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The Sylow theorems and their generalizations

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THE SYLOW THEOREMS AND THEIR GENERALIZATIONS

by

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Introduction

The Sylow theorems and their generalizations are concerned with the following questions: Under what conditions will a group G possess a subgroup of a given order? When will subgroups of the same order be conjugate? If G possesses a subgroup S of a given order h, will all subgroups whose orders divide h be contained in conjugates of S? How many subgroups of G will have order h? The Sylow theorems answer these questions completely if the order being considered is a power of a prime p and is the highest power of p which divides the order of G. They were first proved by Sylow in 1872.* In the case of solvable groups, P. Hall [3] generalized Sylow's theorems to any factor h of the order n of G which is relatively prime to n/h. He later proved [4] that if a group G of order n possesses, for each prime dividing n, a subgroup whose index is a power of p and whose order is prime to p, then G is solvable. Wielandt [6], in 1954, showed that if a group G possesses a nilpotent subgroup N whose order h is prime to its index, then all subgroups of order h are conjugate to N, and any subgroup whose order divides h is contained in some conjugate of N. Using Wielandt's results, P.Hall [5], proved yet more general theorems, concerning groups which are not necessarily

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* "Theorems sur les Group de Substitutions," Math. Annalen (1872), page 584.

solvable. Wielandt [7], then found these results useful in proving theorems concerning the normalizer of a subnormal subgroup.

In section 1, we will define the concepts of p-group and Sylow p-subgroup and state Sylow's theorems. We will then discuss the generalization of p-group to π-group and Sylow p-subgroup to Hall π-subgroup, where π is an arbitrary set of primes. Using these definitions, Hall's generalizations of Sylow's theorems for solvable groups will be stated in section 2, and some examples of groups which do not satisfy these theorems will be discussed. In section 3, we will consider the problem of generalizing the theorems to groups which are not necessarily solvable. Two generalizations of solvability, π-separability and π-solvability, will be discussed in section 4. Finally, in section 5, we will consider Wielandt's application of Hall's results in his paper on subnormal subgroups.

1. p-Groups and π-Groups

A group is said to be a <u>p-group</u> if the order of each of its elements is a power of the prime p. A p-subgroup of a group is a <u>Sylow p-subgroup</u> if it is contained in no larger p-subgroup. The identity is clearly a p-subgroup for any prime p since $p^0 = 1$ is the order of the identity. In some infinite groups such as the additive group of the integers, the identity is the only Sylow p-subgroup. We will, however, restrict ourselves to finite groups, in which case Sylow's three theorems are valid. These are:

Theorem S-1. If G is of order $n = p^m s$ where $(p,s) = 1$, p a prime, then G contains subgroups of orders p^i, $i = 1, \ldots, m$, and each subgroup of order p^i, $i = 1, 2, \ldots, m-1$, is a normal subgroup of at least one subgroup of order p^{i+1}.

Theorem S-2. In a finite group G, the Sylow p-subgroups are conjugate.

Theorem S-3. The number of Sylow p-subgroups of a finite group G is of the form $1 + kp$ and is a divisor of the order of G.

The problem now to be considered concerns the extension of the Sylow theorems and definitions to an arbitrary set of primes. Let π be a nonempty set of primes and let π' be the complementary set consisting of all primes not in π. Every positive integer m can be expressed as

$$m = m_{\pi}m_{\pi'},$$

where m_{π} is the largest divisor of m which has no divisors in π'. Denote the order of G by $|G|$. Using this notation, P. Hall [5] defines a group G to be a <u>π-group</u> if

$$|G|_{\pi} = |G|.$$

A subgroup H of a group G is defined to be a <u>Hall</u> <u>π-subgroup</u> if

$$|H| = |H|_{\pi} = |G|_{\pi}.$$

We will denote a Hall π-subgroup of a group G by G_{π}. Equivalent to Hall's definition of π-group is the following. A finite group G is a <u>π-group</u> if and only if all the prime divisors of the order of each of its elements belong to π. This follows immediately by observing that if h is an element of a group G then the order of h divides $|G|$; and if p is a prime dividing $|G|$, then G has an element of order p. Using this definition we see that a Sylow p-subgroup of a finite group G is a G_p. Also if $|G|_{\pi} = p^a$, for p a prime, then G possesses a G_{π}. If G has a G_{π}, H, then H is clearly contained in no larger π-subgroup. However, not all groups have Hall π-subgroups. Consider the simple group A_5 of order 60.

$$|A_5|_{3,5} = 60_{3,5} = 15.$$

This group has no subgroup of order 15, and hence no Hall 3,5-subgroup.

Two useful properties of Hall groups are the following: If H is a G_π of G and A is any normal subgroup of G, then $H \cap A$ is an A_π and HA/A is a $(G/A)_\pi$. The straightforward proof of these properties is contained in a paper by P. Hall [5].

