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Frequency domain aspects of aggregation

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FREQUENCY DOMAIN ASPECTS OF AGGREGATION

by

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of the periodogram of aggregated series in the nonstationary case for a generalized fractional process. I show that for skip sampling a long memory feature at the zero frequency can arise from the aggregation of a generalized fractional series, while temporal aggregation does not induce such an effect. Simulation results pertaining to the estimates of the memory parameter are included to demonstrate the practical relevance of my theoretical results.

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List of Abbreviations

AMEX	American Stock Exchange
ARFIMA	Autoregressive Fractionally Integrated Moving Average
ARMA	Autoregressive-Moving-Average
DGP	Date Generating Process
DJIA	Dow Jones Industrial Average
dft	Discrete Fourier Transform
LP	Log-Periodogram
RLS	Random Level Shifts
S&P 500	Standard and Poor's 500

Chapter 1

Temporal Aggregation for Stochastic Volatility Model

1.1 Introduction

Long-memory processes, especially the possibility of confusing them with structural changes, are of great interest in the field of time series. Applications of long-memory models are numerous, in particular in relation to stock return volatility in financial markets. Ding, Engle, and Granger (1993) argue that stock return volatility series can be well described by long-memory processes. However, it has also been shown that the estimate of the long-memory parameter, d , is biased away from 0 and the autocovariance function exhibits a slow rate of decay when a stationary short-memory process is contaminated by structural changes in level. In other words, a spurious long-memory process can arise when there are structural changes in a short-memory process. This idea extends that advanced by Perron (1989, 1990), who shows that structural changes and unit roots ($d = 1$) are easily confused in the sense that the estimate of the sum of the autoregressive coefficients is biased towards 1 and that tests of the null hypothesis of a unit root are biased towards non-rejection, with a stationary process contaminated by structural changes. Relevant literature on this issue include Diebold and Inoue (2001), Engle and Smith (1999), Gouriéroux and Jasiak (2001), Granger and Hyung (2004).

Recently, Perron and Qu (2010) analyze the properties of the autocorrelation function, the periodogram, and the log-periodogram (LP) estimate of the long-memory parameter for short-memory processes with random level shifts (RLS). They show that the autocorrelation function, the periodogram, and the LP estimate for the log squared daily returns for the

Standard and Poor's 500 (S&P 500) during 1928-2002 can be explained by a level shift model with a short-memory component, instead of a long-memory process without level shifts. Lu and Perron (2010) present a method to directly estimate the level shift model using an extension of the Kalman filter and apply it to the log absolute returns for the S&P 500, AMEX, DJIA and NASDAQ stock market return indices. Their point estimates imply few level shifts for all series but once these are taken into account there is no evidence of long-memory in the sense that little serial correlation is found in the remaining noise.

McCloskey and Perron (2013) provide simple trimmed versions of the LP estimator, which are consistent and asymptotically normal with the same limiting variance as the standard LP estimator regardless of whether the underlying long/short-memory process is contaminated by level shifts or deterministic trends. Using these robust LP estimators, they study the log-squared daily return series of the S&P 500, the Dow Jones Industrial Average (DJIA), the NASDAQ and the AMEX stock market indices, which are also examined by Lu and Perron (2010). Their robust LP estimator is -0.017, indicating the (near) absence of long-memory for the S&P 500 volatility series, which contradicts the estimate of 0.505, given by the standard LP estimator. Very similar results emerge for the DJIA and the AMEX. An interesting finding is that the robust LP estimator is 0.388, indicating a stationary long-memory process, when using the log of daily realized volatility series constructed from 5-minute returns of the S&P 500 futures index from April 21, 1982 to March 2, 2007. This surprising contradiction between their findings raises a puzzle for econometricians when stock returns volatility is estimated. More recently, Varneskov and Perron (2011) extend the work of Lu and Perron (2010) and adopt a full parametric model including both random level shifts and ARFIMA components. They estimate the long-memory parameter, d , for the realized volatility series of the S&P 500 data generated from 5-minute returns, which is the same data used by McCloskey and Perron (2013). They also estimate the long-memory parameter for the squared daily returns of the S&P 500 data for the time period 1929-2004, using a RLS+ARFIMA $(0, d, 0)$ model. The estimate for the former is 0.3564, indicating the existence of long-memory processes, while it is 0.0532 for the latter, indicating the (near) absence of long-memory processes. They also

consider other RLS+ARFIMA models and the 30-year T-bonds data, the Dollar-AUS exchange rate and the Dollar-AUS exchange rate, all of which present very similar results.

The realized volatility series are essentially the aggregation of the squared high-frequency returns, and the daily return series are the aggregated intraday returns. In this regard, it may be possible to explain the above puzzle by means of temporal aggregation effects. There is a vast amount of papers on temporal aggregation, e.g. Wei (1978) and Chamber (1998), but little attention has been paid to its effects in the frequency domain. Souza (2005, 2007, 2008) develop a series of papers to discuss the effect of temporal aggregation and bandwidth selection in estimating the degree of long memory. Ohanissian et al. (2008) propose a test to distinguish between true and spurious long memory by exploiting the invariance of the long-memory parameter to temporal aggregation. Hassler (2011) investigates whether typical frequency domain assumptions made for semiparametric estimation and inference are closed with respect to aggregation.

It is believed by many that the financial returns are perturbed fractional integrations, which are related to stochastic volatility model (Hassler, 2011). However, the aggregation of stochastic volatility model has been little discussed. Andersen and Bollerslev (2000) study the effect of the temporal aggregation of a long-memory stochastic volatility model in the time domain. Studying the effect of the temporal aggregation of a stochastic volatility model in the frequency domain is also important. First, the log periodogram estimator has become one of the most popular memory parameter estimator among empiricists due to simplicity, intuitiveness and ease of use, as discussed by McCloskey and Perron (2013). Therefore, it is necessary to study the aggregation of the stochastic volatility in the frequency domain. Second, by using a stochastic volatility model, we can better understand the difference between the various measures of volatilities, e.g., the realized volatility and squared daily returns.

In this chapter, we show how the squared low-frequency returns can be expressed in terms of the temporal aggregation of a high-frequency series. We build a bridge between the spectral density function of squared low-frequency returns and that of the squared high-frequency returns. Furthermore, we analyze the properties of the spectral density function of realized

volatility, constructed from the squared returns with different frequencies under temporal aggregation. These will allow us to explain the following puzzles. First, the LP estimates for the log squared daily return series are very close to zero, while they are relatively large (around 0.4) for the log daily realized volatility constructed from high-frequency returns. Second, the LP estimates for the log squared daily return series are near zero, while are very large for the log aggregated squared daily returns. Third, the LP estimates for the log realized return series are very large using high-frequency return data, while they are close to zero for the disaggregated original high-frequency returns when a large bandwidth is used. The theoretical findings are illustrated through the analysis of both low-frequency daily S&P 500 returns from 1928 to 2011 and high-frequency 1-minute S&P 500 returns from 1986 to 2007. We consider the estimation of the long-memory parameter using the standard and the trimmed Log Periodogram (LP) estimate.

The remainder of this chapter is structured as follows. Section 2 introduces the stochastic volatility model and the different aggregation processes for the realized volatility and the squared daily returns. Section 3 analyzes the spectral density of the realized volatility and the squared returns with data of different sampling frequencies. Section 4 provides the rule of equivalence of estimates across aggregation levels for stationary time series, and is also extended to the case with random level shifts (RLS). Section 5 focuses on empirical application of the new theoretical finding to S&P500 data with different sampling frequencies. Section 6 is composed of concluding remarks and a mathematical appendix contains technical derivations.

1.2 Alternative volatility measures

In this section, we first review the nature of stochastic volatility models and the aggregation mechanisms. We then present theoretical results about temporal aggregation in the frequency domain.

1.2.1 Stochastic volatility model

It is evident that the number of observations for stock market returns during a fixed period of time is inversely related to the length of each return interval. That is, with prices fully available, a shorter interval means that the returns are observed more frequently and therefore more observations can be obtained. High-frequency returns are now available since many financial time series are available on a tick-by-tick basis, which is virtually continuous. In this chapter, high-frequency returns are classified according to the numbers of time units included in their intervals. To simplify, the returns obtained every one unit of time period are defined as 1-period returns, and similarly k -period returns denote the returns observed every k units of time periods.

Let $r_{t,n}$ be the n th intraday return at day t such that

$$r_{t,n} = \frac{1}{\sqrt{s}} h_{t,n} \varepsilon_{t,n}$$

$n = 1, \dots, s$ and $t = 1, \dots, T$. Here, s denotes the number of 1-period returns per day and T is the number of days. The component $h_{t,n}$ intends to capture the volatility level, while $\varepsilon_{t,n}$ is i.i.d. standard normal. It is further assumed that $h_{t,n}$ and $\varepsilon_{t,n}$ are mutually independent.

1.2.2 Temporal aggregation

According to the classification of high-frequency returns in section 2.1, a sample of s 1-period returns contains s/k k -period returns. We assume that k is chosen such that $s = kS$ for some integer S . Let $r_{t,p}^{(k)} = \sum_{n=p(k-1)+1}^{pk} r_{t,n} = \sum_{j=0}^{k-1} L^j r_{t,kp}$ denote the continuously compounded k -period return, so that

$$r_{t,p}^{(k)} = \frac{1}{\sqrt{s}} \left(\sum_{j=0}^{k-1} L^j \right) h_{t,kp} \varepsilon_{t,kp}$$

for $p = 1, \dots, S$ and $t = 1, \dots, T$. Here, L is the backshift operator. Therefore, the k -period return $r_{t,p}^{(k)}$ can be written as

$$r_{t,p}^{(k)} = \frac{1}{\sqrt{s}} z_{t,p} \sqrt{\left(\sum_{j=0}^{k-1} L^j \right) h_{t,kp}^2}$$

where $z_{t,p}$ is i.i.d. standard normal. Then the squared k -period return $r_{t,p}^{(k)2}$ is given by

$$r_{t,p}^{(k)2} = \frac{1}{s} z_{t,p}^2 \left(\sum_{j=0}^{k-1} L^j \right) h_{t,kp}^2$$

which can be expressed in term of temporal aggregation, as

$$r_{t,p}^{(k)2} = \left[\frac{1}{s} h_{t,n}^2 z_{t,[n/k]+1}^2 \right]^{(k)}$$

where $[n/k]$ denotes the integer part of n/k and $[\cdot]^{(k)}$ denotes the k -period non-overlapping temporal aggregation. Therefore, the squared daily return

$$r_t^2 = \left[\frac{1}{s} h_{t,n}^2 z_t^2 \right]^{(s)}$$

is the temporal aggregation of $(1/s)h_{t,n}^2 z_t^2$, over a day. Here, z_t is i.i.d. standard normal for $t = 1, \dots, T$. The realized volatility constructed from the k -period return $r_{t,p}^{(k)}$,

$$RV_t = \sum_{p=1}^S r_{t,p}^{(k)2}$$

is the temporal aggregation of the squared k -period return $r_{t,p}^{(k)2}$, such that

$$RV_t = \left[r_{t,p}^{(k)2} \right]^{(S)}$$

1.3 Temporal aggregation in the frequency domain

Assumption 1.1 *We assume that the demeaned squared volatility level $y_{t,n} = h_{t,n}^2 - E(h_{t,n}^2)$ is covariance stationary with integrable spectral density $f_y(\lambda)$.*

Proposition 1.1 *Under Assumption 1, the spectral density of the squared $r_{t,p}^{(k)}$ is*

$$\begin{aligned} f_{r_{t,p}^{(k)2}}(\lambda) &= \frac{1}{s^2 k} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \left| \sum_{j=0}^{k-1} \sin\left(\frac{\lambda + 2j\pi}{2k}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2k}\right) \\ &\quad + \frac{1}{s^2 \pi} \left[k^2 (E(h_{t,n}^2))^2 + \text{Var}(h_{t,n}^{(k)}) \right] \end{aligned} \quad (1.1)$$

Proof. See Appendix. ■

The first term on the right hand side of Equation (1.1) corresponds to the spectral density of the temporal aggregation of the k -period demeaned squared volatility level, $[y_{t,n}]^{(k)}$. The remaining two terms are constant, induced by the noise component.

Remark 1.1 *When $k = s$, we have the spectral density of the squared daily returns*

$$f_{r_t^2}(\lambda) = \frac{1}{s^3} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \sum_{j=0}^{s-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2s}\right) + \frac{1}{s^2\pi} \left[s^2 (E(h_{t,n}^2))^2 + \text{var}(h_{t,n}^{(s)}) \right] \quad (1.2)$$

The first term of (1.2) is such that

$$\frac{1}{s^3} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \sum_{j=0}^{s-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2s}\right) \rightarrow \frac{1}{s} f_y\left(\frac{\lambda}{2s}\right)$$

as $\lambda \rightarrow 0$. Therefore, the spectral density of the demeaned squared daily volatility decreases as s increases. When s is large enough, the spectral density of the squared daily returns $f_{r_t^2}(\lambda)$ will be dominated by the second and third terms of (1.2), which implies that the squared daily return series is dominated by noise.

Proposition 1.2 *The spectral density of the realized volatility obtained from k -period returns, $r_{t,p}^{(k)}$, is*

$$f_{[r_{t,p}^{(k)2}]^{(s)}}(\lambda) = \frac{1}{s^3} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \sum_{j=0}^{s-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2s}\right) + \frac{1}{s^2\pi} \left[sk (Eh_{t,n}^2)^2 + S\text{var}(h_{t,n}^{(k)}) \right] \quad (1.3)$$

Proof. See Appendix. ■

Remark 1.2 *When $S = s$, i.e., the realized volatility is obtained by 1-period returns, $r_{t,n}$, we have the following form of the spectral density*

$$f_{[r_{t,n}^2]^{(s)}}(\lambda) = \frac{1}{s^3} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \sum_{j=0}^{s-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2s}\right) + \frac{1}{s\pi} \left[(Eh_{t,n}^2)^2 + \text{var}(h_{t,n}^2) \right] \quad (1.4)$$

For the spectral density of realized volatility, we have from (1.4)

$$f_{[r_{t,n}^2]^{(s)}}(\lambda) \rightarrow \frac{1}{s\pi} \left[\pi f_y \left(\frac{\lambda}{s} \right) + (E(h_{t,n}^2))^2 + var(h_{t,n}) \right]$$

as $\lambda \rightarrow 0$. This means that the three terms of the spectral density of the realized volatility in equation (1.4) decrease at the same rate when s increases.

Comparing the spectral density of the squared daily returns (1.2) with that for the realized volatility (1.4), note that their first terms are identical, i.e.

$$\frac{1}{s^3} \left| \sin \left(\frac{\lambda}{2} \right) \right|^2 \sum_{j=0}^{s-1} \left| \sin \left(\frac{\lambda + 2j\pi}{2s} \right) \right|^{-2} f_y \left(\frac{\lambda + 2j\pi}{2s} \right)$$

corresponding to the spectral density of the temporal aggregation of the demeaned squared volatility level, $y_{t,n} = h_{t,n}^2 - E(h_{t,n}^2)$, over a day. This is the only part that contains information about the long memory and the remaining terms are simply noise. Therefore, both realized volatility and squared daily returns contain the same information about long memory.

A difference between the spectral density of the squared daily returns and that for the realized volatility series occurs only in the second and the third terms,

$$f_{r_t^2}(\lambda) - f_{[r_{t,p}^{(k)2}]^{(s)}}(\lambda) = \frac{(s-k)}{s\pi} (E(h_{t,n}^2))^2 + \frac{1}{s^2\pi} \left[var(y_{t,n}^{(s)}) - Svar(h_{t,n}^{(k)}) \right]$$

which is independent of the value of λ . Furthermore, $Var([y_{t,n}]^{(s)}) - Svar([h_{t,n}]^{(k)})$ is positive in general because most financial series have positive autocorrelation even at large lag. Therefore, the difference between the spectral density of the squared daily returns and the realized volatility will be positive and larger than $[(s-k)/s\pi] (E(h_{t,n}^2))^2$.

1.4 Long-memory parameter estimates across aggregation levels

In this section, we first review the log-periodogram regressions. We then presents theoretical results about the equivalence of estimates across aggregation level.

1.4.1 Log-periodogram regressions

A long-memory process typically has a spectral density function which is proportional to λ^{-2d-1} as λ goes to zero, where d is the memory parameter. The fractionally integrated model, proposed by Granger and Joyeux (1980) and Hosking (1981), is a long-memory generalization of an ARMA model whose autocorrelations decay exponentially. When $d \in (0, 0.5)$, the autocorrelations decay slowly, a characteristic of long-memory processes. Various estimators of d have been proposed, among which semiparametric estimators have become widely used as they do not require a distributional assumption on the process generating the difference of order d of the series. The most popular semiparametric estimator is the LP regression estimator proposed by Geweke and Porter-Hudak (1983), which uses only frequencies near zero to avoid possible misspecification caused by high frequency movements. The LP regression estimator was analyzed by, among others, Robinson (1995) and Phillips (2007) for the unit root case $d = 1$.

The LP regression estimator is based on the following spectral characterization of a long-memory process:

$$\log f(\lambda) \approx c - 2d \log \lambda$$

as $\lambda \rightarrow 0_+$, where f is the spectral density function of the process. The periodogram of the time series at λ_j is defined as

$$I_x(\lambda_j) \equiv |w_x(\lambda_j)|^2 = w_{x(\lambda_j)} w_x(\lambda_j)^*,$$

where $w_x(\lambda_j)$ is the discrete Fourier transform of the time series $\{x_t\}_{t=1}^T$ evaluated at the Fourier frequency $\lambda_j = 2\pi j/T$, and c^* denotes the complex conjugate of any complex number c . $I_x(\lambda_j)$ can be viewed as a noisy approximation to f . Therefore, the LP regression is given by

$$\log I_x(\lambda_j) = c + dX_j + e_j, \quad j = l, \dots, m,$$

where $X_j = -\log(2 - 2\cos(\lambda_j)) \approx -\log \lambda_j^2$ for $j = l, \dots, m$. The LP regression estimator is

$$\hat{d} = \frac{-0.5 \sum_{j=l}^m (Y_j - \bar{Y}) \log I_j}{\sum_{j=l}^m (Y_j - \bar{Y})^2},$$

where $Y_j = \log |1 - \exp(-i\lambda_j)|$ and $\bar{Y} = (1/(m-l+1)) \sum_{k=l}^m Y_k$. When $l = 1$, this estimator is a standard LP regression estimator. We can trim some of the lower frequencies, as discussed in McCloskey and Perron (2013), to obtain consistency and asymptotic normality with the same limiting variance as the standard LP regression estimator regardless of whether the underlying long/short-memory process is contaminated by level shifts or deterministic trends.

1.4.2 Equivalence of estimates across aggregation levels

Lemma 1.1 *Under Assumption 1, for the spectral densities of the aggregated series and the original squared return series, we have*

$$f_{\left[r_{t,p}^{(k)2} \right]^{(S)}}(\lambda) - S f_{r_{t,p}^{(k)2}}(\lambda/S) \rightarrow 0 \text{ for } \lambda \rightarrow 0.$$

Proof. See Appendix. ■

Lemma 1.1 implies that the spectral densities of the aggregated series and the original squared return series have the same slope near frequency zero. Hence, aggregation does not change the value of the long-memory parameter, consistent with the results of Chamber (1998), Souza (2005) and Hassler (2011).

Corollary 1.1 *The periodogram is a finite sample version of the spectral density; hence, a similar relation holds approximately, i.e.*

$$I_{\left[r_{t,p}^{(k)2} \right]^{(S)},j} - S I_{r_{t,p}^{(k)2},j} \rightarrow 0$$

as $T \rightarrow \infty$ for $\lambda \rightarrow 0$.

A similar result was also obtained for stationary long memory series by Ohanissian et al. (2008).

Remark 1.3 *Lemma 1.1 and Corollary 1.1 do not depend on the stochastic volatility specification. The validity of the results only require that the original process be stationary.*

Remark 1.4 *As discussed by Souza (2008), a better estimator is not necessarily generated by temporally aggregating a time series, because the same estimate can be obtained when the same bandwidths are used on the original time series, which offers a wider choice of bandwidths. That is, the original time series could provide potentially improved estimates in the sense that it allows for more flexible bandwidths selection.*

Remark 1.5 *The microstructure noise is not taken into consideration here. However, adding a microstructure noise process will not change the result in Lemma 1.1, because Assumption 1 will still be satisfied after a stationary noise has been added, and the above results hold as long as the time series is stationary.*

According to Perron and Qu (2010), the autocorrelation function, the periodogram, and the LP estimate for the log squared daily returns for the S&P 500 can be explained by a simple level shift model with a short-memory component, instead of a long-memory process without level shifts. Lu and Perron (2010) estimate a random level shifts model and find that few level shifts are present for financial series, but once these are accounted for, there is no evidence for the existence of long-memory processes in the sense that little serial correlation is found in the remaining noise. Therefore, random level shifts, which have not been included in Assumption 1, should be considered here to generalize Lemma 1.1.

We consider the following random level shift model proposed by Perron and Qu (2010),

$$u_{T,t} = \sum_{j=1}^t \delta_{T,j}, \quad \delta_{T,t} = \pi_{T,t} \eta_t \quad (1.5)$$

where $\eta_t \sim i.i.d. N(0, \sigma_\eta^2)$ and $\pi_{T,t} \sim i.i.d. Bernoulli(p/T, 1)$. It is also assumed that the components $\pi_{T,t}$ and η_t are mutually independent. According to Proposition 3 of Perron and Qu (2010), the limit of the expectation the periodogram has the following form

$$\lim_{T \rightarrow \infty} E \left[\frac{1}{T} I_{u,j} \right] = \frac{p \sigma_\eta^2}{4\pi^3 j^2} \quad \text{as } T \rightarrow \infty.$$

Lemma 1.2 *Under the data generating process (1.5), we have the following relation between the k -period aggregated series and the original random level shift series,*

$$\lim_{T \rightarrow \infty} E \left[I_{u^{(k)},j} \right] - k \lim_{T \rightarrow \infty} E \left[I_{u^{(1)},j} \right] \rightarrow 0 \quad \text{for } \lambda \rightarrow 0.$$

Proof. See Appendix. ■

Therefore, Remark 1.4 still holds when random level shifts are taken into consideration.

1.5 S&P 500 volatility

We consider two series of returns related to the S&P 500 data, namely low-frequency and high-frequency returns. The low-frequency data consist of 22,000 daily return observations for the period from August 13, 1928 to December 30, 2011. Among these daily returns, the observations dated between August 13, 1928 and October 30, 2002 were kindly provided by William Schwert. The source of the data for the period January 4, 1928 through July 2, 1962 is Schwert (1990). From July 3, 1962 to October 30, 2002 it is from the CRSP daily returns file, and the returns for the time period after October 30, 2002 were obtained from the Yahoo Finance website. The high-frequency data pertains to S&P 500 futures. The cleaned version was provided by Ikeda (2010) and includes 1-minute returns from October 7, 1986 to March 2, 2007, amounting to 5,000 trading days in total. In order to eliminate the effect of outliers in the data, we use the logarithmic form of the observations. Since there are some zeros in the original high-frequency and daily data, we demean our data first, as in Deo et al. (2006). Other methods were proposed in the literature, e.g. Perron and Qu (2010) and Lu and Perron (2010), who include adding a small value to the squared returns.

1.5.1 Low Frequency Data

We first start our analysis with low frequency data, i.e., the daily data series. Table 1 shows the LP estimates for the log realized k -day return series, which is simply calculated by cumulating k neighboring squared daily returns that do not overlap. More specifically, $S = 1, 5, 10,$ and 20 stands for the squared daily returns, the realized weekly (every five business days), biweekly, and monthly (every twenty business days) volatilities, respectively. The columns labelled $[\log(r_t^2)]^{(S)}$, $\log(r_t^2)$ and $\log\{[r_t^2]^{(S)}\}$ refer to the S -period aggregation of the logarithmic transformation of squared daily returns, the logarithmic transformation of the original squared daily returns using the bandwidth $N = T/S$, and the logarithmic transformation of the S -period aggregated squared daily returns, respectively, with S denoting the aggregation level.

Both the standard and trimmed LP estimates are presented for purposes of comparisons. For each series, the standard LP estimate is computed using $m = N^{0.5}$, while the trimmed one is constructed using $(l, m) = (N^{0.65}, N^{0.9})$, which performs relatively well according to McCloskey and Perron (2013). In all cases, $N = T/S$. Note that we consider the original daily returns series $\log(r_t^2)$ with different bandwidths to highlight the importance of the bandwidth selection. As the results will show, the empirical estimates are very similar using either the S periods aggregation with bandwidth T/S or the original daily series with the same bandwidth T/S . This is indeed an implication of Lemmas 1.1 and 1.2 so that it should hold whether the process is a RLS or pure long-memory.

Remark 1.6 *Lemmas 1 and 2 apply only to the case $[\log(r_t^2)]^{(S)}$ and $\log(r_t^2)$ but not to $\log\{[r_t^2]^{(S)}\}$ and $\log(r_t^2)$. However, as we shall show in the simulations, all three behave similarly as the aggregation changes. Hence, we conjecture that it would be possible to extend our results to cover the case of $\log\{[r_t^2]^{(S)}\}$ and $\log(r_t^2)$.*

Some interesting results in Table 1 are worth notice. First, with the same bandwidth ($N = T/S$), the estimates for the log realized daily return series, which is indeed the log aggregated squared daily returns, are approximately equal to and always a little bit larger than the estimates for the log squared original returns. This finding confirms Corollary 1.1, i.e., the same long-memory parameter estimate can be obtained for both the aggregated time series and the original series when the same bandwidths are used. Figure 1 shows the periodograms of the squared daily return series (left), the 5-periods aggregation of the squared daily return series/5 (middle), and the 20-periods aggregation of the squared daily return series/20 (right) for frequency indices up to 550. The 550th frequency index corresponds to the frequencies $\pi/20$, $\pi/4$, and π for the squared daily returns, the 5-period aggregation of the squared daily returns, and the 20-period aggregation of the squared daily returns, respectively. Note that they have almost identical pattern near frequency zero. This finding confirms Corollary 1.1 again and implies that we have the same long-memory parameter estimate for the aggregated time series and the original series when the same bandwidths are used.

Second, when $S = 1$, i.e., the daily return series, note that both the standard and trimmed LP estimators are very different. In particular, the trimmed LP estimates are close to zero when $S = 1$, indicating the (near) absence of long-memory processes. On the other hand, the standard LP estimate is 0.57. However, for the realized 5-day return series, the standard and trimmed LP estimators are 0.62 and 0.31, respectively. Also, a feature of interest is the fact that both the standard and trimmed LP estimates increase as S increases. As shown in Remark 1.4, the same estimate of the long-memory parameter for the aggregated series should be obtained when using the same bandwidths as those on the original time series. Therefore, the apparent difference could simply be caused by the bandwidth selection. We actually use a relative small bandwidth for the aggregated series, compared with the original series, because the number of observation in the aggregated series is smaller. The issue of interest is whether this feature is more likely to occur with a RLS model or with a pure long-memory process. As discussed in Perron and Qu (2010), the LP estimate increases as m decreases when considering the RLS plus white noise model. Hence, a larger LP estimate is expected for the aggregated series under RLS. In particular, it is expected to be greater than .5, i.e., in the non-stationary region. No such increase towards values in the non-stationary region is expected if the process is a pure long-memory one as the bandwidth decreases. Hence, these results are more consistent with a RLS being the underlying data-generating process. This will be confirmed via simulations.

1.5.2 High Frequency Data

We now consider high-frequency data, for which the unit of time period is one minute. Table 2 shows the LP estimates for the log realized daily volatility constructed from k -period high-frequency data, and the log squared original returns. Here, $k = 1, 5, 30$, and 330 correspond to the case of 1-minute, 5-minute, 30-minute, and daily returns, respectively. The columns labelled $[\log(r_{t,n}^{(k)2})]^{(S)}$, $\log[r_{t,n}^{(k)2}]$ and $\log[r_{t,n}^{(k)2}]^{(S)}$ refer to the S -period aggregation of the logarithmic transformation of squared k -min returns, the logarithmic transformation of squared k -min returns and logarithmic transformation of the realized daily volatility aggregated by squared k -min returns over a day, respectively. S denotes the number of k -min returns per

day, with $s = kS$. Both the standard and trimmed LP estimates are presented for purposes of comparisons. For each series, the standard LP estimate is constructed using $m = N^{0.5}$, and the trimmed one using $(l, m) = (N^{0.65}, N^{0.9})$. For the log realized return volatility series, we let $N = T$, so that the number of return observations equals the number of days on which prices are available. However, we let $N = TS$ for the log squared return series, which means that the total number of return observations is equal to the product of the number of days and the frequency in each day. For comparison purposes, we also include the estimates for the log squared original returns with $N = T$.

Remark 1.7 *Lemmas 1 and 2 apply only to the case $[\log(r_{t,n}^{(k)2})]^{(S)}$ and $\log[r_{t,n}^{(k)2}]$ but not to $\log[r_{t,n}^{(k)2}]^{(S)}$ and $\log[r_{t,n}^{(k)2}]$. However, as we shall show in the simulations, all three behave similarly as the aggregation changes. Hence, we conjecture that it would be possible to extend our results to cover the case of $\log[r_{t,n}^{(k)2}]^{(S)}$ and $\log[r_{t,n}^{(k)2}]$.*

Some interesting results in Table 2 are worth notice. First, similar to the results in Table 1, with the same bandwidth set to $N = T$, the estimates for the log realized volatility series, which is indeed the aggregated squared returns, are approximately equal to the estimates for the log squared original returns. For instance, when $k = 1$ and $S = 330$, the trimmed LP estimator for the log realized return volatility is 0.48 while the corresponding estimator for the log squared original returns is 0.38.

