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Geodesic Paths for Quantum Many-Body Systems

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We propose a method to obtain optimal protocols for adiabatic ground-state preparation near the adiabatic limit, extending earlier ideas from [D. A. Sivak and G. E. Crooks, Phys. Rev. Lett. **108**, 190602 (2012)] to quantum non-dissipative systems. The space of controllable parameters of isolated quantum many-body systems is endowed with a Riemannian quantum metric structure, which can be exploited when such systems are driven adiabatically. Here, we use this metric structure to construct optimal protocols in order to accomplish the task of adiabatic ground-state preparation in a fixed amount of time. Such optimal protocols are shown to be geodesics on the parameter manifold, maximizing the local fidelity. Physically, such protocols minimize the average energy fluctuations along the path. Our findings are illustrated on the Landau-Zener model and the anisotropic XY spin chain. In both cases we show that geodesic protocols drastically improve the final fidelity. Moreover, this happens even if one crosses a critical point, where the adiabatic perturbation theory fails.

Introduction.—An accurate preparation of quantum states is a fundamental requirement for the realization of emergent quantum technologies such as quantum computers [1], quantum sensors [2], quantum cryptography [3] and quantum simulators [4–7]. To reduce the effects of noise and circumvent decoherence in such quantum devices, it is essential to find the optimal protocol that transforms an experimentally readily available initial state into a desired state with high fidelity, on which the necessary quantum manipulations are then conducted. Quantum optimal control [8] provides powerful methods to cope with this issue and they have been implemented in cold atomic systems [9], atom chips [10], superconducting quantum circuits [11] and are a vital aspect in adiabatic quantum computation [12]. Optimal control algorithms for particular quantum many-body systems have recently been developed in [13, 14], but so far these findings are model specific.

Recently, a new general idea connecting an optimization problem and geometry in dissipative systems was proposed in Refs. [15, 16]. In particular, it was shown that the optimum protocol minimizing heat along a thermodynamic path corresponds to the geodesic associated with the metric given by the friction tensor. These results, however, do not immediately extend to low temperature systems, where the friction tensor vanishes and leading non-adiabatic corrections come from virtual excitations determining the mass renormalization [17]. A time-optimal approach to adiabatic quantum computation was formulated in a differential-geometric framework by A. T. Rezakhani *et al.* [18]. They demonstrated that the optimal strategy, keeping the evolution adiabatic, is given by a geodesic. Although, in their set-up the adiabatic condition is used as a heuristic to define a metric tensor, and therefore there might exist a better definition. In this work, we extend the ideas of Refs. [15, 16, 18] by

using the Fubini-Study quantum metric associated with the quantum fidelity [19]. This metric equips the space of control parameters with a Riemannian structure [20–22].

Let us consider a closed quantum many-body system, described by a Hamiltonian $H(\vec{\lambda}(t))$ depending on time through the control parameters, $\vec{\lambda}(t) = (\lambda^1(t), \dots, \lambda^p(t))^T$, where p is the dimension of the parameter manifold \mathcal{M} . The problem of optimal adiabatic state preparation is then stated as follows: find the optimal protocol $\vec{\lambda}_{\text{opt}}(t)$ that transforms $|\psi_0(0)\rangle$, initial ground-state of $H(\vec{\lambda}(0))$, to the desired state $|\psi_0(t_f)\rangle$, ground-state of $H(\vec{\lambda}(t_f))$. As a measure of similarity between the evolved state $|\psi(t_f)\rangle$ and the target state $|\psi_0(t_f)\rangle$, we use the fidelity

$$\mathcal{F}[\vec{\lambda}(t_f)] = |\langle \psi(t_f) | \psi_0(t_f) \rangle|^2. \quad (1)$$

The task is therefore to find $\vec{\lambda}_{\text{opt}}(t)$ that maximizes \mathcal{F} for a fixed amount of time t_f . It is clear that the formulated problem is highly non-local and, in particular, allows for protocols which strongly deviate from the instantaneous ground-state for intermediate times, but give a very high final fidelity [14, 23]. Here, we focus on a more modest goal of finding protocols optimizing the instantaneous fidelity along the path. An obvious advantage of such protocols is their robustness against any small changes in the couplings or shape of pulses, especially in complex many-particle systems.

Quantum geometry.—A natural way to quantify the distance between two infinitesimally separated ground-states in Hilbert space, is given by $ds^2 \equiv 1 - |\langle \psi_0(\vec{\lambda}) | \psi_0(\vec{\lambda} + d\vec{\lambda}) \rangle|^2 = g_{\mu\nu} d\lambda^\mu d\lambda^\nu$, where the quantum metric tensor $g_{\mu\nu}$ reads

$$g_{\mu\nu} = \text{Re} \left[\langle \psi_0 | \overleftarrow{\partial}_\mu \overleftarrow{\partial}_\nu | \psi_0 \rangle - \langle \psi_0 | \overleftarrow{\partial}_\mu | \psi_0 \rangle \langle \psi_0 | \overleftarrow{\partial}_\nu | \psi_0 \rangle \right], \quad (2)$$

with $\partial_\mu \equiv \partial/\partial\lambda^\mu$ and $\langle \psi_0 | \overleftarrow{\partial}_\mu \equiv \partial_\mu \langle \psi_0 |$. The expansion of

ds^2 in $\{d\lambda^\mu\}$ clearly shows that $g_{\mu\nu}$ induces a metric on \mathcal{M} . This metric tensor was first studied in [19], and became an object of great interest in quantum information theory [24], the study of quantum phase transitions [25] and the characterization of topological phases [26].

