K)$. The factors of K that are not permissible are 3 and (3×7) because they would alter the value of the GCF. This erroneous case is shown in the Venn diagram below. Notice that in this case, the GCF cannot equal $3^2 \times 7^2$ because A and B share an additional factor of 3.



Word Problems: The following two word problems represent typical word problems that students might see in mathematics textbooks. Do not worry about whether these problems represent realistic situations.

7. **Alpaca Rug and Tile Company sells tiles in square sizes. A school wants to buy square tiles for a rectangular shaped cafeteria that is 84 feet by 76 feet. What are the dimensions of the largest square tile that the school can buy to completely tile the cafeteria floor?**

Solution

The dimensions of the largest square tile are 4 feet by 4 feet. Each dimension of the square tile must be a factor of the dimensions of the cafeteria floor in order to fit along the edges of the floor perfectly. Since the tile is square, its length equals its width, so its dimensions must be a common factor of 84 and 76. To determine the largest square tile needed, then, one must identify the greatest common factor of 84 and 76. The prime factorization of 84 is $2^2 \times 3 \times 7$; the prime factorization of 76 is $2^2 \times 19$. The greatest common factor of these two numbers is 2^2 , or 4, because that is the greatest product of prime factors that both prime factorizations share.

8. **Rita and Jean are running around the same race track. They start running from the starting line at the same time. Rita runs one complete lap in 80 seconds. Jean runs one complete lap in 105 seconds. Both runners will stop only when both complete a lap at the exact same time. After how many seconds will both runners stop running?**

Solution

It will take both runners 1680 seconds to stop running. Rita will stop running only at multiples of 80 seconds, since it takes her 80 seconds to finish each lap. Jean will stop running only at multiples of 105 seconds, since it takes her 105 seconds to finish each lap. Therefore, to stop together, the time elapsed must be a multiple of both 80 and 105. To find the first time that both runners stop, the time elapsed must represent the least common multiple of 80 and 105. The prime factorization of 80 is $2^4 \times 5$ and the prime factorization of 105 is $3 \times 5 \times 7$. The least common multiple of these two numbers is $2^4 \times 3 \times 5 \times 7$, or 1680, because that is the least product of prime factors of which both prime factorizations are factors.

APPENDIX E
INFORMED CONSENT FORMS

Participant Informed Consent Form

Dear MA 107 Student,

My name is Ziv Feldman. I am a doctoral student in SED and one of the instructors of MA 107. I am conducting a study to find out how pre-service elementary teachers develop an understanding of elementary number theory topics. Information collected from the study will be used in my doctoral dissertation.

With your permission, I will include you as a participant in my study from October 2011 through November 2011, while you are a student in MA 107. You must be at least 18 years old to participate. By participating in the study you are allowing me to collect data from a Mathematical Background Information Form, a test given at the beginning and at the completion of a 2-3 week unit, and written homework assignments that you complete related to the unit. Participation in this study is completely voluntary, and you can stop participation at any time. If you do not wish to participate, this will not affect your grade in MA 107.

With your permission, you may be videotaped during regular classroom discussions about mathematics. If willing, you may also be asked to participate in audio-taped individual interviews where you will solve and discuss math problems related to the unit. One interview will take place at the beginning of the unit, and a second interview will take place following the conclusion of the unit. The collection of mathematical background information, test data, and video data will not involve any of your time beyond that required for the MA 107 course. For participants who are interviewed, the total time commitment is approximately two hours (two 1-hour interviews). For this time, you will be compensated with a \$30 gift card. There are no known risks associated with participating in this study.

To ensure anonymity and confidentiality, your name will not be used with your test or interview data, but rather a code number will be assigned to you. The code key will be secured in a locked cabinet. During videotaping only your first name will be used. The testing and interview data will be treated confidentially and will not be disclosed unless required by law or regulation. Copies will be made of all study assessments and assignments, and the original documents will be returned to you in a timely manner. Study assessment, interview, and video data will be stored separately from the consent forms in locked files accessible only to me.

There are no direct benefits to you for participating in this study. However, this research may benefit society in the following ways. The findings from this research will be used to inform teacher educators at programs for training elementary and special education teachers about the mathematical knowledge of pre-service teachers. This may help them better cater future instruction of pre-service teachers. Interview transcripts and

videotapes will only be shown to groups of mathematics teacher educators who are involved in pre-service education. A brief summary of the findings will be provided to you.

If you have any questions regarding the study or your participation in it, either now or any time in the future, please feel free to ask me. I may be reached by calling (617) 519-6442 or by emailing zfeld@bu.edu. I will be happy to answer any questions you have. You may also contact my faculty adviser, Dr. Suzanne Chapin at schapin@bu.edu, for further information.

You may obtain further information about your rights as a research participant by calling the BU CRC IRB Office at (617) 358-6115.

Two copies of this form have been included, one for your retention and another to be returned to me. Thank you for your help.

Sincerely,



Ziv Feldman

I have read and understood the above and grant Ziv Feldman permission to include me as a participant in his study. I also acknowledge receipt of a copy of this form.

Signature: _____ Date: _____

Printed Name: _____

I grant ___ or do not grant ___ (initial one) Ziv Feldman permission to access my SAT/ACT scores.

I am ___ or am not ___ (initial one) willing to participate in two 1-hour interviews, for which I will be compensated with a \$30 gift card. I understand that confidentiality will be maintained in the manner described above.

I am ___ or am not ___ (initial one) willing to be videotaped during classroom discussions about the mathematical tasks we will be doing. I understand that confidentiality will be maintained in the manner described above.

Instructor Informed Consent Form

Dear MA 107 Instructor:

As part of my doctoral research, I am studying the developing understanding of pre-service teachers as they study elementary number theory topics as part of their MA 107 coursework. Part of this research involves implementing a two-week instructional unit on number theory and videotaping class sessions surrounding five mathematical tasks. I am requesting your permission to allow me to videotape you during classroom discussions which will occur between October 2011 and November 2011.

The collection of video data will not involve any of your time beyond that required for the course, MA 107. In order to maintain consistency among instructors, I would like to meet with you prior to the number theory unit to discuss implementation strategies, prompts, etc. For this additional preparation time, I will compensate you with a gift card. There are no known risks associated with participating in this videotaping. The videotapes will only be used by me for my doctoral research. It may also be shown to groups of mathematics educators, mathematicians, and teacher educators who are involved in pre-service education. Videotapes will be stored in locked files and/or encrypted electronic files accessible only to me.

