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Negative mass corrections in a dissipative stochastic environment

Luca D'Alessio,^{1,2} Yariv Kafri,³ and Anatoli Polkovnikov²

¹*Department of Physics, The Pennsylvania State University, University Park, PA 16802, USA*

²*Department of Physics, Boston University, Boston, MA 02215, USA*

³*Department of Physics, Technion, Haifa 32000, Israel*

We study the dynamics of a macroscopic object interacting with a dissipative stochastic environment using an adiabatic perturbation theory. The perturbation theory reproduces known expressions for the friction coefficient and, surprisingly, gives an additional *negative* mass correction. The effect of the negative mass correction is illustrated by studying a harmonic oscillator interacting with a dissipative stochastic environment. While it is well known that the friction coefficient causes a reduction of the oscillation frequency we show that the negative mass correction can lead to its *enhancement*. By studying an exactly solvable model of a magnet coupled to a spin environment evolving under standard non-conserving dynamics we show that the effect is present even beyond the validity of the adiabatic perturbation theory.

I. INTRODUCTION

In this article we analyze the dynamics of a macroscopic object coupled to a stochastic environment which consists of many degrees of freedom. It is well known that coupling to an environment leads to dissipation which is typically described by an irreversible friction force. A classical example is the equation of motion describing a single macroscopic particle moving through a medium:

$$m \ddot{\mathbf{X}} + \eta \dot{\mathbf{X}} = -\frac{\partial V}{\partial \mathbf{X}}. \quad (1)$$

Here m and \mathbf{X} are the mass and the position of the particle respectively, η is the friction coefficient and V includes both external potentials and the interaction energy of the particle with the medium. When the potential can be expanded around some equilibrium position $\mathbf{X} = 0$, it becomes $V(\mathbf{X}) \approx \frac{1}{2}m\omega_0^2 \mathbf{X}^2$ and Eq. (1) describes a damped harmonic oscillator. In the under-damped regime $\zeta < 1$, \mathbf{X} oscillates with a *shifted* frequency $\omega = \omega_0 \sqrt{1 - \zeta^2}$ which is always smaller than ω_0 . Here $\zeta = \eta / (2m\omega_0)$ is the damping ratio. The amplitude of the oscillations decays to zero on a time scale $\omega_0 \zeta$. When $\zeta > 1$ the motion becomes over-damped and no oscillations are observed.

Equation (1) can be derived either using phenomenological considerations or through microscopic approaches. For example, Eq. (1) can be obtained if one assumes that the excitations in the medium are small enough so that the medium is only slightly away from equilibrium and linear-response holds [1, 2]. In a different approach, which we follow here, one uses an adiabatic perturbation theory which only assumes that \mathbf{X} changes on a time scale that is longer than the relaxation time in the medium. This approach has been applied successfully to both classical and quantum systems [3–9].

In what follows we use an adiabatic perturbation theory to consider a macroscopic object moving in a *stochastic* environment comprised of many degrees of freedom. The stochastic environment evolves according to Markovian dynamics which obeys detailed balance so that,

if the macroscopic object is stationary, the environment reaches thermal equilibrium with the probability of micro-state s given by $P(s) = Z(\mathbf{X})^{-1} e^{-\beta H(\mathbf{X}, s)}$. Here $H(\mathbf{X}, s)$ is the energy function of the environment, $Z(\mathbf{X})$ is the partition function at fixed value of \mathbf{X} , β is the inverse temperature of the environment and we set the Boltzmann constant to one. Note that by using a Markovian environment we effectively assume that there are two time scales: i) a fast time scale associated with the transitions between microscopic configurations of the environment and ii) a longer time scale associated with the evolution of the macroscopic properties of the environment. The adiabatic perturbation theory is carried out with respect to the second (slower) time scale. For example, for a spin environment the first time scale is associated to individual spin flips while the second one is associated with the evolution of the total magnetization. The adiabatic perturbation theory assumes that the object moves slowly compared to the time scale associated to the evolution of the total magnetization so that the latter is almost equilibrated at every instantaneous position of the object.

The organization of the paper is as follows. In Sec. II we introduce our main result for the friction coefficient and mass renormalization of the macroscopic object in contact with a dissipative stochastic environment. We then apply these results to a simple damped harmonic oscillator to show that, under appropriate conditions, the *negative* mass correction can lead to an *enhancement* of the oscillation frequency. In Sec. III we study a more concrete model of a magnetic oscillator interacting with a spin environment in which our results can be derived explicitly. By comparing the perturbative and the exact solution we show that an enhancement of the oscillation frequency can occur even when the adiabatic perturbation theory fails. In Sec. V we illustrate the consequence of the (negative) mass renormalization on the dynamics of the energy of the macroscopic object and show that, in general, the energy is not a monotonously decreasing function of time despite the fact that the object is in contact with a dissipative environment. Finally in Sec. VI we present a detailed derivation of our results and sys-

tematically introduce the adiabatic perturbation theory. Sec. VII contains our conclusions and future possible research directions.

II. MAIN RESULT

Our main result is that the first (irreversible and reversible) corrections to the motion of \mathbf{X} from the adiabatic perturbation theory give rise to the following equations of motion:

$$(m + \kappa) \ddot{\mathbf{X}} + \eta \dot{\mathbf{X}} = -\frac{\partial V}{\partial \mathbf{X}}, \quad (2)$$

after averaging over histories of the stochastic environment, where

$$\begin{aligned} \eta(t) &= \beta \int_t^\infty dt' \langle \partial_{X_i} H(t) \partial_{X_i} H(t-t') \rangle_{0,c} \quad \text{and} \\ \kappa(t) &= -\beta \int_t^\infty dt' t' \langle \partial_{X_i} H(t) \partial_{X_i} H(t-t') \rangle_{0,c}. \end{aligned} \quad (3)$$

are the friction coefficient and the mass correction respectively. Angular brackets indicate an average over the equilibrium distribution of the medium at inverse temperature β at the instantaneous value of X_i and c indicates a connected correlation function. While in general η and κ are tensors ($\eta_{i,j}$ and $\kappa_{i,j}$), we consider the simple case where there is no preferred direction in space, such that $\eta_{i,j}$ and $\kappa_{i,j}$ are proportional to the identity matrix. Our perturbative treatment is systematic and can be generalized to obtain higher order corrections in $\dot{\mathbf{X}}$ and, in relevant cases, the explicit tensorial behavior of η and κ . The adiabatic perturbation theory carried out here assumes that: i) the environment relaxation time τ is much faster than the bare frequency of motion of the macroscopic object, i.e. $\omega_0 \tau \ll 1$, and that ii) during a relaxation time the object moves a distance smaller than l , where l is the length over which τ and $\frac{\partial^2 V}{\partial X^2}$ change, i.e. $\dot{\mathbf{X}}\tau \ll l$. When these two conditions are satisfied the dynamics of the macroscopic object is described by Eq. (2) and the effect of the environment is contained in just two parameters, η and κ . On the contrary when these conditions break down the dynamics of the object is not given by Eq. (2) and one has to solve a full integral equation which describes both the action of the macroscopic object on the environment and the back-action of the environment on the macroscopic object. The detailed derivation of the adiabatic perturbation theory is given in Sec. VI.

Eq. (3) is identical to the high temperature limit of the expressions recently derived for quantum systems in Ref. [9]. However we note that in [9] the derivation follows a different approach and makes different assumptions about the environment and, as we discuss later, leads to very different predictions. The expression for η is well known from previous works [10, 11], for example, using linear response [12]. It is always non-negative

at positive temperatures. An important and interesting result, which to the best of our knowledge was previously overlooked, is that the mass correction κ can be *negative*. This, for example, happens when the dynamics of the bath is over-damped and the correlation function $\langle \partial_{X_i} H(t) \partial_{X_i} H(t-t') \rangle_{0,c}$ monotonically decreases in time. Both the mass correction and the dissipation originate from driving the medium out of equilibrium through motion of the macroscopic particle. Both also depend on the coupling between the object and the medium through $\partial_X H$. When the coupling is weak, which is the region of main interest here, they are quadratic in the coupling strength, but in general their behavior can be more complicated.

