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Dissipation and energy equipartition

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DISSIPATION AND ENERGY EQUIPARTITION

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ABSTRACT

DISSIPATION AND ENERGY EQUIPARTITION
by Marion Elizabeth Badoian

In the study of general physics, many problems arise in which the process which takes place is such that the energy delivered to a system is equally divided between the energy stored, and later available for work, and the energy dissipated by the system.

This paper constitutes a rather detailed analysis of three of these many processes and an attempt, through the examination of the characteristics of the processes, to develop an underlying physical theorem, or extend an existing theorem that would govern these characteristics. Particular emphasis was placed on the Virial Theorem, on the classical equipartition of energy law, and on the Rayleigh dissipation function in relation to the problems of: a satellite moving in an inverse square central force field and encountering a dissipative medium, a condenser being charged from a constant source of emf through an arbitrary resistor, and a capacitor whose plates are being filled with a dielectric while it is held at constant potential. None of these theorems could be extended to cover all the problems, and a new theorem was not developed.

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Introduction

In an isolated conservative system, one in which any force is a function of position only and is derivable from a potential, the total mechanical energy, expressing the sum of the kinetic and potential energies is constant. Thus any change in one kind of energy is compensated for by an equal but opposite change in the other. That is

$$E = T + V$$

is constant, therefore

$$\Delta E = 0 \qquad \text{and} \qquad \Delta T = -\Delta V \quad .$$

In a non-conservative system, one in which the force is any function of position, velocity and/or time, $\vec{F} = \vec{F}(q, \dot{q}, t)$, and is not derivable from a scalar potential, the previous statement of the principle of conservation of energy is not valid since, in addition to the kinetic and potential energies, other kinds of energy may exist. In order to include these, the principle of conservation of energy may be stated as: The sum of the various kinds of energy in an isolated system, one which does not interact with the rest of the universe, remains constant. Thus,

$$E = \sum_i \mathcal{E}_i$$

is constant, and

$$\Delta E = \Delta \sum_i \mathcal{E}_i = 0 \quad .$$

Therefore, when one kind of energy increases or appears, another kind of energy decreases or disappears, as long as the system is isolated from the rest of the universe.

In a variety of physical processes, where the isolated system is defined so that it includes sources and sinks of energy, it is found that a decrease in the energy of the source, as a result of work done by the source on the rest of the system (defined as negative work), not only satisfies the latter form of the conservation of energy law, but, in fact, is equally divided between the energy stored in the system so that it is later available for work, and the energy dissipated as the process takes place. That is,

$$-\Delta W_{source} = \Delta U_{stored} + \Delta W_{dissipated}$$

where

$$\Delta U_{stored} = -\frac{1}{2}\Delta W_{source}$$

and

$$\Delta W_{dissipated} = -\frac{1}{2}\Delta W_{source} \quad .$$

The rates of change of energy may be similarly related.

The purpose of this study is to investigate the details of some of the problems in which such energy partition occurs to achieve greater understanding of the source of the equal division of energy by finding what, if any, similarities exist among the processes which result in the equipartition, and by determining the conditions required for the equal division of energy to take place. Beyond this, it was hoped that these characteristics would suggest and enable development of an underlying physical theorem, or the extension of an existing theorem, that would govern the partitions in the problems considered. Toward this end, the possible usefulness of the following classical energy concepts were considered: the Virial Theorem, the equipartition of energy law of classical statistical mechanics (kinetic theory), and the Rayleigh dissipation function.

The Virial Theorem relates the time average values of the kinetic and potential energies of a system, subject to central forces, as

$$\overline{T} = \frac{n+1}{2}\overline{V}$$

where n gives the power relation of the force field to the position vector , r . In the case of the inverse square force fields, $n = -2$, and the relation reduces to

$$\overline{T} = -\frac{1}{2}\overline{V} \quad .$$

If, in a time which is long compared to the time for averaging, the total energy of the system is caused to change by altering the state of the system designated by T_0, V_0 to one designated by T_1, V_1 , then

$$\overline{\Delta T} = -\frac{1}{2}\overline{\Delta V} \quad .$$

This indicates that if a system can be devised which uses the potential energy of the system as the source of the energy to the system then the work done by this source is at the expense of the potential energy, and that if one were to calculate the change in this energy in time, only ½ of it could be accounted for in the system after the process had taken place, regardless of the amount of time required for the change to occur. The other ½ is lost as dissipated energy. This result is independent of the process which causes the change to occur and of the initial and final states of the system, other than that they be different from each other. It seems only to depend on the form of the conservative force field dependence on r .

The energy stored, the kinetic energy, which shares the change in potential energy equally with the dissipated energy, is quadratic in form and therefore suggests a possible extension, to non-conservative systems, of the classical equipartition of energy law as governing this partition of energy, into stored and dissipated energy, if the dissipated energy is quadratic in form also.

In classical statistical mechanics, the energy of a single particle in a conservative system is, in most cases, a homogeneous function of a number of position or momentum coordinates, which may be designated as $q_1, q_2, \ldots, q_j$. The total number, j, of parameters upon which the energy depends is called the number of degrees of freedom. The equipartition of energy law states that each degree of freedom which contributes quadratically to the total energy of the system, contributes equally to that energy. This indicates that, if energy is added to this conservative system, each degree of freedom which contributes quadratically to the total energy will share equally the energy being added and that the rates at which the energy associated with each degree of freedom change will also be equal.

For non-conservative systems, it may be possible to extend this idea so that it includes the results of the Virial Theorem, if the dissipated energy as well as the stored energy has a quadratic form. It might be supposed that if the stored and dissipated energies have quadratic forms that they will therefore share equally in any change in the total energy of the system and that their rates of change will also be equal. Furthermore, if the dissipative term is related to the momentum in the form of the Rayleigh dissipation function,

$$\mathcal{F} = \frac{1}{2} \sum_j k_j \dot{q}_j^{\,2} ,$$

then the rate at which energy is lost is also dependent on the quadratic form,

$$\frac{dE}{dt} = -2\mathcal{F} .$$

This study is an attempt to extend these three ideas to the following problems:

A. A satellite of mass m moves about a body of mass M in a near circular orbit which continually decreases in radius due to the presence of a resistive medium. For any change in the orbits the resultant potential energy change is found to be equally divided between the increase in kinetic energy and the increase in the energy dissipated as the satellite moves through the resistive medium.

B. A battery, switch and capacitor are connected in series by a wire of arbitrary but constant resistance. When the switch is closed, the battery delivers energy to the system of capacitor and resiscive wires, but only half of this energy is finally stored in the capacitor.

C. A capacitor is connected to a battery which maintains a constant potential difference between the conducting surfaces. A slab of dielectric, of constant κ , is inserted between the surfaces completely filling the space between them. The work done by the battery during the insertion process is just twice the increase in electrostatic energy between the capacitor plates when the slab completely fills the space between the surfaces.