Figure 2 shows the periodograms of the realized volatility obtained from 1-minute return series (left), s times the periodograms of the squared 1-minute return series (middle), as well as the difference between them (right), with 2500 frequency indices. The 2500th frequency index corresponds to the frequencies π and $\pi/330$ for the realized volatility and the squared 1-minute returns, respectively. The periodogram of the realized volatility (left) and that of the squared 1-minute returns (middle) exhibit very similar values for frequency indices smaller than 1000, especially for frequency indices close to zero. The difference between the periodogram for the realized volatility and the 1-minute returns (right) is slight. These results can be explained by the fact that the realized volatility series here is the 330-period non-overlapping aggregation of the squared 1-minute return series, and their periodograms exhibit approximately the same

values for small frequency indices, as stated in Lemma 1.1.

Second, for the log realized volatility series, the LP estimates decrease as k increases, i.e., smaller when the return interval is longer. With daily returns ($k = 330, S = 1$), so that the log realized volatility series is the log squared daily return series, the standard and trimmed LP estimates are 0.48 and 0.04. In particular, the trimmed LP estimator is close to zero, indicating the (near) absence of long-memory, while the standard LP estimate is large, consistent with a RLS process. Of interest is the fact that for $k = 1, 5, 30$ all estimates are similar when the same bandwidth is used. This accords with the theoretical results that the estimates are invariant to the aggregation level. When $k = 330$, i.e., daily data, the estimates are somewhat smaller. This feature can be explained by the fact that as the aggregation level increases the spectral density function is contaminated by noise, which henceforth reduces the estimate, see Remark 1.1.

Combining the equation for the squared daily returns (1.2) and that for the realized volatility (1.4), we know, as discussed in Section 3, that both the realized volatility and the squared daily returns contain the same information about long memory. However, the squared daily returns contain a larger noise component than the realized volatility. Figure 3 shows the log squared daily return series (left), the log realized volatility obtained from 1-minute return series (middle), and the difference between them (right). We can see that the log squared daily return series (left) exhibits larger variance than the log realized volatility series (middle). No pattern is seen in the difference (right), and it simply seems to be noise. Figure 4 shows the periodograms of the log squared daily returns (left) and the log realized volatility obtained from 1-minute return (middle), as well as the difference between them (right). Note that the periodogram of the log squared daily returns (left) is much larger than that of the log realized volatility constructed from 1-minute returns (middle) except for the first few frequencies near zero. For those frequencies near zero, both periodograms show very large values. Similar to what was shown in Figure 3, the difference (right) appears to be caused by a white noise process. Also, we can see that the white noise process is dominant in the periodogram of the log squared daily return series (left). As shown in Figure 5, similar results occur for the

periodograms of the log realized volatility obtained from 110-minute returns (left), 30-minute returns (middle) and 5-minute returns (right). The periodograms exhibit smaller values when higher frequency data are used, except for the first few frequencies indices near zero. For these, the periodograms may be mainly determined by low-frequency contamination, for example, random level shifts (Perron and Qu, 2010 and McCloskey and Perron, 2011). In general, the values of the periodograms are much larger for frequency indices near zero, which can be explained by the fact that the impact of the random level shifts dominates that of the noise process.

Third, when larger bandwidths ($N = T \times S$) are used to estimate the memory parameter for the log squared original returns, the trimmed LP estimates are close to zero, indicating the (near) absence of long memory, while the standard LP estimates are near the non-stationary region, regardless of the length of the return intervals. These results are consistent with those of Perron and Qu (2009), Lu and Perron (2009), McCloskey and Perron (2011) and Varneskov and Perron (2011). This is an important feature, which shows the importance of the bandwidth selection, in particular in selecting a value large enough. The use of the aggregated squared k minutes returns allow much more flexibility in the possible choice of the bandwidth, so that we can always use $N = TS$, which was suggested by Souza (2008) as leading to improved estimates. Such choices are not possible when using log realized volatility so that the estimator is much more influenced by the low frequencies thereby inducing larger estimates, regardless of whether the true process is RLS or pure long-memory. Hence, we view the estimates obtained with the aggregated squared k minutes returns and a large bandwidth as providing the best estimates indicating the near absence of long-memory. This is re-inforced by the fact that these estimates are (very nearly) the same across aggregation levels, showing robustness to aggregation at any level.

1.6 Monte-Carlo evidence

In this section, we consider various models to simulate "daily" and "1-min" returns to demonstrate the practical relevance of our theoretical results.

1.6.1 Low frequency data

To examine how the empirical features obtained with the daily data can be explained by various models, we consider the following models to generate the demeaned "daily" returns:

$$\text{Model AI: } r_t = \exp(v_t/2 + w_t/2) e_t;$$

$$\text{Model AII: } r_t = \exp(u_{T,t}/2 + w_t/2) e_t;$$

$$\text{Model AIII: } r_t = \exp(u_{T,t}/2 + v_t/2 + w_t/2) e_t,$$

where $e_t \sim i.i.d. N(0, 1)$. Here, the RLS component is $u_{T,t} = \sum_{j=1}^t \delta_{T,t}$, where $\delta_{T,t} = \pi_{T,t} \eta_t$, $\eta_t \sim i.i.d. (0, \sigma_\eta^2)$ and $\pi_{T,t} \sim i.i.d. Bernoulli(p/T, 1)$. Also the long-memory component v_t is such that $(1 - L)^d v_t = \varepsilon_t$, where $\varepsilon_t \sim i.i.d. N(0, \sigma_v^2)$. $w_t \sim i.i.d. N(0, \sigma_w^2)$ is the white noise component. Note that we introduce two noise component w_t and e_t to be better able to control the signal to noise ratio. Models AI, AII and AIII refer to ARFIMA(0, d , 0) plus white noise, RLS plus white noise and RLS plus ARFIMA(0, d , 0) plus white noise, respectively. The model parameters are $T = 22,000$, $p/T = 0.0045$, $d = 0.4564$, $\sigma_w = 1.3121$, $\sigma_v = 0.4603$ and $\sigma_\eta = 0.5081$ in our simulations, which are the estimated values for squared daily returns and realized daily volatility series on S&P 500 from Varneskov and Perron (2011). Throughout, the average of the estimates of d over 100 replications are used.

Table 3 presents the results for model AI, which are much smaller than the corresponding estimates in Table 1, indicating that Model AI is unable to replicate the empirical results. As shown in Tables 4 and 5, Models AII and AIII yield similar results, but Model AIII offers better estimates for aggregation levels $S = 5, 610$ and $S = 20$. To be more specific, for the simulated 5-period series, 10-period series and 20-period series, the average of the standard and trimmed LP estimates for $[\log(r_t^2)]^{(S)}$ are (0.6575, 0.1709) ($S = 5$), (0.7393, 0.2593) ($S = 10$) and (0.7717, 0.3330) ($S = 20$). using AIII, which are close to the corresponding LP estimates obtained with the S&P 500 data shown in Table 1 (0.6313, 0.2570) ($S = 5$), (0.7770, 0.3544) ($S = 10$) and (0.7440, 0.4457) ($S = 20$). For the 1-period series, the LP estimates given by Models AII and AIII are very similar. The similarity between the estimates of Model AII and AIII might be induced by the dominance of the RLS component. When considering the aggregated daily data, the periodogram of the RLS component dominates that of the long-

memory component, and the long-memory component cannot affect the results to a large extent.

1.6.2 High frequency data

We now consider how the empirical features obtained with high frequency data can be explained by the various models. We consider the following models to generate the demeaned "1-min" returns:

$$\text{Model BI: } r_{t,n} = (v_{t,n} + w_{t,n}) e_{t,n}$$

$$\text{Model BII: } r_{t,n} = (u_{T,t,n} + w_{t,n}) e_{t,n}$$

$$\text{Model BIII: } r_{t,n} = (u_{T,t,n} + v_{t,n} + w_{t,n}) e_{t,n}$$

where $e_{t,n} \sim i.i.d. N(0, 1)$. Here, the RLS component is $u_{T,t,n} = \sum_{j=1}^t \delta_{T,t,n}$, with $\delta_{T,t,n} = \pi_{T,t,n} \eta_{t,n}$, $\eta_{t,n} \sim i.i.d. (0, \sigma_\eta^2)$ and $\pi_{T,t,n} \sim i.i.d. Bernoulli(p/Ts, 1)$ and the long-memory component $v_{t,n}$ follows $(1 - L)^d v_{t,n} = \varepsilon_{t,n}$, where $\varepsilon_{t,n} \sim i.i.d. N(0, \sigma_v^2)$ and the white noise component $w_{t,n} \sim i.i.d. N(0, \sigma_w^2)$. Note again that we introduce two noise component w_t and e_t to be better able to control the signal to noise ratio. We adopt a different specification than for the case of low frequency data for the following reason. When dealing with low frequency data, one can use the empirical results of Varneskov and Perron (2012) to calibrate the parameters of the DGPs AI-AIII since these estimates were obtained from either daily data or realized volatility series. When dealing with high frequency data, if one uses the same specification it is not possible to map the empirical results of Varneskov and Perron (2012) to calibrate intra daily parameter values. The specification adopted in models BI-BIII allows one to map in an approximate fashion estimates obtained from realized volatility series into intra daily parameter configurations.

Similar to the case with low frequency data, Models BI, BII and BIII refer to ARFIMA(0, d , 0) plus white noise, RLS plus white noise and RLS plus ARFIMA(0, d , 0) plus white noise, respectively. No estimated values of the parameters for 1-min returns are available. The parameter values are chosen according to the following features. We let the value of the variance of $v_{t,n}$ be $\sigma_{v,n} = \sigma_v / \sqrt{s}$, where σ_v^2 is an estimate of from realized daily volatility data for the S&P

500 from Varneskov and Perron (2011). The persistence parameter d is 0.4564, larger than the estimate from Varneskov and Perron (2011) because $v_{t,n}^2$ has memory parameter less than that of $v_{t,n}$ according to Dittmann and Granger (2002). We use the same value of p as the estimate from Varneskov and Perron (2011). For the white noise component, we just let $\sigma_w = 1/\sqrt{s}$. In summary, the model parameters are $T = 5000$, $s = 330$, $d = 0.4564$, $p/Ts = 0.0123/s$, $\sigma_{v,n} = 0.4603/\sqrt{s}$, $\sigma_w = 1/\sqrt{s}$ and $\sigma_\eta = 4.6554/s$ in our simulations.

Tables 6, 7 and 8 show the average of the estimates of the long-memory parameter over 100 replications for Models BI, BII and BIII, respectively. Comparing the results with those in Table 2, the estimates in Table 6 are small. The estimates in Table 7 are close to those in Table 2, but the bandwidth is set to $N = TS$, the estimates for the original series $\log(r_{t,n}^{(k)2})$ are small and very close to zero for TLP (the periodogram of the RLS component decay faster than that of the ARFIMA component, thus it is more likely dominated by the white noise component for relatively large frequencies and result in the smaller estimates observed). The results in Table 8 are very close to those in Table 2. In this case, the ARFIMA component contributes more with the original long series and a larger bandwidth selection ($N = TS$), while the RLS contributes more when using low frequency data.

1.7 Conclusion

We showed in this chapter that the squared low-frequency returns can be expressed in term of the temporal aggregation of a high-frequency series. We built a bridge between the spectral density function of squared low-frequency and squared high-frequency returns. Furthermore, we analyzed the properties of the spectral density function of realized volatility, constructed from squared returns with different frequencies under temporal aggregation. The theoretical findings were illustrated through the analysis of both low-frequency daily S&P 500 returns from 1928 to 2011 and high-frequency 1-minute S&P 500 returns from 1986 to 2007.

Note: This chapter is based on a joint work with Pierre Perron at Boston University

1.8 Appendix

We first state a lemma that will be used in subsequent proofs.

Lemma A (Souza, 2005 and Hassler, 2011). *Let v_t be a covariance stationary discrete-time process with spectral density function $f_v(\lambda)$, the spectral density of the k -period aggregation, $v^{(k)}$, is*

$$f_{v^{(k)}}(\lambda) = \frac{1}{k} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \sum_{j=0}^{k-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2k}\right) \right|^{-2} f_v\left(\frac{\lambda + 2j\pi}{2k}\right)$$

for $\lambda \in [0, 2\pi]$.

Proof of Proposition 1.1: As shown in Section 2, the squared k -period return $r_{t,p}^{(k)2}$ is given by

$$\begin{aligned} r_{t,p}^{(k)2} &= \frac{1}{s} z_{t,p}^2 \left(\sum_{j=0}^{k-1} L^j \right) h_{kp}^2 \\ &= \frac{1}{s} \left[\left(\sum_{j=0}^{k-1} L^j \right) h_{kp}^2 + (z_{t,p}^2 - 1) \left(\sum_{j=0}^{k-1} L^j \right) h_{kp}^2 \right] \\ &= \frac{1}{s} \left[\left(\sum_{j=0}^{k-1} L^j \right) (h_{t,kp}^2 - E h_{t,kp}^2) + \left(\sum_{j=0}^{k-1} L^j \right) E h_{t,kp}^2 \right] \\ &\quad + \frac{1}{s} \left[(z_{t,p}^2 - 1) \left(\sum_{j=0}^{k-1} L^j \right) (h_{t,kp}^2 - E h_{t,kp}^2) + (z_{t,p}^2 - 1) \left(\sum_{j=0}^{k-1} L^j \right) E h_{t,kp}^2 \right] \\ &= \frac{1}{s} \left[\sum_{j=0}^{k-1} L^j y_{t,kp} + \sum_{j=0}^{k-1} L^j E h_{t,kp}^2 \right] \\ &\quad + \frac{1}{s} \left[(z_{t,p}^2 - 1) \sum_{j=0}^{k-1} L^j y_{t,kp} + (z_{t,p}^2 - 1) \sum_{j=0}^{k-1} L^j E h_{t,kp}^2 \right] \end{aligned}$$

which can be expressed as

$$r_{t,p}^{(k)2} = \frac{1}{s} \left[[y_{t,n}]_p^{(k)} + k E h_{t,n}^2 + [y_{t,n}]_p^{(k)} (z_{t,p}^2 - 1) + k E h_{t,n}^2 (z_{t,p}^2 - 1) \right]$$

From Lemma A, the spectral density of the squared k -period return $r_{t,p}^{(k)2}$ is given by

$$\begin{aligned} f_{r_{t,p}^{(k)2}} &= \frac{1}{s^2 k} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \sum_{j=0}^{k-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2k}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2k}\right) \\ &\quad + \frac{1}{s^2 \pi} [\text{var}\left(y_{t,n}^{(k)}\right) + k^2 (E h_{t,n}^2)^2] \end{aligned}$$

■

Proof of Proposition 1.2: The realized volatility constructed from the k -period returns

$r_{t,p}^{(k)}$ is

$$\begin{aligned}
RV_t &= \sum_{p=1}^S r_{t,p}^{(k)2} \\
&= \frac{1}{s} \sum_{p=1}^S \left[\sum_{j=0}^{k-1} L^j y_{t,kp} + \sum_{j=0}^{k-1} L^j Eh_{t,kp}^2 \right] \\
&\quad + \frac{1}{s} \sum_{p=1}^S \left[(z_{t,p}^2 - 1) \sum_{j=0}^{k-1} L^j y_{t,kp} + (z_{t,p}^2 - 1) \sum_{j=0}^{k-1} L^j Eh_{t,kp}^2 \right] \\
&= \frac{1}{s} \left[\sum_{p=1}^S \sum_{j=0}^{k-1} L^j y_{t,kp} + \sum_{p=1}^S \sum_{j=0}^{k-1} L^j Eh_{t,kp}^2 \right] \\
&\quad + \frac{1}{s} \left\{ \sum_{p=1}^S \left[(z_{t,p}^2 - 1) \sum_{j=0}^{k-1} L^j y_{t,kp} \right] + \sum_{p=1}^S \left[(z_{t,p}^2 - 1) \sum_{j=0}^{k-1} L^j Eh_{t,kp}^2 \right] \right\} \\
&= \frac{1}{s} \left[\sum_{p=1}^s L^j y_{t,n} + sEh_{t,n}^2 + \sum_{p=1}^S (z_{t,p}^2 - 1) \left(y_{t,n}^{(k)} \right) + kEh_{t,n}^2 \sum_{p=1}^S (z_{t,p}^2 - 1) \right]
\end{aligned}$$

which can be expressed as

$$RV_t = \frac{1}{s} \left\{ [y_{t,n}]^{(s)} + sEh_{t,n}^2 + \left[(z_{t,p}^2 - 1) y_{t,n}^{(k)} \right]^{(S)} + kEh_{t,n}^2 [(z_{t,p}^2 - 1)]^{(S)} \right\}.$$

From Lemma A, the spectral density of the realized volatility constructed from the k -period returns $r_{t,p}^{(k)}$, RV_t , is given by

$$\begin{aligned}
f_{RV} &= \frac{1}{s^3} \left| \sin \left(\frac{\lambda}{2} \right) \right|^2 \left| \sum_{j=0}^{s-1} \left| \sin \left(\frac{\lambda + 2j\pi}{2s} \right) \right|^{-2} f_y \left(\frac{\lambda + 2j\pi}{2s} \right) \right. \\
&\quad \left. + \frac{1}{s^2\pi} \left[Svar \left(y_{t,n}^{(k)} \right) + Sk^2 (Eh_{t,n}^2)^2 \right] \right.
\end{aligned}$$

■

Proof of Lemma 1.1: We have the following relation

$$\begin{aligned}
&f_{[r_{t,p}^{(k)2}]^{(s)}}(\lambda) - Sf_{r_{t,p}^{(k)2}}(\lambda/S) \\
&= \frac{1}{s} \left| \sin \left(\frac{\lambda}{2} \right) \right|^2 \sum_{j=0}^{s-1} \left| \sin \left(\frac{\lambda + 2j\pi}{2s} \right) \right|^{-2} f_y \left(\frac{\lambda + 2j\pi}{2s} \right) \\
&\quad - \frac{S}{k} \left| \sin \left(\frac{\lambda}{2S} \right) \right|^2 \sum_{j=0}^{k-1} \left| \sin \left(\frac{\lambda + 2j\pi}{2s} \right) \right|^{-2} f_y \left(\frac{\lambda + 2j\pi}{2s} \right) \\
&= \frac{1}{s^2} [U(\lambda) + V(\lambda)]
\end{aligned}$$

where

$$\begin{aligned} V(\lambda) &= \frac{1}{s} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \sum_{j=1}^{s-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2s}\right) \\ &\quad - \frac{S}{k} \left| \sin\left(\frac{\lambda}{2S}\right) \right|^2 \sum_{j=1}^{k-1} \left| \sin\left(\frac{\lambda + 2j\pi}{2ks}\right) \right|^{-2} f_y\left(\frac{\lambda + 2j\pi}{2ks}\right) \\ &\rightarrow 0 \end{aligned}$$

since $|\sin(\lambda/2)| \rightarrow 0$ for $\lambda \rightarrow 0$, and

$$\begin{aligned} U(\lambda) &= \frac{1}{s} \left| \sin\left(\frac{\lambda}{2}\right) \right|^2 \left| \sin\left(\frac{\lambda}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda}{2s}\right) - \frac{S}{k} \left| \sin\left(\frac{\lambda}{2S}\right) \right|^2 \left| \sin\left(\frac{\lambda}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda}{2ks}\right) \\ &\rightarrow \frac{1}{s} \left(\frac{\lambda}{2}\right)^2 \left| \sin\left(\frac{\lambda}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda}{2s}\right) - \frac{S}{k} \left(\frac{\lambda}{2S}\right)^2 \left| \sin\left(\frac{\lambda}{2s}\right) \right|^{-2} f_y\left(\frac{\lambda}{2ks}\right) \\ &= 0 \end{aligned}$$

since $|\sin(\lambda/2)| \rightarrow \lambda/2$ and $\sin(\lambda/2S) \rightarrow^p \lambda/2S$ for $\lambda \rightarrow 0$. ■

Proof of Lemma 1.2: After the k -period non-overlapping temporal aggregation, the random level shift component, $\eta'_q \sim i.i.d. (0, k^2\sigma_\eta^2)$, and the total number of level shifts is unchanged as long as the sample size is large enough. Therefore,

$$u_{T,q}^{(k)} = \sum_{j=1}^t \delta'_{T,q}, \quad \delta'_{T,q} = \pi'_{T,q} \eta'_q$$

where $\eta'_q \sim iid(0, k^2\sigma_\eta^2)$ and $\pi'_{T,q} \sim i.i.d. Bernoulli(kp/T, 1)$.

According to Proposition 3 of Perron and Qu (2010), the limit of the expectation of the periodogram of the high-frequency time series has the following form

$$\lim_{T \rightarrow \infty} E \left[\frac{1}{T} I_{u^{(1)},j} \right] = \frac{p\sigma_\eta^2}{4\pi^3 j^2}$$

Hence, after k -period non-overlapping temporal aggregation, the limit of the expectation of the periodogram is

$$\lim_{T \rightarrow \infty} E \left[I_{u^{(k)},j} \right] = k \frac{p\sigma_\eta^2}{4\pi^3 j^2}$$

which implies that

$$\lim_{T \rightarrow \infty} E \left[I_{u^{(k)}, j} \right] - k \lim_{T \rightarrow \infty} E \left[I_{u^{(1)}, j} \right] \rightarrow 0 \text{ for } \lambda \rightarrow 0.$$

■

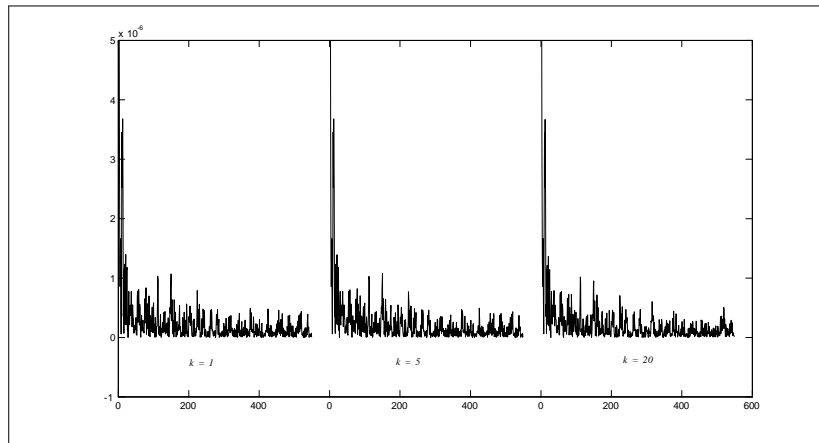


Figure 1.1: The periodograms of the squared daily returns, the 5-period aggregation of the squared daily returns, and the 20-period aggregation of the squared daily returns

Note: The periodogram of the squared daily returns (left), the 5-periods aggregation of the squared daily returns divided by 5 (middle), and the 20-periods aggregation of the squared daily returns divided by 20 (right) for the daily S&P 500 return data. The sample period is from August 13, 1928 to December 30, 2011.

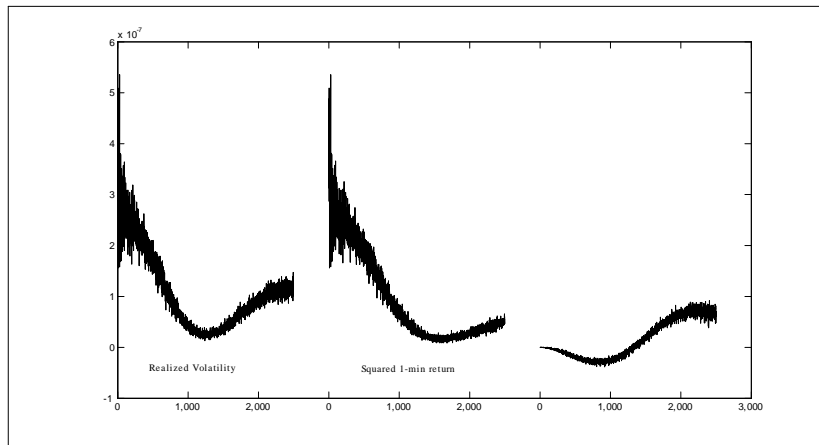


Figure 1.2: The periodograms of the realized volatility obtained from 1-minute returns, the squared 1-minute returns multiplied by 330, and the difference between them

Note: The periodogram of the realized volatility obtained from 1-minute returns (left), the squared 1-minute returns multiplied by 330 (middle), and the difference between them (right) for the high-frequency S&P 500 return data. The sample period is from October 7, 1986 to March 2, 2007.

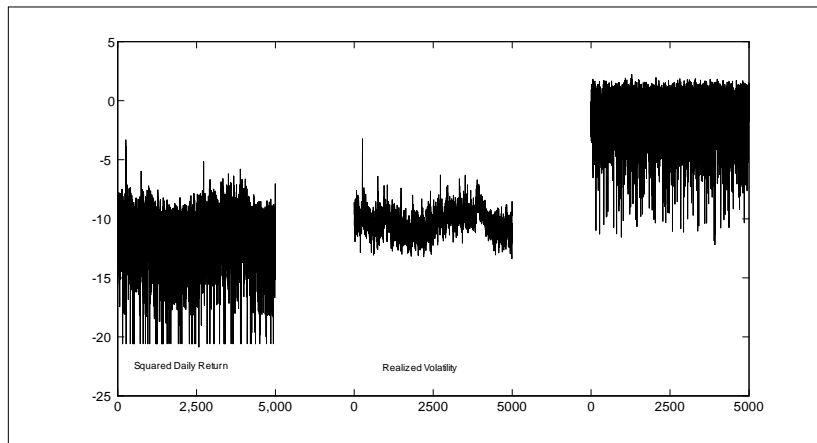


Figure 1.3: The log squared daily returns, log realized volatility and the difference

Note: Log squared daily returns (left), log realized volatility obtained by 1-minute returns (middle) and the difference (right) for the high-frequency S&P 500 return data. The sample period is from October 7th, 1986 to March 2nd, 2007.

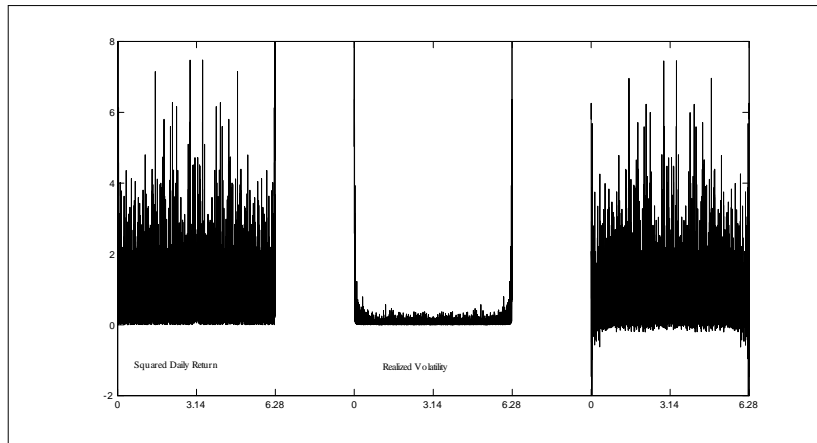


Figure 1.4: The periodograms of the log squared daily returns, the log realized volatility obtained from 1-minute returns, and the difference between them

Note: The periodograms of the log squared daily returns (Left), the log realized volatility obtained from 1-minute returns (middle), and the difference between them (right) for the high-frequency S&P 500 return data. The sample period is from October 7th, 1986 to March 2nd, 2007.

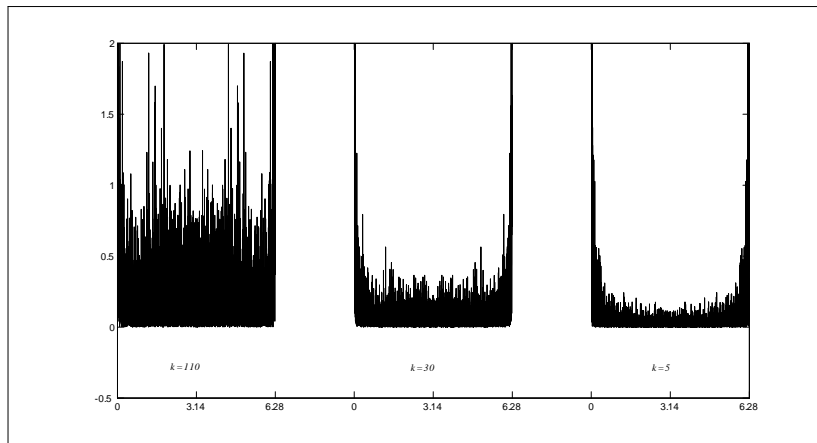


Figure 1.5: The periodograms of the log realized volatility obtained from 110-minute returns, 30-minute returns, and 5-minute returns

Note: The periodogram of the log realized volatility obtained from 110-minutes returns (left), 30-minutes returns (middle), and 5-minutes returns (right) for the high-frequency S&P 500 return data. The sample period is from October 7, 1986 to March 2, 2007.

Table 1.1: Long-memory parameter estimates for daily data

S	$[\log(r_t^2)]^{(S)}$		$\log(r_t^2)$		$\log\{[r_t^2]^{(S)}\}$	
	SLP	TLP	SLP	TLP	SLP	TLP
$S = 1$	0.5659	0.0941	0.5659	0.0941	0.5659	0.0941
$S = 5$	0.6313	0.2570	0.6317	0.2317	0.6236	0.3092
$S = 10$	0.7770	0.3544	0.7762	0.3200	0.7983	0.4359
$S = 20$	0.7440	0.4457	0.7427	0.3915	0.7554	0.4689

Notes: The table reports the log-periodogram regression estimates (SLP) and the trimmed log-periodogram regression estimates (TLP) for the degree of fractional integration in the daily S&P 500 returns data. The sample period is from August 13, 1928 to December 30, 2011. The sample size is 22,000. The rows labelled $S = 1, S = 5, S = 10$ and $S = 20$ refer to the squared daily returns, aggregated 5-day squared daily returns, aggregated 10-day squared daily returns and aggregated 20-day squared daily returns, respectively. The columns labelled $[\log(r_t^2)]^{(S)}$, $\log(r_t^2)$ and $\log\{[r_t^2]^{(S)}\}$ refer to the S -periods of the aggregation of the logarithmic transformation of squared daily returns, the logarithmic transformation of the original squared daily returns, and the logarithmic transformation of the S -periods aggregated squared daily returns, respectively. S denotes the aggregation level. The estimates are based on the bandwidth $m = N^{0.5}$ for the log-periodogram regression and $(l, m) = (N^{0.65}, N^{0.9})$ for the trimmed log-periodogram regression. We set $N = T/S$.