A negative mass correction κ has interesting consequences. First of all we note that any mass renormalization implies that the dynamics of the energy of the macroscopic object is non-Markovian (see Sec. V below). Moreover, for an oscillating macroscopic object a *negative* mass correction can lead to an enhanced oscillation frequency in contrast to the usual suppression. To describe this phenomenon we consider a damped harmonic oscillator, $V(\mathbf{X}) = \frac{1}{2} m \omega_0^2 \mathbf{X}^2$, in the case where the relaxation of the connected “force-force” correlation function is given by:

$$\langle \partial_X H(t) \partial_X H(0) \rangle_{0,c} = g^2 e^{-t/\tau} \quad (4)$$

where τ is the relaxation time of the medium and g is the strength of a simple linear coupling between the macroscopic object and the medium. Substituting Eq. (4) into Eq. (3) we see that $\eta = \beta g^2 \tau$ and $\kappa = -\beta g^2 \tau^2$. Therefore, when the adiabatic perturbation theory is justified, the emergent equation of motion is then given by:

$$(m - \beta g^2 \tau^2) \ddot{\mathbf{X}} + \beta g^2 \tau \dot{\mathbf{X}} = -m \omega_0^2 \mathbf{X}. \quad (5)$$

Naively it might appear that κ is higher order in τ than η and therefore gives a negligible effect when the relaxation of the bath is fast (short τ), but this conclusion can be misleading. It is easy to check that, to leading order, the shift in the frequency due to the mass correction and dissipation are:

$$\delta \omega_\kappa^2 \approx \omega_0^2 \frac{\kappa}{m} = -\omega_0^2 \frac{\beta g^2 \tau^2}{m}, \quad \delta \omega_\eta^2 \approx \frac{\eta^2}{4m^2} = \frac{\beta^2 g^4 \tau^2}{4m^2}.$$

The two corrections have opposite sign and scale in the same way with τ so taking τ small does not guarantee that the frequency shift due to the dissipation dominates. The above expressions suggest that for small coupling between the system and the medium, or at high temperatures, the correction to the frequency due to the mass term $\delta \omega_\kappa$ dominates the frequency shift (see Fig. 1). More generally it is easy to show that the ratio between the renormalized and unperturbed frequency is:

$$\frac{\omega^2}{\omega_0^2} = \frac{1}{1 - \gamma \alpha} \left(1 - \frac{\alpha^2}{4(1 - \gamma \alpha)} \right)$$

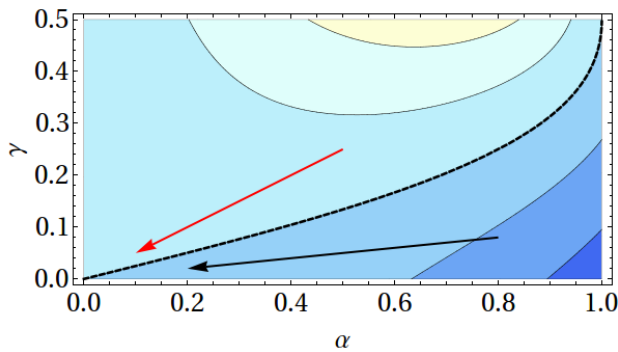


FIG. 1: (Color on-line) Contour plot of the ratio $\frac{\omega^2}{\omega_0^2}$ vs. $\gamma = \omega_0\tau$ and $\alpha = \frac{\beta g^2 \tau}{m\omega_0} = \frac{\eta}{m\omega_0}$ where ω_0 is the unperturbed frequency and $\tau = \frac{|\kappa|}{\eta}$. Along the dashed thick black line the correction due to friction and to the *negative* mass renormalization balance each other and, despite being immersed in a dissipative medium, the object oscillates with its natural frequency, i.e. $\omega = \omega_0$. The red and black arrows represent the limit $\tau \rightarrow 0$, where perturbation theory is best justified, for $\frac{\alpha}{\gamma} = \frac{\beta g^2}{m\omega_0^2} = 2$ and 10 respectively. The contour lines correspond to values from 0.8 to 1.2 (from blue to yellow) in steps of 0.1.

where $\alpha = \frac{\beta g^2 \tau}{m\omega_0} = \frac{\eta}{m\omega_0}$ and $\gamma = \omega_0\tau = \omega_0 \frac{|\kappa|}{\eta}$. For

$$0 < \alpha < \frac{4\gamma}{1 + 4\gamma^2} \quad (6)$$

the motion is under-damped and $\omega > \omega_0$. Specifically, in the small τ limit the frequency ω is enhanced if the ratio $\frac{\alpha}{\gamma} = \frac{\beta g^2}{m\omega_0^2}$, which is proportional to the square of the coupling strength, is smaller than 4. This confirms that the frequency is enhanced in the small coupling limit in agreement with our previous discussion. We note that the validity of the adiabatic perturbation theory requires $\omega\tau \ll 1$. For $\omega > \omega_0$ this also implies $\gamma = \omega_0\tau \ll 1$. In addition, as explained in the Sec. VI, we make the natural assumption that the properties of the environment, such as the relaxation time τ , change slowly with \mathbf{X} so that they can be considered constant while the macroscopic object moves. We also note that the condition $\gamma = \omega_0\tau \ll 1$ together with $\alpha < 4\gamma$ (see Eq. (6)) guarantees that the mass correction is small:

$$\frac{\eta}{m\omega_0} < 4\omega_0\tau \rightarrow |\kappa| < 4m(\omega_0\tau)^2 \ll m$$

where we have used $|\kappa| = \tau\eta$. Therefore when adiabatic perturbation theory applies and Eq. (2) is valid, the total mass $m + \kappa$ is always positive (despite κ being *negative*). Interestingly, the correction due to friction and to the *negative* mass renormalization can balance each other and, despite being immersed in a dissipative medium, the object oscillates with its natural frequency (see Fig. 1).

The above treatment relies on the adiabatic perturbation theory and simple assumptions about the behavior of the medium. Next we will study a more concrete

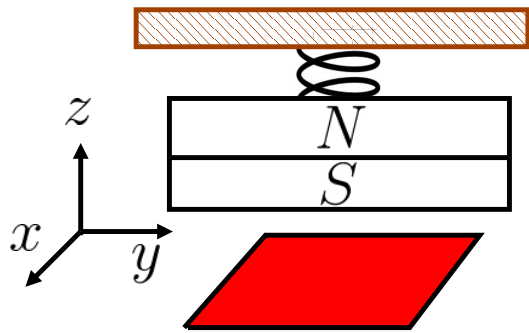


FIG. 2: (Color on-line) Schematic setup considered. A magnet oscillates on top of a two dimensional spin system (indicated in red).

model in which the results above can be derived explicitly. In addition to demonstrating the results derived above in an explicit model, we show that the enhancement of the oscillation frequency can occur even when the adiabatic perturbation theory fails. Finally, this model will also highlight possible shifts in the oscillation frequency which may occur due to the interaction with the medium. These trivial corrections are akin to the Born-Oppenheimer terms in quantum mechanics and are expected to be absent when the oscillator is moving in a translationally invariant medium (see Sec. VI).

III. MAGNETIC OSCILLATOR IN A SPIN ENVIRONMENT

Here we consider a magnetic oscillator which moves in the z direction and which is coupled to a thin two-dimensional spin system which evolves under standard non-conserving spin dynamics (see Fig. 2). The Hamiltonian of the oscillating magnet is:

$$H = \frac{p_z^2}{2m} + \frac{1}{2}m\omega_s^2 z^2 - h(z) \iint dx dy W(x, y) S(x, y) \quad (7)$$

where $S(x, y)$ is the (coarse grained) magnetization of the spin system at position (x, y) , $h(z)$ is the magnetic field at the spin plane which depends on the position z of the magnet, p_z is the momentum conjugate to z and ω_s is the spring oscillation's frequency. The arbitrary weight function $W(x, y)$ describes the spatial variation of the magnetic field in the plane of the spins. When the weight function is flat, i.e. $W(x, y) = \text{const}$, the oscillator is coupled only to the zero Fourier component of the spin configuration while for a generic profile $W(x, y)$ the oscillator is coupled to many Fourier components of the spin configurations. This Hamiltonian is valid when the thickness of the magnetic medium is negligible.

The resulting equation of motion for the magnet is:

$$\begin{aligned} m\ddot{z} &\approx -m\omega_s^2 z + h_1 \iint dx dy W(x, y) S(x, y) \\ &= -m\omega_s^2 z + h_1 \sum_{\mathbf{q}} S_{\mathbf{q}} W_{-\mathbf{q}}. \end{aligned} \quad (8)$$

where $S_{\mathbf{q}}$ and $W_{\mathbf{q}}$ are the Fourier components of the spin configuration and the profile function. In the equation above we have linearized the magnetic field near the equilibrium position: $h(z) \approx h_0 + h_1 z$. This approximation is justified if the amplitude, A , of the motion of the magnetic oscillator is small. In particular we require that $h_1 \gg h_2 A$ where $h_2 \equiv \partial_{zz}^2 h(z)|_{z=0}$. The effect of $h_2 \neq 0$ is discussed in Sec. VI A.

We model the spin environment via over-damped Langevin dynamics [13]:

$$\partial_t S = -\mu \frac{\delta F}{\delta S} + \xi(x, y, t)$$

where μ is the mobility, ξ is a white noise term which satisfies $\langle \xi \rangle = 0$ and

$$\langle \xi(x, y, t) \xi(x', y', t') \rangle = 2 \frac{\mu}{\beta} \delta(x - x') \delta(y - y') \delta(t - t'),$$

where β is the inverse temperature of the spin system and the angular brackets denote an average over noise realizations. F is the Landau-Ginzburg free-energy:

$$F = \iint dx dy \left[\frac{u}{2} |\nabla S|^2 + \frac{r}{2} S^2 - h(z) W(x, y) S(x, y) \right]$$

where we have neglected higher order terms in S . We therefore obtain the following equation of motion for the Fourier components of the spin configuration:

$$\partial_t S_{\mathbf{q}} = \mu \left[-(r + u|\mathbf{q}|^2) S_{\mathbf{q}} + (h_0 + h_1 z) W_{\mathbf{q}} \right] + \xi_{\mathbf{q}}(t) \quad (9)$$

where $\xi_{\mathbf{q}}$ is the Fourier components of the noise.