Chapter I
Satellite

The motion of a satellite about the earth, so that the gravitational force between them is along their line of centers, is referred to as motion under a central force and takes place in a plane. The details of the motion of any satellite about the earth may be found in standard texts on classical mechanics.[1] However, these discussions of orbits and energies are for conservative force fields only, such as the inverse square law forces. These lead to the usual results of conservation of angular momentum and constancy of areal velocity. The closed orbits may be either circular or elliptical.

In a recent paper[2], it is shown that when an artificial satellite travelling, under the influence of a central force, in a circular orbit encounters a drag force which is velocity dependent and directly opposite the direction of the velocity, the path changes adiabatically and the acceleration of the satellite is as if the drag force were pushing the satellite. Actually, the drag force results from a succession of impulses each of which causes an instantaneous decrease in the kinetic energy of the satellite, unbalance in the relationship of the centripetal and gravitational forces, and thus allows the gravitational field to move the satellite into an orbit closer to the source of the gravitational field. This results in an increase in the kinetic energy of the satellite. Thus, the satellite descends gradually in a quasi-circular orbit with the ratio of the change in the angle of descent to the change in the polar angle being much less than unity and nearly constant in time. As the satellite descends, precisely half its loss in potential energy appears as increase in kinetic energy, the other half is dissipated as heat by the drag force.

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1. Goldstein, H., Classical Mechanics, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1957, Ch. 3, for example.

2. Mills, Blake D., Jr.,"Satellite Paradox," Am.J. Phys., 27, February 1959, p.115-117.

This energy division, and other energy relations, can be derived from end-point arguments, from consideration of the Virial Theorem, or from purely dynamical arguments.

First, from consideration of the end-points: The force existing between the satellite of mass, m, and the earth of mass, M, is an attractive force

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} = -\frac{k}{r^2}\hat{r} ,$$

and for any change, dr, in the radius of the orbit, say from r_2 to r_1, which may be shown diagramatically as

we have

$$\vec{F}\cdot d\vec{r} = -dV ,$$

where V is the potential energy. Since the change is in the radial direction,

$$V(r_2) - V(r_1) = \int_{r_1}^{r_2} \frac{k}{r^2}\,dr = -k\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

But,

$$\vec{F} = -\frac{k}{r^2}\hat{r} = -\frac{mv^2}{r}\hat{r} ,$$

where $\frac{mv^2}{r}\hat{r}$ is the centripetal force, must be true for circular orbits. Therefore, $m v^2 = k/r$. Hence, the kinetic energy of the satellite is given by T, where T is $\frac{1}{2} mv^2$ and

and since

$$V(r) = -\frac{k}{r}$$

then

$$T = -\frac{1}{2}V$$

which means that the kinetic and potential energy for a single orbit of radius r_1 are related by $T_1 = -\frac{1}{2}V_1$, and for r_2, $T_2 = -\frac{1}{2}V_2$. Thus for changes in orbits,

$$T_2 - T_1 = -\frac{1}{2}(V_2 - V_1)$$

and defining ΔT as $T_2 - T_1$ and ΔV as $V_2 - V_1$, we get

$$\Delta T = -\frac{1}{2}\Delta V .$$

This result is always true when the initial and final states, as specified by T_1, V_1 and T_2, V_2 respectively, are circular orbits. This result is independent of the form of the velocity dependence of the dissipative force, but does depend on the form of the conservative central force since the assumption made with respect to the force that $\vec{F} = -\frac{k}{r^2}\hat{r} = -\frac{mv^2}{r}\hat{r}$, which is true for circular orbits only, is basic to the derivation of the relation between the kinetic energy and potential energy changes.

Since we can relate the kinetic energy change to the potential energy change, it is also possible to relate the kinetic energy change in time to that of the total mechanical energy, E, of the system, where $E = T + V$. Hence,

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt}$$

but $T = -\frac{1}{2} V$ and $\Delta T = -\frac{1}{2} \Delta V$, a relation which is true for any time interval, infinitesimal or long, so that $\frac{dT}{dt} = -\frac{1}{2}\frac{dV}{dt}$. Making this substitution for $\frac{dV}{dt}$ gives

$$-\frac{dE}{dt} = \frac{dT}{dt} .$$

Thus, for the case of circular orbits, the rate of decrease of total mechanical energy of the system, that is to say the rate of energy dissipation, is exactly equal to the rate of increase of kinetic energy. Since the rates of change are equal, then the integrals of the rates are equal, i.e.

$$\int_0^{\tau} -\frac{dE}{dt}\,dt = \int_0^{\tau} \frac{dT}{dt}\,dt$$

where the limits of integration specify the time interval over which the change in energy occurs and are arbitrary. The change in total energy E in time τ is given by ΔE, and the change in T is ΔT , where

$$-\Delta E = \int_0^{\tau} -\frac{dE}{dt}\,dt = \int_0^{\tau} \frac{dT}{dt}\,dt = \Delta T .$$

Thus, the rate at which energy is stored in the system is equal to the rate at which energy is dissipated by the system, independently of the time interval involved. The net energy dissipated by the system is equal to the net increase in energy stored in the system.

It was suggested by Dr. E. L. O'Neill that a similar result for the energy division, but one which includes the case of elliptical orbits as well, could be obtained by formulating the satellite problem in terms of the Virial Theorem of Clausius.[1]

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1. Goldstein, H., Op. Cit., p.69-71.

The satellite, of mass m and position vectors $\vec{q}$, is acted on by a force $\vec{F}$ such that $\vec{F} = \dot{\vec{p}}$. If we now define a scalar quantity G in terms of components as $G = \vec{p} \cdot \vec{q}$, then the rate of change of G is given by

$$\frac{dG}{dt} = \dot{\vec{q}} \cdot \vec{p} + \dot{\vec{p}} \cdot \vec{q}$$

or

$$\frac{dG}{dt} = 2T + \vec{F} \cdot \vec{q} \quad .$$

Taking the time average,

$$\overline{\frac{dG}{dt}} = \frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt}\, dt = \frac{1}{\tau}\int_0^{\tau} 2T\, dt + \frac{1}{\tau}\int_0^{\tau} \vec{F} \cdot \vec{q}\, dt$$

or

$$\overline{\frac{dG}{dt}} = \frac{1}{\tau}\left[G(\tau) - G(0)\right] = 2\overline{T} + \overline{\vec{F} \cdot \vec{q}} .$$