Table 1.2: Long-memory parameter estimates for high-frequency data

k	$\left[\log \left(r_{t,n}^{(k)2}\right)\right]^{(S)}$		$\log \left[r_{t,n}^{(k)2}\right]$		$\log \left\{\left[r_{t,n}^{(k)2}\right]^{(S)}\right\}$			
	$N = T$		$N = T$		$N = TS$		$N = T$	
	SLP	TLP	SLP	TLP	SLP	TLP	SLP	TLP
$k = 1$	0.6231	0.3949	0.6218	0.3765	0.3964	0.0676	0.6768	0.4842
$k = 5$	0.6336	0.3432	0.6323	0.3260	0.4595	0.0603	0.6497	0.4269
$k = 30$	0.6439	0.2421	0.6443	0.2203	0.5275	0.0679	0.6637	0.3072
$k = 330$	0.4835	0.0415	0.4835	0.0415	0.4835	0.0415	0.4835	0.0415

Notes: The table reports the log-periodogram regression estimates (SLP) and the trimmed log-periodogram regression estimates (TLP) for the degree of fractional integration for the high-frequency S&P 500 returns data. The sample period is from October 7, 1986 to March 2, 2007. The sample size for 1-min returns is 1,650,000 ($T = 5000$ and $s = 330$). The rows labelled $k = 1$, $k = 5$, $k = 30$ and $k = 330$ refer to the 1-min return data, 5-min return data, 30-min return data and daily return data, respectively. The columns labelled $\left[\log \left(r_{t,n}^{(k)2}\right)\right]^{(S)}$, $\log \left[r_{t,n}^{(k)2}\right]$ and $\log \left[r_{t,n}^{(k)2}\right]^{(S)}$ refer to the S -period aggregation of the logarithmic transformation of squared k -min returns, the logarithmic transformation of original squared k -min returns and logarithmic transformation of the realized daily volatility aggregated by squared k -min returns over a day, respectively. S denotes the number of k -min returns per day and we have $s = kS$. The estimates are based on the bandwidth $m = N^{0.5}$ for the log-periodogram regression and $(l, m) = (N^{0.65}, N^{0.9})$ for trimmed log-periodogram regression. For the estimates of aggregated log squared k -min returns and realized volatility, we set $N = T$. We report results for both $N = T \times S$ and $N = T$ for the estimate obtained using the log squared return series.

Table 1.3: Long-memory parameter estimates of simulated daily returns for ARFIMA plus white noise model

S	$[\log(r_t^2)]^{(S)}$		$\log(r_t^2)$		$\log\{[r_t^2]^{(S)}\}$	
	SLP	TLP	SLP	TLP	SLP	TLP
$S = 1$	0.3271	0.0597	0.3271	0.0597	0.3271	0.0597
$S = 5$	0.3631	0.1347	0.3629	0.1214	0.3646	0.1367
$S = 10$	0.3989	0.1868	0.3986	0.1636	0.3922	0.1655
$S = 20$	0.4181	0.2398	0.4171	0.2033	0.4103	0.2042

Notes: The table reports the log-periodogram regression estimates (SLP) and the trimmed log-periodogram regression estimates (TLP) for the degree of fractional integration for the simulated long memory daily return data. The data generating process for the 1-period returns is given by $r_t = \exp(v_t/2 + w_t/2) e_t$, where s is the total number of 1-period returns per day and $e_t \sim N(0, 1)$. Here, v_t and w_t refer to long-memory and white noise components. The model parameters are $T = 22000$, $d = 0.4564$, $\sigma_w = 1.3121$ and $\sigma_v = 0.4603$. We use the same bandwidth for the estimates of the long-memory parameter as in Table 1.1.

Table 1.4: Long-memory parameter estimates of simulated daily returns for RLS plus white noise model

S	$[\log(r_t^2)]^{(S)}$		$\log(r_t^2)$		$\log\{[r_t^2]^{(S)}\}$	
	SLP	TLP	SLP	TLP	SLP	TLP
$S = 1$	0.5636	0.0299	0.5636	0.0299	0.5636	0.0299
$S = 5$	0.7563	0.0944	0.7562	0.0843	0.7649	0.0965
$S = 10$	0.8452	0.1706	0.8452	0.1489	0.8350	0.1655
$S = 20$	0.8850	0.2552	0.8843	0.2206	0.8656	0.2316

Notes: This is the counterpart to Table 1.3, but with the data generating process for the 1-period return given by $r_t = \exp(u_{t,T}/2 + w_t/2) e_t$, where $e_t \sim N(0, 1)$. Here, u_t and w_t refer to RLS and white noise components. The model parameters are $T = 22000$, $p/T = 0.0045$, $\sigma_w = 1.3121$ and $\sigma_\eta = 0.5081$.

Table 1.5: Long-memory parameter estimates of simulated daily returns for RLS plus ARFIMA plus white noise model

S	$[\log(r_t^2)]^{(S)}$		$\log(r_t^2)$		$\log\{[r_t^2]^{(S)}\}$	
	SLP	TLP	SLP	TLP	SLP	TLP
$S = 1$	0.5559	0.0730	0.5559	0.0730	0.5559	0.0730
$S = 5$	0.6575	0.1709	0.6569	0.1518	0.6627	0.1736
$S = 10$	0.7393	0.2593	0.7389	0.2294	0.7353	0.2439
$S = 20$	0.7717	0.3330	0.7722	0.2839	0.7471	0.3004

Notes: This is the counterpart to Table 1.3, but with the data generating process for the 1-period return given by $r_t = \exp(u_{t,T}/2 + v_t/2 + w_t/2) e_t$, where $e_t \sim N(0, 1)$. Here, u_t , v_t and w_t refer to RLS component, long-memory and white noise components. The model parameters are $T = 22000$, $d = 0.4564$, $p/T = 0.0045$, $\sigma_w = 1.3121$, $\sigma_v = 0.4603$ and $\sigma_\eta = 0.5081$.

Table 1.6: Long-memory parameter estimates of simulated high-frequency returns for ARFIMA plus white noise model

k	$\left[\log \left(r_{t,n}^{(k)2}\right)\right]^{(S)}$		$\log \left[r_{t,n}^{(k)2}\right]$		$\log \left\{\left[r_{t,n}^{(k)2}\right]^{(S)}\right\}$			
	$N = T$		$N = T$		$N = TS$		$N = T$	
	SLP	TLP	SLP	TLP	SLP	TLP	SLP	TLP
$k = 1$	0.3646	0.3836	0.3644	0.3350	0.3513	0.0578	0.3632	0.3619
$k = 5$	0.3615	0.2633	0.3614	0.2337	0.3073	0.0473	0.3700	0.3063
$k = 30$	0.2764	0.0977	0.2763	0.0865	0.2147	0.0305	0.3208	0.1571
$k = 330$	0.1029	0.0167	0.1029	0.0167	0.1029	0.0167	0.1029	0.0167

Notes: The table reports the log-periodogram regression estimates (SLP) and the trimmed log-periodogram regression estimates (TLP) for the degree of fractional integration for the simulated high-frequency return data. The data generating process for the 1-period returns is given by $x_{t,n} = (1/\sqrt{s})(v_{t,n} + w_{t,n})e_{t,n}$, where s is the total number of 1-period returns per day and $e_{t,n} \sim N(0, 1)$. Here, $v_{t,n}$ and $w_{t,n}$ refer to long-memory and white noise components. The model parameters are $T = 5000$, $s = 330$, $d = 0.4564$, $\sigma_w = 1/\sqrt{s}$ and $\sigma_v = 0.4603/\sqrt{s}$. We use the same bandwidth for the estimates of the long-memory parameter as in Table 1.2.

Table 1.7: Long-memory parameter estimates of simulated high-frequency returns for RLS plus white noise model

k	$\left[\log \left(r_{t,n}^{(k)2}\right)\right]^{(S)}$		$\log \left[r_{t,n}^{(k)2}\right]$		$\log \left\{\left[r_{t,n}^{(k)2}\right]^{(S)}\right\}$			
	$N = T$		$N = T$		$N = TS$		$N = T$	
	SLP	TLP	SLP	TLP	SLP	TLP	SLP	TLP
$k = 1$	0.9553	0.3959	0.9551	0.3573	0.5018	0.0060	0.9491	0.3610
$k = 5$	0.9146	0.2161	0.9147	0.1938	0.4855	0.0094	0.9355	0.2616
$k = 30$	0.7425	0.0934	0.7425	0.0834	0.4527	0.0155	0.8286	0.1321
$k = 330$	0.4000	0.0254	0.4000	0.0254	0.4000	0.0254	0.4000	0.0254

Notes: This is the counterpart to Table 1.6, but with the data generating process for the 1-period returns given by $r_{t,n} = (1/\sqrt{s})(u_{t,n,T} + w_{t,n})e_{t,n}$. Here, $u_{t,n}$ and $w_{t,n}$ refer to the RLS and white noise components. The model parameters are $T = 5000$, $s = 330$, $p/TS = 0.0123/s$, $\sigma_w = 1/\sqrt{s}$ and $\sigma_\eta = 4.6554/s$.

Table 1.8: Long-memory parameter estimates of simulated high-frequency returns for RLS plus ARFIMA plus white noise model

k	$\left[\log \left(r_{t,n}^{(k)2}\right)\right]^{(S)}$		$\log \left[r_{t,n}^{(k)2}\right]$		$\log \left\{\left[r_{t,n}^{(k)2}\right]^{(S)}\right\}$			
	$N = T$		$N = T$		$N = TS$		$N = T$	
	SLP	TLP	SLP	TLP	SLP	TLP	SLP	TLP
$k = 1$	0.6647	0.4830	0.6646	0.4220	0.4611	0.0743	0.6615	0.4649
$k = 5$	0.6652	0.3492	0.6650	0.3094	0.4556	0.0574	0.6712	0.3941
$k = 30$	0.5815	0.1606	0.5813	0.1432	0.4120	0.0444	0.6191	0.2282
$k = 330$	0.3548	0.0388	0.3548	0.0388	0.3548	0.0388	0.3548	0.0388

Notes: This is the counterpart to Table 1.6, but with the data generating process for the 1-period returns is given by $r_{t,n} = (1/\sqrt{s})(u_{t,n,T} + v_{t,n} + w_{t,n})e_{t,n}$. Here, $u_{t,n}$, $v_{t,n}$ and $w_{t,n}$ refer to RLS, long-memory and white noise components, as discussed in text. The model parameters are $T = 5000$, $s = 330$, $d = 0.4564$, $p/TS = 0.0123/s$, $\sigma_v = 0.4603/\sqrt{s}$, $\sigma_w = 1/\sqrt{s}$ and $\sigma_\eta = 4.6554/s$.

Chapter 2

Discrete Fourier Transforms of Generalized Fractional Processes

2.1 Introduction

In recent years, much attention has focused on the study of the long-memory properties of economic time series, such as the volatilities of financial time series and interest rates. A long-memory process typically has a spectral density function which is proportional to λ^{-2d} as λ goes to zero, where d is the memory parameter. The fractionally integrated model, proposed by Granger and Joyeux (1980) and Hosking (1981), is a long-memory generalization of the ARMA model whose autocorrelations decay exponentially. When $d \in (0, 0.5)$, the autocorrelations decay very slowly, being a characteristic of long-memory processes. The simplest long-memory process is the fractionally integrated model, which is defined by

$$(1 - L)^d x_t = u_t, \quad t = 1, \dots, T \quad (2.1)$$

where u_t is a stationary process with zero mean and spectral density f_u . Equation (2.1) can be rewritten as

$$\sum_{k=0}^T \frac{(-d)_k}{k!} x_{t-k} = u_t.$$

When $|d| < 0.5$, the spectrum of $\{x_t\}$ has the following form

$$f_x(\lambda) = \left|1 - e^{i\lambda}\right|^{-2d} f_u(\lambda) \quad (2.2)$$

which is proportional to $|\lambda|^{-2d}$ as λ goes to zero.

A generalized long-memory process typically has one or more singularities at a frequency v

different from zero, and its spectral density function is proportional to $|\lambda - v|^{-2d}$ as λ goes to v . In this chapter, I consider the following generalized fractional integrated process x_t generated by

$$(1 - 2\eta L + L^2)^d x_t = u_t, \quad (2.3)$$

where $|\eta| < 1$, and u_t is a stationary process with zero mean and continuous spectral density f_u . The generalized fractionally integrated model, proposed by Hosking (1981) and Gray, Zhang and Woodward (1989), is based on Gegenbauer polynomials. I primarily study the case where x_t is nonstationary and $0.5 < d \leq 1$. From formula (6) of Gray, Zhang and Woodward (1989), I have

$$(1 - 2\eta L + L^2)^{-d} = \sum_{n=0}^{\infty} c_{d,n}(\eta, d) L^n, \quad (2.4)$$

where

$$c_{d,n}(\eta, d) = \sum_{k=0}^n \frac{\Gamma(d + 0.5) \Gamma(2d + k)}{k! \Gamma(2d)} \left(\frac{1 - \eta^2}{4} \right)^{0.25 - \lambda/2} P_{k+d-0.5}^{0.5-d}(\eta).$$

In the following part of this chapter, I will assume that the stationary component u_t is a linear process of the form

$$u_t = C(L) \varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j |c_j| < \infty, \quad C(1) \neq 0 \quad (2.5)$$

for all t and with $\varepsilon_t = iid(0, \sigma^2)$ with finite fourth moment.

When $|d| < 0.5$, x_t is a stationary process with spectral density given by

$$f_x(\lambda) = \left| 4 \sin\left(\frac{\lambda + \nu}{2}\right) \sin\left(\frac{\lambda - \nu}{2}\right) \right|^{-2d} f_u(\lambda), \quad (2.6)$$

where $\nu = \cos^{-1}(\eta)$, and $f_u(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} c_j e^{ij\lambda} \right|^2$ as discussed in Chung (1996). A generalized long-memory process typically has a singularity at a frequency v different from zero, and its spectral density function is such that

$$f_x(\lambda) \simeq C |\lambda - v|^{-2d} \quad (2.7)$$

as λ goes to v , where C is a constant. In the case of $0.5 < d < 1$, equation (2.6) can be

interpreted as a spectrum in terms of the limit of the expectation of the periodogram (Solo, 1992). Throughout this chapter, I assume $\lambda \in [0, \pi)$. Taking logarithms of (2.6) produces the equation

$$\ln [I_x(\lambda_s)] = -2d \ln \left| 4 \sin \left(\frac{\lambda_s + \nu}{2} \right) \sin \left(\frac{\lambda_s - \nu}{2} \right) \right| + \ln [f_u(\lambda_s)] + U(\lambda_s), \quad (2.8)$$

where $U(\lambda_s) = \ln [I_x(\lambda_s) / f_x(\lambda_s)]$. Here, I use $I_x(\lambda_s)$ instead of $f_x(\lambda_s)$ because I can not observe $f_x(\lambda_s)$. The analysis of regression of (2.8) is complicated due to the difficulty in characterizing the asymptotic behavior of the dft $w_x(\lambda_s)$, which is the central element in determining the properties of the regression residual $U(\lambda_s)$ in (2.8), as discussed in Phillips (1999).

Phillips (1999) gives an exact representation of the dft in terms of a useful components representation for the case of the fractional processes, as in (2.1) for $d > 0.5$. His representation is particularly useful in analyzing the asymptotic behavior of the dft and periodogram in the nonstationary case when the memory parameter $d > 0.5$. Many applications have been developed in the last decade, such as the log periodogram estimation of memory parameter for the nonstationary and unit root cases (Phillips, 2006, 2007) and the local Whittle estimation of the memory parameter for the nonstationary and unit root cases (Phillips and Shimotsu, 2004, Shimotsu and Phillips, 2002, Shimotsu, 2010). In this chapter, I extend the Phillips' (1999) work to the generalized fractional processes, using a similar structure.

The remainder of this chapter is composed as follows. Section 2 gives the new frequency domain representation of the dft based on the frequency domain decomposition. Section 3 analyzes the asymptotic behavior of the dft. Section 4 describes some statistical applications. Section 5 and 6 are composed of concluding remarks and a mathematical appendix.

2.2 Frequency domain decomposition

For convenience, I express the operator $(1 - 2\eta L + L^2)^d$ in (2.3) as

$$(1 - 2\eta L + L^2)^d = (1 - e^{i\nu} L)^d (1 - e^{-i\nu} L)^d, \quad (2.9)$$

where $\nu = \cos^{-1}(\eta)$. Let

$$D(d, v, L) = D_1(d, v, L) D_2(d, v, L) \quad (2.10)$$

where

$$D_1(d, v, L) = (1 - e^{-iv}L)^d, \quad D_2(d, v, L) = (1 - e^{iv}L)^d,$$

and expanding D_1 and D_2 gives

gives

$$D_1(d, v, L) = \sum_{k=0}^T \frac{(-d)_k}{k!} (e^{-iv}L)^k, \quad D_2(d, v, L) = \sum_{k=0}^T \frac{(-d)_k}{k!} (e^{iv}L)^k.$$

Expanding the polynomial operator about its value at the complex exponential $e^{i\lambda}$, as in Phillips (1999), leads to the following decomposition.

Lemma 2.1

$$\begin{aligned} D(d, v, L) &= D(d, v, e^{i\lambda}) + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \\ &\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \\ &\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1)^2, \end{aligned} \quad (2.11)$$

where

$$\begin{cases} \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) = \sum_{p=0}^{T-1} \tilde{d}_{1\lambda p} e^{-ip\lambda} L^p \\ \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) = \sum_{p=0}^{T-1} \tilde{d}_{2\lambda p} e^{-ip\lambda} L^p \end{cases}$$

and

$$\tilde{d}_{1\lambda p} = \sum_{k=p+1}^T \frac{(-d)_k}{k!} e^{ik(\lambda-v)}, \quad \tilde{d}_{2\lambda p} = \sum_{k=p+1}^T \frac{(-d)_k}{k!} e^{ik(\lambda+v)}.$$

Proof. Lemma 2.1 is an immediate consequence of formula (32) in Phillips and Solo (1992) and Phillips (1999). ■

Using equation (2.11), I can obtain the following representation for u_t

$$\begin{aligned}
u_t &= D(d, v, L) x_t \\
&= D(d, v, e^{i\lambda}) x_t + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1)^2 x_t.
\end{aligned} \tag{2.12}$$

Taking the dft's of both sides of equation (2.12) yields an exact expression of w_u in term of w_x .

Theorem 2.1

$$w_u(\lambda) = w_x(\lambda) D(d, v, e^{i\lambda}) + \frac{1}{\sqrt{2\pi T}} X_T(v, d, \lambda), \tag{2.13}$$

where

$$\begin{aligned}
X_T(v, d, \lambda) &= D_2(d, v, e^{i\lambda}) \left(X_0^{D_1}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1}(d, v, \lambda) \right) \\
&\quad + D_1(d, v, e^{i\lambda}) \left(X_0^{D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_2}(d, v, \lambda) \right) \\
&\quad + e^{-i\lambda} \left(X_0^{D_1 D_2}(d, v, \lambda) - e^{i(T-1)\lambda} X_{T-1}^{D_1 D_2}(d, v, \lambda) \right) \\
&\quad - \left(X_0^{D_1 D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1 D_2}(d, v, \lambda) \right)
\end{aligned}$$

and

$$\begin{aligned}
X_0^{D_1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_0; & X_T^{D_1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_T; \\
X_0^{D_2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_0; & X_T^{D_2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_T; \\
X_0^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_0; \\
X_{T-1}^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_{T-1}; \\
X_T^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_T.
\end{aligned}$$

Proof. See the proof of Theorem 3.2 in Phillips (1999). ■

Since $x_t = 0$ for $t \leq 0$, I have

$$\begin{aligned}
X_T(v, d, \lambda) &= e^{iT\lambda} D_2(d, v, e^{i\lambda}) X_T^{D_1}(d, v, \lambda) + e^{iT\lambda} D_1(d, v, e^{i\lambda}) X_T^{D_2}(d, v, \lambda) \\
&\quad - e^{i(T-2)\lambda} X_{T-1}^{D_1 D_2}(d, v, \lambda) + e^{iT\lambda} X_T^{D_1 D_2}(d, v, \lambda).
\end{aligned}$$

Remark 2.1 Equation (2.13) provides an exact representation of $w_x(\lambda)$. Explicitly,

$$w_x(\lambda) = D^{-1}(d, v, e^{i\lambda}) w_u(\lambda) + \frac{1}{\sqrt{2\pi T}} D^{-1}(d, v, e^{i\lambda}) X_T(d, v, \lambda). \quad (2.14)$$

In terms of the periodogram ordinates, I have the corresponding equation

$$\begin{aligned} I_x(\lambda_s) &= |w_x(\lambda_s)|^2 \\ &= \left| D^{-1}(d, v, e^{i\lambda_s}) w_u(\lambda_s) + \frac{1}{\sqrt{2\pi T}} D^{-1}(d, v, e^{i\lambda_s}) X_T(v, d, \lambda_s) \right|^2 \\ &= \left| D(d, v, e^{i\lambda_s}) \right|^{-2} \left[I_u(\lambda_s) - 2RE \left\{ \frac{1}{\sqrt{2\pi T}} X_T(v, d, \lambda_s) w_u(\lambda_s) \right\} \right. \\ &\quad \left. + \frac{1}{2\pi T} |X_T(v, d, \lambda_s)|^2 \right], \end{aligned} \quad (2.15)$$

which is the periodogram used in the log periodogram regression. Similar to the discussion in Phillips (1999), $|D(d, v, e^{i\lambda_s})|^{-2}$ can be replaced by $|(1 - e^{i(\lambda_s+v)})(1 - e^{i(\lambda_s-v)})|^{-2d}$. However, as will be shown, the residual component $T^{-0.5} X_T(v, d, \lambda_s)$ cannot be neglected in general and its importance grows as d increases.

Remark 2.2 When $d = 1$, the forward factorial $(-d)_k = 0$ for all $k > 1$, so that series involving these coefficients terminate at $k = 1$. In this case, $D(d, v, e^{i\lambda}) = (1 - e^{i(\lambda-v)})(1 - e^{i(\lambda+v)})$, $\tilde{d}_{1\lambda 0} = -e^{ik(\lambda-v)}$, $\tilde{d}_{2\lambda 0} = -e^{ik(\lambda+v)}$, $X_T(v, 1, \lambda) = -e^{2i\lambda} \frac{x_T}{\sqrt{T}}$. Then equation (2.13) can be reduced to

$$w_u(\lambda) = \left(1 - e^{i(\lambda-v)}\right) \left(1 - e^{i(\lambda+v)}\right) w_x + \frac{e^{2i\lambda}}{\sqrt{2\pi T}} x_T(d, v, \lambda), \quad (2.16)$$

and

$$w_x(\lambda) = D^{-1}(d, v, e^{i\lambda}) w_u(\lambda) + \frac{e^{2i\lambda}}{\sqrt{2\pi T}} D^{-1}(d, v, e^{i\lambda}) x_T(d, v, \lambda). \quad (2.17)$$

Remark 2.3 Similar to Phillips (1999), I can derive the dft for the generalized first difference of x_t

$$(1 - 2\eta L + L^2) x_t = (1 - 2\eta L + L^2)^{1-d} u_t. \quad (2.18)$$

As discussed by Gray, Zhang and Woodward (1989), $(1 - 2\eta L + L^2)^{1-d} u_t$ is a stationary process. This is out of the scope of this chapter and will not be discussed here.

2.3 Asymptotic approximations

Without loss of generality, I suppose that $\lambda_s \rightarrow \varphi \neq -v$.

2.3.1 Component approximations

I have the asymptotic representations summarized in the following lemma.

Lemma 2.2 *Suppose $d \in (0.5, 1)$. Then*

(a) *For fixed $\lambda \neq v$*

$$\begin{aligned} & D(d, v, e^{i\lambda}) \\ = & \left(1 - e^{i(\lambda+v)}\right)^d \left(1 - e^{i(\lambda-v)}\right)^d \\ & - \frac{1}{\Gamma(-d)T^{1+d}} \left[\frac{e^{iT(\lambda+v)} \left(1 - e^{i(\lambda-v)}\right)^d}{1 - e^{i(\lambda+v)}} + \frac{e^{iT(\lambda-v)} \left(1 - e^{i(\lambda+v)}\right)^d}{1 - e^{i(\lambda-v)}} \right] \left[1 + O\left(\frac{1}{T}\right)\right] \\ & + \frac{1}{\Gamma(-d)^2 T^{2+2d}} \left[\frac{e^{i2T\lambda}}{\left(1 - e^{i(\lambda+v)}\right) \left(1 - e^{i(\lambda-v)}\right)} \right] \left[1 + O\left(\frac{1}{T}\right)\right]. \end{aligned}$$

(b) *For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and $s \rightarrow \infty$ as $T \rightarrow \infty$*

$$\begin{aligned} D(d, v, e^{i\lambda_s}) &= \left(1 - e^{i(\lambda_s+v)}\right)^d \left(1 - e^{i(\lambda_s-v)}\right)^d \\ &+ \frac{1}{2\pi i \Gamma(-d) T^d s} \left(1 - e^{i(\lambda_s+v)}\right)^d \left[1 + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{T^{1+d}}\right). \end{aligned}$$

(c) *For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and s fixed as $T \rightarrow \infty$*

$$D(d, v, e^{i\lambda_s}) = \frac{\left(1 - e^{i(\lambda_s+v)}\right)^d}{\Gamma(1-d)T^d} F_1^1(1, 1-d; -2\pi is) + O\left(\frac{1}{T^{1+d}}\right).$$

where $F_1^1(a, b; z)$ denotes the confluent hypergeometric function.

(d) *For $\lambda = v$*

$$D(d, v, e^{iv}) = \frac{\left(1 - e^{i2v}\right)^d}{\Gamma(1-d)T^d} \left[1 + O\left(\frac{1}{T}\right)\right],$$

(e) *When $d = 1$,*

$$D(d, v, e^{i\lambda}) = \left(1 - e^{i(\lambda-v)}\right) \left(1 - e^{i(\lambda+v)}\right).$$

holds for any λ .

Proof. See Appendix. ■

Now, I study the correction term $X_T(\lambda, \eta, d)$. The results are given in the following theorem.

Theorem 2.2 *Suppose $d \in (0.5, 1)$. Then*

(a) For fixed $\lambda \neq v$ as $T \rightarrow \infty$

$$\begin{aligned} \frac{X_T(\lambda, \eta, d)}{\sqrt{T}} &= -\frac{e^{i\lambda} (1 - e^{i(\lambda+v)})^d x_T}{(1 - e^{i(\lambda-v)})^{1-d} \sqrt{T}} - \frac{e^{i\lambda} (1 - e^{i(\lambda-v)})^d x_T}{(1 - e^{i(\lambda+v)})^{1-d} \sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda}}{(1 - e^{i(\lambda+v)})^{1-d} (1 - e^{i(\lambda-v)})^{1-d} \sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right) \\ &= O_p\left(\frac{1}{T^{1-d}}\right). \end{aligned}$$

(b) For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and $\frac{s}{T^\alpha} \rightarrow \infty$ as $T \rightarrow \infty$, for some $\alpha \in (0.5, 1)$

$$\begin{aligned} \frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} &= -\frac{e^{i\lambda_s} (1 - e^{i(\lambda_s+v)})^d x_T}{(1 - e^{i(\lambda_s-v)})^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d x_T}{(1 - e^{i(\lambda_s+v)})^{1-d} \sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda_s+v)})^{1-d} (1 - e^{i(\lambda_s-v)})^{1-d} \sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right) \\ &= -\frac{e^{i\lambda_s} (1 - e^{i(\lambda_s+v)})^d x_T}{(-2\pi is)^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d x_T}{(1 - e^{i(\lambda_s+v)})^{1-d} \sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda_s+v)})^{1-d} (-2\pi is)^{1-d} \sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right) \\ &= O_p\left(\frac{1}{s^{1-d}}\right). \end{aligned}$$

(c) For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and s fixed as $T \rightarrow \infty$

$$\begin{aligned} &\frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} \\ &= \left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d \frac{F_1^1(1, 1-d; -2\pi is)}{\Gamma(1-d)} \int_0^1 e^{2\pi is r} X_{n,d}(r) dr \\ &\quad - \left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi is) r^{-d} X_{n,d}(1-r) dr \\ &\quad - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d x_T}{(1 - e^{i(\lambda_s+v)})^{1-d} \sqrt{T}} + O_p\left(\frac{1}{T^{1-d}}\right), \end{aligned}$$

where $X_{n,d}(r, \lambda) = \frac{x_{[nr]}}{T^{d-0.5}}$.