Next, we solve the coupled equations (8) and (9). First, for simplicity, we consider a case in which the weight function is flat so that the magnet is coupled only to the zero Fourier component of the spin configuration [14]. In particular in the equations above we set $W_{\mathbf{q}=0} = 1$ and $W_{\mathbf{q} \neq 0} = 0$ to obtain:

$$\begin{aligned} m\ddot{z} &= -m\omega_s^2 z + h_1 S_0 \\ \partial_t S_0 &= \mu (-r S_0 + h_0 + h_1 z) + \xi_0(t) \end{aligned} \quad (10)$$

It is then convenient to define $Z = \langle z \rangle - \langle z \rangle_{eq}$ and $\sigma_0 = \langle S_0 \rangle - \langle S_0 \rangle_{eq}$ where $\langle z \rangle_{eq}$, $\langle S_0 \rangle_{eq}$ are the stationary solutions of $\langle z \rangle$ and $\langle S_0 \rangle$ respectively:

$$\langle z \rangle_{eq} = \frac{h_0 h_1}{m\omega_s^2 r - h_1^2}, \quad \langle S_0 \rangle_{eq} = \frac{m\omega_s^2 h_0}{m\omega_s^2 r - h_1^2}.$$

Assuming that the system satisfies the stability condition $h_1^2 < m\omega_s^2$, the equations of motion reduce to:

$$m\ddot{Z} = -m\omega_s^2 Z + h_1 \sigma_0 \quad (11a)$$

$$\partial_t \sigma_0 = \mu (h_1 Z - r \sigma_0). \quad (11b)$$

These equations describe the motion of two degrees of freedom with a linear coupling between them. If the relaxation of the spin system is fast ($\mu r \gg \omega_s$) the magnetization σ_0 follows closely the instantaneous equilibrium: $\sigma_0 \approx h_1 Z/r$. This is a direct analogue of the Born-Oppenheimer approximation in quantum mechanics. Substituting $\sigma_0 \approx h_1 Z/r$ into Eq. (11a) we find a trivial shift of the oscillator frequency:

$$\omega_0^2 = \omega_s^2 - \frac{h_1^2}{mr}. \quad (12)$$

where ω_0 includes the interaction effects with the environment and is therefore the direct analogue of the oscillation frequency in Eqs. (5)-(6). Note that that stability condition guarantees that $\omega_0^2 > 0$.

Before turning to the exact solution for $Z(t)$ we consider the adiabatic approximation. It is easy to show that the spin correlation function decays exponentially in time and it is given by $\langle S_0(0) S_0(-t') \rangle_{0,c} = e^{-t\mu r} / (\beta r)$ [13]. The friction and the mass correction are therefore [see Eq. (3)]:

$$\begin{aligned} \eta &= \beta h_1^2 \int_0^\infty dt' \langle S_0(0) S_0(-t') \rangle_{0,c} = \frac{h_1^2}{\mu r^2} \\ \kappa &= -\beta h_1^2 \int_0^\infty dt' t' \langle S_0(0) S_0(-t') \rangle_{0,c} = -\frac{h_1^2}{\mu^2 r^3}. \end{aligned}$$

Therefore, the previous discussion holds if we identify

$$\tau = \frac{|\kappa|}{\eta} = \frac{1}{\mu r}, \quad \beta g^2 = \frac{h_1^2}{r}, \quad \omega_0^2 = \omega_s^2 - \frac{h_1^2}{mr}. \quad (13)$$

Eq. (6) then gives the condition for an oscillation frequency larger than ω_0 . Again we find that in the weak coupling regime, $\frac{h_1^2}{mr} \ll \omega_s^2$, the oscillation frequency is enhanced with respect ω_0 . We note that within this linearized model it is impossible to obtain oscillation frequencies higher than the bare frequency ω_s because the ‘‘Born-Oppenheimer’’ softening (see Eq. (12)) is always larger than the frequency enhancement due to κ . However we stress that the ‘‘Born-Oppenheimer’’ correction will be absent when the oscillator moves in a translationally invariant medium (see Sec. VI) and, in that case, oscillation frequencies which are higher than the bare frequency ω_s can be achieved.

To verify the validity of the adiabatic approximation in the interesting regime where the frequency of oscillation is larger than ω_0 , we now solve the problem exactly. First we solve (11b) to obtain

$$\sigma_0(t) = e^{-\mu r t} \left[\sigma_0(0) + \mu h_1 \int_0^t dt' e^{\mu r t'} Z(t') \right].$$

We now substitute this expression into Eq. (11a) to obtain

$$\begin{aligned} m\ddot{Z}(t) &= -m\omega_s^2 Z(t) + h_1 \sigma_0(0) e^{-\mu r t} \\ &\quad + \mu h_1^2 \int_0^t dt' e^{-\mu r (t-t')} Z(t'). \end{aligned}$$

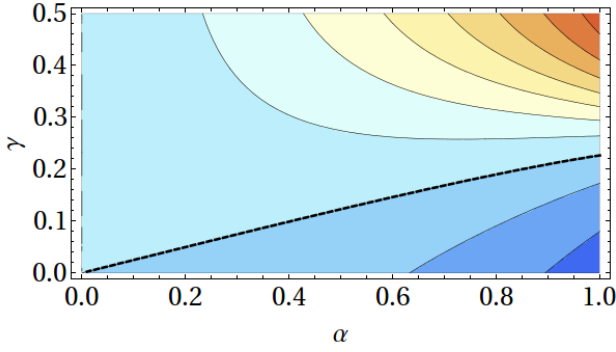


FIG. 3: (Color on-line) As in Fig. (1) we show the contour plot of the ratio $\frac{\omega^2}{\omega_0^2}$ vs α and γ , where now the frequency ω corresponds to the imaginary part of the longest lived normal mode of Eq. (14) (see text). The dashed thick black line marks $\omega = \omega_0$. Contour lines correspond to values from 0.8 to 1.7 (from blue to red) in steps of 0.1.

Note that by expanding $Z(t')$ in the integral around $Z(t)$ the approximate solution described above can be directly obtained. The above equation can be solved exactly using a Laplace transform, $\hat{Z}(s) = \int_0^\infty dt e^{-st} Z(t)$, to give:

$$\hat{Z}(s) = \frac{m(\mu r + s)(sZ(0) + \dot{Z}(0)) + h_1\sigma_0(0)}{m(\mu r + s)(s^2 + \omega_s^2) - h_1^2\mu}.$$

Finally, we assume that the magnet and the spin system are initially at equilibrium, $\sigma_0(0) = 0$, $Z(0) = 0$ and the magnet has an initial velocity $\dot{Z}(0) = v_0$ to obtain:

$$\hat{Z}(s) = v_0 \left[s^2 + \omega_s^2 - \frac{h_1^2}{mr} \frac{1}{(1 + \frac{s}{\mu r})} \right]^{-1}. \quad (14)$$

The frequency of oscillations is given by the imaginary part of the poles of $\hat{Z}(s)$. Clearly, when $h_1^2/(mr) = 0$, i.e. if the magnet and the spin system are decoupled, the poles are at $s = \pm i\omega_s$ and free-oscillations are recovered. In Fig. 3 we show the value of the ratio ω^2/ω_0^2 as a function of α and γ where, as before, $\alpha = \frac{\beta g^2 \tau}{m\omega_0} = \frac{\eta}{m\omega_0}$ and $\gamma = \omega_0 \tau$ (see Eq. (13)). Here ω is given by the imaginary part of the solution of Eq. (14) with the smallest real part, which corresponds to the slowest decaying mode. As expected in the limit $\alpha, \gamma \rightarrow 0$ the behavior of ω^2/ω_0^2 in Figs. 1 and 3 are identical. Note that, in this model, $\omega^2/\omega_0^2 > 1$ well beyond the limit where the adiabatic perturbation theory holds [15].

The results found above for a simplified model, where the magnetic oscillator only couples to the $S_0 \equiv S(\mathbf{q} = 0)$ Fourier component of the magnetization, extend to more complex setups with only minor modifications. In particular, let us consider an arbitrary weight function $W(x, y)$ so that the magnet is coupled to many Fourier components of the spin configuration and show the the effect described above persists. The previous procedure remains valid with minor modifications. For example Eq. (11)

becomes:

$$m\ddot{Z} = -m\omega_s^2 Z + h_1 \sum_{\mathbf{q}} \sigma_{\mathbf{q}} W_{-\mathbf{q}} \quad (15a)$$

$$\partial_t \sigma_{\mathbf{q}} = \mu [W_{\mathbf{q}} h_1 Z - (r + u|\mathbf{q}|^2) \sigma_{\mathbf{q}}] \quad (15b)$$

where, as before, $Z \equiv \langle z \rangle - \langle z \rangle_{eq}$ and $\sigma_{\mathbf{q}} \equiv \langle S_{\mathbf{q}} \rangle - \langle S_{\mathbf{q}} \rangle_{eq}$. However, this time the stationary solutions $\langle z \rangle_{eq}$ and $\langle S_{\mathbf{q}} \rangle_{eq}$ are:

$$\langle z \rangle_{eq} = \frac{h_0 h_1 K}{m\omega_s^2 r - h_1^2 K},$$

$$\langle S_{\mathbf{q}} \rangle_{eq} = \frac{h_0 W_{\mathbf{q}}}{r + u|\mathbf{q}|^2} \frac{m\omega_s^2 r}{m\omega_s^2 r - h_1^2 K}$$

where $K \equiv \sum_{\mathbf{q}} \frac{|W_{\mathbf{q}}|^2}{1 + (u/r)|\mathbf{q}|^2}$ and we have used that $W_{-\mathbf{q}} = W_{\mathbf{q}}^*$. The equations (15) describe the effect of the magnetic oscillator on the spins and the back-action of the spins on the oscillator. Crucially, under our quadratic approximation for the Landau-Ginzburg free-energy F the Fourier components of the spin configuration are decoupled. Moreover we have linearized the magnetic field around the equilibrium position of the oscillator, i.e. $h(z) \approx h_0 + h_1 z$. These two approximations allow to solve the above equation exactly for any profile $W(x, y)$. Repeating the same steps as before, Eqs. (14) and (12) now become:

$$\begin{aligned} \hat{Z}(s) &= v_0 \left[(s^2 + \omega_s^2) - \frac{h_1^2}{mr} f(s/(\mu r)) \right]^{-1} \\ \omega_0^2 &= \omega_s^2 - \frac{h_1^2}{mr} f(0) \end{aligned} \quad (16)$$

where we have defined the function:

$$f(x) \equiv \sum_{\mathbf{q}} \frac{|W_{\mathbf{q}}|^2}{x + 1 + (u/r)|\mathbf{q}|^2}. \quad (17)$$