2. Solvable Groups

Before discussing the conditions under which a group will possess a Hall π-subgroup, it will be useful to define the concepts of composition series, chief series, derived groups, and solvable groups, and to state some results concerning them.* When A is a normal subgroup of G we will write $A \lhd G$. A series of subgroups

$$A = A_k \lhd A_{k-1} \cdots \lhd A_1 \lhd A_0 = G$$

is called a *subnormal series* from G to A. Each of the A_i's is said to be a *subnormal subgroup* of G. When A is a subnormal subgroup of G we will write $A \lhd\lhd G$. If we have $A_i \lhd G$ for each i, then the series is called a *normal series*. A subnormal series from G to A in which each A_i is maximal in A_{i-1} is said to be a *composition series* from G to A. The factor groups A_i/A_{i+1} in a composition series are called *composition factor groups* and their orders are called *composition factors*. A normal series in which each A_i is maximal in A_{i-1} is called a *chief series*. The definitions of *chief factor groups* and *chief factors* are analogous to those for composition factor groups and factors. In a finite group any subnormal series can be extended to a composition series and any normal series can be extended to a chief series.

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* The proofs of these results can be found in *The Theory of Finite Groups* by Marshall Hall, Chapters 8 and 9.[2]

The subgroup G' generated by all commutators $x^{-1}y^{-1}xy$ of a group G is called the derived group of G. A group G is defined to be solvable if the sequence

$$G \supseteq G' \supseteq \cdots \supseteq G^{(i)} \supseteq G^{(i+1)},$$

where each $G^{(i+1)}$ is the derived group of $G^{(i)}$, terminates in the identity in a finite number of steps. Since the derived group is clearly a normal subgroup, a non-abelian simple group such as A_5 cannot be solvable. In a finite group G, the condition that the factor groups in a composition series from G to 1 are cyclic of prime order is equivalent to solvability. From this and Theorem S-1, we see that every p-group is solvable. In a chief series for a solvable finite group G the chief factor groups are elementary Abelian groups; that is they are the direct product of cyclic groups of prime order. Factor groups and subgroups of solvable groups are solvable.

P. Hall [3] has shown that for any solvable group, Sylow's theorems can be generalized in terms of Hall subgroups. Hall's theorems are:

If G is a solvable group of order

$$m = m_\pi m_{\pi'}$$

then the following four statements are true.

Theorem H-1. G possesses at least one G_π (a Hall π-subgroup).

Theorem H-2. Any two G_π's are conjugate.

Theorem H-3. Every π-subgroup of G is contained in some G_π.

Theorem H-4. The number h_{m_π} of G_π's may be expressed as a product of factors, each of which (a) is congruent to 1 modulo some element of π and (b) is a power of a prime and divides one of the chief factors of G.

The following corollaries illustrate the usefulness of Hall's theorems in studying subgroups of solvable groups. Throughout this paragraph, G will be a solvable group with

$$|G| = m_\pi m_{\pi'}.$$

Corollary H-1. If every element of a G_π, S, commutes with every element whose order is relatively prime to m_π, then G is the direct product of a group of order m_π with one of order $m_{\pi'}$.

Proof: By Theorem H-1, G contains at least one $G_{\pi'}$. Let S' be such a subgroup. Since every element of S' commutes with every element of S, we have

(*) $\quad S' \cup S = S'S \subseteq$ the normalizer of $S \subseteq G$.

We know that

$$|S'| = m_{\pi'} \text{ and } |S| = m_\pi$$

which implies

$$|S \cap S'| = 1.$$

Consequently

$$|S' \cup S| = |S'S| = |S|\,|S'| \,/\, |S \cap S'| = |S|\,|S'| = m_\pi m_{\pi'} = |G|.$$

Thus

$$G = S' \cup S.$$

We then see by (*) that S is normal in G. Similarly S' is normal in G. Therefore

$$G \approx S \times S'$$

which proves the corollary.

Corollary H-2,3. No element whose order divides m_π is permutable with a G_π which does not contain it; the normalizer of a G_π is its own normalizer and contains no other G_π.

Proof: Let S be a G_π, N its normalizer and x an element of N whose order divides m_π. N is a subgroup of a solvable group and hence solvable. Applying Theorem H-2 to N, we see that S is conjugate to any other N_π and being normal in N it must be the only N_π in N. Since

$$|N|_\pi = |G|_\pi ,$$

we see that any G_π contained in N is an N_π and must, therefore, be S.

The cyclic group $\{x\}$ is a π-group and is contained in N. From Theorem H-3, it must be contained in S. In particular, x itself must be contained in S.

Let y be an element in the normalizer of N, then $y^{-1}Sy$ is contained in N; but $y^{-1}Sy$ is a G_π, hence

$$y^{-1}Sy = S.$$

This implies that N is its own normalizer.

Corollary H-4. If the solvable group G is of order $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where $p_1 > p_2 > \cdots > p_r$ are primes, and if the chief factors of G are all primes, then G has self conjugate subgroups of orders $p_1^{a_1}$, $p_1^{a_1}p_2^{a_2}$, $\cdots$ $p_1^{a_1} \cdots p_{r-1}^{a_{r-1}}$.

Proof: A subgroup of order $p_1^{a_1}p_2^{a_2} \cdots p_j^{a_j}$ must be a $G_{p_1,p_2,\cdots p_j}$. By Theorem H-1, there is at least one such subgroup, H. The number of conjugates, h, of H is the index of the normalizer of H in G. This implies that h divides $p_{j+1}^{a_{j+1}} \cdots p_r^{a_r}$. By Theorem H-4 and the fact that the chief factors of G are all primes, h is a product of terms of the form $1 + kp_i$ $(i \le j)$, where

$$1 + kp_i = p_t \quad (t > j).$$

However

$$p_t < p_j \le p_i.$$

Hence k = 0 and h = 1. Thus H is a normal subgroup of G.