There are no direct benefits to you for participating in this study. However, this research may benefit society in the following ways. The findings from this research will be used to inform teacher educators at programs for training elementary and special education teachers about the mathematical knowledge of pre-service teachers. This may help them better cater future instruction of pre-service teachers. A brief summary of the findings will be provided to you.

If you have any questions regarding the study or your participation in it, either now or any time in the future, please feel free to ask me. I may be reached by calling (617) 519-6442 or by emailing zfeld@bu.edu. I will be happy to answer any questions you have. You may also contact my faculty adviser, Dr. Suzanne Chapin at schapin@bu.edu, for further information.

You may obtain further information about your rights as a research participant by calling the BU CRC IRB Office at (617) 358-6115.

Two copies of this form have been included, one for your retention and another to be returned to me. Thank you for your help.

Sincerely,



Ziv Feldman

I have read and understood the above and grant Ziv Feldman permission to videotape me as part of his study. By doing so, I am agreeing to meet with Ziv to discuss implementation strategies prior to commencement of the study. I also acknowledge receipt of a copy of this form.

Signature: _____ Date: _____

Printed Name: _____

APPENDIX F
NUMBER THEORY TASKS

Factors – Part I

The purpose of this task is to introduce participants to the concept of *factors*. Distinguishing between the number of factors of prime and composite numbers can help students develop an understanding of the multiplicative structure of number and the role factors play within that structure. This task serves as a beginning step to developing this understanding.

Definitions of Terms (Whole Class)

1. Define the terms, *factor*, *divisor*, and *multiple*.
2. Find all of the distinct factors of the number 234. How many factors does this number have? How do you know that you found all of its factors?

Activity A (Small Group)

3. The *Factor Game* is a strategy game adapted from the *Connected Mathematics* program (Lappan, G., Fey, J. Fitzgerald, W., Friel, S., & Phillips, E., 2006. *Prime Time: Factors and Multiples*. Boston: Pearson Prentice Hall.)

The Factor Game

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

The Factor Game - Instructions

1. Player A chooses a number on the game board and circles it.
2. Using a different color, Player B circles all the proper factors of Player A's number. The **proper factors** of a number are all the factors of that number, except the number itself. For example, the proper factors of 12 are 1,2,3,4, and 6. Although 12 is a factor of itself, it is not a proper factor.
3. Player B circles a new number, and Player A circles all the factors of the number that are not already circled.
4. The players take turns choosing numbers and circling factors.
5. If a player circles a number that has no factors left that have not been circled, then that player does not get the points for the number circled and loses the next turn.
6. The game ends when there are no numbers left with uncircled factors.
7. Each player adds the numbers that are circled with his or her color. The player with the greatest total is the winner.

- a) Play the *Factor Game* at least twice before answering these questions.
- b) What are the best first, second, third, fourth, and fifth moves in the *Factor Game*?
- c) Which characteristics of factors are you using to make your decisions about "best moves?"

Group Discussion Question (Whole Class)

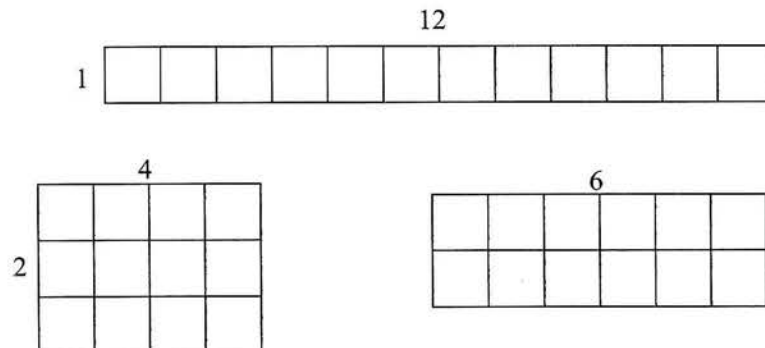
- What strategy should you employ if you want to win the *Factor Game*?

Activity B (Small Group)

4. a) Finish filling out the chart below with numbers that have an odd number of distinct factors. Make sure that at least one of the numbers is greater than 100.

Number	Factors	No. of Factors
4	1, 2, 4	3

- b) Certain elementary curricula have students exploring odd and even factors by representing numbers as rectangular arrays. For example, students may be asked to draw different rectangular arrays to illustrate all of the factors of the number 12, as follows:



Pick one of the numbers in the chart above, except the number 4, and sketch rectangular arrays to illustrate all of its factors.

- c) Why do all of the numbers in your chart have an odd number of factors? Explain.
- d) Identify if each of the following has an odd or an even number of factors. Explain each choice.
- 129
 - $5^2 \times 3^2$
 - 4^6
 - $3^4 \times 11^2 \times 13$

Group Discussion Question (Whole Class)

- How can you tell if a number has an odd or an even number of distinct factors without factoring the number? Explain why your method works.

Summarize and Connect

- Which numbers tend to have a lot of factors and which numbers tend to have few factors? Explain.
- Which type of numbers have an odd number of factors, and why?

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

Factors – Part II

The purpose of this task is to help participants deepen their understanding of the multiplicative structure of number by making connections between a number’s factors and its prime factorization. It also serves as an introduction to the Fundamental Theorem of Arithmetic. This task follows *Factors – Part I* and is adapted from Teppo (2002).

Activity A (Small Group)

1. Find several numbers that have exactly two, three, four, or five distinct factors. Enter your data in the # and Factors columns of each section of the Factors and Prime Factors Table below. It is alright if you cannot fill in every row of the table.

Factors and Prime Factors Table

<u>2 Factors</u>		<u>3 Factors</u>	
#	Factors	#	Factors
<u>4 Factors</u>		<u>5 Factors</u>	
#	Factors	#	Factors

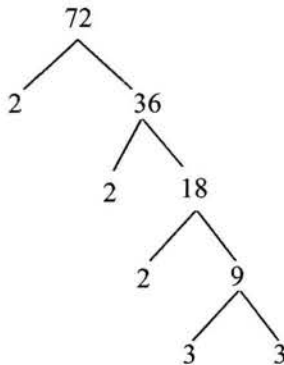
2. Referring to the table above, make conjectures about each of the following questions:
 - a) What type of number has exactly two distinct factors?
 - b) What type of number has exactly three distinct factors?
 - c) What type of number has exactly four distinct factors?
 - d) What type of number has exactly five distinct factors?