Consider the motion of the satellite over a short time interval, for example the period of a single orbit, circular or elliptical. If the angle of descent is small, then over a single period $G(\tau) \approx G(0)$, that is the satellite comes back to approximately the same position and velocity so that

$$G(\tau) - G(0) = 0 \quad .$$

Thus

$$2\overline{T} = -\overline{\vec{F} \cdot \vec{q}} \quad .$$

However, if we examine the satellite in its initial and final states where its position and velocity values are not equal then $G(\tau) \neq G(0)$. For this case, consider the satellite at distances r_2 and r_1 , such that at r_2 it is closer to the earth than at r_1. In passing from r_1 to r_2 under the combined effect of gravitational and dissipative force, the satellite may be considered as passing through a series of orbits for each of which $G(\tau) - G(o) = 0$. Thus, since this is true, the sum of all such terms is also zero, and the time average is zero. Thus, for any time interval which is an integral number of periods (or very much larger than a period)

$$\overline{\frac{dG}{dt}} = 0$$

and again

$$2\overline{T} = -\overline{\vec{F}\cdot\vec{q}} \qquad .$$

We wish to consider the changes in orbits from r_1 to r_2 in detail. Suppose

$$\vec{F} = \vec{F}(q) + \vec{f}(\dot{q}) = -\overrightarrow{\frac{\partial V}{\partial q}} + \overrightarrow{f(\dot{q})}$$

where $\vec{f}$ is proportional to the velocity and is not derivable from a scalar potential, then

$$\overline{T} = \frac{1}{2}\,\overline{\vec{\nabla}V\cdot\vec{q}} - \overline{\vec{f}(\dot{q})\cdot\vec{q}}$$

but $\vec{f}(\dot{q}) \propto \vec{\dot{q}}$, or $\vec{f} = -\beta\vec{\dot{q}}$, giving the velocity dependence explicitly. Therefore,

$$\overline{T} = \frac{1}{2}\,\overline{\vec{\nabla}V\cdot\vec{q}} + \overline{\beta\vec{\dot{q}}\cdot\vec{q}} \qquad .$$

For a circular orbit, the frictional force is always tangent to the path, it is perpendicular to the position vector, and the scalar product between $\vec{f}(\dot{q})$ and $\vec{q}$ vanishes as a result. For elliptical orbits, the velocity is symmetric about the major axis, so that the average over an integral number of periods of $\overline{\vec{f}(\dot{\vec{q}}) \cdot \vec{q}}$ vanishes identically. Thus, there is no contribution to the average value of the kinetic energy from the dissipative force. Thus,

$$2\overline{T} = \overline{\vec{\nabla} V \cdot \vec{q}} \quad .$$

This is the general expression for the time average of the kinetic energy. Since the satellite moves in a central force field, $q = r$, and $\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r}$, hence $\overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r} \cdot \vec{r}}$.

If $V = a\, r^{n+1}$, then

$$\overline{T} = \frac{n+1}{2}\overline{V}$$

and if the force is an inverse square law, which it is for gravitational forces, then $n = -2$, and

$$\overline{T} = -\frac{1}{2}\overline{V} \quad .$$

But

$$\overline{T}_{r_1} = \frac{1}{\tau_1}\int_0^{\tau_1} T_{r_1}\, dt \quad ; \quad \overline{T}_{r_2} = \frac{1}{\tau_2}\int_0^{\tau_2} T_{r_2}\, dt$$

where τ_1 and τ_2 are times used in averaging T_1, V_1 and T_2, V_2, and

$$\overline{V}_{r_1} = \frac{1}{\tau_1}\int_0^{\tau_1} V_{r_1}\, dt \quad ; \quad \overline{V}_{r_2} = \frac{1}{\tau_2}\int_0^{\tau_2} V_{r_2}\, dt$$

so that

$$\overline{T}_{r_2} - \overline{T}_{r_1} = \overline{\Delta T} = -\frac{1}{2}\left(\overline{V}_{r_2} - \overline{V}_{r_1}\right) = -\frac{1}{2}\overline{\Delta V}$$

or

$$\overline{\Delta T} = -\frac{1}{2}\overline{\Delta V} \quad .$$

Thus the time average of the kinetic energy is equal to half the time average of the potential energy, and the average of the increase in the kinetic energy is equal to exactly one-half the average of the decrease of the potential energy. This result is independent of the initial and final states designated by T_{r_1} , V_{r_1} and T_{r_2} , V_{r_2} respectively. Thus, when a satellite moving under the influence of an inverse square central force encounters a dissipative force which produces adiabatic changes in its path, the time average of the loss of total mechanical energy is given by

$$-\overline{\Delta E} = -\overline{\Delta T} - \overline{\Delta V} = +\overline{\Delta T}$$

since

$$\overline{\Delta T} = -\frac{1}{2}\overline{\Delta V}$$

that is, the average energy lost in dissipation is equal to the average energy stored in the system. This relation in the time average of the changes means that for suitable choice of τ ,

$$-\frac{1}{\tau}\int_0^\tau \frac{dE}{dt}\,dt = \frac{1}{\tau}\int_0^\tau \frac{dT}{dt}\,dt$$

regardless of the orbital motion. This does not mean that

$$-\frac{dE}{dt} = +\frac{dT}{dt} ,$$

whereas this latter relationship does hold for circular orbits only. Further relations between these quantities can be obtained by using the Rayleigh dissipation function and the Langrangian formulation. Consider the satellite of mass m to be moving under the influence of a central force field $\vec{F} = -(k/r^2)\hat{r}$ in a circular or elliptical orbit. The position of the satellite is described in temrs of the polar coordinates r and θ . The orbit is caused to change as a result of the satellite passing through a dissipative medium. If the dissipative force is linearly dependent on the velocity, i.e. $\vec{F}_{fr} = -\beta\dot{r}\hat{r} - \beta r^2\dot{\theta}\hat{\theta}$, it is derivable from a dissipation function[1] of the form

$$\mathcal{F} = \frac{1}{2}\sum_j \beta_j \dot{q}_j^{\,2}$$

where the sum is over each degree of freedom of the mass m , or for a homogeneous medium,

$$\mathcal{F} = \frac{1}{2}\beta(\dot{r}^2 + r^2\dot{\theta}^2) \quad .$$

Thus, $\vec{F}_{fr} = -\frac{\partial \mathcal{F}}{\partial \dot{q}}$. The kinetic and potential energies of the satellite are $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and $V = -k/r$ respectively. In the Lagrangian formulation, the Lagrangian of the system is defined as $L = T - V$. The equations of motion, including the dissipative force, are of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = F_{fr}$$

where $q_1 = r$ and $q_2 = \theta$. In terms of the dissipation function, the equations are of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} \quad .$$

Since $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ and $\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{k}{r}$ the equations of motions are in fact

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = -\beta\dot{r}$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -\beta r^2\dot{\theta}$$

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1. Landau and Lifshitz, Mechanics, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1960, p.76.

The latter equation is a torque equation and may be written as the time derivative of the angular momentum

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -\beta r^2\dot{\theta} \quad .$$

Thus, in the presence of an external torque acting on the system, the angular momentum is not conserved, and Kepler's second law demanding that the satellite sweep out equal areas in equal times does not hold.

