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The Sj-test against linear trend

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Dissertation

THE S_j -TEST AGAINST LINEAR TREND

by

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INTRODUCTION

The theory of sequential analysis has been developed since 1943 mainly by Abraham Wald, with the major portion of Wald's results presented in his book, Sequential Analysis (Wiley, 1947). The main feature of sequential analysis is the examination of observations individually, in the order selected, with the sampling being terminated as soon as enough information has been obtained to allow the experimenter to make a decision. Thus the number of observations required is a random variable dependent on the actual outcome of the experiment. In practice the advantage of sequential analysis over the classical approach using a fixed sample size is the saving in number of observations required to reach a decision with fixed probabilities of error. For many common types of problem Wald has shown that the average number of observations required by the best sequential test is considerably smaller than the number required by the best non-sequential test. Since the appearance of Wald's results, articles have appeared in the literature suggesting other sequential tests and studying the properties of sequential procedures in general.

The present paper considers the S_j -test proposed by Noether [5], [7], which is a sequential test designed to test randomness against the alternative of linear trend. In Chapter 2 a chart is presented to facilitate the estimation of the value of j to be used in the test. In Chapter 3 the effect of truncating the test at $2j$ observations is considered. In Chapter 4 the efficiency of the test is considered relative to the best non-parametric fixed-sample-size test against linear trend. In Chapter 5 the S_j -test is extended to the case of two-sided alternatives. In Chapter 6 the effectiveness of the S_j -test against alternatives other than linear trend is considered.

1. PRELIMINARY RESULTS

The sequential probability ratio test (SPRT) introduced by Wald [13], to test a simple hypothesis against a simple alternative is described as follows: Suppose that the distribution of the random variable X depends on the parameter θ and that the frequency function of X is given as $f(x, \theta)$. It is desired to test the null hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1: \theta = \theta_1$. For any sequence x_1, x_2, \dots, x_m of m observations define

$$p_{jm} = \prod_{i=1}^m f(x_i, \theta_j) \quad j = 0, 1$$

and consider the ratio $\frac{p_{1m}}{p_{0m}}$.

After the selection of the m th observation ($m = 1, 2, \dots$) terminate sampling with

$$\text{acceptance of } H_0 \text{ if } \frac{p_{1m}}{p_{0m}} \leq B$$

$$\text{acceptance of } H_1 \text{ if } \frac{p_{1m}}{p_{0m}} \geq A$$

take another observation if $B < \frac{p_{1m}}{p_{0m}} < A$,

where A and B are selected to give desired probabilities α, β of type I and type II error. Wald indicates that

these probabilities of error will be attained approximately by choosing $A = \frac{1-\beta}{\alpha}$ and $B = \frac{\beta}{1-\alpha}$.

In application it is usually simpler to consider the variable

$$z_i = \log \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}$$

which gives

$$\log \frac{p_{1m}}{p_{0m}} = \sum_{i=1}^m z_i = Z_m$$

and results in the revised testing procedure

terminate sampling with

$$(1.1) \quad \begin{aligned} & \text{acceptance of } H_0 \text{ if } Z_m \leq b \\ & \text{acceptance of } H_1 \text{ if } Z_m \geq a \\ & \text{take another observation if } b < Z_m < a, \\ & \text{where } b = \log B, a = \log A. \end{aligned}$$

Generally in this paper the SPRT will be applied to a binomial variable x , i.e., a variable x with probability function $f(x,p)$ given by $f(1,p) = p$, $f(0,p) = 1 - p = q$, to test the null hypothesis $H_0: p = p_0$ against the alternative $H_1: p = p_1 > p_0$. In this case the procedure described by (1.1) is equivalent to the following (cf. Wald [13], Chapter 5). For a given sequence of observations x_1, x_2, \dots let $X_m = \sum_{i=1}^m x_i$.

Terminate sampling with

acceptance of H_0 if $X_m \leq a_m$

acceptance of H_1 if $X_m \geq r_m$

take another observation if $a_m < X_m < r_m$.

a_m and r_m are given by

$$(1.2) \quad \begin{aligned} a_m &= h_0 + sm \\ r_m &= h_1 + sm \end{aligned}$$

where

$$(1.3) \quad h_0 = \frac{\log \frac{\beta}{1-\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}}, \quad h_1 = \frac{\log \frac{1-\beta}{\alpha}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}},$$

$$s = \frac{\log \frac{1-p_0}{1-p_1}}{\log \frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}}$$

Using Wald's notation we will let $L(p')$ represent the operating characteristic (OC) function of the test: $L(p')$ represents the probability of accepting H_0 when $p = p'$. This definition implies that $L(p_0) = 1 - \alpha$, $L(p_1) = \beta$. Also, if n represents the smallest value of m for which either $X_m \leq a_m$ or $X_m \geq r_m$, then the expected number of observations required by the sequential test when $p = p'$ is represented by $E_{p'}(n)$ and is given by

$$(1.4) \quad E_{p'}(n) = \frac{L(p') \log \frac{\beta}{1-\alpha} + (1 - L(p')) \log \frac{1-\beta}{\alpha}}{p' \log \frac{p_1}{p_0} + (1 - p') \log \frac{1-p_1}{1-p_0}}$$

For the above test if the alternative hypothesis specifies that $p = p_1 < p_0$, the testing procedure would be changed as follows.

Terminate sampling with

acceptance of H_0 if $X_m \geq a_m$

acceptance of H_1 if $X_m \leq r_m$

take another observation if $r_m < X_m < a_m$,

where a_m, r_m are defined as before.

The hypothesis of randomness, i.e., that x_1, x_2, \dots, x_n constitute a random sample of size n from a population with distribution function $F(x)$, can be stated as the hypothesis that the joint distribution function $F(x_1, x_2, \dots, x_n)$ of the chance variables X_1, X_2, \dots, X_n is the product of identical distribution functions, i.e.

$$(1.5) \quad F(x_1, x_2, \dots, x_n) = \prod_{i=1}^n F(x_i)$$

The S_j -test suggested by Noether [5], [7] is designed to test this hypothesis against the alternative of linear trend. The hypothesis of linear trend can be stated as the hypothesis that

$$(1.6) \quad F(x_1, x_2, \dots, x_n) = \prod_{i=1}^n F(x_i + i\theta)$$

where θ is a constant.

The S_j -test is carried out by comparing x_i with x_{i+j} for $i = 1, \dots, j$. Under (1.5) $P(x_i > x_{i+j}) = \frac{1}{2}$, while under (1.6) $P(x_i > x_{i+j}) > \frac{1}{2}$ or $< \frac{1}{2}$ depending on whether $\theta > 0$ or < 0 . For a particular alternative value, say $\theta_1 > 0$, each choice of j gives under (1.6) a different value of

$$P(x_i > x_{i+j}) = p_j = \frac{1}{2} + \epsilon_j \quad \epsilon_j > 0$$

The sequential test of randomness is carried out as a Wald sequential binomial test of the hypothesis $H_0: p = \frac{1}{2}$ against the alternative $H_1: p = \frac{1}{2} + \epsilon_j$. Noether suggests that the value of j should be chosen to minimize the expected numbers of observations under H_0 and under H_1 .

In carrying out the sequential test for $p_0 = \frac{1}{2}$, $p_1 = \frac{1}{2} + \epsilon_j$, formulas (1.3) become

$$(1.7) \quad h_0 = \frac{\log \frac{\beta}{1-\alpha}}{\log \frac{1+2\epsilon_j}{1-2\epsilon_j}}, \quad h_1 = \frac{\log \frac{1-\beta}{\alpha}}{\log \frac{1+2\epsilon_j}{1-2\epsilon_j}},$$

$$s = \frac{-\log(1-2\epsilon_j)}{\log \frac{1+2\epsilon_j}{1-2\epsilon_j}}$$

If we let $n_0 = E_{\frac{1}{2}}(n)$, $n_1 = E_{\frac{1}{2} + \epsilon_j}(n)$, then from (1.4)

$$(1.8) \quad n_0 = \frac{2 \left[(1-\alpha) \log \frac{\beta}{1-\alpha} + \alpha \log \frac{1-\beta}{\alpha} \right]}{\log(1-4\epsilon_j^2)}$$

$$n_1 = \frac{2 \left[\beta \log \frac{\beta}{1-\alpha} + (1-\beta) \log \frac{1-\beta}{\alpha} \right]}{\log(1-4\epsilon_j^2) + 2\epsilon_j \log \frac{1+2\epsilon_j}{1-2\epsilon_j}}$$

and the expected number of observations required by the S_j -test will be

$$(1.9) \quad n_0(S_j) = j + n_0, \quad n_1(S_j) = j + n_1$$

under H_0 , H_1 respectively.

To apply the S_j -test an estimate of ϵ_j is needed for fixed θ and j . Noether indicates that probabilities of the type $P(x_i > x_{i+j})$ do not depend strongly on the form of the distribution function $F(x)$ provided that θ is expressed in units of the standard deviation, and therefore the value of such a probability may be estimated by assuming a particular distribution function $F(x)$, e.g. the standard normal distribution. Using the standard notation let

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\Phi(x) = \int_{-\infty}^x \varphi(t) dt$$

Define λ_p by

$$(1.10) \quad \Phi(\lambda_p) = p$$

Noether shows that

$$(1.11) \quad \epsilon_j = \Phi\left(\frac{j\sigma}{\sqrt{2}}\right) - \frac{1}{2}$$

2. DETERMINATION OF j

Noether has suggested that the value of j which minimizes the expected number of observations required by the S_j -test might be obtained by trial and error. It would seem desirable to have a quick method of estimating the value of j required in any particular case. To obtain such an estimate the expressions for $n_0(S_j)$ and $n_1(S_j)$ must be differentiated with respect to j , using (1.8), (1.9), and the estimate of ϵ_j given by (1.11). We have

$$(2.1) \quad n_0(S_j) = j + \frac{2c_0}{\log(1-4\epsilon_j^2)}$$

$$(2.2) \quad n_1(S_j) = j + \frac{2c_1}{\log(1-4\epsilon_j^2) + 2\epsilon_j \log \frac{1+2\epsilon_j}{1-2\epsilon_j}}$$

where $c_0 = (1-\alpha) \log \frac{\beta}{1-\alpha} + \alpha \log \frac{1-\beta}{\alpha}$

$$c_1 = \beta \log \frac{\beta}{1-\alpha} + (1-\beta) \log \frac{1-\beta}{\alpha}$$

We assume in this section that \log represents natural logarithm. By (1.10) and (1.11) we have $\lambda_{\frac{1}{2}+\epsilon_j} = \frac{j\theta}{\sqrt{2}}$

or

$$(2.3) \quad j = \frac{\sqrt{2}}{\theta} \lambda_{\frac{1}{2}+\epsilon_j}$$

From (1.11)

$$\frac{d\epsilon_j}{dj} = \varphi\left(\frac{j\theta}{\sqrt{2}}\right) \cdot \frac{\theta}{\sqrt{2}} = \frac{\theta \varphi\left(\lambda_{\frac{1}{2} + \epsilon_j}\right)}{\sqrt{2}}$$

Assuming $n_0(S_j)$ continuous, from (2.1) we have

$$\frac{dn_0(S_j)}{dj} = 1 - \left(\frac{2c_0}{[\log(1-4\epsilon_j^2)]^2} \right) \left(\frac{-8\epsilon_j}{1-4\epsilon_j^2} \right) \left(\frac{\theta \varphi\left(\lambda_{\frac{1}{2} + \epsilon_j}\right)}{\sqrt{2}} \right)$$

Setting this derivative equal to zero gives

$$(2.4) \quad \theta = \frac{-\sqrt{2} (1-4\epsilon_j^2) [\log(1-4\epsilon_j^2)]^2}{16c_0 \epsilon_j \varphi\left(\lambda_{\frac{1}{2} + \epsilon_j}\right)}$$

Similarly, from (2.2) we have

$$\frac{dn_1(S_j)}{dj} = 1 - \frac{\theta \varphi\left(\lambda_{\frac{1}{2} + \epsilon_j}\right)}{\sqrt{2}} \cdot (2c_1)$$

$$\frac{\frac{-8\epsilon_j}{1-4\epsilon_j^2} + 2\epsilon_j \left(\frac{1-2\epsilon_j}{1+2\epsilon_j} \right) \left(\frac{(1-2\epsilon_j)(2) - (1+2\epsilon_j)(-2)}{(1-2\epsilon_j)^2} \right) + 2 \log \frac{1+2\epsilon_j}{1-2\epsilon_j}}{\left[\log(1-4\epsilon_j^2) + 2\epsilon_j \log \frac{1+2\epsilon_j}{1-2\epsilon_j} \right]^2}$$

$$(2.5) \quad \theta = \frac{\sqrt{2} \left[\log(1-4\epsilon_j^2) + 2\epsilon_j \log \frac{1+2\epsilon_j}{1-2\epsilon_j} \right]^2}{4c_1 Q \left(\lambda_{\frac{1}{2}+\epsilon_j} \right) \log \frac{1+2\epsilon_j}{1-2\epsilon_j}}$$

It can be demonstrated in each case above that the solution presented does give the minimum number of observations for carrying out the test.

Formulas (2.3) and (2.4) are used to obtain the value of j which minimizes $n_0(S_j)$, while (2.3) and (2.5) are used to find the value of j which minimizes $n_1(S_j)$. For example, for given probabilities α and β of type I and type II error, under H_0 θ is given by (2.4) as a function of ϵ_j only. Thus, choosing a value of ϵ_j , the corresponding θ may be calculated and (2.3) then used to obtain the j which minimizes $n_0(S_j)$.