When only the zero Fourier component contributes, i.e. $W_{\mathbf{q}=0} = 1$ and $W_{\mathbf{q} \neq 0} = 0$, we recover $f(x) = (1 + x)^{-1}$ [see Eq. (14)]. Here we consider the case in which many Fourier components contribute. For concreteness we assume that the interaction between the oscillator and the spin system has a Gaussian-like profile with width R from which it follows that $|W_{\mathbf{q}}|^2 = \exp[-(q_x^2 + q_y^2)R^2]$ and (17) becomes (we have replaced $\sum_{\mathbf{q}} \rightarrow (\frac{L}{2\pi})^2 \int d\mathbf{q}$):

$$f(x) = \left(\frac{L^2}{4\pi R^2} \right) y e^{y(x+1)} \Gamma[0, y(x+1)] \quad (18)$$

where $y \equiv R^2 r / u$ and Γ is the incomplete Gamma function. We note that in the limit of a broad profile, i.e. $y = R^2 r / u \gg 1$, $f(x) \sim (1+x)^{-1}$ and we recover the case discussed for a flat weight function while in the narrow profile limit, i.e. $y \ll 1$, $f(x)$ diverges logarithmically. We now substitute the expression above in Eq. (16) and perform the inverse Laplace Transform numerically to obtain the exact trajectory of the magnetic oscillator, $Z(t)$.

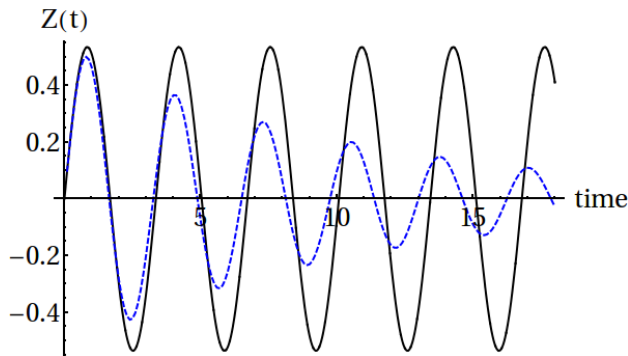


FIG. 4: (Color on-line) Comparison between the exact trajectory $Z(t)$ (dotted blue line) and the “Born-Oppenheimer” approximation $Z_{BO}(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t)$ (solid black line) for the magnetic oscillator interacting with many Fourier modes of the spin systems [see Eqs. (16) and (17)]. The parameters are $u, r, h_1, m, \mu, v_0 = 1$, $R = 0.01$, $L = 2$ and $\omega_s = 2.5$. The value of $\omega_0 \approx 1.87$ is obtained from Eq. (16). We note that the exact solution oscillates at frequency $\omega > \omega_0$.

In Fig. 4 we compare the exact trajectory $Z(t)$ with the “Born-Oppenheimer” approximation $Z_{BO}(t)$ which describes oscillations at frequency ω_0 . For the parameters chosen the mass correction is negative and the oscillation frequency is enhanced, i.e. $\omega > \omega_0$. This shows that the result obtained when the oscillator is coupled only to the zero Fourier component of the spin configuration persists even when the magnet is coupled to many degrees of freedom (many Fourier components) of the environment.

IV. NUMERICAL SIMULATIONS

Let us now verify the results of the analytical calculations by direct numerical simulations of the microscopic equations of motion:

$$m\ddot{z}(t) = -m\omega_s^2 z(t) + h_1 S_0(t).$$

where $S_0(t)$ is the total spin magnetization and we set $m = 1$ and $\omega_s = 2.5$ (in what follows we use arbitrary units). We introduce a time-discretization $\delta t = 0.005$ and numerically compute the values $z_n \equiv z(n\delta t)$ using the velocity-Verlet algorithm [16]. Between two subsequent updates (say n and $n + 1$), the position of the magnetic oscillator is assumed constant. In this time interval the spins are evolved according to Metropolis dynamics governed by the Hamiltonian:

$$H_{\text{Ising}} = - \sum_{\langle i,j \rangle} s_i s_j - (h_0 + h_1 z_n) \sum_i s_i. \quad (19)$$

Here s_i indicates an Ising spin in a $2d$ system of linear size $L = 350$ with periodic boundary conditions and the symbol $\langle i, j \rangle$ indicates a sum over nearest-neighbor. A spin flip leads to a transition from configuration C to a

new configuration C' . This transition is determined by the rates $\Gamma(C \rightarrow C')$:

$$\Gamma(C \rightarrow C') \equiv \frac{N}{\delta t} P(\Delta E) \equiv \frac{N}{\delta t} \min(1, e^{-\beta \Delta E})$$

where β is the inverse temperature of the spin system, ΔE is the energy difference between the two spin configurations and N is the number of attempted spin flips during the a time interval δt . This choice of rates guarantees that the spin system relaxes toward the Gibbs equilibrium distribution at inverse temperature β for any value of z_n . The value of $\beta = 0.05$ is chosen such that the system is in the disordered phase ($\beta_c \approx 0.44$ for $h_0 = h_1 = 0$).

In practice, after each update of the position of the magnetic oscillator, we select at random N spins, flip them according to probability $P(\Delta E)$ and compute the total magnetization S_0 which is then used to update the position of the magnetic oscillator. When N is large the spin magnetization S_0 is able to relax to its equilibrium value, while when N is small the spin magnetization “lags behind” its equilibrium value. Therefore for large N the motion of the magnetic oscillator is slow compared to the relaxation dynamics of the spin magnetization. This numerical scheme is similar to the one described in Ref. [17].

In this setup the dynamics of the spin magnetization is expected to be described by the Langevin equation introduced in the previous section. However the parameters in the Langevin equation (such as μ , r and u) can not be derived rigorously from the microscopic rules introduced here. Therefore, to test our main result (namely that the oscillation frequency is enhanced with respect ω_0) we need to compare three distinct simulations. In the first simulation we set $h_0 = h_1 = 0$ so that the spin system and magnetic oscillator are decoupled and, as expected, we obtain persistent oscillations of the magnetic oscillator at frequency ω_s . In the second simulation, we set $h_0 = h_1 = 7L^{-2}$ and $N = 5L^2$. Now the magnetic oscillator is coupled to the total spin magnetization which is always fully relaxed to its (instantaneous) equilibrium value. Finally, in the third simulation, we keep $h_0 = h_1 = 7L^{-2}$ but we set $N = 0.01L^2$ so that the total spin magnetization is not fully relaxed to its (instantaneous) equilibrium value. Note that the couplings between the magnetic oscillator and the spin systems are proportional to L^{-2} while the total magnetization S_0 is proportional to L^2 . This ensures that the equation of motion for the magnetic oscillator has a well defined thermodynamic limit.

The trajectories of the magnetic oscillator for the simulations described above are shown in Fig. 5(a). A Fourier analysis of these trajectories reveals important quantitative differences, see Fig. 5(b). In the first simulation only the frequency $\omega_s = 2.5$ is present. In the second simulations, in which the magnet and the spin system are coupled and the spin magnetization is fully relaxed to its equilibrium value, the frequencies are narrowly peaked around a *smaller* frequency which, according to the re-

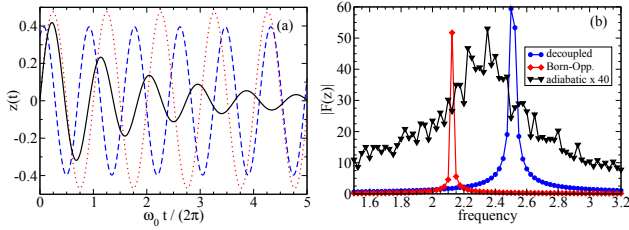


FIG. 5: (Color on-line) Three different microscopic simulations of the magnetic oscillator coupled to a spin system evolving according to Metropolis rates (see text for details). (a) The three trajectories $z(t)$ and (b) their Fourier components. In these simulations $\omega_s = 2.5$, $\omega_0 \approx 2.13$, $\omega \approx 2.35$, $\alpha \approx 0.17$ and $\gamma = 0.74$ where α , γ are defined as in Fig. 1 and Fig. 3. Note that the black curve in panel (b) has been rescaled by a factor of 40.

sult of the previous section, can be interpreted as ω_0 [see Eq. (12)]. In our case $\omega_0 \approx 2.13$. In the third simulation, the frequencies are broadly peaked around at intermediate frequency $\omega_0 < \omega < \omega_s$ with $\omega \approx 2.35$ so that $\omega/\omega_0 \approx 1.1$. The third simulation corresponds to the parameters $\alpha \approx 0.17$ and $\gamma = 0.74$ where α , γ are defined in Fig. 1. This confirms that, the leading correction above adiabaticity, lead to an *enhanced* oscillation frequency with respect to ω_0 .