(d) When $d = 1$,

$$\frac{X_T(\lambda_s, \eta, 1)}{\sqrt{T}} = -e^{2i\lambda_s} \frac{x_T}{\sqrt{T}} = O_p(1).$$

Proof. See Appendix. ■

As in the case of fractional processes (Phillips, 1999), the leading term in the asymptotic approximation of $T^{-0.5}X_T(\lambda_s, \eta, d)$ is the same in parts (a) and (b) of Theorem 2.2. So, although the orders of magnitudes of the remainder differ, I may write, for these cases

$$\begin{aligned} \frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} &= -\frac{e^{i\lambda_s} (1 - e^{i(\lambda_s+v)})^d}{(1 - e^{i(\lambda_s+v)})^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d}{(1 - e^{i(\lambda_s+v)})^{1-d}} \frac{x_T}{\sqrt{T}} \\ &+ \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda_s+v)})^{1-d} (1 - e^{i(\lambda_s-v)})^{1-d}} \frac{x_T}{\sqrt{T}} \\ &+ O_p \left(-\frac{e^{i\lambda_s} (1 - e^{i(\lambda_s+v)})^d}{(1 - e^{i(\lambda_s-v)})^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d}{(1 - e^{i(\lambda_s+v)})^{1-d}} \frac{x_T}{\sqrt{T}} \right. \\ &\left. + \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda_s+v)})^{1-d} (1 - e^{i(\lambda_s-v)})^{1-d}} \frac{x_T}{\sqrt{T}} \right). \end{aligned} \quad (2.19)$$

In part (c), the formula includes $\lambda_s = v$, and

$$\begin{aligned} \frac{X_T(0, \eta, d)}{\sqrt{T}} &= (1 + e^{iv} - e^{-iv}) (1 - e^{i2v})^d \frac{1}{\Gamma(1-d)} \int_0^1 X_{n,d}(r) dr \\ &- (1 + e^{iv} - e^{-iv}) (1 - e^{i2v})^d \frac{1}{\Gamma(1-d)} \int_0^1 r^{-d} X_{n,d}(1-r) dr \\ &+ O_p \left(\frac{1}{T^{1-d}} \right) \end{aligned}$$

2.3.2 Approximations for the dft

Similar to Theorem 3.2 of Phillips (1999), the following asymptotic representations can easily be derived.

(a) **Case** $\lambda_s \rightarrow \phi \neq v$:

From Lemma 2.2 (a), I have

$$\begin{aligned}
& D(d, v, e^{i\lambda_s}) \\
&= \left(1 - e^{i(\lambda_s+v)}\right)^d \left(1 - e^{i(\lambda_s-v)}\right)^d \\
&\quad - \frac{1}{\Gamma(-d)T^{1+d}} \left[\frac{e^{iT(\lambda_s+v)} \left(1 - e^{i(\lambda_s-v)}\right)^d}{1 - e^{i(\lambda_s+v)}} + \frac{e^{iT(\lambda_s-v)} \left(1 - e^{i(\lambda_s+v)}\right)^d}{1 - e^{i(\lambda_s-v)}} \right] \left[1 + O\left(\frac{1}{T}\right)\right] \\
&\quad + \frac{1}{\Gamma(-d)^2 T^{2+2d}} \left[\frac{e^{i2T\lambda_s}}{\left(1 - e^{i(\lambda_s+v)}\right) \left(1 - e^{i(\lambda_s-v)}\right)} \right] \left[1 + O\left(\frac{1}{T}\right)\right] \\
&= \left(1 - e^{i(\lambda_s+v)}\right)^d \left(1 - e^{i(\lambda_s-v)}\right)^d + O\left(\frac{1}{T^{1+d}}\right),
\end{aligned}$$

uniformly for $\lambda_s \in B_\phi = \{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\}$ where $M \rightarrow \infty$ as $T \rightarrow \infty$. Similarly, from Theorem 2.2,

$$\begin{aligned}
\frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} &= -\frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s+v)}\right)^d}{\left(1 - e^{i(\lambda_s-v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s-v)}\right)^d}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \\
&\quad + \frac{1 - e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \left(1 - e^{i(\lambda_s-v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned}$$

uniformly for $\lambda_s \in B_\phi$. It follows that

$$\begin{aligned}
w_x(\lambda_s) &= \left(1 - e^{i(\lambda_s-v)}\right)^{-d} \left(1 - e^{i(\lambda_s+v)}\right)^{-d} w_u(\lambda_s) \\
&\quad - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s+v)}\right)^d}{\left(1 - e^{i(\lambda_s-v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s-v)}\right)^d}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \\
&\quad + \frac{1 - e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \left(1 - e^{i(\lambda_s-v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned} \tag{2.20}$$

uniformly for $\lambda_s \in B_\phi$.

(b) Case $\lambda_s = \frac{2\pi is}{n} \rightarrow v$ and $s \rightarrow \infty$:

From Lemma 2.2 (b), I have

$$\begin{aligned} D(d, v, e^{i\lambda_s}) &= \left(1 - e^{i(\lambda_s+v)}\right)^d \left(1 - e^{i(\lambda_s-v)}\right)^d \\ &\quad + \frac{1}{2\pi i \Gamma(-d) T^d s} \left(1 - e^{i(\lambda_s+v)}\right)^d \left[1 + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{T^{1+d}}\right), \end{aligned}$$

and from Theorem 2.2 (b), I have

$$\begin{aligned} \frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} &= -\frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s+v)}\right)^d x_T}{(-2\pi is)^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s-v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} (-2\pi is)^{1-d} \sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right). \end{aligned}$$

It follows that if $\frac{s}{T} + \frac{T^\alpha}{s} \rightarrow 0$ as $T \rightarrow \infty$, for some $\alpha \in (0.5, 1)$, then

$$\begin{aligned} w_x(\lambda_s) &= \left(1 - e^{i(\lambda_s-v)}\right)^d \left(1 - e^{i(\lambda_s+v)}\right)^d w_u(\lambda_s) \\ &\quad - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s+v)}\right)^d x_T}{(-2\pi is)^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s-v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} (-2\pi is)^{1-d} \sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right). \end{aligned} \tag{2.21}$$

In equations (2.20) and (2.21), the leading term in the asymptotic approximation of $w_x(\lambda_s)$ is the same. Hence, although the orders of magnitudes of the remainder differ, I may write

$$\begin{aligned} w_x(\lambda_s) &= \left(1 - e^{i(\lambda_s-v)}\right)^d \left(1 - e^{i(\lambda_s+v)}\right)^d w_u(\lambda_s) \\ &\quad - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s+v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s-v)}\right)^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s-v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \left(1 - e^{i(\lambda_s-v)}\right)^{1-d} \sqrt{T}} \\ &\quad + o_p\left(-\frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s+v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s-v)}\right)^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s-v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \sqrt{T}}\right. \\ &\quad \left. + \frac{1 - e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s+v)}\right)^{1-d} \left(1 - e^{i(\lambda_s-v)}\right)^{1-d} \sqrt{T}}\right) \end{aligned} \tag{2.22}$$

for both cases, and equation (2.22) is valid for all $\lambda_s = \frac{2\pi is}{n}$ with $\frac{T^\alpha}{s} \rightarrow 0$.

(c) **Case $\lambda_s = \frac{2\pi is}{n} \rightarrow v$ and s fixed**

From Lemma 2.2 (c), I have

$$D(d, v, e^{i\lambda_s}) = \frac{(1 - e^{i(\lambda_s+v)})^d}{\Gamma(1-d)T^d} F_1^1(1, 1-d; -2\pi is) + O\left(\frac{1}{T^{1+d}}\right) \quad (2.23)$$

and it follows that

$$\begin{aligned} \frac{1}{T^d} w_x(\lambda_s) &= \frac{1}{T^d} \left[\frac{(1 - e^{i(\lambda_s+v)})^d}{\Gamma(1-d)T^d} F_1^1(1, 1-d; -2\pi is) + O\left(\frac{1}{T^{1+d}}\right) \right]^{-1} \\ &\quad \times \left[w_u(\lambda_s) + \frac{e^{iT\lambda_s}}{\sqrt{2\pi T}} X_T(v, d, \lambda_s) \right], \end{aligned}$$

giving

$$\frac{w_x(\lambda_s)}{T^d} = \frac{\Gamma(1-d)}{F_1^1(1, 1-d; -2\pi is) (1 - e^{i(\lambda_s+v)})^d} \left[w_u(\lambda_s) + \frac{e^{iT\lambda_s}}{\sqrt{2\pi T}} X_T(v, d, \lambda_s) \right] + O\left(\frac{1}{T}\right). \quad (2.24)$$

Further from Theorem 2.2 (c),

$$\begin{aligned} &\frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} \\ &= \left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d \frac{F_1^1(1, 1-d; -2\pi is)}{\Gamma(1-d)} \int_0^1 e^{2\pi isr} X_{n,d}(r) dr \\ &\quad - \left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi is) r^{-d} X_{n,d}(1-r) dr \\ &\quad - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d}{(1 - e^{i(\lambda_s+v)})^{1-d}} \frac{x_T}{\sqrt{T}} + O_p\left(\frac{1}{T^{1-d}}\right), \end{aligned}$$

so that

$$\begin{aligned}
& \frac{w_x(\lambda_s)}{T^d} \\
&= \frac{\Gamma(1-d)}{F_1^1(1, 1-d; -2\pi is) (1 - e^{i(\lambda_s+v)})^d} w_u(\lambda_s) \\
&+ \frac{1}{\sqrt{2\pi}} \left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d \int_0^1 e^{2\pi isr} X_{n,d}(r) dr \\
&- \frac{1}{\sqrt{2\pi}} \frac{\left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d}{F_1^1(1, 1-d; -2\pi is)} \int_0^1 F_1^1(1, 1-d; -2\pi is) r^{-d} X_{n,d}(1-r) dr \\
&+ O_p\left(\frac{1}{T^{1-d}}\right).
\end{aligned} \tag{2.25}$$

2.3.3 Limit theorems

The partial sums of u_t satisfy the functional law, i.e.

$$X_T(r) = \frac{1}{\sqrt{T}} \sum_{t=0}^{[Tr]} u_t \rightarrow_d B(r), \tag{2.26}$$

where B is a Brownian motion with variance $\omega^2 = \sigma^2 C(1)^2$.

Lemma 2.3 *Suppose $d \in (0.5, 1)$.*

For u_t satisfying (2.5) and with ε_t i.i.d. $(0, \sigma^2)$ and $E|\varepsilon_t|^p < \infty$ for $p > \max\left(\frac{1}{d-0.5}, 2\right)$,

$$X_{T,d}(r) = \frac{x_{[Tr]}}{T^{d-0.5}} \rightarrow^d B_{d-1}(r), \tag{2.27}$$

where $B_{d-1}(r)$ is a sum of fractional Brownian motions, given by

$$B_{d-1}(r) = \frac{\omega_1}{\sqrt{2}} \int_0^r (r-s)^{d-1} dW_1(s) - \frac{\omega_2}{\sqrt{2}} \left(\int_0^r (r-s)^{d-1} dW_2(s) - i \int_0^r (r-s)^{d-1} dW_3(s) \right)$$

with $\omega_1 = \frac{2C(e^{i\lambda})}{\sqrt{\pi}} \frac{\Gamma(d+0.5)}{\Gamma(2d)} \left(\frac{1-\eta^2}{4}\right)^{0.25-0.5d}$ and $\omega_2 = \frac{(T)^{d-0.5}}{\Gamma(d)} (1 - e^{-2iv})^{1-d} \tilde{C}_\lambda(e^{-2i\lambda})$.

When $d = 1$

$$\frac{x_{T,r}}{T^{0.5}} \rightarrow_d \sqrt{2\pi}\omega_3 W_1(r) + \sqrt{2\pi}\omega_4 [W_2(r) - iW_3(r)],$$

where $\omega_3 = \frac{C(e^{i\lambda})}{2\sqrt{\pi} \sin(v)}$ and $\omega_4 = \frac{\tilde{C}_\lambda(e^{-2i\lambda})}{2\sqrt{\pi}}$.

Lemma 2.4 For $\lambda_s = \frac{2\pi is}{n} \rightarrow v$ and s fixed

$$\begin{aligned} \frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} &\rightarrow d \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda_s+v)})^{1-d}} \frac{1}{\Gamma(1-d)} \int_0^1 e^{2\pi isr} B_{d-1} dr F_1^1(1, 1-d; -2\pi is) \\ &\quad - \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda_s+v)})^{1-d}} \int_0^1 e^{2\pi isr} dB(r). \end{aligned} \quad (2.28)$$

Theorem 2.3 For fixed integer s

$$\int_0^r e^{-2\pi is(r-q)} dB(q) = \frac{1}{\Gamma(1-d)} \int_0^r F_1^1(1, 1-d; -2\pi i(r-q)) (r-q)^{-d} B_{d-1}(q) dq, \quad (2.29)$$

and for $s = 0$,

$$B(r) = \frac{1}{\Gamma(1-d)} (r-q)^{-d} B_{d-1}(q) dq. \quad (2.30)$$

Proof. The proof follows from Theorem 3.6 of Phillips (1999). ■

Now I are ready to derive the limit distribution of the dft, as shown in the following theorem, it is very similar to that of Phillips (1999).

Theorem 2.4 Suppose $d \in (0.5, 1)$. The following limit results apply.

(a) Let $\phi \neq v$ and $\lambda_{s_j} \in B_\phi = \{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\}$ for s_j ($j = 1, 2, \dots, J$), where s_j is a finite set of distinct integers. When $M \rightarrow \infty$ as $T \rightarrow \infty$, the quantities $\{w_x(\lambda_{s_j})\}_{j=1}^J$ are asymptotically independently distributed as complex normal $N_c(0, f_x(\phi))$ where

$$f_x(\phi) = \left| 1 - 2\eta e^{i\phi} + e^{2i\phi} \right|^{-2d} f_u(\phi).$$

(b) Let $\{s_j\}_{j=1}^J$ be distinct integers with $0 < l < |s_j - \frac{vT}{2\pi}| < L$ for each j and with $\frac{l}{T} + \frac{T^\alpha}{l} \rightarrow 0$ as $T \rightarrow \infty$, for some $\alpha \in (0.5, 1)$. The quantities $\left\{ (\lambda_{s_j} - v)^d w_x(\lambda_{s_j}) \right\}_{j=1}^J$ are asymptotically independently distributed as complex normal $N_c(0, f_u(\phi))$.

(c) Let $\{s_j - \frac{vT}{2\pi}\}_{j=1}^J$ be a finite set of distinct positive integers which are fixed as $T \rightarrow \infty$. Then, for each j

$$\frac{1}{T^d} w_x(\lambda_{s_j}) \rightarrow_d \frac{1}{\sqrt{2\pi}} \int_0^1 e^{2\pi is_j r} B_{d-1}(r) dr, \quad (2.31)$$

where B_{d-1} is a generalized fractional Brownian motion.

When $d = 1$ and $C(L) = 1$, the following limits apply.

(d) Let $\phi \neq v$ and suppose $\lambda_{s_j} \in B_\phi = \{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\}$ for a finite set of distinct integers s_j ($j = 1, 2, \dots, J$). When $M \rightarrow \infty$ as $T \rightarrow \infty$, the quantities $\{w_x(\lambda_{s_j})\}_{j=1}^J$ are asymptotically distributed as

$$\left\{ \frac{1}{(1 - e^{i(\phi-v)}) (1 - e^{i(\phi+v)})} \xi_j - \frac{e^{2i\phi}}{(1 - e^{i(\phi-v)}) (1 - e^{i(\phi+v)})} \eta \right\}_{j=1}^J,$$

where the $\{\xi_j\}_{j=1}^J$ are i.i.d. $N_c(0, f_u(\phi))$ and independent of

$$\eta = \omega_3 W_1(1) + \omega_4 [W_2(1) - iW_3(1)]$$

with W is a standard brownian motion, $\omega_3 = \frac{C(e^{i\phi})}{2\sqrt{\pi} \sin(v)}$ and $\omega_4 = \frac{\tilde{C}_\lambda(e^{-2i\phi})}{2\sqrt{\pi}}$.

(e) Let $\{s_j\}_{j=1}^J$ be a finite set of distinct integers with $\frac{s_j}{2\pi T} \rightarrow v$ as $T \rightarrow \infty$. The quantities $\{(\lambda_{s_j} - v) w_x(\lambda_{s_j})\}_{j=1}^J$ are asymptotically distributed as

$$i \left(\frac{1}{(1 - e^{2iv})} \xi_j - \frac{e^{2iv}}{(1 - e^{2iv})} \psi \right).$$

(f) Let $\{s_j - \frac{vT}{2\pi}\}_{j=1}^J$ be a finite set of distinct integers which are fixed as $T \rightarrow \infty$. Then, ξ_j in (d) have the representation

$$\xi_j = \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \int_0^1 e^{2\pi i(s_j - \frac{vT}{2\pi})r} d\eta(r) \quad (2.32)$$

and

$$\frac{1}{T} w_x(\lambda_{s_j}) \rightarrow_d \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \int_0^1 e^{2\pi i(s_j - \frac{vT}{2\pi})r} \psi(r) dr,$$

which also holds for $s_j = \frac{vT}{2\pi}$.

Proof. See Appendix. ■

2.4 Some statistical applications

2.4.1 Spectral estimation

The smoothed periodogram estimator of $f_x(\phi)$ is given by

$$\widehat{f}_{xx}(\phi) = \frac{1}{m} \sum_{\lambda_s \in B(\phi)} w_x(\lambda_s) w_x(\lambda_s)^*, \quad (2.33)$$

where $B_m(\phi) = \{\phi - \frac{\pi}{2M}, \phi + \frac{\pi}{2M}\}$, M is the bandwidth parameter that determines the number of frequencies $m = [T/2M]$ used in the smoothing. At the zero frequency $\phi = 0$, I consider a one-sided average of m periodogram ordinates near the origin

$$\widehat{f}_{xx}(\phi) = \frac{1}{m} \sum_{s=0}^{m-1} w_x(\lambda_s) w_x(\lambda_s)^*. \quad (2.34)$$

Then, I have the following theorem.

Theorem 2.5 *Suppose $d \in (0.5, 1)$.*

(a) *For $\phi \neq 0$*

$$\widehat{f}_{xx}(\phi) \rightarrow_p f_x(\phi) = \frac{f_u(\phi)}{|1 - 2\eta e^{i\phi} + e^{2i\phi}|^{2d}}.$$

(b) *For $\frac{m}{T^\alpha} \rightarrow \infty$ with $\alpha \geq \frac{1}{2d}$*

$$\frac{m}{T^{2d}} \widehat{f}_{xx}(0) \rightarrow_d \frac{1}{2\pi} \int_0^1 B_{d-1}(r)^2 dr.$$

Suppose $d = 1$

(a) *For $\phi \neq 0$*

$$\widehat{f}_{xx}(\phi) \rightarrow_p f_x(\phi) + \frac{1}{(1 - e^{i(\lambda-v)})(1 - e^{i(\lambda+v)})} \psi^2.$$

(b) *For $\frac{m}{T^{0.5}} \rightarrow \infty$*

$$\frac{m}{T^2} \widehat{f}_{xx}(0) \rightarrow_d \int_0^1 \psi(r)^2 dr.$$

Proof. See Appendix. ■

2.4.2 Exact log periodogram regression

For the log periodogram regression, I can use the quantity

$$v_x = w_x - \frac{e^{i\lambda} (1 - e^{i(\lambda+v)})^d}{(1 - e^{i(\lambda-v)})^{1-d}} \frac{x_T}{\sqrt{2\pi T}} - \frac{e^{i\lambda} (1 - e^{i(\lambda-v)})^d}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{x_T}{\sqrt{2\pi T}} \\ + \frac{1 - e^{i2\lambda}}{(1 - e^{i(\lambda+v)})^{1-d} (1 - e^{i(\lambda-v)})^{1-d}} \frac{x_T}{\sqrt{2\pi T}}$$

in place of $w_x(\lambda_s)$ in the regression. The usual least squares regression

$$\ln(I_x(\lambda_s)) = \hat{c} - \hat{d} \left| 1 - 2\eta e^{i\lambda_s} + e^{2i\lambda_s} \right| + error$$

is replaced by

$$\ln(I_v(\lambda_s)) = \tilde{c} - \tilde{d} \left(1 - 2\eta e^{i\lambda_s} + e^{2i\lambda_s} \right)^2 + error, \quad (2.35)$$

in which the periodogram ordinates, $I_x(\lambda_s)$, are replaced by $I_v(\lambda_s) = v_x(\lambda_s) v_x(\lambda_s)^*$. Phillips (1999) call this procedure "modified log periodogram regression". I have

$$I_v(\lambda_s) = \left| \left(1 - 2\eta e^{i\lambda_s} + e^{2i\lambda_s} \right)^{-d} w_u(\lambda_s) + o_p \left(\frac{T^d}{s} \right) \right|^2 \\ = \left| 1 - 2\eta e^{i\lambda_s} + e^{2i\lambda_s} \right|^{-3d} I_u(\lambda_s) \left[1 + \left(1 - 2\eta e^{i\lambda_s} + e^{2i\lambda_s} \right)^d w_u(\lambda_s)^{-1} o_p \left(\frac{T^d}{s} \right) \right] \\ = \left| 1 - 2\eta e^{i\lambda_s} + e^{2i\lambda_s} \right|^{-3d} I_u(\lambda_s) \left[1 + o_p \left(\frac{1}{s^{1-d}} \right) \right],$$

which leads to the new regression model

$$\ln(I_v(\lambda_s)) = c - d \left| \left(1 - 2\eta e^{i\lambda_s} + e^{2i\lambda_s} \right)^2 \right| + a(\lambda_s), \quad (2.36)$$

where

$$a(\lambda_s) = \ln(I_u(\lambda_s)/f_u(\lambda_s)) + \ln(f_u(\lambda_s)/f_u(0)) + O_p \left(\frac{1}{s^{1-d}} \right). \quad (2.37)$$

This relation holds for frequencies λ_s satisfying $\frac{s}{T} + \frac{T^\alpha}{s} \rightarrow 0$ as $T \rightarrow \infty$.

I have the same limit theorem as for log periodogram estimator in the stationary case, as

in Robinson (1995),

$$\sqrt{m}(\tilde{d} - d) \rightarrow_d N\left(0, \frac{\pi^2}{24}\right). \quad (2.38)$$

The above methods are based on the assumption that the value of η is known. The estimation method for unknown η needs further study.

2.5 Conclusions

This chapter studies the Discrete Fourier transform (dft) of generalized fractional processes and the exact representation of the dft is given in terms of the component representation. The new representation is particularly useful in analyzing the asymptotic behavior of the dft and the periodogram in the nonstationary case when the memory parameter $d > 0.5$. Many applications can be investigated using the results, such as the log periodogram and local Whittle estimates of the seasonal memory parameter for the nonstationary and seasonal unit root cases.

2.6 Appendix

Proof of Lemma 2.2: Part (a): For fixed $\lambda \neq v$, from Lemma 2.1, I can easily obtain

$$D_1(d, v, e^{i\lambda}) = \left(1 - e^{i(\lambda-v)}\right)^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{iT(\lambda-v)}}{1 - e^{i(\lambda-v)}} \left[1 + O\left(\frac{1}{T}\right)\right]$$

and

$$D_2(d, v, e^{i\lambda}) = \left(1 - e^{i(\lambda+v)}\right)^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{iT(\lambda+v)}}{1 - e^{i(\lambda+v)}} \left[1 + O\left(\frac{1}{T}\right)\right].$$

Then

$$\begin{aligned}
& D(d, v, e^{i\lambda}) \\
&= D_1(d, v, e^{i\lambda}) D_2(d, v, e^{i\lambda}) \\
&= \left(1 - e^{i(\lambda+v)}\right)^d \left(1 - e^{i(\lambda-v)}\right)^d \\
&\quad - \frac{1}{\Gamma(-d) T^{1+d}} \left[\frac{e^{iT(\lambda+v)} \left(1 - e^{i(\lambda-v)}\right)^d}{1 - e^{i(\lambda+v)}} + \frac{e^{iT(\lambda-v)} \left(1 - e^{i(\lambda+v)}\right)^d}{1 - e^{i(\lambda-v)}} \right] \left[1 + O\left(\frac{1}{T}\right)\right].
\end{aligned}$$

Part (b): For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and $s \rightarrow \infty$ as $T \rightarrow \infty$, from Lemma 2.1 I can easily obtain

$$D_1(d, v, e^{i\lambda_s}) = \left(1 - e^{i(\lambda_s-v)}\right)^d + \frac{1}{2\pi i \Gamma(-d) T^d s} \left[1 + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{T^{1+d}}\right)$$

and

$$D_2(d, v, e^{i\lambda_s}) = \left(1 - e^{i(\lambda_s+v)}\right)^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{iT(\lambda_s+v)}}{1 - e^{i(\lambda_s+v)}} \left[1 + O\left(\frac{1}{T}\right)\right].$$

Then

$$\begin{aligned}
& D(d, v, e^{i\lambda_s}) \\
&= D_1(d, v, e^{i\lambda_s}) D_2(d, v, e^{i\lambda_s}) \\
&= \left(1 - e^{i(\lambda_s+v)}\right)^d \left(1 - e^{i(\lambda_s-v)}\right)^d \\
&\quad + \frac{1}{2\pi i \Gamma(-d) T^d s} \left(1 - e^{i(\lambda_s+v)}\right)^d \left[1 + O\left(\frac{1}{s}\right)\right] + O\left(\frac{1}{T^{1+d}}\right).
\end{aligned}$$

Part (c): For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and fixed s as $T \rightarrow \infty$, from Lemma 2.1 I can easily obtain

$$D_1(d, v, e^{i\lambda_s}) = \frac{1}{\Gamma(1-d) T^d} F_1^1(1, 1-d; -2\pi is) + O\left(\frac{1}{T^{1+d}}\right)$$

and

$$D_2(d, v, e^{i\lambda_s}) = \left(1 - e^{i(\lambda_s+v)}\right)^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{iT(\lambda_s+v)}}{1 - e^{i(\lambda_s+v)}} \left[1 + O\left(\frac{1}{T}\right)\right].$$

Then

$$\begin{aligned}
D(d, v, e^{i\lambda_s}) &= D_1(d, v, e^{i\lambda_s}) D_2(d, v, e^{i\lambda_s}) \\
&= \frac{(1 - e^{i(\lambda_s+v)})^d}{\Gamma(1-d) T^d} F_1^1(1, 1-d; -2\pi i s) + O\left(\frac{1}{T^{1+d}}\right),
\end{aligned}$$

where $F_1^1(a, b; z)$ denote the confluent hypergeometric function.

