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# T-EQUIVARIANT DISC POTENTIAL AND SYZ MIRROR CONSTRUCTION

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ABSTRACT. We develop a  $G$ -equivariant Lagrangian Floer theory by counting pearly trees in the Borel construction  $L_G$ . We apply the construction to smooth moment-map fibers of toric semi-Fano manifolds and obtain the  $T$ -equivariant Landau-Ginzburg mirrors. We also apply this to the typical  $S^1$ -invariant SYZ singular fiber, which is the single-pinched torus, and compute its  $S^1$ -equivariant disc potential.

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## 1. INTRODUCTION

Teleman [Tel14] conjectured that the mirror of a Hamiltonian action of a compact Lie group  $G$  on a symplectic manifold  $X$  is a holomorphic map  $\Phi$  from the mirror  $X^\vee$  of  $X$  to the space of conjugacy classes in the Langlands dual group  $G_{\mathbb{C}}^\vee$ . Moreover, the mirrors of the symplectic quotients  $X //_{\lambda} G$  for  $\lambda \in \mathfrak{g}^* / \text{Ad}_G^*$  are related to the fibers of  $\Phi$ . This agrees with the work of Hori-Vafa [HV] on mirror symmetry via gauged linear sigma models.

An important step towards understanding this conjecture is to construct the map  $\Phi$ . In this paper, we develop an equivariant version of the SYZ mirror construction. For the concrete computations we focus on the case where  $G = T$  is a compact torus, and study the  $T$ -equivariant Lagrangian Floer theory of torus fibers of an SYZ fibration preserved by a symplectic  $T$ -action. In general, the  $G$ -equivariant disc potential  $W_L^G$  of a  $G$ -invariant Lagrangian  $L$  gives a fibration of the equivariant mirror space to  $H^2(BG; \mathbb{C})$ .

For instance, let's consider the well-known Landau-Ginzburg mirror of a compact Fano toric  $d$ -fold  $X$  [HV, Giv95, Giv98, LLY99]. It is given by a pair  $((\mathbb{C}^\times)^d, W)$ , where  $W :$

$(\mathbb{C}^\times)^d \rightarrow \mathbb{C}$  is a holomorphic function of the form

$$W(z) = \sum_{i=1}^m q^{\alpha_i} z^{v_i},$$

$v_1, \dots, v_m \in N \cong \mathbb{Z}^d$  are primitive generators of the one-dimensional cones of the fan  $\Sigma$  defining  $X$ ,  $z^{v_i}$  denote the corresponding Laurent monomials, and  $q^{\alpha_1}, \dots, q^{\alpha_m}$  are Kähler parameters associated to certain curves classes  $\alpha_i \in H_2(X; \mathbb{Z})$ .

It was shown [Giv98] that the equivariant quantum cohomology  $QH_{T^d}^\bullet(X)$  of  $X$  is isomorphic to the Jacobian ring of

$$W_\lambda(z) = W(z) + \sum_{i=1}^d \lambda_i \log z_i,$$

where  $\lambda_1, \dots, \lambda_d$  are the equivariant parameters for the torus  $T^d \cong (\mathbb{S}^1)^d$  acting on  $X$ . Iritani [Iri17b, Iri17a] has made an in-depth study of the big equivariant quantum cohomology for toric varieties, which computes the highly non-trivial extension of  $W_\lambda$  in the direction of bulk insertions.

Cho-Oh [CO06] (in the Fano case) and Fukaya-Oh-Ohta-Ono [FOOO10] (in the general compact non-Fano setting) used Lagrangian Floer theory to define the disc potential for the regular fibers  $L$  of the toric moment map and showed that it coincides with the Landau-Ginzburg superpotential  $W$ . From this perspective,  $W$  is viewed as the generating function of open Gromov-Witten invariants, and each term in  $W$  corresponds to a holomorphic disc of Maslov index 2 bounded by  $L$ . This gives an SYZ interpretation of the Landau-Ginzburg mirrors [Aur07, CL10].

It is a natural and interesting question that whether the *equivariant superpotential*  $W_\lambda$  can also be interpreted as a generating function of equivariant disc counts, to which we give an affirmative answer to in this paper.

**Theorem 1.1** (Corollary 3.10). *Let  $X$  be a semi-projective and semi-Fano toric manifold with  $\dim_{\mathbb{C}} X = d$ , and let  $T \subset T^d$  be the subtorus determined by the integral basis  $u_1, \dots, u_\ell \in N \cong \mathbb{Z}^d$ . The  $T$ -equivariant disc potential (Definition 2.18)  $W_T^L$  of a regular fiber  $L$  of the toric moment map given by*

$$W_T^L = \sum_{i=1}^m \exp(g_i(\check{q}(q))) \mathbf{T}^{\omega(\beta_i)} \exp(v_i \cdot (x^1, \dots, x^d)) + \sum_{j=1}^{\ell} (u_j \cdot (x^1, \dots, x^d)) \lambda_j$$

where  $\beta_i$  are the basic disc classes bounded by the toric fiber,  $g_i(\check{q}(q))$  is given by the inverse mirror map in Equation (3.7),  $q^\alpha = \mathbf{T}^{\omega(\alpha)}$  are the Kähler parameters, and  $\mathbf{T}$  is the formal Novikov variable.

In the Fano case, we have  $g_i = 0$ . By taking  $\ell = d$ ,  $u_1, \dots, u_d$  the standard basis of  $\mathfrak{t}^d := \text{Lie}(T^d)$ , and  $x^i = \log z_i$  for  $i = 1, \dots, d$ , the above expression for  $W_T^L$  equals to  $W_\lambda$ .

In general, for a  $T$ -invariant Lagrangian submanifold  $L$  of a Hamiltonian  $T$ -space  $X$  the  $T$ -equivariant disc potential  $W_T^L$  of  $L$  takes the form

$$W_T^L = W(b) + \sum_{i=1}^{\ell} h_i(b) \lambda_i, \quad \ell = \text{rank } T,$$

for weak bounding cochains  $b$ , when  $L$  has minimal Maslov index zero. In relation with the Teleman's conjecture, we can interpret  $W_T^L$  as follows. The equivariant term

$$\Phi := (h_1(b), \dots, h_\ell(b))$$

defines a map from the weak Maurer-Cartan space of  $L$  to  $(t_\Lambda^\ell)^*$ , where  $\Lambda$  is the Novikov field. Then, the non-equivariant term  $W$  can be viewed as a family of potentials on the fibers of  $\Phi$ . It gives a mirror family of the symplectic quotient  $X //_\lambda T$ .

For instance, take  $X = \mathbb{C}^d$ . The non-equivariant part is simply  $W = z_1 + \dots + z_d$ . Suppose  $X //_\lambda T^\ell$  is a semi-Fano toric manifold. A fiber of the above fibration is given by  $z^u_j = c_j$  for some constants  $c_j$  and  $j = 1, \dots, \ell$ . Then  $W$  restricted to fibers give the mirror family of  $X //_\lambda T^\ell$ .

**Remark 1.2.** *The terms  $h_i(b)\lambda_i$  can be understood as obstruction terms for a homotopy equivalence between the equivariant Floer theory of  $L \subset X$  and Floer theory of the quotient  $L/T \subset X //_\lambda T^\ell$ . Daemi-Fukaya [DF17] asserted that this homotopy equivalence can be constructed by Lagrangian correspondence. In our setting we can compute the mirror equations  $h_i(b) = 0$  explicitly.*

*The mirror map  $c(\lambda)$  is crucial to precisely identify which fiber corresponds to  $X //_\lambda T^\ell$ . By the beautiful work of Woodward-Xu [WX], the mirror map can be understood as the change from the gauged Floer theory of  $X$  to the Fukaya category downstairs  $\text{Fuk}(X //_\lambda T^\ell)$ . Gauged Floer theory is formulated in terms of vortex equations. It would be very interesting to investigate the relation with our formulation.*

Equivariant Lagrangian Floer theory has seen major recent developments. In the exact setting and  $G = \mathbb{Z}/2\mathbb{Z}$ , Seidel-Smith [SS10] provided an approach to understanding  $G$ -equivariant Floer theory by combining Lagrangian Floer theory and family Morse theory [Hut08] on  $EG \rightarrow BG$ . Viterbo [Vit99] illustrated some related ideas for  $G = \mathbb{S}^1$ . Hendricks-Lipshitz-Sarkar [HLS16a, HLS16b] developed a homotopy coherent method to build up a  $G$ -equivariant Floer theory. Fukaya [Fuk17] and Daemi-Fukaya [DF17] used  $G$ -equivariant Kuranishi structure to tackle the  $G$ -equivariant transversality problem, and make a formulation using differential forms. There are other interesting works related to this subject [Sei15, LP16].

In this paper we use Morse chains on the Borel construction  $L_T$  for  $L \subset X$ , a  $T$ -invariant compact Lagrangian submanifold in a symplectic manifold  $(X, \omega)$  preserved by a  $T$ -action. This is closest to the approach of Seidel-Smith, although the Lagrangians under our consideration are not exact. They bound non-constant pseudo-holomorphic discs of non-negative Maslov index. These are known as quantum corrections in mirror symmetry, which contribute non-trivial terms to  $W_\lambda$  and make it very interesting.

Since there are only finitely many generators for each finite dimensional approximation of  $L_T$ , it better serves for computations and manifests the equivariant parameters  $\lambda_i$  appeared in the disc potential  $W_\lambda$ . Assuming that  $L$  has minimal Maslov index zero, we construct a curved  $A_\infty$ -algebra

$$\left( C_{\text{Morse}}^\bullet(L, \Lambda_0) \otimes_{\Lambda_0} \Lambda_0[\lambda_1, \dots, \lambda_\ell], \mathfrak{m}^T \right)$$

by counting pearly trees [BC07, FOOO09a] in  $L_T$ , where  $\Lambda_0[\lambda_1, \dots, \lambda_\ell] = H_T^\bullet(\text{pt}; \Lambda_0)$ , and  $\Lambda_0$  is the Novikov ring

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \mid a_i \in \mathbb{C}, A_i \in \mathbb{R}_{\geq 0}; A_i \text{ increases to } +\infty, \right\}.$$

We denote by  $\Lambda_+ \subset \Lambda_0$  the maximal ideal

$$\Lambda_+ = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \mid a_i \in \mathbb{C}, A_i \in \mathbb{R}_{>0}; A_i \text{ increases to } +\infty, \right\}.$$

and also by  $\Lambda$  the Novikov field where the exponents  $A_i \in \mathbb{R}$ .

We will show that (Theorem 2.14)

$$\mathfrak{m}_k^T(X_1, \dots, x_i \otimes \lambda_j, \dots, X_k) = \lambda_j \cdot \mathfrak{m}_k^T(X_1, \dots, x_i \otimes 1, \dots, X_k)$$

and hence we may treat  $\Lambda_0[\lambda_1, \dots, \lambda_\ell]$  as a coefficient ring. Note that the terms  $\mathfrak{m}_k^T(x_1 \otimes 1, \dots, x_k \otimes 1)$  are still series in the equivariant parameters  $\lambda$  and receive non-trivial contributions from pearly trees in  $L_T$ . To achieve this, we use the homotopy unit construction developed by Fukaya-Oh-Ohta-Ono [FOOO09b, Chapter 7]. It has also been used by Charest-Woodward [CW] in a different setup.

We take the boundary deformations  $b = \sum_{i=1}^{\ell} x^i x_i$  for  $x^i \in \Lambda_+$ , where  $x_i$  are degree one critical points of  $L$ . We will see that if  $(L, b)$  is weakly unobstructed, then the  $T$ -equivariant theory  $(L_T, b)$  will also be weakly unobstructed over  $\Lambda[\lambda_1, \dots, \lambda_\ell]$ , namely

$$\mathfrak{m}_0^T = W \cdot 1 \otimes 1 + \sum_{i=1}^{\ell} h_i \cdot 1 \otimes \lambda_i = \left( W + \sum_{i=1}^{\ell} h_i \lambda_i \right) \cdot 1 \otimes 1,$$

where  $h_i$  consists of contributions from pseudo-holomorphic discs of Maslov index zero. As a consequence, the usual disc potential  $W$  is equal to the non-equivariant limit ( $\lambda_i \rightarrow 0$ ) of  $W_\lambda$ .

As a consequence, suppose that we have a Lagrangian torus fibration which admits a free  $T^\ell$ -action. When the fibers are unobstructed, the non-equivariant disc potential is zero by definition. However, from the above consideration, we still have non-trivial equivariant terms and so the equivariant disc potential for a smooth torus fiber is non-zero:

$$W^T = \sum_{i=1}^{\ell} h_i \lambda_i.$$

In Section 4, we study the  $S^1$ -equivariant disc potential for the immersed two-sphere  $S^2$  with a single nodal point. This is also known as the pinched two-torus which is the most typical singular fiber in an SYZ fibration. Even when  $S^2$  does not bound any non-constant discs of Maslov index zero, it still has a non-trivial equivariant disc potential from the contribution of constant polygons at the nodal point. The corresponding moduli have non-trivial obstructions. The gluing technique via isomorphism between smooth and pinch tori in [HKL] is crucial in the computation of the explicit expression of the equivariant disc potential. In [DRET], a different method via Legendrian topology is used to develop the Floer theory of an immersed Lagrangian surface.

**Theorem 1.3** (Theorem 4.6). *The equivariant disc potential of the immersed sphere  $S^2$  is*

$$W = \log(1 - uv) = - \sum_{j=1}^{\infty} \frac{(uv)^j}{j}$$

where  $(u, v) \in \Lambda_0^2 - \{\text{val}(uv) > 0\}$  are the formal deformations corresponding to the degree one immersed generators of  $S^2$ .

The organization of this paper is as follows. We develop a Morse model for the equivariant Lagrangian Floer theory in Section 2. We apply this to formulate and compute the  $T$ -equivariant disc potential for toric semi-Fano manifolds in Section 3. In Section 4, we compute the  $S^1$ -equivariant disc potential for the immersed two-sphere  $S^2$ .

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## 2. A MORSE MODEL FOR EQUIVARIANT LAGRANGIAN FLOER THEORY

There are several approaches to  $G$ -equivariant Lagrangian Floer theory for a pair of  $G$ -invariant Lagrangians in the existing literature. For  $G = \mathbb{Z}_2$ , Seidel and Smith [SS10] used Floer homology coupled with Morse theory on  $EG$  to define  $G$ -equivariant Lagrangian Floer homology of exact Lagrangians. In [HLS16a, HLS16b], Hendricks, Lipshitz, and Sarkar used a homotopy theoretic method to define  $G$ -equivariant Lagrangian Floer homology for  $G$  a compact Lie group. More recently, in [DF17], Daemi and Fukaya defined an equivariant de Rham model using  $G$ -equivariant Kuranishi structures developed in [FOOO09b, Fuk17].

In this section, we develop a Morse model for the  $G$ -equivariant Lagrangian Floer theory focusing on a single  $G$ -invariant Lagrangian. The Lagrangian we study here can be immersed with clean self-intersection. The underlying cochain complexes can be constructed such that it is finite dimensional over the cohomology ring of  $BG$ , and the  $A_\infty$  operations are given by counting pearly trees in the Borel construction  $L_G$ . This suits better for our purpose of computing disc potentials and constructing SYZ mirrors. The  $A_\infty$ -algebra we construct will be unital. This uses the homotopy unit construction in [FOOO09b], which has also been adapted to the stabilizing divisor perturbation scheme by Charest-Woodward [CW] for their (non-equivariant) unital Morse model.

We define the  $G$ -equivariant Lagrangian Floer theory as the (ordinary) Lagrangian Floer theory of  $L_G$  as a Lagrangian submanifold of a certain symplectic manifold, which avoids the issue of equivariant transversality. The Morse model we use makes the theory much more explicit and computable. Since the Lagrangians we study here bounds non-constant pseudo-holomorphic discs, we use the machinery of [FOOO09b] to handle the obstructions. On the other hand, for computing the equivariant disc potential of toric Fano manifolds, virtual technique is not necessary.

**2.1 The non-equivariant singular chain model.** We begin by recalling the  $A_\infty$ -algebra associated to a Lagrangian submanifold  $L$  of a symplectic manifold constructed by Fukaya-Oh-Ohta-Ono [FOOO09b]. We refer to [FOOO09b] for a review on  $A_\infty$ -algebras and Kuranishi structures.

Let  $(X, \omega_X)$  be a symplectic manifold of real dimension  $2d$  and let  $J_X$  be a compatible almost complex structure. We will always assume  $X$  is convex at infinity or geometrically bounded if it is non-compact. Let  $L \subset X$  be a closed, connected, and relatively spin Lagrangian submanifold. In [FOOO09b], a countably generated subcomplex  $C^\bullet(L; \Lambda_0)$  of the singular chain complex  $S^\bullet(L; \Lambda_0)$  (regarded as a cochain complex) and an  $A_\infty$ -algebra structure  $(C^\bullet(L; \Lambda_0), \mathfrak{m})$  were constructed. We briefly recall the construction below, which we will use in constructing the equivariant Morse model.

Let  $\beta \in H_2(X, L; \mathbb{Z})$  and denote by  $\mathcal{M}(\beta; L)$  the moduli space of  $J_X$ -holomorphic stable bordered maps  $u : (D^2, \partial D^2) \rightarrow (X, L)$  of genus 0 representing  $\beta$  and by  $\mathcal{M}_{k+1}(\beta; L)$

the moduli space with  $k + 1$  marked points  $z_0, \dots, z_k$  on  $\partial D^2$ , ordered counter-clockwise.  $\mathcal{M}_{k+1}(\beta; L)$  has virtual dimension  $\mu(\beta) + k - 2 + d$ , where  $\mu(\beta)$  is the Maslov index of  $\beta$ . We have evaluation maps

$$\text{ev} : \mathcal{M}_{k+1}(\beta; L) \rightarrow L^k$$

at the boundary marked points.

Let  $H_2^{\text{eff}}(X, L) \subset H_2(X, L; \mathbb{Z})$  denote the monoid generated by effective ( $J_X$ -holomorphic) disc classes, i.e.

$$H_2^{\text{eff}}(X, L) = \{\beta \in H_2(X, L; \mathbb{Z}) \mid \mathcal{M}(\beta; L) \neq \emptyset\}.$$

We denote by  $\beta_0$  the constant disc class.

For  $\beta \in H_2^{\text{eff}}(X, L)$ , we define

$$\|\beta\| = \begin{cases} \sup\{n \mid \exists \beta_1, \dots, \beta_n \in H_2^{\text{eff}}(X, L) \setminus \{\beta_0\}, \text{ with } \sum_{i=1}^n \beta_i = \beta\} + [\omega(\beta)] - 1, & \text{if } \beta \neq \beta_0; \\ -1, & \text{if } \beta = \beta_0. \end{cases}$$

Let  $\mathfrak{d} : \{1, \dots, k\} \rightarrow \mathbb{Z}_{\geq 0}$ . For a pair  $(\mathfrak{d}, \beta)$  with  $\beta \in H_2^{\text{eff}}(X, L)$ , we define

$$\|(\mathfrak{d}, \beta)\| = \begin{cases} \max_{i \in \{1, \dots, k\}} \{\mathfrak{d}(i)\} + \|\beta\| + k & \text{if } k \neq 0; \\ \|\beta\| & \text{if } k = 0. \end{cases}$$

Let  $\vec{P} = (P_1, \dots, P_k)$  be a  $k$ -tuple of singular chains. We denote by  $\mathcal{M}_{k+1}(\beta; L; \vec{P})$  the fiber product (in the sense of Kuranishi structures) of  $\mathcal{M}_{k+1}(\beta; L)$  with smooth singular chains  $P_i$  on  $L$ ,  $i = 1, \dots, k$ ,

$$\mathcal{M}_{k+1}(\beta; L; \vec{P}) = \mathcal{M}_{k+1}(\beta; L) \times_{L^k} \prod P_i.$$

In [FOOO09b], countable sets  $\mathcal{X}_g(L)$  of smooth singular chains on  $L$  were constructed, and a system of multisections  $\mathfrak{s}_{\mathfrak{d}, k, \beta, \vec{P}}$  for  $\mathcal{M}_{k+1}^{\mathfrak{d}}(\beta; L; \vec{P})$ ,  $\|(\mathfrak{d}, \beta)\| = g$ , were chosen inductively on the generation  $g$ . Here the decoration  $\mathfrak{d}$  emphasizes the generations of the inputs  $P_i \in \mathcal{X}_{\mathfrak{d}(i)}(L)$ .

At each inductive step, new multisections  $\mathfrak{s}_{\mathfrak{d}, k, \beta, \vec{P}}$  are chosen to be transversal to zero section and extend the multisections previously chosen for the boundary strata  $\partial \mathcal{M}_{k+1}(\beta; L; \vec{P})$ . Moreover the zero locus

$$\mathcal{M}_{k+1}^{\mathfrak{d}}(\beta; L; \vec{P})^{\mathfrak{s}_{\mathfrak{d}, k, \beta, \vec{P}}} := \mathfrak{s}_{\mathfrak{d}, k, \beta, \vec{P}}^{-1}(0)$$

is triangulated extending the triangulation on the boundary strata. The new simplices in the triangulation are then regarded as elements of  $\mathcal{X}_g(L)$ . Additional singular simplices are then added to  $\mathcal{X}_g(L)$  so that

$$C_{(g)}(L; \Lambda_0) = \bigoplus_{g' \leq g} \Lambda_0 \cdot \mathcal{X}_{g'}(L)$$

remains a subcomplex of  $S^\bullet(L; \Lambda_0)$  isomorphic on cohomology.

Let  $C^\bullet(L; \Lambda_0) = \lim_{\rightarrow} C_{(g)}(L; \Lambda_0)$ . The  $A_\infty$  operations  $\tilde{\mathfrak{m}}_k : C^\bullet(L; \Lambda_0)^{\otimes k} \rightarrow C^\bullet(L; \Lambda_0)$  are defined by

$$\tilde{\mathfrak{m}}_k(P_1, \dots, P_k) = \sum_{\beta \in H_2^{\text{eff}}(X, L)} \mathbf{T}^{\omega_X(\beta)} \mathfrak{m}_{k, \beta}(P_1, \dots, P_k) \quad (2.1)$$

where

$$\tilde{\mathfrak{m}}_{k, \beta}(P_1, \dots, P_k) = (\text{ev}_0)_* \left( \mathcal{M}_{k+1}(\beta; L; \vec{P})^{\mathfrak{s}} \right) \quad (2.2)$$

for  $(k, \beta) \neq (1, \beta_0)$ , and

$$\tilde{\mathfrak{m}}_{1, \beta_0}(P) = (-1)^d \partial P,$$

where  $\partial$  is the coboundary map on  $C^\bullet(L; \Lambda_0)$ . The maps  $\tilde{\mathfrak{m}}_{k,\beta}$  are of degree  $2 - k - \mu(\beta)$ .

**Remark 2.1.** We do not perturb the input singular chains when choosing perturbation for a fiber product  $\mathcal{M}_{k+1}(\beta; L; \vec{P})$  since the singular chains are fixed during the inductive construction and doing so would destroy the  $A_\infty$ -structure. To achieve transversality, the Kuranishi structures for the moduli spaces  $\mathcal{M}_{k+1}(\beta; L)$  are chosen to be weakly submersive, meaning that the evaluation map  $\text{ev} : V_p \rightarrow L^k$  from the Kuranishi neighborhood  $V_p$  of an element  $p = [\Sigma, \vec{z}, u] \in \mathcal{M}_{k+1}(\beta; L)$  (which parametrizes smooth stable maps close to  $u$ ) to  $L^k$  is a submersion (see [FOOO09b, Lemma 7.1.18]).

**Remark 2.2.** We note that it is only possible to choose finitely many multisections at once while still having them being sufficiently close to the original Kuranishi maps. To deal with this technical difficulty, Fukaya-Oh-Ohta-Ono introduced the notion of  $A_{n,K}$ -structure associated to  $L$  using moduli space of pseudo-holomorphic discs with bounded energy and number of marked points in [FOOO09b, Chapter 7]. The  $A_\infty$ -structure  $\tilde{\mathfrak{m}}$  on  $C^\bullet(L; \Lambda_{0,\text{nov}})$  is obtained from the  $A_{n,K}$ -structures via homological techniques.

**Remark 2.3.** In order to define a  $\mathbb{Z}$ -graded cohomology theory, the conventional grading of the map  $\mathfrak{m}_k$  is  $2 - k$ . This means beyond the Calabi-Yau case, one has to define the Novikov ring  $\Lambda_0$  using an extra grading parameter in order to compensate for the Maslov indices. In this paper, we work on the chain level and do not follow this convention. The grading of  $\mathfrak{m}_{k,\beta}$  is crucial in understanding the vanishing of certain terms when doing computations with the Morse model.

The singular chain model  $(C^\bullet(L; \Lambda_0), \tilde{\mathfrak{m}})$  does not have a strict unit in general. In [FOOO09b, Chapter 7], it was shown that the fundamental cycle  $e$  of  $L$  is a homotopy unit. We briefly describe the key ideas below and refer the readers to [FOOO09b, Chapter 3.3] for the precise definition of a homotopy unit and to [FOOO09b, Chapter 7.3] for the details of the homotopy unit construction.

We enlarge  $(C^\bullet(L; \Lambda_0), \tilde{\mathfrak{m}})$  to a homotopy equivalent unital  $A_\infty$ -algebra  $(C^\bullet(L; \Lambda_0)^+, \tilde{\mathfrak{m}}^+)$  by adding a degree 0 generator  $e^+$  serving as the strict unit and a degree  $-1$  generator  $f$  serving as a homotopy between  $e$  and  $e^+$

$$C^\bullet(L; \Lambda_0)^+ = C^\bullet(L; \Lambda_0) \oplus \Lambda_0 \cdot e^+ \oplus \Lambda_0 \cdot f.$$

The  $A_\infty$  operations  $\tilde{\mathfrak{m}}_k^+$  are defined as follows.

The restriction of  $\tilde{\mathfrak{m}}^+$  to  $C^\bullet(L; \Lambda_0)$  agrees with  $\tilde{\mathfrak{m}}$ .