The fact that \mathcal{M} is a metric space provides us the notion of geodesic curves. On a Riemannian manifold, a geodesic is a path that minimizes the distance functional

$$\mathcal{L}(\vec{\lambda}) = \int_{\vec{\lambda}_i}^{\vec{\lambda}_f} ds = \int_0^{t_f} \sqrt{g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu} dt, \quad (3)$$

between two given points $\vec{\lambda}_i = \vec{\lambda}(0)$ and $\vec{\lambda}_f = \vec{\lambda}(t_f)$, with $\dot{\lambda}^\mu \equiv d\lambda^\mu/dt$. The integrand of \mathcal{L} , which is extremized along the path, corresponds to the fidelity \mathcal{F} between infinitesimally separated ground-states. Therefore a geodesic has the meaning of a path maximizing the local fidelity. In Ref. [21], it was shown that in the leading order of non-adiabatic response, $(g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu)^{1/2}$ gives the mean energy variance δE at any particular point of the protocol. Thus, a geodesic curve also minimizes the energy fluctuations averaged along the path. It is interesting to point out that the energy variance can be interpreted as the time-component of the metric tensor, $\delta E^2 = g_{tt} \equiv \langle \psi(t) | \overleftarrow{\partial}_t \partial_t | \psi(t) \rangle - \langle \psi(t) | \overleftarrow{\partial}_t | \psi(t) \rangle \langle \psi(t) | \partial_t | \psi(t) \rangle$. The equivalence between g_{tt} and the energy variance follows from $i\partial_t | \psi(t) \rangle = H | \psi(t) \rangle$. Near the adiabatic limit, where $| \psi(t) \rangle = | \psi_0 \rangle + \mathcal{O}(\dot{\lambda})$, a geodesic can therefore also be thought of as the curve minimizing the proper time interval along the path. While we focus on the ground-state manifold in this Letter, these ideas apply to excited-states as well. Moreover, as the metric tensor has a well defined classical limit [22], our findings remain valid for classical Hamiltonian systems, where dissipation is very low.

The distance \mathcal{L} along a curve is obviously independent of the parametrization, therefore we may choose $g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu$ to be constant in time. The differential equations for geodesics take then the well known form [27]

$$\ddot{\lambda}^\mu + \Gamma_{\nu\rho}^\mu \dot{\lambda}^\nu \dot{\lambda}^\rho = 0, \quad (4)$$

where the Christoffel symbols are given by $\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\xi} (\partial_\rho g_{\xi\nu} + \partial_\nu g_{\xi\rho} - \partial_\xi g_{\nu\rho})$ and $(g^{\mu\nu}) = (g_{\mu\nu})^{-1}$ is the inverse of the metric tensor [28]. Let us highlight that the conservation of the product $g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu$ along a geodesic, implies that near points where the metric tensor is large, e.g., points corresponding to a small energy gap, the speed $|\dot{\lambda}|$ has to be small. We note that in Ref. [29], geodesics were used to analyze quantum criticalities. Moreover, it has been shown that they correspond to paths minimizing the error in adiabatic and holonomic quantum computation [30]. Below we illustrate how our ideas apply to two specific examples.

The Landau-Zener model.—Let us first illustrate our formalism on a simple two-level system given by the

Landau-Zener Hamiltonian [31],

$$H_{LZ}(t) = x(t)\sigma^x + \epsilon(t)\sigma^z = \begin{pmatrix} \epsilon(t) & x(t) \\ x(t) & -\epsilon(t) \end{pmatrix}. \quad (5)$$

The operators σ^x and σ^z are the usual Pauli matrices and $|\uparrow\rangle = (1, 0)^T$, $|\downarrow\rangle = (0, 1)^T$ denote the eigenstates of σ^z . The parameter x characterizes the coupling between the two levels and ϵ the detuning. The instantaneous eigenstates of this system are given by

$$|\psi_{0,1}\rangle = \mp \frac{1}{\sqrt{2}} \frac{\Omega \mp \epsilon}{\sqrt{\Omega(\Omega \mp \epsilon)}} |\uparrow\rangle + \frac{1}{\sqrt{2}} \frac{x}{\sqrt{\Omega(\Omega \mp \epsilon)}} |\downarrow\rangle, \quad (6)$$

where we defined $\Omega \equiv \sqrt{x^2 + \epsilon^2}$, and the corresponding eigenenergies are $E_{0,1} = \mp\Omega$. Our goal is to obtain the optimal control protocol $\vec{\lambda}_{\text{opt}}(t) = (x_{\text{opt}}(t), \epsilon_{\text{opt}}(t))^T$ maximizing the overlap $\mathcal{F}(t_f) = |\langle \psi(t_f) | \psi_0(t_f) \rangle|^2$, when evolving an initial ground-state $|\psi_0(0)\rangle$ corresponding to $\vec{\lambda}_i = (x_i, \epsilon)^T$, to the target ground-state $|\psi_0(t_f)\rangle$ corresponding to $\vec{\lambda}_f = (x_f, \epsilon)^T$. We assume that $|x_{i,f}| \gg \epsilon$.