Without solving the equations of motion, the time rate of change of the total dynamical energy can be related to the dissipation function which is quadratically dependent on the velocity, i.e. $\mathcal{F} = \frac{1}{2}\beta\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$, and thus, in extension of the classical equipartition of energy law, would be expected to constribute $\frac{1}{2}$ to the rate of decrease of energy. The dynamical energy of the system can be given by considering the constraints of the system independent of time so that $L \neq L(t)$ explicitly. Then

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial t} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial t} \quad .$$

For a conservative system, $\frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$, therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

and

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial t} = \sum \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right)$$

or

$$\frac{d}{dt}\sum_j\left(L - \dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad .$$

Thus,

$$\sum_j \left(L - \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) = \text{constant} = -E$$

where E is called the dynamical energy of the system. Therefore, the rate of change of dynamical energy in time is given by

$$\frac{dE}{dt} = \frac{d}{dt}\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \right)$$

or

$$\frac{dE}{dt} = \sum_j \frac{\partial \dot{q}_j}{\partial t} \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial t} .$$

Substituting for $\frac{dL}{dt}$, the energy expression may be written as

$$\frac{dE}{dt} = \sum_j \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \dot{q}_j \frac{\partial L}{\partial q_j}$$

or

$$\frac{dE}{dt} = \sum_j \dot{q}_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right] .$$

But for non-conservative systems in which the dissipative force is of the form specified in the statement of the problem, Lagrange's equation is of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = - \frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

so that the rate at which the energy changes may be related to the dissipation function by the following relation

$$\frac{dE}{dt} = - \sum_j \dot{q}_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = -2\mathcal{F}$$

since by Euler's theorem of homogeneous functions

$$\sum_i \dot{q}_i \frac{\partial f}{\partial \dot{q}_i} = mf$$

where $f = a\dot{q}_i^m$. Thus the result is wholly dependent on the power dependence of the dissipation function on the velocity of the system. This result is not limited to the case of circular orbits. Thus, for any orbit, in which $\mathcal{F}$ is quadratic in $\dot{q}_j$'s ,

$$\frac{dE}{dt} = -2\mathcal{F}$$

a result wholly dependent on the form of $\mathcal{F}$. If the orbit considered is circular, then, as was shown before,

$$\frac{dT}{dt} = -\frac{1}{2}\frac{dV}{dt}$$

which means that

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt} = -\frac{dT}{dt} = -2\mathcal{F}$$

or

$$\frac{dT}{dt} = 2\mathcal{F}$$

This result is not true for elliptical orbits.

It is possible to derive this relation between the rates of change of the dynamical energy and the kinetic energy from wholly dynamical arguments using $\mathcal{F} = \frac{1}{2}\beta(\dot{r}^2 + r^2\dot{\theta}^2)$, and making the assumption of quasi-circular orbits so that

$$-\frac{k}{r^2}\hat{r} = -\frac{mv^2}{r}\hat{r}$$

From the preceeding section, we have

$$\frac{dL}{dt} = \sum_j \left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial t}\right)$$

which for a non-conservative system becomes

$$\frac{dL}{dt} = \sum_j \left(\dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial t} + \dot{q}_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} \right)$$

or

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) + \dot{q}_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j} .$$

Now referring to the equations of motion on page 14, this is

$$\frac{dL}{dt} = \frac{d}{dt}\left(m\dot{r}^2 + mr\dot{\theta}^2\right) + \beta\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

or

$$\frac{dL}{dt} = 2\left[mr\ddot{r} + rm^2\dot{\theta}\ddot{\theta} + mr\dot{r}\dot{\theta}^2 + 2\mathcal{F}\right].$$

But

$$m r\ddot{r} - m\dot{r}r\dot{\theta}^2 + \frac{k}{r^2}\dot{r} = -\beta\dot{r}^2$$
$$mr^2\dot{\theta}\ddot{\theta} + 2mr\dot{r}\dot{\theta}^2 = -\beta r^2\dot{\theta}^2$$

therefore

$$\begin{aligned} mr\ddot{r} + mr^2\dot{\theta}\ddot{\theta} + mr\dot{r}\dot{\theta}^2 &= -\beta\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r^2}\dot{r} \\ &= -2\mathcal{F} - \frac{\partial V}{\partial r}\frac{\partial r}{\partial t} \\ &= -2\mathcal{F} - dV/dt \end{aligned}$$

and the rate of change of the Lagrangian is given by

$$\frac{dL}{dt} = -2\mathcal{F} - \frac{dV}{dt} .$$

But for quasi-circular orbits, as distinct from elliptical orbits, $-\frac{k}{r^2}\hat{r} = -\frac{mv^2}{r}\hat{r}$, so that $\frac{1}{2}mv^2 = \frac{1}{2}\frac{k}{r}$, and as was shown before

$$T = -\tfrac{1}{2} V$$

and

$$\frac{dT}{dt} = -\frac{1}{2}\frac{dV}{dt} \quad .$$

Therefore

$$\frac{dL}{dt} = \frac{dT}{dt} - \frac{dV}{dt} = -2\mathcal{F} - 2\frac{dV}{dt}$$

which gives

$$\frac{dT}{dt} = -2\mathcal{F} - \frac{dV}{dt} = -2\mathcal{F} + 2\frac{dT}{dt}$$

or

$$-\frac{dT}{dt} = -2\mathcal{F} = \frac{dE}{dt} \quad .$$

Thus, again, for quasi-circular orbits (for example where $\dot{r}^2 < r^2\dot{\theta}^2$),

$$\frac{dT}{dt} = -\frac{dE}{dt} \quad .$$

Note that this relation between E and T is only true for circular orbits, or quasi-circular orbits, whereas the dependence of E on $\mathcal{F}$ is wholly dependent on the power relation between $\mathcal{F}$ and $\dot{q}_j$. The results are independent of the initial values of T , V or $\mathcal{F}$, of the time lapse involved for the energy change to take place.

The relation between the kinetic energy and the dissipation function allows the time dependence of the kinetic energy to be determined explicitly for the case of a dissipative force linear in the velocity (such as a viscous force obeying Stokes' law), and this result can then be extended to give the parametric equations of motion, without having to solve the equations of motion which arise from the

Lagrangian formulation.

Now,

$$\mathcal{F} = \frac{1}{2}\beta(\dot{r}^2 + r^2\dot{\theta}^2)$$

and for quasi-circular orbits,

$$\frac{dT}{dt} = -\frac{dE}{dt} = 2\mathcal{F}$$

or

$$\frac{dT}{dt} = 2\,\beta/m\,T \quad .$$

Thus

$$T = T_0\, e^{2\beta/m\, t} \quad ;$$

the kinetic energy increases exponentially in time.