Group Discussion Questions (Whole Class)

- What is a prime number? What is a composite number?
- Discuss Question 2.

Prime Factorization (Whole Class)

A *factor tree* is a convenient strategy for finding the prime factorization of a composite number. Below is an example of a factor tree for 72:



3. Based on the factor tree above, what is the prime factorization of 72?
4. Create two additional factor trees for 72. Find the prime factorization of 72 based on each of these two factor trees.
5. What is the same about all three factor trees? Can you find a different factor tree for 72 that would not have this in common with the first three? Explain.
6. Consider the following factorizations of N and K : $N = 18 \times 40$ and $K = 16 \times 45$.
 - a) Without multiplying 18×40 and 16×45 , is N greater than, less than, or equal to K ? Explain how you know.

- b) Let $R = 3^2 \times 4^2 \times 5$. Is R greater than, less than, or equal to N? To K? Explain how you know.
- c) What is the difference between a number's factorization and its prime factorization? Use an example to illustrate your point.
7. The Fundamental Theorem of Arithmetic states that any composite number can be decomposed into a product of prime numbers, and that this product is unique. What does this theorem mean?

Activity B (Small Group)

8. Copy the numbers in each # column from the Factors and Prime Factors Table onto the following table. Then find the prime factorization of each number and enter this data into the column labeled Prime Factorization.

<u>2 Factors</u>		<u>3 Factors</u>	
#	Prime Factorization	#	Prime Factorization
<u>4 Factors</u>		<u>5 Factors</u>	
#	Prime Factorization	#	Prime Factorization

9. Can you use the prime factorization of a number to determine its factors? For example, the prime factorization of 10 is 2×5 , and we know that 10 has four distinct factors. Use its prime factorization to construct all four of its factors.

10. Let $M = 2 \times 5 \times 7^2 \times 53$ be the prime factorization of M . List every counting number that is a factor, or divisor, of M . How can you use M 's prime factorization to solve this question? Explain.

Group Discussion Question (Whole Class)

- How can we use prime factors to find all of a number's factors? Describe a general strategy that can be used to find all of the factors of any number.

Summarize and Connect

- How can you determine if a number has exactly two, three, four, or five distinct factors?
- What is the Fundamental Theorem of Arithmetic and what does it mean?

Divisibility

In this activity, we will explore factors and multiples concepts by examining the relationship between prime factorization and divisibility.

Activity A (Small Group)

1. Consider a number, N , whose prime factorization is $N = 2^5 \times 3^2 \times 7 \times 11$. Without using a calculator or making any paper and pencil calculations, decide if N is divisible by each of the following numbers: 2, 5, 6, 7, 8, 13, 27, and 63. Explain each answer.
2. The prime factorization of K is $K = 13 \times 61 \times 127$.
 - a) Without using a calculator or making any paper and pencil calculations, is K divisible by 3, 5, 7, and 11? Explain each answer.
 - b) Is K divisible by 26? By 39? Explain.
 - c) What is K divisible by? How do you know?

Group Discussion Questions (Whole Class)

- How are prime factors related to divisibility?
- Based on Questions 1-2, how might you explain the statement, A is divisible by B ?

Divisibility Rules (Whole Class)

Divisibility rules are efficient tests to determine if one number is divisible by another without actually doing division. The following is a summary of some common divisibility rules.

Divisibility Rule for:	A number is divisible if:
2	Its last digit is even
3	The sum of its digits is divisible by 3
4	Its last two digits form a two-digit number that is divisible by 4
5	Its last digit is 0 or 5

8	Its last three digits form a three-digit number that is divisible by 8
9	The sum of its digits is divisible by 9
10	Its last digit is 0

3. Notice that there is no divisibility rule for 6 in the list above. To create a divisibility rule for 6:
 - a) List the first seven multiples of 6, including 6. What is each multiple's prime factorization?
 - b) What do all of the multiples' prime factorizations have in common?
 - c) Derive a new divisibility for 6 using the divisibility rules above and your work in parts (a) and (b). Justify your rule.

4. Why does the divisibility rule for 10 work? To answer this question:
 - a) List the first several multiples of 10, including 10, in prime factorization form.
 - b) What do all of these multiples' prime factorizations have in common?
 - c) Justify the divisibility rule for 10.

Activity B (Small Group)

5. We can create our own divisibility rules for composite numbers that do not already have well-known rules, like we did for 6. Below are three divisibility rules that students created. For each rule, determine if the rule is correct. If the rule is false, give at least two counter-examples that show the rule is false. If the rule is true, explain why.
 - a) Divisibility rule for 12: check for divisibility by 2 and 6.
 - b) Divisibility rule for 20: check for divisibility by 4 and 5.
 - c) Divisibility rule for 24: check for divisibility by 4 and 6.

Group Discussion Question (Whole Class)

- Develop general guidelines for creating divisibility rules for composite numbers. Use the divisibility rule for 24 to illustrate your guidelines.

Summarize and Connect

Write your responses to the following questions:

- Create a divisibility rule for 36. Use prime factorization to explain why your rule works and why the most likely alternative to your rule cannot work.
- Maddy is looking to determine the divisors, or factors, of $N = 2^3 \times 3 \times 5^2 \times 11 \times 31^4$. When her teacher asks her if N is divisible by 5, Maddy confidently claims that N is divisible by 5 because 5 is one of N 's prime factors. However, when asked if N is divisible by 7, Maddy is unsure. She says "I'm not sure because I don't see 7 in N 's prime factorization. I need to multiply out all of N 's prime factors and then divide by 7 to be sure." How does the Fundamental Theorem of Arithmetic help resolve Maddy's uncertainty about whether N is divisible by 7?

Greatest Common Factor

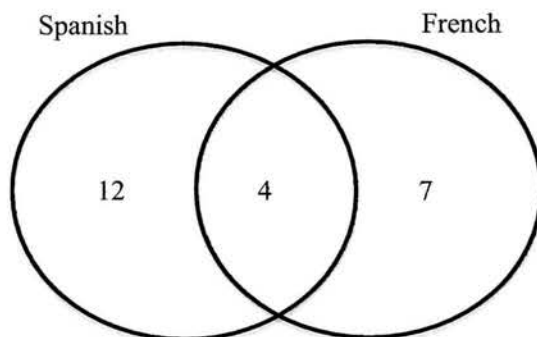
The purpose of this task is to make sense of different methods for finding the greatest common factor (GCF) of two numbers.