Given p any prime, we call a Hall p'-subgroup of a group G a <u>p-complement</u> of G. P. Hall [4] proved that a finite group is solvable if it possesses a p-complement for every prime p. Trivially every group G possesses a p-complement for any prime p which does not divide $|G|$. By Theorem H-1, if G is solvable, since

$$|G| = |G|_p |G|_{p'},$$

it must possess a $G_{p'}$ for any prime p. Consequently we can state

Theorem H-5. A finite group is solvable if and only if it possesses a p-complement for every prime p.

A_5 is clearly insolvable since it possesses no 2-complement.

A_5 also demonstrates that Theorems H-3 and H-4 are not necessarily valid for insolvable groups.* A_5 has Hall 3,2-subgroups of order 12, which are generated by elements A_1, A_2, and B with the following properties:

$B^3 = 1$, $B^{-1}A_1B = A_2$, $B^{-1}A_2B = A_1A_2$, $A_1{}^2 = A_2{}^2 = 1$, $A_1A_2 = A_2A_1$.

Each of these subgroups has only one subgroup of order 4. Hence, A_1, A_2, and A_1A_2 are the only elements of order 2 in $\{A_1, A_2, B\}$. The subgroup generated by (123), (12)(45) is a 2,3-subgroup of order 6. However,

$$(123)(12)(45)(132) = (13)(45).$$

The element (13)(45) is of order 2, but does not commute with (12)(45). Thus $\{(123), (12)(45)\}$ is contained in no Hall 3,2-subgroup and Theorem H-3 is invalid in A_5. There are six Hall 5-subgroups in A_5. The prime factors of six are not congruent to 1 modulo 5. Further, although

$$6 = 1 \text{ modulo } 5,$$

it is not the power of a prime. Thus Theorem H-4 is invalid for A_5.

Theorem H-2 is invalid for the group G of automorphisms of the elementary Abelian group A of order 8. G is a simple group of order 168. It permutes both the seven subgroups of order 2 and the seven subgroups of order 4 transitively. A subgroup F_2 of G which leaves one of the subgroups of A of order 2 fixed cannot be conjugate to a subgroup F_4 of G

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* Burnside [1] discusses A_5 completely in his <u>Theory of Finite Groups</u>, Section 127.

which leaves one of the subgroups of A of order 4 fixed. However, both F_2 and F_4 have index 7 and are Hall 2,3-subgroups.

3. E, C, and D Properties of Groups

We will now investigate conditions under which Hall's first three theorems will be valid for a particular set of primes π in a group which is not necessarily solvable. For ease in stating theorems, we will refer to the following three propositions concerning a finite group G.

E_π: G has at least one G_π.

C_π: G satisfies E_π and any two G_π's are conjugate in G.

D_π: G satisfies C_π and every π-subgroup of G is contained in some G_π.

Sylow's theorems imply

Theorem D-1. If p is a prime, then every finite group satisfies D_p.

Hall's theorems H-1, H-2, H-3, and H-5 can now be stated as a single theorem.

Theorem D-2. G is solvable if and only if G satisfies $D_{p'}$ for all primes p.

The ascending central series of a group G is defined to be the series of subgroups

$$1 = A_0 \subseteq A_1 \subseteq A_2 \cdots$$

where; A_1 is the center of G and, assuming A_i has already been defined, A_{i+1} is defined so that A_{i+1}/A_i is the center of G/A_i. A group is said to be nilpotent if the ascending central series terminates with G. Since the center of any group is a normal abelian subgroup, a nilpotent group is clearly solvable. It can also be shown that subgroups and factor groups of nilpotent groups are nilpotent.* We will now consider the following propositions

E_π^N: G possesses a G_π which is nilpotent.

E_π^S: G possesses a G_π which is solvable.

C_π^S: G satisfies C_π and its G_π's are solvable.

D_π^S: G satisfies D_π and its π-subgroups are solvable.

Wielandt [6] has proved that if the finite group G contains a nilpotent subgroup H, whose order h is prime to its index, and if M is any subgroup whose order divides h, then there exists an element g such that M is contained in $g^{-1}Hg$. Considering that subgroups and conjugates of nilpotent groups are nilpotent, this theorem can be more simply stated as

Theorem D-3. G satisfies E_π^N implies that G satisfies D_π and its π-subgroups are nilpotent.

G satisfies E_π^S is not a sufficient condition for G to satisfy D_π. Consider the simple group of order 168. Its Hall 2,3-subgroups, which we looked at earlier, are solvable, but the group does not satisfy $C_{2,3}$.

The following lemma is useful when working with groups which can be written as direct products of simpler groups.

Lemma 3.1. Let $H = H_1 \times H_2 \times \cdots H_r$. If each H_i satisfies a given one of E_π, C_π, D_π, E_π^N, C_π^S, D_π^S, E_π^S, then so does H.

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* The proof of this statement can be found in The Theory of Finite Groups by Marshall Hall, page 153. [2]

Proof:

(i) Assume each H_i possesses a H_{i_π}, K_i. Then

$$|K_1 \times K_2 \times \cdots \times K_r| = |K_1|\,|K_2| \cdots |K_r| = |H_1|_\pi\,|H_2|_\pi \cdots |H_r|_\pi$$
$$= (|H_1|\,|H_2| \cdots |H_r|)_\pi = |H|_\pi.$$

Thus H satisfies E_π.