Part (d): For $\lambda = v$, from Lemma 2.1 I can easily obtain

$$D_1(d, v, e^{iv}) = \frac{1}{\Gamma(1-d) T^d} \left[1 + O\left(\frac{1}{T}\right) \right]$$

and

$$D_2(d, v, e^{iv}) = (1 - e^{2iv})^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{2iv}}{1 - e^{2iv}} \left[1 + O\left(\frac{1}{T}\right) \right].$$

Then

$$D(d, v, 1) = \frac{(1 - e^{2iv})^d}{\Gamma(1-d) T^d} \left[1 + O\left(\frac{1}{T}\right) \right].$$

■

Proof of Lemma 2.1: From Equation (3.12), I have

$$\begin{aligned}
u_t &= D(d, v, e^{i\lambda}) x_t + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) (e^{-i\lambda L} - 1) x_t \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) (e^{-i\lambda L} - 1) x_t \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) (e^{-i\lambda L} - 1)^2 x_t \\
&= D(d, v, e^{i\lambda}) x_t + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) (e^{-i\lambda L} - 1) x_t \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) (e^{-i\lambda L} - 1) x_t \\
&\quad + e^{-i\lambda} \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) (e^{-i\lambda L} - 1) x_{t-1} \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) (e^{-i\lambda L} - 1) x_t.
\end{aligned}$$

Taking the dft of both sides of (3.18), and following the steps in the proof of Lemma 3.3,

I obtain

$$\begin{aligned}
w_u(\lambda) &= w_x(\lambda) D(d, v, e^{i\lambda}) + \frac{1}{\sqrt{2\pi T}} D_2(d, v, e^{i\lambda}) \left(X_0^{D_1}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1}(d, v, \lambda) \right) \\
&+ \frac{1}{\sqrt{2\pi T}} D_1(d, v, e^{i\lambda}) \left(X_0^{D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_2}(d, v, \lambda) \right) \\
&+ \frac{1}{\sqrt{2\pi T}} \left(e^{-i\lambda} X_0^{D_1 D_2}(d, v, \lambda) - e^{i(T-2)\lambda} X_{T-1}^{D_1 D_2}(d, v, \lambda) \right) \\
&- \frac{1}{\sqrt{2\pi T}} \left(e X_0^{D_1 D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1 D_2}(d, v, \lambda) \right)
\end{aligned}$$

where

$$\begin{aligned}
X_0^{D_1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) x_0; & X_T^{D_1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) x_T; \\
X_0^{D_2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) x_0; & X_T^{D_2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) x_T; \\
X_0^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) x_0; \\
X_{T-1}^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) x_{T-1}; \\
X_T^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) x_T.
\end{aligned}$$

■

Proof of Theorem 2.2: Part (a): For fixed $\lambda \neq v$ as $T \rightarrow \infty$, from Lemma 2.1 I can easily obtain

$$\begin{aligned}
\frac{X_T^{D_1}(d, v, \lambda)}{\sqrt{T}} &= \frac{\tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) x_t}{\sqrt{T}} \\
&= \frac{e^{i\lambda}}{(1 - e^{i(\lambda-v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right), \\
\frac{X_T^{D_2}(d, v, \lambda)}{\sqrt{T}} &= \frac{\tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) x_t}{\sqrt{T}} \\
&= \frac{e^{i\lambda}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{X_T^{D_1 D_2}(d, v, \lambda)}{\sqrt{T}} &= \frac{\tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_t}{\sqrt{T}} \\
&= \frac{e^{i\lambda}}{(1 - e^{i(\lambda-v)})^{1-d}} \frac{\tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right) \\
&= \frac{e^{i2\lambda}}{(1 - e^{i(\lambda+v)})^{1-d} (1 - e^{i(\lambda-v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{X_T(\lambda, \eta, d)}{\sqrt{T}} &= -\frac{e^{i\lambda} (1 - e^{i(\lambda+v)})^d x_T}{(1 - e^{i(\lambda-v)})^{1-d} \sqrt{T}} - \frac{e^{i\lambda} (1 - e^{i(\lambda-v)})^d x_T}{(1 - e^{i(\lambda+v)})^{1-d} \sqrt{T}} \\
&\quad + \frac{1 - e^{i2\lambda}}{(1 - e^{i(\lambda+v)})^{1-d} (1 - e^{i(\lambda-v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right) \\
&= O_p\left(\frac{1}{T^{1-d}}\right).
\end{aligned}$$

Part (b): For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and $\frac{s}{T^\alpha} \rightarrow \infty$ as $T \rightarrow \infty$, for some $\alpha \in (0.5, 1)$, I have already shown of that

$$\begin{aligned}
\frac{X_T^{D_1}(d, v, \lambda)}{\sqrt{T}} &= \frac{\tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_t}{\sqrt{T}} \\
&= \frac{e^{i\lambda}}{(1 - e^{i(\lambda-v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{X_T^{D_2}(d, v, \lambda)}{\sqrt{T}} &= \frac{\tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_t}{\sqrt{T}} \\
&= \frac{e^{i\lambda}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{X_T^{D_1 D_2}(d, v, \lambda)}{\sqrt{T}} &= \frac{\tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_t}{\sqrt{T}} \\
&= \frac{e^{i2\lambda}}{(1 - e^{i(\lambda+v)})^{1-d} (1 - e^{i(\lambda_s-v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right).
\end{aligned}$$

I also have

$$\frac{e^{i\lambda}}{(1 - e^{i(\lambda-v)})^{1-d}} \frac{x_T}{\sqrt{T}} = \frac{e^{i\lambda_s}}{(-2\pi is)^{1-d}} \frac{x_T}{T^{d-0.5}} + o_p\left(\frac{1}{s^{1-d}}\right).$$

Then, I have

$$\begin{aligned} \frac{X_T^{D_1 D_2}(d, v, \lambda)}{\sqrt{T}} &= \frac{e^{i2\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d} (1 - e^{i(\lambda_s-v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right) \\ &= \frac{e^{i2\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d} (-2\pi is)^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right). \end{aligned}$$

and

$$\begin{aligned} \frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} &= -\frac{e^{i\lambda_s} (1 - e^{i(\lambda+v)})^d}{(1 - e^{i(\lambda_s-v)})^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{x_T}{\sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda}}{(1 - e^{i(\lambda+v)})^{1-d} (1 - e^{i(\lambda_s-v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right) \\ &= -\frac{e^{i\lambda_s} (1 - e^{i(\lambda+v)})^d}{(-2\pi is)^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s-v)})^d}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{x_T}{\sqrt{T}} \\ &\quad + \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d} (-2\pi is)^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{s^{1-d}}\right) \\ &= O_p\left(\frac{1}{s^{1-d}}\right) \end{aligned}$$

Part (c): For $\lambda = \lambda_s = \frac{2\pi is}{n} \rightarrow v$ and fixed s as $T \rightarrow \infty$, I have already shown of that

$$\begin{aligned} &\frac{X_T^{D_1}(d, v, \lambda)}{\sqrt{T}} \\ &= \frac{\tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_t}{\sqrt{T}} \\ &= \frac{F_1^1(1, 1-d; -2\pi is)}{\Gamma(1-d)} \int_0^1 e^{2\pi is r} X_{n,d}(r) dr \\ &\quad - \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi is) r^{-d} X_{n,d}(1-r) dr + o_p\left(\frac{1}{T^{1-d}}\right), \end{aligned}$$

$$\begin{aligned} \frac{X_T^{D_2}(d, v, \lambda)}{\sqrt{T}} &= \frac{\tilde{D}_{2\lambda}(d, v, e^{-i\lambda L}) x_t}{\sqrt{T}} \\ &= \frac{e^{i\lambda}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{x_T}{\sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right), \end{aligned}$$

and

$$\begin{aligned} &\frac{X_T^{D_1 D_2}(d, v, \lambda)}{\sqrt{T}} \\ &= \left(1 - e^{i(\lambda+v)}\right)^d \frac{F_1^1(1, 1-d; -2\pi i s)}{\Gamma(1-d)} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr \\ &\quad - \left(1 - e^{i(\lambda+v)}\right)^d \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi i s) r^{-d} X_{n,d}(1-r) dr \\ &\quad + o_p\left(\frac{1}{T^{1-d}}\right). \end{aligned}$$

where $X_{n,d}(r, \lambda) = \frac{x_{[nr]}}{T^{d-0.5}}$. From Lemma 2.1, I can easily obtain

$$\begin{aligned} \frac{\tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) x_T}{\sqrt{T}} &= \frac{F_1^1(1, 1-d; -2\pi i s)}{\Gamma(1-d)} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr \\ &\quad - \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi i s) r^{-d} X_{n,d}(1-r) dr \\ &\quad + O_p\left(\frac{1}{T^{1-d}}\right), \end{aligned}$$

where $X_{n,d}(r, \lambda) = \frac{x_{[nr]}}{T^{d-0.5}}$. Then I have

$$\begin{aligned}
& \frac{X_T(\lambda_s, \eta, d)}{\sqrt{T}} \\
= & \left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d \frac{F_1^1(1, 1-d; -2\pi i s)}{\Gamma(1-d)} \int_0^1 e^{2\pi i s r} X_{n,d}(r) dr \\
& - \left(1 + e^{i\lambda_s} - e^{-i\lambda_s}\right) \left(1 - e^{i(\lambda_s+v)}\right)^d \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi i s) r^{-d} X_{n,d}(1-r) dr \\
& - \frac{e^{i\lambda} (1 - e^{i(\lambda_s-v)})^d}{(1 - e^{i(\lambda_s+v)})^{1-d}} \frac{x_T}{\sqrt{T}} + O_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned}$$

■

Proof of Lemma 2.3: Suppose $d \in (0.5, 1)$, from (2.10), I have

$$D(d, v, L) x_t = D_1(d, v, L) D_2(d, v, L) x_t$$

and I have the following expression for x_t

$$D_2(d, v, L) x_t = (1 - e^{-iv} L)^{-d} u_t.$$

According to Lemma 1 of Phillips (1999), by expanding the polynomial operator about its value at the complex exponential e^{iv} , I have the following decomposition

$$u_t = C(e^{iv}) \varepsilon_t - (1 - e^{-iv} L) \tilde{\varepsilon}_t.$$

Then I have the following equation

$$D_2(d, v, L) x_t = C(e^{iv}) (1 - e^{-iv} L)^{-d} \varepsilon_t - (1 - e^{-iv} L)^{1-d} \tilde{\varepsilon}_t,$$

where $\tilde{\varepsilon}_t = \tilde{C}_\lambda(e^{-iv} L) \varepsilon_t$, $\tilde{C}_\lambda(e^{-iv} L) = \sum_{p=0}^{T-1} \tilde{c}_{\lambda p} e^{-ipv} L^p$, and $\tilde{c}_{\lambda p} = \sum_{k=p+1}^T c_k$. Under (2.5), $\tilde{\varepsilon}_t$ is stationary with zero mean and finite variance. I have

$$x_t = C(e^{i\lambda}) (1 - e^{iv} L)^{-d} (1 - e^{-iv} L)^{-d} \varepsilon_t - (1 - e^{iv} L)^{-d} G(d, v, L) \varepsilon_t,$$

where

$$G(d, v, L) = (1 - e^{-iv}L)^{1-d} \tilde{C}_\lambda(e^{-i\lambda}L).$$

Expanding the polynomial operator about its value at the complex exponential e^{-iv} , I have the following decomposition

$$G(d, v, L) = G(d, v, e^{-i\lambda}) - \tilde{G}(d, v, e^{i\lambda})(e^{iv}L - 1).$$

Then,

$$\begin{aligned} x_t &= C(e^{i\lambda})(1 - e^{iv}L)^{-d}(1 - e^{-iv}L)^{-d}\varepsilon_t - (1 - e^{iv}L)^{-d}G(d, v, e^{-i\lambda})\varepsilon_t + \hat{u}_t \quad (2.39) \\ &= C(e^{i\lambda})z_{1t} - G(d, v, e^{-i\lambda})z_{2t} + \hat{u}_t, \end{aligned}$$

where $\hat{u}_t = (1 - e^{iv}L)^{1-d}\tilde{G}(d, v, e^{i\lambda})\varepsilon_t$ is stationary with zero mean and finite variance, with

$$z_{1t} = (1 - e^{iv}L)^{-d}(1 - e^{-iv}L)^{-d}\varepsilon_t$$

and

$$z_{2t} = (1 - e^{iv}L)^{-d}\varepsilon_t.$$

For z_{2t} , by equation (25) of Rosenblatt (1976), I have

$$z_{2,r} \rightarrow_d \frac{(T)^{d-0.5}}{\Gamma(d)} \int_0^r (r-s)^{d-1} e^{-ivs} dW(s).$$

For z_{1t} , I can write it as

$$z_{1t} = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}.$$

From equation (13) of Gray, Zhang and Woodward (1989), I know that

$$b_j \simeq \frac{2}{\sqrt{\pi}} \frac{\Gamma(d+0.5)}{\Gamma(2d)} \left(\frac{1-\eta^2}{4} \right)^{0.25-0.5d} \cos[(k+d)v - (d\pi/2)] (k)^{d-1}.$$

As discussed on page 3 of Dacidsen and De Long (1999), the coefficients satisfy

$$b_j \sim L(j)j^{d-1},$$

where $L(j)$ denotes any slow varying component, have essentially the same asymptotic properties as the fractional process. Therefore,

$$\begin{aligned}
& \frac{z_{1t}(r)}{(T)^{d-0.5}} \\
&= \frac{z_{1,[Tr]}}{(T)^{d-0.5}} \\
&= \frac{2}{\sqrt{\pi}} \frac{\Gamma(d+0.5)}{\Gamma(2d)} \left(\frac{1-\eta^2}{4}\right)^{0.25-0.5d} \int_0^r (r-s)^{d-1} \cos[(k+d)v - (d\pi/2)] dW(s).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& X_{T,d}(r) \\
&= \frac{x_{[Tr]}}{T^{d-0.5}} \\
&\rightarrow {}_dC(e^{i\lambda}) \frac{2}{\sqrt{\pi}} \frac{\Gamma(d+0.5)}{\Gamma(2d)} \left(\frac{1-\eta^2}{4}\right)^{0.25-0.5d} \\
&\quad \times \int_0^r (r-s)^{d-1} \cos[(k+d)v - (d\pi/2)] dW(s) \\
&\quad - G(d, v, e^{-i\lambda}) \frac{(T)^{d-0.5}}{\Gamma(d)} \int_0^r (r-s)^{d-1} e^{-ivs} dW(s) \\
&= \omega_1 \int_0^r (r-s)^{d-1} \cos[(k+d)v - (d\pi/2)] dW_1(s) - \omega_2 \int_0^r (r-s)^{d-1} e^{-ivs} dW_2(s) \\
&= \frac{\omega_1}{\sqrt{2}} \int_0^r (r-s)^{d-1} dW_1(s) - \frac{\omega_2}{\sqrt{2}} \left(\int_0^r (r-s)^{d-1} dW_2(s) - i \int_0^r (r-s)^{d-1} dW_3(s) \right) \\
&= B_{d-1}(r),
\end{aligned}$$

where

$$\omega_1 = \frac{2C(e^{i\lambda})}{\sqrt{\pi}} \frac{\Gamma(d+0.5)}{\Gamma(2d)} \left(\frac{1-\eta^2}{4}\right)^{0.25-0.5d}$$

and

$$\omega_2 = \frac{(T)^{d-0.5}}{\Gamma(d)} (1 - e^{-2iv})^{1-d} \tilde{C}_\lambda(e^{-2i\lambda}).$$

When $d = 1$, (2.39) is

$$\begin{aligned} x_t &= C(e^{i\lambda}) (1 - e^{iv}L)^{-1} (1 - e^{-iv}L)^{-1} \varepsilon_t - (1 - e^{iv}L)^{-1} G(d, v, e^{-i\lambda}) \varepsilon_t + \widehat{u}_t \\ &= C(e^{i\lambda}) z_{1t}(1) - G(d, v, e^{-i\lambda}) z_{2t}(1) + \widehat{u}_t. \end{aligned}$$

Then $z_{1t}(1)$ can be written as

$$z_{1t}(1) = \frac{1}{\sin(v)} \sum_{j=1}^{\infty} \sin(jv) \varepsilon_{t-j+1}.$$

Assuming $\varepsilon_t = 0$ for $t \leq 0$, then by Chan and Wei's Theorem 2.2 I have

$$\begin{aligned} \frac{y_T}{T^{0.5}} &= \frac{1}{\sin(v)} \sum_{j=1}^t \sin[(t-j+1)v] \varepsilon_t \\ &= \frac{1}{\sin(v)} \left\{ \sin[(T+1)v] \sum_{j=1}^t \cos(jv) \varepsilon_j - \cos[(T+1)v] \sum_{j=1}^t \sin(jv) \varepsilon_j \right\} \\ &\rightarrow d \frac{1}{\sqrt{2} \sin(v)} \{ \sin[(T+1)v] W_1(1) - \cos[(T+1)v] W_2(1) \} \\ &= \frac{1}{\sqrt{2} \sin(v)} W_1(1). \end{aligned}$$

For $z_{2t}(1)$, by equation (25) of Rosenblatt, I have

$$\begin{aligned} \frac{z_{2,r}}{T^{0.5}} &\rightarrow d \int_0^r e^{-ivs} dW(s) \\ &= \frac{1}{\sqrt{2}} [W_2(1) - iW_3(1)], \end{aligned}$$

where $W_i(1)$ ($i = 1, 2, 3$) are standard Brownian motion. Then

$$\frac{x_{T,r}}{T^{0.5}} \rightarrow_d \frac{C(e^{i\lambda})}{\sqrt{2} \sin(v)} W_1(r) + \frac{\widetilde{C}_\lambda(e^{-2i\lambda})}{\sqrt{2}} [W_2(r) - iW_3(r)].$$

■

Proof of Lemma 2.4: For $\lambda_s = \frac{2\pi is}{n} \rightarrow v$ and s fixed, from equation (2.25), I have

$$\begin{aligned} & \frac{X_T^{D_1 D_2}(d, v, \lambda)}{\sqrt{T}} \\ = & \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{F_1^1(1, 1-d; -2\pi is)}{\Gamma(1-d)} \int_0^1 e^{2\pi is r} X_{n,d}(r) dr \\ & - \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi is) r^{-d} X_{n,d}(1-r) dr \\ & + O_p\left(\frac{1}{T^{1-d}}\right). \end{aligned}$$

Applying Lemma 2.3, I have

$$\begin{aligned} \frac{X_T^{D_1 D_2}(d, v, \lambda)}{\sqrt{T}} & \rightarrow d \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{F_1^1(1, 1-d; -2\pi is)}{\Gamma(1-d)} \int_0^1 e^{2\pi is r} B_{d-1}(r) dr \\ & - \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{1}{\Gamma(1-d)} \int_0^1 F_1^1(1, 1-d; -2\pi is) r^{-d} B_{d-1}(r) dr. \end{aligned}$$

Lemma E of Phillips (1999) shows that

$$\Gamma(1-d)^{-1} \int_0^1 F_1^1(1, 1-d; -2\pi i(1-q)) (1-q)^{-d} B_{d-1}(q) dq = \int_0^1 e^{-2\pi is(1-q)} dB(q).$$

Then

$$\begin{aligned} \frac{X_T^{D_1 D_2}(d, v, \lambda)}{\sqrt{T}} & = \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{F_1^1(1, 1-d; -2\pi is)}{\Gamma(1-d)} \int_0^1 e^{2\pi is(1-r)} B_{d-1}(1-r) dr \\ & - \int_0^1 e^{2\pi is(1-q)} dB(q) \\ & = \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d}} \frac{1}{\Gamma(1-d)} \int_0^1 e^{2\pi is r} B_{d-1}(r) dr F_1^1(1, 1-d; -2\pi is) \\ & - \frac{e^{i\lambda_s}}{(1 - e^{i(\lambda+v)})^{1-d}} \int_0^1 e^{2\pi is r} dB(r). \end{aligned}$$

■

Proof of Theorem 2.3: Part(a):

From (2.20), I have

$$\begin{aligned}
w_x(\lambda_s) &= \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \\
&\quad - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s + v)})^d x_T}{(-2\pi i s)^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s - v)})^d x_T}{(1 - e^{i(\lambda_s + v)})^{1-d} \sqrt{T}} \\
&\quad + \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda_s + v)})^{1-d} (-2\pi i s)^{1-d} \sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right) \\
&= \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \left[1 + O_p\left(\frac{1}{M}\right)\right] \\
&\quad + O_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned}$$

where the error magnitudes hold uniformly for $\lambda_{s_j} \in B_\phi = \{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\}$. Theorem 3 of Hannan (1973) implies that the quantities $\{w_u(\lambda_{s_j})\}_{j=1}^J$ are asymptotically independent and distributed with the same complex normal distribution $N_c(0, f_u(\phi))$ as $T \rightarrow \infty$. The stated result for the quantities $\{w_x(\lambda_{s_j})\}_{j=1}^J$ follows directly.

Part (b): From (2.21), I have

$$\begin{aligned}
w_x(\lambda_s) &= \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \\
&\quad - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s + v)})^d x_T}{(1 - e^{i(\lambda_s - v)})^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s - v)})^d x_T}{(1 - e^{i(\lambda_s + v)})^{1-d} \sqrt{T}} \\
&\quad + \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda_s + v)})^{1-d} (1 - e^{i(\lambda_s - v)})^{1-d} \sqrt{T}} \\
&\quad + o_p\left(-\frac{e^{i\lambda_s} (1 - e^{i(\lambda_s + v)})^d x_T}{(1 - e^{i(\lambda_s - v)})^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} (1 - e^{i(\lambda_s - v)})^d x_T}{(1 - e^{i(\lambda_s + v)})^{1-d} \sqrt{T}}\right) \\
&\quad + \frac{1 - e^{i2\lambda_s}}{(1 - e^{i(\lambda_s + v)})^{1-d} (1 - e^{i(\lambda_s - v)})^{1-d} \sqrt{T}}
\end{aligned}$$

Then

$$\begin{aligned}
& (\lambda_{s_j} - v)^d w_x(\lambda_s) \\
= & (\lambda_{s_j} - v)^d \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \\
& - \frac{(\lambda_{s_j} - v)^d e^{i\lambda_s} \left(1 - e^{i(\lambda_s + v)}\right)^d}{\left(1 - e^{i(\lambda_s - v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{(\lambda_{s_j} - v)^d e^{i\lambda_s} \left(1 - e^{i(\lambda_s - v)}\right)^d}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \\
& + \frac{(\lambda_{s_j} - v)^d \left(1 - e^{i2\lambda_s}\right)}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d} \left(1 - e^{i(\lambda_s - v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \\
& + o_p \left(- \frac{(\lambda_{s_j} - v)^d e^{i\lambda_s} \left(1 - e^{i(\lambda_s + v)}\right)^d}{\left(1 - e^{i(\lambda_s - v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{(\lambda_{s_j} - v)^d e^{i\lambda_s} \left(1 - e^{i(\lambda_s - v)}\right)^d}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \right. \\
& \left. + \frac{(\lambda_{s_j} - v)^d \left(1 - e^{i2\lambda_s}\right)}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d} \left(1 - e^{i(\lambda_s - v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \right)
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
& (\lambda_{s_j} - v)^d w_x(\lambda_s) \\
= & (\lambda_{s_j} - v)^d \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \\
& - \frac{(\lambda_{s_j} - v)^d e^{i\lambda_s} \left(1 - e^{i(\lambda_s + v)}\right)^d}{\left(1 - e^{i(\lambda_s - v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} - \frac{(\lambda_{s_j} - v)^d e^{i\lambda_s} \left(1 - e^{i(\lambda_s - v)}\right)^d}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \\
& + \frac{(\lambda_{s_j} - v)^d \left(1 - e^{i2\lambda_s}\right)}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d} \left(1 - e^{i(\lambda_s - v)}\right)^{1-d}} \frac{x_T}{\sqrt{T}} \\
& + o_p \left(\frac{(\lambda_{s_j} - v)^d \left(1 - e^{i(\lambda_{s_j} - v)}\right)^{-d}}{T^d s_j^{1-d}} \right)
\end{aligned}$$

and

$$\begin{aligned}
& (\lambda_{s_j} - v)^d w_x(\lambda_{s_j}) \\
&= \left(-\frac{1}{i}\right)^d (1 - e^{2iv})^d w_u(\lambda_{s_j}) \left[1 + O_p\left(\frac{L}{T}\right)\right] \\
&\quad + \left(\frac{2\pi s_j}{T}\right)^d \left[\frac{T}{2\pi i s_j}\right] \left[1 + O\left(\frac{L}{T}\right)\right] \frac{e^{i2v}}{(1 - e^{2iv})} \frac{1}{\sqrt{2\pi} T^{1-d}} \frac{x_T}{T^{d-0.5}} \\
&\quad + o_p\left(\frac{1}{T^{\alpha(1-d)}}\right) \\
&= e^{\frac{\pi di}{2}} (1 - e^{2iv})^d w_u(\lambda_{s_j}) + O\left(\frac{L}{T}\right) + o_p\left(\frac{1}{T^{\alpha(1-d)}}\right)
\end{aligned}$$

uniformly over s_j . It follows that the quantities $\left\{(\lambda_{s_j} - v)^d w_x(\lambda_{s_j})\right\}_{j=1}^J$ are asymptotically distributed as $\left\{e^{\frac{\pi di}{2}} (1 - e^{2iv})^d w_u(\lambda_{s_j})\right\}_{j=1}^J$.

Part (c): Note that for each j

$$\begin{aligned}
\frac{1}{T^d} w_x(\lambda_{s_j}) &= \frac{1}{\sqrt{2\pi}} \frac{1}{T} \sum_{t=1}^T \frac{x_t}{T^{d-0.5}} e^{2\pi s_j i \frac{t}{T}} \\
&= \frac{1}{\sqrt{2\pi}} \int_0^1 e^{2\pi s_j i r} X_{T,d}(r) dr + o_p(1)
\end{aligned}$$

and so, by the continuous mapping theorem,

$$\frac{1}{T^d} w_x(\lambda_{s_j}) \rightarrow_d \frac{1}{\sqrt{2\pi}} \int_0^1 e^{2\pi i s_j r} B_{d-1}(r) dr.$$

Part (d): From equation (2.17), I have

$$\begin{aligned}
& w_x(\lambda_{s_j}) \\
&= \left(1 - 2\eta e^{i\lambda_{s_j}} + e^{i2\lambda_{s_j}}\right)^{-1} w_u(\lambda_{s_j}) + \frac{e^{2i\lambda} e^{iT\lambda}}{\sqrt{2\pi T}} \left(1 - 2\eta e^{i\lambda_{s_j}} + e^{i2\lambda_{s_j}}\right)^{-1} x_T(v, d, \lambda_{s_j}) \\
&= \left[\left(1 - 2\eta e^{i\phi} + e^{i2\phi}\right)^{-1} w_u(\lambda) + \frac{e^{2i\phi}}{\sqrt{2\pi T}} \left(1 - 2\eta e^{i\phi} + e^{i2\phi}\right)^{-1} x_T(v, d, \phi)\right] \\
&\quad \times \left[1 + O_p\left(\frac{1}{M}\right)\right] \\
&\rightarrow \left(1 - 2\eta e^{i\phi} + e^{i2\phi}\right)^{-1} \xi_j - \frac{e^{2i\phi}}{\sqrt{2\pi T}} \psi.
\end{aligned}$$

Part (e):

$$\begin{aligned}
& (\lambda_{s_j} - v) w_x(\lambda_{s_j}) \\
&= (\lambda_{s_j} - v) \left(1 - 2\eta e^{i\lambda_{s_j}} + e^{2i\lambda_{s_j}}\right)^{-1} w_u(\lambda_{s_j}) \\
&\quad - \frac{e^{2i\lambda} e^{iT\lambda_{s_j}}}{\sqrt{2\pi T}} \left(1 - 2\eta e^{i\lambda_{s_j}} + e^{2i\lambda_{s_j}}\right)^{-1} x_T(v, d, \lambda_{s_j}) \\
&= -\frac{1}{i} \frac{1}{(1 - e^{2iv})} w_u(\lambda_{s_j}) \left[1 + O\left(\frac{1}{T}\right)\right] + \frac{1}{i} \left[1 + O\left(\frac{1}{T}\right)\right] \frac{x_T(v, d, \lambda_{s_j})}{\sqrt{2\pi T}} \\
&= i \left(\frac{1}{(1 - e^{2iv})} \xi_j - \frac{e^{2iv}}{(1 - e^{2iv})} \eta \right).
\end{aligned}$$

Part (f):

$$\begin{aligned}
\frac{1}{T} w_x(\lambda_{s_j}) &= \frac{1}{\sqrt{2\pi}} \frac{1}{T} \sum_{t=1}^T \frac{x_t}{T^{d-0.5}} e^{2\pi s_j i \frac{t}{T}} \\
&\rightarrow d \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \int_0^1 e^{2\pi i (s_j - \frac{vT}{2\pi}) r} B(r) dr \\
\xi_j &= \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \int_0^1 e^{2\pi i (s_j - \frac{vT}{2\pi}) r} dB(r)
\end{aligned}$$

and, by integration by part,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \int_0^1 e^{2\pi i (s_j - \frac{vT}{2\pi}) r} B(r) dr &= \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \left[\left. \frac{e^{2\pi i (s_j - \frac{vT}{2\pi}) r} B(r)}{2\pi i (s_j - \frac{vT}{2\pi})} \right|_0^1 \right. \\
&\quad \left. - \frac{1}{2\pi i (s_j - \frac{vT}{2\pi})} \int_0^1 e^{2\pi i (s_j - \frac{vT}{2\pi}) r} dB(r) \right] \\
&= \xi_j \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \int_0^1 e^{2\pi i (s_j - \frac{vT}{2\pi}) r} dB(r)
\end{aligned}$$

$$\frac{1}{T} w_x(\lambda_{s_j}) \rightarrow_d \frac{1}{\sqrt{2\pi}} \frac{1}{(1 - e^{2iv})} \int_0^1 e^{2\pi i (s_j - \frac{vT}{2\pi}) r} \eta(r) dr,$$

which also holds for $s_j = \frac{vT}{2\pi}$. ■

Proof of Theorem 2.4:

Part (a): For $\phi \neq 0$, from (3.16)

$$\begin{aligned}
w_x(\lambda_s) &= \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \\
&\quad - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s + v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s - v)}\right)^{1-d} \sqrt{T}} - \frac{e^{i\lambda_s} \left(1 - e^{i(\lambda_s - v)}\right)^d x_T}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d} \sqrt{T}} \\
&\quad + \frac{1 - e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s + v)}\right)^{1-d} \left(1 - e^{i(\lambda_s - v)}\right)^{1-d} \sqrt{T}} + o_p\left(\frac{1}{T^{1-d}}\right) \\
&= \left(1 - e^{i(\phi - v)}\right)^{-d} \left(1 - e^{i(\phi + v)}\right)^{-d} w_u(\lambda_s) \left[1 + O_p\left(\frac{1}{M}\right)\right] + O_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned}$$

where the error magnitudes hold uniformly for $\lambda_{s_j} \in B_\phi = \{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\}$. Then, as $T \rightarrow \infty$ with $M/T \rightarrow 0$, I have

$$\begin{aligned}
\widehat{f}_{xx}(\phi) &= \frac{1}{m} \sum_{\lambda_s \in B(\phi)} w_x(\lambda_s) w_x(\lambda_s)^* \tag{2.40} \\
&= \frac{1}{\left(1 - e^{i(\phi - v)}\right)^d \left(1 - e^{i(\phi + v)}\right)^d} \frac{1}{m} \sum_{\lambda_s \in B(\phi)} w_u(\lambda_s) w_u(\lambda_s)^* \\
&\quad + O_p\left(\frac{1}{M}\right) + O_p\left(\frac{1}{T^{1-d}}\right) \\
&\rightarrow \frac{1}{\left(1 - e^{i(\phi - v)}\right)^d \left(1 - e^{i(\phi + v)}\right)^d} f_u(\phi) \\
&= \frac{f_u(\phi)}{|1 - 2\eta e^{i\phi} + e^{2i\phi}|^{2d}} \\
&= f_x(\phi).
\end{aligned}$$

Part (b): For $\frac{m}{T^\alpha} \rightarrow \infty$ with $\alpha \geq \frac{1}{2d}$

$$\begin{aligned}
\frac{m}{T^{2d}} \widehat{f}_{xx}(0) &= \sum_{s=0}^{m-1} \frac{w_x(\lambda_s) w_x(\lambda_s)^*}{T^d} \frac{w_x(\lambda_s)^*}{T^d} \\
&= \sum_{s=0}^{T-1} \frac{w_x(\lambda_s) w_x(\lambda_s)^*}{T^d} \frac{w_x(\lambda_s)^*}{T^d} - \sum_{s=m}^{T-1} \frac{w_x(\lambda_s) w_x(\lambda_s)^*}{T^d} \frac{w_x(\lambda_s)^*}{T^d} \\
&= \frac{1}{2\pi} \sum_{t=1}^T \left(\frac{x_t}{T^d} \right)^2 - \sum_{s=m}^{T-1} \frac{w_x(\lambda_s) w_x(\lambda_s)^*}{T^d} \frac{w_x(\lambda_s)^*}{T^d} \\
&= \frac{1}{2\pi} \frac{1}{T} \sum_{t=1}^T \left(\frac{x_t}{T^d - 0.5} \right)^2 - \sum_{s=m}^{T-1} \frac{w_x(\lambda_s) w_x(\lambda_s)^*}{T^d} \frac{w_x(\lambda_s)^*}{T^d}.
\end{aligned} \tag{2.41}$$

Since $m/T^\alpha \rightarrow \infty$, I have

$$\begin{aligned}
\frac{1}{T^d} w_x(\lambda_s) &= \frac{1}{T^d} \left(1 - e^{i(\lambda_s - v)}\right)^d \left(1 - e^{i(\lambda_s + v)}\right)^d w_u(\lambda_s) \\
&\quad - \frac{1}{T^d} \frac{e^{i2\lambda_s}}{\left(1 - e^{i(\lambda_s + v)}\right) \left(1 - e^{i(\lambda_s - v)}\right)} \frac{x_T}{\sqrt{T}} + o_p \left(\frac{1}{T^d} \frac{\left(1 - e^{i(\lambda_s - v)}\right)^{-d}}{s^{1-d}} \right) \\
&= O_p \left(\frac{1}{m^d} \right)
\end{aligned}$$

uniformly for $s \geq m$. When m is such that $m/T^\alpha \rightarrow \infty$, it follows that

$$\frac{1}{T^d} w_x(\lambda_s) = o_p \left(\frac{1}{T^{\alpha d}} \right)$$

and then

$$\sum_{s=m}^{T-1} \frac{w_x(\lambda_s) w_x(\lambda_s)^*}{T^d} \frac{w_x(\lambda_s)^*}{T^d} = o_p \left(\frac{1}{T^{2\alpha d}} \right) = o_p(1)$$

for α chosen such $\alpha \geq 1/2d$. Then I have

$$\begin{aligned}
\frac{m}{T^{2d}} \widehat{f}_{xx}(0) &= \frac{1}{2\pi} \frac{1}{T} \sum_{t=1}^T \left(\frac{x_t}{T^{d-0.5}} \right)^2 + o_p(1) \\
&\rightarrow \frac{1}{2\pi} \int_0^1 B_{d-1}(r)^2 dr.
\end{aligned}$$

■

Chapter 3

Some Consequences of Aggregation in the Frequency Domain

3.1 Introduction

Aggregation of time series implies that a process is observed at a frequency lower than that it is generated at. In this chapter, I consider some problems raised when a time series is skip sampled, i.e., it is observed every m th period or temporally aggregated by cumulating m nonoverlapping neighboring observations. This method is frequently used in economics and finance studies, such as using end-of-month observations to analyze interest rate behavior (see, e.g. Granger and Siklos (1995)), and measuring stock market volatility by realized volatility which is essentially the temporal aggregation of squared high-frequency returns (see, e.g. Andersen and Bollerslev (1998)).