Venn Diagrams (Whole Class)

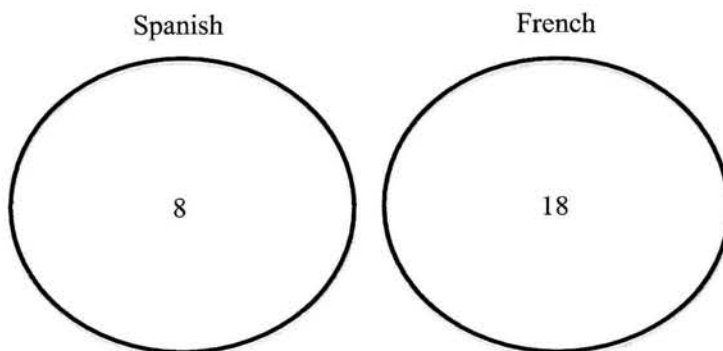
1. What does the term, *greatest common factor*, mean?

A *Venn Diagram* is a very useful tool for helping determine the greatest common factor (GCF) of two numbers. It can also be used in non-mathematical situations. Here are two examples:

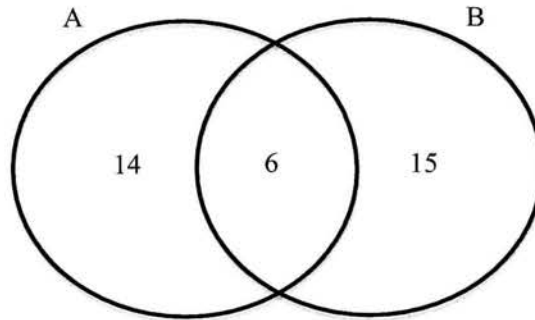
- A. In Ms. Shelby's class, 16 students speak Spanish and 11 speak French. Of these students, 4 speak both Spanish and French.



- B. In Mr. Roth's class, 8 students speak Spanish and 18 speak French. There are no students who speak both Spanish and French.



2. a) Leora draws the following Venn diagram to find the greatest common factor of two numbers, A and B. Where in her diagram do you find the GCF? Why?



- b) What do the 14 and 15 represent in her diagram?
- c) Does her diagram imply that $A = 20$ and $B = 21$? Explain why or why not.
- d) Alter Leora's diagram to show the prime factors of A and B. What are the values of A and B?

Activity A (Small Group)

3. Make a Venn diagram for 48 and 96.
- a) What is the greatest common factor of 48 and 96?
- b) Where in the diagram do you find the GCF?
4. Make a Venn diagram for 56 and 33.
- a) What is the greatest common factor of 56 and 33?
- b) Where in the diagram do you find the GCF?
5. Mariela finds the GCF of 252 and 270 using prime factorization instead of a Venn diagram. She compares their prime factorizations in the following way:

$$\begin{array}{l}
 252 = 2^2 \times 3^2 \times 7 \\
 270 = 2 \times 3^3 \times 5
 \end{array}
 \left. \vphantom{\begin{array}{l} 252 \\ 270 \end{array}} \right\} \text{GCF} = 2 \times 3^2 = 18$$

- a) Explain Mariela's method. Why does it work?
- b) Use her method to find the GCF of 550 and 968.

Group Discussion Question (Whole Class)

- How is Mariela's method similar to and different from the Venn diagram method?

Activity B (Small Group)

Show how you solve each word problem below:

6. Barney is creating gift bags for a party he is hosting. Each gift bag must have cufflinks and neckties. He has 90 cufflinks and 75 neckties with which to make the bags. If each bag must be identical and every cufflink and necktie must be used, how many gift bags can Barney possibly make?
7. Robin is making balloon arrangements for a birthday party. Each arrangement must have red and blue balloons. Robin has a total of 32 red and 24 blue balloons. What is the greatest number of balloon arrangements she can make if each arrangement must be identical and every balloon must be used?
8. Make up your own word problem whose solution requires finding the GCF of two numbers.

Group Discussion Questions (Whole Class)

- Discuss Questions 6-8.

Summarize and Connect

Write your responses to the following question:

- What is the relationship between the Venn diagram and prime factorization methods for finding the greatest common factor of two numbers? Use the GCF of 168 and 180 to illustrate your ideas.

Least Common Multiple

The purpose of this task is to develop participants' understanding of the least common multiple (LCM) of two numbers, and to relate this concept to prime factorization.

Definition (Whole Class)

1. What does the term, *least common multiple*, mean?

Activity A (Small Group)

2. Bella is trying to find the LCM of 10 and 18. Her work is below:

The first few multiples of 10 are 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120,

The first few multiples of 18 are 18, 36, 54, 72, 90, 108, 126, 144, 162, 180,

The LCM of 10 and 18 must be 90!

Describe Bella's method. Does it work? Why or why not.

3. Let's analyze Bella's method a little more carefully. Fill out the table below by finding the prime factorization of each multiple of 10 and 18.

Multiples of 10	Prime Factorization	Multiples of 18	Prime Factorization
10		18	
20		36	
30		54	
40		72	
50		90	
60		108	
70		126	
80		144	
90		162	
100		180	

- What does the prime factorization of every multiple of 10 have in common?
- What does the prime factorization of every multiple of 18 have in common?
- What does the prime factorization of 90 have in common with the multiples of 10 and the multiples of 18?
- Why is 90 the LCM of 10 and 18?

Group Discussion Questions (Whole Class)

- How is the prime factorization of the LCM of two numbers connected to the prime factorization of each of those two numbers?
- Can you remove one of the prime factors from either 10 or 18 and still preserve the same LCM?
- Can a Venn diagram help illustrate the LCM of two numbers?

Activity B (Small Group)

4. Jessie is also trying to find the LCM of 10 and 18. Her work is below:

$$\begin{array}{l} 18 = 2 \times 3^2 \\ 10 = 2 \times 5 \end{array} \left. \vphantom{\begin{array}{l} 18 = 2 \times 3^2 \\ 10 = 2 \times 5 \end{array}} \right\} LCM = 2 \times 3^2 \times 2 \times 5 = 180.$$

Is Jessie's method valid? Explain.

5. Mariela uses prime factorization to find the LCM of 84 and 90. Her work is below:

$$\begin{array}{l} 84 = 2^2 \times 3 \times 7 \\ 90 = 2 \times 3^2 \times 5 \end{array} \left. \vphantom{\begin{array}{l} 84 = 2^2 \times 3 \times 7 \\ 90 = 2 \times 3^2 \times 5 \end{array}} \right\} LCM = 84 \times 3 \times 5 = 1260$$

- Does Mariela's method work? Why or why not?
- Use her method to find the LCM of 550 and 968.