The equation of deviation from the path, $r(\theta)$, may be derived from this equation for $T(t)$. From the previous arguments we have, $T = -\frac{1}{2}V$, for near circular orbits, and $T = T_0 e^{2\beta/m\, t}$. But $V = -k/r$, therefore $T = \frac{1}{2}k/r$ and $T_0 = \frac{1}{2}k/r_0$. Substituting into the equation for $T(t)$ gives

$$r = r_0\, e^{-2\beta/m\, t} \quad .$$

The radius decreases exponentially with an increase in time, the rate of decrease depending on the ratio β/m. If $\dot{r}^2 << r^2\dot{\theta}^2$, which is true for small angles of descent, then

$$T = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2}m\,r_0^2\dot{\theta}_0^2\, e^{-2\beta/m\, t}$$

or

$$r^2\dot{\theta}^2 = r_0^2\dot{\theta}_0^2\, e^{-2\beta/m\, t}$$

so that

$$\dot{\theta}^2 = \dot{\theta}_0^2 \, e^{6\beta/m \, t}$$

or

$$\dot{\theta} = \dot{\theta}_0 \, e^{3\beta/m \, t}$$

Thus,

$$\theta = \frac{m}{3\beta} \dot{\theta}_0 \, e^{3\beta/m \, t}$$

but

$$r = r_0 \, e^{-2\beta/m \, t}$$

and

$$r^{-3/2} = r_0^{-3/2} \, e^{3\beta/m \, t}$$

therefore

$$\theta = \frac{m}{3\beta} \dot{\theta}_0 \frac{r_0^{3/2}}{r^{3/2}}$$

or

$$\theta r^{3/2} = \frac{m}{3\beta} \dot{\theta}_0 r_0^{3/2}$$

Since

$$T_0 = \frac{1}{2} m r_0^2 \dot{\theta}_0^2 = \frac{k}{2r_0} \qquad \text{and} \qquad T = \frac{1}{2} m r^2 \dot{\theta}^2 = \frac{k}{2r}$$

then

$$\dot{\theta}_0^2 = \frac{k}{m r_0^3}$$

which gives

$$\theta_0 = \sqrt{\frac{k}{m}}\; r_0^{-3/2} \qquad \text{and} \qquad \theta = \sqrt{\frac{k}{m}}\; r^{-3/2}$$

with the result being

$$r^{3/2} = \frac{m}{3\beta}\sqrt{\frac{k}{m}}\;\theta^{-1} \quad .$$

This is the equation of the path of the satellite moving in near circular orbits through a homogeneous dissipative medium, whenever the dissipative force is linearly dependent on the velocity of the satellite and acts tangentially to the direction of motion of the satellite.

If the dissipative force is of the form

$$\vec{F} = -\beta v^2 \vec{v} = -\beta\left[\dot{r}^2 + r^2\dot{\theta}^2\right]\left[r\,\hat{r} + r\dot{\theta}\,\hat{\theta}\right]$$

then in order that $\vec{F} = -\nabla\mathcal{F}$ where $\mathcal{F}$ is the dissipation function, $\mathcal{F}$ must be of the form $\mathcal{F} = \frac{\beta}{4}\left[\dot{r}^2 + r^2\dot{\theta}^2\right]^2$. And for this relation between $\mathcal{F}$ and the velocities, the time rate of change of total energy becomes

$$\frac{dE}{dt} = -4\mathcal{F} \quad .$$

Now, regardless of the form of the dissipative force, for quasi-circular orbits, the following relation is always true:

$$-\frac{dE}{dt} = \frac{dT}{dt} \quad .$$

Thus we have that

$$\frac{dT}{dt} = 4\mathcal{F}$$

but

$$\mathcal{F} = \frac{\beta}{m^2} T^2$$

so that

$$\frac{dT}{dt} = 4 \frac{\beta}{m^2} T^2$$

or

$$\frac{dT}{T^2} = 4 \frac{\beta}{m^2} dt$$

Integrating and setting the integration constant equal to the value of T at $t = 0$, i.e. $T = T_0$, we have that

$$\frac{1}{T} = \frac{1}{T_0} - \frac{4\beta}{m^2} t$$

or

$$T = \frac{1}{\frac{1}{T_0} - \frac{4\beta}{m^2} t}$$

Thus, as t increases, $\frac{1}{T_0} - \frac{4\beta}{m^2} t$ decreases, and T decreases in time from T_0.

The path deviation $r(t)$ is derivable from

$$T = \frac{1}{2} \frac{k}{r} = \frac{1}{\frac{1}{T_0} - \frac{4\beta}{m^2} t}$$

or

$$\frac{1}{T} = \frac{2r}{k} = \frac{1}{T_0} - \frac{4\beta}{m^2} t$$

and results in the following expression for $r(t)$:

$$r = r_0 - \frac{2\beta}{m^2} kt$$

which indicates that as t increases, r(t) decreases linearly from r_0 . As for the quadratic dissipation function, these equations could be extended to give $\theta(t)$ and then $r(\theta)$. It seems unnecessary to do so since it adds little information to the energy discussion at hand.

In summary, when a satellite moves in an inverse square field, in a quasi-circular orbit, and is simultaneously influenced by a velocity dependent dissipative force, causing an orbital change from one circular orbit to another, end-point arguments show that the resultant increase in kinetic energy is equal to ½ the potential energy decrease, and, that the rate of change of the total dynamical energy of the system equals the negative of the rate of change of kinetic energy, i.e. of the energy stored in the system.

More importantly, discussion of the problem in terms of the Virial Theorem removes the restriction to circular orbits, and gives time averaged energy relations which are applicable to elliptical as well as circular orbits. The central force field result of $\overline{T} = -\frac{1}{2}\overline{V}$ was extended to give the average of the increase in the kinetic energy equal to exactly ½ the average of the decrease of potential energy, independently of the boundary conditions. That is, the average energy lost in dissipation is equal to the average energy stored in the system. This result is wholly dependent on the form of the conservative force field only, not the dissipative force. Since the result is true for any dissipative force-velocity relation, this discourages the desire to extend the equipartition of energy concept, as stated in the Introduction, to cover the dissipative processes.

Although energy expressions, similar to those found via the end-point arguments, are found when considering the problem in terms of Lagrange's equations and a dissipation function derived from a homogeneous Stokes' law force, the elegance of this treatment of the problem is in being able to determine the rate of change of dynamical energy in terms of the dissipation, to relate it to the time rate of change of energy stored in the system, and to determine the path equation of the satellite moving in quasi-circular orbits. Unfortunately, this is not extended to the elliptical case, and thus is not as general a treatment of the satellite problem as that using the Virial Theorem.

The equipartition is wholly dependent on the form of the central force field. Neither the initial and final conditions, the time interval required for the change, nor the form of the dissipation function effect any of the energy relations. The only change due to a different dissipation function is in the factor multiplying β / m in the equation for the rate at which energy is dissipated, and correspondingly changes the path equation derived from this as a basis. The ratio of the rate of dissipation and of kinetic energy increase is not affected by the change.

Chapter II
Capacitor Charging

When a battery charges a capacitor, through a series resistor, the energy delivered by the battery in the charging process is divided equally between the electrostatic energy stored in the capacitor, and later available for work, and the energy dissipated as joule heat in the resistor, regardless of how small the resistor may be. This problem seems similar to the satellite problem, as far as energy divisions are concerned, but it is distinct in many characteristics, which will be discussed in the following text, and in application of the arguments which yield energy partition. In this problem the Virial Theorem does not apply. However, the energy division may be derived from end-point arguments or from purely dynamical arguments, including an "electrical dissipation function".