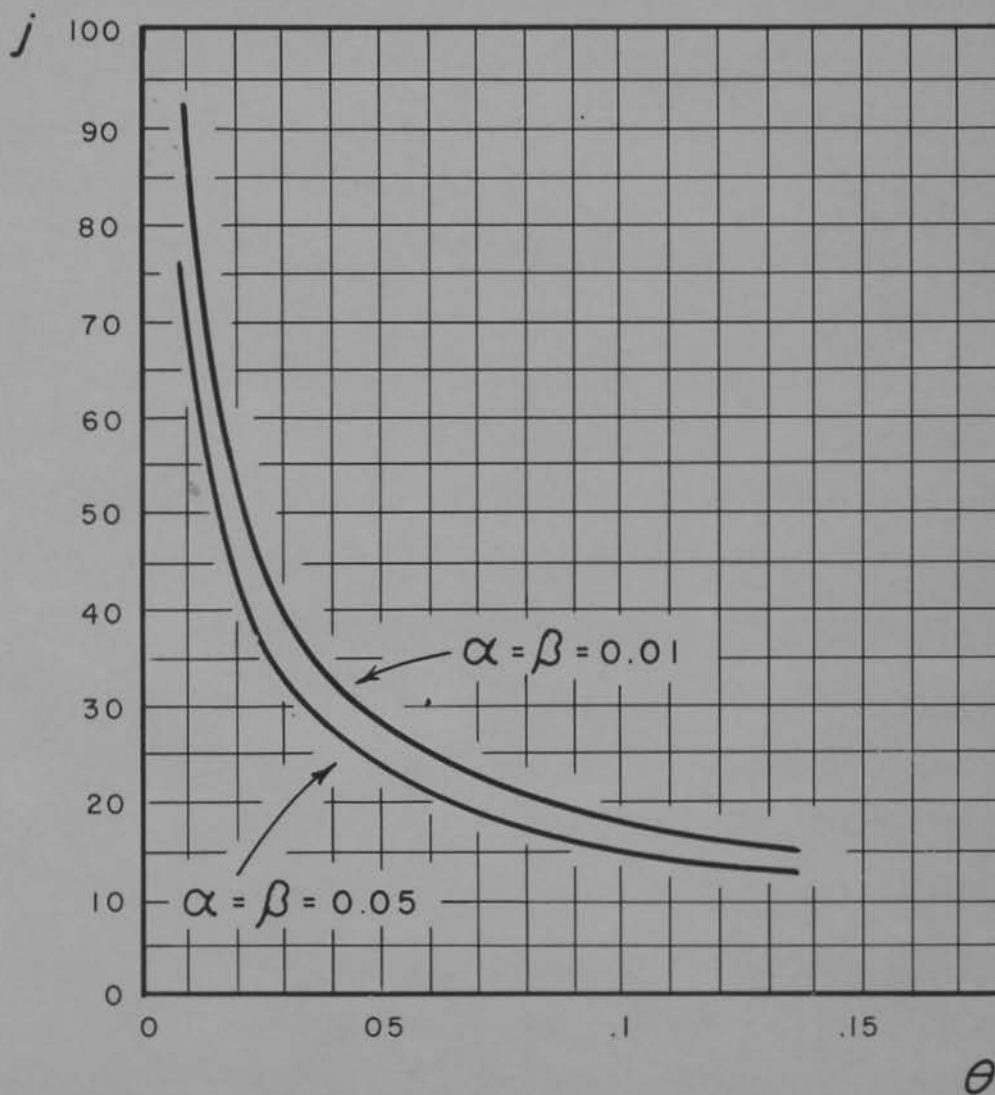
Such computations were carried out for four cases:

(i) $\alpha = \beta = 0.05$, (ii) $\alpha = \beta = 0.01$,

(iii) $\alpha = 0.05$, $\beta = 0.01$, (iv) $\alpha = 0.01$, $\beta = 0.05$.

It was found that when $\alpha = \beta$, the same value of j could be used to minimize both $n_0(S_j)$ and $n_1(S_j)$. Fig. 2.1 shows curves which enable one to estimate this j for given θ for cases (i) and (ii). In case (iii) the j required to minimize $n_0(S_j)$ is a little smaller than that shown for case (ii) while the j required to minimize $n_1(S_j)$ is a little larger than that shown for case (i). In case (iv)

FIGURE 2-1

VALUE OF j FOR APPLICATION OF THE S_j -TEST

the j required to minimize $n_0(S_j)$ is a little larger than that shown for case (i) while the j required to minimize $n_1(S_j)$ is a little smaller than that shown for case (ii). Since different values of j would be required in cases (iii) and (iv) to minimize $n_0(S_j)$ and $n_1(S_j)$, the corresponding curves are not shown on the chart. It is suggested that in either of these cases a value of j between those shown for cases (i) and (ii) would be appropriate.

After j is chosen, we may calculate

$$\lambda_{\frac{1}{2} + \epsilon_j} = \frac{j\theta}{\sqrt{2}}$$

and use tables of the standard normal distribution to obtain

$$\frac{1}{2} + \epsilon_j = \Phi\left(\lambda_{\frac{1}{2} + \epsilon_j}\right)$$

The sequential binomial test is then carried out as the test of $H_0: p = \frac{1}{2}$ against $H_1: p = \frac{1}{2} + \epsilon_j$ as indicated in the introduction.

3. EFFECT OF TRUNCATION

As described, the S_j -test is truncated with j comparisons ($2j$ observations). Thus, the expected numbers of observations required by the test will not generally be given by $n_0(S_j)$ and $n_1(S_j)$, and the probabilities of type I and type II errors will be changed. It is desirable to ascertain the effect of truncation on the test.

Wald states that truncation will have no great effect on α and β if the truncation number (j) is two or three times as large as the expected number of observations (n_0 or n_1). To determine the relationship between j and n_0 , n_1 , by using formulas (1.8), (2.3), (2.4) and (2.5) we find

$$\frac{n_0}{j} = \frac{-(1-4\epsilon_j^2) \log(1-4\epsilon_j^2)}{8 \epsilon_j \lambda_{\frac{1}{2}+\epsilon_j} \Phi\left(\lambda_{\frac{1}{2}+\epsilon_j}\right)}$$

$$\frac{n_1}{j} = \frac{\log(1-4\epsilon_j^2) + 2\epsilon_j \log \frac{1+2\epsilon_j}{1-2\epsilon_j}}{2 \lambda_{\frac{1}{2}+\epsilon_j} \Phi\left(\lambda_{\frac{1}{2}+\epsilon_j}\right) \log \frac{1+2\epsilon_j}{1-2\epsilon_j}}$$

As $\epsilon_j \rightarrow 0$, $\frac{n_0}{j} \rightarrow \frac{1}{2}$ and $\frac{n_1}{j} \rightarrow \frac{1}{2}$

Note that the ratios $\frac{n_0}{j}$ and $\frac{n_1}{j}$ depend on ϵ_j only, not on α and β , although for fixed θ , ϵ_j of course depends on α and β . In practice if alternatives θ such that $|\theta| < 0.1$ are of interest, computations using the formulas of Chapter 2 indicate that for $\alpha, \beta \geq 0.01$, $\epsilon_j \leq 0.40$. For ϵ_j in this range Table 3.1 shows the values of the ratios $\frac{n_0}{j}$ and $\frac{n_1}{j}$.

From this table we see that under H_0 the truncation number is approximately twice the expected number of observations throughout the range of interest, while under H_1 the situation is apparently less favorable, and it might be expected that there would be more effect on the OC function.

The effect of truncation can be determined more precisely by using the procedure described in Stockman and Armitage [10] to obtain the probability that a decision will not be reached in j comparisons ($2j$ observations). Consider again the Wald SPRT for the binomial hypothesis $H_0: p = \frac{1}{2}$ against the alternative $H_1: p = \frac{1}{2} + \epsilon_j$

Table 3.1.

Ratios of Expected Numbers of Observations to the
Truncation Number

ϵ_j	$\frac{n_0}{j}$	$\frac{n_1}{j}$
0.05	0.500	0.502
0.10	0.500	0.507
0.15	0.501	0.517
0.20	0.502	0.533
0.25	0.503	0.556
0.30	0.505	0.590
0.35	0.508	0.645
0.40	0.511	0.745

described in Chapter 1. If we let $Y_m = m - X_m$, the testing procedure can be described as

terminate sampling with

$$\text{acceptance of } H_0 \text{ if } Y_m \geq \frac{1-s}{s} X_m - \frac{h_0}{s}$$

$$\text{acceptance of } H_1 \text{ if } Y_m < \frac{1-s}{s} X_m - \frac{h_1}{s}$$

take another observation if

$$\frac{1-s}{s} X_m - \frac{h_1}{s} < Y_m < \frac{1-s}{s} X_m - \frac{h_0}{s}$$

where h_0 , h_1 , s are given by (1.7).

In particular, considering the S_j -test for the special case $\theta = 0.04$, and setting $\alpha = \beta = 0.05$, it is found from Fig. 2.1 that $j = 28$, and thus from formula (1.11)

$\epsilon_j = 0.29$. Using formulas (1.7) we calculate

$$-h_0 = h_1 = 2.222345, \quad s = 0.6547549$$

Therefore,

$$\frac{1-s}{s} = 0.527289, \quad \frac{h}{s} = 3.39416$$

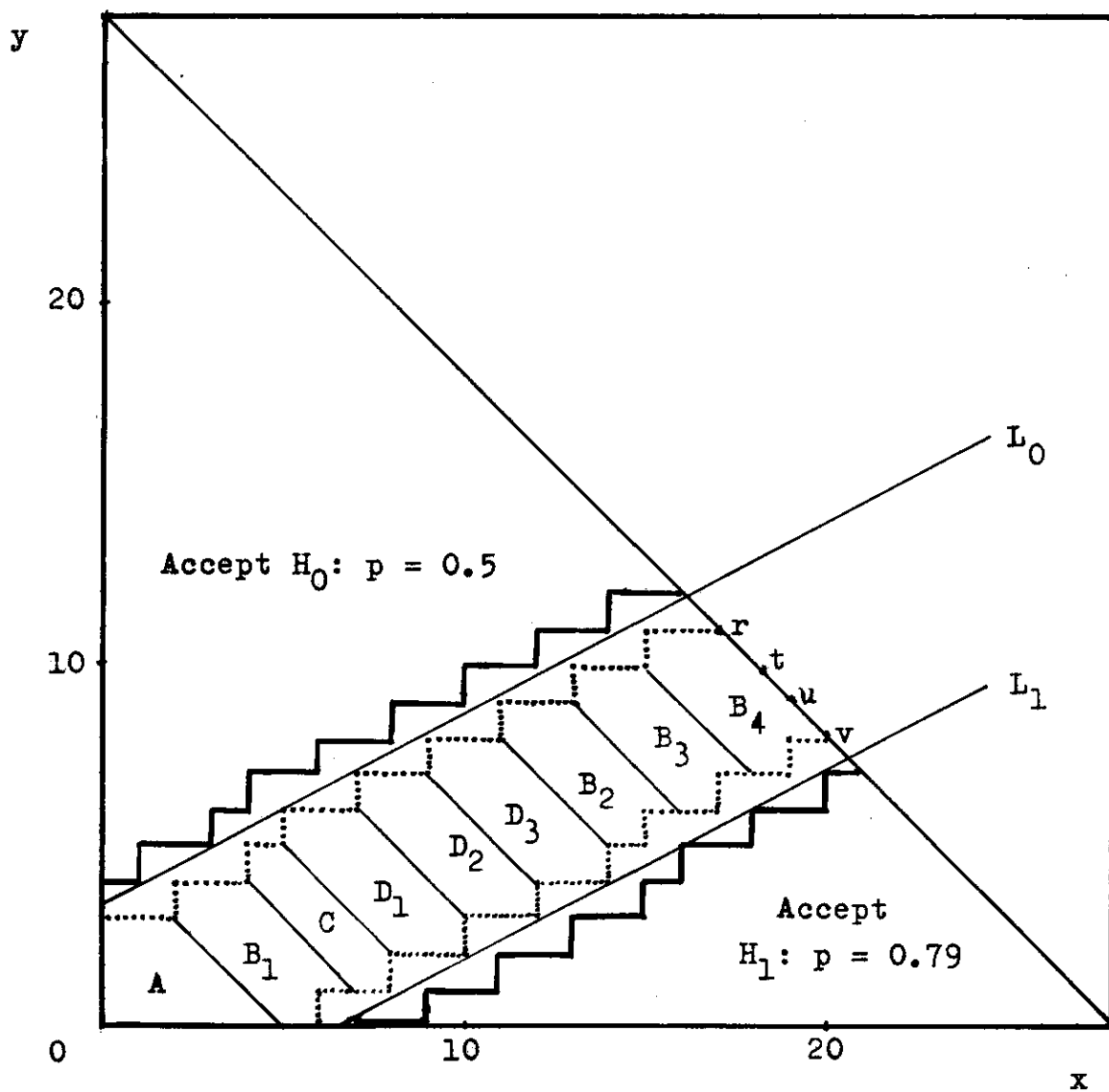
The lines corresponding to the equations

$$L_0: y = 0.527289x + 3.39416 \text{ and } L_1: y = 0.527289x - 3.39416,$$

which define the acceptance and rejection regions of the testing procedure are shown in Fig. 3.1.

Fig. 3.1.

Acceptance and Rejection Regions for the S_j -Test: $\theta = 0.04$



The solid boundary lines in this diagram indicate the smallest integer values of X_m and Y_m for which H_0 or H_1 may be accepted. The dotted boundary lines are drawn at unit distance from outer boundary lines, and the region enclosed divided into blocks by diagonal lines corresponding to various fixed values of m .

To find the probability that the test will not terminate with $n \leq 28$, one must find the number of admissible paths, i.e. paths in the closed region bounded by the dotted lines, which lead to each of the points on the right boundary of block B_4 , since the probability of being at the point $P:(x,y)$ with the test not terminated is equal to

$$p^x q^y \text{ (no. of admissible paths to } P\text{)}$$

For each block set up a matrix such that the element in the i th row, j th column represents the number of admissible paths from the i th point on the left boundary to the j th point on the right boundary of the block (numbering points from the top of the block). Take '0' as the only point on the left boundary of block A. Then, in the product of two matrices corresponding to adjacent blocks the element in the i th row, j th column will represent the number of admissible paths from the i th point on the left

boundary of the first block to the j th point on the right boundary of the second block. Thus, the numbers of admissible paths from 0 to points of the right boundary of B_4 are given by the elements of the matrix $ABCD^3B^3$. If we let $C(n,r)$ represent the number of combinations of n things taken r at a time, the elements of these matrices may be written

$$A = (C(5,3) \quad C(5,2) \quad C(5,1) \quad C(5,0))$$

$$= (10 \quad 10 \quad 5 \quad 1)$$

$$B = \begin{pmatrix} C(3,1) & C(3,0) & 0 & 0 \\ C(3,2) & C(3,1) & C(3,0) & 0 \\ C(3,3) & C(3,2) & C(3,1) & C(3,0) \\ 0 & C(3,3) & C(3,2) & C(3,1) - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} C(2,1) & C(2,0) & 0 & 0 \\ C(2,2) & C(2,1) & C(2,0) & 0 \\ 0 & C(2,2) & C(2,1) & C(2,0) \\ 0 & 0 & C(2,2) & C(2,1) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} C(3,1) & C(3,0) & 0 & 0 \\ C(3,2) & C(3,1) & C(3,0) & 0 \\ C(3,3) & C(3,2) & C(3,1) & C(3,0) \\ 0 & C(3,3) & C(3,2) & C(3,1) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 3 \end{pmatrix}$$

Using the above we have

$$ABCD^3B^3 = (10,863,774 \quad 9,022,602 \quad 4,569,182 \quad 1,230,305)$$

and the probability of not terminating with $n \leq 28$, i.e.

the probability of staying within the region bounded by the dotted lines and arriving at one of the points on the right boundary of B_4 is given by

$$10,863,774p^{17}q^{11} + 9,022,602p^{18}q^{10} + 4,569,182p^{19}q^9 \\ + 1,230,305p^{20}q^8$$

Since under H_0 , $p = q = \frac{1}{2}$, while under H_1 , $p = 0.79$, $q = 0.21$, we calculate

$$P(\text{not term. with } n \leq 28 \mid H_0) = 0.09568$$

$$P(\text{not term. with } n \leq 28 \mid H_1) = 0.11144$$

Thus, as implied by Table 3.1, the probability that the test has not terminated by the truncation point is greater under H_1 than under H_0 .