Group Discussion Questions (Whole Class)

- Discuss Question 5.

- Can you use a Venn diagram to make sense of Mariela's method? Explain

Activity C (Small Group)

6. Marshall and Lily decide to open a lemonade stand together. Both work today. Marshall works every 12 days, while Lily works every 8 days. How many days must they wait until they both work on the same day again?
7. Ted is buying materials for the skyscraper scale model he is constructing. He needs screws and plugs, but screws come in bags of 24 while plugs come in bags of 15. If he needs to buy the same number of screws and plugs, what is the least number of screws he must buy?
8. Make up your own word problem whose solution requires finding the LCM of two numbers.

Group Discussion Questions (Whole Class)

- Discuss Questions 6-8.

Summarize and Connect

Write your responses to the following questions:

- Describe two different methods of using multiples and prime factorization to find the LCM of two numbers. Use $N = 2^4 \times 3^2$ and $K = 2^2 \times 3^3 \times 5$ to illustrate your ideas.
- Given the three clues below, identify the two numbers in the mystery number pair.
Clue 1: The greatest common factor of the mystery pair is 9.
Clue 2: The least common multiple of the mystery pair is 126.
Clue 3: Both of the numbers in the mystery pair have two digits.

APPENDIX G
SOLUTIONS TO NUMBER THEORY TASKS

Factors – Part I

1. The factors and divisors of a number are identical. Both terms refer to any integer, z , such that if z divides evenly into some integer, y , then z is a factor or divisor of y . Another way to think about factors or divisors is that if some integer is a factor of y , then it can be multiplied by another integer to create the product, y . The term factor is usually used when referring to multiplication, while the term divisor is used when discussing division.

A multiple of some integer, z , is any number that is the product of z and some integer. For example, consider the equation $z = d \times a$, where a , d , and z are all integers. In this equation, z is a multiple of a and d , while a and d are factors, or divisors, of z .

2. 234 has twelve distinct factors: 1, 2, 3, 6, 9, 13, 18, 26, 39, 78, 117, and 234.

I found them all because I determined all the factor pairs whose product is 234. Since the square root of 234 is slightly greater than 15, each factor pair of 234 must include one factor less than or equal to 15 and one factor greater than or equal to 15. I started building factor pairs by dividing 234 by the counting numbers and skipping over counting numbers that are not factors of 234. Once I discovered that 13 is a factor of 234, all I had to check was if 14 and 15 are factors of 234 (they are not).

3. b) Best first move: 47 (gain of 46 points, 47-1)
Best second move: 49 (gain of 42 points, 49-7)
Best third move: 35 (gain of 30 points, 35-5)
Best fourth move: 39 (gain of 23 points, 39-16)
Best fifth move: 26 (gain of 24 points, 26-2)
- c) You need to look for numbers that have relatively few factors. The best first move is to pick the greatest prime (47) because your opponent will only get 1 point. After that, you should not pick any primes because you will lose a turn. Squares of primes, such as 49, and products of two primes, such as 46, are all good candidates. However, you need to pay attention to what factors are still available on the board. As more factors are circled, certain numbers (for example, 25 and 21) can no longer be circled.

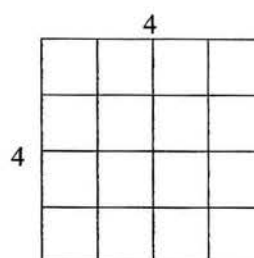
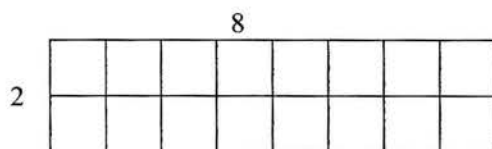
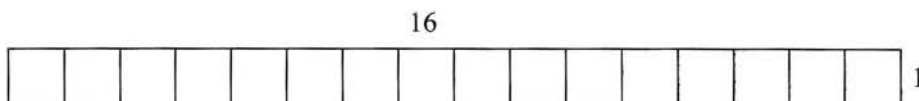
Group Discussion Question (Whole Class)

- The key to the factor game is recognizing which composite numbers have lots of factors and which have few. For example, picking a number that is the product of two primes is a good early move especially if one of the factors has already been chosen. One useful strategy is to calculate the point gains when selecting numbers for your turn. For example, if you pick 49 and your opponent circles 7, then you have just accumulated 42 (49-7) points!

4. a)

Number	Factors	No. of Factors
4	1, 2, 4	3
9	1, 3, 9	3
16	1, 2, 4, 8, 16	5
36	1, 2, 3, 4, 6, 9, 12, 18, 36	9
121	1, 11, 121	3

b) The rectangular arrays for 16 are below:



c) Some numbers have an odd number of factors because they are squares, so one of their factors is used twice to make a factor pair (e.g., the factor pairs of 4 are 1×4 and 2×2).

d) 129 has an even number of factors since it is not a square number.

$5^2 \times 3^2$ has an odd number of factors since it can be written as a square number (15^2).

4^6 has an odd number of factors since it can be written as a square number (64^2).

$3^4 \times 11^2 \times 13$ has an even number of factors since it is not a square number (there is only one group of 13 in its prime factorization, and each prime factor must appear an even number of times for a square number).

Group Discussion Question (Whole Class)

- Square numbers have an odd number of factors. All factors can be grouped in pairs that multiply to the original number but in the case of square numbers, the factors in one factor pair are identical and only counted/listed once (the square root of the number).

Factors – Part II

1. Students do not need to fill out every row of this table; just enough to allow them to begin recognizing patterns.

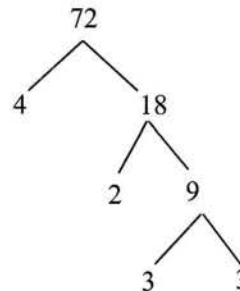
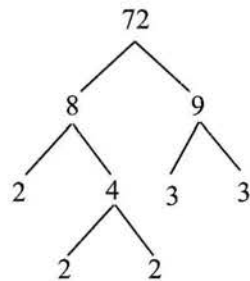
<u>2 Factors</u>		<u>3 Factors</u>	
#	Factors	#	Factors
2	1, 2	4	1, 2, 4
3	1, 3	9	1, 3, 9
5	1, 5	25	1, 5, 25
7	1, 7	49	1, 7, 49
11	1, 11	121	1, 11, 121
<u>4 Factors</u>		<u>5 Factors</u>	
#	Factors	#	Factors
6	1, 2, 3, 6	16	1, 2, 4, 8, 16
8	1, 2, 4, 8	81	1, 3, 9, 27, 81
15	1, 3, 5, 15	625	1, 5, 25, 125, 625
27	1, 3, 9, 27	2401	1, 7, 49, 343, 2401
35	1, 5, 7, 35	14641	1, 11, 121, 1331, 14641

2. a) A prime number has exactly two distinct factors.
 b) A number that is the square of a prime number has exactly three distinct factors.
 c) A number that is either the product of two distinct primes or is the cube of a prime has exactly four distinct factors.
 d) A number that is the fourth power of a prime has exactly five distinct factors.