(ii) Assume each H_i satisfies C_π. This implies, by (i) that H possesses at least one H_π. Let R and M be any two H_π's. Then we can write

$$M = M_1 \times M_2 \times \cdots \times M_r$$

and

$$R = R_1 \times R_2 \times \cdots \times R_r,$$

where M_i and R_i are subgroups of H_i. Each M_i is clearly a π-subgroup of H, since

$$|M| = |M_1|\,|M_2| \cdots |M_i| \cdots |M_r| = |H|_\pi.$$

If M_i were not an H_{i_π}, we could find a π-subgroup of H_i, K_i, such that $|K_i|$ would be greater that $|M_i|$. In which case the subgroup

$$M' = M_1 \times M_2 \times \cdots \times K_i \times \cdots \times M_r$$

would be a π-subgroup with order greater that $|H|_\pi$, which is clearly impossible. Similarly each R_i is a H_{i_π}. For each i, we can find an element of H_i, x_i, such that

$$x_i^{-1}M_ix_i = R_i.$$

Consider the element of H,

$$x = (x_1, x_2, \cdots x_r).$$

$$x^{-1}Mx = (x_1^{-1}, \cdots, x_r^{-1})(M_1 \times \cdots \times M_r)(x_1, \cdots, x_r)$$
$$= (x_1^{-1}M_1x_1 \times \cdots \times x_r^{-1}M_rx_r) = (R_1 \times R_2 \times \cdots \times R_r).$$

Thus H satisfies C_π.

(iii) Assume each H_i satisfies D_π. Let

$$Y = (Y_1 \times Y_2 \times \cdots \times Y_r)$$

be a π-subgroup of H. Then each Y_i is a π-subgroup of H_i, which implies there is an $H_{i\pi}$, M_i, such that Y_i is contained in M_i.

$$Y = (Y_1 \times \cdots \times Y_r) \subseteq (M_1 \times \cdots \times M_r).$$

By the proof of (i), $M_1 \times M_2 \times \cdots \times M_r$ is an H_π. Therefore H satisfies D_π.

(iv) It can easily be shown that

$$H' = H_1' \times H_2' \times \cdots \times H_r',$$

where H' is the derived group of H and H_i' is the derived group of H_i for each i. This implies that H is solvable if and only if H_i is solvable for each i. Therefore, by (i), (ii), and (iii), if each H_i satisfies a given one of E_π^S, C_π^S, D_π^S, so does H.

(v) If N is a normal subgroup of H,

$$N = N_1 \times \cdots \times N_r,$$

then

$$H/N \approx H_1/N_1 \times H_2/N_2 \times \cdots \times H_r/N_r.$$

If z is in the center of H,

$$z = (z_1, z_2, \cdots, z_r),$$

then z_i is in the center of H_i for each i and conversely. Therefore, H is nilpotent if and only if H_i is nilpotent

for each i. Combining this last statement with (i) completes the proof of the lemma.

An immediate consequence of this lemma is that a group will satisfy both D_π and $D_{\pi'}$ if it is the direct product of π and π' groups. By a well known theorem concerning composition series, if G is a finite group, then each of its chief factor groups is the direct product of isomorphic simple groups.*
In particular if H is a minimal normal subgroup of a finite group G, then there is some chief series from 1 to G containing H,

$$1 \lhd H \lhd H_n \cdots \lhd H_0 = G.$$

Consequently $H/1 = H$ is the direct product of simple groups,

$$H = A_1 \times \cdots \times A_m.$$

Each of the A_i's is subnormal in G, since it is normal in H. Being simple, they must be minimal subnormal subgroups of G, which implies they are composition factor groups of G. From the above, we see that H is the direct product of its composition factor groups. Combining this with Lemma 3.1, we have

Lemma 3.2. If all the composition factor groups of G satisfy a given one of E_π, C_π, D_π, E_π^S, C_π^S, D_π^S, E_π^N, then so does every minimal normal subgroup of G.

A sufficient condition for G to satisfy E_π is that G possess a normal $G_{\pi'}$.** An immediate consequence of this is

Theorem E-1.*** If K is a normal subgroup of G such that K satisfies C_π and G/K satisfies E_π, then G satisfies E_π.

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* Theorem 8.6.1, The Theory of Finite Groups, Marshall Hall [2].
** Theorem 25, Chapter 4, The Theory of Groups, Zassenhaus [8].
*** The proof of this can be found in a paper by P. Hall [5].

Proceeding by induction on the order of G and using Lemma 3.2, we have as a corollary

Corollary E-1.1. If all the composition factor groups of G satisfy C_{π}, then G satisfies E_{π}.

Another useful result concerning composition series is the following.

Lemma 3.3. If there is a subnormal series from G to 1,

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_r = 1,$$

so that each G_i/G_{i+1} satisfies E_{π}^{N}, then every composition factor group of G satisfies E_{π}^{N}.