V. ENERGY ABSORPTION FROM THE ENVIRONMENT AND NEGATIVE MASS

To get some intuition for the negative mass correction, note that *any* mass correction implies that the dynamics of the energy of the macroscopic object is non-Markovian. To see this we use Eq. 2 to write the time-derivative of its energy as:

$$\frac{dE}{dt} = \frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{X}}^2 + V(\mathbf{X}) \right] = -\dot{\mathbf{X}} \left(\kappa \ddot{\mathbf{X}} + \eta \dot{\mathbf{X}} \right) \quad (20)$$

and note that it depends on $\ddot{\mathbf{X}}$. The potential $V(\mathbf{X})$ includes the interaction energy with the environment and therefore $E(t)$ is conserved in the adiabatic limit. While the dissipative term $(-\eta \dot{\mathbf{X}}^2)$ is always negative, the non-Markovian term $(-\kappa \ddot{\mathbf{X}} \dot{\mathbf{X}})$ can change sign and become dominant close to turning points where $\dot{\mathbf{X}} = 0$. When $\kappa < 0$ the macroscopic object transiently absorbs energy from the bath as it recedes from the turning points. This holds when the bath is dissipative (or when its dissipative dynamics dominate over its inertial dynamics). In cases when $\kappa > 0$ the system absorbs energy from the bath as it approaches the turning point. This is expected when the bath has a strong enough inertia so that, as the macroscopic object slows down at the turning points, the bath still “pushes” it in the direction of the turning point.

To understand the fact that, when $\kappa < 0$, the macroscopic object transiently absorbs energy from the bath

as it approaches the turning point in Fig. 6 we show, on the same plot, the exact time evolution of $Z(t)$ and $\sigma_0(t)$ together with the instantaneous value of the equilibrium energy of the magnetic oscillator

$$E(t) = \frac{1}{2} m \dot{Z}^2 + \frac{1}{2} m \omega_s^2 Z^2 - \frac{h_1^2}{r} Z^2.$$

The first two terms are the kinetic and elastic potential energies while the last term represent the “Born-Oppenheimer” interaction potential energy. The last term is obtained by substituting in the interaction energy, $U_{int} = -h_1 \sigma_0 Z$, σ_0 with its equilibrium value $\sigma_0^{eq} = \frac{h_1}{r} Z$ (see Eq. (11b)).

The trajectories for $Z(t)$ and $\sigma_0(t)$ are obtained by solving Eqs. (11) numerically. We observe that, for a short time after the object leaves the turning point, the magnet absorbs energy from the environment. This is due to the fact that away from the adiabatic limit σ_0 is not in equilibrium at the instantaneous value of $Z(t)$ but instead it “lags” behind its equilibrium value. When the system approaches the $Z > 0$ turning point $\sigma_0 < \sigma_0^{eq} \equiv \frac{h_1}{r} Z$, and the environment applies a force (proportional to the lag) which decreases $|\dot{Z}|$. At the turning point, Z changes direction but σ_0 is still smaller than σ_0^{eq} . This force is now increasing $|\dot{Z}|$ which causes the object to transiently absorb energy from the bath. To see this more explicitly we combine Eqs. (11a) and (11b) to write the equation of motion for the macroscopic object as:

$$m \ddot{Z}(t) = -m \omega_0^2 Z - \frac{h_1}{\mu r} \partial_t \sigma_0$$

This shows explicitly that an additional force, proportional to $\partial_t \sigma_0$, acts on the magnet because the magnetization σ_0 has not reached equilibrium at the instantaneous value of Z . If this was the case, $\partial_t \sigma_0$ would be zero and the equation above would simply describe oscillations at frequency ω_0 . Consider the object moving towards the positive Z turning point. As it approaches the turning point σ_0 trails behind Z so that $\partial_t \sigma_0$ is positive (see Eq. (11b)). The force experienced by the magnet pulls it away from the turning point, slowing it down. Right after the turning point, since σ_0 still trails behind Z , $\partial_t \sigma_0$ is still positive pulling it away from the turning point causing it to absorb energy from the environment. This chain of events can be clearly seen in Fig. 6.

We stress that Eq. (20) only holds in the long time limit when η and κ have reached their asymptotic values (see Eq. (3)). In particular if the environment is initially (at $t = 0$) in equilibrium then $E(t) < E(0)$ for any $t > 0$ and any initial condition of the oscillator (including $\dot{Z}(0) = 0$ and $Z(0)$ at a turning point). This is required by the second law of thermodynamics. However, we note that neither the second law of thermodynamics nor the condition $E(t) < E(0)$ imply that the function $E(t)$ decreases monotonically. In fact the opposite is true as shown in Fig. 6.

The initial decrease of the energy can be understood from the short time behavior of the force acting on the

macroscopic object which is given by (see Eq. (39) below)

$$F(t=0) = -\left\langle \frac{\partial H}{\partial X} \right\rangle = -\left\langle \frac{\partial H}{\partial X} \right\rangle_0 - \beta \dot{X}(t) \langle \partial_X H(t=0) \partial_X H(t=0) \rangle_{0,c} dt.$$

where, for simplicity, we have assumed that the macroscopic object is described by a single degree of freedom and, as in Eq. (3), the suffix “0” indicates that the averages are over the equilibrium distribution of the environment at inverse temperature β and c indicates a connected correlation function. This implies that, to leading order in dt , the energy change (including the trivial “Born-Oppenheimer” term) is given by:

$$\begin{aligned} \frac{dE}{dt} &= F(0) \dot{X}(0) \\ &= -\beta \dot{X}^2(0) \langle 0 | \partial_X H(0) \partial_X H(0) | 0 \rangle_c < 0 \end{aligned}$$

which is always negative (for an initial state with positive temperature). The statement that, if the environment is initially (at $t=0$) in equilibrium then $E(t) < E(0)$ can be made rigorous for any $t > 0$ [18].

In Fig. 6 the parameters are chosen so that the mass correction is negative and the oscillation frequency is enhanced, i.e. $\omega > \omega_0$. This can be seen clearly in Fig. 7 where we compare the exact trajectory $Z(t)$ with the “Born-Oppenheimer” approximation $Z_{BO}(t)$ which describes oscillations at frequency ω_0 and Z_η which includes both the “Born-Oppenheimer” correction and the effect of the dissipation. We note that Z_η oscillates, as expected, at a reduced frequency $\omega < \omega_0$. We stress that the two approximations, $Z_{BO}(t)$ and $Z_\eta(t)$, become asymptotically exact in the adiabatic limit. Finally, we stress that when the inertia dominates the dynamics of the environment we expect the mass correction to be positive.

VI. ADIABATIC PERTURBATION THEORY FOR MARKOV PROCESSES

Finally, in this Section we present a detailed derivation of the main results Eqs. (3). We consider a Markov process whose rates change as a function of a set of external parameters $\mathbf{X}(t)$ which evolve in time. We are interested in developing an adiabatic perturbation theory in the rate of change of these external parameters. To this end, start with the master equation

$$\partial_t |P(t)\rangle = \mathbf{M}(\mathbf{X}(t)) |P(t)\rangle \quad (21)$$

where $\mathbf{M}(\mathbf{X}(t))$ is a Markov matrix. The vector $|P\rangle$ specifies the probability of being in a given configuration. The off diagonal elements of the matrix \mathbf{M} specify transition rates and are positive. The diagonal elements are set by conservation of probability, $M_{ii} = -\sum_{j \neq i} M_{ji}$. The matrix \mathbf{M} is in general not symmetric and therefore admits separate left and right eigenvectors (for more details see

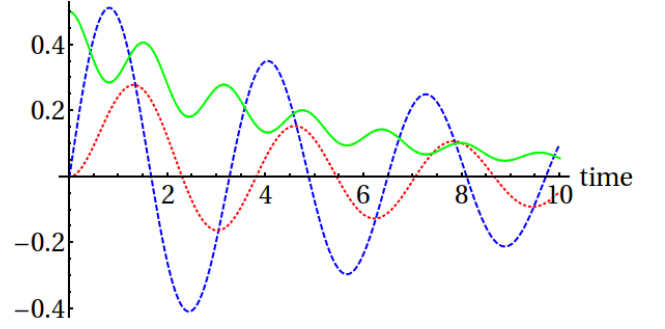


FIG. 6: (Color on-line) Exact time dependence of $Z(t)$ (dashed blue line) $\sigma_0(t)$ (dotted red line) and the energy (solid green line) of the magnetic oscillator $E(t) = \frac{1}{2}m\dot{Z}^2 + \frac{1}{2}m\omega_s^2 Z^2 - \frac{h_1^2}{r} Z^2$. The trajectories have been obtained by solving Eq. (11) exactly with the initial conditions $Z(0) = 0$, $\sigma_0(0) = 0$ and $\dot{Z}(0) = v_0 = 1$. The parameters m , μ , h_1 are set to one while $r = 1.1$ and $\omega_s = 2$. These parameters correspond to $\tau \approx 0.9$, $\omega_0 \approx 1.76$, $\gamma \approx 1.59$ and $\alpha \approx 0.47$. In all the above we use arbitrary units. We note that for these parameters the mass correction is negative.