Group Discussion Question (Whole Class)

- A prime number is a counting number that has exactly two distinct factors, 1 and the number itself. A composite number is a counting number that has more than two distinct factors.
 - Refer to Question 2 solutions above.
3. Based on the given factor tree, the prime factorization of 72 is $2^3 \times 3^2$.

4. The prime factorization of 72 is $2^3 \times 3^2$ based on each of the following two factor trees.



5. The prime factorization of 72 remains the same for all three factor trees. No, it is not possible to find a different factor tree of 72 that illustrates a different prime factorization for 72. The reason is that the prime factorization of 72 is always the same no matter how 72 is factored.
6. a) $N = K$. If one decomposes the factors of N and K further into their prime factors, one finds that N and K both equal $2 \times 3 \times 3 \times 2 \times 2 \times 2 \times 5 = 3^2 \times 2^4 \times 5$.
- b) $R = N = K$ because $3^2 \times 4^2 \times 5 = 3^2 \times (2^2)^2 \times 5 = 3^2 \times 2^4 \times 5$. This is the same prime factorization as N and K .
- c) A number can have several different factorizations, but only one possible prime factorization. For example, 2×18 , 4×9 , 12×3 represent three possible factorizations of 36 since they represent any combination of factors whose product is 36. However, the prime factorization of 36 must be $2 \times 2 \times 3 \times 3$, since this is the only combination of prime factors whose product is 36. Any factorization of 36 can be further factored to eventually result in the prime factorization.
7. The Fundamental Theorem of Arithmetic means that any composite number has a single, unique prime factorization. No matter how many factorizations a number might have, each of its factorizations can be decomposed further into the same exact product of prime factors. This theorem is important because it says that every counting number can be constructed from prime factors in a unique way – every composite number has its own unique “genetic make-up.”

8.

<u>2 Factors</u>		<u>3 Factors</u>	
#	Prime Factorization	#	Prime Factorization
2	2	4	2×2
3	3	9	3×3
5	5	25	5×5
7	7	49	7×7
11	11	121	11×11
<u>4 Factors</u>		<u>5 Factors</u>	
#	Prime Factorization	#	Prime Factorization
6	2×3	16	$2 \times 2 \times 2 \times 2$
8	$2 \times 2 \times 2$	81	$3 \times 3 \times 3 \times 3$
15	3×5	625	$5 \times 5 \times 5 \times 5$
27	$3 \times 3 \times 3$	2401	$7 \times 7 \times 7 \times 7$
35	5×7	14641	$11 \times 11 \times 11 \times 11$

9. Yes, you can use the prime factorization of a number to determine all of its factors. Since $10 = 2 \times 5$, its factors are 1, 2, 5, and 2×5 , or 10.

10. Find all combinations of one, two, three, four, and five prime factor products of M .

Its factors are:

1, 2, 5, 7, 53 (single prime factors)

2×5 , 2×7 , 2×53 , 5×7 , 5×53 , 7×7 , 7×53 (product of two prime factors)

$2 \times 5 \times 7$, $2 \times 5 \times 53$, $2 \times 7 \times 7$, $2 \times 7 \times 53$, $5 \times 7 \times 7$, $5 \times 7 \times 53$, $7 \times 7 \times 53$ (product of three prime factors)

$2 \times 5 \times 7 \times 7$, $2 \times 5 \times 7 \times 53$, $2 \times 7 \times 7 \times 53$, $5 \times 7 \times 7 \times 53$ (product of four prime factors)

$2 \times 5 \times 7 \times 7 \times 53$ (product of five prime factors)

Simplified, the factors are:

1, 2, 5, 7, 53

10, 14, 106, 35, 265, 49, 371
70, 530, 98, 742, 245, 1855, 2597
490, 3710, 5194, 12985
25970

Group Discussion Question (Whole Class)

- You can use the prime factors of a number to find the number's factors by listing all combinations of one, two, three, four, etc. prime factor products. First, list the prime factors of the number; then all products of two prime factors; then all products of three prime factors; etc...

Divisibility

- N is divisible by 2, 6, 7, 8, and 63 because each divisor is represented in the prime factorization of N . Of these, the prime divisors are visible in the prime factorization (7 and 2), while the composite divisors are the product of a combination of N 's prime factors ($6 = 2 \times 3$; $8 = 2^3$; $63 = 3^2 \times 7$). N is not divisible by 5, 13, and 27 because none of these numbers are part of the prime factorization of N , either visibly or as the product of N 's prime factors.
- K is not divisible by 3, 5, 7, or 11 since none are represented in K 's prime factorization.
 - K is not divisible by 26 because $26 = 2 \times 13$ and 2 is not a factor of K . K is not divisible by 39 because $39 = 3 \times 13$ and 3 is not a factor of K . In general, a number is divisible by another number if all of the divisor's factors are also factors of the dividend.
 - K is divisible by all product combinations of its prime factors: 1, 13, 61, 127, 13×61 , 13×127 , 61×127 , and K itself.

Group Discussion Questions (Whole Class)

- The prime factors of a number can help identify a number's factors. A number is only divisible by some product combination of its prime factors, including 1 and the number itself.
 - A is divisible by B if and only if B 's prime factorization is represented in the prime factorization of A .
-

Multiples of 6	Prime Factorization
6	2×3
12	$2^2 \times 3$
18	2×3^2
24	$2^3 \times 3$
30	$2 \times 3 \times 5$
36	$2^2 \times 3^2$
42	$2 \times 3 \times 7$

- All of the multiples of 6 have a (2×3) in their prime factorizations.
- A number is divisible by 6 if it is divisible by 2 and by 3. If these requirements are met, then the prime factorization of the number will have a 2×3 in it, which makes it a multiple of 6.