First, consider the simplest standard protocol, where ϵ is time-independent and $x(t)$ linearly depends on time [32]: $x_{\text{lin}}(t) = x_i + (x_f - x_i)t/t_f$. This protocol corresponds to the paradigmatic Landau-Zener problem [33], and the initial adiabatic ground-state tunnels to the excited-state during the evolution with a finite probability, which yields a final fidelity given by $\mathcal{F}(t_f) \approx 1 - \exp\left[-\pi \frac{\epsilon^2 t_f}{(x_f - x_i)}\right]$. An intuitive way to improve this protocol would be to simply adjust the speed $\dot{x}(t)$ during the evolution, slowing down near the avoided level-crossing, thereby reducing transitions to the excited-state.

Next, let us fix ϵ and consider $x(t)$ as an arbitrary time-dependent parameter in the system. The quantum metric tensor is very easy to compute using the ground-state (6):

$$g_{xx} = \frac{\epsilon^2}{4(x^2 + \epsilon^2)^2}, \quad (7)$$

and thus the geodesic protocol, determined by $g_{xx} \dot{x}^2 = \text{const}$, reads $x_{\text{geo}}(t) = \epsilon \tan[\alpha_i + (\alpha_f - \alpha_i)t/t_f]$, where $\alpha_{i,f} = \arctan(x_{i,f}/\epsilon)$. The geodesic protocol slows down close to the avoided level-crossing (Fig. 1(a)), and hence minimizes the tunneling probability to the excited-state during the evolution (Fig. 1(c)). In the context of quantum adiabatic search algorithms, a similar protocol is discussed in Ref. [34], obtained by enforcing adiabatic evolution on each infinitesimal time interval. In Ref. [35], such protocol was implemented experimentally, using a two-level quantum system consisting of Bose-Einstein condensates in optical lattices, achieving a higher fidelity than a linear driving protocol.

It is intuitively clear that one can further optimize the protocol by increasing the number of control parameters. Mathematically, this is reflected in the fact that by choosing the parameter manifold, we consider a subset of the

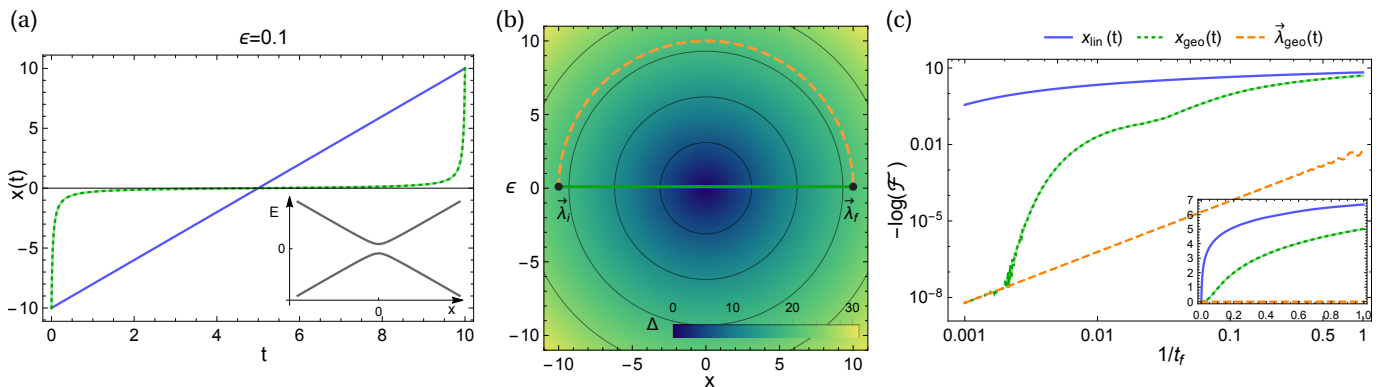


Figure 1. (Color online) Geodesic for the passage through an avoided level crossing: (a) The linear (blue solid) and the geodesic (green dotted) Landau-Zener protocols are depicted for the initial $x_i = -10$ and final $x_f = 10$ points, with $\epsilon = 0.1$ and a total evolution time of $t_f = 10$. (b) The energy gap, $\Delta = E_1 - E_0 = 2\sqrt{x^2 + \epsilon^2}$, is plotted in the (x, ϵ) parameter space. The dashed orange and the straight green lines correspond to the circular and constant ϵ geodesic protocols, respectively (see text for details). (c) $-\log(\mathcal{F})$ as a function of $1/t_f$, for the three different protocols $x_{\text{in}}(t)$ (blue solid), $x_{\text{geo}}(t)$ (green dotted) and $\vec{\lambda}_{\text{geo}}(t) = \Omega_i(\sin \theta_{\text{geo}}(t), \cos \theta_{\text{geo}}(t))^T$ (orange dashed), is shown on a logarithmic scale, for the values used in (a). The inset shows the same on a linear scale.