To determine the effect of truncation on the OC function, consider for the i th point on the left boundary of each block the probability that the test will terminate with a number of observations less than or equal to the number required to reach the right boundary of the block. These probabilities are shown in Table 3.2. Then we have

Table 3.2.

Probability That a Path Starting at the i th Point on the Left Boundary of a Block Will Lead to a Decision Without Crossing the Right Boundary of the Block

Block	Probability of Accepting H_0	Probability of Accepting H_1
A	$q^4 + 4pq^4$	
B	$i=1 \quad q^2 + 2pq^2$ $i=2 \quad q^3$	$i=4 \quad p^2$
C	$i=1 \quad q^2$	$i=4 \quad p^2$
D	$i=1 \quad q^2 + 2pq^2$ $i=2 \quad q^3$	$i=4 \quad p^3$

$$\begin{aligned}
(3.1) \quad P(\text{accepting } H_0 \text{ with } n \leq 28) &= q^4 + 4pq^4 + q^2 n_1(C) p^4 q^4 \\
&+ (q^2 + 2pq^2) \left(n_1(B_1) p^2 q^3 + n_1(D_1) p^5 q^5 + n_1(D_2) p^7 q^6 \right. \\
&\quad \left. + n_1(D_3) p^9 q^7 + n_1(B_2) p^{11} q^8 + n_1(B_3) p^{13} q^9 \right. \\
&\quad \left. + n_1(B_4) p^{15} q^{10} \right) \\
&+ q^3 \left(n_2(B_1) p^3 q^2 + n_2(D_1) p^6 q^4 + n_2(D_2) p^8 q^5 + n_2(D_3) p^{10} q^6 \right. \\
&\quad \left. + n_2(B_2) p^{12} q^7 + n_2(B_3) p^{14} q^8 + n_2(B_4) p^{16} q^9 \right)
\end{aligned}$$

$$\begin{aligned}
(3.2) \quad P(\text{accepting } H_1 \text{ with } n \leq 28) &= \\
&p^2 \left(n_4(B_1) p^5 + n_4(C) p^7 q + n_4(B_2) p^{14} q^5 + n_4(B_3) p^{16} q^6 \right. \\
&\quad \left. + n_4(B_4) p^{18} q^7 \right) \\
&+ p^3 \left(n_4(D_1) p^8 q^2 + n_4(D_2) p^{10} q^3 + n_4(D_3) p^{12} q^4 \right)
\end{aligned}$$

where $n_i(B_j)$, $n_i(C)$, $n_i(D_k)$ represent the number of admissible paths leading to the i th point on the left boundary of B_j , C , D_k respectively. The values of these are shown in Table 3.3.

Formulas (3.1) and (3.2) give

$$P(\text{accepting } H_0 \text{ with } n \leq 28 \mid H_0) = 0.87013$$

$$P(\text{accepting } H_1 \text{ with } n \leq 28 \mid H_0) = 0.03418$$

Table 3.3.

Number of Admissible Paths Leading to Points on
Block Boundaries

The number of admissible paths to the i th point on the left boundary of each block is given by the i th element of the matrix shown.

Block	Matrix Giving Number of Admissible Paths
B_1	$A = (10 \quad 10 \quad 5 \quad 1)$
C	$AB = (65 \quad 56 \quad 28 \quad 7)$
D_1	$ABC = (186 \quad 205 \quad 119 \quad 42)$
D_2	$ABCD = (1292 \quad 1200 \quad 688 \quad 245)$
D_3	$ABCD^2 = (8164 \quad 7201 \quad 3999 \quad 1423)$
B_2	$ABCD^3 = (50,094 \quad 43,187 \quad 23,467 \quad 8268)$
B_3	$ABCD^3B = (303,310 \quad 258,324 \quad 138,392 \quad 40,003)$
B_4	$ABCD^3B^2 = (1,823,294 \quad 1,533,461 \quad 793,509 \quad 218,398)$

$$P(\text{accepting } H_0 \text{ with } n \leq 28 \mid H_1) = 0.03299$$

$$P(\text{accepting } H_1 \text{ with } n \leq 28 \mid H_1) = 0.85555$$

To determine the actual probabilities of type I and type II error for the truncated test, we must consider the points on the right boundary of B_4 and determine a decision rule for sequences of observations which would correspond to admissible paths leading to one of these points. The rule generally used is to accept H_0 if the

likelihood function $\frac{P_{1m}}{P_{0m}} < 1$ and to accept H_1 if

$\frac{P_{1m}}{P_{0m}} > 1$. If we label the points on the right boundary of B_4 r, t, u, v (See Fig. 3.1), we find that this rule leads to acceptance of H_0 for r and t and to acceptance of H_1 for u and v . Thus, for the truncated test, the probabilities of type I and type II error, α' and β' , will be given by

$$\begin{aligned} \alpha' &= P(\text{accepting } H_1 \text{ with } n \leq 28 \mid H_0) + P(u, v \mid H_0) \\ &= 0.05578 \end{aligned}$$

$$\begin{aligned} \beta' &= P(\text{accepting } H_0 \text{ with } n \leq 28 \mid H_1) + P(r, t \mid H_1) \\ &= 0.06153 \end{aligned}$$

In the calculation of probabilities by formulas (3.1) and (3.2) one obtains the probability that the sequential test will terminate with exactly n observations for $n \leq 28$. Using these probabilities we find that for the truncated binomial test the expected number of observations is 13.79 under H_0 and 15.41 under H_1 . Thus for the truncated S_j -test, with $j = 28$, we have the expected numbers of observations equal to $n_0'(S_j) = 41.79$ and $n_1'(S_j) = 43.41$. Comparing these numbers with those obtained by Wald's formulas for the non-truncated test, we find $n_0(S_j) = 40.92$, $n_1(S_j) = 42.79$. Certainly the expected numbers of observations should be smaller for the truncated than for the non-truncated test; the apparent discrepancy here is due to the fact that Wald's formulas do not take into account the excess over boundaries on termination, i.e. they assume that a path terminates exactly on one of the lines L_0 or L_1 (see Fig. 3.1), rather than on the solid boundary line.

The procedure described above could be carried out for various values of θ and for other choices of α , β to get a better idea of the general effect of truncation. Judging from the above, it would seem that the expected numbers of observations for the truncated test will

generally be at least as large as the values given by Wald's formulas, and these formulas might provide reasonable approximations. The probabilities of error, α' and β' , of the truncated test were in this case a little larger than the chosen values of α and β . This might be true in general, but the relation between these probabilities would probably depend strongly on the slopes of the lines L_0 and L_1 and the shapes of the blocks bounded by the lines.

To ascertain the effect of varying the value of j chosen for any fixed θ , the computations described in this chapter were carried out also for $j = 27, 29, \text{ and } 30$. These results, together with those shown previously, are presented in Table 3.4.

Table 3.4.

Probabilities of Error (α' and β') and Expected Numbers of Observations ($n_0'(S_j)$ and $n_1'(S_j)$) of the Truncated S_j -Test, with $\theta = 0.04$, $\alpha = \beta = 0.05$.

$n_0(S_j)$ and $n_1(S_j)$ represent the estimated expected numbers of observations of the non-truncated test (obtained by using Wald's approximation formulas).

j	ϵ_j	α'	β'	$n_0'(S_j)$	$n_1'(S_j)$	$n_0(S_j)$	$n_1(S_j)$
27	0.28	0.0695	0.0623	41.65	42.98	41.09	42.94
28	0.29	0.0558	0.0615	41.79	43.41	40.92	42.79
29	0.29	0.0726	0.0444	42.89	44.52	41.92	43.79
30	0.30	0.0575	0.0428	43.17	45.07	41.88	43.75

4. COMPARISON OF THE S_j -TEST WITH MANN'S T TEST

4.1. Efficiency of the S_j -Test Relative to the T Test.

Noether [5] has compared the S_j -test with several non-parametric fixed-sample-size tests of randomness and has found that the S_j -test will in general require fewer observations on the average for given probabilities α , β of type I and type II error. The best non-parametric test of randomness against the alternative of linear trend is the T test proposed by Mann [4]. The definition of T is given as follows. For a given sequence of n observations x_1, x_2, \dots, x_n , let

$$\begin{aligned} y_{ik} &= 1 && \text{if } x_i < x_k \\ &= 0 && \text{if } x_i > x_k \end{aligned}$$

$$1 \leq i < k \leq n$$

and define

$$(4.1) \quad T = \sum_{i=1}^{n-1} \sum_{k=i+1}^n y_{ik}$$

Mann has shown that

$$(4.2) \quad E_0(T) = \frac{n(n-1)}{4}$$

$$(4.3) \quad \sigma_0^2(T) = \frac{n(n-1)(2n+5)}{72}$$

$$(4.4) \quad E_1(T) = \frac{n(n-1)}{4} + \sum_{i=1}^{n-1} \sum_{k=i+1}^n \epsilon_{ik}$$

where $\epsilon_{ik} = P(y_{ik} = 1) - \frac{1}{2}$

The comparison of the S_j -test with the T test is complicated by the fact that there is no expression available for the variance of T under H_1 . Noether has stated that if $\sigma_1^2(T)$ satisfies a specified condition, it then follows that at least for sufficiently small values of θ the S_j -test requires fewer observations on the average (is more efficient) than the T test.

Two different methods have been used to try to answer this question of efficiency of the S_j -test relative to the T test: (1) the average number of observations required by the S_j -test is compared with the number of observations required by its fixed-sample-size analogue, the B_j -test; the B_j -test is in turn compared with the T test by finding the asymptotic relative efficiency of the B_j -test with respect to the T test. (2) A formula estimating $\sigma_1^2(T)$ is obtained and used to approximate the number of observations required by the T test, this number then being compared with the expected number of observations required by the S_j -test.

To compare first the B_j -test with the T test the definition of asymptotic relative efficiency given by Pitman is used (see, for example, Noether [6]). The procedure and notation is similar to that used by Stuart [11], [12].

The asymptotic relative efficiency may be obtained as follows. Suppose the hypothesis $H_0: \theta = \theta_0$ is being tested against the alternative $H_1: \theta = \theta_1 > \theta_0$ and two test statistics t_1 and t_2 are being considered. For integral $k \geq 1$, suppose

$$\begin{aligned} \left. \frac{d^k E(t_i)}{d\theta^k} \right|_{\theta=\theta_0} &= 0 && \text{for } k < m_i \\ &\neq 0 && \text{for } k = m_i \end{aligned}$$

Let

$$E'(t_i) = \left. \frac{d^{m_i} E(t_i)}{d\theta^{m_i}} \right|_{\theta=\theta_0}, \quad \sigma_0^2(t_i) = \text{var}(t_i | \theta=\theta_0)$$

$$I_i = \frac{(E'(t_i))^2}{\sigma_0^2(t_i)}$$

and define c_i by

$$(4.5) \quad I_i = c_i^2 n^{2m_i} \delta_i$$

Then the asymptotic relative efficiency of t_1 compared to t_2 is

$$(4.6) \quad \text{A.R.E.}(t_1, t_2) = 0 \quad \text{if } \delta_1 < \delta_2$$

$$= \left[\frac{c_1}{c_2} \right]^{\frac{1}{m\delta}} \quad \text{if } \delta_1 = \delta_2 = \delta$$

and $m_1 = m_2 = m$

In carrying out a fixed-sample-size B_j -test the first problem, as in the S_j -test, is the choice of j for any given alternative θ . If N represents the number of observations required by the test, it will also be determined whether we should take $j \geq \frac{N}{2}$, or whether a choice $j < \frac{N}{2}$ would correspond to a smaller value of N for given θ . This latter choice would correspond to the situation where x_i is compared with x_{i+j} for $i \leq j$, then x_i is compared with x_{i+j} for $2j+1 \leq i \leq 3j$, ..., x_i is compared with x_{i+j} for $2(g-1)j+1 \leq i \leq 2(g-1)j+k$, where $k \leq j$; i.e., there are g groups of observations to be compared, with j pairs of observations in each of the first $g-1$ groups, k pairs of observations (but total of $j+k$ observations) in the g th group. The total number of observations required by the test would be $N = (2g-1)j + k$.

Thus letting

$$\begin{aligned} H_{is} &= 1 && \text{if } x_i > x_s \\ &= 0 && \text{if } x_i < x_s \end{aligned}$$

we can represent the statistic $y = \sum H_{is}$ to be used in the B_j -test as

$$(4.7) \quad y = \sum_{h=0}^{g-2} \sum_{i=1}^j H_{2hj+i, 2hj+j+i} + \sum_{i=0}^k H_{2(g-1)j+i, 2(g-1)j+j+i}$$

where $j \geq 1$, $k \leq j$, $(2g-1)j + k = N$

Under the assumption that x_i , x_s are normally distributed with variance 1, and therefore that $x_i - x_s$ is normally distributed with mean $(i-s)\theta$, variance 2, Stuart [11] shows that

$$\left. \frac{dE(H_{is})}{d\theta} \right|_{\theta=0} = \frac{i-s}{2\sqrt{\pi}}$$

From this we find that

$$E'(y) = \left. \frac{dE(y)}{d\theta} \right|_{\theta=0} = \frac{-j}{2\sqrt{\pi}}$$

where summation is to be taken for all terms in (4.7). The number of these terms is seen to be $\frac{N-j+k}{2}$. Therefore

$$E'(y) = \frac{-j(N-j+k)}{4\sqrt{\pi}}$$

Since y is binomially distributed, with $p = \frac{1}{2}$ under H_0 ,

$$\sigma_0^2(y) = \frac{N-j+k}{2} \cdot \frac{1}{4} = \frac{N-j+k}{8}$$

Thus

$$\begin{aligned} I &= \frac{(E'(y))^2}{\sigma_0^2(y)} = \frac{j^2(N-j+k)^2}{16\pi} \cdot \frac{8}{N-j+k} \\ &= \frac{j^2(N-j+k)}{2\pi} \end{aligned}$$

Since the efficiency of a test increases with I , then for any fixed sample size N , j should be chosen to maximize I .

$$\text{Let } j = aN, \quad k = bN$$

$$\text{where } 0 < a < 1, \quad b \leq a, \quad b \leq 1-a$$

Then we have

$$I = \frac{a^2(1-a+b)N^3}{2\pi}$$

If $0 < a \leq \frac{1}{2}$, since $b \leq a$, we must have $a^2(1-a+b) \leq a^2$.

Thus maximum value of $a^2(1-a+b)$ is $\frac{1}{4}$ (for $a = b = \frac{1}{2}$).

If $\frac{1}{2} \leq a < 1$, since $b \leq 1-a$, we must have $a^2(1-a+b) \leq 2a^2(1-a)$. Thus maximum value of $a^2(1-a+b)$ is $\frac{8}{27}$ (for $a = \frac{2}{3}$, $b = \frac{1}{3}$).

This gives

$$\max I = \frac{4N^3}{27\pi}$$

The test obtained in this way, taking $j = \frac{2N}{3}$, corresponds to the test S_3 defined by Cox and Stuart [3].