$e^+$  is the strict unit, i.e.

$$\tilde{\mathfrak{m}}_2^+(e^+, x) = (-1)^{|x|} \tilde{\mathfrak{m}}_2^+(x, e^+) = x, \quad (2.3)$$

for  $x \in C^\bullet(L; \Lambda_0)^+$ , and

$$\tilde{\mathfrak{m}}_k^+(\dots, e^+, \dots) = 0 \quad (2.4)$$

for  $k \neq 2$ .

$f$  is the homotopy between  $e$  and  $e^+$  in the following sense. Let  $\vec{a} = \{a_1, \dots, a_{|\vec{a}|}\} \subset \{1, \dots, k\}$ ,  $a_1 < \dots < a_{|\vec{a}|}$ . For  $\vec{P} = (P_1, \dots, P_{k-|\vec{a}|})$  with  $P_i \in \mathcal{X}_{\partial(i)}(L)$ , let  $\vec{P}^+$  be the  $k$ -tuple obtained by inserting  $e$  into the  $a_1, \dots, a_{|\vec{a}|}$ -th places of  $\vec{P}$ . We will assume  $L \in \mathcal{X}_0(L)$  and set  $\partial^+(i) = g_i^+$ , if  $\vec{P}^+ = (P_1^+, \dots, P_k^+)$  and  $P_i^+ \in \mathcal{X}_{g_i^+}(L)$ . We have the map

$$\text{forget}_{\vec{a}} : \mathcal{M}_{k+1}(\beta; L; \vec{P}^+) \rightarrow \mathcal{M}_{k+1-|\vec{a}|}(\beta; L; \vec{P}), \quad (2.5)$$

which forgets the  $a_1, \dots, a_{|\vec{a}|}$ -th marked points (and stabilizes).

Let  $\vec{a}^1 \sqcup \vec{a}^2 = \vec{a}$  be a splitting of  $\vec{a}$ . Denote by  $\vec{P}'$  the  $(k - |\vec{a}^1|)$ -tuple given by removing  $e$  from the  $a_1^1, \dots, a_{|\vec{a}^1|}^1$ -th places of  $\vec{P}^+$ . For the choices of perturbations for  $\mathcal{M}_{k+1}(\beta; L; \vec{P}^+)$ ,

we have the perturbation  $\mathfrak{s}_{\partial^+, k, \beta, \bar{P}^+}$  obtained by just inserting  $e$  into the  $a_1, \dots, a_{|\bar{a}|}$ -th place and the perturbations pulled back via the map  $\text{forget}_{\bar{a}^1, \bar{a}^2} : \mathcal{M}_{k+1}(\beta; L; \bar{P}^+) \rightarrow \mathcal{M}_{k+1-|\bar{a}^1|}(\beta; L; \bar{P}')$  which forgets the  $a_1^1, \dots, a_{|\bar{a}^1|}^1$ -th marked points and stabilizes. Moreover, we have a perturbation  $\mathfrak{s}_{\partial^+, k, \beta, \bar{P}^+}^+$  (transversal to the zero section) on  $[0, 1]^{|\bar{a}|} \times \mathcal{M}_{k+1}(\beta; L; \bar{P}^+)$  interpolating between them.

The operations  $m_{k, \beta}^+(k, \beta) \neq (1, \beta_0)$ , with inputs  $f$  inserted into  $a_1, \dots, a_{|\bar{a}|}$ -th place of  $\bar{P}$  are defined by

$$(\text{ev}_0)_* \left( \left( [0, 1]^{|\bar{a}|} \times \mathcal{M}_{k+1}(\beta; L; \bar{P}^+) \right)^{\mathfrak{s}_{\partial^+, k, \beta, \bar{P}^+}^+} / \sim \right). \quad (2.6)$$

Here  $\sim$  is the equivalence relation collapsing fibers of the forgetful maps (see [FOOO09b, Definition 7.3.28]). The singular simplices which appear in (2.6) are also included in the inductive construction of  $\mathcal{X}_g(L)$ .

We also set

$$\tilde{m}_{1, \beta_0}^+(f) = e^+ - e$$

and therefore

$$\tilde{m}_1^+(f) = e^+ - e + h, \quad (2.7)$$

$$h = \sum_{\beta \neq \beta_0} T^{\omega_X(\beta)} (\text{ev}_0)_* \left( ([0, 1] \times \mathcal{M}_2(\beta; L; e))^{\mathfrak{s}^+} / \sim \right) \equiv 0 \pmod{\Lambda_+ \cdot C^\bullet(L; \Lambda_0)}.$$

The terms

$$\sum_{\mu(\beta) \geq 2} (\text{ev}_0)_* \left( ([0, 1] \times \mathcal{M}_2(\beta; L; e))^{\mathfrak{s}^+} / \sim \right)$$

are of degrees at most  $-2$  and therefore are degenerate singular chains.

**2.2 The non-equivariant Morse model with a strict unit.** Pearl complex (Morse model of Lagrangian Floer theory) was developed by Biran and Cornea [BC07, BC09] for monotone Lagrangians (a similar idea also appeared in an earlier work of Oh [Oh96]). It has many important applications, including the proof of homological mirror symmetry for Fermat Calabi-Yau hypersurfaces by Sheridan [She15].

A Morse model of Lagrangian Floer theory was constructed in [FOOO09a] based on their singular chain model. Note that degenerate chains are not mod out in this singular chain model. We follow their construction, with a modification that we add certain degenerate chains as summands in the definition of unstable chains  $\Delta_p$  of critical points  $p$  and the chains of forward orbits  $G(\tau)$  of singular chains  $\tau$ . The main reason is that there are unwanted degenerate chains appeared in  $\partial\Delta_p$  and  $\partial G(\tau)$ , and we ‘contract them away’ by adding the cones over the unwanted terms.

Moreover, by adapting the homotopy unit construction in [FOOO09b, Chapter 7], the Morse model we construct will be unital. In Charest-Woodward [CW], a unital Morse model was constructed also using the idea of the homotopy unit construction.

Let  $f : L \rightarrow \mathbb{R}$  be Morse function. Let  $V$  be a negative pseudo-gradient vector field for  $f$ , i.e.  $df(V)|_p \leq 0$ , with equality if and only if  $p \in \text{Crit}(f)$ ; and for each  $p \in \text{Crit}(f)$ ,  $V$  coincides with the negative gradient vector field for the Euclidean metric on a Morse

chart of  $p$ . Let  $\Phi_t$  be the flow of  $V$ . For each  $p \in \text{Crit}(f)$ , we denote by  $W^s(f; p)$  and  $W^u(f; p)$  its stable and unstable manifolds respectively. Namely,

$$W^s(f; p) = \left\{ q \in L \mid \lim_{t \rightarrow +\infty} \Phi_t(q) = p \right\},$$

and

$$W^u(f; p) = \left\{ q \in L \mid \lim_{t \rightarrow -\infty} \Phi_t(q) = p \right\}.$$

We define the degree  $|p|$  of  $p$  by

$$|p| := d - \text{ind}(p),$$

where  $\text{ind}(p)$  is the Morse index of  $p$ . Then

$$|p| = \dim W^s(f; p) = \text{codim} W^u(f; p).$$

We will assume that  $V$  satisfies the Smale conditions and call such a pair  $(f, V)$  Morse-Smale. For  $p, q \in \text{Crit}(f)$ , let  $\mathcal{M}(p, q)$  be the moduli space of flow lines from  $p$  to  $q$ . By the Smale conditions, we have

$$\dim \mathcal{M}(p, q) = |q| - |p| - 1.$$

It is well known that (see e.g. [Hut08])  $\mathcal{M}(p, q)$  and  $W^u(f; p)$  have natural compactifications to smooth manifolds with corners  $\overline{\mathcal{M}}(p, q)$  and  $\overline{W}^u(f; p)$  whose codimension  $k$  stratum consists of  $k$ -times broken flow lines

$$\overline{\mathcal{M}}(p, q)_k = \bigcup_{r_0=p, r_1, \dots, r_k, r_{k+1}=q \text{ distinct}} \mathcal{M}(r_0, r_1) \times \dots \times \mathcal{M}(r_k, r_{k+1}).$$

and

$$\overline{W}^u(f; p)_k = \bigcup_{r_0=p, r_1, \dots, r_k \text{ distinct}} \mathcal{M}(r_0, r_1) \times \dots \times \mathcal{M}(r_{k-1}, r_k) \times W^u(f; r_k),$$

In particular, we have

$$\partial \overline{\mathcal{M}}(p, q) = \bigcup_r \mathcal{M}(p, r) \times \mathcal{M}(r, q),$$

and

$$\partial \overline{W}^u(f; p) = \bigcup_r \mathcal{M}(p, r) \times W^u(f; r). \quad (2.8)$$

Let  $C^\bullet(f; \Lambda_0)$  be the cochain complex

$$C^\bullet(f; \Lambda_0) = \bigoplus_{p \in \text{Crit}(f)} \Lambda_0 \cdot p,$$

with grading given by  $|p|$ .

In [FOOO09a], a (non-unital)  $A_\infty$ -algebra structure was constructed on  $C^\bullet(f; \Lambda_0)$  assuming  $L$  has a triangulation whose simplices are the closures of  $W^u(f; p)$ . To establish an isomorphism between Morse and singular cohomology, we have to associate to each critical point  $p$  of degree  $|p|$  a singular chain  $\Delta_p \in S^{|p|}(L; \mathbb{Z})$ . A natural candidate for  $\Delta_p$  is a fundamental chain for  $\overline{W}^u(f; p)$ . Suppose we choose such a chain  $\Delta_p$ . From (2.8), we can see that in general  $\overline{W}^u(f; p)$  has boundary components of the form  $\overline{\mathcal{M}}(p, r) \times \overline{W}^u(f; r)$  with  $|r| \geq |p| + 2$ . Since  $\dim \overline{\mathcal{M}}(p, r) \geq 1$ , and the image of  $\overline{\mathcal{M}}(p, r) \times \overline{W}^u(f; r)$  in  $L$  is supported on  $\overline{W}^u(f; r)$ , the facet of  $\Delta_p$  representing  $\overline{\mathcal{M}}(p, r) \times \overline{W}^u(f; r)$  is a degenerate chain on  $L$ . Thus, for the assignment  $p \mapsto \Delta_p$  to be a chain map, the standard approach is

to mod out by degenerate chains. On the other hand,  $\mathcal{X}_g(L)$  are constructed including degenerate chains since the  $\tilde{m}_k$  product of degenerate chains may no longer be a degenerate chain.

We overcome this difficulty by adding certain degenerate chains to the fundamental chains (of the compactified unstable submanifolds) and define a map  $\iota : C^\bullet(f; \mathbb{Z}) \rightarrow S^\bullet(L; \mathbb{Z})$ ,  $\iota(p) = \Delta_p$  accordingly such that it is a chain map which induces an isomorphism on cohomology. See Figure 1.

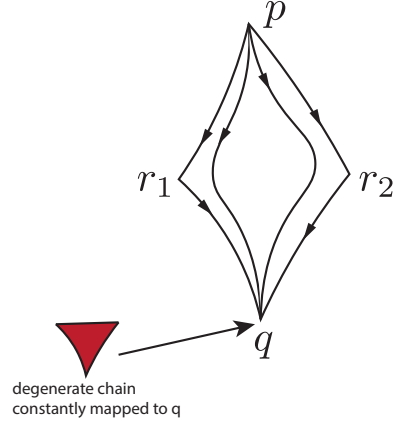


FIGURE 1. An example of the added degenerate chain.

We choose smooth cubical singular chains  $d_p \in C_{d-|p|}^{\square}(\overline{W}^u(f; p); \mathbb{Z})$  and  $m_{p,q} \in C_{|q|-|p|-1}^{\square}(\overline{\mathcal{M}}(p, q); \mathbb{Z})$  representing the fundamental classes of  $\overline{W}^u(f; p)$  and  $\overline{\mathcal{M}}(p, q)$ , respectively, satisfying

$$\partial d_p = \sum_r (-1)^{|r|-|p|-1} m_{p,r} \times d_r, \quad (2.9)$$

and

$$\partial m_{p,q} = \sum_r (-1)^{|r|-|p|-1} m_{p,r} \times m_{r,q}. \quad (2.10)$$

To regard  $d_p$  as simplicial singular chains, we choose triangulations for (the domain of)  $d_p$  and  $m_{p,q}$  inductively on  $\text{ind}(p) = d - |p|$ . For  $\text{ind}(p) = 0$ ,  $d_p$  and  $m_{p,q}$  are 0-simplices. Suppose we have chosen triangulations for  $d_p$  and  $m_{p,q}$  for  $\text{ind}(p) < \ell$ . For  $\text{ind}(p) = \ell$ ,  $m_{p,q}$  with  $|q| = |p| + 1$  are 0-simplices. Suppose we have chosen triangulation for  $m_{p,q}$  for  $|q| - |p| = 1, \dots, m'$ ,  $m' < m$ . For  $|q| - |p| = m$ , we first triangulate  $\partial m_{p,q}$  as follows: each  $m_{p,r} \times m_{r,q}$  in  $\partial m_{p,q}$  is a product of simplicial complexes. We choose linear orders on vertices of  $m_{p,r}$  and  $m_{r,q}$  (compatible with their orientations). Then, there exists a unique triangulation  $m_{p,r} \otimes m_{r,q}$  of  $m_{p,r} \times m_{r,q}$  such that the vertices of  $m_{p,r} \otimes m_{r,q}$  are pairs  $(x, y)$ , where  $x$  is a vertex of  $m_{p,r}$  and  $y$  is a vertex of  $m_{r,q}$ , and an  $n$ -simplex in  $m_{p,r} \otimes m_{r,q}$  is defined by a set  $\{(x_0, y_0), \dots, (x_n, y_n)\}$  such that  $x_0 \leq \dots \leq x_n, y_0 \leq \dots \leq y_n$ ,  $\{x_0, \dots, x_n\}$  defines a simplex  $\tau_1$  on  $m_{p,r}$  and  $\{y_0, \dots, y_n\}$  defines a simplex  $\tau_2$  on  $m_{r,q}$  with  $\dim \tau_1 + \dim \tau_2 \geq n$ . Notice that this is the standard product structure in the category of simplicial sets. We then triangulate  $m_{p,q}$  extending the triangulation on  $\partial m_{p,q}$ . Finally, we triangulate  $d_p$  extending the triangulation on  $\partial d_p$  given by  $m_{p,r} \otimes d_r$ .

For each  $m_{p,r} \times d_r$  in the boundary of  $d_p$ , let  $Cm_{p,r}$  be the (simplicial) cone over  $m_{p,r}$  with vertices

$$\{\text{vertices of } Cm_{p,r}\} = \{\text{vertices of } m_{p,r}\} \cup \{*\},$$

and simplices

$\{n\text{-simplices of } Cm_{p,r}\} = \{n\text{-simplices of } m_{p,r}\} \cup \{\text{cones of } n-1\text{-simplices of } m_{p,r} \text{ with the vertex } \{*\}\},$

for  $n \geq 1$ . In particular, this means  $\{*\}$  is the vertex of the cone. As a singular chain on  $L$ , the map  $Cm_{p,r} \rightarrow L$  is defined by composing the contraction  $Cm_{p,r} \rightarrow m_{p,r}$  with the map  $m_{p,r} \rightarrow L$ . Note that for two simplicial complexes  $A$  and  $B$ , we have  $C(A \otimes B) = CA \otimes CB$ .

We construct  $\Delta_p$  inductively on  $\text{ind}(p)$  as follows: For  $\text{ind}(p) = 0$ , we set  $\Delta_p = d_p$ . Suppose we have constructed  $\Delta_p$  for  $\text{ind}(p) < \ell$ . For  $\text{ind}(p) = \ell$ , we define

$$d_p^\diamond = d_p + \sum_{m_{p,r} \otimes d_r} m_{p,r} \otimes \Delta_r,$$

and set

$$\Delta_p = d_p^\diamond + \sum_{m_{p,r} \otimes \Delta_r} Cm_{p,r} \otimes \Delta_r. \quad (2.11)$$

By construction, we have

$$\partial \Delta_p = \sum_{|r|=|p|+1} m_{p,r} \times \Delta_r. \quad (2.12)$$

**Theorem 2.4.** *The map  $\iota : C^\bullet(f; \mathbb{Z}) \rightarrow S^\bullet(L; \mathbb{Z})$ ,  $\iota(p) = \Delta_p$ , with  $\Delta_p$  defined as in (2.11), is a chain map which induces an isomorphism on cohomology  $H^\bullet(C^\bullet(f; \mathbb{Z}); \delta) \cong H^\bullet(L; \mathbb{Z})$ . Here  $\delta$  denotes the Morse differential.*

If we were to consider  $S^\bullet(L; \mathbb{Z})$  modded out by the subcomplex of degenerate chains, a proof can be found in [HL99]. Since we do not wish to mod out by degenerate chains, we would have to modify their arguments. Let's first introduce some terminologies.

We say a smooth singular chain is *generic* if every face of every simplex intersects the stable submanifolds of the critical points of  $f$  transversely. Let  $C_{gen}^\bullet(L; \mathbb{Z}) \subset S^\bullet(L; \mathbb{Z})$  be the subcomplex of generic singular chains. It is easy to see that  $C_{gen}^\bullet(L; \mathbb{Z})$  and  $S^\bullet(L; \mathbb{Z})$  are isomorphic on cohomology.

For a generic simplex  $\tau$ , we denote by  $\mathcal{M}(\tau, q)$  the moduli space of flow lines from  $\tau$  to a critical point  $q$ . Its dimension is given by  $\dim \mathcal{M}(\tau, q) = \dim \tau - \text{ind}(q)$ . Let  $\tau(j)$  denote the codimension  $j$  stratum of  $\tau$ .  $\mathcal{M}(\tau, q)$  has a natural compactification to a smooth manifold with corners  $\overline{\mathcal{M}}(\tau, q)$ , whose codimension  $k$  stratum is

$$\overline{\mathcal{M}}(\tau, q)_k = \bigcup_{j=0}^k \bigcup_{p_1, \dots, p_j, q \text{ distinct}} \mathcal{M}(\tau(k-j), p_1) \times \mathcal{M}(p_1, p_2) \times \dots \times \mathcal{M}(p_{j-1}, p_j) \times \mathcal{M}(p_j, q).$$

For  $k = 1$ , we have

$$\partial \overline{\mathcal{M}}(\tau, q) = \mathcal{M}(\partial \tau, q) \bigcup_{p \neq q} (-1)^{\dim \tau + \text{ind}(q)} \mathcal{M}(\tau, p) \times \mathcal{M}(p, q).$$

We define the *forward orbit*  $\mathcal{F}(\tau)$  of  $\tau$  to be the set

$$\mathcal{F}(\tau) := [0, \infty) \times \tau$$

together with the map  $e : \mathcal{F}(\tau) \rightarrow L$  defined by

$$e(t, x) = \Phi_t(\tau(x)).$$

$\mathcal{F}(\tau)$  has a natural compactification to a smooth manifold with corners  $\overline{\mathcal{F}}(\tau)$ , whose codimension  $k$  stratum is

$$\overline{\mathcal{F}}(\tau)_k = \mathcal{F}(\tau(k)) \bigcup_{j=1}^k \bigcup_{r_1, \dots, r_j, q \text{ distinct}} \mathcal{M}(\tau(k-j), r_1) \times \mathcal{M}(r_1, r_2) \times \dots \times \mathcal{M}(r_{j-1}, r_j) \times W^u(r_j).$$

When  $k = 1$ , we have

$$\partial \overline{\mathcal{F}}(\tau) = -\tau \cup -\mathcal{F}(\partial\tau) \cup \bigcup_{p \in \text{Crit}(f)} \mathcal{M}(\tau, p) \times W^u(p).$$

*Proof.* The assertion that  $\iota$  is a chain map follows from (2.12). To show that  $\iota$  induces an isomorphism on cohomology, we use a modification of the proof found in [HL99]. We have a chain map  $A : C_{\text{gen}}^\bullet(L; \mathbb{Z}) \rightarrow C^\bullet(f; \mathbb{Z})$  defined by

$$A(\tau) = \sum_{\text{ind}(q) = \dim \tau} \#\mathcal{M}(\tau, q) \cdot q.$$

In particular,  $\mathcal{M}(\tau, p)$  is empty when  $\tau$  is a degenerate simplex. Clearly,  $A \circ \iota = id : C^\bullet(f; \mathbb{Z}) \rightarrow C^\bullet(f; \mathbb{Z})$ . We also have a map  $G : C_{\text{gen}}^\bullet(L; \mathbb{Z}) \rightarrow S^{\bullet-1}(L; \mathbb{Z})$  defined as follows. Let  $m_{\tau, q} \in C_{\dim \tau - \text{ind}(q)}^\square(\overline{\mathcal{M}}(\tau, q); \mathbb{Z})$  be smooth cubical singular chains representing the fundamental classes of  $\overline{\mathcal{M}}(\tau, q)$  satisfying

$$\partial m_{\tau, q} = m_{\partial\tau, q} + \sum_{p \neq q} (-1)^{\dim \tau + \text{ind}(q)} m_{\tau, p} \times m_{p, q}.$$

We again triangulate  $m_{\tau, q}$  inductively on the dimension to regard them as simplicial singular chains. We also choose smooth singular chains  $F(\tau)$  representing the fundamental classes of  $\overline{\mathcal{F}}(\tau)$  satisfying

$$\partial F(\tau) = -\tau - F(\partial\tau) + \sum_r m_{\tau, r} \times d_r. \quad (2.13)$$

We then construct  $G(\tau)$  inductively on  $\dim \tau$  by first replacing  $m_{\tau, r} \times d_r$  with  $m_{\tau, r} \times \Delta_r$  on  $\partial F(\tau)$ , then glue  $Cm_{\tau, r} \times \Delta_r$  to the boundary of the resulting chain. This ensures that  $G$  is a chain homotopy between  $\iota \circ A$  and the identity, i.e.,  $G$  satisfies

$$\Delta_{A(\tau)} - \tau = \partial G(\tau) + G(\partial\tau). \quad (2.14)$$

□

Below we explain the construction of the strict unit in our setting. For simplicity, we will always assume  $f$  has a unique maximum point  $\mathbf{1}^\blacktriangledown$ , so that  $\Delta_{\mathbf{1}^\blacktriangledown} = e$  is the fundamental cycle.

Let

$$CF^\bullet(L; \Lambda_0) = C^\bullet(f; \Lambda_0) \oplus \Lambda_0 \cdot \mathbf{1}^\blacktriangledown \oplus \Lambda_0 \cdot \mathbf{1}^\blacktriangledown \quad (2.15)$$

with  $|\mathbf{1}^\blacktriangledown| = 0$  and  $|\mathbf{1}^\blacktriangledown| = -1$ . The superscripts  $\blacktriangledown$ ,  $\blacktriangledown$  and  $\blacktriangledown$  are borrowed from [CW]. We extend the Morse differential  $\delta$  to  $CF^\bullet(L; \Lambda_0)$  by setting

$$\delta(\mathbf{1}^\blacktriangledown) = 0, \quad \delta(\mathbf{1}^\blacktriangledown) = (-1)^d (\mathbf{1}^\blacktriangledown - \mathbf{1}^\blacktriangledown). \quad (2.16)$$

We now construct a unital  $A_\infty$ -algebra structure on  $CF^\bullet(L; \Lambda_0)$ . Let

$$\mathcal{X}_{-1}(L) = \{\Delta_p | p \in \text{Crit}(f)\}.$$

For  $g \geq 0$ , suppose  $\mathcal{X}_{g'}(L)$  had been constructed for  $g' < g$ . The perturbations  $\mathfrak{s}_{\partial, \ell, \beta, \bar{p}}$  and  $\mathfrak{s}_{\partial^+, k, \beta, \bar{p}^+}$  in Section 2.1 can be chosen to have the following property:

Each face  $\tau$  of a simplex in the triangulation of

$$(\text{ev}_0)_* (\mathcal{M}_{\ell+1}(\beta; L; P_1, \dots, P_\ell)^{\mathfrak{s}_{\partial, \ell, \beta, \bar{P}}}),$$

$\|(\partial, \beta)\| = g$ , or in the triangulation of

$$(\text{ev}_0)_* \left( [0, 1]^{|\bar{a}|} \times \mathcal{M}_{k+1}(\beta; L; \bar{P}^+)^{\mathfrak{s}_{\partial^+, k, \beta, \bar{P}^+}} \right),$$

$\|(\partial^+, \beta)\| = g$  is transversal to the stable submanifold  $W^s(f; p)$  for all  $p \in \text{Crit}(f)$ . Moreover, for each  $\tau$  of dimension at most  $d$ , there exists at most one critical point  $p(\tau) \in \text{Crit}(f)$  such that  $W^s(f; p(\tau))$  is of complementary dimension to  $\tau$  and intersects  $\tau$  at a unique point.