full Hilbert space. Thus, the geodesics found within this manifold will generally have non-vanishing geodesic curvature. By introducing extra parameters, i.e., by increasing the dimensionality of the subset, we can find geodesics with smaller and smaller geodesic curvature, which correspond to shorter geodesics and hence higher final fidelity. In the example discussed here, the geodesic we found has zero geodesic curvature, so introducing more parameters will not affect the length. To illustrate this, let us expand the parameter manifold and allow both x and ϵ to depend on time: $\vec{\lambda}(t) = (x(t), \epsilon(t))^T = \Omega(t)(\sin \theta(t), \cos \theta(t))^T$. In coordinates $\mu, \nu \in \{\Omega, \theta\}$, the quantum metric tensor shortens to $(g_{\mu\nu}) = \text{diag}(0, 1/4)$. Obviously, the metric tensor has zero components with respect to Ω , as changing the overall energy scale does not affect the eigenstates. In turn, this implies that we are free to choose the arbitrary protocol $\Omega(t)$. Let us choose the circular protocol $\Omega_{\text{geo}}(t) = \Omega_i$. The geodesic equation for $\theta(t)$ reduces then to $\ddot{\theta} = 0$, which yields $\theta_{\text{geo}}(t) = \theta_i + (\theta_f - \theta_i)t/t_f$, with $\theta_{i,f} = \arctan(x_{i,f}/\epsilon_{i,f})$. This protocol $\vec{\lambda}_{\text{geo}}(t)$ is nothing but a great circle in the full $SU(2)$ manifold of the two-level system, and thus has zero geodesic curvature. Therefore introducing the only remaining independent parameter ϕ , which defines the magnetic field angle in the xy -plane, will not affect the geodesic [27]. It is easy to see that the circular protocol is equivalent to the one with constant ϵ , discussed earlier, up to an overall rescaling of Ω .

Despite the formal equivalence between the constant ϵ and circular geodesic protocols leading to the same distance, there is an important physical difference between them. In the limit of small ϵ , the former protocol corresponds to crossing a small gap region, while the latter corresponds to the time-independent gap (Fig. 1(b)). The slightly counterintuitive equivalence between these

two geodesic protocols is hidden in their very different velocity profiles. In the former case, one first moves very fast to the small gap region $x \sim \epsilon$ and then slowly crosses it. In the latter case, one changes θ with a uniform velocity without changing the gap. It is intuitively clear that the circular protocol is more robust against introducing additional degrees of freedom, e.g., introducing a third excited-state. These extra degrees of freedom should also break the degeneracy between the geodesics. Even in the two-level case the circular protocol generally performs better, since the adiabatic approximation breaks down at much smaller velocities for the constant ϵ -protocol. Except for very large t_f , where the two protocols are equivalent, they give the same fidelity (c.f. green and orange lines in Fig. 1(c)).

The anisotropic XY spin chain.—Let us now apply our analysis to a quantum many-body system. For this purpose, we consider the illustrative example of the anisotropic XY spin chain in a transverse magnetic field [36], given by the Hamiltonian

$$H_{XY} = - \sum_{j=1}^N \left[\frac{1+\gamma}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2} \sigma_j^y \sigma_{j+1}^y + h \sigma_j^z \right], \quad (8)$$

where σ_j^α , with $\alpha = x, y, z$, are the Pauli matrices describing the spin on the j -th site of the chain. We assume periodic boundary conditions, $\sigma_{N+1}^\alpha = \sigma_1^\alpha$, and fix the overall energy scale to unity. The parameters of the model are the anisotropy γ of the nearest neighbor spin-spin exchange interaction along the x and y direction, and the transverse magnetic field h . We add an additional tuning parameter ϕ , describing a simultaneous rotation of all spins around the z axis by an angle $\phi/2$. The corresponding Hamiltonian is obtained by $\tilde{H}_{XY}(h, \gamma, \phi) = R_z(\phi) H_{XY} R_z^\dagger(\phi)$, where $R_z(\phi) = \prod_{l=1}^N \exp(-i\frac{\phi}{2} \sigma_l^z)$. We