Proof: Since G is finite, we can extend the subnormal series to a composition series by inserting a finite number of subnormal subgroups between G_i and G_{i+1} for each i. We will then have for each i,

$$G_i = G_{i,0} \triangleright G_{i,1} \triangleright \cdots \triangleright G_{i,n} = G_{i+1}.$$

Let $G_{i,m}$ be any subgroup from the composition series we have constructed. Let H/G_{i+1} be a nilpotent Hall π-subgroup of G_i/G_{i+1}. Since $G_{i,m}$ is subnormal in G_i, $(G_{i,m} \cap H)/G_{i+1}$ is a nilpotent Hall π-subgroup of $G_{i,m}/G_{i+1}$. From this and the fact that $G_{i,m+1}/G_{i+1}$ is normal in $G_{i,m}/G_{i+1}$, it follows that

$$(G_{i,m+1}/G_{i+1})(G_{i,m} \cap H/G_{i+1})/(G_{i,m+1}/G_{i+1}),$$

which we will denote by F, is a π-Hall subgroup of

$$(G_{i,m}/G_{i+1})/(G_{i,m+1}/G_{i+1}).$$

Now

$$F \approx (G_{i,m} \cap H/G_{i+1})/(G_{i,m+1} \cap H/G_{i+1})$$

and $(G_{i,m} \cap H/G_{i+1})/(G_{i,m+1} \cap H/G_{i+1})$, being a factor group of a nilpotent group, is nilpotent. Hence

$$(G_{i,m}/G_{i+1})/(G_{i,m+1}/G_{i+1})$$

satisfies E_{π}^{N}. Also

$$(G_{i,m}/G_{i+1})/(G_{i,m+1}/G_{i+1}) \approx G_{i,m}/G_{i,m+1}.$$

By the Jordan Holder theorem, given any composition factor group C of G, there is some i and m such that

$$C \approx G_{i,m}/G_{i,m+1}.$$

Therefore every composition factor group of G satisfies E_{π}^{N}. The principle theorem of P. Hall's paper, "Theorems Like Sylow's" [5] is stated below.

Theorem D-4. If K is a normal subgroup of G such that K satisfies E_{π}^{N} and G/K satisfies D_{π}^{S}, then G satisfies D_{π}^{S}. Using this theorem, Lemmas 3.2 and 3.3, and proceeding by induction on the order of G, we have

Corollary D-4.1. If G has a subnormal series

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_r = 1$$

such that each of the factor groups G_i/G_{i+1} satisfies E_{π}^{N}, then G satisfies D_{π}^{S}.

4. π-Separable and π-Solvable Groups

We found in section 2 that for π any set of primes, all solvable groups satisfy D_π^S. For π a particular set of primes, we can define a property of groups which is similar to solvability in terms of D_π. We say that a finite group is <u>π-separable</u> if its composition factor groups are π',p-groups for various primes p in π. By Theorem S-1, all the composition factor groups of a π-separable group satisfy E_π^N. We then have, as a special case of Corollary D-4.1,

Corollary D-4.2. All π-separable groups satisfy $D_{\pi_1}^S$ for any subset π_1 of π.

The following theorem is analogous to a theorem concerning solvable groups.

Theorem 4.1. A group G is π-separable, if it has a normal subgroup H such that both H and G/H are π-separable.

Proof: Let

$$G/H = A_0/H \triangleright \cdots \triangleright A_{r-1}/H \triangleright A_r/H = H/H$$

and

$$H = B_0 \triangleright B_1 \triangleright \cdots \triangleright B_{s-1} \triangleright B_s = 1$$

be composition series for G/H and H respectively. Then by the hypothesis of the theorem, $(A_i/H)/(A_{i+1}/H)$ and and B_i/B_{i+1} are π',p-subgroups. Also

$$(A_i/H)/(A_{i+1}/H) \approx A_i/A_{i+1}.$$

Hence the composition series

$$G = A_0 \triangleright A_1 \triangleright \cdots \triangleright H \triangleright B_1 \triangleright \cdots \triangleright B_s = 1$$

satisfies the conditions for G to be π-separable.

The following theorem is similar to Theorem H-5. It gives a sufficient condition for a group to be π-separable.

Theorem 4.2. If G satisfies E_π and $E_{p'}$ for all primes p in π, then G is π-separable.

Before proving the theorem, it will be useful to prove the following lemma.

Lemma 4.2.1. If $|G| = m_1 m_2 \cdots m_n$, $(m_i, m_j) = 1$, $i \neq j$; and G satisfies $E_{\pi_i'}$, $i = 2, 3, \cdots, n$, where π_i denotes the set of all primes dividing m_i; then G satisfies E_{π_1}.

Proof: This is trivial in the cases $n = 1$ and $n = 2$. Consider the case of $n \geq 3$. For $i = 2, 3, \cdots, n$, let S_i be one of the $G_{\pi_i'}$'s guaranteed by the hypothesis. Then

$$|S_i| = |G|/m_i.$$

We will show by induction on j that

$$\left|\bigcap_{i=2}^{j} S_i\right| = |G|/\prod_{i=2}^{j} m_i,$$

for all j. In particular we will have

$$\left|\bigcap_{i=2}^{n} S_i\right| = |G|/\prod_{i=2}^{n} m_i = m_1,$$

so that G possesses a G_{π_1}.