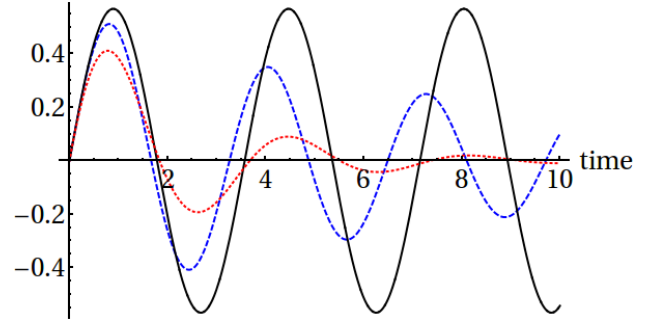


FIG. 7: (Color on-line) Comparison between the exact trajectory $Z(t)$ (dotted blue line), the “Born-Oppenheimer” approximation $Z_{BO}(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t)$ (solid black line) and the “Born-Oppenheimer” approximation with the inclusion of dissipation Z_η (dotted red line) obtained by solving the equation $m\ddot{Z} = -m\omega_0^2 Z - \eta \dot{Z}$. The parameters are as in Fig. 6. On one hand, $Z_\eta(t)$ oscillates at a reduced frequency, $\omega < \omega_0$, and quickly decay to zero as expected for a damping ratio $\zeta = \eta / (2m\omega_0) \approx 0.23$. On the other hand, the exact solution oscillates at frequency $\omega > \omega_0$ and decay to zero more slowly. When $\gamma = \omega_0 \tau \ll 1$ the difference between these trajectories decreases.

[19, 20]). It is easy to see that there is a trivial left eigenvector $\langle 0|$ with eigenvalue zero and all entries equal to one:

$$\langle 0| = \langle 1, 1, \dots, 1|, \quad \langle 0|\mathbf{M} = 0. \quad (22)$$

With these definitions the average of a given observable, \mathcal{O} is given by:

$$\langle \mathcal{O} \rangle = \langle 0|\mathcal{O}|P\rangle = \sum_{ij} \mathcal{O}_{ij} P_j \quad (23)$$

The diagonal values of the operator \mathcal{O} specify the values of the observable \mathcal{O} in configuration $|j\rangle$. Off diagonal terms are can also occur, for example when \mathcal{O} measures currents.

Next, we define the adiabatic basis of the instantaneous eigenvectors of $\mathbf{M}(\mathbf{X}(t))$:

$$\mathbf{M}(\mathbf{X}(t))|n(\mathbf{X}(t))\rangle = \epsilon_n(\mathbf{X}(t))|n(\mathbf{X}(t))\rangle \quad (24)$$

where the normalization of the vectors is such that $\langle n(\mathbf{X})|m(\mathbf{X})\rangle = \delta_{n,m}$. $\epsilon_n(\mathbf{X}(t))$ is the eigenvalue at a fixed value of $\mathbf{X}(t)$. Eigenvalues of Markov matrices are known to have obey $\text{Re}[\epsilon_n(\mathbf{X}(t))] \leq 0$ [19, 20]. We now rewrite the master equation in the instantaneous basis, see Eq. (24). (This step is analogue to using a moving reference frame in classical mechanics.) Using

$$|P(t)\rangle = \sum_n a_n(t)|n(\mathbf{X}(t))\rangle, \quad (25)$$

in Eq. (21) gives:

$$\sum_n \left(\dot{a}_n |n(\mathbf{X})\rangle + a_n \frac{\partial}{\partial t} |n(\mathbf{X})\rangle \right) = \sum_n a_n \epsilon_n(\mathbf{X}) |n(\mathbf{X})\rangle.$$

where, if necessary, the time-derivative of $|n(\mathbf{X})\rangle$ can be evaluated using the chain rule:

$$\frac{\partial}{\partial t} |n(\mathbf{X})\rangle \equiv \dot{X}_i \frac{\partial}{\partial X_i} |n(\mathbf{X})\rangle.$$

To simplify notations we have suppressed, for now, the explicit dependence of the quantities on time. We will reintroduce the time-dependence explicitly when necessary to avoid confusion. Projecting on $\langle m(\mathbf{X})|$ gives the set of coupled equations:

$$\dot{a}_m = \epsilon_m(\mathbf{X})a_m - \sum_n \langle m(\mathbf{X})|\partial_t|n(\mathbf{X})\rangle a_n. \quad (26)$$

Next, it is useful to define

$$a_m(t) = \tilde{a}_m(t) \exp \left[\int_0^t \epsilon_m(t') dt' \right], \quad (27)$$

where the dynamics start at time $t = 0$. Then using Eq. (26) we have

$$\dot{\tilde{a}}_m(t) = - \sum_n \tilde{a}_n(t) e^{\int_0^t (\epsilon_n(t') - \epsilon_m(t')) dt'} \langle m(\mathbf{X})|\partial_t|n(\mathbf{X})\rangle,$$

which in turn is equivalent to the integral equation

$$\tilde{a}_m(t) = - \int_0^t dt' \left[\sum_n \tilde{a}_n(t') e^{\int_0^{t'} (\epsilon_n(t'') - \epsilon_m(t'')) dt''} \langle m(\mathbf{X}(t'))|\partial_{t'}|n(\mathbf{X}(t'))\rangle \right] + \tilde{a}_m(0). \quad (28)$$

In terms of the original a_m we then have

$$a_m(t) = e^{\int_0^t \epsilon_m(t') dt'} a_m(0) - \int_0^t dt' \sum_n a_n(t') e^{\int_{t'}^t \epsilon_m(t'') dt''} \langle m(\mathbf{X}(t'))|\partial_{t'}|n(\mathbf{X}(t'))\rangle. \quad (29)$$

Recall that Markov matrices always have an eigenstate $|0(\mathbf{X})\rangle$ with zero eigenvalue, i.e. $\epsilon_0(\mathbf{X}) = 0$ [19, 20]. This eigenvector corresponds to the steady-state of the system at fixed \mathbf{X} . For this state we then have

$$\begin{aligned} a_0(t) &= - \int_0^t dt' \sum_n a_n(t') \langle 0(\mathbf{X}(t'))|\partial_{t'}|n(\mathbf{X}(t'))\rangle + a_0(0) \\ &= a_0(0) = 1. \end{aligned} \quad (30)$$

The equality follows from differentiating the orthogonality relation

$$\partial_t \langle 0(\mathbf{X}(t))|n(\mathbf{X}(t))\rangle = 0$$

and noting that

$$\partial_t \langle 0(\mathbf{X}(t))| = \partial_t \langle 1, 1, \dots, 1| = 0.$$

so that

$$\langle 0(\mathbf{X}(t))|\partial_t|n(\mathbf{X}(t))\rangle = 0 \quad \forall n.$$

Equation (30) simply reflects the conservation of probability. Up to this point all the expressions are exact. In particular Eq. (29) is a rewriting, in a particular (co-moving) basis, of the master equation Eq. (21). We now develop a formal perturbation theory in the time derivatives of \mathbf{X} . This is possible because, by assumption, the eigenvalues ϵ_m and eigenvectors $|n\rangle$ depends on time only through $\mathbf{X}(t)$. For example the energies which appear in Eq. (29) are:

$$\int_{t'}^t \epsilon_m(t'') dt'' \equiv \int_0^{t-t'} \epsilon_m(t-q) dq \quad (31)$$

Expanding in a Taylor series in q and using the chain rule $\partial_t = \dot{\mathbf{X}} \frac{\partial}{\partial \mathbf{X}}$ we arrive at:

$$\begin{aligned} \epsilon_m(t-q) &\approx \epsilon_m(t) - q \frac{\partial \epsilon_m}{\partial t} + \dots \\ &\approx \epsilon_m(\mathbf{X}(t)) - q \left(\dot{\mathbf{X}} \frac{\partial \epsilon_m}{\partial \mathbf{X}} \right) \Big|_t + \dots \end{aligned}$$