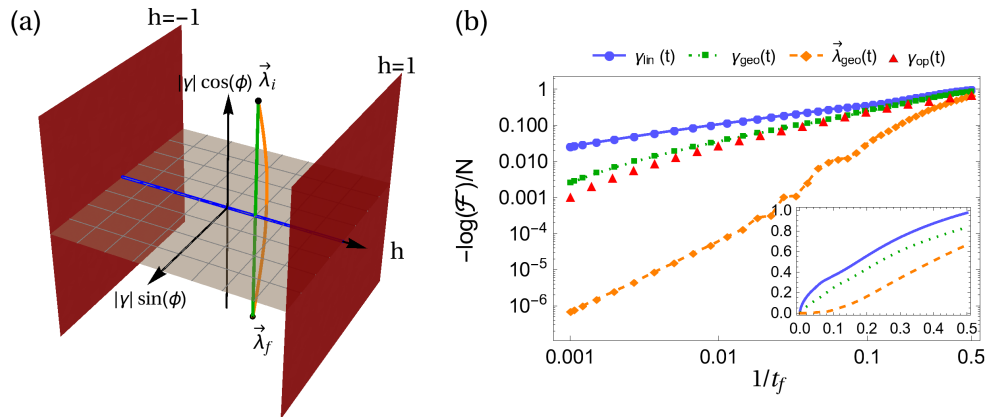


Figure 2. (Color online) Geodesic passage through a quantum phase transition: (a) The phase diagram of the rotated XY spin chain in cylindrical coordinates $(|\gamma| \cos \phi, |\gamma| \sin \phi, h)$ is depicted. The two red planes ($|h| = 1$) indicate the Ising criticality, where the system undergoes a continuous transition between a paramagnetic and a ferromagnetic phase. The blue line ($\gamma = 0$) marks the anisotropic transition, separating the two different aligned ferromagnetic phases. The green and orange lines illustrate the driving protocols for crossing and avoiding the quantum criticality, respectively. (b) $-\log(\mathcal{F})/N$ for the three different driving protocols is shown on a logarithmic scale, with $\gamma_i = 1$, $\gamma_f = -1$, $h = 0.5$ and $N = 900$. The inset shows the same on a linear scale. For comparison we also plotted $-\log(\mathcal{F})/N$ for the optimal power-law protocol $\gamma_{\text{op}}(t) = \text{sgn}[\gamma_{\text{lin}}(t)] |\gamma_{\text{lin}}(t)|^{\tau_{\text{op}}}$ (red triangles).

note that such a rotation of the whole system by ϕ does not affect its spectrum, but it modifies the eigenstates. \hat{H}_{XY} can be mapped to non-interacting fermions using the standard Jordan-Wigner and Fourier transformations [37], providing a unique ground-state $|\text{GS}\rangle$ throughout the entire parameter space, once a particular fermion parity is fixed [38]. The phase diagram of the model is illustrated in Fig. 2(a).

We focus on the $|h| \leq 1$ region of the parameter space and study the fidelity $\mathcal{F}(t_f) = |\langle \psi(t_f) | \text{GS}(t_f) \rangle|^2$ for the preparation of a target ground-state $|\text{GS}(t_f)\rangle$ from an initial ground-state $|\text{GS}(0)\rangle$, lying in a different phase region than the target state. Let us analyze the passage through the anisotropic transition line with fixed $h = 0.5$, for the initial $\gamma_i = 1$ and final $\gamma_f = -1$ points. As in the previously studied case, we compare the three different protocols: a linear $\gamma_{\text{lin}}(t)$, a geodesic $\gamma_{\text{geo}}(t)$ and a geodesic one which avoids the quantum phase transition $\vec{\lambda}_{\text{geo}}(t) = |\gamma(t)| (\cos \phi(t), \sin \phi(t))^T$. The corresponding quantum metric tensor was calculated in Ref. [20, 21] and reads

$$g_{\gamma\gamma} = \frac{1}{16|\gamma|(1+|\gamma|)^2}, \quad g_{\phi\phi} = \frac{|\gamma|}{8(|\gamma|+1)}, \quad g_{\gamma\phi} = 0. \quad (9)$$

In Fig. 2(b), we plot the resulting final fidelities. For the linear protocol $\gamma_{\text{lin}}(t) = \gamma_i + (\gamma_f - \gamma_i)t/t_f$, the observed result, $-\frac{1}{N} \log(\mathcal{F}) \sim 1/\sqrt{t_f}$, is in perfect agreement with the general Kibble-Zurek predictions for linear quenches [39]. Crossing the quantum phase transition along a geodesic protocol clearly yields a higher final fidelity, as can be seen in Fig. 2(b). The energy gap vanishes at the quantum phase transition $\gamma = 0$ and consequently the metric diverges, which imposes $\dot{\gamma}_{\text{geo}} \rightarrow 0$

on the velocity when approaching the transition. This is due to the fact that the product $g_{\gamma\gamma} \dot{\gamma}^2$ has to be constant along a geodesic. The corresponding geodesic takes then the form $\gamma_{\text{geo}}(t) = \text{sgn}[X(t)] \tan^2[X(t)]$, where $X(t) = \chi_i + (\chi_i - \chi_f)t/t_f$ and $\chi_{i,f} = \text{sgn}(\gamma_{i,f}) \arctan(\sqrt{|\gamma_{i,f}|})$. In Ref. [40, 41], the optimal adiabatic crossing of a quantum critical point has been analyzed. More specifically, they found that in order to minimize the number of excitations, the driving protocol should be given by a power-law, where the exponent serves as a minimization parameter. However, this optimization of the exponent yields only an incremental improvement of the final fidelity compared to the geodesic protocol (see Fig. 2(b)). And thus the geodesic still gives a nearly optimal protocol despite the breakdown of the adiabatic perturbation theory.

Finally, let us study the final fidelity when tuning both γ and ϕ simultaneously. In this case, the metric tensor can be expressed by $g_{\mu\nu} = \frac{1}{4} \text{diag}(1, \sin^2 \eta)$, where $\mu, \nu \in \{\eta, \varphi\}$, defined by $\gamma = \tan^2 \eta$ and $\phi = \sqrt{2} \varphi$. The resulting geodesic protocol $\vec{\lambda}_{\text{geo}}(t)$ is thus given by a great circle on the sphere defined by $\{\eta, \varphi\}$. This geodesic protocol gives significantly better final fidelity than the linear one as it avoids the critical point (c.f. Fig. 2(b)).