If the capacitor is initially uncharged, the battery, of constant emf $\mathcal{E}$, does work in depositing each increment of charge dq on the capacitor so that

$$-\Delta W_{battery} = \int_0^Q \mathcal{E}\, dq = \mathcal{E} Q \quad .$$

That is, the energy of the battery decreases by an amount $\mathcal{E}Q$. The energy of the capacitor changes from zero energy to an energy determined from

$$\Delta W_{capacitor} = \int_0^Q V_c\, dq$$

where V_c is the potential drop across the capacitor and is equal to q/C. This results in

$$\Delta W_{capacitor} = \frac{Q^2}{2C}$$

Thus, the energy gained by the capacitor is exactly $\frac{1}{2}$ that delivered by the battery in the charging process. The other half is dissipated in the resistor.

For a more detailed discussion, consider a perfect capacitor, C, initially uncharged, connected through a series resistor, R, and a switch, to a constant source of emf, $\mathcal{E}$. The resistor is considered to be the wire-resistance, but may include the internal resistance of the battery, or of the capacitor thereby assuring perfection in the capacitor. Diagramatically:

As soon as the switch is closed, current flows in the circuit and at any instant the potential drops in the circuit satisfy the equation

$$\mathcal{E} = V_R + V_C$$

where V_R is the voltage drop across the resistor, and V_C is the voltage drop across the capacitor. At any instant of time, $q = CV_C$ is the charge on the capacitor and $i = V_R / R$ is the current through the resistor. Hence

$$\mathcal{E} = iR + q/C$$

where $i = dq/dt$, $\mathcal{E}$ is constant, and the dissipation force is an Ohm's law force for which V_R is linearly related to i.

Note that if q is considered a generalized coordinate, then $i = \dot{q}$ may be considered a generalized velocity and the circuit equations relating these quantities to the potential drops in the circuit, and the potential drops to each other, should be vectorial. For example, they shoul be written in the form $\vec{V}_R = -R\vec{i}$, where the negative sign indicates that the potential drop $\vec{V}_R$ opposes the direction of the current. (It is the constant applied emf which pushes the current against the retardations.) In this form, it is immediately noted that the dissipation force is an Ohm's law force for which $\vec{V}_R$ is linearly related to $\vec{i}$ in exactly the same form $\vec{F}_{fr} = -\beta\vec{\dot{q}}$ as it was for the satellite, but here $\beta \longrightarrow R$, a positive constant. Therefore, $\vec{F}_{fr} \longrightarrow \vec{V}_R \longrightarrow -R\vec{\dot{q}} \longrightarrow -R\vec{i}$ [1], and may be derived from a dissipation function of the form $\mathcal{F}_R = \frac{1}{2}Ri^2$. However, it seems more useful to maintain the standardized notation of electricity for the problem and to omit the vector notation, without loss of generality.

Now,

$$\frac{d\mathcal{E}}{dt} = 0 = R\frac{di}{dt} + \frac{1}{C}\frac{dq}{dt}$$

or

$$R\frac{di}{dt} = -\frac{1}{C}\frac{dq}{dt} = -\frac{1}{C}i$$

so that

$$\frac{di}{i} = -\frac{1}{RC}\,dt$$

which gives

$$i = Ae^{-t/RC}$$

At $t=0$, $q=0$, therefore $\mathcal{E} = i_0R$ or $i_0 = \mathcal{E}/R = A$ so that

$$i = \frac{\mathcal{E}}{R}e^{-t/RC} = i_0e^{-t/RC}$$

and the current decreases as t increases.

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1. The full arrow ($\longrightarrow$) is used to denote "corresponds to"; the half arrow ($\rightharpoonup$) is used to denote vector notation.

Since $\mathcal{E} = iR + q/C$, substituting for $i(t)$ gives $q(t)$, or

$$q = C\mathcal{E}(1 - e^{-t/RC}) = Q(1 - e^{-t/RC})$$

where $Q = C\mathcal{E}$ is the final charge on the capacitor since at equilibrium $i = 0$ and $V_R = iR = 0$ which gives $V_C = \mathcal{E}$.

The energy associated with the capacitor can now be found since the instantaneous power input to the circuit from the battery is defined as $P = \mathcal{E} i$, or

$$P = Ri^2 + \frac{qi}{C}$$

so that

$$P = i^2R + i\frac{Q}{C}(1 - e^{-t/RC})$$

where $\frac{iQ}{C}(1 - e^{-t/RC})$ is the instantaneous rate of power input into the capacitor, and i^2R is the joule heat, the instantaneous rate at which electrical energy is converted into heat energy in the resistor. Since i may be considered a degree of freedom of the system, and since it enters quadratically into the total energy of the system, the classical equipartition of energy law extended to this case might lead to the assumption that this should contribute ½ the total energy of the system, i.e. that the quadratic i will cause a ½ factor to appear in the energy contribution to the system from R so that $Ri^2 = \frac{1}{2}\mathcal{E} i$.

Since the power is defined as the rate at which work is done, the energy given the system by the battery may be determined from

$$-dW_{battery} = \mathcal{E}dq \quad ,$$

where the minus sign indicates work done on the system, which gives

$$-\Delta W_{battery} = \int_0^Q \mathcal{E}dq = C\mathcal{E}^2$$

since $Q = C\mathcal{E}$. The energy stored in the capacitor is obtained from

$$P_c = \frac{dW_c}{dt} = \frac{iQ}{C}\left(1 - e^{-t/RC}\right)$$

or

$$\frac{dW_c}{dt} = \frac{\mathcal{E}Q}{CR}\left(e^{-t/RC} - e^{-2t/RC}\right)$$

since $i = \frac{\mathcal{E}}{R} e^{-t/RC}$. Since it takes an infinite time to charge the capacitor, the change in electrostatic energy stored in the field between the capacitor plates is

$$\Delta W_c = \frac{\mathcal{E}Q}{CR}\left[\int_0^\infty e^{-t/RC}\,dt - \int_0^\infty e^{-2t/RC}\,dt\right]$$

or

$$\Delta W_c = \frac{\mathcal{E}Q}{CR}\left[RC - \frac{RC}{2}\right] = \frac{1}{2}C\mathcal{E}^2$$

which is exactly ½ the energy delivered by the battery.

Since the only remaining circuit element is R, the other half of the battery's initial available energy must go into heat loss in the resistor. That is,

$$P_R = i^2 R = \frac{dW_R}{dt}$$

so that

$$\Delta W_R = \int_0^\infty \frac{\mathcal{E}^2}{R} e^{-2t/RC}\,dt$$

where the limits of integration are the same for the ΔW_C integral since the current flows only as long as charge is being deposited on the capacitor. Integrating, this gives

$$\Delta W_R = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\mathcal{E}^2 .$$

This is equal to ΔW_C , and is exactly $\frac{1}{2}$ the energy delivered to the system by the battery. Since $\Delta W_R = \Delta W_C$, then

$$\Delta W_R = \int_0^\infty \frac{dW_R}{dt} dt$$

and

$$\Delta W_C = \int_0^\infty \frac{dW_C}{dt} dt$$

are also equal. Although the integrals are equal, this does not mean that the integrands are equal, i.e. $\frac{dW_C}{dt} \neq \frac{dW_R}{dt}$. Thus, this relation is not as general as that found for the satellite problem where at any instant of time the energy being added to the system was equal to the rate of change of the energy stored in the system.