Using (4.5) we have for the B_j -test

$$m = 1, \quad c^2 = \frac{4}{27\pi}, \quad \delta = \frac{3}{2}$$

Stuart [12] has found for the T test

$$m = 1, \quad c^2 = \frac{1}{4\pi}, \quad \delta = \frac{3}{2}$$

Thus, from (4.6)

$$\text{A.R.E.}(B_j, T) = \left[\frac{4}{27\pi} \bigg/ \frac{1}{4\pi} \right]^{\frac{1}{3}} = \left[\frac{16}{27} \right]^{\frac{1}{3}} \approx 0.84$$

To compare the number of observations required by the B_j -test with the expected number of observations required by the S_j -test, first of all we see that for the S_j -test

$$n_0(S_j) = j + \frac{2c_0}{\log(1-4\epsilon_j^2)}$$

where j is found from formulas (2.3) and (2.4) or by using Fig. 2.1.

For given θ the value of j , say j' , required by the B_j -test will be different from the value of j required by the S_j -test. The number of observations required by the B_j -test will be $N = 3r$, where $r = \frac{j'}{2}$. An approximation to the value of N may be made by noticing that r represents the number of observations required by the fixed sample binomial test for the hypothesis $H_0: p = \frac{1}{2}$ against the alternative $H_1: p_r = \frac{1}{2} + \epsilon_{2r}$ with α, β as probabilities of type I and type II error.

If we let

$$y = \sum_{i=1}^r x_i$$

represent the sum of the r binomial observations, then we have

$$\text{under } H_0: \quad E_0(y) = \frac{r}{2}, \quad \sigma_0^2(y) = \frac{r}{4}$$

$$\text{under } H_1: \quad E_1(y) = rp_r, \quad \sigma_1^2(y) = rp_r(1-p_r)$$

where $p_r = P(x_i > x_{i+2r}) = \frac{1}{2} + \epsilon_{2r}$

As in (1.11) we can approximate

$$p_r = \Phi\left(\frac{2r\theta}{\sqrt{2}}\right) = \Phi(\sqrt{2} r\theta)$$

Then, approximating the distribution of y with the nor-

mal distribution, and using formula (2.3) of Noether [5] ,

$$r = \left[\frac{\lambda_{1-\alpha} + 2 \lambda_{1-\beta} \sqrt{(\Phi(\sqrt{2} r\theta))(1-\Phi(\sqrt{2} r\theta))}}{2 \Phi(\sqrt{2} r\theta) - 1} \right]^2$$

The comparison of $n(S_j)$ with N of the B_j -test is thus complicated by the fact that in neither case is the required number of observations given as a function of θ alone, but is expressed as a function of j and θ .

A rough approximation to the ratio $\frac{N}{n(S_j)}$ under H_0 may be obtained by using the fact that a sequential test uses on the average approximately one-half as many observations as the corresponding non-sequential test, and determining the value of this ratio, first using the j of the S_j -test, then using the j of the B_j -test.

Consider the j of the S_j -test, which should give an upper bound for the ratio. Since we have seen in Chapter 3 that the ratio $\frac{n_0}{j} \sim \frac{1}{2}$, we have

$$n_0(S_j) \sim j + \frac{1}{2} = \frac{3j}{2}$$

$$N \sim j + j = 2j$$

$$\frac{N}{n_0(S_j)} \sim \frac{4}{3} = 1.33$$

Consider the j of the B -test, which should give a lower bound for the ratio.

$$N = 2r + r = 3r$$

$$n_0(S_j) \sim 2r + \frac{r}{2} = \frac{5r}{2}$$

$$\frac{N}{n_0(S_j)} \sim \frac{6}{5} = 1.2$$

Computations carried out for values of θ equal to 0.08, 0.04, 0.02, 0.01, 0.002, for $\alpha = \beta = 0.05$, yielded in each case a value for the ratio slightly larger than 1.24.

If we can consider the value of A.R.E.(B_j, T) as indicating approximately, at least for θ small, the ratio of the number of observations required by the T test to the number required by the B_j -test, then we might consider the efficiency of the S_j -test relative to the T test as given by

$$\frac{N}{n_0(S_j)} \text{ A.R.E.}(B_j, T) \sim (1.24) \left(\frac{16}{27} \right)^{\frac{1}{3}} \sim 1.04$$

at least for the values of θ investigated. This indicates that the S_j -test probably requires slightly fewer observations than the T test.

The fact that there is only a slight difference between the two test was demonstrated also by the selection of 18 samples for the example of Chapter 3, i.e. for the alternative $\theta = 0.04$, using 44 observations (which is the approximate value of $n_1^!(S_j)$) for the T test. All samples led to rejection of H_0 by both tests, with the mean number of observations for the S_j -test equal to 44.6.

4.2. Variance of T Under H₁.

An alternative method of comparing the S_j -test with the T test is to obtain an estimate of $\sigma_1^2(T)$. Such an estimate was found under the assumption that X follows the rectangular distribution. The expression obtained might be expected to approximate $\sigma_1^2(T)$ for other types of distributions provided that probabilities of the type $P(x_i < x_j < x_k)$, $P(x_j < x_i < x_k)$, etc., do not depend strongly on the type of distribution. (For convenience it is assumed in this section that the X_i follow an upward trend.)

From (4.1) and the definition of variance

$$(4.8) \quad \sigma^2(T) = \sum_{i < k} \sigma^2(y_{ik}) + 2 \sum_{\substack{i < k \\ j < k}} \text{cov}(y_{ik}y_{jk}) \\ + 2 \sum_{i < j < k} \text{cov}(y_{ij}y_{jk})$$

Since y_{ik} is a binomial variable

$$\sum_{i < k} \sigma^2(y_{ik}) = \sum_{i < k} \left(\frac{1}{4} - \epsilon_{ik}^2 \right) \\ = \frac{n(n-1)}{2} \cdot \frac{1}{4} - \sum_{i < k} \epsilon_{ik}^2$$

where

$$(4.9) \quad \epsilon_{ik} = \frac{(k-i)\theta}{2\sqrt{3}} \left(1 - \frac{(k-i)\theta}{4\sqrt{3}} \right)$$

$$\begin{aligned} \sum_{i < k} \sigma^2(y_{ik}) &= \frac{n(n-1)}{8} - \frac{\theta^2}{12} \sum_{i < k} (k-i)^2 \\ &+ \frac{\theta^3}{24\sqrt{3}} \sum_{i < k} (k-i)^3 - \frac{\theta^4}{576} \sum_{i < k} (k-i)^4 \end{aligned}$$

Since

$$\sum_{i < k} (k-i)^2 = \frac{n^2(n^2-1)}{12}$$

$$\sum_{i < k} (k-i)^3 = \frac{n(n^2-1)(3n^2-2)}{60}$$

$$\sum_{i < k} (k-i)^4 = \frac{n^2(n^2-1)(2n^2-3)}{60}$$

we obtain

$$\begin{aligned} (4.10) \quad \sum_{i < k} \sigma^2(y_{ik}) &= \frac{n(n-1)}{8} - \frac{n^2(n^2-1)}{144} \theta^2 \\ &+ \frac{n(n^2-1)(3n^2-2)}{1440\sqrt{3}} \theta^3 - \frac{n^2(n^2-1)(2n^2-3)}{34560} \theta^4 \end{aligned}$$

To find the covariances,

$$\text{cov}(y_{ij}y_{jk}) = P(x_i < x_j < x_k) - P(x_i < x_j) P(x_j < x_k)$$

Assume that X_1 follows the rectangular distribution in the interval $0 \leq X_1 \leq 1$, that X_2 follows the rectangular distribution in the interval $\theta_1 \leq X_2 \leq 1+\theta_1$, that X_3 follows the rectangular distribution in the interval

$\theta_1 + \theta_2 \leq X_3 \leq 1 + \theta_1 + \theta_2$ where $0 < \theta_1 < \theta_1 + \theta_2 < 1$. This will give as distribution functions of X_1, X_2, X_3

$$(4.11) \quad F_1(x) = x, \quad F_2(x) = x - \theta_1, \quad F_3(x) = x - (\theta_1 + \theta_2)$$

$$\begin{aligned} P(x_1 < x_2 < x_3) &= \int_{\theta_1}^{\theta_1 + \theta_2} x-1 \, dx + \int_{\theta_1 + \theta_2}^1 x(1 + \theta_1 + \theta_2 - x) \, dx \\ &\quad + \int_1^{1 + \theta_1} 1(1 + \theta_1 + \theta_2 - x) \, dx \\ &= \frac{1 + 6\theta_1\theta_2 + 3(\theta_1 + \theta_2) - (\theta_1 + \theta_2)^3}{6} \end{aligned}$$

Now let

$$(4.12) \quad \theta_1 = \frac{\theta}{\sqrt{12}}(j-i), \quad \theta_2 = \frac{\theta}{\sqrt{12}}(k-j), \quad \theta_1 + \theta_2 = \frac{\theta}{\sqrt{12}}(k-i)$$

$$P(x_i < x_j < x_k) = \frac{1}{6} \left(1 + \frac{\theta^2}{2}(j-i)(k-j) + \frac{3\theta}{\sqrt{12}}(k-i) - \frac{\theta^3}{24\sqrt{3}}(k-i)^3 \right)$$

$$\text{cov}(y_{ij}y_{jk}) = \frac{1}{6} \left(1 + \frac{\theta^2}{2}(j-i)(k-j) + \frac{3\theta}{2\sqrt{3}}(k-i) - \frac{\theta^3}{24\sqrt{3}}(k-i)^3 \right)$$

$$- \left(\frac{1}{2} + \frac{(j-i)\theta}{2\sqrt{3}} \left(1 - \frac{(j-i)\theta}{4\sqrt{3}} \right) \right) \left(\frac{1}{2} + \frac{(k-j)\theta}{2\sqrt{3}} \left(1 - \frac{(k-j)\theta}{4\sqrt{3}} \right) \right)$$

$$= -\frac{1}{12} + \frac{\theta^2}{48}((k-j)^2 + (j-i)^2) - \frac{\theta^3}{144\sqrt{3}}((k-j)^3 + (j-i)^3) \\ - \frac{\theta^4}{576}(j-i)^2(k-j)^2$$

Note that this formula can be applied for $\frac{k\theta}{\sqrt{12}} < 1$.

$$\text{cov}(y_{ik}y_{jk}) = P(x_i < x_k, x_j < x_k) - P(x_i < x_k) P(x_j < x_k)$$

$$\text{where } P(x_i < x_k, x_j < x_k) = P(x_i < x_j < x_k) + P(x_j < x_i < x_k)$$

Using (4.11)

$$P(x_2 < x_1 < x_3) = \int_{\theta_1}^{\theta_1 + \theta_2} (x - \theta_1) \cdot 1 \, dx + \int_{\theta_1 + \theta_2}^1 (x - \theta_1)(1 + \theta_1 + \theta_2 - x) \, dx \\ = \frac{1 + 2\theta_1^3 - 3\theta_1^2 - 6\theta_1\theta_2 + 3\theta_1^2\theta_2 + 3\theta_2 - \theta_2^3}{6}$$

$$P(x_1 < x_3, x_2 < x_3) = P(x_1 < x_2 < x_3) + P(x_2 < x_1 < x_3)$$

$$= \frac{2 + \theta_1^3 - 3\theta_1^2 + 3\theta_1 + 6\theta_2 - 3\theta_1\theta_2^2 - 2\theta_2^3}{6}$$

Making the substitution given by (4.12)

$$\text{cov}(y_{ik}y_{jk}) = \frac{1}{3} + \frac{\theta^3}{72\sqrt{12}}(j-i)^3 - \frac{\theta^2}{24}(j-i)^2 + \frac{\theta}{2\sqrt{12}}(j-i) \\ + \frac{\theta}{\sqrt{12}}(k-j) - \frac{\theta^3}{24\sqrt{12}}(j-i)(k-j)^2 - \frac{\theta^3}{36\sqrt{12}}(k-j)^3$$

$$\begin{aligned}
& - \left(\frac{1}{2} + \frac{(k-i)\theta}{2\sqrt{3}} \left(1 - \frac{(k-i)\theta}{4\sqrt{3}} \right) \right) \left(\frac{1}{2} + \frac{(k-j)\theta}{2\sqrt{3}} \left(1 - \frac{(k-j)\theta}{4\sqrt{3}} \right) \right) \\
& = \frac{1}{12} - \frac{\theta^2}{48} \left((k-i)^2 + (k-j)^2 \right) + \frac{\theta^3}{144\sqrt{3}} \left((k-i)^3 + 3(k-j)^2(k-i) \right) \\
& \quad - \frac{\theta^4}{576} (k-i)^2 (k-j)^2
\end{aligned}$$

$$\text{Let } \sum_1 = \sum_{i < j < k}, \quad \sum_2 = \sum_{\substack{i < k \\ j < k}}$$

$$\begin{aligned}
\text{Then } & \sum_1 \text{cov}(y_{ij}y_{jk}) + \sum_2 \text{cov}(y_{ik}y_{jk}) \\
& = \sum_1 \left(-\frac{1}{12} \right) + \frac{\theta^2}{48} \left(\sum_1 (k-j)^2 + \sum_1 (j-i)^2 \right) \\
& \quad - \frac{\theta^3}{144\sqrt{3}} \left(\sum_1 (k-j)^3 + \sum_1 (j-i)^3 \right) - \frac{\theta^4}{576} \sum_1 (j-i)^2 (k-j)^2 \\
& \quad + \sum_2 \left(\frac{1}{12} \right) - \frac{\theta^2}{48} \left(\sum_2 (k-i)^2 + \sum_2 (k-j)^2 \right) \\
& \quad + \frac{\theta^3}{144\sqrt{3}} \left(\sum_2 (k-i)^3 + 3 \sum_2 (k-j)^2 (k-i) \right) \\
& \quad - \frac{\theta^4}{576} \sum_2 (k-i)^2 (k-j)^2
\end{aligned}$$

Using the results

$$\begin{aligned}\sum_1 ((k-j)^2 + (j-i)^2) &= \sum_{g=1}^{n-2} g^2 (n-g-1)(n-g) \\ &= \frac{n(n-1)(n-2)(2n-1)(n+1)}{60}\end{aligned}$$

$$\begin{aligned}\sum_1 ((k-j)^3 + (j-i)^3) &= \sum_{g=1}^{n-2} g^3 (n-g-1)(n-g) \\ &= \frac{n(n-1)(n-2)(n+1)(n^2-n+1)}{60}\end{aligned}$$

$$\begin{aligned}\sum_1 (j-i)^2 (k-j)^2 &= \sum_{k=1}^{n-2} \sum_{g=1}^{n-1-k} k^2 g^2 (n-k-g) \\ &= \frac{n(n-1)(n-2)(n+1)(2n^3+4n^2+3n+6)}{2520}\end{aligned}$$

$$\begin{aligned}\sum_2 (k-i)^2 &= \sum_2 (k-j)^2 = \sum_{y=1}^{n-1} \frac{y^2 (n^2 - 3n + 3y - y^2)}{2} \\ &= \frac{n(n-1)(n-2)(8n+1)(n+1)}{120}\end{aligned}$$

$$\begin{aligned}\sum_2 (k-i)^3 &= \sum_{y=1}^{n-1} \frac{y^3 (n^2 - 3n + 3y - y^2)}{2} \\ &= \frac{n(n-1)(n-2)(n+1)(5n^2+n-3)}{120}\end{aligned}$$

$$\begin{aligned}\sum_2 (k-j)^2 (k-i) &= \sum_{y=1}^{n-2} \sum_{z=y+1}^{n-1} (y^2 z (n-z) + y z^2 (n-z)) \\ &= \frac{n(n-1)(n-2)(n+1)(5n^2+n-3)}{180}\end{aligned}$$

$$\begin{aligned}\sum_2 (k-i)^2 (k-j)^2 &= 2 \sum_{y=1}^{n-2} \sum_{z=y+1}^{n-1} y^2 z^2 (n-z) \\ &= \frac{n(n-1)(n-2)(n+1)(20n^3-2n^2-33n-3)}{1260}\end{aligned}$$

we obtain

$$\begin{aligned}(4.13) \quad \sum_1 \text{cov}(y_{ij} y_{jk}) + \sum_2 \text{cov}(y_{ik} y_{jk}) \\ &= \frac{n(n-1)(n-2)}{72} - \frac{n(n-1)(n-2)(n+1)(3n+1)}{1440} \theta^2 \\ &\quad + \frac{n(n-1)(n-2)(n+1)(13n^2+5n-7)}{17280\sqrt{3}} \theta^3 \\ &\quad - \frac{n^2(n-1)(n-2)(n+1)(2n^2-3)}{69120} \theta^4\end{aligned}$$

Then, from (4.8), (4.10), and (4.13) we have

$$\begin{aligned}(4.14) \quad \sigma^2(\mathbb{T}) &= \frac{n(n-1)(2n+5)}{72} - \frac{n(n^2-1)(3n^2-2)}{720} \theta^2 \\ &\quad + \frac{n(n^2-1)(13n^3-3n^2-17n+2)}{8640\sqrt{3}} \theta^3 \\ &\quad - \frac{n^2(n-1)(n^2-1)(2n^2-3)}{34560} \theta^4\end{aligned}$$

This formula can be applied for $\frac{n\theta}{\sqrt{12}} < 1$.