Denote by  $\mathcal{X}_g^\circ(L)$  the set of these singular simplices  $\tau$ . Define  $\Pi(\tau) = \epsilon \Delta_{p(\tau)}$  where  $\epsilon = \pm 1$  is given by

$$\tau \times_L W^s(f; p(\tau)) = \epsilon W^u(f; p(\tau)) \cap W^s(f; p(\tau)),$$

if there exists a unique  $p(\tau)$  such that  $W^s(f; p(\tau))$  intersects  $\tau$  at a unique point, and  $\Pi(\tau) = 0$ , otherwise. In particular, we have  $\Pi(\tau) = 0$  whenever  $\tau$  is a degenerate simplex. We further add simplices  $G(\tau)$  (defined as in Theorem 2.4),  $\tau \in \mathcal{X}_g^\circ(L)$  to obtain  $\mathcal{X}_g(L)$ . For chains of the form  $G(\tau)$ , we put  $G(G(\tau)) = 0$ . The maps  $\Pi : C^\bullet(L; \Lambda_0) \rightarrow C_{(-1)}^\bullet(L; \Lambda_0)$  and  $G : C^\bullet(L; \Lambda_0) \rightarrow C^{\bullet-1}(L; \Lambda_0)$  satisfy

$$\Pi(\tau) - \tau = (\partial G(\tau)) + G(\partial\tau), \quad (2.17)$$

for  $\tau \in C^\bullet(L; \Lambda_0)$ . Let

$$C_{(-1)}^\bullet(L; \Lambda_0)^+ = C_{(-1)}^\bullet(L; \Lambda_0) \oplus \Lambda_0 \cdot e^+ \oplus \Lambda_0 \cdot f.$$

We recall that  $e^+$  is the unit and  $f$  is the homotopy between  $e^+$  and the fundamental class  $e$ . We extend the coboundary operator  $\partial$  to  $C_{(-1)}^\bullet(L; \Lambda_0)^+$  by setting

$$\partial e^+ = 0, \quad \partial f = (-1)^d (e^+ - e).$$

We then extend  $\Pi$  and  $G$  to maps  $\Pi : C_{(-1)}^\bullet(L; \Lambda_0)^+ \rightarrow C_{(-1)}^\bullet(L; \Lambda_0)^+$  and  $G : C_{(-1)}^\bullet(L; \Lambda_0)^+ \rightarrow C_{(-1)}^\bullet(L; \Lambda_0)^+$  respectively by setting

$$\Pi(e^+) = e, \quad \Pi(f) = 0,$$

and

$$G(e^+) = -f, \quad G(f) = 0,$$

so that they satisfy (2.17) for  $\tau \in C_{(-1)}^\bullet(L; \Lambda_0)^+$ .

Homological perturbation can then be applied to reduce the  $A_\infty$ -algebra structure on  $C_{(-1)}^\bullet(L; \Lambda_0)^+$  to  $C_{(-1)}^\bullet(L; \Lambda_0)^+$  (see [FOOO09a, Theorem 3.3]). We identify the latter with  $CF^\bullet(L; \Lambda_0)$  via the map  $\iota : CF^\bullet(L; \Lambda_0) \rightarrow C_{(-1)}^\bullet(L; \Lambda_0)^+$

$$\iota(p) = \Delta_p, \quad \iota(\mathbf{1}^\nabla) = e^+, \quad \iota(\mathbf{1}^\vee) = f. \quad (2.18)$$

We will denote the resulting unital  $A_\infty$ -algebra by  $(CF^\bullet(L; \Lambda_0), \mathfrak{m})$ .

Similar to Theorem 5.1 in [FOOO09a], we have

**Theorem 2.5.**  *$(CF^\bullet(L; \Lambda_0), \mathfrak{m})$  is a unital  $A_\infty$ -algebra.  $\mathfrak{f}$  is a unital homotopy equivalence between  $(CF^\bullet(L; \Lambda_0), \mathfrak{m})$  and  $(C_{(-1)}^\bullet(L; \Lambda_0)^+, \tilde{\mathfrak{m}}^+)$ .*

Explicitly, the operations  $\mathfrak{m}_k$  are given in terms of maps  $\mathfrak{m}_\Gamma$  and  $\mathfrak{f}_\Gamma$  associated to *decorated planar rooted trees*.

**Definition 2.6** ([FOOO09a]). A decorated planar rooted tree is a quintuple  $\Gamma = (T, \iota, v_0, V_{\text{tad}}, \eta)$  consisting of

- $T$  is a tree;
- $\iota : T \rightarrow D^2$  is an embedding into the unit disc;
- $v_0$  is the root vertex and  $\iota(v_0) \in \partial D^2$ ;
- $V_{\text{tad}}$  is the set of interior vertices with valency 1;
- $\eta : V(\Gamma)_{\text{int}} \rightarrow \mathbb{Z}_{\geq 0}$ .

where  $V(\Gamma)$  is the set of vertices;  $V(\Gamma)_{\text{ext}} = \iota^{-1}(\partial D^2)$  is the set of exterior vertices and  $V(\Gamma)_{\text{int}} = V(\Gamma) \setminus V(\Gamma)_{\text{ext}}$  is the set of interior vertices. For  $k \geq 0$ , denote by  $\Gamma_{k+1}$  the set of isotopy classes represented by  $\Gamma = (T, \iota, v_0, \eta)$  with  $|V(\Gamma)_{\text{ext}}| = k + 1$  and  $\eta(v) > 0$  if the valency  $\ell(v)$  of  $v$  is 1 or 2. In other words, the elements of  $\Gamma_{k+1}$  are stable.

We will refer to the elements of  $\Gamma_{k+1}$  as stable trees.

Denote by  $\Gamma^0 \in \Gamma_2$  the unique tree with no interior vertices. For this tree  $\Gamma_0$ , we define

$$\mathfrak{m}_{\Gamma_0} := \tilde{\mathfrak{m}}_{1, \beta_0} : CF^\bullet(L; \Lambda_0) \rightarrow CF^\bullet(L; \Lambda_0),$$

and

$$\mathfrak{f}_{\Gamma_0} : CF^\bullet(L; \Lambda_0) \rightarrow C^\bullet(L; \Lambda_0)^+$$

the inclusion map.

For each  $k \geq 0$ ,  $\Gamma_{k+1}$  contains a unique element that has a single interior vertex  $v$ , which is denoted by  $\Gamma_{k+1}$ . We fix a labeling  $\{\beta_0, \beta_1, \dots\}$  of elements of  $H_2^{\text{eff}}(X, L)$ , with  $\beta_0$  the constant disc class and define

$$\mathfrak{m}_{\Gamma_{k+1}} = \Pi \circ \tilde{\mathfrak{m}}_{k, \beta_{\eta(v)}}^+,$$

and

$$\mathfrak{f}_{\Gamma_{k+1}} = G \circ \tilde{\mathfrak{m}}_{k, \beta_{\eta(v)}}^+.$$

For general  $\Gamma$ , we cut it at the vertex  $v$  closest to  $v_0$  so that  $\Gamma$  is decomposed into  $\Gamma^{(1)}, \dots, \Gamma^{(\ell)}$  and an interval adjacent to  $v_0$  in the counter-clockwise order. We define

$$\mathfrak{m}_\Gamma = \Pi \circ \tilde{\mathfrak{m}}_{\ell, \beta_{\eta(v)}}^+ \circ (\mathfrak{f}_{\Gamma^{(1)}} \otimes \dots \otimes \mathfrak{f}_{\Gamma^{(\ell)}})$$

and

$$\mathfrak{f}_\Gamma = G \circ \tilde{\mathfrak{m}}_{\ell, \beta_{\eta(v)}}^+ \circ (\mathfrak{f}_{\Gamma^{(1)}} \otimes \dots \otimes \mathfrak{f}_{\Gamma^{(\ell)}}).$$

At last, we define the  $A_\infty$ -operations  $\mathfrak{m}_k : CF^\bullet(L; \Lambda_0)^{\otimes k} \rightarrow CF^\bullet(L; \Lambda_0)$  by

$$\mathfrak{m}_k = \sum_{\Gamma \in \Gamma_{k+1}} \mathbf{T}^{\omega_X(\Gamma)} \mathfrak{m}_\Gamma, \quad (2.19)$$

where  $\omega_X(\Gamma) = \sum_{v \in V_{\text{int}}} \omega_X(\beta_{\eta(v)})$ .

The maps  $\mathfrak{m}_\Gamma$  restricted to  $C^\bullet(f; \Lambda_0)$  are given by counting pearly trees as depicted in Figure 2 (This observation was made in [FOOO09a]. See also [CW] for a description of  $\mathfrak{m}_\Gamma$  with general inputs in terms of pearly trees.). For a decorated tree  $\Gamma \in \Gamma_{k+1}$ , the exterior vertices  $v_0, \dots, v_k$  are labeled respecting the counter-clockwise orientation. Each edge is oriented in the direction from the  $k$  input vertices  $v_1, \dots, v_k$  to the root vertex  $v_0$ . We denote by  $e_i$  the edge attached to  $v_i$  and by  $v^\pm(e)$  the vertices such that  $e$  is the edge from  $v^-(e)$  to  $v^+(e)$ . For  $p_1, \dots, p_k, q \in \text{Crit}(f)$ , consider the moduli space  $\mathcal{M}_\Gamma(f; p_1, \dots, p_k, q)$  consisting of the following configurations:

- for each interior vertex  $v$ , a bordered stable map  $u_v$  representing the class  $\beta_{\eta(v)}$  with  $\ell(v)$  boundary marked points. We denote by  $p(e, v)$  the marked point corresponding to the edge  $e$  attached to  $v$ ;

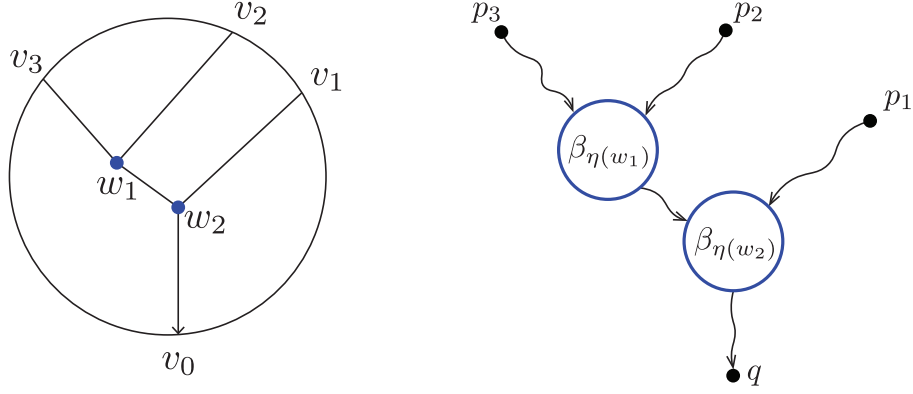


FIGURE 2. Pearl trees

- for  $i = 1, \dots, k$ , the input edge  $e_i$  corresponds to a flow line  $\gamma_i$  from  $p_i$  to  $u_{v^+(e_i)}(p(e_i, v^+(e_i)))$ ;
- the output edge  $e_0$  corresponds to a flow line  $\gamma_0$  from  $u_{v^-(e_0)}(p(e_0, v^-(e_0)))$  to  $q$ ;
- an interior edge  $e$  corresponds to a flow line  $\gamma_e$  from  $u_{v^-(e)}(p(e, v^-(e)))$  to  $u_{v^+(e)}(p(e, v^+(e)))$ .

The virtual dimension of  $\mathcal{M}_\Gamma(f; p_1, \dots, p_k, q)$  is given by

$$\dim \mathcal{M}_\Gamma(f; p_1, \dots, p_k, q) = k - 2 + \mu(\Gamma) - \sum_{i=1}^k |p_i| + |q|, \quad (2.20)$$

where  $\mu(\Gamma) = \sum_{v \in V_{int}} \mu(\beta_\eta(v))$ . The maps  $m_\Gamma : C^\bullet(f; \Lambda_0)^{\otimes k} \rightarrow C^\bullet(f; \Lambda_0)$  are given by

$$m_\Gamma(p_1, \dots, p_k) = \sum_{q \in \text{Crit}(f)} \# \mathcal{M}_\Gamma(f; p_1, \dots, p_k, q) \cdot q,$$

where  $\# \mathcal{M}_\Gamma(f; p_1, \dots, p_k, q)$  is the signed count of the moduli space of virtual dimension 0.

Let us now consider the weak Maurer-Cartan equation for  $(CF^\bullet(L; \Lambda_0), m)$ .

Let  $(A, m)$  be an  $A_\infty$ -algebra over  $\Lambda_0$  with the strict unit  $e_A$ , and set

$$A_+ = \{x \in A \mid x \equiv 0 \pmod{\Lambda_+ \cdot A}\}.$$

The weak Maurer-Cartan equation for an element  $b \in A_+$  is given by

$$m_0^b(1) = m_0(1) + m_1(b) + m_2(b, b) + \dots \in \Lambda_0 \cdot e_A. \quad (2.21)$$

The condition that  $b \in A_+$  ensures the convergence of  $m_0^b(1)$ . We denote by

$$MC(A) = \{b \in A_+^{odd} \mid m_0^b(1) \in \Lambda_0 \cdot e_A\},$$

the space of odd solution to the weak Maurer-Cartan equation.

We say an  $A_\infty$ -algebra  $A$  is *weakly unobstructed* if  $MC(A)$  is nonempty, in which case we have  $(m_1^b)^2 = 0$  for any  $b \in MC(A)$ , thus defining a cohomology theory  $H^\bullet(A, m_1^b)$ .

For any  $b \in MC(A)$ , we put

$$e^b := 1 + b + b \otimes b + \dots$$

We define the deformation the  $A_\infty$  structure  $m$  by  $b$  by

$$m_k^b(x_1, \dots, x_k) = m(e^b, x_1, e^b, x_2, e^b, \dots, e^b, x_k, e^b).$$

Notice that  $m_0^b(1) = m(e^b)$ .

The following lemma concerns the weakly unobstructedness of  $(CF^\bullet(L; \Lambda_0), \mathfrak{m})$ . This technique of finding weak bounding cochains in the presence of a homotopy unit was introduced in [FOOO09b, Chapter 7] and [CW, Lemma 2.44].

**Lemma 2.7.** *Let  $b \in CF^1(L; \Lambda_+)$ . Suppose  $\mathfrak{m}_0^b = W(b)\mathbf{1}^\nabla$  and the minimal Maslov index of  $L$  is nonnegative. Then there exists  $b^+ \in CF^1(L; \Lambda_+)$  such that  $\mathfrak{m}_0^{b^+}(1) = W^\nabla(b)\mathbf{1}^\nabla$ , i.e.,  $(CF^\bullet(L; \Lambda_0), \mathfrak{m})$  is weakly unobstructed. In particular, if the minimal Maslov index of  $L$  is at least two, then,  $W^\nabla(b) = W(b)$ .*

*Proof.* By (2.7), we have

$$\mathfrak{m}_1(\mathbf{1}^\nabla) = \mathbf{1}^\nabla - \mathbf{1}^\nabla + \Pi(h),$$

where

$$\Pi(h) = \Pi \left( \sum_{\substack{\beta \neq \beta_0 \\ \mu(\beta)=0}} \mathbf{T}^{\omega(\beta)}(\text{ev}_0)_*([0, 1] \times \mathcal{M}_2(\beta; L; \mathbf{e}))^{s^+} / \sim \right)$$

is a  $\Lambda_+$  multiple of  $\mathbf{1}^\nabla$ . The vanishing of the contribution of higher Maslov index discs follows from the fact that the output singular chains are of degrees at most  $-2$ , whose projection to  $CF^\bullet(L; \Lambda_0)$  vanishes. Hence we can write (2.2) as

$$\mathfrak{m}_1(\mathbf{1}^\nabla) = \mathbf{1}^\nabla - (1 - h)\mathbf{1}^\nabla, \quad (2.22)$$

where  $h \in \Lambda_+$ .

Let's tentatively take  $b' = b + \mathbf{1}^\nabla$ . We have

$$\mathfrak{m}_0^{b'}(1) = \mathfrak{m}_0(1) + \mathfrak{m}_1(b + \mathbf{1}^\nabla) + \mathfrak{m}_2(b + \mathbf{1}^\nabla, b + \mathbf{1}^\nabla) + \dots = W(b)\mathbf{1}^\nabla + \sum_{\mu(\beta)=0} \mathfrak{m}_{1,\beta}^b(\mathbf{1}^\nabla).$$

The second equality is due to the vanishing of the  $\mathfrak{m}_k$  operations with more than one  $\mathbf{1}^\nabla$  as an input since the outputs are of degrees at most  $-2$ . Also, notice that the terms  $\mathfrak{m}_{1,\beta}^b(\mathbf{1}^\nabla)$  are of degree 0 and therefore are multiples of  $\mathbf{1}^\nabla$ .

Using the notation of (2.22), we can write

$$\sum_{\mu(\beta)=0} \mathfrak{m}_{1,\beta}^b(\mathbf{1}^\nabla) = \mathbf{1}^\nabla - (1 - h(b))\mathbf{1}^\nabla = \mathbf{1}^\nabla - (1 - h)\mathbf{1}^\nabla + \sum_{\mu(\beta)=0, \ell_1+\ell_2 \geq 1} \mathfrak{m}_{1+\ell_1+\ell_2, \beta}(\underbrace{b, \dots, b}_{\ell_1}, \mathbf{1}^\nabla, \underbrace{b, \dots, b}_{\ell_2}).$$

Since there exist no gradient flow line from  $b$  to the maximum  $\mathbf{1}^\nabla$ , the contribution of the terms

$$\sum_{\ell_1+\ell_2 \geq 1} \mathfrak{m}_{1+\ell_1+\ell_2, \beta_0}(\underbrace{b, \dots, b}_{\ell_1}, \mathbf{1}^\nabla, \underbrace{b, \dots, b}_{\ell_2})$$

are zero, and therefore  $h(b) \in \Lambda_+$ .

Now, set  $b^+ = b + \frac{W}{1-h(b)}\mathbf{1}^\nabla$ . We have

$$\mathfrak{m}_0^{b^+}(1) = W(b)\mathbf{1}^\nabla + \frac{W}{1-h(b)}(\mathbf{1}^\nabla - (1-h(b))\mathbf{1}^\nabla) = \frac{W(b)}{1-h(b)}\mathbf{1}^\nabla =: W^\nabla(b)\mathbf{1}^\nabla. \quad (2.23)$$

This means  $b^+ \in MC(CF^\bullet(L; \Lambda_0))$ .

If the minimal Maslov index of  $L$  is at least two, then,  $h(b) = 0$  and therefore  $W^\nabla(b) = W(b)$ .  $\square$

**Definition 2.8.** *Let  $x_1, \dots, x_d$  be a basis of  $CF^1(L; \mathbb{Q})$ , and write  $b = \sum_{i=1}^d x^i x_i$ ,  $x^i \in \Lambda_+$ . If  $\mathfrak{m}_0^b = W(x^1, \dots, x^d)\mathbf{1}^\nabla$ , then we will call  $W^\nabla(x^1, \dots, x^d)$  the disc potential of  $(CF^\bullet(L; \Lambda_0), \mathfrak{m})$ .*

**2.3 The equivariant Morse model.** Let  $G$  be a compact Lie group. We choose smooth finite dimensional approximations  $EG(N)$  and  $BG(N)$  for the universal bundle  $EG$  and classifying space  $BG$ ,

$$EG(0) = G \hookrightarrow EG(1) \hookrightarrow EG(2) \hookrightarrow \dots \quad (2.24)$$

and

$$BG(0) = \text{pt} \hookrightarrow BG(1) \hookrightarrow BG(2) \hookrightarrow \dots \quad (2.25)$$

The maps above are taken to be smooth embeddings. We view  $T^*EG(N)$  and  $T^*BG(N)$  as symplectic manifolds equipped with the canonical symplectic forms. Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and let  $\mu_N : T^*EG(N) \rightarrow \mathfrak{g}^*$  be a moment map for the Hamiltonian  $G$ -action on  $T^*EG(N)$  lifted from the  $G$ -action on  $EG(N)$ . Since  $G$  acts on  $T^*EG(N)$  freely, we have

$$T^*EG(N) // G := \mu_N^{-1}(0) / G \cong T^*BG(N),$$

canonically, as symplectic manifolds. We denote by  $J_{T^*BG(N)}$  the almost complex structure induced by a  $G$ -invariant compatible almost complex structure on  $T^*EG(N)$ .

Let  $(X, \omega_X)$  be a symplectic manifold equipped with a symplectic  $G$ -action and let  $J_X$  be a  $G$ -invariant almost complex structure compatible with  $\omega_X$ . We again assume  $X$  is convex at infinity or geometrically bounded if it is non-compact. Let  $L \subset X$  be a  $G$ -invariant, closed, connected, relatively spin Lagrangian submanifold.

Let  $X_N = X \times_G \mu_N^{-1}(0)$  and set  $X_G = \lim_{\rightarrow} X_N$ . We note that  $\pi : X_N \rightarrow T^*BG(N)$  is a fiber bundle with fibers  $X$ . Let  $\iota : X \rightarrow X_N$  be the inclusion of a fiber. By construction,  $X_N$  is endowed with a symplectic form  $\omega_{X_N}$  and a compatible almost complex structure  $J_{X_N}$  satisfying

$$\omega_{X_N}|_{X_{N-1}} = \omega_{X_{N-1}}, \quad \iota^* \omega_{X_N} = \omega_X, \quad (2.26)$$

and

$$J_{X_N}|_{X_{N-1}} = J_{X_{N-1}}, \quad D\iota \circ J_X = J_{X_N} \circ D\iota, \quad D\pi \circ J_{X_N} = J_{T^*BG(N)} \circ D\pi. \quad (2.27)$$

Let  $L_N = L \times_G EG(N)$  be the finite dimensional approximations of the Borel construction  $L_G = L \times_G EG$ . We note that  $L_N \rightarrow BG(N)$  is a fiber bundle with fibers  $L$ . (2.24) and (2.25) induce a commutative diagram

$$\begin{array}{ccccccc} L_0 = L & \hookrightarrow & L_1 & \hookrightarrow & L_2 & \hookrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_0 = X & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & \dots \end{array} \quad (2.28)$$

where the vertical arrows are embeddings of Lagrangian submanifolds.

**Proposition 2.9** (effective disc classes). *The induced map  $\iota_* : H_2(X, L; \mathbb{Z}) \rightarrow H_2(X_N, L_N; \mathbb{Z})$  restricts to a bijection*

$$\iota_* : H_2^{\text{eff}}(X, L) \xrightarrow{\sim} H_2^{\text{eff}}(X_N, L_N). \quad (2.29)$$

*Proof.* Suppose  $u : (D^2, \partial D^2) \rightarrow (X_N, L_N)$  be a  $J_{X_N}$ -holomorphic disc. Then,  $\pi \circ u$  is a  $J_{T^*BG(N)}$ -holomorphic disc with Lagrangian boundary condition  $BG(N)$ . Since  $BG(N) \subset T^*BG(N)$  is an exact Lagrangian, it does not bound any non-trivial pseudo-holomorphic discs. This means  $\pi \circ u$  is necessarily constant and  $\text{Im}(u)$  is contained in a fiber of  $\pi$  over  $BG(N)$ .  $\square$

**Corollary 2.10** (Maslov index). *The Maslov index of  $\beta \in H_2^{\text{eff}}(X, L)$  is equal to the Maslov index of  $\iota_* \beta \in H_2^{\text{eff}}(X_N, L_N)$ .*

For simplicity, we will denote the disc classes  $\beta \in H_2^{\text{eff}}(X, L)$  and  $\iota_*\beta \in H_2^{\text{eff}}(X_N, L_N)$  both by  $\beta$ . We also denote their Maslov index by  $\mu(\beta)$  and symplectic area by  $\omega(\beta)$ .

**Corollary 2.11** (regularity). *A  $J_{X_N}$ -holomorphic disc  $u : (D^2, \partial D^2) \rightarrow (X_N, L_N)$  is regular if it is regular as a disc in the corresponding fiber of  $\pi$ .*

*Proof.* Let  $E := u^*TX_N$  and  $F := (\partial u)^*TL_N$ . Denote by  $A^0(E, F)$  the space of smooth global sections of  $E$  with boundary values in  $F$  and by  $A^1(E)$  the space of smooth global  $(0, 1)$ -forms with coefficient in  $E$ . Consider the two term elliptic complex

$$A^0(E, F) \xrightarrow{\bar{\partial}} A^1(E). \quad (2.30)$$

Here  $\bar{\partial}$  is the Cauchy-Riemann map linearized at  $u$ . Pulling-back the following exact sequences

$$0 \rightarrow TX \rightarrow TX_N \rightarrow TT^*BG(N) \rightarrow 0,$$

$$0 \rightarrow TL \rightarrow TL_N \rightarrow TBG(N) \rightarrow 0.$$

via  $u$ , we can choose splittings

$$E = E_1 \oplus E_2 := u^*TX \oplus (\pi \circ u)^*TT^*BG(N),$$

and

$$F = F_1 \oplus F_2 := (\partial u)^*TL \oplus (\pi \circ \partial u)^*TBG(N),$$

such that  $F_1 \subset E_1$  and  $F_2 \subset E_2$  are subbundles. This gives a decomposition of (2.30)

$$A^0(E_1, F_1) \oplus A^0(E_2, F_2) \xrightarrow{\bar{\partial}_1 \oplus \bar{\partial}_2} A^1(E_1) \oplus A^1(E_2).$$

By Proposition 2.9,  $\pi \circ u$  is a constant map and hence  $\bar{\partial}_2$  is surjective. Therefore,  $\bar{\partial}$  is surjective if and only if  $\bar{\partial}_1$  is surjective.  $\square$

Let  $\{(f_N, \mathcal{V}_N)\}_{N \geq 0}$  be a collection such that each pair  $(f_N, \mathcal{V}_N)$  is a Morse-Smale pair on  $L_N$ . Let  $(C\mathcal{F}^\bullet(L_N; \Lambda_0), \mathfrak{m}^N)$  denote be the unital Morse model for  $L_N$  constructed following Section 2.2. We denote by  $\tilde{\mathfrak{m}}^N, \tilde{\mathfrak{m}}^{N,+}$  the  $A_\infty$  structure on the singular chain models  $C^\bullet(L_N; \Lambda_0)$  and  $C^\bullet(L_N; \Lambda_0)^+$ , respectively, and by  $\mathfrak{m}_\Gamma^N$  the maps associated to the decorated planar rooted trees  $\Gamma$  in the definition of  $\mathfrak{m}^N$ .