Substituting back into Eq. (31) we have:

$$\int_0^{\delta t} \epsilon_m(t-q) dq \approx \epsilon_m(t) \delta t - \frac{\delta t^2}{2} \left(\dot{\mathbf{X}} \frac{\partial \epsilon_m}{\partial \mathbf{X}} \right) + \dots \quad (32)$$

where we have defined $\delta t \equiv t - t'$. This expression shows that if

$$\gamma_1 = \frac{\delta t \dot{\mathbf{X}}}{\epsilon_m} \frac{\partial \epsilon_m}{\partial \mathbf{X}} \ll 1$$

then only the first order on the RHS of Eq. (32) contributes. The dimensionless parameter γ_1 describes the relative change of the eigenvalues ϵ_m during a time-interval δt . Specifically it measures the change in relaxation time during a time δt . This is naturally expected to be small, in particular when the coupling to the bath is weak. This dimensionless parameter can be made arbitrary small by decreasing the velocity $\dot{\mathbf{X}}$ and we can therefore conclude that at small velocities:

$$\int_{t'}^t \epsilon_m(t'') dt'' \approx \epsilon_m(t)(t - t')$$

where, importantly, the energies are evaluated at the final time t . A similar argument applies to the matrix elements in Eq. (29). In fact the relative change of the matrix element during a time interval δt

$$\gamma_2 \equiv \delta t \dot{X}_i \frac{\frac{\partial}{\partial X_i} \left(\langle m(\mathbf{X}) | \frac{\partial}{\partial X_j} | n(\mathbf{X}) \rangle \right)}{\langle m(\mathbf{X}) | \frac{\partial}{\partial X_j} | n(\mathbf{X}) \rangle}$$

can be made arbitrary small by reducing the velocity \dot{X}_i . Therefore, at sufficiently small velocity, $\gamma_1, \gamma_2 \ll 1$ and both the energies and the matrix elements can be considered constant and can be evaluated at the final time t . Then Eq. (29) reduces to:

$$a_m(t) = e^{\epsilon_m(t)t} a_m(0) - \int_0^t dt' \sum_n a_n(t') \times e^{\epsilon_m(t)(t-t')} \dot{X}_i(t') \langle m(\mathbf{X}(t)) | \frac{\partial}{\partial X_i} | n(\mathbf{X}(t)) \rangle.$$

We now assume that we start at the steady-state of the system at $\mathbf{X}(t=0)$ so that a_m with $m \neq 0$ are small quantities. Then, it is easy to see that the lowest order contribution to the amplitude a_m is linear in the velocity and originates from the term $n=0$ (the contribution from a_n with $n \neq 0$ is, at least, of order \dot{X}^2):

$$a_m(t) = - \int_0^t dt' e^{\epsilon_m(t)(t-t')} \dot{X}_i(t') \langle m(\mathbf{X}(t)) | \frac{\partial}{\partial X_i} | 0(\mathbf{X}(t)) \rangle \quad (33)$$

where we used $a_0 = 1$.

The above derivations apply to any Markov process. We now restrict ourselves to processes which begin in equilibrium and whose dynamics satisfy detailed balance for any fixed value of \mathbf{X} . We stress that the detailed balance condition constraint the eigenvalues of the Markov matrix to be real. Then the steady-state eigenvector is given by

$$|0(\mathbf{X})\rangle = \frac{1}{Z(\mathbf{X})} \sum_s e^{-\beta H(\mathbf{X},s)} |s\rangle, \quad (34)$$

where $|s\rangle$ is a vector with a 1 entry at configuration s and zero otherwise, $H(\mathbf{X}, s)$ is the energy of state s at a given value of \mathbf{X} and β is the inverse temperature. Z is

the partition function of the model at a given value of \mathbf{X} . Then it is straightforward to see that

$$\begin{aligned} \langle m(\mathbf{X}) | \frac{\partial}{\partial X_i} | 0(\mathbf{X}) \rangle &= \\ - \frac{1}{Z} \sum_s \langle m(\mathbf{X}) | (\partial_{X_i}(\beta H) - \langle \partial_{X_i}(\beta H) \rangle) e^{-\beta H(\mathbf{X},s)} | s \rangle. \end{aligned} \quad (35)$$

Here the angular brackets denote an average with respect to the equilibrium measure at a given value of \mathbf{X} . Next, note that from the orthogonality relation $\langle m(\mathbf{X}) | 0(\mathbf{X}) \rangle = 0$ for $m \neq 0$ the second term in the relation above drops out, so that

$$\begin{aligned} \langle m(\mathbf{X}) | \frac{\partial}{\partial X_i} | 0(\mathbf{X}) \rangle &= - \frac{1}{Z} \sum_s \langle m(\mathbf{X}) | \partial_{X_i}(\beta H) e^{-\beta H(\mathbf{X},s)} | s \rangle \\ &= - \langle m(\mathbf{X}) | \partial_{X_i}(\beta H) | 0(\mathbf{X}) \rangle. \end{aligned} \quad (36)$$

Therefore

$$a_m(t) = \int_0^t dt' e^{\epsilon_m(t)(t-t')} \dot{X}_i(t') \langle m(\mathbf{X}(t)) | \partial_{X_i}(\beta H) | 0(\mathbf{X}(t)) \rangle$$

In the paper our interest is in calculating thermodynamic generalized forces $\langle \partial_{X_j}(\beta H) \rangle$. While in the paper we assume that β is constant, here we also allow for it to serve as an externally varying parameter (see for example [6]). Using

$$|P(t)\rangle = |0(\mathbf{X}(t))\rangle + \sum_{m \neq 0} a_m(t) |m(\mathbf{X}(t))\rangle \quad (37)$$

we have

$$\begin{aligned} \langle 0 | \partial_{X_j}(\beta H) | P(\mathbf{X}(t)) \rangle &= \langle 0 | \partial_{X_j}(\beta H) | 0(\mathbf{X}(t)) \rangle \\ &+ \int_0^t dt' e^{\epsilon_m(t)(t-t')} \dot{X}_i(t') \cdot \\ \sum_{m \neq 0} \langle 0 | \partial_{X_j}(\beta H) | m(\mathbf{X}(t)) \rangle \langle m(\mathbf{X}(t)) | \partial_{X_i}(\beta H) | 0(\mathbf{X}(t)) \rangle. \end{aligned} \quad (38)$$

where we have used that the left zero eigenvalue is independent on $\mathbf{X}(t)$ (see Eq. (22)). Using the relation $e^{\epsilon_m(t)(t-t')} |m\rangle = e^{\mathbf{M}(t)(t-t')} |m\rangle$ this can be rewritten as:

$$\begin{aligned} \langle 0 | \partial_{X_j}(\beta H) | P(\mathbf{X}(t)) \rangle &= \langle 0 | \partial_{X_j}(\beta H) | 0(\mathbf{X}(t)) \rangle \\ &+ \int_0^t dt' \dot{X}_i(t') \sum_{m \neq 0} \langle 0 | \partial_{X_j}(\beta H) e^{\mathbf{M}(t)(t-t')} | m(\mathbf{X}(t)) \rangle \\ &\times \langle m(\mathbf{X}(t)) | \partial_{X_i}(\beta H) | 0(\mathbf{X}(t)) \rangle = \langle \partial_{X_j}(\beta H)(t) \rangle_0 \\ &+ \int_0^t dt' \dot{X}_i(t') \langle \partial_{X_j}(\beta H)(t') \partial_{X_i}(\beta H)(t) \rangle_{0,c} \\ &= \langle \partial_{X_j}(\beta H)(t) \rangle_0 + \int_0^t dt' \dot{X}_i(t-t') \mathcal{K}_{0,c}(t-t', t) \end{aligned} \quad (39)$$

where the subscript 0 indicates that the expectation value are evaluated in the instantaneous equilibrium at fixed $\mathbf{X}(t)$ and $\mathcal{K}_{0,c}$ is a short-hand notation for the two-times force-force connected correlation function evaluated at equilibrium. In the long time limit (t much longer than the relaxation time τ of $\mathcal{K}_{0,c}$) the integral above is dominated by short $t' \leq \tau$ implying that one can extend the upper limit of integration to ∞ . Note that the existence of a finite τ essentially assumes that the spectrum of the Markov matrix is gapped. Finally we have

$$\langle \partial_{X_j}(\beta H)(t) \rangle = \langle \partial_{X_j}(\beta H)(t) \rangle_0 + \int_0^\infty dt' \dot{X}_i(t-t') \mathcal{K}_{0,c}(t-t', t) \quad (40)$$

For a particle moving in a stochastic environments the equations of motion (averaged over histories of the stochastic environment) are given by

$$m \ddot{\mathbf{X}} = - \left\langle \frac{\partial H}{\partial \mathbf{X}} \right\rangle. \quad (41)$$

Using the above results and keeping β fixed we have

$$\begin{aligned} m \ddot{\mathbf{X}} &= - \langle \partial_{X_j} H(t) \rangle_0 \\ &\quad - \beta \int_0^\infty dt' \dot{X}_i(t-t') \mathcal{K}_{0,c}(t-t', t) \\ &= F_{\text{BO}} - \eta \dot{\mathbf{X}} - \kappa \ddot{\mathbf{X}} + O(\ddot{\mathbf{X}}), \end{aligned} \quad (42)$$

where we have identified the “Born-Oppenheimer” contribution to the force

$$F_{\text{BO}} = - \frac{\partial V}{\partial \mathbf{X}} = - \langle \partial_{\mathbf{X}} H(t) \rangle_0 = - \langle 0 | \partial_{\mathbf{X}} H | 0(\mathbf{X}(t)) \rangle$$

and expanded $\dot{X}_i(t-t')$ around $t' = 0$ to obtain η and κ defined as in Eq. (3). The “Born-Oppenheimer” force can be easily computed from the following identity [see Eq. (34)]:

$$\partial_{\mathbf{X}} \langle 0 | 0(\mathbf{X}) \rangle = - \langle 0 | \partial_{\mathbf{X}} (\beta H) | 0(\mathbf{X}) \rangle - \frac{1}{Z(\mathbf{X})} \frac{\partial Z(\mathbf{X})}{\partial \mathbf{X}} \langle 0 | 0(\mathbf{X}) \rangle$$

where $Z(\mathbf{X})$ is the partition function at fixed value of \mathbf{X} . Using the orthogonality relation $\langle 0 | 0(\mathbf{X}) \rangle = 1$ for any \mathbf{X} we obtain:

$$F_{\text{BO}} = \frac{1}{\beta Z(\mathbf{X})} \frac{\partial Z(\mathbf{X})}{\partial \mathbf{X}} \quad (43)$$

which shows that the “Born-Oppenheimer” force depends exclusively on the energy spectrum and vanish if $Z(\mathbf{X}) = \text{const.}$. This is the case, for example, when the unperturbed system is translationally invariant.