Table 4.1. shows the ratio of $\sigma_0^2(T)$ to $\sigma_1^2(T)$ for various values of θ and n .

4.3. Estimated Number of Observations Required by the T Test for Fixed θ .

From (4.4) and (4.9) we find

$$E_1(T) = \frac{n(n-1)}{4} - \frac{n(n^2-1)\sqrt{3}}{36} \theta + \frac{n^2(n^2-1)}{288} \theta^2$$

Using this result, the formulas for $E_0(T)$ and $\sigma_0^2(T)$ given by (4.2) and (4.3), and the estimate of $\sigma_1^2(T)$ given by (4.14), it is now possible to determine the sample size required for the T test to obtain desired probabilities α , β of type I and type II error.

Assuming the distribution of T to be approximately normal under H_0 and under H_1 ,

$$\lambda_{1-\beta} = \frac{E_0(T) - \lambda_{1-\alpha} \sigma_0(T) - E_1(T)}{\sigma_1(T)}$$

or

$$\begin{aligned} \lambda_{1-\beta}^2 \sigma_1^2(T) &= (E_0(T) - E_1(T))^2 + \lambda_{1-\alpha}^2 \sigma_0^2(T) \\ &\quad - 2 \lambda_{1-\alpha} \sigma_0(T) (E_0(T) - E_1(T)) \end{aligned}$$

Table 4.1.

Value of $\frac{\sigma_0^2(\tau)}{\sigma_1^2(\tau)}$ for selected values of θ and n .

$\theta \backslash n$	20	40	60	80	100
0.01	1.01	1.02	1.05	1.09	1.14
0.02	1.02	1.08	1.19	1.36	1.61
0.03	1.05	1.19	1.47	1.97	2.98
0.04	1.08	1.35	1.96	3.53	—
0.06	1.18	1.94	—	—	—
0.08	1.34	3.38	—	—	—
0.10	1.56	—	—	—	—
σ_0^2	237.5	1842	6146	14483	28188

Substituting into this equation the expressions for $E_0(T)$, $E_1(T)$, $\sigma_0^2(T)$ and $\sigma_1^2(T)$ in terms of n and θ and simplifying gives

$$\begin{aligned} & \theta^4 \left(\frac{n^4(n^2-1)^2}{(288)^2} + \frac{\lambda_{1-\beta}^2 n(n-1)(n^2-1)(2n^2-3)}{34560} \right) \\ & - \theta^3 \sqrt{3} \left(\frac{2n^3(n^2-1)^2}{(36)(288)} + \frac{\lambda_{1-\beta}^2 n(n^2-1)(13n^3-3n^2-17n+2)}{25920} \right) \\ & + \theta^2 \left(\frac{2\lambda_{1-\alpha} \sigma_0 n^2(n^2-1)}{288} + \frac{\lambda_{1-\beta}^2 n(n^2-1)(3n^2-2)}{720} + \frac{3n^2(n^2-1)^2}{(36)^2} \right) \\ & - \theta \sqrt{3} \left(\frac{2\lambda_{1-\alpha} \sigma_0 n(n^2-1)}{36} \right) \\ & + \frac{\lambda_{1-\alpha}^2 n(n-1)(2n+5)}{72} - \frac{\lambda_{1-\beta}^2 n(n-1)(2n+5)}{72} = 0 \end{aligned}$$

If $\alpha = \beta$, the constant term in this equation is zero. Thus, taking $\alpha = \beta$, eliminating the root $\theta = 0$, and simplifying further gives

$$\begin{aligned} & \theta^3 \left(\frac{n^3(n^2-1)}{82944} + \frac{\lambda_{1-\alpha}^2 (n-1)(2n^2-3)}{34560} \right) \\ & - \theta^2 \sqrt{3} \left(\frac{n^2(n^2-1)}{5184} + \frac{\lambda_{1-\alpha}^2 (13n^3-3n^2-17n+2)}{25920} \right) \\ & + \theta \left(\frac{\lambda_{1-\alpha} \sigma_0 n}{144} + \frac{\lambda_{1-\alpha}^2 (3n^2-2)}{720} + \frac{n(n^2-1)}{432} \right) \\ & - \frac{\sqrt{3} \lambda_{1-\alpha} \sigma_0}{18} = 0 \end{aligned}$$

with
$$\sigma_0 = \sqrt{\frac{n(n-1)(2n+5)}{72}}.$$

This equation was used (by choosing various fixed values of n , solving the equation for θ) to determine the number of observations required by the T test for $\alpha = \beta = 0.05$. The results are shown graphically in Fig. 4.1, along with curves showing the corresponding numbers of observations required under H_0 and under H_1 for the S_j -test.

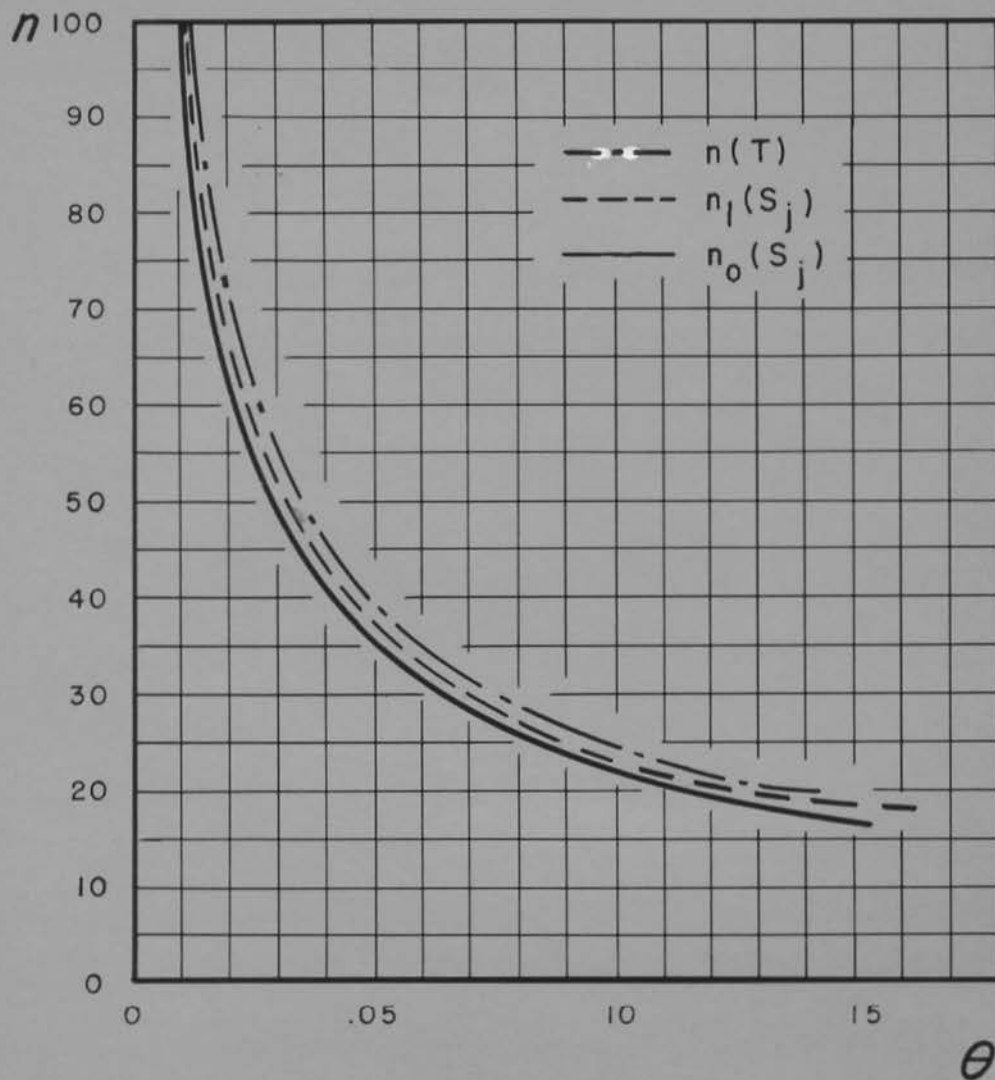
From this diagram we see that in terms of number of observations required, the S_j -test has an advantage over the T test. We should note, however, the following.

First, the estimates of $E_1(T)$, $\sigma_1^2(T)$ obtained from the rectangular distribution do not hold exactly for other distributions; hence, the value of n required for the T test will be different from the estimate shown.

Second, although the estimates of $n_0(S_j)$, $n_1(S_j)$ were obtained by taking $\alpha = \beta = 0.05$, since the test is truncated in application, these probabilities will be changed. In the example of Chapter 3 it was found that the actual probabilities of error for the truncated test were larger than 0.05; if this is true in general, the S_j -test would require a number of observations larger than the estimate shown in order to achieve the nominal values of α and β .

FIGURE 4.1

NUMBER OF OBSERVATIONS REQUIRED BY MANN'S
T TEST AND BY THE S_j -TEST ($\alpha = \beta = 0.05$)



5. TWO-SIDED S_j -TEST

5.1. Two-Sided Binomial SPRT

Suppose it is desired to test the hypothesis $H_0: p = \frac{1}{2}$ against the alternative $H_1: |p - \frac{1}{2}| = \epsilon$. To obtain an approximate test consider the following two one-sided binomial SPRT's: (The method used is similar to that used in Sobel and Wald [9] in connection with a different problem.)

Test 1 $H_0: p = \frac{1}{2}$ against $H_1: p = \frac{1}{2} + \epsilon$

$$\epsilon > 0$$

Test 2 $H_0: p = \frac{1}{2}$ against $H_1: p = \frac{1}{2} - \epsilon$

Let the probabilities of type I and type II error be represented by α_1, β_1 , for Test 1 and α_2, β_2 , for Test 2. The acceptance and rejection numbers of Test 1, from formulas (1.2) and (1.7), are given by

$$a_{1m} = -h_{10} + s_1 m$$

$$r_{1m} = h_{11} + s_1 m$$

where

$$h_{10} = \frac{-\log \frac{\beta_1}{1-\alpha_1}}{\log \frac{1+2\epsilon}{1-2\epsilon}}, \quad h_{11} = \frac{\log \frac{1-\beta_1}{\alpha_1}}{\log \frac{1+2\epsilon}{1-2\epsilon}}, \quad s_1 = \frac{-\log(1-2\epsilon)}{\log \frac{1+2\epsilon}{1-2\epsilon}}$$

Similarly for Test 2,

$$a_{2m} = -h_{20} + s_2 m$$

$$r_{2m} = h_{21} + s_2 m$$

where

$$h_{20} = \frac{-\log \frac{\beta_2}{1-\alpha_2}}{\log \frac{1-2\epsilon}{1+2\epsilon}}, \quad h_{21} = \frac{\log \frac{1-\beta_2}{\alpha_2}}{\log \frac{1-2\epsilon}{1+2\epsilon}}, \quad s_2 = \frac{-\log(1+2\epsilon)}{\log \frac{1-2\epsilon}{1+2\epsilon}}$$

Consider the case $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$.

Let $h_{10} = h_0$, $h_{11} = h_1$, $s_1 = s$. ($\frac{1}{2} < s < 1$). Then

$$h_{20} = -h_0, \quad h_{21} = -h_1, \quad s_2 = 1-s.$$

Thus formulas for the acceptance and rejection numbers of the two tests become

$$(5.1) \quad \begin{array}{l} \text{Test 1} \quad \begin{cases} a_{1m} = -h_0 + sm \\ r_{1m} = h_1 + sm \end{cases} \\ \text{Test 2} \quad \begin{cases} a_{2m} = h_0 + (1-s)m \\ r_{2m} = -h_1 + (1-s)m \end{cases} \end{array}$$

If $\alpha \leq \beta < \frac{1}{2}$ (which is reasonable in applications of the test), then since function $y = x(1-x)$ is increasing for $0 \leq x < \frac{1}{2}$, we will have

$$\alpha(1-\alpha) \leq \beta(1-\beta)$$

from which

$$\frac{1-\alpha}{\beta} \leq \frac{1-\beta}{\alpha}$$

$$\log \frac{1-\alpha}{\beta} \leq \log \frac{1-\beta}{\alpha}$$

$$-\log \frac{\beta}{1-\alpha} \leq \log \frac{1-\beta}{\alpha}$$

This implies that $h_0 < h_1$ and therefore the acceptance and rejection numbers will correspond to lines such as those shown in Fig. 5.1, where $h_0 = h_1$ if $\alpha = \beta$.

Let the two-sided binomial test be referred to as

Test 3 $H_0: p = \frac{1}{2}$ against $H_1: |p - \frac{1}{2}| = \epsilon$

To carry out this test, carry out Tests 1 and 2 simultaneously until a decision is reached for each test, and then use the following decision rule for Test 3.

Accept H_0 only if both Tests 1 and 2 accept H_0 .

Reject H_0 if either Test 1 or Test 2 rejects H_0 .