The equipartition result found here is independent of the non-zero value of R , and exists regardless of how small R may be. That is, R , although it may be small, may not be considered negligible since it is R in conjunction with C (as the time constant $T = RC$) which determines the rapidity of response of a given system. For a given battery and capacitor, the value of R determines the rate of charging the capacitor, and therefore determines the initial maximum value of the current, $i_o = \mathcal{E}/R$, and the rate at which it falls off to zero, $i \propto e^{-t/RC}$. At $t=0$, $q = Q_o(1 - e^{-t/RC})$ is zero, and $i_o = \mathcal{E}/R$ is a maximum. Thus, all the energy is *initially* dissipated in the resistor, regardless of its small but non-zero value, so that $P_R = V_R i = i^2 R$ is not negligible.

That the joule heat loss is $\frac{1}{2}$ the total energy depends only on the size of the capacitor and battery used, and on the choice of initial and final conditions as we shall now show.

For example, if the capacitor is considered to be initially charged to some value Q_o , but with a potential difference $V_C < \mathcal{E}$, a more physically real case than the preceeding, the energy expressions which result from the calculations are

$$-\Delta W_b = \mathcal{E}(Q - Q_o) = \mathcal{E}(C\mathcal{E} - Q_o) \quad ,$$

for the work done by the battery in charging the capacitor;

$$\Delta W_c = \frac{1}{2C}(C\mathcal{E}-Q_o)(C\mathcal{E}-Q_o) \quad ,$$

for the change in electrostatic potential energy of the capacitor; and

$$\Delta W_R = \frac{1}{2C}(C\mathcal{E}-Q_o)(C\mathcal{E}-Q_o) \quad ,$$

for the energy dissipated in the resistor. Thus

$$\frac{|\Delta W_R|}{|\Delta W_b|} = \frac{1}{2}\left(1-\frac{Q_o}{Q_f}\right)$$

and

$$\frac{|\Delta W_C|}{|\Delta W_b|} = \frac{1}{2}\left(1+\frac{Q_o}{Q_f}\right) \quad .$$

The two ratios are equal only when $Q_o = 0$. Thus, the equipartition of energy does depend on the initial conditions.

As another change in boundary conditions, consider a change in the time interval. That is, instead of considering the process in a time interval $0 \leq t \leq \infty$, consider the process in the time interval $0 \leq t \leq \tau$ where τ is arbitrary. Note that since the energy storage rate is an exponential function which increases in time, i.e. $\frac{dW_c}{dt} = \frac{\mathcal{E}Q}{CR}\left(e^{-t/RC} - e^{-2t/RC}\right)$, to its final value $\left.\frac{dW_c}{dt}\right]_{t=\infty} = \frac{\mathcal{E}^2}{R}$ and the energy dissipation is an exponentially decreasing function of time, i.e. $\frac{dW_R}{dt} = \frac{\mathcal{E}^2}{R}e^{-2t/RC}$, from its initial value $\left.\frac{dW_R}{dt}\right]_{t=0} = \mathcal{E}^2/R$, then the rates are obviously not equal for an arbitrary time interval. Therefore, as was noted on page 31,

$$\frac{dW_C}{dt} \neq \frac{dW_R}{dt}$$

and this is true for any time interval $0 \leq t \leq \tau$.

If one carried through the details of this argument , one would find that

$$\Delta W_b = -C\mathcal{E}^2\left(1-\frac{V_o}{\mathcal{E}}\right)\left(1-e^{-t/RC}\right),$$

$$\Delta W_c = \frac{1}{2}C\mathcal{E}^2\left(1-\frac{V_o}{\mathcal{E}}\right)\left(1-e^{-t/RC}\right)\left[\left(1+\frac{V_o}{\mathcal{E}}\right)-\left(1-\frac{V_o}{\mathcal{E}}\right)e^{-t/RC}\right],$$

and

$$\Delta W_R = \frac{1}{2}C\mathcal{E}^2\left(1-V_o/\mathcal{E}\right)^2\left(1-e^{-t/RC}\right)\left(1+e^{-t/RC}\right).$$

The sum of ΔW_c and ΔW_R is equal to $-\Delta W_b$ as it should be, but $\Delta W_R \neq \Delta W_c$ for arbitrary time intervals, so that $\frac{dW_c}{dt} \neq \frac{dW_R}{dt}$. Assigning different values to the ratio of $\frac{V_o}{\mathcal{E}} = \alpha$ will give different values of ΔW_R, ΔW_c, $-\Delta W_b$, and $\Delta W_R/\Delta W_b$ and $\Delta W_c/\Delta W_b$, and it can there be seen that only for the case where the time interval is $0 \leq t \leq \infty$ and the value of α is zero so that the capacitor is initially uncharged does the "equipartition" occur.

Thus the equipartition depends on the choice of initial voltage of the capacitor and occurs only in the case for which $V_o=0$. It is also dependent on the time intervals involved, and on the final state of the system.

To show that the result is independent of the form of the dissipative force, consider the following. Instead of the Ohm's law, which, like Stokes' law, is a linear force-velocity relation, assume the dissipative force dependent on some other power of the current. Let the force be expressed in terms of odd powers, to avoid difficulty with the sign of the dissipation term, i.e. power loss, upon reversal of the sign of the emf. That is,

$$F_{fr} = R_{2n+1}\,\mathcal{E}^{2n+1} = V_R$$

or, in terms of the current,

$$F_{fr} = -R_{2n+1}\, i^{2n+1} = V_R$$

where the minus sign again indicates that the potential drop opposes the direction of the current, as was pointed out on page 27 bottom. The potential drops in the circuit satisfy the equation

$$\mathcal{E} = R_{2n+1}\, i^{2n+1} + q/C$$

or, taking the time derivative,

$$0 = R_{2n+1}(2n+1)\, i^{2n}\frac{di}{dt} + i/C \quad .$$

This gives

$$i^{2n-1}\, di = -\frac{dt}{(2n+1)R_{2n+1}C}$$

or

$$i^{2n} = -\frac{2nt}{(2n+1)R_{2n+1}C} + (C')^{2n} \qquad , n>0$$

and since at $t = 0$, $q=0$, $i^{2n+1} = \dfrac{\mathcal{E}}{R_{2n+1}} = I_o$, then

$$i^{2n} = -\frac{2nt}{(2n+1)R_{2n+1}C} + I_o^{2n} \quad ,$$

or

$$i = \left[-\frac{2nt}{(2n+1)R_{2n+1}C} + I_o^{2n}\right]^{1/2n} \quad .$$

The power dissipation term in the resistor is calculated as before from

$$P = V_R i = R_{2n+1}\, i^{2n+1}\, i = \frac{dW_R}{dt} \quad .$$

Thus,

$$\Delta W_R = \int_0^{t,i=0} R_{2n+1}\, i^{2n+2}\, dt \qquad ;$$

where the upper limit of integration t,i=0 indicates complete charging of the capacitor since $i = \left[\frac{-2nt}{(2n+1)R_{2n+1}C} + I_0^{2n}\right]^{1/2n}$ becomes equal to zero at that instant of time. This will obviously occur at a value of $t < \infty$.