Let's recall that for an element  $p = [\Sigma, \vec{z}, u] \in \mathcal{M}_{k+1}(\beta; L)$ , a Kuranishi chart around  $u$  is a quintuple  $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ , where  $\Gamma_p$  is the finite automorphism group of  $(\Sigma, \vec{z}, u)$  acting on the vector bundle  $E_p \rightarrow V_p$ ,  $V_p$  is smooth a manifold with corners parameterizing smooth stable maps close to  $u$ ,  $s_p = \bar{\partial}_{J_X}$  is a  $\Gamma$ -equivariant smooth section of  $E_p \rightarrow V_p$ , and  $\psi_p$  is a homeomorphism from  $s_p^{-1}\{0\}/\Gamma_p$  to a neighborhood of  $p$ . By Proposition 2.9,  $u$  is contained in a fiber of  $X_N \rightarrow T^*BG(N)$  over  $BG(N)$ . For every  $N \geq 0$  and  $q \in BG(N)$ , we choose a contractible neighborhood  $U_q$  of  $q$  in  $BG(N)$  and a trivialization of  $\pi^{-1}(U_q) \cong X \times U_q$  (and similarly  $L_N|_{U_q} \cong L \times U_q$ ) compatible with the diagram (2.28). This gives an trivialization  $\mathcal{M}_{k+1}(\beta; L) \times U_q$  of  $\mathcal{M}_{k+1}(\beta; L_N)$ . Then, for an element  $p$  in a fiber of  $\mathcal{M}_{k+1}(\beta; L) \times U_q \rightarrow U_q$ , we can take the Kuranishi chart around  $p$  to be  $(V_p \times U_q^{k+1}, E_p \oplus T_q BG(N)^{\oplus k}, \Gamma_p, (\psi'_p, id), \bar{\partial}_{J_{X_N}})$ , where  $\psi'_p$  is the pullback of  $\psi_p$  via the projection  $s_p^{-1}(0)/\Gamma_p \times U_q \rightarrow s_p^{-1}(0)/\Gamma_p$  and  $id : U_q \rightarrow U_q$  is the identity. Then, we have

$$\mathcal{M}_{k+1}(\beta; L_{N-1}) = \mathcal{M}_{k+1}(\beta; L_N) \times_{L_N} L_{N-1},$$

as Kuranishi structures. Moreover, we can orient the moduli spaces using the relative spin structure on  $L$  such that the orientations are compatible with the restriction above.

We now describe a Morse model for equivariant Lagrangian Floer theory.

**Condition 2.12.** We choose Morse-Smale pairs  $\{(f_N, \mathcal{V}_N)\}$  satisfying the following conditions.

- (1) For  $N \geq 1$ , we have  $f_N|_{L_{N-1}} = f_{N-1}$  and  $\mathcal{V}_N|_{L_{N-1}} = \mathcal{V}_{N-1}$ . This means  $\text{Crit}(f_{N-1}) \subset \text{Crit}(f_N)$ , and

$$W^u(f_N; p) \times_{L_N} L_{N-1} = W^u(f_{N-1}; p),$$

for  $p \in \text{Crit}(f_{N-1})$ .

- (2) For  $N \geq 1$ , and  $q \in L_N \setminus L_{N-1}$ , we have  $\Phi_t(q) \notin L_{N-1}$  for all  $t \geq 0$ . This implies

$$W^u(f_N; p) \times_{L_N} L_{N-1} = \emptyset$$

for  $p \in \text{Crit}(f_N) \setminus \text{Crit}(f_{N-1})$ , and  $\iota(W^s(f_{N-1}; p)) = W^s(f_N; p)$  in  $L_N$  for  $p \in \text{Crit}(f_{N-1})$ .

- (3) For every  $\ell \geq 0$ , there exists an integer  $N(\ell) > 0$  such that  $|p| > \ell$  for all  $N \geq N(\ell)$  and  $p \in \text{Crit}(f_N) \setminus \text{Crit}(f_{N-1})$ .

- (4) The Morse function  $f_N$  has a unique maximum  $\mathbf{1}_{L_N}^\bullet$  and the inclusion of critical points  $\text{Crit}(f_N) \hookrightarrow \text{Crit}(f_{N+1})$  maps  $\mathbf{1}_{L_N}^\bullet$  to  $\mathbf{1}_{L_{N+1}}^\bullet$ . This means we have  $CF^\bullet(L_N; \Lambda_0) \subset CF^\bullet(L_{N+1}; \Lambda_0)$ . We will abuse notation and denote  $\mathbf{1}_{L_N}^\bullet$ ,  $\mathbf{1}_{L_N}^\nabla$  and  $\mathbf{1}_{L_N}^\vee$  simply by  $\mathbf{1}^\bullet$ ,  $\mathbf{1}^\nabla$ , and  $\mathbf{1}^\vee$ .

The Morse-Smale pairs  $\{(f_N, \mathcal{V}_N)\}$  can be constructed as follows. We choose  $EU(k)$  to be the infinite Stiefel manifold  $V_k(\mathbb{C}^\infty)$  and  $BU(k)$  to be the infinite Grassmanian  $G_k(\mathbb{C}^\infty) = V_k(\mathbb{C}^\infty)/U(k)$ . We have an embedding of  $G_n(\mathbb{C}^\infty)$  into the skew-Hermitian matrices on  $\mathbb{C}^\infty$  by identifying  $V \in G_k(\mathbb{C}^\infty)$  with the orthogonal projection  $P_V$  on  $V$ . Let  $A$  be the diagonal matrix with entries  $\{1, 2, 3, \dots\}$ , the map

$$F : G_k(\mathbb{C}^\infty) \rightarrow \mathbb{R}, \quad F(V) = -\text{Re}(\text{tr}(AV)),$$

is a perfect Morse function decreasing along the finite dimension strata  $G_k(\mathbb{C}^N) \subset G_k(\mathbb{C}^\infty)$  as  $k$  increases.

We fix an embedding  $G \rightarrow U(n)$  of Lie groups for sufficiently large  $n$ , and choose  $EG = EU(n)$  and  $BG = EU(n)/G$ . This induces a fibration  $\pi : BG \rightarrow BU(k)$  with fibers  $U(k)/G$ . We choose a partition of unity  $\{\rho_i\}$  subordinate to open cover  $\{U_i\}$  of  $G_k(\mathbb{C}^\infty)$ .

We remark that the restrictions of a typical Morse function on  $BG$  to  $BG(N)$  satisfy the above properties. We can use a partition of unity on  $BG$  together with Morse functions on  $L$  to construct Morse functions on the total space of the fibration  $L_G \rightarrow BG$  so that their restrictions to  $L_N \rightarrow BG(N)$  satisfy the above properties.

**Proposition 2.13.** Let  $\iota_{N-1}^N : CF^\bullet(L_{N-1}; \Lambda_0) \rightarrow CF^\bullet(L_N; \Lambda_0)$  be the inclusion map. Under Condition 2.12, the singular chain models  $(C^\bullet(L_N; \Lambda_0)^+, \mathfrak{m}^{N,+})$  can be constructed such that the resulting Morse models  $(CF^\bullet(L_N; \Lambda_0), \mathfrak{m}^N)$  satisfy the following property:

$$\iota_{N-1}^N \left( \mathfrak{m}_\Gamma^{N-1}(p_1, \dots, p_k) \right) = \mathfrak{m}_\Gamma^N(p_1, \dots, p_k)|_{L_{N-1}} \quad (2.31)$$

for  $p_1, \dots, p_k \in CF^\bullet(L_{N-1}; \Lambda_0)$  and  $\Gamma \in \mathbf{\Gamma}_{k+1}$ . The RHS means that we only consider the outputs of  $\mathfrak{m}_\Gamma^N$  in  $CF^\bullet(L_{N-1}; \Lambda_0)$ .

*Proof.* We proceed by induction on  $N$ . For  $N = 0$ , the statement of the proposition is void. Suppose we have constructed  $A_\infty$ -algebras  $(CF^\bullet(L_{N'}; \Lambda_0), \mathfrak{m}^{N'})$ ,  $N' < N$ , satisfying (2.31). Let  $\mathcal{X}'_g(L_N) \subset \mathcal{X}_g(L_N)$  be the subset

$$\mathcal{X}'_g(L_N) = \{P \in \mathcal{X}_g(L_N) | P \times_{L_N} L_{N-1} \neq \emptyset\}.$$

Denote by  $\Delta_{p,N}$  unstable chain of  $p \in \text{Crit}(f_N)$  defined as in (2.11). For  $g = -1$ , we have

$$\mathcal{X}'_{-1}(L_N) = \{\Delta_{p,N} | p \in \text{Crit}(f_{N-1})\},$$

by Condition 2.12. We define  $r_{-1} : \mathcal{X}'_{-1}(L_N) \rightarrow \mathcal{X}_{-1}(L_{N-1})$  by  $r_{-1}(\Delta_{p,N}) = \Delta_{p,N-1}$ .

Let  $\vec{a} = \{a_1, \dots, a_{|\vec{a}|}\} \subset \{1, \dots, k\}$ ,  $a_1 < \dots < a_{|\vec{a}|}$ . For  $p_1, \dots, p_{k-|\vec{a}|} \in \text{Crit}(f_{N-1})$ , let  $\vec{\Delta}_N = (\Delta_{p_1,N}, \dots, \Delta_{p_{k-|\vec{a}|},N})$ , and let  $\vec{\Delta}_N^+$  be the  $k$ -tuple obtained by inserting  $e = \Delta_{\mathbf{1}}$  into the  $a_1, \dots, a_{|\vec{a}|}$ -th places of  $\vec{\Delta}_N$ . Since the holomorphic discs are contained in the fibers of  $X_N \rightarrow T^*BG(N)$  over  $BG(N)$ , by Condition 2.12, we have

$$\mathcal{M}_{k+1-|\vec{a}|}(\beta; L_N; \vec{\Delta}_N) \times_{L_N} L_{N-1} = \mathcal{M}_{k+1}(\beta; L_{N-1}; \vec{\Delta}_{N-1}), \quad (2.32)$$

and

$$\left([0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N; \vec{\Delta}_N^+)\right) \times_{L_N} L_{N-1} = [0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_{N-1}; \vec{\Delta}_{N-1}^+), \quad (2.33)$$

as compact subsets of  $\mathcal{M}_{k+1-|\vec{a}|}(\beta; L_N; \vec{\Delta}_N)$  and  $[0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N; \vec{\Delta}_N^+)$ , respectively. Note that the restriction of the natural Kuranishi structure for the fiber product  $\mathcal{M}_{k+1-|\vec{a}|}(\beta; L_N)$  (resp.  $[0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N)$ ) to the LHS of (2.32) (resp. (2.33)) agrees with the Kuranishi structure of the RHS of (2.32) (resp. (2.33)).

For  $g \geq 0$ , suppose  $\mathcal{X}_{g'}(L)$  had been constructed for  $g' < g$ . We construct the set of singular simplices  $\mathcal{X}_g(L)$ , and choose perturbations  $\mathfrak{s}_{\mathfrak{d}, \beta, \ell, \vec{P}_N}$ ,  $\|(\mathfrak{d}, \beta)\| = g$ , and  $\mathfrak{s}_{\mathfrak{d}^+, k, \beta, \vec{P}_N^+}$ ,  $\|(\mathfrak{d}^+, \beta)\| = g$  with following additional properties:

- (1) There exists a bijection  $r_g : \mathcal{X}'_g(L_N) \rightarrow \mathcal{X}_g(L_{N-1})$ .
- (2) Let  $\vec{P}_N = (P_{1,N}, \dots, P_{\ell,N})$ ,  $P_{i,N} \in \mathcal{X}'_{\mathfrak{d}(i)}(L_N)$ , and  $\mathfrak{d}(i) \leq g$ , we have

$$\mathcal{M}_{\ell+1}(\beta; L_N; \vec{P}_N) \times_{L_N} L_{N-1} = \mathcal{M}_{\ell+1}(\beta; L_{N-1}; P_{1,N-1}, \dots, P_{\ell,N-1}) \quad (2.34)$$

as Kuranishi structures, where  $P_{i,N-1} = r_{\mathfrak{d}(i)}(P_{i,N}) \in \mathcal{X}_{\mathfrak{d}(i)}(L_{N-1})$ .

- (3) Let  $\vec{P}_N^+ = (P_{1,N}^+, \dots, P_{k,N}^+)$ ,  $P_{i,N}^+ \in \mathcal{X}'_{\mathfrak{d}^+(i)}(L_N)$  with  $\mathfrak{d}^+(i) \leq g$ , we have

$$\left([0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N; \vec{P}_N^+)\right) \times_{L_N} L_{N-1} = [0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_{N-1}; P_{1,N-1}^+, \dots, P_{k,N-1}^+) \quad (2.35)$$

as Kuranishi structures, where  $P_{i,N-1}^+ = r_{\mathfrak{d}^+(i)}(P_{i,N}^+) \in \mathcal{X}_{\mathfrak{d}^+(i)}(L_{N-1})$ . In particular, if  $P_{i,N}^+ = \Delta_{\mathbf{1}_{L_N}^+}$ , then  $P_{i,N-1}^+ = \Delta_{\mathbf{1}_{L_{N-1}}^+}$ .

- (4) Let  $P_{i,N} \in \mathcal{X}'_{\mathfrak{d}(i)}(L_N)$ ,  $i = 1, \dots, \ell$ . Let  $\mathfrak{s}_{\mathfrak{d}, \beta, \ell, \vec{P}_{N-1}}$  be the perturbation chosen at the previous inductive step on  $N$  for  $\mathcal{M}_{\ell+1}(\beta; L_{N-1}; \vec{P}_{N-1})$ . If  $\|(\mathfrak{d}, \beta)\| = g$ , then for each  $p \in \text{Crit}(f_{N-1})$ , there is a bijective correspondence between the simplices in the triangulation of

$$(\text{ev}_0)_* \left( \mathcal{M}_{\ell+1}(\beta; L_{N-1}; \vec{P}_{N-1})^{\mathfrak{s}_{\mathfrak{d}, \beta, \ell, \vec{P}_{N-1}}} \right)$$

which are of complementary dimension to the stable submanifold  $W^s(f_{N-1}; p)$  in  $L_{N-1}$  and intersects  $W^s(f_{N-1}; p)$  at a unique point, and simplices in the triangulation of

$$(\text{ev}_0)_* \left( \mathcal{M}_{\ell+1}(\beta; L_N; \vec{P}_N)^{\mathfrak{s}_{\mathfrak{d}, \beta, \ell, \vec{P}_N}} \right)$$

which are of complementary dimension to the stable submanifold  $W^s(f_N; p)$  in  $L_N$  and intersects  $W^s(f_N; p)$  at a unique point. Moreover, the intersection points for the corresponding simplices have the same orientation.

- (5) The analogue of (4) holds for the perturbation  $\mathfrak{s}_{\partial^+, k, \beta, \vec{P}_N^+}$  and the simplices in the triangulation of

$$(\text{ev}_0)_* \left( \left( [0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N; \vec{P}_N^+) \right)^{\mathfrak{s}_{\partial^+, k, \beta, \vec{P}_N^+}} \right).$$

It is easy to see that these properties together would imply that  $(CF^\bullet(N; \Lambda_0), \mathfrak{m}^N)$  satisfies property (2.31).

Let  $P_{i,N} \in \mathcal{X}'_{\partial(i)}(L_N)$ ,  $i = 1, \dots, \ell$ . Suppose  $\|(\partial, \beta)\| = g$ , then  $\partial(i) < g$ . By (2), we have

$$\mathcal{M}_{\ell+1}(\beta; L_N; \vec{P}_N) \times_{L_N} L_{N-1} = \mathcal{M}_{\ell+1}(\beta; L_{N-1}; \vec{P}_{N-1})$$

as Kuranishi structures. We choose perturbation  $\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_N}$  to extend the perturbation  $\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_{N-1}}$  and the perturbation for the boundary strata  $\partial \mathcal{M}_{\ell+1}(\beta; L_N; \vec{P}_N)$ . Note that  $\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_{N-1}}$  is compatible with the perturbation for  $\partial \mathcal{M}_{\ell+1}(\beta; L_N; \vec{P}_N)$ , since the latter is extended from the perturbation for  $\partial \mathcal{M}_{\ell+1}(\beta; L_N; \vec{P}_N) \times_{L_N} L_{N-1}$  at previous inductive steps on  $g$ .

We triangulate  $\mathcal{M}_{\ell+1}(\beta; L_N; \vec{P}_N)^{\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_N}}$  extending the triangulation on  $\mathcal{M}_{\ell+1}(\beta; L_{N-1}; \vec{P}_{N-1})^{\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_{N-1}}}$  and triangulation on the boundary strata. We can choose the perturbation  $\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_N}$  such that any simplex in this triangulation away from  $\mathcal{M}_{\ell+1}(\beta; L_{N-1}; \vec{P}_{N-1})^{\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_{N-1}}}$  is transverse to the stable submanifolds of  $f_N$  (via the evaluation map  $\text{ev}_0$ ). Suppose  $\tau^N$  is a simplex in this triangulation and  $\tau^N \times_{L_N} L_{N-1} \neq \emptyset$ . By subdividing if necessary, we can assume that  $\tau^N$  has a face  $\tau^{N-1}$  in  $\mathcal{M}_{\ell+1}(\beta; L_{N-1}; \vec{P}_{N-1})^{\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_{N-1}}}$  and no other faces of  $\tau^N$  of dimension greater or equal to  $\dim \tau^{N-1}$  are in  $\mathcal{M}_{\ell+1}(\beta; L_{N-1}; \vec{P}_{N-1})^{\mathfrak{s}_{\partial, \beta, \ell, \vec{P}_{N-1}}}$ . Notice that  $\tau^N$  is not transverse to all stable submanifolds of  $f_N$ , since the stable submanifolds remain unchanged by Condition 2.12 but the ambient space  $L_N$  is of higher dimension than  $L_{N-1}$ . We can however move the vertices of  $\tau^N$  so that it is transverse to the stable submanifolds of  $f_N$ . We choose the perturbation  $\mathfrak{s}_{\partial^+, k, \beta, \vec{P}_N^+}$  and the triangulation for

$\left( [0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N; \vec{P}_N^+) \right)^{\mathfrak{s}_{\partial^+, k, \beta, \vec{P}_N^+}}$  in a similar manner.

Finally, we define  $r_g : \mathcal{X}'_g(L_N) \rightarrow \mathcal{X}_g(L_{N-1})$  by  $r_g(\tau^N) = \tau^{N-1}$ , and  $r_g(G(\tau^N)) = G(\tau^{N-1})$ .  $\square$

Let  $CF_G^\bullet(L; \Lambda_0)$  be the direct limit

$$CF_G^\bullet(L; \Lambda_0) = \lim_{\rightarrow} CF^\bullet(L_N; \Lambda_0).$$

For  $k \geq 0$  and  $\Gamma \in \Gamma_{k+1}$ , we define the map  $\mathfrak{m}_\Gamma^G : CF_G^\bullet(L; \Lambda_0)^{\otimes k} \rightarrow CF_G^\bullet(L; \Lambda_0)$  as follows. Let  $p_1, \dots, p_k \in CF_G^\bullet(L; \Lambda_0)$  and suppose  $p_1, \dots, p_k \in CF^\bullet(L_{N_0}; \Lambda_0)^+$  for some  $N_0$ . Set  $\ell = \sum_{i=1}^k |p_i| + 2 - k - \mu(\Gamma)$ , and let  $N(\ell) \geq N_0$  be as in Condition 2.12 (3). We define  $\mathfrak{m}_\Gamma^G$  by

$$\mathfrak{m}_\Gamma^G(p_1, \dots, p_k) = \mathfrak{m}_\Gamma^{N(\ell)}(p_1, \dots, p_k). \quad (2.36)$$

By Proposition 2.13, (2.36) is defined independent of the choice of  $N(\ell)$ . We define the maps  $\mathfrak{m}_k^G : CF_G^\bullet(L; \Lambda_0)^{\otimes k} \rightarrow CF_G^\bullet(L; \Lambda_0)$  by

$$\mathfrak{m}_{k, \beta}^G = \sum_{\Gamma \in \Gamma_{k+1}, \beta = \sum \beta_{\eta(v)}} \mathbf{T}^{\omega(\beta)} \mathfrak{m}_\Gamma^G. \quad (2.37)$$

$$\mathfrak{m}_k^G = \sum_{\beta} \mathfrak{m}_{k, \beta}^G. \quad (2.38)$$

It is easy to see that  $(CF_G^\bullet(L; \Lambda_0), \mathfrak{m}^G)$ ,  $\mathfrak{m}^G = (\mathfrak{m}_K^G)_{k \geq 0}$ , is an  $A_\infty$ -algebra with a strict unit  $\mathbf{1}^\nabla$ . For fixed inputs, the  $A_\infty$ -relations can be checked on  $(CF^\bullet(L_N; \Lambda_0), \mathfrak{m}^N)$  for sufficiently large  $N$ .

$(CF_G^\bullet(L; \Lambda_0), \mathfrak{m}^G)$  will be our Morse model for the  $G$ -equivariant Lagrangian Floer theory ( $G$ -equivariant pearl complex) of  $L$ .

**2.4 Homotopy partial units and the equivariant disc potentials.** The main goal of this paper is to recover the equivariant toric super potentials

$$W_\lambda = W + \sum_{i=1}^d x^i \lambda_i,$$

as the disc potentials of a  $T = (\mathbb{S}^1)^d$ -equivariant Lagrangian Floer theory  $(CF_T^\bullet(L; \Lambda_0), \mathfrak{m}^T)$  of the regular moment map fibers  $L$  of  $2d$ -dimensional compact semi-Fano toric manifolds. The terms  $\lambda_1, \dots, \lambda_d$  are the equivariant parameters generating the cohomology ring

$$H_T^\bullet(\text{pt}) = H^\bullet(BT; \mathbb{C}) = \mathbb{C}[\lambda_1, \dots, \lambda_d].$$

Clearly, arbitrary choices of Morse functions  $f_N$  would not suffice for our purpose since the equivariant parameters do not manifest as elements of  $CF_T^\bullet(L; \Lambda_0)$ . Moreover, since  $\lambda_1, \dots, \lambda_d$  have cohomological degree 2, the expression of  $W_\lambda$  suggests that the deformation  $\mathfrak{m}_0^{T,b}$  is, a priori, obstructed. For these reasons, we construct in this section a homotopy equivalent alternative to our equivariant Morse model and define its equivariant disc potential in the case when  $BG$  admits a perfect Morse function. This has the advantage of being more suitable for applications.

To begin with, we fix a Morse function  $f$  on  $L$  with a unique maximum point and set  $f_0 := f$ . We choose the Morse functions  $f_N$  on  $L_N$  for  $N \geq 1$  to be of the form

$$f_N = \pi_N^* \varphi_N + \phi_N$$

where

- $\varphi_N$  is a perfect Morse functions on  $BG(N)$ . The existence of such Morse functions is well-known when  $G$  is a torus or  $U(n)$ .
- $\phi_N$  is a (generically) fiberwise Morse function over  $BG(N)$  such that  $\phi_N$  restricted to the fiber  $L = \pi_N^{-1}(\{\lambda\})$  over each critical point  $\lambda \in \text{Crit}(\varphi_N)$  agrees with  $f$ .

With such choices of Morse functions, we have

$$C^\bullet(f_N; \Lambda_0) = C^\bullet(f; \Lambda_0) \otimes H^\bullet(BG(N); \Lambda_0).$$

and

$$CF_G^\bullet(L; \Lambda_0) = (C^\bullet(f; \Lambda_0) \otimes H_G^\bullet(\text{pt}; \Lambda_0)) \oplus \Lambda_0 \cdot \mathbf{1}^\nabla \oplus \Lambda_0 \cdot \mathbf{1}^\vee.$$

It is well known that  $H_G^\bullet(\text{pt}) = H_T^\bullet(\text{pt})^W$  where  $T \subset G$  is a maximal torus and  $W$  is the Weyl group. This means  $H_G^\bullet(\text{pt})$  is a polynomial ring with generators in even degrees.

Let  $\mathbf{1}_L^\vee$  denote the maximum point of  $f$ . We enlarge  $C^\bullet(f_N; \Lambda_0)$  fiberwise over  $\text{Crit}(\varphi_N)$  to  $CF^\bullet(L_N; \Lambda_0)^\dagger$

$$CF^\bullet(L_N; \Lambda_0)^\dagger = CF^\bullet(L; \Lambda_0) \otimes H^\bullet(BG(N); \Lambda_0).$$

For  $\lambda \in \text{Crit}(\varphi_N)$ , we put

$$\lambda^\vee = \mathbf{1}^\vee \otimes \lambda, \quad \lambda^\nabla = \mathbf{1}^\nabla \otimes \lambda, \quad \lambda^\vee = \mathbf{1}^\vee \otimes \lambda.$$

We also denote the maximum point of  $\text{Crit}(\varphi_N)$  by  $\mathbf{1}_{BG(N)}$  and put

$$\mathbf{1}^\vee = \mathbf{1}^\vee \otimes \mathbf{1}_{BG(N)}, \quad \mathbf{1}^\nabla = \mathbf{1}^\nabla \otimes \mathbf{1}_{BG(N)}, \quad \mathbf{1}^\vee = \mathbf{1}^\vee \otimes \mathbf{1}_{BG(N)}.$$

Let

$$C^\bullet(L_N; \Lambda_0)^\dagger = C^\bullet(L_N; \Lambda_0)^+ \otimes H^\bullet(BG(N); \Lambda_0).$$

We will abuse notations and put

$$\lambda^\nabla = e \otimes \lambda, \quad \lambda^\nabla = e^+ \otimes \lambda, \quad \lambda^\nabla = f \otimes \lambda.$$

Using the idea of the homotopy unit construction, we can construct an  $A_\infty$ -structure  $\tilde{m}^{N, \dagger}$  on  $C^\bullet(L_N; \Lambda_0)^\dagger$  with the following properties.

The restriction of  $\tilde{m}^{N, \dagger}$  to  $C^\bullet(L_N; \Lambda_0)^+$  agrees with  $\tilde{m}^{N, +}$ . In particular,  $e^+$  is the strict unit, and  $f$  is the homotopy between  $e$  and  $e^+$ .