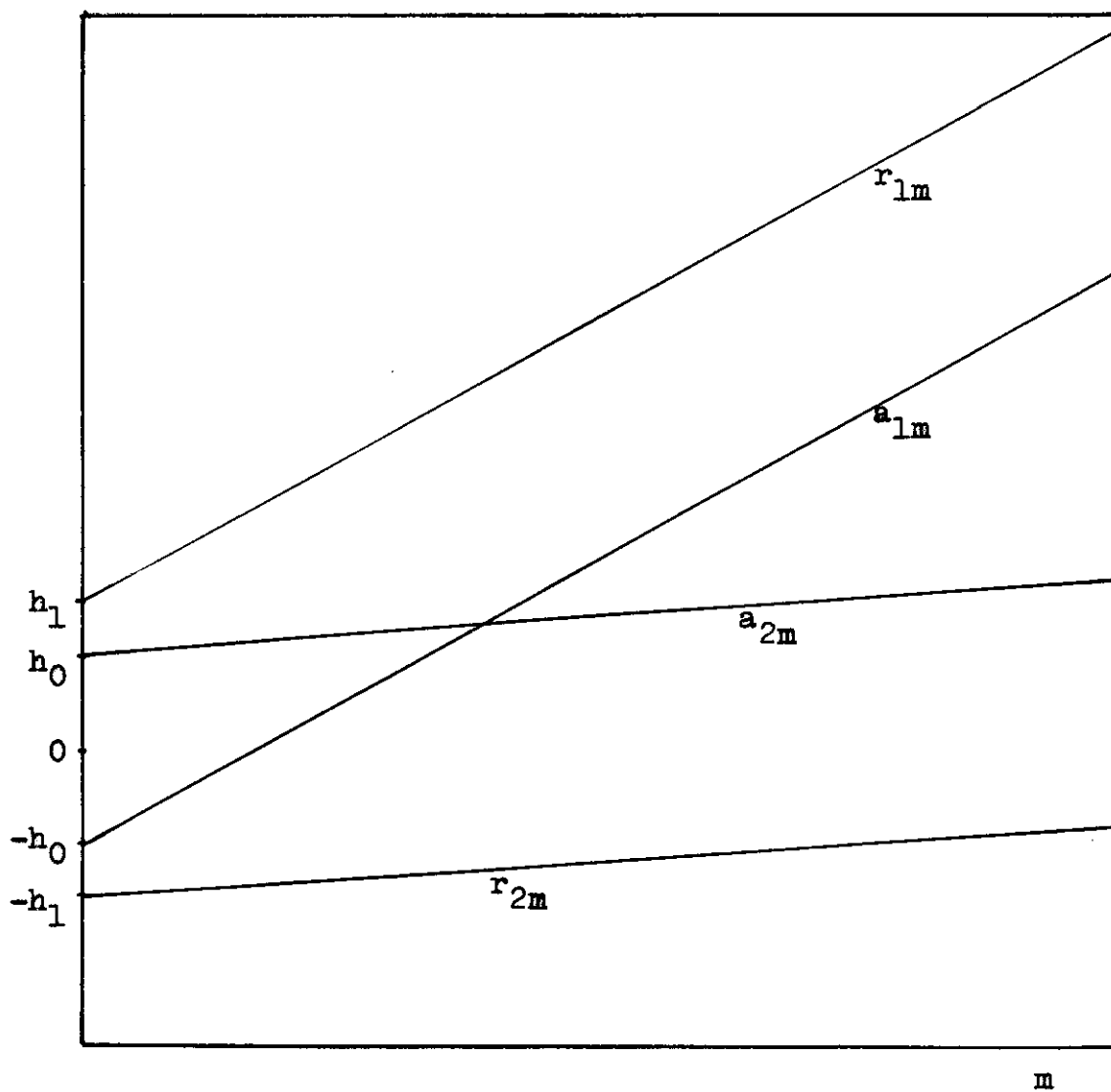
(i.e. accept $p = \frac{1}{2} + \epsilon$ if Test 1 rejects H_0 ,
accept $p = \frac{1}{2} - \epsilon$ if Test 2 rejects H_0 .)

From Fig. 5.1 it can be seen that both tests cannot lead to rejection of H_0 .

The two-sided test described here is the same as that given by Romani [8], pp. 73-75, but his formulas for determining the probabilities of error for the test are not correct.

Fig. 5.1.

Acceptance and Rejection Lines of Tests 1 and 2



5.2. OC Function of the Two-Sided Test.

If we let $L_1(p)$, $L_2(p)$, $L(p)$ represent the OC functions of Tests 1, 2, 3 respectively, then $L(p)$ can be given in terms of $L_1(p)$ and $L_2(p)$.

For any given p' note that for Test 3:

$$\begin{aligned} P(\text{accepting } p = \frac{1}{2} + \epsilon \mid p') &= P(\text{accepting } p = \frac{1}{2} + \epsilon \mid p'; \text{Test 1}) \\ &= 1 - L_1(p') \end{aligned}$$

$$\begin{aligned} P(\text{accepting } p = \frac{1}{2} - \epsilon \mid p') &= P(\text{accepting } p = \frac{1}{2} - \epsilon \mid p'; \text{Test 2}) \\ &= 1 - L_2(p') \end{aligned}$$

and since both Tests 1 and 2 terminate with probability one, the OC function of Test 3 is

$$\begin{aligned} L(p') &= P(\text{accepting } p = \frac{1}{2} \mid p') \\ &= 1 - \left[1 - L_1(p') \right] - \left[1 - L_2(p') \right] \end{aligned}$$

$$(5.2) \quad L(p') = L_1(p') + L_2(p') - 1$$

To have probabilities of type I and type II error equal to α and β respectively for Test 3 would require that

$$L\left(\frac{1}{2}\right) = 1 - \alpha, \quad L\left(\frac{1}{2} + \epsilon\right) = L\left(\frac{1}{2} - \epsilon\right) = \beta.$$

By symmetry of the testing procedures we have

$$L_1\left(\frac{1}{2} - \epsilon\right) = L_2\left(\frac{1}{2} + \epsilon\right).$$

From Wald [13], p. 51,

$$L_1(p^*) = \frac{A^{h(p^*)} - 1}{A^{h(p^*)} - B^{h(p^*)}}$$

where $h(p^*) \neq 0$ is found from

$$p^*(1 + 2\epsilon)^{h(p^*)} + (1 - p^*)(1 - 2\epsilon)^{h(p^*)} = 1$$

To evaluate $L_1\left(\frac{1}{2} - \epsilon\right)$ we must find the value of h for which

$$\left(\frac{1}{2} - \epsilon\right)(1 + 2\epsilon)^h + \left(\frac{1}{2} + \epsilon\right)(1 - 2\epsilon)^h = 1$$

The function

$$g(h) = \left(\frac{1}{2} - \epsilon\right)(1 + 2\epsilon)^h + \left(\frac{1}{2} + \epsilon\right)(1 - 2\epsilon)^h$$

has a minimum at $h = h^* > 0$, is decreasing for $h < h^*$, is increasing for $h > h^*$, and $\rightarrow \infty$ as $h \rightarrow \infty$ (cf. Wald [13], p. 158). Since $g(0) = 1$, there exists $h_0 > 0$ for which $g(h_0) = 1$. Since

$$g(3) = 1 - 16\epsilon^4 < 1$$

we see that $h_0 > 3$.

Since $L(p^*)$ is an increasing function of h (cf. Wald [13], p. 96)

$$L_1\left(\frac{1}{2} - \epsilon\right) > \frac{A^3 - 1}{A^3 - B^3} \quad \text{for } 0 < \epsilon < \frac{1}{2}$$

For $\alpha = \beta = 0.05$ we have $A = 19$, $B = \frac{1}{19}$, and

$$L_1\left(\frac{1}{2} - \epsilon\right) > 0.9998$$

For $\alpha = 0.025$, $\beta = 0.05$, we have $A = 38$, $B = \frac{2}{39}$, and

$$L_1\left(\frac{1}{2} - \epsilon\right) > 0.99998$$

Thus we may reasonably assume

$$L_1\left(\frac{1}{2} - \epsilon\right) = L_2\left(\frac{1}{2} + \epsilon\right) \sim 1$$

For each of Tests 1 and 2 take probabilities of type I and type II error equal to $\frac{\alpha}{2}$ and β respectively, i.e.

$$L_1\left(\frac{1}{2}\right) = L_2\left(\frac{1}{2}\right) = 1 - \frac{\alpha}{2}$$

$$L_1\left(\frac{1}{2} + \epsilon\right) = L_2\left(\frac{1}{2} - \epsilon\right) = \beta$$

Then from formula (5.2) we have

$$\begin{aligned} L\left(\frac{1}{2}\right) &= L_1\left(\frac{1}{2}\right) + L_2\left(\frac{1}{2}\right) - 1 \\ &= \left(1 - \frac{\alpha}{2}\right) + \left(1 - \frac{\alpha}{2}\right) - 1 \\ &= 1 - \alpha \end{aligned}$$

$$L\left(\frac{1}{2} + \epsilon\right) = L_1\left(\frac{1}{2} + \epsilon\right) + L_2\left(\frac{1}{2} + \epsilon\right) - 1$$

$$\sim \beta + 1 - 1 = \beta$$

$$L\left(\frac{1}{2} - \epsilon\right) = L_1\left(\frac{1}{2} - \epsilon\right) + L_2\left(\frac{1}{2} - \epsilon\right) - 1$$

$$\sim 1 + \beta - 1 = \beta$$

5.3. Bounds for the ASN Function.

Let n_1, n_2, n represent the number of observations required by Tests 1, 2, 3 respectively. Then

$$n = \max(n_1, n_2)$$

and

$$E(n) > \max(E(n_1), E(n_2))$$

Note that for $p = \frac{1}{2}$, $E(n_1) = E(n_2)$.

To get an upper bound for $E(n)$ consider (as in Sobel and Wald [9]) the tests

T_1^* : continue Test 1 until H_0 is rejected

T_2^* : continue Test 2 until H_0 is rejected

Then

$$(5.3) \quad E(n) \leq E(n | T_1^*)$$

$$(5.4) \quad E(n) \leq E(n | T_2^*)$$

A close upper bound for $E(n)$ can be obtained from (5.3) when $p \geq \frac{1}{2} + \epsilon$ and from (5.4) when $p \leq \frac{1}{2} - \epsilon$. Because of symmetry only one of the above cases need be considered, e.g. (5.3).

Since T_1^i terminates with probability one ($p \geq \frac{1}{2} + \epsilon$)

$$E(n | T_1^i) = \frac{E(Z_n)}{E(z)} = \frac{\log A}{E(z)}$$

where

$$E(z) = p \log(1 + 2\epsilon) + (1 - p) \log(1 - 2\epsilon)$$

Thus,

$$E(n) \leq \frac{\log A}{\left(\frac{1}{2} + \epsilon'\right)\log(1 + 2\epsilon) + \left(\frac{1}{2} - \epsilon'\right)\log(1 - 2\epsilon)}$$

where $\epsilon' \geq \epsilon$

$$\leq \frac{2 \log A}{(1 + 2\epsilon)\log(1 + 2\epsilon) + (1 - 2\epsilon)\log(1 - 2\epsilon)}$$

This will give a close upper bound for $E(n)$ under H_1 , but a different bound is needed under H_0 . To obtain this bound compare $E(n)$ with $E(n_1)$. Let m_a represent the value of m for which $a_{1m} = a_{2m}$. It is seen from formulas (5.1) that such a value always exists.

If Test 1 leads to rejection of H_0 with n_1 observations, Test 2 must have previously accepted H_0 ($n_2 < n_1$) and therefore $n = n_1$.

If Test 1 accepts H_0 with $n_1 \geq m_a$ then Test 2 must have accepted H_0 with $n_2 \leq n_1$ and no further observations are needed for Test 3; i.e., $n = n_1$.

If Test 1 accepts H_0 with $n_1 < m_a$, then n_2 and hence n may be greater than n_1 .

Thus n can be greater than n_1 only in the case where H_0 is accepted by Test 1 with $n_1 < m_a$. For any given value of n_1 , say $n_1 = k$, let $I(k)$ represent the conditional expected number of observations required to complete Test 3, i.e.

$$I(k) = E(n_2 - k \mid n_1 = k < m_a , n_2 > k)$$

Then

$$\begin{aligned} E(n) &= E(n_1) + \sum_{k < m_a} P(n_1 = k) I(k) \\ &\leq E(n_1) + P(n_1 < m_a; H_0 \text{ accepted}) I(m_a) \end{aligned}$$

where

$$I(m_a) = \max_{k < m_a} I(k)$$

$I(m_a)$ will be no greater than the maximum expected number of observations required to reach a decision by Test 2 starting at any point between the acceptance and rejection lines a_{2m} and r_{2m} .

To determine this maximum number let $Z_n = \sum_{i=0}^n z_i$, with the sequential test terminating at stage n with acceptance of H_0 if $Z_n \leq b$ and with rejection of H_0 if $Z_n \geq a$. If the initial value $z_0 = 0$, then the relation between the values of a and b and the probabilities of type I and type II error, α and β , is given approximately by the formulas

$$a = \log \frac{1-\beta}{\alpha}, \quad b = \log \frac{\beta}{1-\alpha}$$

We assume in this section that \log indicates natural logarithm.

Now suppose the initial value $z_0 = c$. The sequential procedure in this case is equivalent to one in which the initial value of z , say $z'_0 = 0$, but with acceptance and rejection values equal to $b' = b - c$ and $a' = a - c$. The relation between a' , b' and the corresponding probabilities of type I and type II error, α' and β' , will be given approximately as

$$a' = \log \frac{1-\beta'}{\alpha'}, \quad b' = \log \frac{\beta'}{1-\alpha'}$$

In the first situation ($z_0 = 0$) the expected numbers of observations under H_0 and under H_1 are given approximately as

$$E_0(n) = \frac{\alpha a + (1-\alpha)b}{E_0(z)}, \quad E_1(n) = \frac{(1-\beta)a + \beta b}{E_1(z)}$$

and in the second situation ($z_0 = c$) corresponding formulas will be

$$E_0(n') = \frac{\alpha' a' + (1-\alpha')b'}{E_0(z)}, \quad E_1(n') = \frac{(1-\beta')a' + \beta' b'}{E_1(z)}$$

To determine the maximum value of $E_0(n')$ as a function of c , $E_0(n')$ must be expressed in terms of α , β , c .