There are numerous papers on aggregation in the time domain (see, e.g. Silvestrini and Veredas (2008) for a recent survey), but little attention has been paid to the effects in the frequency domain. Souza (2005, 2007, 2008), in a series of papers, discusses the effect of aggregation and bandwidth selection in estimating long memory parameter. Ohanissian et al. (2008) propose a test to distinguish between true and spurious long memory by exploiting the invariance of the memory parameter from temporal aggregation of the series. Hassler (2011) investigates whether the typical frequency domain assumptions made for the semiparametric estimation and inference are closed with respect to aggregation.

This literature mainly relies on the spectrum which is not observed and may not be well defined. Instead of studying the spectrum of aggregated stationary series, I analyze the properties of the periodogram under aggregation for both stationary and nonstationary cases. More

specifically, I use the periodogram ordinates $I_x(\lambda_s)$ evaluated at the fundamental frequencies $\lambda_s = 2\pi s/T$, $s = 0, 1, \dots, T - 1$ as an approximation to the spectrum $f_x(\lambda_s)$ at frequency λ_s . The adoption of the periodogram has many advantages in practice. For example, unlike the spectrum, the periodogram can be directly used in many semiparametric estimation methods, such as the log periodogram and local Whittle estimators of the long memory parameter. Also, in the nonstationary case, the spectrum is not integrable, while the periodogram is still well defined. I derive the discrete Fourier transform (dft) and the periodogram of aggregated series in terms of the dft of the original series. Here, my results do not rely on any additional assumptions, i.e., the dft and the periodogram of the aggregated series can be obtained as long as the original dft exists. Such an approach enriches the literature on the treatment of aggregation, which is usually concerned with deriving the representation of the aggregated variable under certain assumptions, such as assuming the underlying variables to be some form of autoregressive moving average, or ARMA process (see, e.g. Wei (1978), Weiss (1984)).

Following results pertaining to the periodogram under aggregation, I then explore the interaction between aggregation and persistence, which is of great interest. It is documented that a process with a unit root at the zero frequency can arise because of the aggregation of series having a unit root at some seasonal frequency, as discussed by Granger and Siklos (1995). Many researchers, e.g. Hylleberg (1994), argue that, instead of being deterministic, seasonality is governed by stochastic trends. Hence, modeling seasonality has advantages compared to removing seasonality using some seasonal adjustment methods. This motivates me to study generalized long memory processes, which allow for singularities in the spectrum at any frequency, in particular at seasonal frequencies. Note that, for nonstationary processes, the spectrum can be defined in terms of the limit of the expectation of the periodogram, as in Solo (1992). The usual fractional processes are also discussed in this chapter as a special case.

Phillips (1999) gives a new representation for a fractional process in the frequency domain, that is particularly useful in analyzing the asymptotic behavior of the dft and the periodogram in the nonstationary case. In light of his results, I derive the dft of a generalized fractional process and give an exact representation of the dft via a useful components representation. A

long memory process typically has a spectral density function which is proportional to λ^{-2d} as λ goes to zero, where d is the memory parameter. The fractional model, proposed by Granger and Joyeux (1980) and Hosking (1981), is a long memory generalization of the ARMA model whose autocorrelations decay exponentially. When $d \in (0, 0.5)$, the autocorrelations decay very slowly, exhibiting a characteristic of stationary long memory processes. I consider the generalized fractional model, proposed by Gray, Zhang and Woodward (1989), which is based on Gegenbauer polynomials. A generalized fractional process typically has a singularity in the spectrum at a frequency v which may be different from zero, and its spectral density function is proportional to $|\lambda - v|^{-2d}$ as λ approaches v .

As in Solo (1992), I use the limit of the expectation of the periodogram to be the analogue of the spectrum in the nonstationary case. For both the original series before aggregation and the aggregated series, I analyze the limit of the expectation of the periodogram in the nonstationary case for generalized fractional processes. I explore the interaction between long memory and aggregation, based on the obtained limits. While my focus is on the case $d \in (0.5, 1)$, the methods introduced here are also applicable when $d > 1$. Since the limit of the expectation of the periodogram of a stationary variable is just the spectrum, the results obtained are also valid for the stationary case as well, i.e., $d \in (0, 0.5)$.

The remainder of this chapter is as follows. Section 2 derives the dft and the periodogram of the aggregated series in light of the aliasing effect. Section 3 derives the dft of a generalized fractional process, and analyzes the limit of the expectation of the periodogram of the aggregated series for a generalized fractional process in the nonstationary case. The interaction between the aggregation and persistence is also studied. Section 4 presents simulation results pertaining to the estimates of the memory parameter to demonstrate the practical relevance of my theoretical results. Section 5 offers concluding remarks. A mathematical appendix contains technical derivations.

3.2 Aggregation in the frequency domain

3.2.1 Notation

Let $\{x_t, t = 1, \dots, T\}$, denote the time series to be aggregated and m be the level of aggregation. Here, I adopt Tanaka's (1999) assumption that $x_t = 0$ for $t \leq 0$ and further assume that m is chosen such that $T = mS$ for some integer S . Here, aggregation implies that a process is observed at a frequency lower than at which it is generated, and includes skip sampling and temporal aggregation. For skip sampling, observations are recorded every m th period, while temporal aggregation refers to series that are the accumulation of m nonoverlapping neighboring observations. For stock variables, aggregation refers to skip sampling, i.e.,

$$\bar{x}_p^{(m)} = x_{mp}, \quad p = 1, \dots, S,$$

where p is the new time scale. In the case of flow variables, aggregation refers to m -component nonoverlapping sums:

$$\begin{aligned} \tilde{x}_p^{(m)} &= \sum_{j=0}^{m-1} x_{mp-j} \\ &= G(L, m)x_{mp}, \quad p = 1, \dots, S, \end{aligned}$$

where L is the backshift operator and $G(L, m) = \sum_{j=0}^{m-1} L^j$. Hence, $\{\tilde{x}_p^{(m)}\}$ is obtained by skip sampling the overlapping moving average process $\{G(L, m)x_t\}$, i.e.

$$\tilde{x}_p^{(m)} = \bar{z}_p^{(m)},$$

where

$$z_t = G(L, m)x_t.$$

These definition are standard in the literature on temporal aggregation (see, e.g. Souza (2008) and Hassler (2011)).

Souza (2005) and Hassler (2011) derive the spectral density function of the aggregation of long memory variables. However, the spectrum cannot be directly observed in empirical

research. In practice, the periodogram is used as an approximation to the spectrum. Furthermore, the spectrum $f_x(\lambda)$ at frequency λ exists only if x_t is stationary. In the nonstationary case, it is not integrable, but it is still feasible to study the periodogram ordinates $I_x(\lambda_s)$ evaluated at the fundamental frequencies $\lambda_s = 2\pi s/T$, $s = 0, 1, \dots, T-1$. Therefore, instead of examining the spectrum, I analyze the properties of the periodogram under aggregation. As usual, the periodogram is defined as

$$I_x(\lambda_s) = w_x(\lambda_s) w_x(\lambda_s)^* \quad (3.1)$$

where $w_x(\lambda_s)$ is the dft

$$w_x(\lambda_s) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^T x_t e^{it\lambda_s} \quad (3.2)$$

of $\{x_t\}$, and c^* denotes the complex conjugate value of any complex number c .

The following notation is used throughout: " \rightarrow " stands for the limit as T goes to infinity; " $x \sim y$ " means that $x/y \rightarrow 1$.

3.2.2 Results and discussion

As shown in e.g., Granger and Siklos, (1995), Souza, (2005) and Hassler, (2011), skip sampling causes the well-known aliasing phenomenon for a continuous-time processes observed at a discrete-time interval. It follows that the dft of the skip sampled series equals the sum of the dfts of the original series at the aliased frequencies divided by \sqrt{m} . This result is stated as follows.

Lemma 3.1 (*Aliasing*) *The dft of the skip sampled series $\{\bar{x}_p^{(m)}\}$, is given by:*

(a) *If m is an odd number:*

$$w_{\bar{x}}(\lambda_s) = \frac{1}{\sqrt{m}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} w_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right);$$

(b) If m is an even number:

$$w_{\bar{x}}(\lambda_s) = \begin{cases} \frac{1}{\sqrt{m}} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), & -\pi < \lambda_s \leq 0 \\ \frac{1}{\sqrt{m}} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), & 0 < \lambda_s \leq \pi. \end{cases}$$

Lemma 3.1 gives the relationship between the dft of the original series and the skip sampled one. Roughly speaking, the dft after skip sampling can be obtained by folding the original dft m times and then dividing the sum by \sqrt{m} . Here, the sum of the dfts of the original series is divided by \sqrt{m} instead of m , because there is a prefactor $1/\sqrt{2\pi T}$ in the definition of the dft, as shown in Equation (3.2). After aggregation, the total number of observations is T/m , and the prefactor becomes $1/\sqrt{2\pi T/m}$. Hence, the ratio of the original prefactor and the prefactor after aggregation is $1/\sqrt{m}$. Regardless of the prefactor, the aliasing effect holds in Equation (3.2), i.e.

$$S(\lambda_s) = \sum_{t=1}^T x_t e^{it\lambda_s}$$

exhibits an aliasing effect.

Inserting $w_{\bar{x}}(\lambda_s)$ obtained in Lemma 3.1 into (3.1), one can obtain the periodogram of the skip sampled series, $I_{\bar{x}}(\lambda_s)$, as a function of the periodogram and the dft of the original series. The outcome is formalized in the following theorem.

Theorem 3.1 *The periodogram of the skip sampled series $\{\bar{x}_p^{(m)}\}$, is given by*

(a) If m is an odd number:

$$I_{\bar{x}}(\lambda_s) = \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} I_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) + \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k \neq j} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* ;$$

(b) If m is an even number:

$$I_{\bar{x}}(\lambda_s) = \begin{cases} \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} I_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) + \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} \sum_{k \neq j} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* & \text{if } -\pi < \lambda_s < 0, \\ \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} I_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) + \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} \sum_{k \neq j} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* & \text{if } 0 < \lambda_s \leq \pi. \end{cases}$$

Using the periodogram instead of the spectrum has many advantages. First, the periodogram is used in many semiparametric estimation methods, such as the log periodogram and local Whittle estimators of the long memory parameter. Second, in the nonstationary case, the spectrum does not exist, while the periodogram is still well defined and can be used in estimation. In the stationary case, the analysis of the spectrum can be considered as a special case, as stated in the following remark.

Remark 3.1 *When $\{x_t\}$ is stationary and ergodic, as discussed by Peligrad and Wu (2010), the spectrum is the limit of the expectation of the periodogram. In the limit, w_x are independent and identically normally distributed random variable, and the limit of the expectation of the second term in Theorem 3.1 goes to zeros. The limit of the expectation of the periodogram, i.e., the spectrum, for the aggregated series $\{\bar{x}_t\}$ is given by:*

(a) *If m is an odd number:*

$$\begin{aligned} \lim_{T \rightarrow \infty} E [I_{\bar{x}}(\lambda_s)] &= f_{\bar{x}}(\lambda_s) \\ &= \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} f_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right); \end{aligned}$$

(b) *If m is an even number:*

$$\begin{aligned} \lim_{T \rightarrow \infty} E [I_{\bar{x}}(\lambda_s)] &= f_{\bar{x}}(\lambda_s) \\ &= \begin{cases} \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} f_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), & -\pi < \lambda_s < 0 \\ \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} f_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), & 0 < \lambda_s \leq \pi. \end{cases} \end{aligned}$$

This special case of my results was also considered in Souza (2005) and Hassler (2011).

The periodogram under temporal aggregation is also subject to aliasing. The temporally aggregated variable, $\{\hat{x}_p^{(m)}\}$, is obtained by skip sampling the overlapping moving average process $\{G(L, m)x_t\}$. Hence, in order to obtain the periodogram of the temporally aggregated series, one needs first to calculate the dft of the overlapping moving average process $\{G(L, m)x_t\}$. It is convenient to manipulate the operator $G(L, m)$ in a form that more readily accommodates dfts. I expand the polynomial operator, $G(L, m)$, about its value at the complex exponential $e^{i\lambda}$, as in Solo (1992), leading to the following decomposition.

Lemma 3.2

$$G(L, m) = G(e^{i\lambda}, m) + \tilde{G}_\lambda(e^{-i\lambda}L, \lambda, m) \left(e^{-i\lambda}L - 1 \right), \quad (3.3)$$

where

$$\tilde{G}(e^{-i\lambda}L, \lambda, m) = \sum_{k=0}^{m-2} \tilde{g}_{k\lambda} e^{-i\lambda k} L^k, \quad \tilde{g}_{k\lambda} = \sum_{s=k+1}^{m-1} e^{i\lambda s}.$$

The representation (3.3) is an immediate consequence of formula (32) in Phillips and Solo (1992). Using the operator (3.3), I rewrite the overlapping moving average process z_t in the following form for all $t \leq T$

$$\begin{aligned} z_t &= G(L, m)x_t \\ &= G(e^{i\lambda}, m)x_t + \tilde{G}(e^{-i\lambda}L, \lambda, m) \left(e^{-i\lambda}L - 1 \right) x_t. \end{aligned} \quad (3.4)$$

Taking the dft of both sides of equation (3.4) now yields an exact expression of w_z in terms of w_x . The result is given in the following Lemma.

Lemma 3.3 *The dft of the overlapping moving average process $\{G(L, m)x_t\}$ is*

$$w_z(\lambda) = w_x(\lambda) G(e^{i\lambda}, m) - \frac{e^{iT\lambda}}{\sqrt{2\pi T}} X_T^G(\lambda, m), \quad (3.5)$$

where $G(e^{i\lambda}, m) = \sum_{k=0}^{m-1} e^{ik\lambda}$,

$$X_T^G(\lambda, m) = \tilde{G}(e^{-i\lambda}L, \lambda, m)x_T$$

and

$$\tilde{G}(e^{-i\lambda}L, \lambda, m) = \sum_{k=0}^{m-2} \tilde{g}_{k\lambda} e^{-i\lambda k} L^k, \quad \tilde{g}_{k\lambda} = \sum_{s=k+1}^{m-1} e^{i\lambda s}.$$

Equation (3.5) provides an exact representation of $w_z(\lambda)$ in terms of $w_x(\lambda)$ and a residual component involving $X_T^G(\lambda, m)$. Then, using Lemma 3.1 pertaining to the sum of the dfts of the original series at the aliased frequencies, I obtain the dft of the skip sampled series $\{G(L, m)x_t\}$, i.e., the dft of the temporal aggregation of $\{x_t\}$, as follows.

Lemma 3.4 (Aliasing) *The dft of the temporally aggregated series $\{\tilde{x}_p^{(m)}\}$ is given by:*

(a) *If m is an odd number:*

$$w_{\tilde{x}}(\lambda_s) = \frac{1}{\sqrt{m}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \left[G\left(e^{i\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right)}, m\right) w_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) - \frac{e^{iS\lambda_s}}{\sqrt{2\pi T}} X_T^G\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m\right) \right];$$

(b) *If m is an even number:*

$$w_{\tilde{x}}(\lambda_s) = \begin{cases} \frac{1}{\sqrt{m}} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} \left[G\left(e^{i\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right)}, m\right) w_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) - \frac{e^{iS\lambda_s}}{\sqrt{2\pi T}} X_T^G\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m\right) \right] \\ \text{if } -\pi < \lambda_s \leq 0, \\ \frac{1}{\sqrt{m}} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} \left[G\left(e^{i\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right)}, m\right) w_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) - \frac{e^{iS\lambda_s}}{\sqrt{2\pi T}} X_T^G\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m\right) \right] \\ \text{if } 0 < \lambda_s \leq \pi. \end{cases}$$

Inserting $w_{\tilde{x}}(\lambda_s)$ obtained in Lemma 3.4 into (3.1), I obtain the periodogram of the skip sampled series $\{G(L, m)x_t\}$, i.e., the periodogram of the temporal aggregation of $\{x_t\}$, as follows.

Theorem 3.2 *The periodogram of the temporal aggregation of $\{x_t\}$ is given by:*

(a) *If m is an odd number:*

$$\begin{aligned} I_{\tilde{x}}(\lambda_s) &= \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} T_m\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) I_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) \\ &+ \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k \neq j} G\left(e^{i\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right)}, m\right) G\left(e^{-i\left(\frac{\lambda_s}{m} + \frac{2k\pi}{m}\right)}, m\right) \\ &\times w_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) w_x\left(\frac{\lambda_s}{m} + \frac{2k\pi}{m}\right)^* \\ &+ \frac{1}{2\pi m T} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} X_T^G\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m\right) X_T^G\left(\frac{\lambda_s}{m} + \frac{2k\pi}{m}, m\right)^* \\ &- \frac{2}{m\sqrt{2\pi T}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} RE \left[\left(e^{iS\lambda_s} X_T^G\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m\right) \right) \right. \\ &\left. \times w_x\left(\frac{\lambda_s}{m} + \frac{2k\pi}{m}\right)^* G\left(e^{-i\left(\frac{\lambda_s}{m} + \frac{2k\pi}{m}\right)}, m\right) \right]; \end{aligned}$$

(b) If m is an even number: when $-\pi < \lambda_s \leq 0$,

$$\begin{aligned}
& I_{\tilde{x}}(\lambda_s) \\
= & \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} T_m \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) I_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) \\
& + \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} \sum_{k \neq j} G \left(e^{i \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right)}, m \right) G \left(e^{-i \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)}, m \right) \\
& \times w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* \\
& + \frac{1}{2\pi m T} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} \sum_{k=-\frac{m}{2}+1}^{\frac{m}{2}} X_T^G \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m \right) X_T^G \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m}, m \right)^* \\
& - \frac{2}{m\sqrt{2\pi T}} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} \sum_{k=-\frac{m}{2}+1}^{\frac{m}{2}} RE \left[\left(e^{iS\lambda_s} X_T^G \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m \right) \right) \right. \\
& \left. \times w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* G \left(e^{-i \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)}, m \right) \right];
\end{aligned}$$

when $0 < \lambda_s \leq \pi$,

$$\begin{aligned}
& I_{\tilde{x}}(\lambda_s) \\
= & \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} T_m \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) I_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) \\
& + \frac{1}{2\pi m T} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} \sum_{k \neq j} G \left(e^{i \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right)}, m \right) G \left(e^{-i \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)}, m \right) \\
& \times w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* \\
& + \frac{1}{2\pi m T} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} \sum_{k=-\frac{m}{2}}^{\frac{m}{2}-1} X_T^G \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m \right) X_T^G \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m}, m \right)^* \\
& - \frac{2}{m\sqrt{2\pi T}} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} \sum_{k=-\frac{m}{2}}^{\frac{m}{2}-1} RE \left[\left(e^{iS\lambda_s} X_T^G \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}, m \right) \right) \right. \\
& \left. \times w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* G \left(e^{-i \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)}, m \right) \right],
\end{aligned}$$

where

$$T_m(\lambda) = \frac{\sin^2\left(\frac{m\lambda}{2}\right)}{\sin^2\left(\frac{\lambda}{2}\right)}$$

and $T_m(\lambda)/2\pi T$ is the so-called Fejer Kernel, as discussed in Souza (2005).

The analysis of the spectrum for temporal aggregation can be obtained as a special case, as stated in the following remark.

Remark 3.2 *When the process is stationary, except for the first term in the expressions of $I_{\tilde{x}}(\lambda_s)$ in Theorem 3.2, the expectations of all other terms go to zeros as the sample size goes to infinity. Hence, the limit of the expectation of the periodogram for the aggregated series $\{\tilde{x}_t\}$ is given by:*

(a) *If m is an odd number:*

$$\begin{aligned} \lim_{T \rightarrow \infty} E[I_{\tilde{x}}(\lambda_s)] &= f_{\tilde{x}}(\lambda_s) \\ &= \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} T_m\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) f_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right); \end{aligned}$$

(b) *If m is an even number: when $-\pi < \lambda_s < 0$,*

$$\begin{aligned} \lim_{T \rightarrow \infty} E[I_{\tilde{x}}(\lambda_s)] &= f_{\tilde{x}}(\lambda_s) \\ &= \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} T_m\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) f_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right); \end{aligned}$$

when $0 < \lambda_s \leq \pi$,

$$\begin{aligned} \lim_{T \rightarrow \infty} E[I_{\tilde{x}}(\lambda_s)] &= f_{\tilde{x}}(\lambda_s) \\ &= \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} T_m\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right) f_x\left(\frac{\lambda_s}{m} + \frac{2j\pi}{m}\right), \end{aligned}$$

where

$$T_m(\lambda) = \frac{\sin^2\left(\frac{m\lambda}{2}\right)}{\sin^2\left(\frac{\lambda}{2}\right)}.$$

In this case, the term $T_m(\lambda)$ appears in the expressions of the limit of the expectation, which is not present under skip sampling, because \tilde{x}_t is constructed from temporal aggregation, i.e.,

skip sampling a series of moving averages. The moving average induces $T_m(\lambda)$ in the frequency domain.

Theorems 2.1 and 2.2 provide the periodogram for the skip sampling and the temporal aggregation of some series $\{x_t\}$. Souza (2005, 2007 and 2008) and Hassler (2011) study the spectral density of the aggregation of a stationary long memory series and investigate whether the typical frequency domain assumptions made for semiparametric estimation and inference still hold after aggregation. Hence, studying the periodogram allows to investigate semiparametric estimation and inference with respect to aggregation. This is, however, outside the scope of this chapter and will be discussed elsewhere.

3.3 Discrete fourier transforms of generalized fractional processes

This section explores the interaction between aggregation and persistence, based on the results obtained about the periodogram under aggregation. I aim at examining whether aggregating series which possess a long memory property at some seasonal frequency could yield a process with long memory at frequency zero. The model for the generalized long memory processes includes the case with seasonality when a singularity in the spectrum occurs at a frequency different from zero.

Consider the following generalized fractional process $\{x_t\}$ generated by:

$$(1 - 2\eta L + L^2)^d x_t = u_t, \quad (3.6)$$

where $|\eta| \leq 1$. Throughout, I suppose that $u_t = 0$ for all $t \leq 0$ and more explicit conditions on u_t are given in the following assumption.

Assumption 3.1 (*Error Condition*) For all $t > 0$, u_t has Wold representation

$$u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j c_j < \infty, \quad C(1) \neq 0,$$

where $\varepsilon_t \sim i.i.d. (0, \sigma^2)$ with $E(|\varepsilon_t|^p) < \infty$ for some $p > 4$.

Note that when $\eta = 1$, the generalized fractionally integrated process reduces to the stan-

ard fractional process

$$(1 - L)^{2d} x_t = u_t, \quad t = 1, \dots, T. \quad (3.7)$$

When $|d| < 0.5$ and $|\eta| < 1$, $\{x_t\}$ is a stationary process and its spectral density is given by

$$f_x(\lambda) = \left| 4 \sin\left(\frac{\lambda + \nu}{2}\right) \sin\left(\frac{\lambda - \nu}{2}\right) \right|^{-2d} f_u(\lambda), \quad (3.8)$$

where $\nu = \cos^{-1}(\eta)$ and f_u is the spectral density of $\{u_t\}$. When $|d| < 1/4$ and $\eta = 1$, the spectrum of $\{x_t\}$ has the following form

$$f_x(\lambda) = \left| 1 - e^{i\lambda} \right|^{-4d} f_u(\lambda). \quad (3.9)$$

I shall focus on the case where $\{x_t\}$ is nonstationary, i.e., $0.5 < d < 1$ and $|\eta| < 1$ or $0.25 < d < 0.5$ and $\eta = 1$. In these cases, the spectrum does not exist because it is not integrable. However, when $\eta = 1$ and $1/4 < d < 3/4$, Solo (1992) provides a formal justification of $f_x(\lambda)$ as specified in (3.9) as a spectrum in terms of the limit of the expectation of the periodogram for fractional processes of the form (3.7). As will be shown in the next section, $f_x(\lambda)$ in (3.8) is the limit of the expectation of the periodogram for generalized fractional processes specified by (3.6).

An alternative expression for x_t is obtained by the inversion of (3.6), giving

$$x_t = (1 - 2\eta L - L^2)^{-d} u_t = \sum_{n=0}^t c_n(\eta, d) u_{t-n},$$

where

$$c_n(\eta, d) = \frac{\Gamma(d + 0.5) \Gamma(2d + n)}{n! \Gamma(2d)} \left(\frac{1 - \eta^2}{4} \right)^{0.25 - \lambda/2} P_{k+d-0.5}^{0.5-d}(\eta),$$

and $P_k^\mu(\eta)$ is the Legendre function of the first kind. See, for example, Gray, Zhang and Woodward (1989) and Chung (1996). As stated in formula (9) of Chung (1996), the coefficients in the power-series expansion of $(1 - 2\eta L - L^2)^{-d}$ can be approximated by

$$c_n \sim \frac{\cos[(n + d)v - (d\pi/2)]}{\Gamma(d) \sin^d(v)} \left(\frac{2}{n} \right)^{1-d} = O_p(n^{d-1}) \quad \text{as } n \rightarrow \infty.$$

Therefore, when $d < 1$, the coefficients decrease at a hyperbolic rate as $n \rightarrow \infty$, which is the same rate as for fractional processes.

3.3.1 Frequency domain decomposition

Phillips (1999) gives a new representation in the frequency domain for fractional processes to analyze the asymptotic behavior of the dft in the nonstationary case. In light of his work, I propose a new frequency domain representation of the dft via a useful components representation. For the case with $\eta = 1$ and $1/4 < d < 1/2$, the dft has been studied by Phillips (1999). Here, I primarily examine the case $0.5 < d < 1$ when $|\eta| < 1$.

Expanding $(1 - 2\eta L - L^2)^{-d}$ in Equation (3.6), I have

$$\sum_{l=0}^T \frac{(-d)_l}{l!} (e^{-ivL})^l \sum_{m=0}^T \frac{(-d)_m}{m!} (e^{ivL})^m x_t = u_t. \quad (3.10)$$

It is convenient to define

$$D(d, v, L) = \sum_{l=0}^T \frac{(-d)_l}{l!} (e^{-ivL})^l \sum_{m=0}^T \frac{(-d)_m}{m!} (e^{ivL})^m$$

and let

$$D(d, v, L) = D_1(d, v, L) D_2(d, v, L),$$

where

$$D_1(d, v, L) = \sum_{k=0}^T \frac{(-d)_k}{k!} (e^{-ivL})^k$$

and

$$D_2(d, v, L) = \sum_{k=0}^T \frac{(-d)_k}{k!} (e^{ivL})^k.$$

I expand the polynomial operator about its value at the complex exponential $e^{i\lambda}$ and get the following decomposition.