Conclusion.—We used a geometric approach to obtain optimal protocols for the adiabatic preparation of ground-states in quantum many-body systems close to the adiabatic limit. Those are geodesics in the space of control parameters, maximizing the overlap between the evolved state and the target state, while simultaneously keeping the quantity $g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu$, which is equal to the energy variance, stationary along the path. Further, we showed that by increasing the number of control param-

eters and tuning them along geodesic paths on the extended parameter space can provide a further increase in the final fidelity. This method can be applied to various optimization problems like finding best quantum annealing protocols, optimum adiabatic path for quantum simulation or minimization of heating in experiments with ultra-cold atoms.

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Supplementary Material for the paper “Geodesic paths for quantum many-body systems”

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QUANTUM GEOMETRIC TENSOR

In this section we show that the quantum metric tensor $g_{\mu\nu}$, introduced in the main text, is the symmetric part of the more general *quantum geometric tensor*. The quantum geometric tensor was introduced by Provost and Vallee [1], but the term itself first appeared in a work from M. Berry [2]. For a ground-state $|\psi_0\rangle$ of a generic quantum system, it is given by

$$\chi_{\mu\nu} \equiv \langle \psi_0 | \overleftarrow{\partial}_\mu \partial_\nu | \psi_0 \rangle - \langle \psi_0 | \overleftarrow{\partial}_\nu | \psi_0 \rangle \langle \psi_0 | \partial_\mu | \psi_0 \rangle. \quad (\text{S1})$$

Alternatively, it can also be expressed as a sum over all the eigenstates $|\psi_m\rangle$, by

$$\chi_{\mu\nu} = \sum_{m \neq 0} \frac{\langle \psi_0 | \partial_\mu \hat{H} | \psi_m \rangle \langle \psi_m | \partial_\nu \hat{H} | \psi_0 \rangle}{(E_0 - E_m)^2}, \quad (\text{S2})$$

where the resolution of identity $\sum_m |\psi_m\rangle \langle \psi_m| = \hat{1}$ and $\langle \psi_m | \partial_\mu | \psi_n \rangle = \langle \psi_m | \partial_\mu \hat{H} | \psi_n \rangle / (E_n - E_m)$, valid for $m \neq n$, were used.

The symmetric part of the quantum geometric tensor

$$g_{\mu\nu} \equiv \frac{1}{2}(\chi_{\mu\nu} + \chi_{\nu\mu}) = \text{Re}(\chi_{\mu\nu}), \quad (\text{S3})$$

corresponds to the quantum metric tensor used in the main text. It defines a Riemannian metric in the parameter manifold \mathcal{M} with respect to the local coordinates $\{\lambda^\mu\}$, and therefore also a measure of distances between different ground-states, identified as points in \mathcal{M} by the map $(\lambda^\mu) \in \mathcal{M} \longleftrightarrow |\psi_0(\lambda^\mu)\rangle$. The distance ds between two ground-states differing by an infinitesimal variation of coordinates in \mathcal{M} is then given by

$$ds^2 \equiv 1 - |\langle \psi_0(\vec{\lambda}) | \psi_0(\vec{\lambda} + d\vec{\lambda}) \rangle|^2 = g_{\mu\nu} d\lambda^\mu d\lambda^\nu, \quad (\text{S4})$$

where Einstein summation convention over repeated indices is implied.

The anti-symmetric part of the quantum geometric tensor defines the Berry curvature

$$F_{\mu\nu} \equiv i(\chi_{\mu\nu} - \chi_{\nu\mu}) = -2 \text{Im}(\chi_{\mu\nu}), \quad (\text{S5})$$

which gives rise to the Berry phase and a topological invariant known as the first Chern number.

RELATIONSHIP BETWEEN THE QUANTUM METRIC TENSOR AND THE ENERGY VARIANCE

In the following, we present the relationship between the energy fluctuations δE and the quantum metric tensor $g_{\mu\nu}$. The energy fluctuations are defined by

$$\delta E^2(t) \equiv \langle \psi(t) | H^2 | \psi(t) \rangle - \langle \psi(t) | H | \psi(t) \rangle^2. \quad (\text{S6})$$

Within adiabatic perturbation theory [3], we can compute $|\psi(t)\rangle$ in powers of the driving velocities $\dot{\lambda}^\mu$

$$|\psi(t_f)\rangle = |\psi_0\rangle - i\dot{\lambda}^\mu \sum_{m \neq 0} \frac{\langle \psi_m | \partial_\mu H | \psi_0 \rangle}{(E_m - E_0)^2} |\psi_m\rangle + \dots, \quad (\text{S7})$$

where we assumed that $|\dot{\lambda}| \ll 1$. Inserting this expansion into the expression of the energy fluctuations yields

$$\delta E^2 \approx \chi_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu = \left(g_{\mu\nu} - \frac{i}{2} F_{\mu\nu} \right) \dot{\lambda}^\mu \dot{\lambda}^\nu = g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu, \quad (\text{S8})$$

showing that the metric tensor defines the leading non-adiabatic correction of the energy fluctuations $\delta E^2 \approx g_{\mu\nu} \dot{\lambda}^\mu \dot{\lambda}^\nu$. This result was shown in Ref. [4]. We note that due to energy conservation in a closed quantum system, the energy fluctuations are equal to the fluctuations of the work done on the system, δW^2 , and therefore the quantum metric tensor can also be measured through the work fluctuations.