(i) $|S_2 \cap S_3|$ divides both

$$|G|/m_2 = m_3 m_1 m_4 \cdots m_n$$

and

$$|G|/m_3 = m_2 m_1 m_4 \cdots m_n,$$

and

$$(m_2, m_3) = 1.$$

Consequently $|S_2 \cap S_3|$ divides

$$m_1 m_4 \cdots m_n = |G|/m_2 m_3.$$

(ii) We have

$$|G| \geq |S_2 S_3| = |S_2||S_3|/|S_2 \cap S_3|$$

$$= (|G|/m_2)(|G|/m_3)/|S_2 \cap S_3|.$$

This implies that

$$|S_2 \cap S_3| \geq |G|/m_2 m_3.$$

Combining (i) and (ii) we have

$$\left|\bigcap_{i=2}^{3} S_i\right| = |G|/\prod_{i=2}^{3} m_i.$$

Now assume this relationship is true for all $i < j$. Then

$$\left|\bigcap_{i=2}^{j-1} S_i\right| = |G|/\prod_{i=2}^{j-1} m_i = m_1 m_j m_{j+1} \cdots m_n.$$

Consequently $|S_j \cap (\bigcap_{i=2}^{j-1} S_i)|$ divides both

$$|G|/m_j = m_2 \cdots m_{j-1} m_1 m_{j+1} \cdots m_n$$

and

$$|G|/\prod_{i=2}^{j-1} m_i = m_j m_1 m_{j+1} \cdots m_n.$$

Also

$$\left(\prod_{i=2}^{j-1} m_i, m_j\right) = 1.$$

This implies that $|S_j \cap (\bigcap_{i=2}^{j-1} S_i)|$ divides

$$|G|/m_j\left(\prod_{i=2}^{j-1} m_i\right) = |G|/\prod_{i=2}^{j} m_i.$$

By reasoning similar to that in (ii),

$$\left|S_j \cap \left(\bigcap_{i=2}^{j-1} S_i\right)\right| \geq |G|/\prod_{i=2}^{j} m_i.$$

Therefore, for all values of j,

$$\left|\bigcap_{i=2}^{j} S_i\right| = \left|S_j \cap \left(\bigcap_{i=2}^{j-1} S_i\right)\right| = |G| / \prod_{i=2}^{j-1} m_i,$$

which concludes the proof of the lemma.

We will also use a lemma which was proved by P. Hall [4].

Lemma 4.2.2. If G is of order $p^a q^b m$, where p and q are different primes, which do not divide m, and a and b are positive; and if G has a subgroup H of order $p^a q^b$, and also two proper subgroups of index p^a and q^b respectively, then G cannot be simple.

We are now ready to prove the theorem.

If G is a p-group, the theorem is trivial. Consider then that $|G|$ is the product of powers of two or more primes. Proceeding by induction on the order of G, assume the theorem has been proved for all groups whose orders are less than the order of G. We can write

$$|G| = m p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where $p_i \in \pi$, $i = 1, \cdots, r$, and m has no prime divisors from π. Then letting $p_1^{a_1} p_2^{a_2}$ be m_1 in Lemma 4.2.1, we see that G possesses a G_{p_1, p_2}. The hypothesis of the theorem states that G possesses a $G_{p_1'}$, and a $G_{p_2'}$. Hence by Lemma 4.2.2, G contains a proper normal subgroup, M. For each i, let S_i' be one of the $G_{p_i'}$'s guaranteed by the hypothesis of the theorem, and let S be one of the G_π's. Then $S \cap M$ is an M_π, $S_i' \cap M$ is an $M_{p_i'}$; SM/M is a Hall π-subgroup of G/M, and $S_i'M/M$ is a Hall p_i'-subgroup of G/M.

Also $|M| < |G|$ and $|G/M| < |G|$. Hence by the induction hypothesis, M and G/M are π-separable. Therefore, by Theorem 4.1, G is π-separable.

We shall call a finite group G π-serial if every composition factor group of G is either a π-group or a π′-group. A group which is both π-separable and π-serial is said to be π-solvable. That is, a group is π-solvable if and only if all of its composition factor groups are either π′-groups or p-groups for various primes p in π. Every solvable group is clearly π-solvable for any set of primes π, since all of its composition factor groups are p-groups. If π_1 is a subset of π and G is π-solvable, then G is π_1-solvable. This follows by noting that $\pi_1' \supset \pi'$. An immediate consequence of the following theorem is that any π-subgroup of a π-solvable group is solvable.

Theorem 4.3. If H is a π-subgroup of a π-separable group G, then H is solvable.

Proof: Let

$$H = A_0 \rhd A_1 \rhd \cdots \rhd A_n = 1$$

be a composition series for H. Then, since G is π-separable each composition factor group is a π′,p-group. Also $|A_i/A_{i+1}|$ divides H . Since H is a π-group, A_i/A_{i+1} is a p-group. For A_i/A_{i+1} to be a composition factor group, it must be simple. By Theorem S-1, the only simple p-groups are the cyclic groups of prime order. Hence the composition factor groups of H are all of prime order, which implies that H is solvable.

As a special case of Corollary D-4.2, we have

Theorem 4.4. Every π-solvable group satisfies $D_{\pi_1}^S$ for any subset π_1 of π.

A useful corollary of this is the following.

Corollary 4.4. If G has a normal $G_{\pi'}$, K, such that G/K is solvable, then G satisfies D_π.

Proof: Since K is a π'-group, it is π-solvable. The solvability of G/K, implies that it is π-solvable. By similar reasoning to that used in the proof of Theorem 4.1, since K and G/K are π-solvable groups, G is π-solvable. Therefore, by Theorem 4.4, G satisfies $D_{\pi_1}^S$ for any subset π_1 of π.