The  $\tilde{m}_k^{N, \dagger}$  operations with  $\lambda^\nabla$  as an input are given by

$$\tilde{m}_2^{N, \dagger}(\lambda_i^\nabla, \lambda_j^\nabla) = e^+ \otimes \lambda_i \cup \lambda_j, \quad \tilde{m}_2^{N, \dagger}(\lambda_i^\nabla, \lambda_j^\nabla) = f \otimes \lambda_i \cup \lambda_j, \quad (2.39)$$

where  $\cup$  is the cup product of  $H^\bullet(BG(N))$ , and

$$\tilde{m}_2^{N, \dagger}(\lambda^\nabla, P) = \tilde{m}_{2, \beta_0}^{N, \dagger}(\lambda^\nabla, P), \quad \tilde{m}_2^{N, \dagger}(P, \lambda^\nabla) = \tilde{m}_{2, \beta_0}^{N, \dagger}(P, \lambda^\nabla) \quad (2.40)$$

for  $P \in C^\bullet(L_N; \Lambda_0)^\dagger$  a singular chain. Moreover,

$$\tilde{m}_k^{N, \dagger}(\dots, \lambda^\nabla, \dots) = 0 \quad (2.41)$$

for  $k \neq 2$ .

$\lambda^\nabla$  is the homotopy between  $\lambda^\nabla$  and  $\lambda^\nabla$  in the following sense.

Let  $\vec{a} = \{a_1, \dots, a_{|\vec{a}|}\} \subset \{1, \dots, k\}$ ,  $a_1 < \dots < a_{|\vec{a}|}$ . For  $\vec{P} = (P_1, \dots, P_{k-|\vec{a}|})$  with  $P_i \in \mathcal{X}_{\partial(i)}(L_N)$ , let  $\vec{P}^\dagger$  be the  $k$ -tuple obtained by inserting  $\lambda^\nabla$  into the  $a_1, \dots, a_{|\vec{a}|}$ -th places of  $\vec{P}$ . We put  $\partial^\dagger = g_i^\dagger$ , if  $\vec{P}^\dagger = (P_1^\dagger, \dots, P_k^\dagger)$  and  $P_i^\dagger \in \mathcal{X}_{g_i^\dagger}(L)$ .

Let  $\vec{a}^1 \amalg \vec{a}^2 = \vec{a}$  be a splitting of  $\vec{a}$ . Denote by  $\vec{P}'$  the  $(k - |\vec{a}^1|)$ -tuple given by removing  $\lambda^\nabla$  from the  $a_1^1, \dots, a_{|\vec{a}^1|}^1$ -th places of  $\vec{P}^\dagger$ . For the choices of perturbations for  $\mathcal{M}_{k+1}(\beta; L_N; \vec{P}^\dagger)$ . We have the perturbation  $s_{\partial^\dagger, k, \beta, \vec{P}^\dagger}$  obtained by just inserting  $\lambda^\nabla$  into the  $a_1, \dots, a_{|\vec{a}|}$ -th place, and the perturbations  $s^\nabla$  pulled back via  $\text{forget}_{\vec{a}^1, \vec{a}^2} : \mathcal{M}_{k+1}(\beta; L_N; \vec{P}^\dagger) \rightarrow \mathcal{M}_{k+1-|\vec{a}^1|}(\beta; L_N; \vec{P}')$ . Moreover, we have a perturbation  $s_{\partial^\dagger, k, \beta, \vec{P}^\dagger}^\dagger$  on  $[0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N; \vec{P}^\dagger)$  interpolating between them.

The  $\tilde{m}_{k, \beta}^{N, \dagger}$  operations,  $(k, \beta) \neq (1, \beta_0)$ , with inputs  $\lambda^\nabla$  inserted into  $a_1, \dots, a_{|\vec{a}|}$ -th place of  $\vec{P}$  are defined by

$$(\text{ev}_0)_* \left( \left( [0, 1]^{|\vec{a}|} \times \mathcal{M}_{k+1}(\beta; L_N; \vec{P}^\dagger) \right)^{s_{\partial^\dagger, k, \beta, \vec{P}^\dagger}^\dagger} / \sim \right) \quad (2.42)$$

Here  $\sim$  is once again the equivalence relation collapsing fibers of the forgetful maps. Since  $W^u(f_N; \lambda^\nabla)$  is the restriction of the bundle  $L_N \rightarrow BG(N)$  over  $W^u(\varphi_N, \lambda)$ , our choice of Kuranishi structures for the disc moduli (see Section 2.3) ensures that

$$(\text{ev}_0)_* \left( \mathcal{M}_{k+1}(\beta; L_N; \vec{P}^\dagger)^{s^\nabla} \right)$$

are degenerate singular chains which become zero in the quotient.

The definitions of  $\tilde{m}_{k, \beta}^{N, \dagger}$  operations,  $(k, \beta) \neq (1, \beta_0)$ , with distinct  $\lambda^\nabla$  (and  $\mathbf{1}^\nabla$ ) as inputs are similar to what we have described above, and hence omitted.

We also set

$$\tilde{m}_{1, \beta_0}^{N, \dagger}(\lambda^\nabla) = \lambda^\nabla - \lambda^\nabla,$$

and therefore

$$\tilde{m}_1^{N,\dagger}(\lambda^\nabla) = \lambda^\nabla - \lambda^\bullet + h_\lambda, \quad (2.43)$$

$$h_\lambda = \sum_{\beta \neq \beta_0} T^{\omega_X(\beta)} (\text{ev}_0)_* ([0,1] \times \mathcal{M}_2(\beta; L_N; \lambda^\bullet))^{\sharp} / \sim \equiv 0 \pmod{\Lambda_+ \cdot C^\bullet(L_N; \Lambda_0)^\dagger}. \quad (2.44)$$

We again use homological perturbation to reduce  $(C^\bullet(L_N; \Lambda_0)^\dagger, \tilde{m}^{N,\dagger})$  to a  $A_\infty$ -algebra  $(CF^\bullet(L_N; \Lambda_0)^\dagger, m^{N,\dagger})$  as in Section 2.2. Aside from the obvious properties inherited from  $(C^\bullet(L_N; \Lambda_0)^\dagger, \tilde{m}^{N,\dagger})$ , it has the following additional property: Let  $X = \sum x_i \otimes \lambda_j \in CF^\bullet(L_N; \Lambda_0)^\dagger$ , and denote  $\lambda \cdot X = \sum x_i \otimes (\lambda \cup \lambda_j)$  and  $X \cdot \lambda = \sum x_i \otimes (\lambda_j \cup \lambda)$ . We have

$$m_2^{N,\dagger}(\lambda^\nabla, X) = \lambda \cdot X, \quad (2.45)$$

$$m_2^{N,\dagger}(X, \lambda^\nabla) = X \cdot \lambda, \quad (2.46)$$

and therefore

$$m_2^{N,\dagger}(\lambda^\nabla, X) = (-1)^{|X|} m_2^{N,\dagger}(X, \lambda^\nabla). \quad (2.47)$$

Let

$$CF_G^\bullet(L; \Lambda_0)^\dagger = \lim_{\rightarrow} CF^\bullet(L_N; \Lambda_0)^\dagger = CF^\bullet(L; \Lambda_0) \otimes H_G^\bullet(\text{pt}; \Lambda_0).$$

We define the alternative equivariant Morse model  $(CF_G^\bullet(L; \Lambda_0)^\dagger, m^{G,\dagger})$  using the finite dimensional approximations  $(CF^\bullet(L_N; \Lambda_0)^\dagger, m^{N,\dagger})$  as in Section 2.3. We summarize its properties below.

The restriction of  $m^{G,\dagger}$  to  $CF_G^\bullet(L; \Lambda_0)$  agrees with  $m^G$ .

We have

$$m_2^{G,\dagger}(\lambda^\nabla, X) = \lambda \cdot X = (-1)^{|X|} m_2^{G,\dagger}(X, \lambda^\nabla) \quad (2.48)$$

for  $X \in C_G^\bullet(L; \Lambda_0)^\dagger$ , and

$$m_k^{G,\dagger}(\dots, \lambda^\nabla, \dots) = 0, \quad (2.49)$$

for  $k \neq 2$ .

When the minimal Maslov index of  $L$  is nonnegative, the elements  $\mathbf{1}^\nabla$  and  $\lambda^\nabla$  satisfy

$$m_1^{G,\dagger}(\mathbf{1}^\nabla) = \mathbf{1}^\nabla - (1-h)\mathbf{1}^\bullet, \quad (2.50)$$

for some  $h$ , and

$$m_1^{G,\dagger}(\lambda^\nabla) = m_1^{G,\dagger}(m_2^{G,\dagger}(\lambda^\nabla, \mathbf{1}^\nabla)) = -(\lambda^\nabla - (1-h)\lambda^\bullet), \quad (2.51)$$

where the last inequality follows from  $A_\infty$ -relations.

Since the properties  $\lambda^\nabla$  satisfy are similar to those of a strict unit, we will call  $\lambda^\nabla$  *partial units* and  $\lambda^\bullet$  *homotopy partial units*.

The following theorem shows that  $(CF_G^\bullet(L; \Lambda_0)^\dagger, m^{G,\dagger})$  can be defined over the graded coefficient ring  $H_G^\bullet(\text{pt}; \Lambda_0)$ , namely

$$(CF_G^\bullet(L; \Lambda_0)^\dagger, m^{G,\dagger}) = (CF^\bullet(L; H_G^\bullet(\text{pt}; \Lambda_0)), m^{G,\dagger}), \quad (2.52)$$

It is important to point out that the  $A_\infty$ -structure of RHS cannot be determined by  $m$  since the latter does not reflect structure of the fiber bundle  $L_G \rightarrow BG$ .

**Theorem 2.14.** *Let  $X_1, \dots, X_k \in CF_G^\bullet(L; \Lambda_0)^\dagger$ . Let  $\ell \in \{1, \dots, k\}$  and write  $X_\ell = \sum x_i \otimes \lambda_j$ . We have*

$$m_k^{G,\dagger}(X_1, \dots, X_k) = (-1)^\ell \sum \lambda_j \cdot m_k^{G,\dagger}(X_1, \dots, X_{\ell-1}, x_i \otimes 1, X_{\ell+1}, \dots, X_k). \quad (2.53)$$

*Proof.* By (2.48), we have

$$\mathfrak{m}_k^{G,\dagger}(X_1, \dots, X_{\ell-1}, x_i \otimes \lambda_j, X_{\ell+1}, \dots, X_k) = \mathfrak{m}_k^{G,\dagger}(X_1, \dots, X_{\ell-1}, \mathfrak{m}_2^{G,\dagger}(\lambda^\nabla, x_i \otimes 1), X_{\ell+1}, \dots, X_k),$$

Then, by (2.49) and  $A_\infty$ -relations, we have

$$\begin{aligned} \mathfrak{m}_k^{G,\dagger}(X_1, \dots, X_{\ell-1}, \mathfrak{m}_2^{G,\dagger}(\lambda^\nabla, x_i \otimes 1), X_{\ell+1}, \dots, X_k) &= (-1)^{\ell-1} \mathfrak{m}_k^{G,\dagger}(\mathfrak{m}_2^{G,\dagger}(\lambda^\nabla, X_1) \dots, X_{\ell-1}, x_i \otimes 1, X_{\ell+1}, \dots, X_k) \\ &= (-1)^\ell \mathfrak{m}_2^{G,\dagger}(\lambda^\nabla, \mathfrak{m}_k^{G,\dagger}(X_1, \dots, X_{\ell-1}, x_i \otimes 1, X_{\ell+1}, \dots, X_k)). \end{aligned}$$

Applying (2.48) to the last term above, we get

$$\mathfrak{m}_k^{G,\dagger}(X_1, \dots, X_{\ell-1}, x_i \otimes \lambda_j, X_{\ell+1}, \dots, X_k) = (-1)^\ell \lambda_j \cdot \mathfrak{m}_k^{G,\dagger}(X_1, \dots, X_{\ell-1}, x_i \otimes 1, X_{\ell+1}, \dots, X_k).$$

□

Let's now consider a modified version of weak Maurer-Cartan equation for  $(CF_G^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}^{G,\dagger})$ . Let  $b \in CF_G^\bullet(L; \Lambda_+)^{\dagger, \text{odd}}$  and let  $\mathfrak{m}_k^{G,\dagger,b}$  be the deformation of  $\mathfrak{m}_k^{G,\dagger}$  by  $b$ . Suppose

$$\mathfrak{m}_0^{G,\dagger,b}(1) \in H_G^\bullet(\text{pt}; \Lambda_0) \cdot \mathbf{1}^\nabla. \quad (2.54)$$

From (2.48), (2.49), we have  $(\mathfrak{m}_1^{G,\dagger,b})^2 = 0$ , which gives rise to a well-defined cohomology theory  $H^\bullet(CF_G^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}_1^{G,\dagger,b})$ . We therefore define the *equivariant weak Maurer-Cartan space* of  $CF_G^\bullet(L; \Lambda_0)^\dagger$  to be

$$MC_G \left( CF_G^\bullet(L; \Lambda_0)^\dagger \right) = \left\{ b \in CF_G^\bullet(L; \Lambda_+)^{\dagger, \text{odd}} \mid b \text{ satisfies (2.54)} \right\}.$$

**Definition 2.15.** We say that  $(CF_G^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}^{G,\dagger})$  is *weakly unobstructed* if  $MC_G(CF_G^\bullet(L; \Lambda_0)^\dagger)$  is nonempty.

Similar to Lemma 2.7, we have

**Lemma 2.16.** Let  $b \in CF_G^1(L; \Lambda_+)$ . Suppose  $\mathfrak{m}_0^{G,\dagger,b}(1) = W(b)\mathbf{1}^\nabla + \sum_{\text{deg } \lambda=2} \phi_\lambda(b)\lambda^\nabla$ , and the minimal Maslov index of  $L$  is nonnegative, then there exists  $b^\dagger \in CF_G^1(L; \Lambda_+)^{\dagger}$  such that  $\mathfrak{m}_0^{G,\dagger,b^\dagger}(1) = W^\nabla(b)\mathbf{1}^\nabla + \sum_{\text{deg } \lambda=2} \phi_\lambda^\nabla(b)\lambda^\nabla$ , i.e.,  $(CF_G^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}^{G,\dagger})$  is weakly unobstructed. In particular, if the minimal Maslov index of  $L$  is at least two, then  $W^\nabla = W(b)$ ,  $\phi_\lambda^\nabla(b) = \phi_\lambda(b)$ .

*Proof.* Let's tentatively take  $b' = b + W(b)\mathbf{1}^\nabla + \sum_{\lambda \neq 1} \phi_\lambda(b)\lambda^\nabla$ . We also put  $b_1 = b + W(b)\mathbf{1}^\nabla$  and  $b_2 = W(b)\mathbf{1}^\nabla + \sum_{\lambda \neq 1} \phi_\lambda(b)\lambda^\nabla$ . By (2.51) we have

$$\begin{aligned} \mathfrak{m}_0^{G,\dagger,b'}(1) &= W(b)\mathbf{1}^\nabla + \sum_{\lambda \neq 1} \phi_\lambda(b)\lambda^\nabla + W(b)(\mathbf{1}^\nabla - (1-h)\mathbf{1}^\nabla) - \sum_{\lambda \neq 1} \phi_\lambda(b)(\lambda^\nabla - (1-h)\lambda^\nabla) + \\ &\quad \sum_{k=2} \left( \mathfrak{m}_k^{G,\dagger}(b_1, \dots, b_1) + \mathfrak{m}_k^{G,\dagger}(b_2, \dots, b_2) + \sum_{\lambda \neq 1} \phi_\lambda(b) \sum \mathfrak{m}_k^{G,\dagger}(\dots, b, \lambda^\nabla, \dots) \right). \end{aligned}$$

Here  $\sum \mathfrak{m}_k^{G,\dagger}(\dots, b, \lambda^\nabla, \dots)$  is the summation over the terms which have both  $b$  and  $\lambda^\nabla$  as inputs.

By the proof of Lemma 2.7, we have

$$W(b)(\mathbf{1}^\nabla - (1-h)\mathbf{1}^\nabla) + \sum_{k=2} \mathfrak{m}_k^{G,\dagger}(b_1, \dots, b_1) = W(b)(\mathbf{1}^\nabla - (1-h(b))\mathbf{1}^\nabla)$$

for some  $h(b) \in \Lambda_+$ . We also have  $\sum_{k \geq 2} \mathfrak{m}_k^{G,\dagger}(b_2, \dots, b_2) = 0$  by the proof of Lemma 2.16.

Let's now consider the terms  $m_k^{G,\dagger}(\dots, b, \lambda^\nabla, \dots)$ . By (2.48), (2.49) and  $A_\infty$ -relations, we have

$$m_k^{G,\dagger}(\dots, b, \lambda^\nabla, \dots) = (-1)^{\text{sgn}} m_2^{G,\dagger} \left( \lambda^\nabla, m_2^{G,\dagger} \left( \lambda^\nabla, \dots m_2^{G,\dagger} \left( \lambda^\nabla, m_k^{G,\dagger}(\dots, b, \mathbf{1}^\nabla, \dots) \right) \right) \right),$$

where the inner most term  $m_k^{G,\dagger}(\dots, b, \mathbf{1}^\nabla, \dots)$  has only  $\mathbf{1}^\nabla$  and  $b$  as inputs. By degree reason  $m_k^{G,\dagger}(\dots, b, \mathbf{1}^\nabla, \dots) = 0$  when it has more than one  $\mathbf{1}^\nabla$  as inputs. This means  $m_k^{G,\dagger}(\dots, b, \lambda^\nabla, \dots) = 0$  when it has two  $\lambda^\nabla$  or both  $\mathbf{1}^\nabla$  and  $\lambda^\nabla$  as inputs. Therefore,

$$\begin{aligned} \sum m_k^{G,\dagger}(\dots, b, \lambda^\nabla, \dots) &= \sum m_k^{G,\dagger}(b, \dots, b, \lambda^\nabla, b, \dots, b) \\ &= (-1)^{\text{sgn}} \sum m_2^{G,\dagger} \left( \lambda^\nabla, m_k^{G,\dagger}(b, \dots, b, \mathbf{1}^\nabla, b, \dots, b) \right). \end{aligned}$$

Recall from the proof of Lemma 2.7 that  $m_k^{G,\dagger}(b, \dots, b, \mathbf{1}^\nabla, b, \dots, b)$  is a  $\Lambda_+$  multiple of  $\mathbf{1}^\nabla$ . Since  $m_2^{G,\dagger}(\lambda^\nabla, \mathbf{1}^\nabla) = \lambda^\nabla$ , the RHS above is a  $\Lambda_+$  multiple of  $\lambda^\nabla$ . We can therefore write

$$\begin{aligned} - \sum_{\lambda \neq 1} \phi_\lambda(b) (\lambda^\nabla - (1-h)\lambda^\nabla) + \sum_{k \geq 2, \lambda \neq 1} \phi_\lambda(b) m_k^{G,\dagger}(\dots, b, \lambda^\nabla, \dots) \\ = - \sum_{\lambda \neq 1} \phi_\lambda(b) (\lambda^\nabla - (1-h_\lambda(b))\lambda^\nabla) \end{aligned}$$

for some  $h_\lambda(b) \in \Lambda_+$ .

Finally, let

$$b^\dagger = b + \frac{W(b)}{1-h(b)} \mathbf{1}^\nabla + \sum_{\lambda \neq 1} \frac{\phi_\lambda(b)}{h_\lambda(b) - 1} \lambda^\nabla. \quad (2.55)$$

We have

$$\begin{aligned} m_0^{G,\dagger,b^\dagger}(1) &= W(b) \mathbf{1}^\nabla + \sum_{\lambda \neq 1} \phi_\lambda(b) \lambda^\nabla + \frac{W(b)}{1-h(b)} (\mathbf{1}^\nabla - (1-h(b)) \mathbf{1}^\nabla) + \sum_{\lambda \neq 1} \frac{\phi_\lambda(b)}{1-h_\lambda(b)} (\lambda^\nabla - (1-h_\lambda(b)) \lambda^\nabla) \\ &= \frac{W(b)}{1-h(b)} \mathbf{1}^\nabla + \sum_{\lambda \neq 1} \frac{\phi_\lambda(b)}{1-h_\lambda(b)} \lambda^\nabla =: W^\nabla(b) \mathbf{1}^\nabla + \sum_{\lambda \neq 1} \phi_\lambda^\nabla(b) \lambda^\nabla. \end{aligned}$$

Since  $W(b), \phi_\lambda(b) \in \Lambda_+$ , we get  $b^\dagger \in MC_G(CF_G^\bullet(L; \Lambda_0)^\dagger)$ .

If the minimal Maslov index of  $L$  is at least two, then  $h(b) = 0$ ,  $h_\lambda(b) = 0$  and therefore  $W^\nabla(b) = W(b)$ ,  $\phi_\lambda^\nabla(b) = \phi_\lambda(b)$ .  $\square$

Equation (2.50) implies  $[\mathbf{1}^\nabla] = [\mathbf{1}^\nabla]/(1-h) \in HF_G^0(L; \Lambda_0)$  (in the weakly unobstructed case so that the equivariant Floer cohomology is well-defined). Since  $h \in \Lambda_+$ ,  $\mathbf{1}^\nabla$  is a cohomological unit. This is important, for instance when we consider isomorphisms of objects in the Fukaya category.

**Corollary 2.17.** *In the setting of Lemma 2.16, if  $b \in C^1(f; \Lambda_0)$  and  $m_0^b(1) \in \Lambda_0 \cdot \mathbf{1}^\nabla$ , then  $(CF_G^\bullet(L; \Lambda_0)^\dagger, m^{G,\dagger})$  is weakly unobstructed.*

*Proof.* If  $b \in C^1(f; \Lambda_0)$  and  $m_0^b(1) \in \Lambda_0 \cdot \mathbf{1}^\nabla$ , then  $m_0^{G,\dagger,b}(1)$  is of the form  $m_0^{G,\dagger,b}(1) = W(b) \mathbf{1}^\nabla + \sum_{\lambda \neq 1} \phi_\lambda(b) \lambda^\nabla$ .  $\square$

Here we remark that if  $b \in CF_G^1(L; \Lambda_0)$ , then in fact  $b \in C^1(f; \Lambda_0)$ , since the generators of  $H_G^\bullet(\text{pt}; \Lambda_0)$  are of (even) degrees at least 2 (other than the fundamental class).

**Definition 2.18.** Let  $X_1, \dots, X_d$  be a basis of  $CF_G^1(L; \mathbb{Q})$ , and write  $b = \sum_{i=1}^d x^i X_i$ ,  $x^i \in \Lambda_+$ . If  $\mathfrak{m}_0^{G, \dagger, b}(1) = W(x^1, \dots, x^d) \mathbf{1}^\vee + \sum_{\lambda \neq 1} \phi_\lambda(x^1, \dots, x^d) \lambda^\vee$ , then we will call

$$W^G(x^1, \dots, x^d) := W^\nabla(x^1, \dots, x^d) + \sum_{\lambda} \phi_\lambda^\nabla(x^1, \dots, x^d) \lambda$$

the equivariant disc potential of  $(CF_G^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}^{G, \dagger})$ .

**2.5 Equivariant Floer theory for Lagrangian immersions.** In this section we describe a generalization of the equivariant Morse model to immersed Lagrangians with clean self-intersections.

The singular chain model of Lagrangian Floer theory for immersed Lagrangians with transverse self-intersections has been developed by Akaho-Joyce [AJ10]. Their construction can be generalized to the case of clean self-intersections when combined with Lagrangian Floer theory for a pair of cleanly intersecting Lagrangians in [FOOO09b]. We begin by explaining this singular chain model.

Let  $(X, \omega_X)$  be as in Section 2.1 and let  $L$  be a closed, connected, relatively spin, immersed Lagrangian submanifold with clean self-intersections. We denote by  $\iota : \tilde{L} \rightarrow X$  the immersion  $L_0$  and by  $\mathcal{I}$  the self-intersection. We assume the preimage of  $\mathcal{I}$  consists of two sets of connected components  $\mathcal{I}^-$  and  $\mathcal{I}^+$ , each of which is diffeomorphic to  $\mathcal{I}$ . We write

$$\tilde{L} \times_L \tilde{L} = \coprod_{j=-1,0,1} R_j,$$

where  $R_0$  is the diagonal and

$$R_1 = \{(p_-, p_+) \in \tilde{L} \times \tilde{L} \mid p_- \in \mathcal{I}^-, p_+ \in \mathcal{I}^+, \iota(p_-) = \iota(p_+)\},$$

$$R_{-1} = \{(p_+, p_-) \in \tilde{L} \times \tilde{L} \mid p_+ \in \mathcal{I}^+, p_- \in \mathcal{I}^-, \iota(p_+) = \iota(p_-)\},$$

We have canonical isomorphisms  $R_0 \cong \tilde{L}$  and  $R_{-1} \cong R_1 \cong \mathcal{I}$ .  $R_{-1}$  and  $R_1$  are the two branches of the immersed loci.

Let  $J_X$  be a compatible almost complex structure of  $(X, \omega)$ . Let  $\alpha : \{0, \dots, k\} \rightarrow \{-1, 0, 1\}$ . Let  $(\Sigma, \vec{z}, u, \tilde{u})$  be a quadruple, where  $\Sigma$  is a genus 0 bordered Riemann surface,  $\vec{z} = (z_0, \dots, z_k)$  are distinct counter-clockwise ordered smooth points on  $\partial\Sigma$ ,  $u : (\Sigma, \partial\Sigma) \rightarrow (X, L)$  is a  $J$ -holomorphic map with  $(\Sigma, \vec{z}, u)$  stable, and  $\tilde{u} : \partial\Sigma \setminus \{z_i \mid \alpha(i) \neq 0\} \rightarrow \tilde{L}$  is a continuous map, such that  $\tilde{u}$  is locally a lift of  $u|_{\partial\Sigma}$ , i.e.,

$$\iota \circ \tilde{u} = u|_{\partial\Sigma}$$

and

$$\left( \lim_{z \rightarrow z_i^-} \tilde{u}(z), \lim_{z \rightarrow z_i^+} \tilde{u}(z) \right) \in R_{\alpha(i)}, \quad \alpha(i) \neq 0.$$

Let  $\beta \in H_2(X, L; \mathbb{Z})$ . We denote by  $\mathcal{M}_{k+1}(\alpha, \beta; L)$  the moduli space of such quadruples  $[\Sigma, \vec{z}, u, \tilde{u}]$  satisfying  $u_*([\Sigma]) = \beta$ .