4. a)

Multiples of 10	Prime Factorization
10	2×5
20	$2^2 \times 5$
30	$2 \times 3 \times 5$
40	$2^3 \times 5$
50	2×5^2

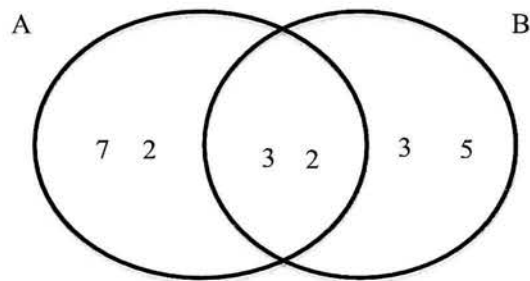
- b) All of the multiples of 10 have a (2×5) in their prime factorizations.
- c) A number is divisible by 10 if it is divisible by 2 and by 5. If these requirements are met, then the prime factorization of the number will have a 2×5 in it, which makes it a multiple of 10. All multiples of 10 end in a zero.
5. a) This rule is incorrect. 6 and 18 are two counter-examples since both are divisible by 2 and by 6 but neither is divisible by 12. If a number is divisible by 6, then it must contain 2×3 in its prime factorization. Thus, checking to see if this number is divisible by 2 is redundant, since 2 is already known to be one of its prime factors. Thus, the greatest number this rule can detect is divisibility by 2×3 , or 6. To properly check for divisibility by 12, one must check for divisibility by 3 and by 4.
- b) This rule is correct. If a number is divisible by 4 and 5, then it must have 2^2 and 5 in its prime factorization. Thus, it must have $(2^2 \times 5)$, or 20, as a factor.
- c) This rule is incorrect. 12 and 36 are two counter-examples since both are divisible by 4 and 6 but neither is divisible by 24. If a number is divisible by 6, then it must contain 2×3 in its prime factorization. If a number is divisible by 4, then it must contain 2×2 in its prime factorization. However, since divisibility by 6 already ensures that one 2 is in the prime factorization of the number, divisibility by 4 can only inform us of a second 2 in the prime factorization. Thus, the greatest number this rule can detect is divisibility by $2 \times 3 \times 2$, or 12, not 24. To properly check for divisibility by 24, one must check for divisibility by 3 and by 8.

Group Discussion Question (Whole Class)

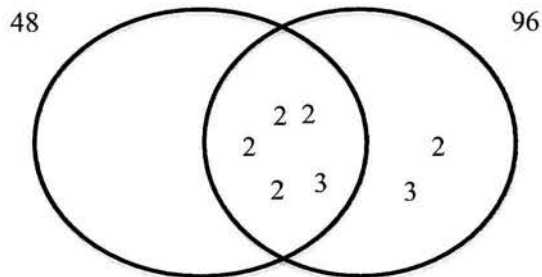
- Checking divisibility by a given number requires one to check divisibility by factors of that number that are relatively prime to each other. For example, divisibility by 24 requires checking divisibility by 3 and 8, since they do not share any factors. Divisibility by 3 ensures a 3 is one of the prime factors; divisibility by 8 ensures that 2^3 is also in the prime factorization of the number. Thus, divisibility by both 3 and 8 ensures that 3×2^3 , or 24, is in the prime factorization of the number. Thus, the number is divisible by 24.

Greatest Common Factor

1. The greatest common factor of two numbers is the greatest number that is a factor of both original numbers.
2. a) The 14 and 15 represent factors of A and B, respectively.
b) The greatest common factor of A and B is located in the overlapping region of the Venn diagram. The reason is that each circle represents the factors of A and B, and so the overlapping region represents all of the common factors of A and B. The GCF of A and B is 6.
c) $A = 7 \times 2 \times 3 \times 2 = 84$; $B = 3 \times 2 \times 3 \times 5 = 90$.

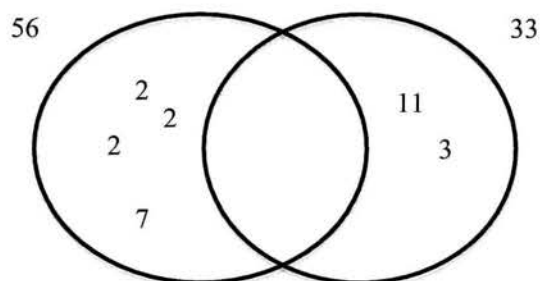


3.



- a) The greatest common factor of 48 and 96 is 48.
- b) The GCF is the product of the prime factors in the overlapping section of the diagram.

4.



- a) The greatest common factor of 56 and 33 is 1.
- b) The GCF is in the overlapping section of the diagram. Since no prime factors reside in the overlapping region, the GCF equals 1.
5. a) Mariela finds the GCF by identifying the common prime factors in the prime factorization of both numbers. Since both 252 and 270 contain one 2 and two 3's, their GCF is $2 \times 3 \times 3$, or 18.
- b) $550 = 2 \times 5^2 \times 11$
 $968 = 2^3 \times 11^2$
GCF = $2 \times 11 = 22$

Group Discussion Question (Whole Class)

- Mariela's method is similar to the Venn diagram method because both require one to find the prime factorization of each number and identify which prime factors are common to both numbers. Mariela's method is different from the Venn diagram because it is a symbolic representation of prime factorization, not a pictorial representation like the Venn diagram.
6. Barney can make fifteen gift bags because 15 is the GCF of 90 and 75. Since each gift bag must be identical with no items left over, the number of cufflinks in each bag must be a factor of 90 and the number of neckties in each bag must be a factor of 75. Thus, the number of gift bags must be a factor of both 90 and 75. The only choices are 1, 3, 5, and 15, of which 15 is the greatest.
7. Robin can make a maximum of eight balloon arrangements since 8 is the GCF of 32 and 24. Since all balloons must be used and each arrangement must be identical, the number of arrangements must be a factor of both 32 and 24. The only choices are 1, 2, 4, and 8. 8 is the greatest of the common factors.
8. Student responses may vary.

Group Discussion Questions (Whole Class)

- See answers to Questions 6-8 above.

Least Common Multiple

1. The least common multiple of two numbers represents the smallest number that is a multiple of both original numbers.
2. Bella lists the multiples of 10 and 18 in two separate lists. She then looks for multiples that are on both lists. The first one she finds is the one she circles as the GCF of 10 and 18. This method works because Bella is finding the common multiples of both 10 and 18 and picking the smallest one.
- 3.