Another property of π-solvable groups is given by the following theorem.*

Theorem 4.5. Every π-solvable group satisfies $D_{\pi_1'}$ for any subset π_1 of π.

This theorem is not necessarily true for groups which are π-separable, but not π-solvable. The simple group A_5 of order 60 is 2-separable, but not 2-solvable. It does not satisfy $D_{2'}$, since a 2'-subgroup would have order 15. As a corollary to Theorem 4.5, we have

Corollary 4.5. If G has a normal $G_{\pi'}$, K, such that K is solvable, then G satisfies D_π.

Proof: K is solvable, hence K is π'-solvable. G/K, being a π-group, is also π'-solvable. Thus G is π'-solvable. Therefore, by Theorem 4.5, G satisfies D_π.

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* The proof of this theorem can be found in P. Hall's paper "Theorems Like Sylow's." [5].

5. An Application of π-groups

Wielandt [7] uses Hall π-subgroups in studying the normalizer of a subnormal subgroup. Given a set of primes π, he denotes the subgroup generated by all Hall π-subgroups of a group G by πG. πG is clearly a characteristic subgroup of G, since any G_π is mapped by an automorphism into some other G_π. The normalizer of a subgroup A is denoted NA. In studying NA, Wielandt shows that if $\pi_1, \pi_2, \cdots, \pi_n$ are sets of primes with the property that the index of each maximal normal subgroup of A contains at least one prime factor from $\bigcup_{i=1}^{n} \pi_i$; then

$$NA = \bigcap_{i=1}^{n} N\pi_i A.$$

In this way, he reduces the problem of studying NA to studying $N\pi A$.

The <u>subnormal closure</u> of A in G is the intersection of all subnormal subgroups of G which contain A. We will denote this by $A^{\cdot\cdot G}$. That is

$$A^{\cdot\cdot G} = \bigcap S \;(A \subseteq S \lhd \lhd G).$$

Wielandt [7] shows that πA is the intersection of those subnormal subgroups B of A for which

$$|A:B|_\pi = 1.$$

Using this fact, it follows that

$$\pi A = K^{\cdot\cdot A} \leq K^{\cdot\cdot G},$$

where K is an A_π. A consequence of this equality is

Theorem 5.1. If K is an A_π, then

$$NK \subseteq N\pi A.$$

If certain restrictions are made on A_π, we can obtain a more useful expression for $N\pi A$.

Theorem 5.2. Let $A \triangleleft \triangleleft G$. If A contains an A_π, K, with the property that for each g in $N\pi A$ there exists an a in πA, such that

$$g^{-1}Kg = a^{-1}Ka;$$

then

$$N\pi A = \pi A \cdot NK = NK \cdot \pi A.$$

Theorems 5.1 and 5.2 are due to Wielandt [7].

It is not always true that an A_π with the properties hypothesized by Theorem 5.2 exists, even when A satisfies E_π. Consider the simple group G of order 168. We saw in section 2, that G possesses two distinct conjugate sets of Hall 2,3-subgroups. There is an automorphism α of order 2 which exchanges the two sets. Let M be the normal product of G by the group of automorphisms $\{1, \alpha\}$.* Then M is the set of all symbols $[t, g]$, where t is an element of $\{1, \alpha\}$ and g is an element of G. Multiplication in M is defined by

$$[t_1, g_1][t_2, g_2] = [t_1t_2, t_1(g_1) \cdot g_2].$$

The subgroup of symbols of the type $[t, 1]$ is isomorphic to $\{1, \alpha\}$ and the subgroup of symbols of the type $[1, g]$ is isomorphic to G. We will denote the latter subgroup by $[1, G]$.

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* A discussion of the Normal Product can be found in Chapter 6 of The Theory of Groups by Marshall Hall [2].

$[1,G]$ is a normal subgroup of M, and $(2,3)G$ is a characteristic subgroup of G. Consequently $(2,3)[1,G]$ is a normal subgroup of M. Let $(2,3)[1,G]$ be the A of Theorem 5.2. We then have

$$(2,3)A = (2,3)(2,3\,[1,G]) = (2,3)\,[1,G] = A.$$

From the above equalities and the fact that A is normal in M, it follows that

$$N(2,3)A = M.$$

Thus $[\alpha,1]$ is an element of $N(2,3)A$. Let F be a $G_{3,2}$. We have

$$[\alpha,1]^{-1}[1,F][\alpha,1] = [\alpha^{-1},1][1,F][\alpha,1] = [\alpha^{-1}\cdot 1,1(1)\cdot F][\alpha,1]$$
$$= [\alpha^{-1}\alpha,\alpha(F)] = [1,\alpha(F)].$$

Now α exchanges the two classes of Hall groups. Hence for no x in G is

$$x^{-1}Fx = \alpha(F).$$

Consequently, for no $[1,x]$ in $(2,3)A$ is

$$[1,x]^{-1}[1,F][1,x] = [1,\alpha(F)].$$

Therefore M has a subnormal subgroup A, which contains no $A_{2,3}$ satisfying the hypothesis of Theorem 5.2.