For the convenience of writing, we introduce the following terminology: We will refer to elements of  $\mathcal{M}_{k+1}(\alpha, \beta; L)$  as stable polygons if  $\alpha(i) \neq 0$  for some  $i \in \{0, \dots, k\}$ . The corners of a polygon are the boundary marked points  $z_i$  with  $\alpha(i) \neq 0$ . If  $\alpha(i) = 0$  for all  $i$ , we will simply refer to the elements of  $\mathcal{M}_{k+1}(\alpha, \beta; L)$  as stable discs.

Let  $\sigma : R_{-1} \coprod R_1 \rightarrow R_{-1} \coprod R_1$  be the involution swapping the two immersed sectors, i.e.,  $\sigma(p_-, p_+) = (p_+, p_-)$ . We have evaluation maps  $\text{ev}_i : \mathcal{M}_{k+1}(\alpha, \beta; L) \rightarrow \tilde{L} \times_L \tilde{L}$  defined

by

$$\mathrm{ev}_i([\Sigma, \vec{z}, u, \tilde{u}]) = \begin{cases} \tilde{u}(z_i) \in R_0 & \alpha(i) = 0 \\ \left( \lim_{z \rightarrow z_i^-} \tilde{u}(z), \lim_{z \rightarrow z_i^+} \tilde{u}(z) \right) \in R_{\alpha(i)} & \alpha(i) \neq 0, \end{cases}$$

at the input marked points  $i = 1, \dots, k$ , and

$$\mathrm{ev}_0([\Sigma, \vec{z}, u, \tilde{u}]) = \begin{cases} \tilde{u}(z_0) \in R_0 & \alpha(0) = 0 \\ \sigma \left( \lim_{z \rightarrow z_0^-} \tilde{u}(z), \lim_{z \rightarrow z_0^+} \tilde{u}(z) \right) \in R_{-\alpha(0)} & \alpha(0) \neq 0, \end{cases}$$

at the output marked point.

The Kuranishi structures on  $\mathcal{M}_{k+1}(\alpha, \beta; L)$  are taken to be weakly submersive. For  $\vec{P} = (P_1, \dots, P_k), P_1, \dots, P_k \in S^\bullet(\tilde{L} \times_L \tilde{L}; \mathbb{Q})$ , we denote by  $\mathcal{M}_{k+1}(\alpha, \beta, \vec{P})$  the fiber product

$$\mathcal{M}_{k+1}(\alpha, \beta, \vec{P}) = \mathcal{M}_{k+1}(\alpha, \beta, L) \times_{(\tilde{L} \times_L \tilde{L})^k} \vec{P}$$

in the sense of Kuranishi structures. We can then construct an  $A_\infty$ -structure  $\tilde{\mathfrak{m}}$  on a countably generated subcomplex  $C^\bullet(L; \Lambda_0)$  as in Section 2.1. We define the maps  $\tilde{\mathfrak{m}}_{k, \beta} : C^\bullet(L; \Lambda_0)^{\otimes k} \rightarrow C^\bullet(L; \Lambda_0) \subset S^\bullet(\tilde{L} \times_L \tilde{L}; \Lambda_0)$  by

$$\tilde{\mathfrak{m}}_{1, \beta_0}(P) = (-1)^n \partial P,$$

and by

$$\begin{aligned} \tilde{\mathfrak{m}}_{k, (\alpha, \beta)}(P_1, \dots, P_k) &= (\mathrm{ev}_0)_* \left( \mathcal{M}_{k+1}(\alpha, \beta; \vec{P})^s \right), \\ \tilde{\mathfrak{m}}_{k, \beta}(P_1, \dots, P_k) &= \sum_{\alpha} \tilde{\mathfrak{m}}_{k, (\alpha, \beta)}(P_1, \dots, P_k) \end{aligned}$$

for  $(k, \beta) \neq (1, \beta_0)$ . Notice that  $\tilde{\mathfrak{m}}_{k, (\alpha, \beta)}(P_1, \dots, P_k) = \emptyset$  unless  $P_i$  is a singular chain on  $R_{\alpha(i)}$ . The  $A_\infty$ -operations  $\tilde{\mathfrak{m}}_k$  are given by

$$\tilde{\mathfrak{m}}_k(P_1, \dots, P_k) = \sum_{\beta} \mathbf{T}^{\omega(\beta)} \tilde{\mathfrak{m}}_{k, \beta}(P_1, \dots, P_k).$$

Similarly, we have a unital  $A_\infty$ -algebra  $(C^\bullet(L; \Lambda_0)^+, \tilde{\mathfrak{m}}^+)$  homotopy equivalent to  $((C^\bullet(L; \Lambda_0), \tilde{\mathfrak{m}}))$  obtained via the homotopy unit construction.

Let  $f : \tilde{L} \times_L \tilde{L} \rightarrow \mathbb{R}$  be a Morse function with a unique maximum point  $\mathbf{1}^\nabla$  on  $\tilde{L}$ . Let  $C^\bullet(f; \Lambda_0)$  be the cochain complex generated by the critical points of  $f$

$$C^\bullet(f; \Lambda_0) = \bigoplus_{p \in \mathrm{Crit}(f)} \Lambda_0 \cdot p,$$

and set

$$CF^\bullet(L; \Lambda_0) = C^\bullet(f; \Lambda_0) \oplus \Lambda_0 \cdot \mathbf{1}^\nabla \oplus \Lambda_0 \cdot \mathbf{1}^\nabla.$$

Let

$$\mathcal{X}_{-1}(L) = \{\Delta_p | p \in \mathrm{Crit}(f)\},$$

where  $\Delta_p$  is defined by (2.11). Let  $C_{(-1)}^\bullet(L; \Lambda_0) \subset S^\bullet(\tilde{L} \times_L \tilde{L}; \Lambda_0)$  be the subcomplex generated elements in  $\mathcal{X}_{-1}(L)$ , and set

$$C_{(-1)}^\bullet(L; \Lambda_0)^+ = C_{(-1)}^\bullet(L; \Lambda_0) \oplus \Lambda_0 \cdot \mathbf{e}^+ \oplus \Lambda_0 \cdot f.$$

We again identify  $CF^\bullet(L; \Lambda_0)$  with  $C_{(-1)}^\bullet(L; \Lambda_0)^+$  via the assignment  $p \mapsto \Delta_p$  and equip  $CF^\bullet(L; \Lambda_0)$  with a unital  $A_\infty$ -structure  $\mathfrak{m}$  via homological perturbation as in Section 2.2.

The  $A_\infty$  operations  $\mathfrak{m}_k$  counts stable pearly trees: the interior vertices of a tree are decorated by stable polygons. The edges of a tree correspond to gradient flow lines of the Morse function  $f$  (on one of the connected components  $R_j$ ). The connected component  $R_i$

that the flow line is contained in has to match with the branch jump label of the corner of the disc that the flow line is attached to, or otherwise the fiber product is empty. See Figure 3 for some examples.

For a pair  $(L_1, L_2)$  of closed, connected, relatively spin, embedded Lagrangian submanifolds intersecting cleanly, the union  $L = L_1 \cup L_2$  is a immersed Lagrangian with clean self-intersections with  $\tilde{L} = L_1 \amalg L_2$ . We choose the splitting  $\iota^{-1}(\mathcal{I}) = \mathcal{I}^- \amalg \mathcal{I}^+$  so that  $\mathcal{I}^- \subset L_1$  and  $\mathcal{I}^+ \subset L_2$ .

We can define a pearl complex  $(CF^\bullet(L_1, L_2; \Lambda_0), m_1^{L_1, L_2})$  for the Lagrangian intersection Floer theory of  $(L_1, L_2)$  as follows:

$$CF^\bullet(L_1, L_2; \Lambda_0) = \bigoplus_{p \in \text{Crit}(f|_{R_1})} \Lambda_0 \cdot p$$

is the subcomplex of  $CF^\bullet(L; \Lambda_0)$  generated by critical points of  $f$  in  $R_1$ , and  $m_1^{L_1, L_2}$  counts stable pearly trees  $\Gamma \in \Gamma_2$  with input and output vertices in  $R_1$ .

For the equivariant Morse model, we consider a Lagrangian immersion  $\iota : L \rightarrow X$  which is  $G$ -equivariant. We have the Borel construction  $L_G = L \times_G EG \rightarrow BG$  which has the self-clean intersection  $\mathcal{I}_G \subset L_G$ , and normalization  $\iota : \tilde{L}_G \rightarrow L_G$ . We can choose a suitable collection of Morse-Smale pairs  $(f_N, \mathcal{V}_N)$  on the finite dimensional approximation  $\tilde{L}_N \times_{L_N} \tilde{L}_N$  of  $\tilde{L}_G \times_{L_G} \tilde{L}_G$  and proceed to construct  $A_\infty$ -algebras  $(CF_G^\bullet(L; \Lambda_0), m^G)$  and  $(CF_G^\bullet(L; \Lambda_0)^\dagger, m^{G, \dagger})$  following Section 2.2 and 2.4. This give rise to interesting equivariant disc potentials even in the case when  $L$  is unobstructed as we will see in Section 4.

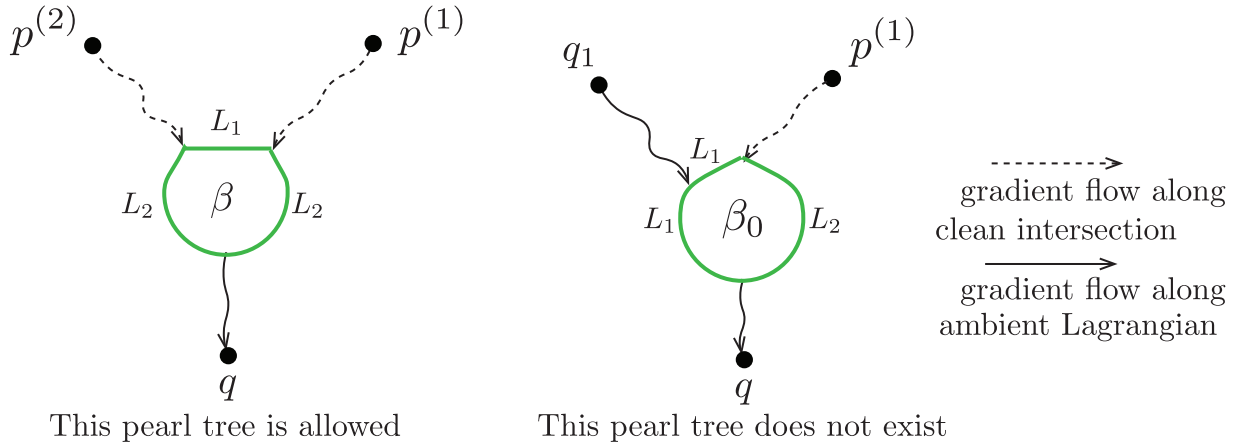


FIGURE 3. In the figure,  $\beta_0$  is a constant polygon. The pearly tree shown on the right actually does not exist, due to inconsistency of Lagrangian boundary labels.

### 3. THE T-EQUIVARIANT DISC POTENTIALS OF TORIC MANIFOLDS

In this section, we study the equivariant Morse model in the case of a torus  $T = (\mathbb{S}^1)^\ell$  acting on a closed, connected, relative spin Lagrangian submanifold  $L = (\mathbb{S}^1)^\ell \times P$  of product type of a  $2d$ -dimensional symplectic  $T$ -manifold  $X$ , such that  $T$  acts freely on the first factor of  $L$  and trivially on the second factor. We choose a suitable choice of Morse-Smale pairs in order to construct  $(CF_T^\bullet(L; \Lambda_0)^\dagger, m^{T, \dagger})$ . When applied to the regular

moment map fibers  $L = (\mathbb{S}^1)^d$  of a compact semi-Fano toric manifold, we recover the equivariant toric superpotential  $W_\lambda$  as the equivariant disc potential.

**3.1 Morse theory on the approximation spaces.** We begin by describing our choice of Morse-Smale pairs  $(f_N, \mathcal{V}_N)$  on  $L_N$ .

The models we choose for the universal bundle  $ET$  and the classifying space  $BT$  are  $ET = (\mathbb{S}^\infty)^\ell$  and  $BT = (\mathbb{C}\mathbb{P}^\infty)^\ell$ , respectively. We also have the finite dimensional approximations  $ET(N) = (\mathbb{S}^{2N+1})^\ell$  and  $BT(N) = (\mathbb{C}\mathbb{P}^N)^\ell$ . Since the  $T$  acts trivially on  $P$ , we have  $L_T = (\mathbb{S}^1)_T^\ell \times (P \times (\mathbb{C}\mathbb{P}^\infty)^\ell)$  and  $L_N = (\mathbb{S}^1)_N^\ell \times (P \times (\mathbb{C}\mathbb{P}^N)^\ell)$ .

For  $N \geq 1$ , let  $[z_{i,0}, \dots, z_{i,\ell}]$  be the homogeneous coordinates on the  $i$ -th component of  $BT(N) = (\mathbb{C}\mathbb{P}^N)^\ell$ . Let  $\pi_N : L_N \rightarrow (\mathbb{C}\mathbb{P}^N)^\ell$  be the projection map. Let  $\{U_{j_1 \dots j_\ell}\}$  be the open cover of  $(\mathbb{C}\mathbb{P}^N)^\ell$  defined by

$$U_{j_1 \dots j_\ell} = \left\{ [z_{i,0}, \dots, z_{i,N}]_{i=1, \dots, \ell} \in (\mathbb{C}\mathbb{P}^N)^\ell \mid z_{i,j_i} \neq 0 \right\}.$$

Set  $\tilde{U}_{j_1 \dots j_\ell} = \pi_N^{-1}(U_{j_1 \dots j_\ell}) \cong L \times (\mathbb{C}^N)^\ell$ . We will be working with the atlas  $\{\tilde{U}_{j_1 \dots j_\ell}\}$  for  $L_N$  with local coordinates

$$\left( \left( \theta_1^{(j_1 \dots j_\ell)}, \dots, \theta_\ell^{(j_1 \dots j_\ell)}, \vec{p}^{(j_1 \dots j_\ell)} \right), \left( z_{i,0}^{(j_1 \dots j_\ell)}, \dots, z_{i,j_i}^{(j_1 \dots j_\ell)} = 1, \dots, z_{i,N}^{(j_1 \dots j_\ell)} \right)_{i=1, \dots, \ell} \right)$$

on  $\tilde{U}_{j_1 \dots j_\ell}$ , where  $\theta_i^{(j_1 \dots j_\ell)} \in [0, 2\pi)$  are angular coordinates on  $(\mathbb{S}^1)^\ell$ ,  $\vec{p}^{(j_1 \dots j_\ell)}$  is any coordinate system on  $P$ , and the term under “ $\widehat{\phantom{x}}$ ” is omitted. Set  $\vartheta_{i,j}^{(j_1 \dots j_\ell)} = \text{Arg}(z_{i,j}^{(j_1 \dots j_\ell)})$ . The transition map  $\tilde{U}_{j_1 \dots j_\ell}$  to  $\tilde{U}_{j'_1 \dots j'_\ell}$  is given by

$$\left( \left( \theta_1^{(j_1 \dots j_\ell)}, \dots, \theta_\ell^{(j_1 \dots j_\ell)}, \vec{p}^{(j_1 \dots j_\ell)} \right), \left( z_{i,0}^{(j_1 \dots j_\ell)}, \dots, z_{i,j_i}^{(j_1 \dots j_\ell)} = 1, \dots, z_{i,N}^{(j_1 \dots j_\ell)} \right) \right) \mapsto \left( \left( \theta_1^{(j_1 \dots j_\ell)} + \vartheta_{1,j'_1}^{(j_1 \dots j_\ell)}, \dots, \theta_\ell^{(j_1 \dots j_\ell)} + \vartheta_{\ell,j'_\ell}^{(j_1 \dots j_\ell)}, \vec{p}^{(j_1 \dots j_\ell)} \right), \left( \frac{z_{i,0}^{(j_1 \dots j_\ell)}}{z_{i,j'_i}^{(j_1 \dots j_\ell)}}, \dots, \frac{z_{i,j_i}^{(j_1 \dots j_\ell)}}{z_{i,j'_i}^{(j_1 \dots j_\ell)}} = 1, \dots, \frac{z_{i,j_i}^{(j_1 \dots j_\ell)}}{z_{i,j'_i}^{(j_1 \dots j_\ell)}}, \dots, \frac{z_{i,N}^{(j_1 \dots j_\ell)}}{z_{i,j'_i}^{(j_1 \dots j_\ell)}} \right) \right).$$

Let's fix the inclusion  $(\mathbb{C}\mathbb{P}^N)^\ell \hookrightarrow (\mathbb{C}\mathbb{P}^{N+1})^\ell$  to be

$$([z_{i,0}, \dots, z_{i,N}]) \mapsto ([z_{i,0}, \dots, z_{i,N}, 0]). \quad (3.1)$$

Which in turn, fixes the inclusions  $X_N \hookrightarrow X_{N+1}$  and  $L_N \hookrightarrow L_{N+1}$ .

We have a perfect Morse function  $\varphi : (\mathbb{C}\mathbb{P}^\infty)^\ell \rightarrow \mathbb{R}$

$$\varphi([z_{i,0}, z_{i,1}, \dots]) = - \sum_{i=1}^{\ell} \frac{\sum_{k=1}^{\infty} k |z_{i,k}|^2}{\sum_{k=0}^{\infty} |z_{i,k}|^2}.$$

decreasing along the finite dimensional strata. The critical points of  $\varphi$  are of the form

$$\text{Crit}(\varphi) = \{([0, \dots, 0, z_{i,j_i} \neq 0, 0, \dots]_{i=1, \dots, \ell}) \mid (j_1 \dots j_\ell) \in \mathbb{Z}_{\geq 0}^\ell\}.$$

with degrees given by  $\sum_{i=1}^{\ell} 2j_i$ . We denote the degree 2 critical points with  $j_i = 1$ , and  $j_k = 0$  for  $k \neq i$  by  $\lambda_i$ . We also denote the critical point where  $\varphi$  attains the maximum by  $\mathbf{1}_{BT}$ , i.e.

$$\mathbf{1}_{BT} = ([z_{i,0} \neq 0, 0, 0, \dots]_{i=1, \dots, \ell}).$$

We set  $\varphi_N = \varphi|_{BT(N)}$ . Note that  $\varphi_N$  is a perfect Morse function on  $(\mathbb{C}\mathbb{P}^N)^\ell$ . We will abuse notation and denote the degree two critical points and the maximum of  $\varphi_N$  again by  $\mathbf{1}_{BT}$  and  $\lambda_i$ , respectively. Note that we have

$$H^\bullet(BT(N); \mathbb{Z}) = H^\bullet((\mathbb{C}\mathbb{P}^N)^\ell; \mathbb{Z}) = \mathbb{Z}[\lambda_1, \dots, \lambda_\ell] / (\lambda_i^{N+1}),$$

and

$$H_T^\bullet(\text{pt}; \mathbb{Z}) = \mathbb{Z}[\lambda_1, \dots, \lambda_\ell].$$

On the other hand, let  $f_P$  be a Morse function on  $P$  with a unique maximum  $\mathbf{1}_P$ , and let  $f_{(\mathbb{S}^1)^\ell} : (\mathbb{S}^1)^\ell \rightarrow \mathbb{R}$  be the perfect Morse function

$$f_{(\mathbb{S}^1)^\ell}(\theta_1, \dots, \theta_\ell) = \sum_{i=1}^{\ell} \cos(\theta_i).$$

The critical points of  $f_{(\mathbb{S}^1)^\ell}$  are of the form

$$\text{Crit}(f_{(\mathbb{S}^1)^\ell}) = \{(\theta_1, \dots, \theta_\ell) \in L \mid \theta_i = 0 \text{ or } \theta_i = \pi\}.$$

Let  $f = (f_{(\mathbb{S}^1)^\ell}, f_P)$  be the Morse function on  $L$ . We denote by  $\mathbf{1}_L$  the critical point where  $f$  attains the maximum, *i.e.*  $\mathbf{1}_L = ((0, \dots, 0), \mathbf{1}_P)$ . We also denote by  $x_1, \dots, x_\ell$  the degree 1 critical points, and  $x_i \wedge x_j$ ,  $1 \leq i < j \leq \ell$  the degree 2 critical points of  $f$  of the following forms

$$x_i = ((0, \dots, \theta_i = \pi, \dots, 0), \mathbf{1}_P),$$

and

$$x_i \wedge x_j = ((0, \dots, \theta_i = \pi, \dots, \theta_j = \pi, \dots, 0), \mathbf{1}_P).$$

Let  $\{D_j\}$  be the open cover of  $\mathbb{C}\mathbb{P}^\infty$  given by

$$D_j = \left\{ (z_0^{(j)}, \dots, z_N^{(j)}) \in U_j : |z_k^{(j)}| < 2 \text{ for all } k \right\}. \quad (3.2)$$

Let  $\{\rho^{(j)}\}$  be a smooth partition of unity subordinate to  $\{D_j\}$ , and denote by  $\rho_i^{(j)}$  the post-composition of  $\rho^{(j)}$  with the projection from  $L_T$  to the  $i$ -th component of  $(\mathbb{C}\mathbb{P}^\infty)^\ell$ . We define  $\rho^{(j_1 \dots j_\ell)} : L_T \rightarrow \mathbb{R}$  by

$$\rho^{(j_1 \dots j_\ell)} = \prod_{i=1}^{\ell} \rho_i^{(j_i)}.$$

Let  $\rho_N^{(j_1 \dots j_\ell)} = \rho^{(j_1 \dots j_\ell)}|_{BT(N)}$ , then  $\{\rho_N^{(j_1 \dots j_\ell)}\}$  is a smooth partition of unity on  $L_N$ .

Let  $\phi_N^{(j_1 \dots j_\ell)} : \tilde{U}_{j_1 \dots j_\ell} \rightarrow \mathbb{R}$  be the fiber-wise Morse function

$$\phi_N^{(j_1 \dots j_\ell)} \left( (\theta_1^{(j_1 \dots j_\ell)}, \dots, \theta_\ell^{(j_1 \dots j_\ell)}, \vec{p}^{(j_1 \dots j_\ell)}), (z_{i,0}^{(j_1 \dots j_\ell)}, \dots, z_{i,N}^{(j_1 \dots j_\ell)}) \right) = f \left( \theta_1^{(j_1 \dots j_\ell)}, \dots, \theta_\ell^{(j_1 \dots j_\ell)}, \vec{p}^{(j_1 \dots j_\ell)} \right),$$

and set

$$\phi_N = \sum_{(j_1 \dots j_\ell)} \rho_N^{(j_1 \dots j_\ell)} \phi_N^{(j_1 \dots j_\ell)}.$$

Notice that  $\phi_N = \phi_N^{(j_1 \dots j_\ell)}$  in a neighborhood of  $\pi_N^{-1}(\lambda)$ ,  $\lambda \in \text{Crit}(\varphi_N)$ . Then, for  $\epsilon > 0$  sufficiently small, the function  $f_N : L_N \rightarrow \mathbb{R}$  defined by

$$f_N := \epsilon \phi_N + \pi_N^* \varphi_N$$

is a Morse function, with critical points

$$\text{Crit}(f_N) = \left\{ (x, \lambda) \mid x \in \text{Crit}(\phi_N|_{\pi_N^{-1}(\lambda)}), \lambda \in \text{Crit}(\varphi_N) \right\} = \text{Crit}(f) \times \text{Crit}(\varphi_N).$$

The degree of  $(x, \lambda) \in \text{Crit}(f_N)$  is given by

$$|(x, \lambda)| = |x| + |\lambda|,$$

where  $|x|$  and  $|\lambda|$  are the degrees of  $x$  and  $\lambda$  as critical points of  $f$  and  $\varphi_N$ , respectively.

Let  $V_P$  be a negative pseudo-gradient vector field for  $f_P$ . Let  $V_N$  be the vector field given by

$$\begin{aligned} V_N|_{\tilde{U}_{j_1 \dots j_\ell}} = & 4 \sum_{i=1}^{\ell} \sum_{j \neq j_i} (j - j_i) \left| z_{i,j}^{(j_1 \dots j_\ell)} \right|^2 \frac{\partial}{\partial |z_{i,j}^{(j_1 \dots j_\ell)}|^2} \\ & + \sum_{i=1}^{\ell} \left( \rho_{N,i}^{(j_i)} \sin(\theta_i^{(j_1 \dots j_\ell)}) + \sum_{j \neq j_i} \rho_{N,i}^{(j)} \sin(\theta_i^{(j_1 \dots j_\ell)} + \theta_{i,j}^{(j_1 \dots j_\ell)}) \right) \frac{\partial}{\partial \theta_i^{(j_1 \dots j_\ell)}}. \end{aligned}$$

Here  $\rho_{N,i}^{(j)} = \rho_i^{(j)}|_{BT(N)}$ . Then

$$\mathcal{V}_N = V_N + V_P$$

is a negative pseudo-gradient for  $f_N$ . For  $N = 0$ , we set  $f_0 = f$ , and  $\mathcal{V}_0 = \sum_{i=1}^{\ell} \sin(\theta_i) \frac{\partial}{\partial \theta_i} + V_P$ .

**Proposition 3.1.** *For generic choices of  $V_P$ , the pairs  $\{(f_N, \mathcal{V}_N)\}$  are Morse-Smale and satisfy Condition 2.12.*

For  $p, q \in \text{Crit}(f_N)$ , let  $\mathcal{M}(p, q)$  be the moduli space of gradient flow lines from  $p$  to  $q$ , modulo reparametrizations. Let  $(C^\bullet(f_N; \mathbb{Z}), \delta)$  to be the Morse cochain complex

$$C^\bullet(f_N; \mathbb{Z}) := \bigoplus_{p \in \text{Crit}(f_N)} \mathbb{Z} \cdot p$$

We have

$$C^\bullet(f_N; \mathbb{Z}) = C^\bullet(f; \mathbb{Z}) \otimes \mathbb{Z}[\lambda_1, \dots, \lambda_\ell] / (\lambda_i^{N+1}). \quad (3.3)$$

For convenience, we will write the elements of  $\text{Crit}(f_N)$  as  $x \otimes \lambda$ , where  $x \in \text{Crit}(f)$  and  $\lambda \in \text{Crit}(\varphi_N)$ . The following result will be used in the computation of the equivariant disc potentials.