Let

$$c = \log k$$

Since

$$a' = \log \frac{1-\beta'}{\alpha'}$$

$$a - c = \log \frac{1-\beta'}{\alpha'}$$

$$\log \frac{1-\beta}{\alpha} - \log k = \log \frac{1-\beta'}{\alpha'}$$

$$\frac{1}{k} \cdot \frac{1-\beta}{\alpha} = \frac{1-\beta'}{\alpha'}$$

$$\frac{1-\beta}{k\alpha} \alpha' + \beta' = 1$$

Similarly, from the expression for b'

$$\frac{\beta}{k(1-\alpha)} \alpha' + \beta' = \frac{\beta}{k(1-\alpha)}$$

Solving the last two equations simultaneously gives

$$\alpha' = \frac{\alpha(k(1-\alpha) - \beta)}{1 - \alpha - \beta}, \quad \beta' = \frac{\beta(1 - k\alpha - \beta)}{k(1 - \alpha - \beta)}$$

Thus

$$E_0(n') = \frac{\left(\frac{\alpha(k(1-\alpha) - \beta)}{1 - \alpha - \beta}\right)(a - \log k) + \left(1 - \frac{\alpha(k(1-\alpha) - \beta)}{1 - \alpha - \beta}\right)(b - \log k)}{E_0(z)}$$

To maximize $E_0(n')$ with respect to k (or c) it is sufficient to minimize the numerator (since both numerator and denominator are negative). This numerator can be re-written as

$$\frac{\alpha(a-b)(k(1-\alpha) - \beta)}{1 - \alpha - \beta} + b - \log k$$

Differentiating with respect to k , setting the derivative equal to zero, gives

$$\frac{\alpha(a-b)(1-\alpha)}{1 - \alpha - \beta} - \frac{1}{k} = 0$$

or

$$k = \frac{1 - \alpha - \beta}{\alpha(a-b)(1-\alpha)}$$

Substituting this value of k in the formula for $E_0(n')$ and simplifying gives

$$\max E_0(n') = \frac{1 - \frac{\alpha\beta(a-b)}{1 - \alpha - \beta} + \log \frac{\alpha\beta(a-b)}{1 - \alpha - \beta}}{E_0(z)}$$

where $E_0(z) = \frac{1}{2} \log (1-4\epsilon^2)$.

Note that the relation between $\max E_0(n)$ starting with any initial value $z_0 = c$ and $E_0(n)$ starting with $z_0 = 0$ depends only on α and β (not on parameter values under H_0 and H_1 .)

Using this upper bound we may now write as the upper bound of the number of observations required by Test 3 under H_0

$$(5.5) \quad E(n) \leq E(n_1) + P(n_1 < m_a; H_0 \text{ accepted}) \max E_0(n^*)$$

In order to make use of this upper bound, a bound must also be set for $P(n_1 < m_a; H_0 \text{ accepted})$. In the application considered later this probability is approximately equal to $\frac{1}{2}$ but no general expression for the probability or its upper bound has yet been found.

5.4. Extension to the S_j -Test.

To set up a two-sided S_j -test, i.e. a test of the hypothesis

$$p = P(X_i > X_{i+j}) = \frac{1}{2}$$

against the alternative

$$\left| p_j - \frac{1}{2} \right| = \epsilon_j$$

it would seem desirable as in the one-sided test to choose j in such a way that the expected number of observations under H_0 and under H_1 would be minimized. Since upper bounds are available for these expected numbers, one possible procedure might be to choose j to minimize these upper bounds. This would require determining an expression for the probability given in formula (5.5) for the upper bound under H_0 . However, since minimizing the two upper bounds might lead to different values of j and only one value can be used in application, a solution might be to use the value of j obtained in minimizing the upper bound under H_1 . This, however, would still leave unanswered the question of the effect of truncation on the values of α , β , and $E(n)$.

Another possible procedure is to make use of an approximation given by Armitage [1]. Armitage gives a closed sequential procedure based on divergent lines; in the only practical application mentioned, Armitage's divergent lines correspond to the rejection lines, r_{1m} and r_{2m} , of the two one-sided Wald tests considered above. He has no acceptance lines, accepting H_0 only if it is not

rejected at point of truncation. The truncation number, N for the Armitage procedure is chosen to make the probabilities of type I and type II errors equal to α , β respectively for the truncated test.

In his Table 5 Armitage gives the values of N corresponding to various ϵ_N for $\alpha = 0.025$ (one-sided), $\beta = 0.05$. Armitage's table has been extended to give values of N for additional values of ϵ_N in column 2 of Table 5.1. In column 3 of this table are shown the values of $\lambda_{\frac{1}{2} + \epsilon_N}$, and in column 4, the values of $\theta =$

$\frac{\sqrt{2}}{N} \lambda_{\frac{1}{2} + \epsilon_N}$. Fig. 5.2. shows graphically the relation

between θ and N .

In application given that

$$F(x_1, \dots, x_n) = \prod_{i=1}^n F(x_i + i\theta)$$

the S_j -test will be a test of the hypothesis $\theta = 0$ against the alternative $|\theta| = \theta_1$.

To set up a testing procedure for any alternative θ_1 (for $\alpha = \beta = 0.05$), Fig. 5.2. may be used to obtain the appropriate value of N . The corresponding value of ϵ_N is then found (with the aid of standard normal tables) by

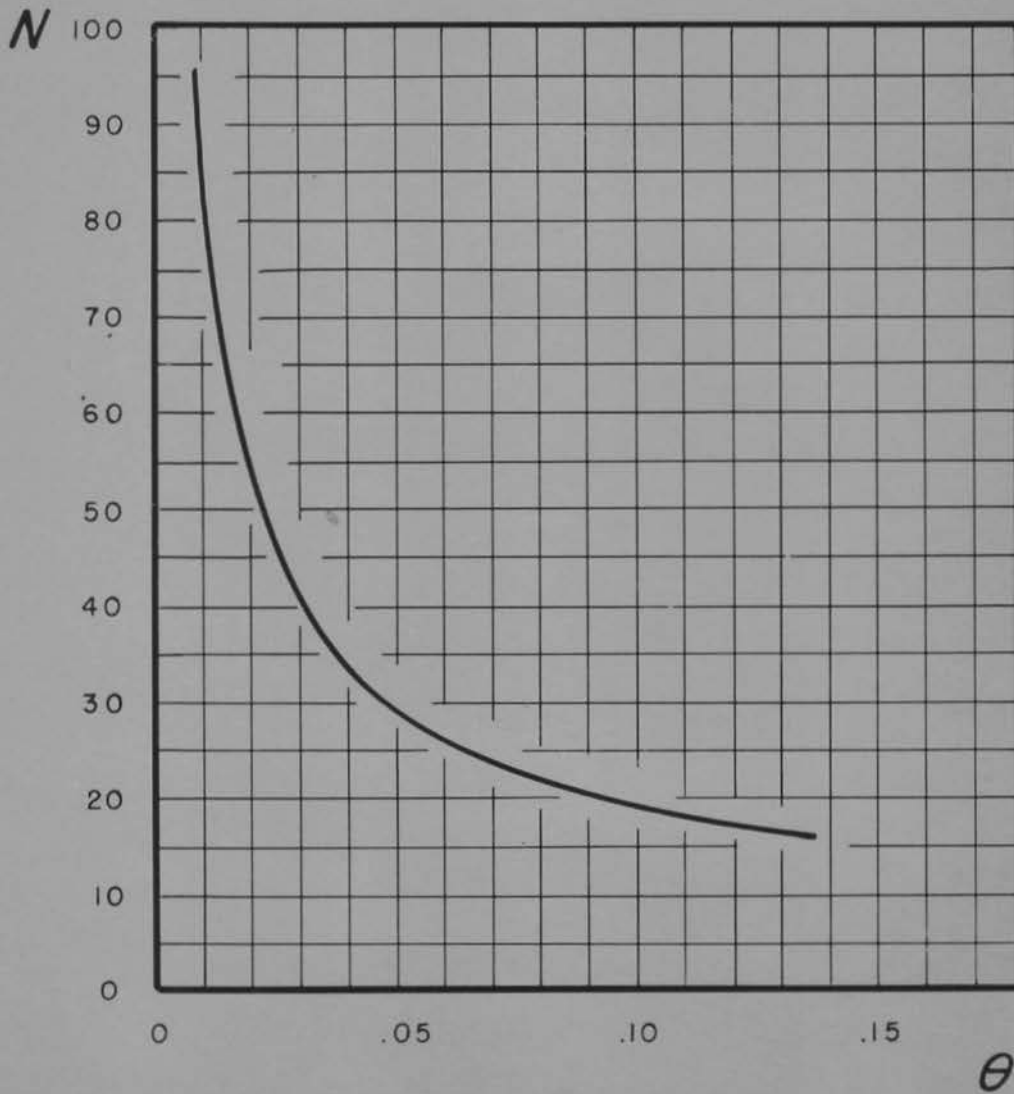
Table 5.1.

Value of Truncation Number N for the Two-Sided S_j -Test, $\alpha = \beta = 0.05$. Underlined values of N are taken from Table 5 of Armitage [1]; remaining values of N are obtained by Armitage's approximation formula.

ϵ_N	N	$\lambda_{\frac{1}{2} + \epsilon_N}$	$\theta = \frac{\sqrt{2}}{N} \lambda_{\frac{1}{2} + \epsilon_N}$
0.05	<u>1778</u>	0.126	0.00010
0.10	<u>438</u>	0.253	0.00082
0.15	<u>192</u>	0.385	0.00284
0.20	<u>105</u>	0.524	0.00706
0.21	95	0.553	0.00823
0.22	86	0.583	0.00959
0.23	78	0.613	0.0111
0.24	71	0.643	0.0128
0.25	<u>65</u>	0.674	0.0147
0.26	60	0.706	0.0166
0.27	55	0.739	0.0190
0.28	51	0.772	0.0214
0.29	47	0.806	0.0243
0.30	<u>43</u>	0.842	0.0277
0.31	40	0.878	0.0310
0.32	37	0.915	0.0350
0.33	35	0.954	0.0385
0.34	32	0.994	0.0439
0.35	<u>30</u>	1.036	0.0488
0.36	28	1.080	0.0545
0.37	26	1.126	0.0612
0.38	24	1.175	0.0692
0.39	23	1.227	0.0754
0.40	<u>21</u>	1.282	0.0863
0.41	20	1.341	0.0948
0.42	18	1.405	0.1104
0.43	17	1.476	0.1228
0.44	16	1.555	0.1374
0.45	<u>14</u>	1.645	0.1662

FIGURE 5.2

APPLICATION OF ARMITAGE TEST PROCEDURE
TO THE S_j -TEST, TAKING $j = N$ ($\alpha = \beta = 0.05$)



$$\lambda_{\frac{1}{2} + \epsilon_N} = \frac{N\theta}{\sqrt{2}}$$

A sequential binomial test is then used for the hypothesis $p = \frac{1}{2}$ against the alternative $|p - \frac{1}{2}| = \epsilon_N$, truncating the test at N comparisons ($2N$ observations). For example, to test against the alternative $|\theta| = 0.04$, from Fig. 5.2 $N = 34$, and

$$\lambda_{\frac{1}{2} + \epsilon_N} = \frac{34(0.04)}{\sqrt{2}} = 0.962$$

From tables of the standard normal distribution $\epsilon_N = 0.332$. Thus, the test becomes a sequential binomial test of the hypothesis $p = \frac{1}{2}$ against the alternative $|p - \frac{1}{2}| = 0.332$. As rejection lines use the rejection lines corresponding to the alternative values $p = 0.832$ and $p = 0.168$ for one-sided tests. With the Armitage procedure the test will be terminated at 34 comparisons, H_0 being accepted at that stage if it has not been previously rejected.

Since the rejection lines of the Armitage procedure correspond to the rejection lines of the two-sided binomial test described earlier, while the acceptance rules differ, it would seem appropriate to compare these two procedures with respect to α , β , $E(n)$. For this

purpose the example above is considered, with the two-sided binomial test being set up using the values of N , ϵ_N obtained from the Armitage procedure (probably different from the values which would be obtained by minimizing the bounds on the expected numbers of observations). Since $\alpha = \beta = 0.05$, each one-sided test is set up with $\alpha' = 0.025$, $\beta' = 0.05$, and from formulas (5.1) we find as equations of the acceptance and rejection lines

$$\begin{aligned} a_{1m} &= -1.857 + 0.6817m & a_{2m} &= 1.857 + 0.3183m \\ r_{1m} &= 2.274 + 0.6817m & r_{2m} &= -2.274 + 0.3183m \end{aligned}$$

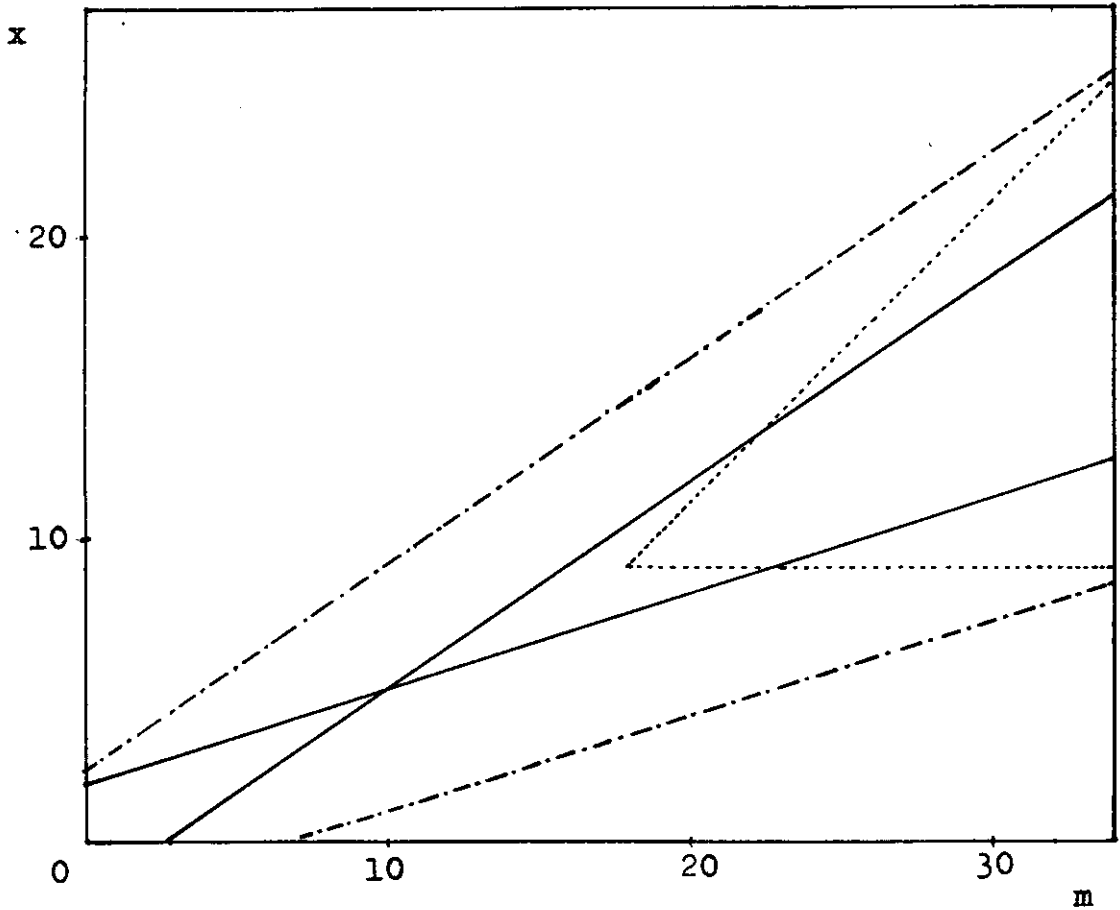
These lines are shown in Fig. 5.3. From this diagram we see, as Armitage points out, that some saving of observations is possible in applying his procedure, since once a path reaches one of the dotted lines, rejection of H_0 before truncation is impossible and H_0 may be accepted. Thus the dotted lines become the acceptance lines of the Armitage procedure.