We can now consider the problem of finding sufficient conditions for the existence of a Hall π-subgroup which does satisfy the hypothesis of Theorem 5.2. Before doing this, it will be useful to introduce the concept of a Sylow series of Complexion. Let $p_1, p_2, \cdots, p_r$, be distinct primes. We say that a finite group H has a <u>Sylow Series of Complexion</u> $\{p_1, \cdots, p_r)$ if H is divisible by no primes other than $p_1, \cdots, p_r$ and if for each $i = 1, 2, \cdots, r-1$ H has a normal $H_{p_1, \cdots, p_i}$. We should note that the order of the primes is important. For example, a Sylow series of complexion (2,3,5) would consist of a normal Sylow 2-subgroup and a normal Hall 2,3-subgroup, while a Sylow series of complexion (5,3,2) would consist of a normal Sylow 5-subgroup and a normal Hall 5,3-subgroup.

We can also define a Sylow series of complexion in terms of factor groups of a normal series. Let K_j be a normal $H_{p_1, \cdots, p_j}$. Since K_j is normal and

$$|K_j| = |H|_{p_1, \cdots, p_j};$$

it follows that K_t is contained in K_j for any $t \leq j$. Let S_j be a Sylow p_j-subgroup of K_j. Then

$$K_j/K_{j-1} \approx S_j.$$

Also S_j is a Sylow p_j-subgroup of H. From the above statements, we see that a group H with a Sylow series of complexion will possess a normal series in which the factor groups of each neighboring element are isomorphic to the Sylow subgroups of H. The converse is easily seen to be true also.

P. Hall [5] proves the following theorem.

Theorem 5.3. Let $p_1, \cdots, p_r$ be the distinct primes which divide $|G|_\pi$, then any two G_π's, both of which have a Sylow series of complexion $(p_1, \cdots, p_r)$ must be conjugate in G.

By the above theorem, if ΠA contains an A_π, M, with a Sylow series of complexion, then M is conjugate to any other A_π with the same series of complexion. Two subgroups which do not have the same series of complexion cannot be isomorphic. Consequently, M satisfies the hypothesis of Theorem 5.2.

The principal theorem of Wielandt's paper [7] "Der Normalisator einer Subnormalen Untergruppe," gives necessary and sufficient conditions for πA to be normal, when A satisfies certain conditions.

Theorem 5.4. Let $A \lhd \lhd G$. If in the normal subgroup of G

$$H = \pi(\bigcup_{g \in G} g^{-1}Ag)$$

there is an H_π, K, with the following property (*) that for each g in G there exists an h in H for which

$$g^{-1}Kg = h^{-1}Kh;$$

and letting M be the A_π $K \cap A$, any two of the following are equivalent.

(a) πA is normal in G.

(b) If U is a subgroup of G with the property that M is a π-group, where

$$M = \bigcup_{u \in U} (u^{-1}Mu),$$

then

$$M^U = M.$$

(c) It is true that NK is contained in NM.

(d) There is a group U contained in NK with the property that

$$HU^G = G.$$

Again by Theorem 5.3, if H possesses an H_π which has a Sylow series of complexion, then this H_π satisfies property (*) of the above theorem. Wielandt [7] proves that if A is a subnormal subgroup of G, and

$$H = \pi A^G;$$

and if A possesses an A_π, M, with a Sylow series of complexion, then H possesses an H_π, K, with a Sylow series of complexion and $A \cap K = M$. This implies that a sufficient condition for statements a - d of Theorem 5.4 to be equivalent is that A possess an A_π with a Sylow series of complexion. If H satisfies C_π or D_π, it will clearly satisfy property (*). Using this fact, and Corollaries D-4.1, 4.4, and 4.5, Wielandt [7] proves the following theorem.

Theorem 5.5. Let $A \triangleleft \triangleleft G$ and let

$$H = \pi A^G.$$

Let at least one of the following two conditions be true:

(a) Each composition factor group of A satisfies E_π^N.

(b) A contains a normal A_π, L, with the property that L or A/L is solvable.

Then A satisfies E_π; for each A_π, M, there is an H_π, K, with $A \cap K = M$; and each H_π satisfies property (*) of Theorem 5.4.

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Abstract

The thesis treats the generalization of Sylow's theorems about p-subgroups to theorems concerning Hall π-subgroups. The Hall π-subgroup is an extension of the Sylow p-subgroup to a set of primes π instead of the single prime p. In the first section Sylow's theorems are stated and the Hall π-subgroup is defined. Hall [3] generalized Sylow's theorems completely in the case of solvable groups. In particular he showed that every solvable group possesses a Hall π-subgroup for any set of primes . He also showed [4] that if a group G is not solvable there is at least one set of primes π such that G possesses no Hall π-subgroup. His results are discussed in section 2 and some examples of insolvable groups for which his generalized theorems are invalid are given. For a particular set of primes π it is possible to generalize Sylow's theorems even when the groups being considered are not necessarily solvable. Some theorems of this type are considered in section 3. For example, one of the theorems gives sufficient conditions, depending on the set of primes π, for a group to possess a Hall π-subgroup. For a particular set of primes π, solvability can be generalized to π-separability and π-solvability. Theorems similar to Hall's theorems for solvable groups can be proved for π-separable and π-solvable

groups. These are discussed in section 4. A theorem concerning sufficient conditions for a group to be π-separable is proved. Finally in section 5, some of the results of the earlier sections are applied to the problem of subnormal subgroups and their normalizers.