**Proposition 3.2.** *For  $N \geq 1$  and  $x_1, \dots, x_\ell$ , we have*

$$\delta(x_i \otimes \mathbf{1}_{BT}) = \mathbf{1}_L \otimes \lambda_i.$$

*In particular, there exists a unique gradient flow line from  $x_i \otimes \mathbf{1}_{BT}$  to  $\mathbf{1}_L \otimes \lambda_i$ .*

*Proof.* The possible outputs of  $\delta(x_i \otimes \mathbf{1}_{BT})$  are of degree 2 critical points of the form  $x_j \wedge x_k \otimes \mathbf{1}_{BT}$ , and  $\mathbf{1}_L \otimes \lambda_j$ . It is easy to see that  $\mathcal{M}(x_i \otimes \mathbf{1}_{BT}, x_j \wedge x_k \otimes \mathbf{1}_{BT}) = \emptyset$  if both  $j \neq i$  and  $k \neq i$ , and  $\mathcal{M}(x_i \otimes \mathbf{1}_{BT}, x_j \wedge x_k \otimes \mathbf{1}_{BT})$  consists of two points of opposite orientation if either  $j = i$  or  $k = i$ .

Suppose  $\Phi : \mathbb{R} \rightarrow L_N$  is a flow line from  $x_i \otimes \mathbf{1}_{BT}$  to  $\mathbf{1}_L \otimes \lambda_j$ . Its projection  $\pi_N \circ \Phi : \mathbb{R} \rightarrow (\mathbb{C}\mathbb{P}^N)^\ell$  is a flow line for the vector field  $V$  on  $(\mathbb{C}\mathbb{P}^N)^\ell$  given by

$$V|_{U_{j_1 \dots j_\ell}} = 4 \sum_{i=1}^{\ell} \sum_{j \neq j_i} (j - j_i) \left| z_{i,j}^{(j_1 \dots j_\ell)} \right|^2 \frac{\partial}{\partial |z_{i,j}^{(j_1 \dots j_\ell)}|^2}$$

from  $\mathbf{1}_{BT} \rightarrow \lambda_j$ , whose image is contained in  $U_{0\dots 0}$ . This means the image of  $\Phi$  is contained in  $\tilde{U}_{0\dots 0}$ , and we can therefore write

$$\Phi(t) = \left( \left( \theta_1^{(0\dots 0)}(t), \dots, \theta_\ell^{(0\dots 0)}(t) \right), \left( z_{\mu,1}^{(0\dots 0)}(t), \dots, z_{\mu,N}^{(0\dots 0)}(t) \right)_{\mu=1,\dots,\ell} \right),$$

in term of the coordinates on  $\tilde{U}_{0\dots 0}$ . For  $\Phi(t)$  to have the correct asymptotics, we must have  $z_{\mu,\nu}^{(0\dots 0)}(t) = 0$  for  $\mu \neq i$  and  $\nu \neq 1$ ;  $\theta_\mu^{(0\dots 0)}(t) = 0$  for  $\mu \neq i$ ;  $z_{\ell,1}^{(0\dots 0)}(t) = e^{2t+i\vartheta}$  where  $\vartheta \in [0, 2\pi)$ ;  $\theta_i^{(0\dots 0)}(t)$  satisfies

$$\lim_{t \rightarrow -\infty} \theta_i^{(0\dots 0)}(t) = \pi,$$

and

$$\lim_{t \rightarrow +\infty} \theta_i^{(0\dots 0)}(t) + \delta_{ij}\vartheta = 0,$$

in addition to

$$\frac{d\theta_i^{(0\dots 0)}(t)}{dt} = a(t) \sin\left(\theta_i^{(0\dots 0)}(t)\right) + b(t) \sin\left(\theta_i^{(0\dots 0)}(t) + \vartheta\right).$$

Here  $a(t) = \rho_1^{(0)}|_{U_0}(e^{2t+i\vartheta})$  and  $b(t) = \rho_1^{(1)}|_{U_0}(e^{2t+i\vartheta})$ , and  $\{\rho_1^{(0)}, \rho_1^{(1)}\}$  is the partition of unity for  $\mathbb{C}\mathbb{P}^1$ .

For existence of a flow line when  $i = j$ , we note that the flow line with  $\theta_i^{(0\dots 0)}(t) = \pi$  and  $\vartheta = 0$  has the desired asymptotics. As for uniqueness, assume without loss of generality that  $\Phi(0)$  is in a neighborhood of  $x_i \otimes \mathbf{1}_{BT}$  such that  $\rho_N^{(0)}(t) = 1$  and  $\rho_N^{(1)}(t) = 0$ . In this neighborhood, we have explicitly

$$\theta_i^{(0\dots 0)}(t) = 2 \cot^{-1} \left( e^{-t} \cot \left( \frac{\theta_i^{(0\dots 0)}(0)}{2} \right) \right).$$

It is easy to see that

$$\lim_{t \rightarrow -\infty} \theta_i^{(0\dots 0)}(t) = \pi,$$

if  $\theta_i^{(0\dots 0)}(0) = \pi$ , and

$$\lim_{t \rightarrow -\infty} \theta_i^{(0\dots 0)}(t) = 0,$$

otherwise. Solving for

$$\frac{d\theta_i^{(0\dots 0)}(t)}{dt} = 0$$

gives

$$\tan \theta_i^{(0\dots 0)}(t) = \frac{b(t) \sin(\vartheta)}{a(t) + b(t) \cos(\vartheta)}.$$

This means we have

$$\lim_{t \rightarrow +\infty} \theta_i^{(0\dots 0)}(t) + \delta_{ij}\vartheta = \pi,$$

unless  $\vartheta = 0$  and  $i = j$ , in which case

$$\lim_{t \rightarrow +\infty} \theta_i^{(0\dots 0)}(t) + \delta_{ij}\vartheta = 0.$$

□

**3.2 Computing the T-equivariant disc potentials.** By applying the construction in Section 2.3 and 2.4 with the choice of Morse-Smale pairs made in Section 3.1, one obtains

$$(CF_T^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}^{T, \dagger}) = (CF^\bullet(f; \Lambda_0) \otimes \Lambda_0[\lambda_1, \dots, \lambda_\ell], \mathfrak{m}^{T, \dagger})$$

associated to the pair  $(L, T)$ . In this section, we compute the equivariant disc potential of  $(CF_T^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}^{T, \dagger})$  assuming that every holomorphic disc bounded by  $L$  has non-negative Maslov index.

For simplicity of notations, we will suppress “ $\blacktriangledown$ ” and denote the unique maximum  $\mathbf{1}_L^\blacktriangledown$  on  $L$  by  $\mathbf{1}_L$ . Let  $x_1, \dots, x_\ell, y_1, \dots, y_\nu$  be a basis of  $C^1(f; \mathbb{Q})$ , and put  $X_i := x_i \otimes \mathbf{1}_{BT}$  and  $y_i := y_i \otimes \mathbf{1}_{BT}$ . We also denote  $\lambda_i := \mathbf{1}_L \otimes \lambda_i$ , and  $\mathbf{1} := \mathbf{1}_L \otimes \mathbf{1}_{BT}$ .

Let  $b = \sum_{i=1}^\ell x^i X_i + \sum_{i=1}^\nu y^i y_i$ , where  $x^i, y^i \in \Lambda_+$ . We consider the boundary deformation of  $\mathfrak{m}_0^{T, \dagger}$  by  $b$

$$\mathfrak{m}_0^{T, \dagger, b}(1) = \mathfrak{m}_0^{T, b}(1) = \mathfrak{m}_0^T(1) + \mathfrak{m}_1^T(b) + \mathfrak{m}_2^T(b, b) + \dots$$

The first equality above follows from the fact that the restriction of  $\mathfrak{m}^{T, \dagger}$  to  $CF_T^\bullet(L; \Lambda_0)$  agrees with  $\mathfrak{m}^T$ .

We compute the obstruction  $\mathfrak{m}_0^{T, b}(1)$  by counting pearly trees in  $L_T \subset X_T$ <sup>1</sup>. Since  $\mathfrak{m}_{k, \beta}^T(b, \dots, b)$  is of degree  $2 - \mu(\beta)$ , the outputs of  $\mathfrak{m}_0^{T, b}(1)$  have degree either 0 or 2 depending on the Maslov indices of the contributing disc classes. The possible outputs are of the following forms:

- (Degree zero)  $\mathbf{1}$ ,
- (Degree two)  $p \otimes \mathbf{1}_{BT}$ , where  $p \in \text{Crit}(f)$  is a degree 2 critical point,
- (Degree two)  $\lambda_i$ .

Notice that all the critical points above are contained in  $\text{Crit}(f_1)$ . Thus,  $\mathfrak{m}_0^{T, b}(1)$  can be computed by counting pearly trees in  $L_1$ .

**Proposition 3.3.** *Suppose  $T = (\mathbb{S}^1)^\ell$  acts on  $(X, \omega_X)$  preserving  $\omega_X$ . Let  $L \subset X$  be a  $T$ -invariant closed Lagrangian submanifold of product type  $L = (\mathbb{S}^1)^\ell \times P$  such that the  $T$ -action on  $L$  is standard, i.e.,  $T$  acts freely on  $(\mathbb{S}^1)^\ell$  and trivially on  $P$ . Suppose  $L$  has non-negative minimal Maslov index, then*

$$\mathfrak{m}_0^{T, b}(1) = \mathfrak{m}_0^b(1) \otimes \mathbf{1}_{BT} + \sum_{i=1}^\ell h_i(\vec{x}, \vec{y}) \lambda_i \quad (3.4)$$

where  $\mathfrak{m}$  is the  $A_\infty$ -structure on  $CF^\bullet(L; \Lambda_0)$ , and  $h_i(\vec{x}, \vec{y}) \in \Lambda_0$ . Moreover, if  $\mathfrak{m}_0^b(1) = W(\vec{x}, \vec{y}) \mathbf{1}_L$  for some  $W(\vec{x}, \vec{y}) \in \Lambda_+$ , then  $(CF_T^\bullet(L; \Lambda_0)^\dagger, \mathfrak{m}^{T, \dagger})$  is weakly unobstructed.

*Proof.* The first two types of outputs are contributed by pearly trees contained in the fiber  $L$  over the critical point  $\mathbf{1}_{BT}$ , and coincide with  $\mathfrak{m}_0^b(1)$  by our construction of  $\mathfrak{m}^T$ . Thus, we have the expression (3.4).

The last assertion follows from Corollary 2.17, namely, if  $\mathfrak{m}_0^b(1) = W(\vec{x}, \vec{y}) \mathbf{1}_L$ , then

$$\mathfrak{m}_0^{T, b}(1) = W(\vec{x}, \vec{y}) \mathbf{1} + \sum_{i=1}^\ell h_i(\vec{x}, \vec{y}) \lambda_i. \quad (3.5)$$

□

<sup>1</sup>To be more precise, we should perform the calculation on the finite dimensional approximations  $L_N \subset X_N$  and then take its limit (see (2.36), (2.37) and (2.38)). By abuse of notations, the  $A_\infty$ -operations on  $L_T$  are regarded as the  $A_\infty$ -operations on the approximation spaces.

In the following, we shall compute  $m_0^b(1)$  and  $h_i(\vec{x}, \vec{y})$  explicitly under additional assumptions. We begin by simplifying  $h_i(\vec{x}, \vec{y})$  under the condition that the minimal Maslov index of  $L$  is at least 2.

**Lemma 3.4.** *In the situation of Proposition 3.3, if, in addition, the minimal Maslov index of  $L$  is at least 2, then  $h_i(\vec{x}, \vec{y}) = x^i$  in (3.4).*

*Proof.* The terms  $h_i(\vec{x}, \vec{y})$  are contributed by disc classes of Maslov index 0 by degree reason. Since  $L$  has minimal Maslov index at least two, the only contribution comes from the trivial disc class  $\beta_0$ , hence the computation of the terms  $m_k^T(b, \dots, b)$  reduces to  $m_\Gamma^T(b, \dots, b)$ , where  $\Gamma \in \Gamma_{k+1}$  is a stable planar rooted tree with all interior vertices decorated by  $\beta_0$ . In other words, the configurations  $\Gamma$  we are counting are Morse flow trees in  $L_1$ .

By Proposition 3.2, we have  $m_1^T(X_i) = \lambda_i$ . We will show momentarily that the coefficients of  $\lambda_i$  in the terms  $m_k^T(X_{i_1}, \dots, X_{i_k})$  are zero for  $k \geq 2$  and  $(i_1, \dots, i_k) \in \{1, \dots, \ell\}^k$ . First, let's consider the case when  $i_j \neq i$ , for some  $j$ . In the proof of Proposition 3.2, we showed that there exists no flow line from  $X_{i_j}$  to  $\lambda_i$ . This means for generic small perturbations, the moduli spaces  $\mathcal{M}_\Gamma(f_1; X_{i_1}, \dots, X_{i_k}; \lambda_i)$  (which are responsible for the coefficient of  $\lambda_i$  in  $m_\Gamma^T(X_{i_1}, \dots, X_{i_k})$ ) are empty.

Now, we consider  $m_k^T(X_i, \dots, X_i)$ ,  $k \geq 2$ . Since rotations on the  $i$ -th circle factor of  $L$  commutes with the structure group of  $L_1$ , we have a global  $S^1$ -action on  $L_1$  rotating the  $i$ -th circle factor of the fibers  $L$ . To achieve transversality for the moduli spaces  $\mathcal{M}_\Gamma(f_1; X_i, \dots, X_i; \lambda_i)$ , we can perturb the the flow lines from the first  $k - 1$  inputs using the  $S^1$ -action. We have the unique flow line  $\gamma$  from  $X_i$  to  $\lambda_i$  as in Proposition 3.2. For a perturbed flow line from the first  $k - 1$  inputs to intersect with  $\gamma$  (in order to form a flow tree), its projection in  $BT(1)$  must coincide with  $\pi(\gamma)$ . However, over  $\pi(\gamma)$ , the flow lines from the first  $k - 1$  inputs are simply the rotations of  $\gamma$  by  $S^1$ , which do not intersect with  $\gamma$ . Thus,  $\mathcal{M}_\Gamma(f_1; X_i, \dots, X_i; \lambda_i)$  are empty for generic small perturbations.

Similarly, since the fiber bundle  $P_T = P \times BT$  is trivial, there exists no flow line from  $y_j$  to  $\lambda_i$ , and therefore the coefficients of  $\lambda_i$  in the terms  $m_k^T(b, \dots, y_j, \dots, b)$  are zero.  $\square$

Thus, in the setting of Lemma 3.4, we have

$$m_0^{T,b}(1) = m_0^b(1) \otimes \mathbf{1}_{BT} + \sum_{i=1}^{\ell} x^i \lambda_i.$$

In [FOOO10], it was shown that the moment map fibers of a compact toric manifold are weakly unobstructed in the de Rham model. The following is the corresponding statement in the Morse model (we restrict ourselves to semi-Fano case to ensure that minimal Maslov index is 2).

**Lemma 3.5.** *Let  $L$  be a regular moment map fiber of a compact semi-Fano toric manifold. We have  $m_0^b(1) \in \Lambda_0 \cdot \mathbf{1}_L$ .*

*Proof.* The only possible outputs of  $m_0^b(1)$  are multiples of  $\mathbf{1}_L$  and degree two critical points of  $f$  (which are of the form  $X_j \wedge X_k$ ), contributed by Maslov index zero and two disc classes, respectively. Since  $L$  has minimal Maslov index two, only Morse flow trees contribute to a degree two critical points. We have  $m_1(X_i) = 0$ , since  $f$  is a perfect Morse function on  $L$ . For  $m_k(X_{i_1}, \dots, X_{i_k})$ ,  $k \geq 2$ , if there are two repeated inputs  $X_i$ , then by perturbing unstable hypertori  $\Delta_{X_i}$  using the  $S^1$ -action rotating the  $i$ -th circle factor of  $L$ ,

the perturbed hypertori do not intersect and hence  $\mathfrak{m}_k(X_{i_1}, \dots, X_{i_k}) = 0$ . In the case of distinct inputs, we have

$$\sum_{\sigma \in S_k} \mathfrak{m}_k \left( X_{i_{\sigma(1)}}, \dots, X_{i_{\sigma(k)}} \right) = 0, \quad k \geq 2,$$

due to the orientations on the corresponding moduli spaces.  $\square$

**Remark 3.6.** *Before proceeding, a remark is in order addressing the perturbations used in the proof of Lemma 3.4, and 3.5. Recall from Remark 2.1 that we do not perturb the input singular chains when choosing perturbation for a fiber product  $\mathcal{M}_{k+1}(\beta; L_N; \bar{P})$  since the singular chains are fixed during the inductive construction and doing so would destroy the  $A_\infty$ -structure. On the other hand, since the Kuranishi structure on the disc moduli are weakly submersive, we can realize the perturbation of input singular chains by perturbing the respective evaluation maps.*

Combining the above lemmas, to show that the equivariant toric superpotentials coincide with the equivariant disc potentials, it remains to compute the (non-equivariant) disc potential.

Recall that the holomorphic disc classes are generated by the basic disc classes  $\beta_i$  for  $1 \leq i \leq m$  [CO06]. Moreover, by [FOOO10], a stable disc class of Maslov index two must be of the form  $\beta_i + \alpha$  for some curve class  $\alpha$  with  $c_1(\alpha) = 0$ . Let  $n_1(\beta_i + \alpha)$  be the degree of the virtual fundamental class  $\text{ev}_*[\mathcal{M}_1(\beta_i + \alpha)]$ . In the Morse model, it is given by counting the number of times  $\text{ev}_*[\mathcal{M}_1(\beta_i + \alpha)]$  hits the maximal point.

**Theorem 3.7** (Equivariant toric superpotential). *For compact semi-Fano toric manifolds  $X$  of dimension  $2d$ , let  $v_1, \dots, v_m \in \mathbb{Z}^d$  be the primitive generators of the one-dimensional cones of the fan  $\Sigma$  defining  $X$ . We have*

$$\mathfrak{m}_0^{T,b}(1) = W^{\text{disc}}(x^1, \dots, x^d) \mathbf{1} + \sum_{i=1}^d x^i \lambda_i = \left( W^{\text{disc}}(x^1, \dots, x^d) + \sum_{i=1}^d x^i \lambda_i \right) \mathbf{1},$$

where

$$W^{\text{disc}} = \sum_{i=1}^m \left( \sum_{\alpha: c_1(\alpha)=0} n_1(\beta_i + \alpha) \mathbf{T}^{\omega_X(\alpha)} \right) \mathbf{T}^{\omega_X(\beta_i)} \exp(v_i \cdot (x^1, \dots, x^d)). \quad (3.6)$$

In particular in the Fano case,

$$\mathfrak{m}_0^{T,b} = \left( W^{\text{HV}}(e^{x^1}, \dots, e^{x^d}) + \sum_{i=1}^d x^i \lambda_i \right) \mathbf{1},$$

where  $W^{\text{HV}}$  is the Hori-Vafa superpotential [HV, HKK<sup>+</sup>03].

*Proof.* Recall from Proposition 3.3 that  $W^{\text{disc}} = \mathfrak{m}_0^b(1)$ . By degree reason, a stable rooted tree  $\Gamma \in \Gamma_{k+1}$ , contributing to the term  $\mathfrak{m}_k(b, \dots, b)$ ,  $k \geq 1$ , must have exactly one interior vertex decorated by a Maslov index two disc class and the remaining interior vertices are decorated by the trivial disc class  $\beta_0$ .

We take the perfect Morse function  $f$  on  $L$  such that the boundaries of the finitely many basic holomorphic discs passing through the maximal point do not intersect with  $\Delta_{X_\mu} \cap \Delta_{X_\nu}$  for any  $\mu \neq \nu$ , where  $\Delta_{X_j}$  denotes the unstable hypertorus of the degree-one critical point  $X_j$ . This means the only  $\Gamma \in \Gamma_{k+1}$ , contributing to  $\mathfrak{m}_k(b, \dots, b)$  has exactly one interior vertex, which is decorated by  $\beta_i + \alpha$ . It remains to consider contributions from the moduli spaces of the form  $\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{X_{\mu_1}}, \dots, \Delta_{X_{\mu_k}})$ ,  $(\mu_1, \dots, \mu_k) \in \{1, \dots, d\}^k$ .

Since  $L$  has minimal Maslov index two and  $\beta_i + \alpha$  has Maslov index two, the moduli spaces  $\mathcal{M}_{k+1}(\beta_i + \alpha; L)$  have no (non-constant) disc bubbles. We have  $\mathcal{M}_{k+1}(\beta_i + \alpha; L) = \mathcal{M}_1(\beta_i + \alpha; L) \times \hat{C}_k$  (as topological spaces) via the forgetful map  $\text{forget} : \mathcal{M}_{k+1}(\beta_i + \alpha; L) \rightarrow \mathcal{M}_1(\beta_i + \alpha; L)$ , where  $\hat{C}_k$  is certain iterated blow up of

$$C_k = \{(t_1, \dots, t_k) | 0 \leq t_1 \leq \dots \leq t_k \leq 1\},$$

defined in [FOOO10, Lemma 11.8] (see also [FOOO16, Remark 4.1.1]). The output evaluation of the fundamental class  $[\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{X_{\mu_1}}, \dots, \Delta_{X_{\mu_k}})]$  intersects the maximal point only if the boundary of the basic holomorphic disc, which represents  $\beta_i$  and passes through the maximal point, intersects all the input hypertori  $\Delta_{X_{\mu_i}}$ .

Suppose the output evaluation of the fundamental class  $[\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{X_{\mu_1}}, \dots, \Delta_{X_{\mu_k}})]$  intersects the maximal point. Let  $r_j$  be the number of times  $\Delta_{X_j}$  appears in the inputs. Let  $\psi_1, \dots, \psi_k : L \rightarrow L$  be small rotations in the direction of  $\partial\beta_i$  with increasing angles and denote  $\Delta_{X_j, \ell} = \psi_\ell(\Delta_{X_j})$ ,  $1 \leq \ell \leq k$ . We consider all possible configurations  $\Delta_{X_{\mu_1}, \ell_1}, \dots, \Delta_{X_{\mu_k}, \ell_k}$  where  $\{\ell_1, \dots, \ell_k\}$  is a cyclic permutation of  $\{1, \dots, k\}$ . For each  $u \in \mathcal{M}_1(\beta_i + \alpha; L)$ , the fiber  $\text{forget}^{-1}(u)$  contains  $\prod_{j=1}^d m_j^{r_j}$  many discs intersecting these configurations (determined by the location of  $z_0$ ), where  $m_j$  is the multiplicity of  $\partial\beta_i \cap \Delta_{X_j}$  (see Figure 4 for an illustration).

Let  $r_j$  be the number of times  $\Delta_{X_j}$  appears in the inputs. We denote by  $X_{j,1}, \dots, X_{j,r_j}$  the copies of  $\Delta_{X_j}$  in the order they appear in the inputs. Let  $X_j^{(1)}, \dots, X_j^{(r_j)}$  be disjoint small perturbations of  $\Delta_{X_j}$  along the direction of  $\partial\beta_i$  (and ordered along this direction). We also denote by  $X_{j,\ell}^{(j_\ell)} = X_j^{(j_\ell)}$ ,  $j_\ell \in \{1, \dots, r_j\}$  the corresponding perturbation for the ordered copies of  $\Delta_{X_j}$ . Let's consider all possible configurations of perturbed input hypertori  $X_{j,\ell}^{(j_\ell)}$ , where  $\{j_1, \dots, j_{r_j}\}$  is a cyclic permutation of  $\{1, \dots, r_j\}$ . Then, for each  $u \in \mathcal{M}_1(\beta_i + \alpha; L)$ , the fiber  $\text{forget}^{-1}(u)$  contains  $\prod_{j=1}^d m_j^{r_j}$  many discs intersecting these configurations (determined by the location of  $z_0$ ), where  $m_j$  is the multiplicity of  $\partial\beta_i \cap \Delta_{X_j}$  (see Figure 4 for an illustration).

We denote by  $E$  the obstruction bundle of  $\mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{X_{\mu_1}}, \dots, \Delta_{X_{\mu_k}})$  and let  $s : \mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{X_{\mu_1}}, \dots, \Delta_{X_{\mu_k}}) \rightarrow E^{\otimes k}$  be a section corresponding to the summation of all possible configurations of perturbed input hypertori described above. We choose the multisection  $\varepsilon : \mathcal{M}_{k+1}(\beta_i + \alpha; L; \Delta_{X_{\mu_1}}, \dots, \Delta_{X_{\mu_k}}) \rightarrow E^{\otimes k}/S_k$  to be  $s/S_k$ , resulting in the coefficient  $\frac{n_1(\beta_i + \alpha) \prod_{j=1}^d m_j^{r_j}}{k!}$ . Summing over all the possible inputs of  $k$  hypertori and the  $\frac{k!}{\prod_{j=1}^d (r_j!)}$  orderings among each them, we obtain the term  $n_1(\beta_i + \alpha) \exp(v_i \cdot (x^1, \dots, x^d))$ .  $\square$

**Remark 3.8.** For the above theorem, we have chosen the perturbations for the fiber products to be the average of all configurations of the perturbed unstable hypertori of  $X_i$ . Indeed, we can choose other perturbations, which gives different expressions of the (non-equivariant) disc potential.