GEODESICS AND THE EULER-LAGRANGE EQUATIONS

In general, the quantum distance between two ground-states situated at $\vec{\lambda}_i$ and $\vec{\lambda}_f$, connected by a path $\vec{\lambda}$ in parameter space, can be written as

$$\mathcal{L}(\vec{\lambda}) = \int_{\vec{\lambda}_i}^{\vec{\lambda}_f} ds = \int_{\vec{\lambda}_i}^{\vec{\lambda}_f} \sqrt{g_{\mu\nu} d\lambda^\mu d\lambda^\nu}. \quad (\text{S9})$$

The curve $\vec{\lambda}$ may be parametrized by t such that $\vec{\lambda} \equiv \vec{\lambda}(t)$ with $\vec{\lambda}(t_i) = \vec{\lambda}_i$ and $\vec{\lambda}(t_f) = \vec{\lambda}_f$, and consequently the previous equation becomes

$$\mathcal{L}(\vec{\lambda}) = \int_{t_i}^{t_f} \sqrt{g_{\mu\nu} \frac{d\lambda^\mu}{dt} \frac{d\lambda^\nu}{dt}} dt. \quad (\text{S10})$$

We note that the above functional \mathcal{L} is invariant under any reparametrization of the parameter t , i.e., $\tau = \tau(t)$. Further, the stationary curve $\vec{\lambda}_{\text{geo}}(t)$ of \mathcal{L} will naturally inherit this property and is referred to as the *geodesic* connecting the boundary points. In case $t_i = 0$, a convenient parametrization is $t = t_f \tau$, $dt = t_f d\tau$, and then Eq. (S10) becomes

$$\mathcal{L}(\vec{\lambda}) = \int_0^1 \sqrt{g_{\mu\nu} \frac{d\lambda^\mu}{d\tau} \frac{d\lambda^\nu}{d\tau}} d\tau. \quad (\text{S11})$$

Such choice is often referred to as the *proper parametrization*.

The path $\vec{\lambda}_{\text{geo}}(t)$ that minimizes the distance between the fixed endpoints is obtained by

$$\frac{\delta \mathcal{L}}{\delta \vec{\lambda}} = 0. \quad (\text{S12})$$

It turns out, the variation of $\mathcal{L}(\vec{\lambda})$ can be calculated in an easier way. To this end, let us introduce the action functional \mathcal{E} , defined by

$$\mathcal{E} = \frac{1}{2} \int_0^1 \left(g_{\mu\nu} \frac{d\lambda^\mu}{d\tau} \frac{d\lambda^\nu}{d\tau} \right) d\tau, \quad (\text{S13})$$

which has a much simpler integrand. The Cauchy-Schwarz inequality for square-integrable functions,

$$\left(\int_a^b f(t) h(t) dt \right)^2 \leq \left(\int_a^b f^2(t) dt \right) \left(\int_a^b h^2(t) dt \right), \quad (\text{S14})$$

for $f(t) = 1$, $h(t) = \sqrt{g_{\mu\nu} (d\lambda^\mu/d\tau) (d\lambda^\nu/d\tau)}$, $a = 0$ and $b = 1$, implies then that

$$(\mathcal{L})^2 \leq 2 \mathcal{E}, \quad (\text{S15})$$

where the equality holds if and only if h is constant. Hence, if we apply the principle of stationary action to the functional \mathcal{E} , we also obtain the stationary solutions of \mathcal{L} , with one very important caveat: the functional \mathcal{E} is not invariant under change of parametrizations, as one can easily verify by Eq. (S13). Consequently, the stationary curve of \mathcal{E} will also be stationary for \mathcal{L} , provided that the solution curve $\vec{\lambda}_{\text{geo}}(t)$ is parametrized only by linear functions of t [6, 7].

The stationary solutions of \mathcal{E} are then found by the standard procedure [6], and follow from the Euler-Lagrange equations, which in local coordinates $\{\lambda^\mu\}$ read

$$\frac{d^2 \lambda^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu \frac{d\lambda^\nu}{d\tau} \frac{d\lambda^\rho}{d\tau} = 0, \quad (\text{S16})$$

where $\Gamma_{\mu}^{\nu\rho}$ are the Christoffel symbols of the second kind, defined by

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\xi} (\partial_\rho g_{\xi\nu} + \partial_\nu g_{\xi\rho} - \partial_\xi g_{\nu\rho}), \quad (\text{S17})$$

with $\partial_\mu \equiv \partial/\partial\lambda^\mu$ and $g^{\mu\xi}$ are the components of the inverse of the metric tensor $g_{\mu\xi}$, i.e., $(g^{\mu\xi}) = (g_{\mu\xi})^{-1}$.

We note that the integrand of \mathcal{L} corresponds to $1 - \mathcal{F}$, for infinitesimally separated ground-states, as can be seen from Eq. (S4). Therefore, geodesics are paths maximizing the local fidelity. Moreover, the integrand of \mathcal{E} gives the energy fluctuations δE in the leading order of non-adiabatic response [4], at any particular point of the protocol. Thus, geodesics are curves that also minimizes energy fluctuations averaged along the path.