The two procedures were first compared by the selection of random samples. As might be expected, the results showed little difference between the procedures

Fig. 5.3.

Acceptance and Rejection Lines for Two-Sided Binomial

Test. $H_0: p = 0.5$, $H_1: |p - 0.5| = 0.332$



- Acceptance lines for Wald two-sided test (path must cross both lines)
- Acceptance lines for Armitage procedure
- - - - Rejection lines (same for both methods)

under H_1 . For 30 samples selected with $p = 0.832$, the mean number of observations was 14.77 for the Armitage procedure, 14.93 for the Wald two-sided test. 29 of these samples led to rejection of H_0 by both procedures; 1 sample led to acceptance of H_0 with 28 observations for the Armitage procedure, 34 observations for the Wald test.

For 100 samples selected with $p = \frac{1}{2}$, the mean number of observations for the Armitage procedure was 21.51, with one sample leading to rejection of H_0 , while the mean number of observations for the Wald two-sided test was 17.83, with two samples leading to rejection of H_0 .

For 20 samples selected for an intermediate value $p = 0.7$, the mean number of observations for the Armitage procedure was 20.8, with 13 samples leading to rejection of H_0 , while the mean number of observations for the Wald two-sided test was 19.0, with 12 samples leading to rejection of H_0 .

The two procedures may be more precisely compared by the computation of exact probabilities, using a method of calculation similar to that used in Chapter 3. Since we expect little difference between the procedures under H_1 , these computations were carried out only under H_0 .

For the Armitage procedure

$$P(\text{rejecting } H_0 \mid H_0) = 0.03696, \quad E_0(n) = 20.85$$

while for the Wald two-sided test

$$P(\text{rejecting } H_0 \mid H_0) = 0.04568, \quad E_0(n) = 16.71$$

The expected number of observations is thus considerably larger for the Armitage procedure than for the Wald two-sided test. This would probably be true for other alternatives also (and would be true even if the probabilities of type I error could be equalized). Note that the Armitage procedure cannot accept H_0 with $n < 18$, while for the Wald two-sided test $P(n < 18) = 0.627$.

The difference between the probabilities of type I error for the two procedures is due mainly to the treatment of samples which have not led to termination with $n \leq 34$ in the Wald test; i.e., for the Wald two-sided test $P(\text{rejecting } H_0 \text{ with } n \leq 34 \mid H_0) = 0.03646$ as compared to the Armitage value $P(\text{rejecting } H_0 \mid H_0) = 0.03696$. For the four possible positions between the acceptance and rejection lines when $m = 34$, the procedure followed for the Wald two-sided test has been to accept H_0 for the two points nearer the acceptance line ($\frac{P_{1m}}{P_{0m}} < 1$) and to reject H_0 for the two points nearer the rejection line ($\frac{P_{1m}}{P_{0m}} > 1$).

6. EFFECTIVENESS OF THE S_j -TEST AGAINST ALTERNATIVES OTHER THAN LINEAR TREND

The S_j -test might conceivably be useful in testing the hypothesis of randomness against the alternative of either a downward trend or an upward trend, not necessarily linear, or against the alternative of linear trend when the observations are not equally spaced (e.g. when the trend may be linear with respect to time, but observations are not made at equal time intervals). Under each of the above alternatives the value of $P(X_i > X_{i+j}) = p_{ij}$ would not be constant for fixed j . Thus it might be of interest to consider a sequential binomial test of the hypothesis $H_0: p = \frac{1}{2}$ against the alternative $H_1: p = p_1^* = \frac{1}{2} + \epsilon$, $\epsilon > 0$, and try to determine the effect on the test, i.e. on the OC function and the expected numbers of observations, if p_i varies with successive observations. We want then to compare the properties of the test under the following conditions

$$(6.1) \quad p_i = \frac{1}{2} + \epsilon'$$

$$(6.2) \quad p_i = \frac{1}{2} + \epsilon' + \epsilon_i$$

Using the generalization of Wald's fundamental identity given by Blom [2], noting that the conditions for the identity and for its differentiation are satisfied, we have

$$E(e^{tZ_n} (\varphi_1(t) \cdots \varphi_n(t))^{-1}) = 1$$

where $\varphi_i(t) = E(e^{tz_i})$.

Differentiating with respect to t

$$E(-e^{tZ_n} \sum_{i=1}^n \frac{\varphi_1 \cdots \varphi_{i-1} \varphi'_i \varphi_{i+1} \cdots \varphi_n}{(\varphi_1 \cdots \varphi_n)^2}$$

$$+ Z_n e^{tZ_n} (\varphi_1 \cdots \varphi_n)^{-1}) = 0$$

where $\varphi_i = \varphi_i(t)$, $\varphi'_i = \frac{d\varphi_i}{dt}$.

Setting $t = 0$, we have

$$E(- \sum_{i=1}^n E(z_i) + Z_n) = 0$$

from which

$$E(Z_n) = E(\sum_{i=1}^n E(z_i))$$

In the application of the sequential test we take

$$z_i = \log \frac{f_1(x_i)}{f_0(x_i)}$$

which gives in the case of the binomial distribution

$$z_i = \log(1 + 2\epsilon) \quad \text{with probability } p_i$$

$$z_i = \log(1 - 2\epsilon) \quad \text{with probability } q_i = 1 - p_i$$

Therefore, under condition (6.2)

$$\begin{aligned}
 E_2(z_i) &= \left(\frac{1}{2} + \epsilon' + \epsilon_i \right) \log(1 + 2\epsilon) \\
 &\quad + \left(\frac{1}{2} - \epsilon' - \epsilon_i \right) \log(1 - 2\epsilon) \\
 &= \frac{1}{2} \log(1 - 4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon} + \epsilon_i \log \frac{1+2\epsilon}{1-2\epsilon} \\
 \sum_{i=1}^n E_2(z_i) &= n \left(\frac{1}{2} \log(1 - 4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon} \right) \\
 &\quad + \left(\sum_{i=1}^n \epsilon_i \right) \log \frac{1+2\epsilon}{1-2\epsilon}
 \end{aligned}$$

while under (6.1)

$$\sum_{i=1}^n E_1(z_i) = n \left(\frac{1}{2} \log(1 - 4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon} \right)$$

Thus under (6.2) we have

$$\begin{aligned}
 E_2(Z_n) &= (E_2(n)) \left(\frac{1}{2} \log(1 - 4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon} \right) \\
 &\quad + (E_2 \left(\sum_{i=1}^n \epsilon_i \right)) \left(\log \frac{1+2\epsilon}{1-2\epsilon} \right)
 \end{aligned}$$

or

$$E_2(n) = \frac{E_2(Z_n) - (E_2 \left(\sum_{i=1}^n \epsilon_i \right)) \left(\log \frac{1+2\epsilon}{1-2\epsilon} \right)}{\frac{1}{2} \log(1 - 4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon}}$$

Under (6.1) we have

$$E_1(n) = \frac{E_1(Z_n)}{\frac{1}{2} \log(1-4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon}}$$

If we let L_1 represent the probability of accepting H_0 under condition (6.1) and L_2 represent the probability of accepting H_0 under condition (6.2), we have

$$E_1(n) = \frac{L_1 b + (1-L_1)a}{\frac{1}{2} \log(1-4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon}}$$

$$E_2(n) = \frac{L_2 b + (1-L_2)a - (E_2(\sum_{i=1}^n \epsilon_i))(\log \frac{1+2\epsilon}{1-2\epsilon})}{\frac{1}{2} \log(1-4\epsilon^2) + \epsilon' \log \frac{1+2\epsilon}{1-2\epsilon}}$$

Taking $\epsilon' = \epsilon$, which corresponds to the alternative hypothesis H_1 , the denominators of $E_1(n)$, $E_2(n)$ will be positive.

If $E_2(\sum_{i=1}^n \epsilon_i) > 0$, then we can have $E_1(n) = E_2(n)$ only if $L_2 < L_1$, or if $L_1 = L_2$, then we must have $E_2(n) < E_1(n)$.

If $E_2(\sum_{i=1}^n \epsilon_i) < 0$, then we can have $E_1(n) = E_2(n)$ only if $L_2 > L_1$, or if $L_1 = L_2$, then we must have $E_2(n) > E_1(n)$.

Taking $\epsilon' = 0$, which corresponds to the null hypothesis H_0 , the denominators of $E_1(n)$, $E_2(n)$ will be negative.

If $E_2\left(\sum_{i=1}^n \epsilon_i\right) > 0$, then we can have $E_1(n) = E_2(n)$

only if $L_2 < L_1$, or if $L_1 = L_2$, then we must have

$$E_2(n) > E_1(n).$$

If $E_2\left(\sum_{i=1}^n \epsilon_i\right) < 0$, then we can have $E_1(n) = E_2(n)$

only if $L_2 > L_1$, or if $L_1 = L_2$, then we must have

$$E_2(n) < E_1(n).$$

From the above we see that if $E_2\left(\sum_{i=1}^n \epsilon_i\right)$ is numerically small, and there is little effect on the OC function, i.e., if $L_1 \sim L_2$, then also $E_1(n) \sim E_2(n)$. Also, under H_1 the test may be poorer (either $L_2 > L_1$ or $E_2(n) > E_1(n)$) if $E_2\left(\sum_{i=1}^n \epsilon_i\right) < 0$, while under H_0 the test may be poorer (either $L_2 < L_1$ or $E_2(n) > E_1(n)$) if $E_2\left(\sum_{i=1}^n \epsilon_i\right) > 0$. To judge the effect of different values of $E_2\left(\sum_{i=1}^n \epsilon_i\right)$ on $E_2(n)$, it is necessary to judge the corresponding effect on L_2 . One difficulty is that the value of L_2 will probably depend not only on $E_2\left(\sum_{i=1}^n \epsilon_i\right)$ but also on individual terms of the sequence $\left\{\sum_{i=1}^m \epsilon_i\right\}$. The problem has been considered only for

a fairly simple case, that for which

$$\begin{aligned}\epsilon_i &= \epsilon_1 && \text{for } i \text{ odd} \\ &= -\epsilon_1 && \text{for } i \text{ even}\end{aligned}$$

Then

$$\begin{aligned}\sum_{i=1}^n \epsilon_i &= \epsilon_1 && \text{for } n \text{ odd} \\ &= 0 && \text{for } n \text{ even}\end{aligned}$$

and

$$\begin{aligned}E_2\left(\sum_{i=1}^n \epsilon_i\right) &= \epsilon_1 P(n \text{ odd}) > 0 && \text{if } \epsilon_1 > 0 \\ &< 0 && \text{if } \epsilon_1 < 0\end{aligned}$$

Computations have been carried out for the binomial hypothesis: $H_0: p = \frac{1}{2}$ against the alternative $H_1: p = 0.79$ (which is the example considered in Chapter 3), using $\epsilon = 0.29$, $\epsilon_1 = \pm 0.1, \pm 0.2$. The results are shown in Table 6:1. From this table we see that in this case, which corresponds to H_1 , for $E_2\left(\sum_{i=1}^n \epsilon_i\right) > 0$, the probability of error is smaller than for the case $\epsilon_i = 0$, with no great effect on $E(n)$, while for $E_2\left(\sum_{i=1}^n \epsilon_i\right) < 0$, the value of $E(n)$ has increased, and the probability of error is again smaller than for the case $\epsilon_i = 0$.

Table 6.1.

Values of Probability of Accepting H_0 and $E(n)$ Given
 That $p_i = p_1^* + \epsilon_i$, where $H_0: p = 0.5$ is being tested
 against $H_1: p = p_1^* = 0.79$ ($\alpha = \beta = 0.05$) and
 $\epsilon_i = \epsilon_1$ for i odd, $\epsilon_i = -\epsilon_1$ for i even.

ϵ_1	$E\left(\sum_{i=1}^n \epsilon_i\right)$	$P(\text{acc } H_0)$	$E(n)$	$P(\text{not term. with } n \leq 28)$
-0.2	-0.0891	0.0306	16.83	0.1400
-0.1	-0.0497	0.0562	15.87	0.1216
0	0	0.0615	15.41	0.1114
0.1	0.0577	0.0520	15.35	0.1071
0.2	0.1230	0.0284	15.64	0.1039

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ABSTRACT

The S_j -test proposed by Noether [7] is a sequential test of the hypothesis of randomness against the alternative of linear trend, which can be expressed as the hypothesis that the joint distribution of X_1, X_2, \dots, X_n is given by

$$F(x_1, x_2, \dots, x_n) = \prod_{i=1}^n F(x_i + i\theta)$$

The S_j -test is carried out by comparing X_i with X_{i+j} , for fixed $j > 0$, and using the sequential probability ratio test of the hypothesis $H_0: p = P(X_i > X_{i+j}) = \frac{1}{2}$ against the alternative $H_1: p = \frac{1}{2} + \epsilon_j$, the test being truncated with j comparisons if a decision has not been reached by that point.

To carry out the S_j -test the value of j is chosen to minimize the expected numbers of observations required by the sequential test. The dissertation provides a chart from which one can read the value of j required for given alternatives θ for $\alpha = \beta = 0.05$ and $\alpha = \beta = 0.01$, where α, β represent the probabilities of type I, type II error respectively. Since the S_j -test is carried out as a truncated sequential test, the effect of truncation on the probabilities of error and on the average sample number is considered.

The S_j -test is compared with Mann's T test, which is the most efficient non-parametric test against trend. It is found that the expected numbers of observations required by the S_j -test are slightly smaller than the number of observations required by the T test. In view of the ease of carrying out the S_j -test compared to the T test, it would seem that the S_j -test represents a useful test against linear trend. In the course of this comparison of the two tests a formula is obtained for the variance of T under the hypothesis of linear trend, for samples selected from the rectangular distribution.

A two-sided sequential binomial test is set up, using two one-sided sequential probability ratio tests, and the method is then applied to set up a two-sided S_j -test. The choice of j is made by considering a type of closed sequential scheme described by Armitage [1], and a chart is presented to allow estimation of j for any given alternative θ .

In the last section the question of effectiveness of the S_j -test against upward or downward trends which are not necessarily linear and against linear trends with unequally spaced observations is briefly discussed.

AUTOBIOGRAPHY

Elizabeth Shuhany was born on April 9, 1925 in Chelmsford, Mass., the daughter of Andrew M. and Anna (Klatka) Shuhany. She attended public schools in Chelmsford, graduating from Chelmsford High School in 1943. She entered Boston University in January 1944. She was elected to membership in Phi Beta Kappa in her senior year and received the A.B. degree, with honor and with distinction in mathematics, in June 1947. She entered Boston University Graduate School in the fall of 1947 and received the A.M. degree in June 1949. Since 1947 she has been employed by Boston University, holding the rank of Assistant Professor of Mathematics since 1956.