A. Subleading corrections in the small system-bath coupling limit

We now compute the first subleading correction in the limit of small system-bath coupling. We stress that our

adiabatic perturbation theory does not rely on the small system-bath coupling however, in this limit, it is easy to explicitly compute the subleading corrections (which are non-linear in the velocity) and show that they are indeed negligible at small velocities.

We start from the exact equation Eq. (29) and assume that the system is at steady-state so that a_m with $m \neq 0$ are small quantities. It is easy to see that the lowest non-vanishing contribution to a_m is of first order in the system-bath coupling. This contribution is obtained by substituting in Eq. (29) the right eigenvectors computed to first order in the system-bath coupling and the unperturbed eigenvalues and left eigenvectors. Noting that the unperturbed eigenvalues and eigenvectors are independent on X and are therefore time-independent we arrive at:

$$a_m(t) = - \int_0^t dt' e^{\epsilon_m(t-t')} \dot{X}(t') \langle m | \frac{\partial}{\partial X} | 0(X(t')) \rangle$$

where we used $a_0 = 1$ (the contributions from a_n with $n \neq 0$ are higher order in the system-bath coupling).

We now apply the expression:

$$f(X(t')) = f(X(t-t'')) = f(X) - t'' \dot{X} \frac{\partial f}{\partial X} + \dots,$$

where $t'' \equiv t - t'$ and the RHS is evaluated at t to obtain:

$$\begin{aligned} a_m(t) &= - \int_0^t dt'' e^{\epsilon_m t''} \dot{X}(t-t'') \times \\ &\quad \left[\langle m | \frac{\partial}{\partial X} | 0(X) \rangle - t'' \dot{X}(t) \langle m | \frac{\partial^2}{\partial X^2} | 0(X) \rangle + \dots \right] \end{aligned} \quad (44)$$

Using the form of the steady state eigenvector [see Eq. (34)] and manipulations similar to the one in Eqs. (34)–(36) we obtain that to lowest order in the system-bath coupling:

$$\begin{aligned} \langle m | \frac{\partial}{\partial X} | 0(X) \rangle &= - \langle m | \frac{\partial}{\partial X} (\beta H)(t) | 0 \rangle \\ \langle m | \frac{\partial^2}{\partial X^2} | 0(X) \rangle &= - \langle m | \frac{\partial^2}{\partial X^2} (\beta H)(t) | 0 \rangle \end{aligned} \quad (45)$$

where $|0\rangle$ is the unperturbed stationary state. Then we compute

$$\langle 0 | \partial_X (\beta H) | P(X) \rangle, \quad |P(X)\rangle = |0(X)\rangle + \sum_{m \neq 0} |m(X)\rangle$$

which, to lowest order in the system-bath coupling, gives [we have followed the same manipulations as in Eq. (39)]:

$$\begin{aligned} \langle 0 | \partial_X (\beta H) | P \rangle &= \langle 0 | \partial_X (\beta H) | 0 \rangle + \int_0^\infty dt'' \dot{X}(t-t'') \times \\ &\quad \left[\langle 0 | \partial_X (\beta H)(t-t'') \partial_X (\beta H)(t) | 0 \rangle \right. \\ &\quad \left. - \dot{X}(t) t'' \langle 0 | \partial_X (\beta H)(t-t'') \partial_{XX}^2 (\beta H)(t) | 0 \rangle \right] \end{aligned} \quad (46)$$

where we have extended the upper limit of integration to infinity by noting that this integral is dominated by $t'' \lesssim \tau$ (where τ is the correlation time).

In the example of the magnetic oscillator considered above the interaction energy between the macroscopic degrees of freedom and the environment is:

$$H_{int} = -h(X) S_0 \approx -(h_0 + h_1 X + h_2 X^2 + \dots) S_0$$

where S_0 is the zero Fourier component of the magnetization and we have expanded the magnetic field $h(X)$ to second order around the equilibrium position (in Sec. III we considered only the first order). We now see that both correlation functions appearing in Eq. (29) are proportional to $\langle 0|S_0(t - t'')S_0(t)|0\rangle = \frac{e^{-t''/\tau}}{\beta r}$. More importantly by expanding $\dot{X}(t - t'') \approx \dot{X}(t) - t''\ddot{X}(t)$ in Eq. (46) we can explicitly read the coefficients which multiply \dot{X} , \ddot{X} and $\dot{X}\ddot{X}$. These coefficients are respectively the friction coefficient η , mass renormalization κ and the non-linear coefficient η_2 :

$$\eta = \frac{\beta h_1^2}{r} \tau, \quad \kappa = -\frac{\beta h_1^2}{r} \tau^2, \quad \eta_2 = -\frac{\beta h_1 h_2}{r} \tau^2.$$

As expected we reproduce the expressions for η and κ in the Sec. III [recall that $\tau = 1/(\mu r)$]. We can now explicitly see when the quadratic contribution in the velocity is negligible compared to the friction and the mass renormalization:

$$\begin{aligned} |\dot{X}\dot{X}\eta_2| &\ll |\eta\dot{X}| \rightarrow (\omega_0\tau)A/l \ll 1 \\ |\dot{X}\dot{X}\eta_2| &\ll |\kappa\ddot{X}| \rightarrow A/l \ll 1 \end{aligned} \quad (47)$$

where we have used that for an oscillating object $\dot{X} \sim \omega_0 A$, $\ddot{X} \sim \omega_0^2 A$ and $l \equiv h_1/h_2$ is the characteristic length over which the system bath coupling changes. The two conditions in Eq. (47) can be satisfied when both dimensionless parameters $\omega_0\tau$ and A/l are small. The first condition simply states that the system is slow compared to the relaxation time of the environment. Naively, the second condition, $A \ll l$, limits the validity of our results to the small amplitude oscillation regime. This is misleading since A should be understood as (at most) the distance covered during a single relaxation time, i.e. $A \sim \dot{X}\tau$. This is obvious if one realizes that the environment relaxes to equilibrium on the time-scale τ and therefore the dynamics of the system can not depend on phenomena which occurred too far in the past. Therefore the condition $A \ll l$ can be satisfied (for any τ) at sufficiently small velocity signifying that the expressions derived in the Sec. III become asymptotically exact in the limit of small velocity. In particular, for the magnetic oscillator example, the quadratic contribution to the velocity can be neglected for $\dot{X} \ll v_{cr}$ where:

$$v_{cr} \sim \frac{l}{\tau} = \frac{\mu r h_1}{h_2}. \quad (48)$$

VII. CONCLUSION

In conclusion we have studied the dynamics of a macroscopic object coupled to a stochastic environment using an adiabatic perturbation theory. The dynamics of the stochastic environment are taken to obey detailed balance. Surprisingly we find that the emergent equations of motion contain, besides the usual dissipative term, a *negative* mass correction. Naively this term is higher order in perturbation theory than the dissipative term and therefore negligible. However, as we demonstrated, this can be misleading. For example, the frequency shift in a damped harmonic oscillator resulting from both terms is of the same order (in τ).

Moreover, the dissipative term breaks time-reversal symmetry while the mass correction does not. This means that these two quantities capture fundamentally different properties and should be analyzed separately. This is exemplified in Sec. V where we have analyzed the dynamics of the energy of the macroscopic object. While dissipation would lead to a monotonic energy decrease the mass correction leads to the non-Markovian dynamics. In fact the macroscopic object can transiently absorb energy from the environment and its energy is, in general, a non-monotonically decreasing function of time despite the fact that the object is coupled to a dissipative environment.

Correction to the mass in emergent equations of motion are by no means uncommon. For example in hydrodynamics it is well known (see for example [21]) that interaction of, say, a ball with a fluid lead to both a dissipative term and a mass correction. However, these mass corrections are always positive in contrast to the *negative* mass correction discussed above. Interestingly, when the coupling between the environment and the object is via the momentum it is possible to find a negative mass correction for a particle in the strong coupling regime [22, 23]. However, that setup is rather different from the situation described here where we focus on real space interactions and the negative mass correction appears in the weak coupling limit.

Remarkably, within a similar adiabatic perturbation theory in quantum systems the mass correction can be shown to be strictly positive [9]. It remains unclear how the two approaches can be made to agree and what this implies either on the adiabatic perturbation theory in quantum mechanics or the validity of stochastic dynamics for classical systems with many degrees of freedom. We plan to address this question in a future publication.

Finally, we note that the above treatment has been carried out to an order which is formally identical to a linear-response regime [12]. It will be interesting to see if higher order terms can lead to other interesting effects.

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