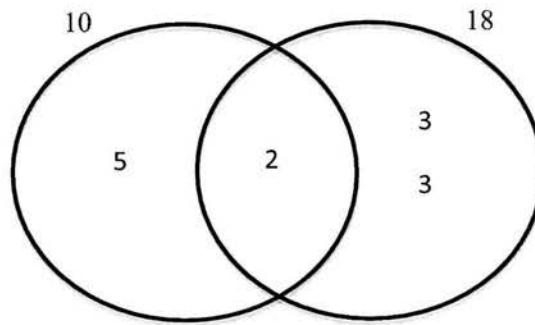
Multiples of 10	Prime Factorization	Multiples of 18	Prime Factorization
10	2×5	18	2×3^2
20	$2^2 \times 5$	36	$2^2 \times 3^2$
30	$2 \times 3 \times 5$	54	2×3^3
40	$2^3 \times 5$	72	$2^3 \times 3^2$
50	2×5^2	90	$2 \times 3^2 \times 5$
60	$2^2 \times 3 \times 5$	108	$2^2 \times 3^3$
70	$2 \times 5 \times 7$	126	$2 \times 3^2 \times 7$
80	$2^4 \times 5$	144	$2^4 \times 3^2$
90	$2 \times 3^2 \times 5$	162	2×3^4
100	$2^2 \times 5^2$	180	$2^2 \times 3^2 \times 5$

- a) The prime factorization of every multiple of 10 has a (2×5) in common.
- b) The prime factorization of every multiple of 18 has a (2×3^2) in common.
- c) The prime factorization of 90 has a (2×5) and a (2×3^2) in common with the multiples of 10 and 18, respectively.
- d) 90 is the LCM of 10 and 18 because it contains both the prime factors of 10 and of 18 and is the least number to do so.

Group Discussion Question (Whole Class)

- The prime factorization of the LCM of two numbers contains the prime factors of both numbers. It does so without repeating any shared prime factors more times than is necessary in order to contain the complete prime factorizations of both numbers. For example, since 10 and 18 share a common prime factor of 2, their LCM needs to contain only one 2 in its prime factorization.
- If you remove any of the non-shared prime factors (3 or 5), the LCM will change. If you remove one of the shared prime factors (2), the LCM will remain the same.

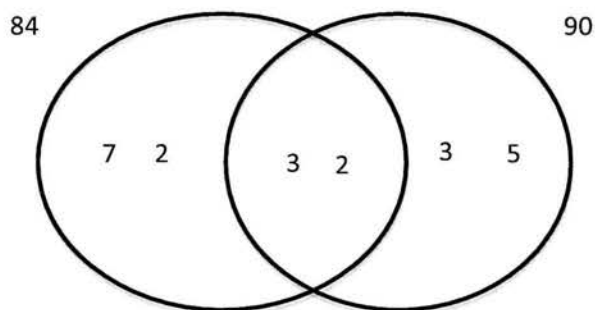
- Yes, a Venn diagram is one tool that can help illustrate the LCM of two numbers. The LCM is represented by multiplying together all of the prime factors in the entire diagram (without double-counting the overlapping region). The following is a Venn diagram illustrating the LCM of 10 and 18:



4. No, Jessie's method is not valid. She is finding a common multiple, but not the least common multiple. The reason is that she is simply multiplying together all of the prime factors of both 10 and 18 without accounting for the shared prime factor of 2. Since 2 is shared by both 10 and 18, it needs only to be represented once, not twice, in the prime factorization of the LCM.
5. a) Yes, Mariela's method works because she is finding the smallest multiple of 84 that contains all of the prime factors of 90. This is a slightly different way of finding the LCM than shown in Question 3. Here, Mariela knows that the LCM of 84 and 90 must be a multiple of either number. So, she starts with 84 and multiplies it by the prime factors of 90 that 84 does not already contain (i.e., 5 and a second 3).
 b) $550 = 2 \times 5^2 \times 11$
 $968 = 2^3 \times 11^2$
 $\text{LCM} = 550 \times 2^2 \times 11 = 24200$, or $\text{LCM} = 968 \times 5^2 = 24200$

Group Discussion Question (Whole Class)

- See answer for Question 5 above. Note that Mariela's method can be done in two ways: start with the lesser number and multiply it by the prime factors of the greater number that it does not contain; or start with the greater number and multiply it by the prime factors of the lesser number that it does not contain.
- Yes, a Venn diagram can provide a visual representation of Mariela's method.



In the diagram above, the LCM can be viewed as the product of the prime factors in the left-hand circle (representing 84's prime factorization) multiplied by the prime factors in the right-hand circle that are not contained in left-hand circle. Thus, 84 is multiplied by 3 and 5, the only prime factors of 90 not included in the prime factorization of 84. This is another way to see why the LCM of two numbers is the product of all of the prime factors within a Venn diagram (and why one does not double-count the overlapping region).

6. Marshall and Lily must wait 24 days because 24 is the LCM of 12 and 8. Since Marshall works every 12 days, he will only work on days that are multiples of 12 (i.e., 12, 24, 36, etc.). Since Lily works every 8 days, she will only work on days that are multiples of 8 (i.e., 8, 16, 24, 32, etc.). The first day their work schedules coincide is on day 24.
7. Ted must buy at least 120 screws because 120 is the LCM of 24 and 15. Since screws come in bags of 24, he can only buy a multiple of 24. Since plugs come in bags of 15, he can only buy a multiple of 15. Since 120 is the smallest number that is both a multiple of 24 and 15, he can buy 120 screws and 120 plugs.
8. Student responses may vary.

Group Discussion Questions (Whole Class)

- See answers to Questions 6-8 above.

APPENDIX H

MATHEMATICAL BACKGROUND INFORMATION FORM

Mathematical Background Information Form

Participant Background Information

Name: _____ Gender: _____ M _____ F

Major/Concentration (check all that apply):

____ Elementary Education ____ Special Education
____ Early Childhood Education ____ Other (specify: _____)

Year in School: ____ FR ____ SO ____ JR ____ SR ____ GRAD

Previous Math Courses Taken (check all that apply):

____ Algebra I ____ Geometry ____ Calculus I
____ Algebra II ____ Pre-calculus ____ Other (specify: _____)

AP Mathematics Credit Received (check all that apply):

____ AP Statistics ____ Calculus AB ____ Calculus BC ____ None

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