ADDITIONAL PARAMETER SPACE EXTENSION FOR THE TWO-LEVEL SYSTEM

In this section we compute the geodesics when tuning the magnetic field in the xy-plane of the two-level system. Let us consider the Hamiltonian

$$H = \begin{pmatrix} \epsilon & x(t) - iy(t) \\ x(t) + iy(t) & -\epsilon \end{pmatrix}. \quad (\text{S18})$$

We will use the following coordinates to simplify the calculations

$$x(t) = h(t) \cos \phi(t), \quad y(t) = h(t) \sin \phi(t), \quad (\text{S19})$$

with the inverse given by

$$h^2(t) = x^2(t) + y^2(t), \quad \tan \phi(t) = \frac{y(t)}{x(t)}. \quad (\text{S20})$$

The Hamiltonian reduces therefore to

$$H = \begin{pmatrix} \epsilon & h(t)e^{-i\phi(t)} \\ h(t)e^{i\phi(t)} & -\epsilon \end{pmatrix}. \quad (\text{S21})$$

The eigenstates are given by

$$|\psi_{0,1}\rangle = \mp \frac{1}{\sqrt{2}} \sqrt{1 \mp \frac{\epsilon}{\sqrt{h^2 + \epsilon^2}}} |\uparrow\rangle + \frac{1}{\sqrt{2}} \frac{h e^{i\phi}}{\sqrt{(h^2 + \epsilon^2) \mp \epsilon \sqrt{h^2 + \epsilon^2}}} |\downarrow\rangle, \quad (\text{S22})$$

with the corresponding eigenenergies $E_{0,1} = \mp \sqrt{h^2 + \epsilon^2}$. The metric tensor with respect to h and ϕ reads

$$(g_{\mu\nu}) = \begin{pmatrix} g_{hh} & g_{h\phi} \\ g_{\phi h} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \frac{\epsilon^2}{(h^2 + \epsilon^2)^2} & 0 \\ 0 & \frac{1}{4} \frac{h^2}{(h^2 + \epsilon^2)} \end{pmatrix}. \quad (\text{S23})$$

We note that for $\epsilon = 0$, the metric element g_{hh} equals to 0 and hence we are left with a pseudo-Riemannian metric. This is a consequence of the fact that the ground-state $|\psi_0\rangle$ is independent of h for $\epsilon = 0$ and therefore there is no notion of distance along the h direction. Consequently, we are free to choose $h(t)$, such that $h > 0$, since we want to avoid the degeneracy point $E_0 = E_1$. The most simple function $h(t)$ satisfying this is the constant one. The remaining geodesic equation for $\phi(t)$, obtained by lowering the index in the geodesic equations (Eq. (4) of the main text) in order to avoid the use of the inverse metric, is then simply

$$\phi'' = 0, \quad (\text{S24})$$

with the solution $\phi(t) = (\phi_f - \phi_i) \frac{t}{t_f} + \phi_i$.

Let us come back to the case $\epsilon \neq 0$. We observe that when rescaling $h = \tilde{\epsilon} \tilde{h}$, the metric can be simplified to

$$(g_{\mu\nu}) = \begin{pmatrix} g_{\tilde{h}\tilde{h}} & g_{\tilde{h}\phi} \\ g_{\phi\tilde{h}} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \frac{1}{(\tilde{h}^2 + 1)^2} & 0 \\ 0 & \frac{1}{4} \frac{\tilde{h}^2}{(\tilde{h}^2 + 1)} \end{pmatrix}. \quad (\text{S25})$$

This metric simplifies even further, by introducing

$$\tilde{h}(t) = \tan \vartheta(t), \quad (\text{S26})$$

and hence we obtain

$$(g_{\mu\nu}) = \begin{pmatrix} g_{\vartheta\vartheta} & g_{\vartheta\phi} \\ g_{\phi\vartheta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \sin^2 \vartheta \end{pmatrix}. \quad (\text{S27})$$

The corresponding Christoffel symbols are

$$\begin{aligned} \Gamma_{\vartheta\vartheta}^{\vartheta} &= 0, & \Gamma_{\vartheta\phi}^{\vartheta} &= 0, & \Gamma_{\phi\vartheta}^{\vartheta} &= 0, & \Gamma_{\phi\phi}^{\vartheta} &= -\cos \vartheta \sin \vartheta, \\ \Gamma_{\vartheta\vartheta}^{\phi} &= 0, & \Gamma_{\vartheta\phi}^{\phi} &= \cot \vartheta, & \Gamma_{\phi\vartheta}^{\phi} &= \cot \vartheta, & \Gamma_{\phi\phi}^{\phi} &= 0, \end{aligned} \quad (\text{S28})$$

and the geodesic equations are given by

$$-\cos \vartheta \sin \vartheta \phi'^2 + \vartheta'' = 0, \quad 2 \cot \vartheta \vartheta' \phi' + \phi'' = 0. \quad (\text{S29})$$

The resulting geodesics are thus great circles on the sphere defined by the coordinates $\{\vartheta, \phi\}$. A detailed derivation of this is given in [8].

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