Take  $\mathbb{P}^1$  as an example. Let  $L$  be the equator. Let's perturb  $X$  in the counterclockwise order with respect to the left hemisphere. Then the non-equivariant disc potential will read

$$W = \mathbf{T}^{A/2} \left( \frac{1}{1-x} + (1-x) \right)$$

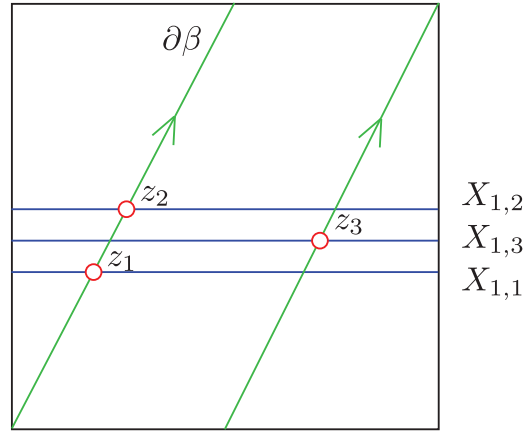


FIGURE 4. The circles correspond to a choice of intersection points which determines the configuration of perturbed hypertori.

instead of the well-known expression  $\mathbf{T}^{A/2} (e^x + e^{-x})$ .

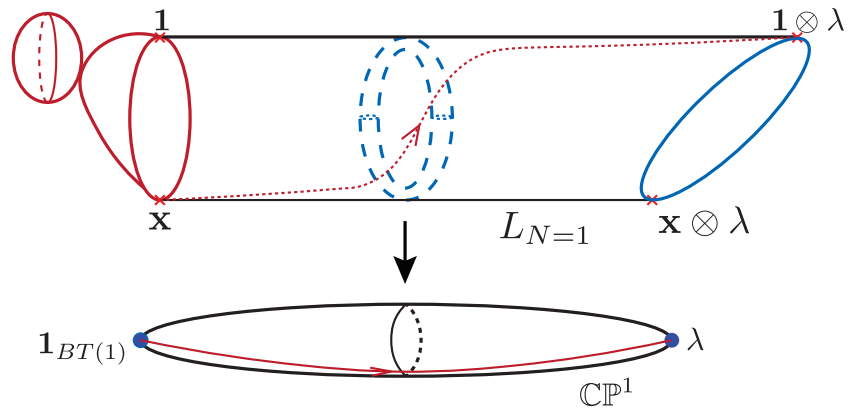


FIGURE 5. The geometric inputs of  $W^{equiv}$ . The non-constant discs (with sphere bubbles) contribute to  $\mathbf{1}^\vee$ , and the flow line contributes to  $\lambda^\vee$ .

In the Fano case,  $n_1(\beta_i + \alpha) = 0$  whenever  $\alpha \neq 0$  by dimension reason. Moreover  $n_1(\beta_i) = 1$  [CO06]. More generally, the (non-equivariant) disc potential  $W^{disc}$  for compact semi-Fano toric manifold has been computed by [CLLT17] using Seidel representations. The coefficients are given by the inverse mirror map. We recall it in the following theorem.

**Theorem 3.9** ([CLLT17]). *Let  $X$  be a compact semi-Fano toric manifold. Then*

$$\sum_{\alpha: c_1(\alpha)=0} n_1(\beta_i + \alpha) \mathbf{T}^{\omega(\alpha)} = \exp(g_i(\check{q}(q))),$$

where

$$g_i(\check{q}) := \sum_C \frac{(-1)^{(D_i \cdot C)} (- (D_i \cdot C) - 1)!}{\prod_{j \neq i} (D_j \cdot C)!} \check{q}^C, \tag{3.7}$$

the summation is over all effective curve classes  $C \in H_2^{\text{eff}}(X)$  satisfying

$$-K_X \cdot C = 0, D_i \cdot C < 0 \text{ and } D_j \cdot C \geq 0 \text{ for all } j \neq i$$

and  $\check{q} = \check{q}(q)$  is the inverse of the mirror map  $q = q(\check{q})$ .

A similar result also holds for toric semi-Fano Gorenstein orbifold  $[\text{CCL}^+]$ . However we stick with the manifold case for simplicity. Moreover, compactness of  $X$  can be replaced by requiring  $X$  to be semi-projective, so that the disc moduli spaces are compact. A toric manifold  $X$  is said to be semi-projective if it has a torus-fixed point, and the natural map  $X \rightarrow \text{Spec } H^0(X, \mathcal{O}_X)$  is projective [CLS11, Section 7.2]. Combining, we get the following.

**Corollary 3.10.** *For a toric fiber of a semi-projective and semi-Fano toric manifold, the equivariant disc potential equals to*

$$W^{\text{equiv}} = \sum_{i=1}^m \exp(g_i(\check{q}(q))) \mathbf{T}^{\omega_X(\beta_i)} \exp(v_i \cdot (x^1, \dots, x^d)) + \sum_{i=1}^d x^i \lambda_i$$

where  $g_i(\check{q}(q))$  is given by the inverse mirror map in Equation (3.7).

#### 4. $S^1$ -EQUIVARIANT DISC POTENTIAL FOR THE IMMERSSED 2-SPHERE

In this section, we consider the  $S^1$ -invariant immersed two-sphere  $\mathcal{S}^2$  with a single nodal point. This is a typical singular fiber in a Lagrangian torus fibration, which has an important application in the SYZ mirror construction.

First, we make the following observation which is important for finding isomorphisms between objects in the Fukaya category.

**Lemma 4.1.** *Let  $L_1, L_2$  be two graded Lagrangians in the Fukaya category that cleanly intersect with each other, and  $\alpha \in CF^0(L_1, L_2)$ . Suppose that  $CF^j(L_1, L_2) = 0$  for all  $j < 0$ . For any  $b_i \in CF^1(L_i, \Lambda_+)$  and  $w_i \in \Lambda$ ,*

$$\mathbf{m}_1^{b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee, b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee}(\alpha) = \mathbf{m}_1^{b_1, b_2}(\alpha).$$

Similarly, if further  $CF^j(L_2, L_1) = 0$  for all  $j < 0$  and  $\beta \in CF^0(L_2, L_1)$ , then

$$\mathbf{m}_2^{b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee, b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee, b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee}(\beta, \alpha) = \mathbf{m}_2^{b_1, b_2, b_1}(\beta, \alpha).$$

*Proof.* This is simply due to degree reason. The degree of  $\mathbf{1}_{L_i}^\vee$  is  $(-1)$ . The extra terms are  $m_k^{b_1, \dots, b_1, b_2, \dots, b_2}(\mathbf{1}_{L_1}^\vee, \dots, \mathbf{1}_{L_1}^\vee, \alpha, \mathbf{1}_{L_2}^\vee, \dots, \mathbf{1}_{L_2}^\vee) \in CF^*(L_1, L_2)$  for  $k \geq 2$ , which has degree  $2 - k + (-1) \cdot (k - 1) = 3 - 2k < 0$  and hence must vanish. The extra terms in  $\mathbf{m}_2^{b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee, b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee, b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee}(\beta, \alpha)$  also vanish for the same degree reason.  $\square$

An isomorphism pair between  $(L_1, b_1 + w_1 \cdot \mathbf{1}_{L_1}^\vee)$  and  $(L_2, b_2 + w_2 \cdot \mathbf{1}_{L_2}^\vee)$ , where  $b_i + w_i \cdot \mathbf{1}_{L_i}^\vee$  are weak bounding cochains, consists of  $\alpha \in CF^0(L_1, L_2)$  and  $\beta \in CF^0(L_2, L_1)$  satisfying

$$\begin{aligned} \mathbf{m}_1^{b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee, b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee}(\alpha) &= 0, \\ \mathbf{m}_1^{b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee, b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee}(\beta) &= 0, \\ \mathbf{m}_2^{b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee, b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee, b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee}(\beta, \alpha) &= \mathbf{1}_{L_1}^\vee + \mathbf{m}_1^{b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee}(\gamma_1) \text{ for some } \gamma_1, \\ \mathbf{m}_2^{b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee, b_1+w_1 \cdot \mathbf{1}_{L_1}^\vee, b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee}(\alpha, \beta) &= \mathbf{1}_{L_2}^\vee + \mathbf{m}_1^{b_2+w_2 \cdot \mathbf{1}_{L_2}^\vee}(\gamma_2) \text{ for some } \gamma_2, \end{aligned}$$

for some  $\gamma_1 \in CF^\bullet(L_1)$  and  $\gamma_2 \in CF^\bullet(L_2)$ . By the above lemma, it suffices to check  $m_1^{b_1, b_2}(\alpha) = 0$ ,  $m_1^{b_2, b_1}(\beta) = 0$ ,  $m_2^{b_1, b_2, b_1}(\beta, \alpha) = \mathbf{1}_{L_1}^\nabla$  and  $m_2^{b_2, b_1, b_2}(\alpha, \beta) = \mathbf{1}_{L_2}^\nabla$ . Note that since  $m_1^{b_i, b_i}(\mathbf{1}_{L_i}^\nabla) = \mathbf{1}_{L_i}^\nabla - \mathbf{1}_{L_i}^\nabla + h(b_i)\mathbf{1}_{L_i}^\nabla$  for some  $h(b_i) \in \Lambda_+$ , by rescaling  $\beta$  if necessary, these equations ensure we have an isomorphism pair.

The above statements have a direct  $G$ -equivariant analog, by replacing  $m_k$  by  $m_k^G$  and  $w_i = w_i^0 + \sum_\lambda \phi_\lambda \cdot \lambda$  where the summation is over degree two equivariant parameters  $\lambda$ ,  $w_i^0, \phi_\lambda \in \Lambda$ .

We consider the following immersed sphere  $\mathcal{S}^2$  in

$$X := \{(a, b, z) \in \mathbb{C}^2 \times \mathbb{C}^\times \mid ab = 1 + z\}.$$

Equip  $X$  with the symplectic form inherited from the standard symplectic form on  $\mathbb{C}^3$ . Let  $\Pi: X \rightarrow \mathbb{C}^\times$  be the projection to the  $z$ -component. Regarding  $\mathbb{S}^1$  as the unit circle in the complex plane  $\mathbb{C}$ , the space  $X$  has a fiberwise Hamiltonian  $\mathbb{S}^1$ -action given by  $\eta \cdot (a, b, z) \mapsto (\eta \cdot a, \eta^{-1} \cdot b, z)$ . One obtains the special Lagrangian torus fibration

$$(|a|^2 - |b|^2, |z|) \tag{4.1}$$

with respect to the holomorphic volume form  $\Omega_X = (da \wedge db)/z$ , see [HL82, Go101, Gro01]. In particular, a singular fiber which will be denoted by  $\mathcal{S}$  occurs over the point  $(|a|^2 - |b|^2 = 0, |z| = 1)$  and intersects transversally at the point  $(0, 0, -1)$ .

Following the construction in Section 2.3, we construct the Borel model for  $\mathcal{S}$  and its finite dimensional approximations. In this case, the clean intersection  $\mathcal{I} \subset \mathcal{S}$  is simply a point and invariant under the  $\mathbb{S}^1$ -action. Let  $T = \mathbb{S}^1$  and  $\tilde{L} = \mathbb{S}^2$ . We have Lagrangian immersions

$$\tilde{L}_N := \tilde{L} \times_{\mathbb{S}^1} EG(N) \rightarrow X_N \tag{4.2}$$

cleanly intersecting at  $\mathcal{I}_N = BT(N) = \mathbb{C}\mathbb{P}^N$  for each  $N \in \mathbb{N}$ . Each finite dimensional approximation can be understood by toric geometry topologically. For instance, the domain of the lowest dimensional approximation (4.2) is regarded as the Hirzebruch surface  $\mathbb{F}_1 \rightarrow \mathbb{C}\mathbb{P}^1$  whose fiber is  $\tilde{L}$ , see Figure 6 (b).

Let  $f_T: \tilde{L}_T \rightarrow \mathbb{R}$  be the sum of a horizontal lift of the perfect Morse function  $\varphi: BT \rightarrow \mathbb{R}$  (see Section 3.1) and function on  $\tilde{L}_T \rightarrow BT$  which restricts to a perfect Morse function  $\phi: \mathbb{S}^2 \rightarrow \mathbb{R}$  on generic fibers. It has critical points  $\mathbf{1}_{\mathbb{S}^2} \otimes \lambda^\ell$  and  $\text{pt}_{\mathbb{S}^2} \otimes \lambda^\ell$  for  $\ell \geq 0$ .

Recall that  $L$  is a special Lagrangian with respect to  $\Omega_X$  in  $X$ . Let  $\Omega_{BG(N)}$  be a holomorphic volume form on  $T^*BG(N)$  with respect to which the zero section is a special Lagrangian. We take a positive linear combination  $\Omega_N$  of  $\Omega_X$  and  $\Omega_{BG(N)}$  as a holomorphic volume form of the  $X$ -bundle over  $T^*BG(N)$ .

**Lemma 4.2.** *The Lagrangian  $\tilde{L}_N$  in (4.2) is special with respect to  $\Omega_N$ . In particular,  $\tilde{L}_N$  is  $\mathbb{Z}$ -graded.*

Following Section 2.5, we obtain an  $A_\infty$ -algebra  $(CF_T^\bullet(L, \Lambda_0)^\dagger, m^{T, \dagger})$  associated to  $(L, T)$ . Note that there are immersed generators  $U \otimes \lambda_T^\ell, V \otimes \lambda_T^\ell$  of  $\tilde{L}_N$  where  $\lambda_T^\ell$  for  $\ell \geq 0$  labels critical points of the Morse function on  $\mathcal{I}_T = \mathbb{C}\mathbb{P}^\infty$ . By abuse of notations, degree one immersed generators of  $L_T$  will be simply denoted by  $U := U \otimes \mathbf{1}_{\mathcal{I}_T}, V := V \otimes \mathbf{1}_{\mathcal{I}_T}$ . Then we take the formal deformations  $b = uU + vV$  of  $\mathcal{S}_T$  for  $(u, v) \in (\Lambda_+ \times \Lambda_0) \cup (\Lambda_0 \times \Lambda_+)$ .

**Lemma 4.3.** *The formally deformed immersed Lagrangian  $(\mathcal{S}_T, b = uU + vV)$  is weakly unobstructed.*

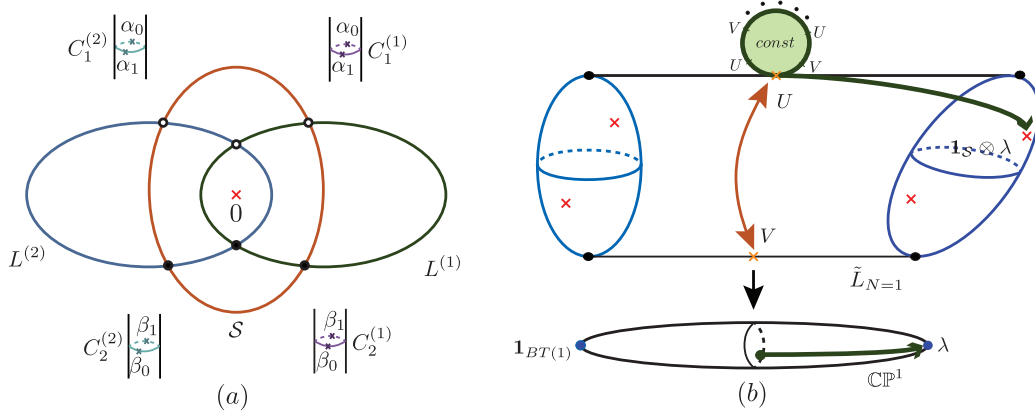


FIGURE 6. Comparison of three Lagrangians and constant disc contribution.

*Proof.* Since  $\mathcal{S}_T$  has minimal Maslov index  $\geq 2$ ,  $m_0^{\mathcal{S}_T, b}$  can be expressed as

$$m_0^{\mathcal{S}_T, b}(1) = m_0^{\mathcal{S}, b}(1) \otimes \mathbf{1}_{BT} + W(u, v) \cdot \mathbf{1}_{\mathcal{S}} \otimes \lambda$$

by the same argument in Proposition 3.3. Unlike the smooth case, extra care for  $m_0^{\mathcal{S}, b}$  is necessary because constant discs might bubble off at the immersed loci. Consider the anti-symplectic involution  $\sigma: X_T \rightarrow X_T$  given by  $(a, b, z) \mapsto (\bar{b}, \bar{a}, \bar{z})$ , which acts on  $X_N$  fiberwise and swaps two immersed generators  $U$  and  $V$ . In [HKL], by taking orientations respecting the involution, the constant disc contributions are canceled out. Thus  $(\mathcal{S}, b = uU + vV)$  is unobstructed, that is,  $m_0^{\mathcal{S}, b}(1) = 0$ . Therefore,

$$m_0^{\mathcal{S}_T, b}(1) = W(u, v) \cdot \mathbf{1}_{\mathcal{S}} \otimes \lambda.$$

□

More precisely, the  $\mathbf{1}_{\mathcal{S}} \otimes \lambda$  appeared above is  $\lambda^\vee$  in the notation of Section 2.4. By Lemma 2.16, the above  $b$  can be modified (by adding terms involving  $\mathbf{1}^\vee$ ) to  $b'$  such that  $m_0^{\mathcal{S}_T, b'}(1) = W^\vee(u, v) \cdot \lambda^\vee$ . By abuse of notations, we shall simply replace  $b$  by  $b'$  and denote it by  $b$  again,  $\lambda^\vee$  by  $\lambda$ ,  $W^\vee$  by  $W$ , and  $\mathbf{1}^\vee$  by  $\mathbf{1}$ .

**Remark 4.4.** *This line of thoughts is similar to the work in [HKL]. In this example by the maximal principle we do not have non-constant holomorphic discs. However, there are still very interesting non-trivial contributions coming from constant polygons with corners at  $U, V$ . Note that for each constant polygon,  $U$  and  $V$  corners must occur in pair (to go back to the original branch). This is the reason that the Floer theory is convergent for  $\{(u, v) \in \Lambda_0^2 : \text{val}(uv) > 0\}$ .*

In order to compute  $W(u, v)$ , we compare  $\mathcal{S}$  with the  $T$ -equivariant theory of Chekanov and Clifford tori  $L^{(1)}$  and  $L^{(2)}$  respectively. For the special Lagrangian torus fibration (4.1), the Lagrangian torus over a point in the chamber  $|z| < 1$  (resp.  $|z| > 1$ ) is called *Chekanov* (resp. *Clifford*) type. Since any pair of distinct fibers in (4.1) does *not* intersect, they cannot be isomorphic. We apply Lagrangian isotopies to Chekanov torus and Clifford torus without intersecting the wall  $z = 0$  to make them intersect as in Figure 6 (a). Those isotoped Lagrangians are still  $\mathbb{Z}$ -graded and called Chekanov torus and Clifford torus respectively.

The equivariant theory for the tori  $L^{(i)}$  is understood in the same way as in the toric case above, except that in this local case  $L^{(i)}$ 's do not bound any non-constant holomorphic disc.

We equip  $L^{(i)}$  with the *non-trivial* spin structure along the  $T$ -orbit direction. This will give extra systematic contributions to the orientation bundle of the moduli space of strips bounded by  $L^{(i)}$  and other Lagrangians. The reason for doing this will be seen below.

Fix a perfect Morse function on each  $L^{(i)}$ , such that the (compactified) unstable submanifolds of the degree one generators  $X^{(i)}, Y^{(i)}$  are the hypertori chosen in [HKL]. The unstable submanifold of  $X^{(i)}$  is transverse to the  $T = S^1$  orbits in  $L^{(i)}$ . We also have a perfect family Morse theory for  $L_T^{(i)}$  as in Section 3, whose critical points are labeled by  $\eta \otimes (\lambda^{(i)})^\ell$  where  $\eta = \mathbf{1}^{(i)}, X^{(i)}, Y^{(i)}, X^{(i)} \wedge Y^{(i)}$ . The formal deformations are taken as  $b^{(i)} = x^{(i)}X^{(i)} + y^{(i)}Y^{(i)}$  (where  $X^{(i)}, Y^{(i)}$  denotes  $X^{(i)} \otimes \mathbf{1}^{(i)}, Y^{(i)} \otimes \mathbf{1}^{(i)}$ ). By Proposition 3.3, the equivariant disc potential equals to

$$W^{(i)} = x^{(i)}\lambda^{(i)}.$$

Since  $L^{(i)}$  does not bound any non-constant Maslov-zero disc,  $(W^{(i)})^\nabla = W^{(i)}$  and again we will denote  $(\lambda^{(i)})^\nabla$  by  $\lambda^{(i)}$ .

**Lemma 4.5.** *The three formally deformed Lagrangians  $(L^{(1)}, b^{(1)})$ ,  $(L^{(2)}, b^{(2)})$ , and  $(\mathcal{S}, b)$  with a non-standard spin structure along the  $T$ -orbit direction are isomorphic if*

$$\begin{cases} uv = 1 - \exp x^{(i)}, & \text{for } i = 1, 2 \\ u = \exp y^{(1)}, \\ v = \exp(-y^{(2)}). \end{cases} \quad (4.3)$$

*Proof.*  $L^{(i)}$  intersects with  $\mathcal{S}$  at two circles  $C_j^{(i)}$  for  $j = 1, 2$ , which are invariant under the  $T$ -action (which is free on  $C_j^{(i)}$ ). Thus we have the equivariant spaces  $(C_j^{(i)})_T$ , which are the clean intersections (denoted by  $\alpha$  and  $\beta$ ) between  $L_T^{(i)}$  and  $\mathcal{S}_T$  in  $X_T$ . We have the morphisms  $\alpha_k \otimes \lambda_{C_1^{(i)}}^\ell, \bar{\beta}_k \otimes \lambda_{C_2^{(i)}}^\ell$  for  $k = 0, 1$  from  $L^{(i)}$  to  $\mathcal{S}$ , and  $\beta_k \otimes \lambda_{C_2^{(i)}}^\ell, \bar{\alpha}_k \otimes \lambda_{C_1^{(i)}}^\ell$  for  $k = 0, 1$  from  $\mathcal{S}$  to  $L^{(i)}$  where  $\alpha_\bullet$ 's and  $\beta_\bullet$ 's are depicted in Figure 6 (a) and the subscripts denote the degrees. They are given by critical points of a perfect family Morse function on  $(C_j^{(i)})_T \rightarrow BT$ . Denote  $\alpha_0 \otimes \mathbf{1}_{C_1^{(i)}}, \beta_0 \otimes \mathbf{1}_{C_2^{(i)}}$  simply by  $\alpha_0, \beta_0$  which has degree zero. Then

$$(\mathbf{m}_1^T)^{(b^{(i)}, b)}(\alpha_0) = \mathbf{m}_1^{(b^{(i)}, b)}(\alpha_0), (\mathbf{m}_1^T)^{(b, b^{(i)})}(\beta_0) = \mathbf{m}_1^{(b, b^{(i)})}(\beta_0), (\mathbf{m}_2^T)^{(b^{(i)}, b, b^{(i)})}(\beta_0, \alpha_0) = \mathbf{m}_2^{(b^{(i)}, b, b^{(i)})}(\beta_0, \alpha_0)$$

simply by degree reason (namely  $\lambda$  has degree two and hence it does not appear in the above equations).

This implies that we have exactly the same gluing formula for the equivariant theories (see [HKL, Theorem 3.7] for the proof of the gluing formula in nonequivariant theory):

$$uv = 1 - \exp x^{(i)}, u = \exp y^{(1)}, v = \exp(-y^{(2)}).$$

(The minus sign in front of  $\exp x^{(i)}$  is exactly due to the non-trivial spin structure we fix for  $L^i$ .) Under these relations,  $(\mathbf{m}_1^T)^{(b^{(i)}, b)}(\alpha_0) = 0, (\mathbf{m}_1^T)^{(b, b^{(i)})}(\beta_0) = 0, (\mathbf{m}_2^T)^{(b^{(i)}, b, b^{(i)})}(\beta_0, \alpha_0) = \mathbf{1}_{L^i} \otimes \mathbf{1}_{BT}$ .  $\square$

**Theorem 4.6.** *The equivariant potential function of the immersed sphere  $S^2$  is*

$$W = \lambda \log(1 - uv) = -\lambda \sum_{j=1}^{\infty} \frac{(uv)^j}{j}.$$

*Proof.* Consider the  $A_{\infty}$  equation

$$((\mathfrak{m}_1^T)^{b^{(1)},b})^2(\alpha_0) = \pm (\mathfrak{m}_2^T)^{b,b^{(1)},b^{(1)}}((\mathfrak{m}_0^T)^{L^{(1)},b^{(1)}}(\alpha_0) \pm (\mathfrak{m}_2^T)^{b,b,b^{(1)}}(\alpha_0, (\mathfrak{m}_0^T)^{S,b}).$$

(4.3) makes (LHS) zero. Hence

$$x^{(1)} \cdot (\mathfrak{m}_2^T)^{b,b^{(1)},b^{(1)}}(\lambda^{(1)}, \alpha_0) \pm W(u, v) \cdot (\mathfrak{m}_2^T)^{b,b,b^{(1)}}(\alpha_0, \lambda) = 0.$$

On the other hand, we have

$$(\mathfrak{m}_2^T)^{b,b^{(1)},b^{(1)}}(\lambda^{(1)}, \alpha_0) = (\mathfrak{m}_2^T)^{b,b,b^{(1)}}(\alpha_0, \lambda) = \alpha_0 \otimes \lambda_{C_1^1}$$

(recall that  $\lambda$  and  $\lambda^{(i)}$  are partial units). Hence we have

$$W(u, v) = \pm \lambda x^{(1)}.$$

By  $\exp x^{(1)} = 1 - uv$ , we have

$$W = \lambda \log(1 - uv) = -\lambda \sum_{j=1}^{\infty} \frac{(uv)^j}{j}.$$

□

**Remark 4.7.** *Note that if we did not take the non-trivial spin structure, then the gluing formula reads  $\exp x^{(1)} = -1 + uv$ . However, the left hand side lies in  $1 + \Lambda_+$ , while the right hand side lies in  $-1 + \Lambda_+$ , and hence there is no common intersection. It means the formal deformations of  $L_T^{(1)}$  in the trivial spin structure cannot be isomorphic with the formal deformations of  $\mathcal{S}_T$ . This is why we take the torus  $L^{(i)}$  equipped with the non-trivial spin structure in the very beginning.*

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