Lemma 3.5

$$\begin{aligned}
D(d, v, L) &= D(d, v, e^{i\lambda}) + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1)^2,
\end{aligned} \tag{3.11}$$

where

$$\begin{cases} \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) = \sum_{p=0}^{T-1} \tilde{d}_{1\lambda p} e^{-ip\lambda} L^p \\ \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) = \sum_{p=0}^{T-1} \tilde{d}_{2\lambda p} e^{-ip\lambda} L^p \end{cases}$$

and

$$\tilde{d}_{1\lambda p} = \sum_{k=p+1}^T \frac{(-d)_k}{k!} e^{ik(\lambda-v)}, \quad \tilde{d}_{2\lambda p} = \sum_{k=p+1}^T \frac{(-d)_k}{k!} e^{ik(\lambda+v)}.$$

Lemma 3.5 is an immediate consequence of formula (32) in Phillips and Solo (1992) and Phillips (1999). Using equation (3.11), I obtain the following representation for u_t

$$\begin{aligned}
u_t &= D(d, v, L) x_t \\
&= D(d, v, e^{i\lambda}) x_t + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1)^2 x_t.
\end{aligned} \tag{3.12}$$

Taking the dft of both sides of equation (3.12) now yields an exact expression of w_u in terms of w_x .

Lemma 3.6

$$w_u(\lambda) = w_x(\lambda) D(d, v, e^{i\lambda}) + \frac{1}{\sqrt{2\pi T}} X_T^D(v, d, \lambda), \tag{3.13}$$

where

$$\begin{aligned}
X_T^D(v, d, \lambda) &= D_2(d, v, e^{i\lambda}) \left(X_0^{D_1}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1}(d, v, \lambda) \right) \\
&\quad + D_1(d, v, e^{i\lambda}) \left(X_0^{D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_2}(d, v, \lambda) \right) \\
&\quad + \left(e^{-i\lambda} X_0^{D_1 D_2}(d, v, \lambda) - e^{i(T-1)\lambda} X_{T-1}^{D_1 D_2}(d, v, \lambda) \right) \\
&\quad - \left(e X_0^{D_1 D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1 D_2}(d, v, \lambda) \right)
\end{aligned}$$

and

$$\begin{aligned}
X_0^{D1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_0; & X_T^{D1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_T; \\
X_0^{D2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_0; & X_T^{D2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_T; \\
X_0^{D1D2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_0; \\
X_{T-1}^{D1D2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_{T-1}; \\
X_T^{D1D2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_T.
\end{aligned}$$

Using Lemma 3.6, I have

$$w_x(\lambda) = D^{-1}(d, v, e^{i\lambda}) w_u(\lambda) + \frac{e^{iT\lambda}}{\sqrt{2\pi T}} D^{-1}(d, v, e^{i\lambda}) X_T^D(d, v, \lambda). \quad (3.14)$$

The representation (3.14) holds for all fundamental frequencies $\lambda_s = 2\pi s/T$. This components representation is helpful in deriving the asymptotic representation of $w_x(\lambda_s)$. Here, I focus on the case with $\lambda_s \rightarrow \phi \neq v$.

In terms of the periodogram ordinates, I have the corresponding representation

$$\begin{aligned}
I_x(\lambda_s) &= |w_x(\lambda)|^2 & (3.15) \\
&= \left| D^{-1}(d, v, e^{i\lambda}) w_u(\lambda) + \frac{e^{iT\lambda}}{\sqrt{2\pi T}} D^{-1}(d, v, e^{i\lambda}) X_T^D(v, d, \lambda) \right|^2 \\
&= \left| D(d, v, e^{i\lambda}) \right|^{-2} \left[I_u(\lambda_s) - 2RE \left\{ \frac{1}{\sqrt{2\pi T}} X_T^D(v, d, \lambda) w_u(\lambda) \right\} \right. \\
&\quad \left. + \frac{1}{2\pi T} |X_T^D(v, d, \lambda)|^2 \right]
\end{aligned}$$

which is the periodogram used in log periodogram regression to estimate the long memory parameter. Following the discussion in Phillips (1999), $|D(d, v, e^{i\lambda})|^{-2}$ can be replaced by $|(1 - e^{i(\lambda+v)})(1 - e^{i(\lambda-v)})|^{-2d}$. However, the residual component $T^{-1/2} X_T^D(v, d, \lambda)$ cannot be neglected in general and its importance grows as d increases.

3.3.2 Asymptotic approximations

In this subsection, I develop some asymptotic approximations to simplify the representation of $D(d, v, e^{i\lambda})$ and $X_T^D(d, v, \lambda)$ in (3.14). The asymptotic representations are summarized in the following lemma.

Lemma 3.7 *Suppose $d > 0$. For any fixed $\lambda \neq v$:*

$$\begin{aligned} & D(d, v, e^{i\lambda}) \\ = & \left(1 - e^{i(\lambda+v)}\right)^d \left(1 - e^{i(\lambda-v)}\right)^d \\ & - \frac{1}{\Gamma(-d) T^{1+d}} \left[\frac{e^{iT(\lambda+v)} \left(1 - e^{i(\lambda-v)}\right)^d}{1 - e^{i(\lambda+v)}} + \frac{e^{iT(\lambda-v)} \left(1 - e^{i(\lambda+v)}\right)^d}{1 - e^{i(\lambda-v)}} \right] \left[1 + O\left(\frac{1}{T}\right)\right] \\ & + \frac{1}{\Gamma(-d)^2 T^{2+2d}} \left[\frac{e^{i2T\lambda}}{\left(1 - e^{i(\lambda+v)}\right) \left(1 - e^{i(\lambda-v)}\right)} \right] \left[1 + O\left(\frac{1}{T}\right)\right]. \end{aligned}$$

and

$$\frac{X_T^D(d, v, \lambda)}{\sqrt{T}} = O_p\left(\frac{1}{T^{1-d}}\right).$$

Then, one can easily derive the following asymptotic representations. For $\lambda \rightarrow \phi \neq v$, from Lemma 3.7, I have

$$D(d, v, e^{i\lambda}) = \left(1 - e^{i(\lambda+v)}\right)^d \left(1 - e^{i(\lambda-v)}\right)^d + O\left(\frac{1}{T^{1+d}}\right)$$

uniformly for $\lambda_s \in B_\phi = \{\phi - \pi/M, \phi + \pi/M\}$ where $M \rightarrow \infty$ as $T \rightarrow \infty$. Similarly,

$$\frac{X_T^D(\lambda, \eta, d)}{\sqrt{T}} = O_p\left(\frac{1}{T^{1-d}}\right)$$

uniformly for $\lambda_s \in B_\phi$. It follows that

Lemma 3.8

$$w_x(\lambda_s) = \left(1 - e^{i(\lambda_s-v)}\right)^{-d} \left(1 - e^{i(\lambda_s+v)}\right)^{-d} w_u(\lambda_s) + O_p\left(\frac{1}{T^{1-d}}\right) \quad (3.16)$$

uniformly for $\lambda_s \in B_\phi$.

The following limit results applies.

Lemma 3.9 *Suppose $d \in (0.5, 1)$. Let $\phi \neq v$ and $\lambda_{s_j} \in B_\phi = \{\phi - \pi/M, \phi + \pi/M\}$ for a finite set of distinct integers s_j ($j = 1, \dots, J$). When $M \rightarrow \infty$ as $T \rightarrow \infty$, the elements of $\{w_x(\lambda_{s_j})\}_{j=1}^J$ are asymptotically independently distributed as complex normal $N_c(0, f_x(\phi))$ where $f_x(\phi) = |1 - 2\eta e^{i\phi} + e^{2i\phi}|^{-2d} f_u(\phi)$.*

Lemma 3.9 extends Phillips' (1999) result for the limit theory of the dft of nonstationary fractional processes to the case of nonstationary generalized fractional processes. Solo (1992) defines the limit of the expectation of the periodogram as a spectrum in the nonstationary case, which motivates me to study this limit.

Theorem 3.3 *The limit of the expectation of the periodogram of $\{x_t\}$ is given by*

$$\begin{aligned} f_x(\lambda) &= \lim_{T \rightarrow \infty} E[I(\lambda_s)] \\ &= \begin{cases} |4 \sin\left(\frac{\lambda+\nu}{2}\right) \sin\left(\frac{\lambda-\nu}{2}\right)|^{-2d} f_u(\lambda), & \text{for } |\eta| < 1 \text{ and } 1/2 < d < 1; \\ |\sin\left(\frac{\lambda}{2}\right)|^{-4d} f_u(\lambda), & \text{for } \eta = 1 \text{ and } 1/4 < d < 1/2. \end{cases} \end{aligned}$$

In Theorem 3.3, $f_x(\lambda)$ has the same expression as the spectrum in the stationary case. Solo (1992) proposes a justification of $f_x(\lambda)$ as the limit of the expectation of the periodogram for fractional processes. My results extend the case of Solo (1992) in the sense that $f_x(\lambda)$ can be also interpreted as a spectrum in terms of the limit of the expectation of the periodogram for generalized fractional processes with $1/2 < d < 1$.

3.3.3 Interaction between aggregation and persistence

In the nonstationary case, the memory parameter cannot be defined via the spectrum. which is not integrable. Hence, I define the memory parameter in terms of the limit of the expectation of the periodogram as follows.

Definition 3.1 *Let $\{x_t\}$ be a nonstationary process such that*

$$\lim_{T \rightarrow \infty} E[I_x(\lambda - \lambda_0)] \sim |\lambda - \lambda_0|^{-2d(\lambda_0)} f_e(\lambda) \quad \text{as } \lambda \rightarrow 0$$

where f_e is assumed to be continuous, bounded above, and bounded away from zero and $\lambda, \lambda_0 \in (0, \pi]$. Then $d(\lambda_0)$ is the memory parameter of $\{x_t\}$ at frequency λ_0 .

I shall focus on $d > 0$, which is the case of interest in economic and business applications. In order to study the interaction between aggregation and persistence, I derive results about the limit of the expectation of the periodogram under aggregation.

Theorem 3.4 Under the data generation process (3.6), the limit of the expectation of the periodogram of aggregated series is given by

(a) If m is an odd number:

$$\lim_{T \rightarrow \infty} E[I(\lambda_s)] = \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} T_m \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) f_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right);$$

(b) If m is an even number:

$$\lim_{T \rightarrow \infty} E[I(\lambda_s)] = \begin{cases} \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} T_m \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) f_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), \\ \quad \text{if } -\pi < \lambda_s < 0, \\ \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} T_m \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) f_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), \\ \quad \text{if } 0 < \lambda_s \leq \pi, \end{cases}$$

where

$$f_x(\lambda) = \begin{cases} |4 \sin\left(\frac{\lambda+\nu}{2}\right) \sin\left(\frac{\lambda-\nu}{2}\right)|^{-2d} f_u(\lambda), & \text{for } |\eta| < 1 \text{ and } 1/2 < d < 1, \\ |\sin\left(\frac{\lambda}{2}\right)|^{-4d} f_u(\lambda), & \text{for } \eta = 1 \text{ and } 1/4 < d < 1/2 \end{cases}$$

and

$$T_m(\lambda) = \begin{cases} 1, & \text{in case of skip sampling;} \\ \frac{\sin^2\left(\frac{m\lambda}{2}\right)}{\sin^2\left(\frac{\lambda}{2}\right)}, & \text{in case of temporal aggregation.} \end{cases}$$

The results in Theorem 3.4 are similar to those obtained by Souza (2005) and Hassler (2011), although they study the spectral density of stationary variables. Theorem 3.4 can be extended to cover cases with multiple singularities, as discussed by Ooms (1995).

Based on the result for the limit of the expectation of periodogram, I am able to examine the interaction between aggregation and persistence.

Theorem 3.5 Under the data generation process (3.6), for the skip sampling case, the memory parameter at frequency zero satisfies

$$d(0) = \max_m \{d(\lambda_m)\}$$

where $\lambda_m = 0, \pm 2\pi/m, \pm 4\pi/m, \dots$ and $|\lambda_m| \leq \pi$. For the temporal aggregation case, the memory parameter at frequency zero is not affected by the memory parameters at other frequencies.

It is clear from Theorem 3.5 that if the memory parameter of the original series is d at frequency zero, the memory parameter of the skip sampled series is $d^* \geq d$ for the generalized fractional process and $d^* = d$ for a fractional process. However, temporal aggregation does not affect the memory parameter at frequency zero.

3.4 Simulations

In this section, I introduce the log-periodogram (LP) regression, and then simulate and estimate the memory parameter of the fractional series and the generalized fractional series to demonstrate the practical relevance of my theoretical results.

3.4.1 Log-periodogram based estimator

Various estimators of the memory parameter d have been proposed, among which the semiparametric estimators became widely used because they do not require distributional assumptions on the original series. The most popular semiparametric estimator is the log-periodogram (LP) regression estimator proposed by Geweke and Porter-Hudak (1983), which uses only frequencies near zero to avoid possible misspecification caused by high-frequency movements. The LP estimator has been analyzed by, among others, Robinson (1995). Perron and Qu (2007 and 2010) present theoretical results about the limit distributions of the autocorrelation function and the periodogram as well as the LP estimate of d in the presence of random level shifts. We can trim some of the lower frequencies, as discussed in McCloskey and Perron (2013), to obtain consistency and asymptotic normality with the same limiting variance as the standard LP regression estimator regardless of whether the underlying long/short-memory process is contaminated by level shifts or deterministic trends.

The LP regression estimator of the memory parameter d proposed by Geweke and Porter-Hudak (1983) is based on the spectral characterization of a long memory process which implies the following relationship:

$$\log f(\lambda) \approx c - 2d \log \lambda,$$

as $\lambda \rightarrow 0_+$, where f is the spectral density function of the process. The periodogram of the time series at frequency λ_j is defined as

$$I_x(\lambda_j) \equiv |w_x(\lambda_j)|^2 = w_x(\lambda_j)w_x(\lambda_j)^*,$$

where $w_x(\lambda_j)$ is the dft of the time series $\{x_t\}_{t=1}^T$ evaluated at the Fourier frequency $\lambda_j = 2\pi j/T$. The LP regression can be obtained by

$$\log I_x(\lambda_j) = c + dX_j + e_j, \quad j = l, \dots, \tau,$$

where $X_j = -\log(2 - 2\cos(\lambda_j)) \approx -\log \lambda_j^2$ for $j = l, \dots, \tau$. The LP estimator is

$$\hat{d} = \frac{-0.5 \sum_{j=l}^{\tau} (Y_j - \bar{Y}) \log I_j}{\sum_{j=l}^{\tau} (Y_j - \bar{Y})^2},$$

where $Y_j = \log |1 - \exp(-i\lambda_j)|$ and $\bar{Y} = (1/(\tau - l + 1)) \sum_{k=l}^{\tau} Y_k$. When $l = 1$, it corresponds to the standard LP estimator. Here, I use $l = 10$ because the distribution of the periodogram is abnormal around frequency zero, as discussed by Robinson (1995).

Velasco (1999) extends the LP regression to the nonstationary case and shows, under Gaussianity, the consistency of a LP estimator obtained by trimming low-frequency ordinates when $d \in (0.5, 1)$. Kim and Phillips (2006) and Phillips (2007) consider the unit root case where $d = 1$.

3.4.2 Fractional integration

I consider the fractional process $\{x_t\}$ generated by

$$(1 - L)^d x_t = u_t, \quad t = 1, \dots, T,$$

where u_t follows an ARMA(1,1) process

$$(1 + aL)u_t = (1 + bL)e_t$$

and $e_t \sim i.i.d.N(0, 1)$.

Figure 3.1 plots the limit of the expectation of the periodogram for the skip sampled series and the periodogram averaged over 100 realizations of the process, as a function of the frequency index. The total number of observations for the original series is $T = 2048$. The aggregation degree is $m = 4$. Thus, the length of the aggregated series is 512. The aggregated series are generated using (a) $d = 0.6$, $a = b = 0$, (b) $d = 0.8$, $a = b = 0$, (c) $d = 0.8$, $a = 0.8$, $b = 0$ and (d) $d = 0.8$, $a = 0.5$, $b = 0$. Figure 3.2 shows the corresponding limit and periodogram for the temporally aggregated series. As one can see from the figures, for both skip sampling and temporal aggregation, the averaged periodograms are scattered around the solid lines, representing the limits of the expectations of the periodograms.

Table 3.1 shows the LP estimates averaged over 100 realizations of the process for skip sampling. Here, I use $\tau = T^{0.8}$ for the LP regression. The total number of observations for the original series is $T = 8192$. The aggregation degrees are $m = 1, 2, 4, 8$ and 16. The memory parameter is set to $d = 0.6, 0.7, 0.8$ and 0.9. Table 3.2 presents the corresponding estimates for temporal aggregation. As shown, the estimates for both skip sampling and temporal aggregation are near the true values of the memory parameter of the original series, regardless of the aggregation degree m . Therefore, neither skip sampling or temporal aggregation changes the memory parameter for fractional processes.

3.4.3 Generalized fractional integration

I consider the generalized fractional process $\{x_t\}$ generated by

$$(1 - 2\eta L + L^2)^d x_t = u_t,$$

where $\eta = \cos(2\pi/n)$, and $u_t \sim i.i.d.N(0, 1)$. Here, I apply the method used in Gray, Zhang and Woodward (1989) to generate the realizations of $\{x_t\}$.

Figure 3.3 plots the limit of the expectation of the periodogram for the skip sampled series and the periodogram averaged over 100 realizations of the process, as a function of the frequency index. The total number of observations for the original series is $T = 2048$. The aggregation degree is $m = 4$. Thus, the length of the aggregated series is 512. The aggregated

series are generated using (a) $d = 0.6$, $n = 4$, (b) $d = 0.8$, $n = 4$, (c) $d = 0.6$, $n = 16$ and (d) $d = 0.8$, $n = 16$. Figure 3.4 shows the corresponding limit and periodogram for the temporally aggregated series. For both skip sampling and temporal aggregation, the averaged periodograms are scattered around the solid line, which indicates the limit of the expectation of the periodogram.

It is interesting to note from Theorem 3.3 that there is no pole at frequency zero in the limit of the expectation of the periodogram of the original series. However, as shown in Figure 3.3 (a) and (b), there are peaks at frequency zero for both the limit of the expectation of the periodogram and the averaged periodogram when $n = 4$, which verifies Theorem 3.5. That is, when the periodogram of the original series has a pole at $2\pi/4$, skip sampling, which folds the periodogram 4 times, moves the pole to frequency zero. This finding further confirms that a process with long memory at zero frequency can arise due to the skip sampling of a series which virtually possesses long memory property at seasonal frequency. When $n = 16$, there is no pole at frequency zero for the skip-sampled series since the periodogram of the original series has a pole at frequency $2\pi/16$ but here I only fold it 4 times. In this case, the pole at frequency $2\pi/16$ can be moved to frequency zero if and only if the aggregation degree is set to an exact multiple of 16. Furthermore, for the temporal aggregation case shown in Figure 3.4, no pole is found at frequency zero in the periodograms regardless of the aggregation degree, which is caused by the fact that the periodogram of the original series does not have peaks at the origin. This finding confirms Theorem 3.5, i.e., temporally aggregating a series does not change the memory parameter.

Table 3.3 presents the LP estimates averaged over 100 realizations of the process for skip sampling. I use $\tau = T^{0.5}$ for the LP regression. Here, I adopt a smaller value of τ because the estimate might be affected by the seasonal frequency. The total number of observations for the original series is $T = 8192$. The aggregation degrees are $m = 1, 2, 4, 6, 8, 16$. The memory parameter is set to $d = 0.6, 0.7, 0.8$ and 0.9 ; $n = 4, 5, 6, 7$ and 8 . One should note that $d(0)$ equals $d(2\pi/m)$ if $2\pi/n \in \{\lambda_m\}$, such as when $m = 4, 8, 16$ and $n = 4$, $m = 8, 16$ and $n = 8$. To be more specific, when $n = 4$, the long memory parameter of the original series is nonzero at the

seasonal frequency $2\pi/4$; the long memory parameter of the skip-sampled series at frequency zero will equal that of the original series at frequency $2\pi/4$ when the aggregation degree is set to an exact multiple of 4, i.e., cases with $m = 4, 8, 16$ bolded in Table 3.3. Similarly, a nonzero long memory parameter for the skip-sampled series can be observed when (a) $n = 8, m = 8$, (b) $n = 8, m = 16$. In this regard, the series generated by the generalized fractional process might have nonzero memory parameter at frequency zero after skip sampling. Table 3.4 shows the corresponding estimate for the temporally aggregated series. However, the estimators for the temporally aggregated series are zero because temporal aggregation does not change the memory parameter at frequency zero and the original series has a memory parameter of 0 at frequency zero.

3.5 Conclusions

In this chapter, I analyze the properties of aggregation in the frequency domain. The dft and the periodogram of aggregated series are derived with respect to the aliasing effect. I then study the limit of the expectation of the periodogram of aggregated series in the nonstationary case for generalized fractional processes. Based on the limit theory, I show that, for skip sampling, a long memory process at frequency zero can arise when the series has a long memory at some seasonal frequency. However, temporal aggregation does not change the memory parameter for a general fractional integrated process.

Future work might look into the following issues: (1) A particularly interesting case to study is $d = 1$. As demonstrated by Granger and Siklos (1995), a unit root found in low frequency data could be caused by a seasonal unit root in high frequency data. The analysis of the aggregation of generalized fractional processes with $d = 1$ will be helpful to further examine this property. (2) I will use above results for the periodogram under aggregation to study the properties of semiparametric estimators, such as the log-periodogram and local Whittle estimators with respect to aggregation. (3) Hassler and Tsai (2012) use the spectrum to discuss the asymptotic behavior of aggregation for stationary processes in the frequency domain as the aggregation level m approaches infinity. Extensions to the nonstationary case

can be made based on my study of the periodogram in this chapter. (4) Hassler (2011) studies the property of the spectrum of multiple time series for stationary processes under aggregation, while the corresponding property of the periodogram is still unknown.

3.6 Appendix

Proof of Lemma 3.1: In the case of stock variables, aggregation refers to skip sampling where observations are obtained every m th period,

$$\bar{x}_p^{(m)} = x_{mp}.$$

Using the inverse Fourier transform, I obtain

$$x_{mt} = \sqrt{2\pi T} \int_{-\pi}^{\pi} w_x(\lambda) e^{-i\lambda mp} d\lambda.$$

Suppose that m is an odd number. Then

$$\bar{x}_p^{(m)} = \sqrt{2\pi T} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \int_{(2j-1)\pi/m}^{(2j+1)\pi/m} w_x(\lambda) e^{-i\lambda mp} d\lambda.$$

Let $\lambda' = \lambda - 2\pi j/m$. Then

$$\begin{aligned} \bar{x}_p^{(m)} &= \sqrt{2\pi T} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \int_{-\pi/m}^{\pi/m} w_x(\lambda' + 2\pi j/m) e^{-i\lambda' mp} e^{-i2j\pi p} d\lambda' \\ &= \sqrt{2\pi T} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \int_{-\pi/m}^{\pi/m} w_x(\lambda' + 2\pi j/m) e^{-i\lambda' mp} d\lambda'. \end{aligned}$$

with $\omega = m\lambda'$,

$$\begin{aligned}\bar{x}_p^{(m)} &= \frac{\sqrt{2\pi T}}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \int_{-\pi}^{\pi} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) e^{-i\omega p} d\omega \\ &= \frac{\sqrt{2\pi T}}{m} \int_{-\pi}^{\pi} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) e^{-i\omega p} d\omega.\end{aligned}$$

I also have

$$\bar{x}_p^{(m)} = \sqrt{2\pi T/m} \int_{-\pi}^{\pi} w_{\bar{x}}(\omega) e^{-i\omega p} d\omega.$$

Therefore, the dft of the skip sampled series has the following form

$$w_{\bar{x}}(\omega) = \frac{1}{\sqrt{m}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right).$$

A similar result can be obtained when m is an even number:

$$w_{\bar{x}}(\lambda_s) = \begin{cases} \frac{1}{\sqrt{m}} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), & -\pi < \lambda_s \leq 0 \\ \frac{1}{\sqrt{m}} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right), & 0 < \lambda_s \leq \pi \end{cases}.$$

■

Proof of Theorem 3.1: When m is an odd number, the periodogram of the skip sampled

series, $\{\bar{x}_p^{(m)}\}$, can be written as follows,

$$\begin{aligned}
I_{\bar{x}} &= |w_{\bar{x}}(\omega)|^2 \\
&= \frac{1}{m} \left[\sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) \right] \left[\sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) \right]^* \\
&= \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right)^* \\
&\quad + \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k \neq j} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) w_x \left(\frac{\omega}{m} + \frac{2\pi k}{m} \right)^* \\
&= \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} I_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) + \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k \neq j} w_x \left(\frac{\omega}{m} + \frac{2\pi j}{m} \right) w_x \left(\frac{\omega}{m} + \frac{2\pi k}{m} \right)^*.
\end{aligned}$$

A similar result can be obtained for the case where m is an even number:

$$I_{\bar{x}}(\lambda_s) = \begin{cases} \frac{1}{m} \sum_{j=-\frac{m}{2}+1}^{\frac{m}{2}} I_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) + \frac{1}{m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{k \neq j} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^*, & \text{for } -\pi < \lambda_s \leq 0; \\ \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} I_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) + \frac{1}{m} \sum_{j=-\frac{m}{2}}^{\frac{m}{2}-1} \sum_{k \neq j} w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^*, & \text{for } 0 < \lambda_s \leq \pi. \end{cases}$$

Proof of Lemma 3.2: See Phillips and Solo (1992, eq. (32)). ■

Proof of Lemma 3.3: The proof follows that of Theorem 2.2 of Phillips (1999). From Equation (3.4), I have the following alternative expression for the overlapping moving average process z_t

$$\begin{aligned}
z_t &= G(L, m)x_t \\
&= G(e^{i\lambda}, m)x_t + \tilde{G}_\lambda(e^{-i\lambda}L, \lambda, m)(e^{-i\lambda}L - 1)x_t.
\end{aligned}$$

Note that

$$\tilde{G}_\lambda \left(e^{-i\lambda} L, \lambda, m \right) \left(e^{-i\lambda} L - 1 \right) x_t = \left(e^{-i\lambda} L - 1 \right) X_t^G(\lambda, m) = e^{-i\lambda} X_{t-1}^G(\lambda, m) - X_t^G(\lambda, m), \quad (3.17)$$

where $X_t^G(\lambda, m) = \tilde{G}_\lambda \left(e^{i\lambda} L, \lambda, m \right) x_t$. Since $e^{-i\lambda} X_{t-1}^G(\lambda, m) - X_t^G(\lambda, m)$ is a telescoping Fourier sum, taking dft of (3.17) leaves us with $(X_0^G(\lambda, m) - e^{in\lambda} X_T^G(\lambda, m)) / \sqrt{2\pi T}$. It follows that

$$w_z(\lambda) = w_x(\lambda) G(e^{i\lambda}, m) - \frac{e^{iT\lambda}}{\sqrt{2\pi T}} X_T^G(\lambda, m).$$

■

Proof of Lemma 3.4: The proof is straightforward using (3.14) and Lemma 3.6. ■

Proof of Theorem 3.2: Note that

$$G(e^{i\lambda})G(e^{-i\lambda}) = \frac{\sin^2\left(\frac{m\lambda}{2}\right)}{\sin^2\left(\frac{\lambda}{2}\right)}.$$

Then, the rest of the proof follows directly. ■

Proof of Lemma 3.5: From Phillips and Solo (1992, eq. (32)), I have

$$D_1(d, v, L) = D_1(d, v, e^{i\lambda}) + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda} L) \left(e^{-i\lambda} L - 1 \right)$$

and

$$D_2(d, v, L) = D_2(d, v, e^{i\lambda}) + \tilde{D}_{2\lambda}(d, v, e^{-i\lambda} L) \left(e^{-i\lambda} L - 1 \right),$$

where

$$\begin{cases} \tilde{D}_{1\lambda}(d, v, e^{-i\lambda} L) = \sum_{p=0}^{T-1} \tilde{d}_{1\lambda p} e^{-ip\lambda} L^p \\ \tilde{D}_{2\lambda}(d, v, e^{-i\lambda} L) = \sum_{p=0}^{T-1} \tilde{d}_{2\lambda p} e^{-ip\lambda} L^p \end{cases}$$

and

$$\tilde{d}_{1\lambda p} = \sum_{k=p+1}^T \frac{(-d)_k}{k!} e^{ik(\lambda-v)}, \quad \tilde{d}_{2\lambda p} = \sum_{k=p+1}^T \frac{(-d)_k}{k!} e^{ik(\lambda+v)}.$$

Then,

$$\begin{aligned}
D(d, v, L) &= D_1(d, v, L) D_2(d, v, L) \\
&= \left[D_1(d, v, e^{i\lambda}) + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \right] \\
&\quad \times \left[D_2(d, v, e^{i\lambda}) + \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \right] \\
&= D(d, v, e^{i\lambda}) + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1)^2.
\end{aligned}$$

■

Proof of Lemma 3.6: From Equation (3.12), I have

$$\begin{aligned}
u_t &= D(d, v, e^{i\lambda}) x_t + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \quad (3.18) \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1)^2 x_t \\
&= D(d, v, e^{i\lambda}) x_t + D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \\
&\quad + D_1(d, v, e^{i\lambda}) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t \\
&\quad + e^{-i\lambda} \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_{t-1} \\
&\quad + \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) (e^{-i\lambda}L - 1) x_t.
\end{aligned}$$

Taking the dft of both sides of (3.18), and following the steps in the proof of Lemma 3.3,

I obtain

$$\begin{aligned}
w_u(\lambda) &= w_x(\lambda) D(d, v, e^{i\lambda}) + \frac{1}{\sqrt{2\pi T}} D_2(d, v, e^{i\lambda}) \left(X_0^{D_1}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1}(d, v, \lambda) \right) \\
&\quad + \frac{1}{\sqrt{2\pi T}} D_1(d, v, e^{i\lambda}) \left(X_0^{D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_2}(d, v, \lambda) \right) \\
&\quad + \frac{1}{\sqrt{2\pi T}} \left(e^{-i\lambda} X_0^{D_1 D_2}(d, v, \lambda) - e^{i(T-2)\lambda} X_{T-1}^{D_1 D_2}(d, v, \lambda) \right) \\
&\quad - \frac{1}{\sqrt{2\pi T}} \left(e X_0^{D_1 D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1 D_2}(d, v, \lambda) \right)
\end{aligned}$$

where

$$\begin{aligned}
X_0^{D_1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_0; & X_T^{D_1}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) x_T; \\
X_0^{D_2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_0; & X_T^{D_2}(d, v, \lambda) &= \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_T; \\
X_0^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_0; \\
X_{T-1}^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_{T-1}; \\
X_T^{D_1 D_2}(d, v, \lambda) &= \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}L) \tilde{D}_{2\lambda}(d, v, e^{-i\lambda}L) x_T.
\end{aligned}$$

■

Lemma A1 (Phillips, 1999) Suppose $d > 0$ and $\lambda \neq 0$. Then

$$\sum_{k=0}^T \frac{(-d)_k}{k!} e^{ik\lambda} = \left(1 - e^{i\lambda}\right)^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{iT\lambda}}{1 - e^{i\lambda}} \left[1 + O\left(\frac{1}{T}\right)\right].$$

Proof of Lemma 3.7: For some fixed $\lambda \neq v$, from Lemma A1, I obtain

$$D_1(d, v, e^{i\lambda}) = \left(1 - e^{i(\lambda-v)}\right)^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{iT(\lambda-v)}}{1 - e^{i(\lambda-v)}} \left[1 + O\left(\frac{1}{T}\right)\right]$$

and

$$D_2(d, v, e^{i\lambda}) = \left(1 - e^{i(\lambda+v)}\right)^d - \frac{1}{\Gamma(-d) T^{1+d}} \frac{e^{iT(\lambda+v)}}{1 - e^{i(\lambda+v)}} \left[1 + O\left(\frac{1}{T}\right)\right].$$

Then

$$\begin{aligned}
& D(d, v, e^{i\lambda}) \\
&= D_1(d, v, e^{i\lambda}) D_2(d, v, e^{i\lambda}) \\
&= \left(1 - e^{i(\lambda+v)}\right)^d \left(1 - e^{i(\lambda-v)}\right)^d \\
&\quad - \frac{1}{\Gamma(-d) T^{1+d}} \left[\frac{e^{iT(\lambda+v)} \left(1 - e^{i(\lambda-v)}\right)^d}{1 - e^{i(\lambda+v)}} + \frac{e^{iT(\lambda-v)} \left(1 - e^{i(\lambda+v)}\right)^d}{1 - e^{i(\lambda-v)}} \right] \left[1 + O\left(\frac{1}{T}\right)\right] \\
&\quad + \frac{1}{\Gamma(-d)^2 T^{2+2d}} \left[\frac{e^{i2T\lambda}}{\left(1 - e^{i(\lambda+v)}\right) \left(1 - e^{i(\lambda-v)}\right)} \right] \left[1 + O\left(\frac{1}{T}\right)\right].
\end{aligned}$$

■

Proof of Lemma 3.8: For a fixed $\lambda \neq v$ as $T \rightarrow \infty$, from Lemma 3.6, I have

$$\begin{aligned} X_T^D(v, d, \lambda) &= D_2(d, v, e^{i\lambda}) \left(X_0^{D_1}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1}(d, v, \lambda) \right) \\ &\quad + D_1(d, v, e^{i\lambda}) \left(X_0^{D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_2}(d, v, \lambda) \right) \\ &\quad + \left(e^{-i\lambda} X_0^{D_1 D_2}(d, v, \lambda) - e^{i(T-1)\lambda} X_{T-1}^{D_1 D_2}(d, v, \lambda) \right) \\ &\quad - \left(e X_0^{D_1 D_2}(d, v, \lambda) - e^{iT\lambda} X_T^{D_1 D_2}(d, v, \lambda) \right). \end{aligned}$$

One should notice that

$$X_0^{D_1}(d, v, \lambda) = X_0^{D_2}(d, v, \lambda) = X_0^{D_1 D_2}(d, v, \lambda) = 0$$

due to the assumption that $x_t = 0$ for $t \leq 0$. Also,

$$e^{iT\lambda} D_2(d, v, e^{i\lambda}) X_T^{D_1}(d, v, \lambda) = e^{iT\lambda} D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda L}) x_T$$

where,

$$D_2(d, v, e^{i\lambda}) \tilde{D}_{1\lambda}(d, v, e^{-i\lambda}) x_T = D_2(d, v, e^{i\lambda}) \sum_{p=0}^{T-1} \left(\sum_{k=p+1}^T \frac{(-d)_k}{k!} e^{ik(\lambda-v)} \right) e^{-ip\lambda} x_{T-p}.$$

Similar to the proof of Theorem 3.2 of Phillips (1999), I have

$$\frac{1}{\sqrt{T}} D_2(d, v, e^{i\lambda}) e^{iT\lambda} X_T^{D_1}(d, v, \lambda) x_T = O_p\left(\frac{1}{T^{1-d}}\right)$$

and

$$\begin{aligned} \frac{1}{\sqrt{T}} D_1(d, v, e^{i\lambda}) e^{iT\lambda} X_T^{D_2}(d, v, \lambda) x_T &= O_p\left(\frac{1}{T^{1-d}}\right); \\ \frac{1}{\sqrt{T}} e^{i(T-2)\lambda} X_{T-1}^{D_1 D_2}(d, v, \lambda) x_T &= O_p\left(\frac{1}{T^{1-d}}\right); \\ \frac{1}{\sqrt{T}} e^{iT\lambda} X_T^{D_1 D_2}(d, v, \lambda) x_T &= O_p\left(\frac{1}{T^{1-d}}\right). \end{aligned}$$

Therefore,

$$\frac{1}{\sqrt{T}} X_T^D(v, d, \lambda) = O_p\left(\frac{1}{T^{1-d}}\right).$$

■

Proof of Lemma 3.9: From (3.16), I have

$$\begin{aligned}
w_x(\lambda_s) &= \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \\
&\quad - \frac{e^{i2\lambda}}{(1 - e^{i(\lambda + v)})(1 - e^{i(\lambda - v)})} \frac{x_T}{\sqrt{2\pi T}} + o_p\left(\frac{1}{T^{1-d}}\right) \\
&= \left(1 - e^{i(\lambda_s - v)}\right)^{-d} \left(1 - e^{i(\lambda_s + v)}\right)^{-d} w_u(\lambda_s) \left[1 + O_p\left(\frac{1}{M}\right)\right] \\
&\quad + O_p\left(\frac{1}{T^{1-d}}\right),
\end{aligned}$$

where the error magnitudes hold uniformly for $\lambda_{s_j} \in B_\phi = \{\phi - \frac{\pi}{M}, \phi + \frac{\pi}{M}\}$. Theorem 3 of Hannan (1973) implies that the quantities $\{w_u(\lambda_{s_j})\}_{j=1}^J$ are asymptotically independent and distributed with the same complex normal distribution $N_c(0, f_u(\phi))$ as $T \rightarrow \infty$. The result for the quantities $\{w_x(\lambda_{s_j})\}_{j=1}^J$ then follows directly. ■

Proof of Theorem 3.3: From Lemma 3.9, I know that the family $\{w_x(\lambda_{s_j})\}_{j=1}^J$ are asymptotically independently distributed as complex normal $N_c(0, f_x(\phi))$ where

$$f_x(\phi) = \left|1 - 2\eta e^{i\phi} + e^{2i\phi}\right|^{-2d} f_u(\phi).$$

Then, the limit of the expectation of the periodogram of $\{x_t\}$ is given by

$$\begin{aligned}
f_x(\lambda) &= \lim_{T \rightarrow \infty} E[I_\lambda(\lambda_s)] \\
&= \begin{cases} |4 \sin(\frac{\lambda + \nu}{2}) \sin(\frac{\lambda - \nu}{2})|^{-2d} f_u(\lambda), & \text{for } |\eta| < 1 \text{ and } 1/2 < d < 1; \\ |\sin(\frac{\lambda}{2})|^{-4d} f_u(\lambda), & \text{for } \eta = 1 \text{ and } 1/4 < d < 1/2. \end{cases}
\end{aligned}$$

■

Proof of Theorem 3.4: When $|\eta| < 1$ and $1/2 < d < 1$, from Lemma 3.9, I know that $w_x(\lambda_s)$ are independent with mean zero for $\lambda \neq 0$ and $0.5 < d < 1$. Therefore,

$$\lim_{T \rightarrow \infty} E \left[w_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) w_x \left(\frac{\lambda_s}{m} + \frac{2k\pi}{m} \right)^* \right] = 0 \text{ if } j \neq k$$

From Lemma 3.8, I have

$$\frac{1}{\sqrt{T}} X_T^D(v, d, \lambda) = O_p\left(\frac{1}{T^{1-d}}\right).$$

I also know that $w_u(\lambda) = O(1)$. Hence,

$$\frac{1}{\sqrt{2\pi T}} X_T^D(v, d, \lambda) w_u(\lambda) = o(1).$$

and

$$\frac{1}{2\pi T} |X_T(v, d, \lambda)|^2 = o(1).$$

From Lemma 3.7, I know that

$$D^{-1}(d, v, e^{i\lambda}) \rightarrow^p \left(1 - e^{i(\lambda+v)}\right)^{-d} \left(1 - e^{i(\lambda-v)}\right)^{-d}.$$

Since u_t is stationary, I have

$$E \lim_{T \rightarrow \infty} I_u(\lambda_s) = f_u(\lambda_s).$$

Then, using Theorem 3.2, I obtain the required result. ■

Proof of Theorem 3.5: From Theorem 3.4, I know that the limit of the expectation of the periodogram of the skip sampled series equals the sum of the limit of the expectation of the periodogram of the original series at the aliased frequencies $\lambda_m = 0, \pm 2\pi/m, \pm 4\pi/m, \dots$ and $|\lambda_m| \leq \pi$ as $\lambda_s = 0$. Therefore, $d(0) = \max_m \{d(\lambda_m)\}$. For temporal aggregation, I have

$$\begin{aligned} T_m \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) &= \frac{\sin^2 \left(\frac{\lambda_s}{2} + j\pi \right)}{\sin^2 \left(\frac{\lambda_s}{2m} + \frac{j\pi}{m} \right)} \\ &= \frac{\sin^2 \left(\frac{\lambda_s}{2} \right)}{\sin^2 \left(\frac{\lambda_s}{2m} + \frac{j\pi}{m} \right)} \\ &= \begin{cases} m^2, & \text{as } \lambda_s \rightarrow 0, \text{ and } j = 0 \\ \frac{\lambda_s^2}{\sin^2 \left(\frac{j\pi}{m} \right)}, & \text{as } \lambda_s \rightarrow 0, \text{ and } j \neq 0 \end{cases} \\ &= \begin{cases} O(1), & \text{as } \lambda_s \rightarrow 0, \text{ and } j = 0 \\ O(\lambda_s^2), & \text{as } \lambda_s \rightarrow 0 \text{ and } j \neq 0 \end{cases}. \end{aligned}$$

Therefore,

$$\begin{aligned}
 T_m \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) f_x \left(\frac{\lambda_s}{m} + \frac{2j\pi}{m} \right) &= \begin{cases} O \left(\lambda_s^{-2d(0)} \right), & \text{as } \lambda_s \rightarrow 0, \text{ and } j = 0 \\ O \left(\lambda_s^{2-2d(\frac{j\pi}{m})} \right), & \text{as } \lambda_s \rightarrow 0, \text{ and } j \neq 0 \end{cases} \\
 &= \begin{cases} O \left(\lambda_s^{-2d(0)} \right), & \text{as } \lambda_s \rightarrow 0, \text{ and } j = 0 \\ o(1), & \text{as } \lambda_s \rightarrow 0, \text{ and } j \neq 0 \end{cases}
 \end{aligned}$$

Then, I know that the the limit of the expectation of periodogram of the temporally aggregated series equals the sum of $T_m(\lambda_s) f_x(\lambda_s)$ at the aliased frequencies $\lambda_m = 0, \pm 2\pi/m, \pm 4\pi/m, \dots$ and $|\lambda_m| \leq \pi$ as $\lambda_s = 0$. Hence, for the temporal aggregation case, the memory parameter is invariant at frequency zero. ■

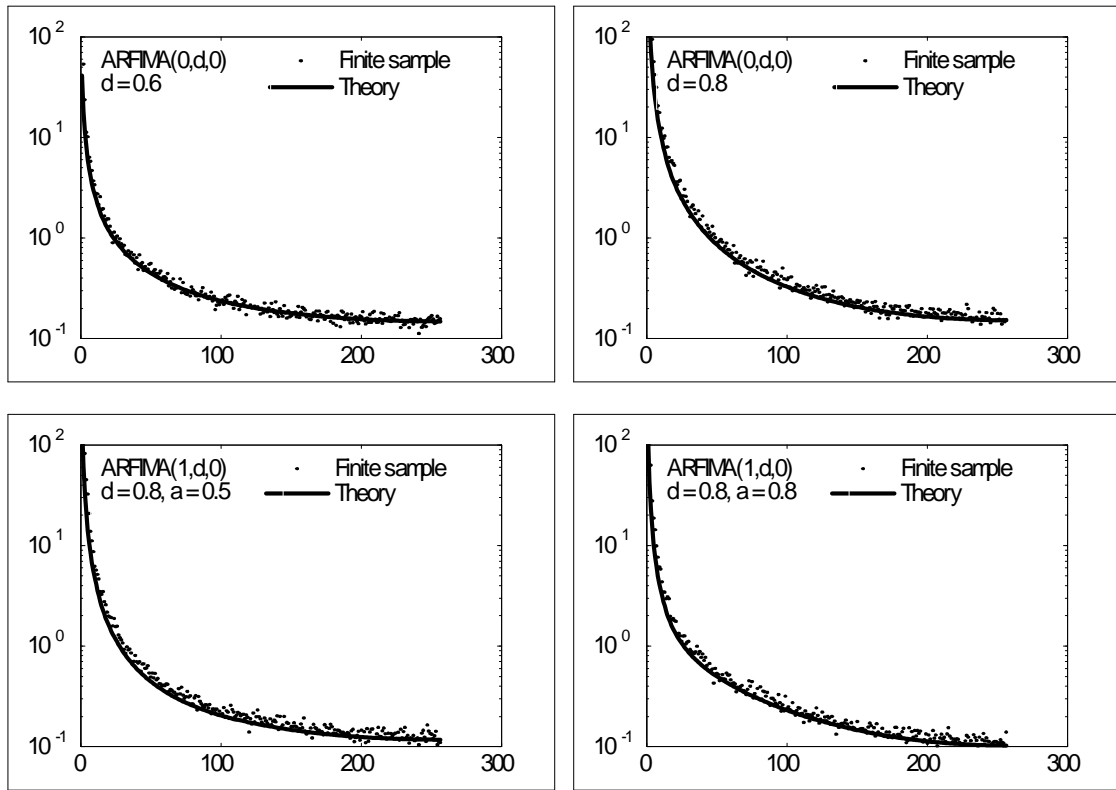


Figure 3-1: The limit of the expectation of the periodogram and the periodogram for skip sampled ARFIMA processes

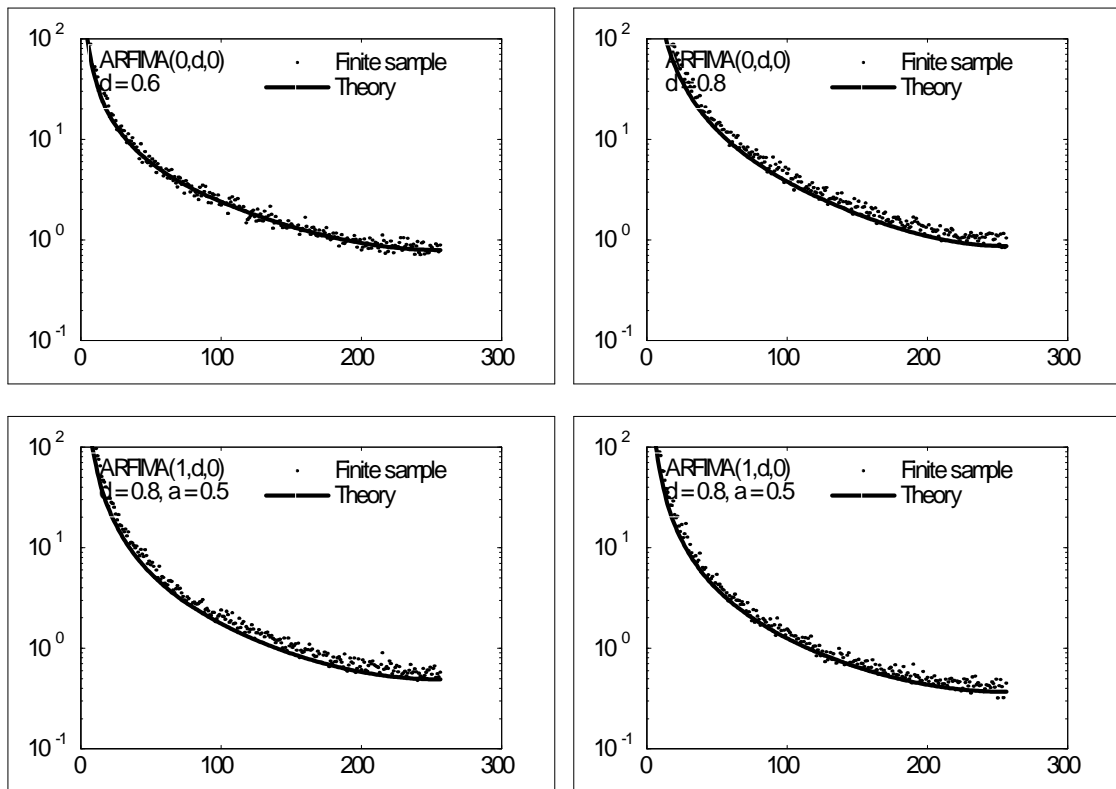


Figure 3.2: The limit of the expectation of the periodogram and the periodogram for temporally aggregated ARFIMA processes

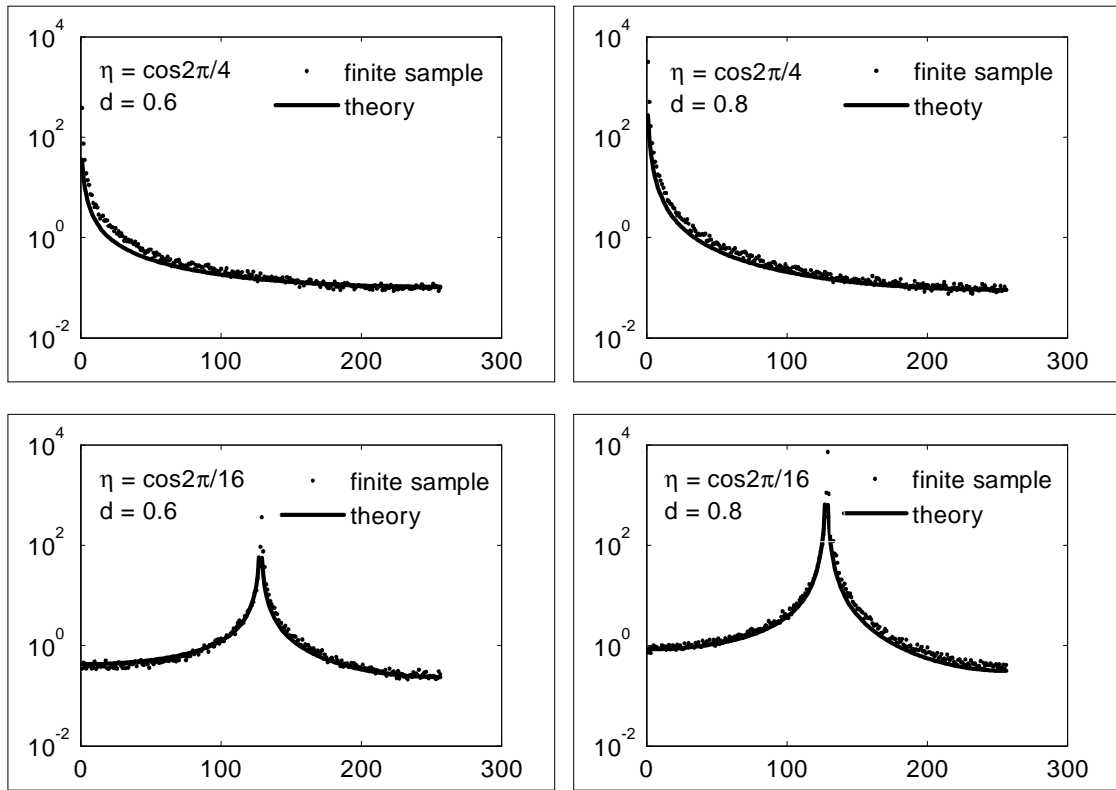


Figure 3.3: The limit of the expectation of the periodogram and the periodogram for skip sampled generalized fractional integrated processes

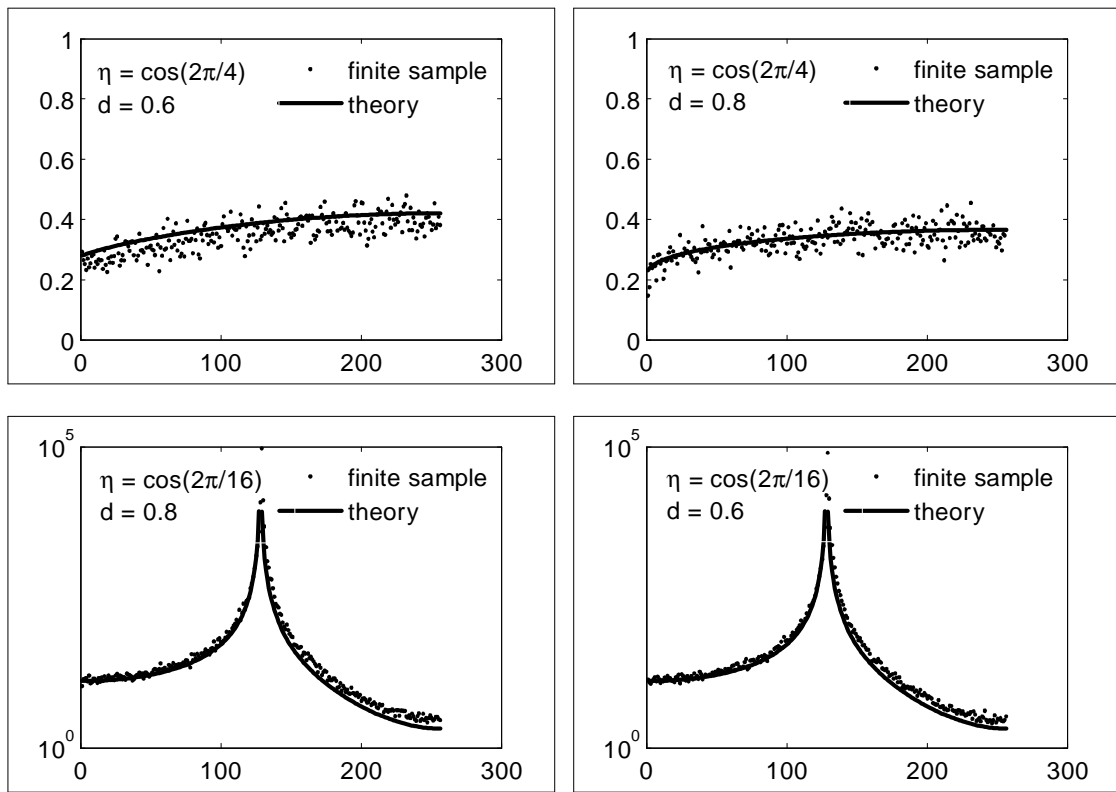


Figure 3.4: The limit of the expectation of the periodogram and the periodogram for temporal aggregated generalized fractional integrated processes

Table 3.1: LP estimates of skip-sampled ARFIMA processes

m	1	2	4	8	16
d		$a = 0$	$b = 0$		
0.6	0.6205	0.6071	0.5996	0.6095	0.5724
0.7	0.7257	0.7144	0.7207	0.7451	0.7277
0.8	0.8265	0.8268	0.8409	0.8854	0.8419
0.9	0.9296	0.9297	0.9444	0.9938	0.9301
		$a = 0.5$	$b = 0$		
0.6	0.5905	0.4991	0.5081	0.5087	0.5067
0.7	0.6973	0.6239	0.6416	0.6605	0.6526
0.8	0.8016	0.7328	0.7689	0.8167	0.7970
0.9	0.9017	0.8324	0.8754	0.9468	0.9115
		$a = 0$	$b = 0.5$		
0.6	0.6527	0.6452	0.6369	0.6457	0.6030
0.7	0.7555	0.7516	0.7505	0.7829	0.7330
0.8	0.8605	0.8644	0.8682	0.9080	0.8650
0.9	0.9642	0.9635	0.9769	1.0216	0.9453
		$a = 0.5$	$b = 0.5$		
0.6	0.6247	0.6072	0.6057	0.6021	0.5683
0.7	0.7295	0.7151	0.7227	0.7405	0.7259
0.8	0.8281	0.8284	0.8412	0.8851	0.8421
0.9	0.9324	0.9281	0.9504	1.0030	0.9369

Table 3.2: LP estimates of temporally aggregated ARFIMA processes

m	1	2	4	8	16
d		$a = 0$	$b = 0$		
0.6	0.6231	0.6501	0.6876	0.7566	0.8030
0.7	0.7249	0.7538	0.7926	0.8819	0.8732
0.8	0.8308	0.8689	0.9028	1.0037	0.9936
0.9	0.9326	0.9700	1.0201	1.1432	1.0669
		$a = 0.5$	$b = 0$		
0.6	0.5912	0.6274	0.6799	0.7433	0.7903
0.7	0.6953	0.7373	0.7864	0.8784	0.8870
0.8	0.7985	0.8433	0.8963	1.0142	0.9955
0.9	0.9029	0.9464	1.0133	1.1288	1.0670
		$a = 0$	$b = 0.5$		
0.6	0.6541	0.6607	0.6914	0.7575	0.7818
0.7	0.7547	0.7666	0.8018	0.8815	0.8952
0.8	0.8608	0.8771	0.9201	1.0126	0.9893
0.9	0.9645	0.9897	1.0282	1.1418	1.0740
		$a = 0.5$	$b = 0.5$		
0.6	0.6216	0.6518	0.6890	0.7549	0.8175
0.7	0.7264	0.7564	0.7894	0.8836	0.8992
0.8	0.8312	0.8699	0.9061	1.0089	0.9763
0.9	0.9309	0.9744	1.0226	1.1408	1.0803

Table 3.3: LP estimates of skip-sampled generalized fractional processes

m	1	2	4	8	16
n	$d = 0.6$				
4	-0.0075	-0.0008	0.7001	0.6863	0.6782
5	-0.0199	0.0162	-0.0276	-0.0117	-0.0519
6	0.0032	0.0080	0.0020	-0.1016	-0.1193
7	-0.0032	-0.0012	0.0246	-0.0089	0.0835
8	-0.0028	0.0014	-0.0049	0.6173	0.7327
	$d = 0.7$				
4	-0.0121	-0.0071	0.7622	0.7443	0.7708
5	-0.0052	0.0167	0.0189	-0.0917	0.0160
6	-0.0125	-0.0007	0.0209	0.0607	0.0349
7	-0.0071	0.0084	-0.0193	0.0237	-0.0162
8	-0.0037	-0.0178	-0.0390	0.7287	0.8096
	$d = 0.8$				
4	-0.0121	0.0188	0.8838	0.8315	0.7868
5	-0.0080	0.0119	-0.0176	-0.0326	-0.0570
6	-0.0035	-0.0038	-0.0318	0.0342	0.0195
7	-0.0199	-0.0290	0.0296	-0.0379	-0.0620
8	-0.0007	-0.0315	0.0123	0.8458	0.7775
	$d = 0.9$				
4	-0.0062	-0.0398	0.9806	0.9469	1.0358
5	0.0049	-0.0098	-0.0032	-0.0370	-0.0967
6	-0.0023	0.0029	-0.0522	-0.0064	0.0076
7	0.0042	0.0097	0.0258	-0.0454	0.0522
8	-0.0024	-0.0032	0.0282	1.0228	1.0330

Table 3.4: LP estimates of temporally aggregated generalized fractional processes

m	1	2	4	8	16
n	$d = 0.6$				
4	0.0048	-0.0046	-0.0055	-0.0479	-0.0316
5	-0.0131	0.0273	-0.0330	0.0375	0.0243
6	0.0107	-0.0367	-0.0192	0.0273	-0.0732
7	-0.0006	-0.0058	0.0254	0.0033	-0.0967
8	-0.0053	0.0140	-0.0177	-0.0965	-0.1278
	$d = 0.7$				
4	0.0157	-0.0230	-0.0235	-0.0390	-0.0140
5	0.0185	0.0239	-0.0399	-0.0064	-0.0200
6	-0.0094	-0.0052	-0.0038	0.0204	-0.0194
7	0.0058	0.0258	0.0440	-0.0064	-0.0460
8	0.0006	-0.0195	-0.0077	-0.0260	-0.1533
	$d = 0.8$				
4	0.0046	0.0194	-0.0083	-0.0261	-0.0259
5	-0.0041	-0.0206	0.0194	0.0187	-0.1031
6	0.0024	-0.0059	0.0357	-0.0120	-0.1055
7	0.0008	-0.0122	-0.0082	-0.0143	-0.0013
8	-0.0010	-0.0171	0.0044	0.0094	-0.0495
	$d = 0.9$				
4	-0.0098	-0.0237	-0.0294	-0.0638	-0.0261
5	-0.0103	-0.0231	-0.0275	0.0220	-0.0785
6	0.0002	0.0030	-0.0135	0.0066	0.0592
7	-0.0229	-0.0006	-0.0087	-0.1026	-0.0532
8	0.0116	0.0040	-0.0441	0.0179	-0.0585

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