Now,

$$\Delta W_R = R_{2n+1} \int_0^{t,\,i=0} \left[\frac{-2nt}{(2n+1)R_{2n+1}C} + I_0^{2n}\right]^{\frac{2(n+1)}{2n}} dt$$

$$= -\frac{1}{2} R_{2n+1}^2 C \left[\frac{-2nt}{(2n+1)R_{2n+1}C} + I_0^{2n}\right]_0^{\frac{(2n+1)R_{2n+1}C I_0^{2n}}{2n}}$$

$$= \frac{1}{2} R_{2n+1}^2 C \left[\left(I_0^{2n}\right)\right]^{\frac{2n+1}{n}} = \frac{1}{2} C R_{2n+1}^2 I_0^{2(2n+1)} \quad .$$

But

$$V_R = \mathcal{E} = R_{2n+1} I_0^{2n+1} \qquad ,$$

thus

$$\Delta W_R = \frac{1}{2} C \mathcal{E}^2 \qquad .$$

Therefore, equipartition of energy occurs whenever the dissipation force is of the form $F = -R_{2n+1}\, i^{2n+1}$. This includes all odd powers of i.

Thus, the dependence of the equipartition result is wholly on the constancy of $\mathcal{E}$, on the initial state of the capacitor, and, through the time interval requirement, on the final conditions. If we desired to charge the capacitor with smaller loss of energy, i.e. with an efficiency greater than 50%, we would replace the single source by N equal sources such that the emf of the cells used in each step is related to the final potential by $V_n = \frac{n}{N} V_o$. The V_n is the potential of the battery used in the nth step of the process. The efficiency η for an N step charging of a capacitor is

$$\eta = \frac{N}{N+1} .$$

In review, when a capacitor is charged through a series resistor from a single source of constant emf, the process is 50% efficient, independently of the magnitude of the resistance. The efficiency, as in the case of the satellite, is independent of the form of the dissipative function, i.e. does not depend on Ohm's law being satisfied by the resistor, but does depend on the initial and final conditions, unlike the satellite, and on the time intervals involved. The rates of change of energy stored and dissipated are not equal.

An initial expectation that the Stokes' law type dissipative force, derivable from a dissipative function $\mathcal{F} = \frac{R i^2}{2}$, might allow extension of the classical equipartition of energy law was not realized since this was not the case. The equipartition, as we have said, is independent of the form of the dissipative force, and it does not apply to the rates of dissipation and storage of energy either. It was shown that even when the dissipative force is of the form $V_R = -R_{2n+1}\, i^{2n+1}$ the equipartition of energy occurs. That the equipartition is independent of the magnitude of the resistor and of the form of the dissipative force is one of the most important results in Chapter II.

Chapter III

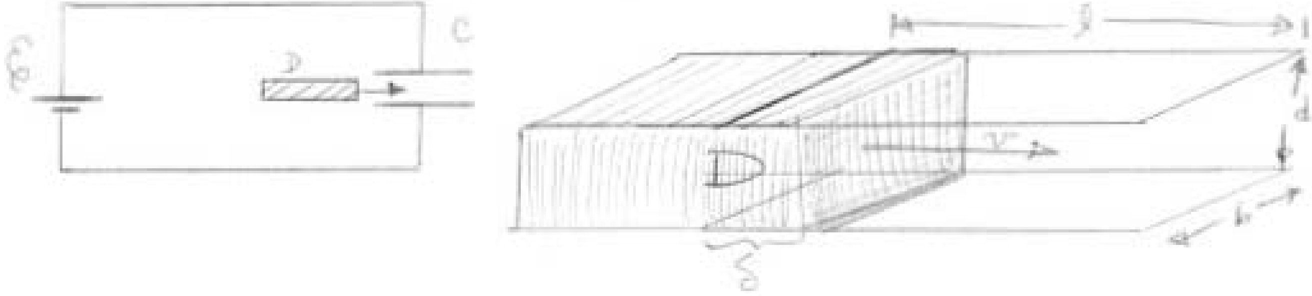
Capacitor-Dielectric

Consider the case of a capacitor being filled with an ideal dielectric, i.e. one which is isotropic, homogeneous, and linear so that $\vec{D}$, $\vec{E}$, and $\vec{P}$ are linearly proportional. Choosing the MKS system of units, the relation between $\vec{D}$ and $\vec{E}$ is written $\vec{D} = \epsilon\vec{E} = \epsilon_0 \kappa \vec{E}$ where κ is the dielectric constant, and is unity for air. The process can be carried out by one of two basic methods, either by keeping the charge Q on the plates constant, or by keeping the potential difference, V_C , constant which is the case to be considered in detail. For either case, the final value of the capacitance is greater than that of the air-filled capacitor, i.e. $C_d > C_{air}$.

For the constant voltage case, the potential between the capacitor plates is kept constant by connecting the capacitor to a battery of constant emf, $\mathcal{E}$. The connection is made by low-valued resistance wire so that the joule heat loss may be so small as to be neglected. The resistance of the wire, R , is small and the current, i ,through the resistor, which depends wholly on the velocity of the dielectric, is also small since the dielectric moves slowly. Thus, since the current is small, $P = i^2R$ is very small and may be neglected in comparison to the power losses associated with the capacitor. That is, at any instant of time, $V_R = iR < V_C = q/C$.

Note that for the case of the capacitor problem discussed in Chapter II, i depends on $\mathcal{E}$ and is not small, although R may be, so that $P = i^2R(= \mathcal{E}^2/R$ initially) is quite large, and is not negligible.

The air filled capacitor of this section is considered after it is fully charged (hereafter designated as $t = 0$) so that the potential difference across the capacitor is equal to the emf of the battery and a current no longer flows until the dielectric is introduced.



The initial configuration is an equilibrium configuration with $q = Q_0$, $C = C_0 = \frac{\epsilon_0 A}{d}$ and $V_C = \mathcal{E} = V_0$ and $i = 0$. The potential energy stored in the field between the air filled capacitor plates is

$$\mathcal{U}_i = \frac{1}{2} C_0 V_0^2 = \frac{1}{2} C_0 \mathcal{E}^2 \quad .$$

A dielectric, moving through a resistive medium which causes it to move without acceleration, is introduced into the field region a distance $\delta = x_0 << \ell$. The potential difference across the plates tends to drop, due to the change in E, the field between the plates, caused by the presence of the dielectric of mass m and its polarization charges, and a current flow is set up by the battery which acts to maintain V_C at its equilibrium value $\mathcal{E}$, and this is done by an increase in the amount of charge by an amount dq. The rate at which charge is deposited depends on the rate at which the dielectric moves into the field region. When the dielectric is wholly in the region between the plates, the capacitance has increased from $C_0 = \epsilon_0 A/d$ to $C = \epsilon A/d = \kappa C_0$. But

$$C = \frac{|Q|}{|\Delta V|} = \kappa C_0 = \kappa \left[\frac{|Q_0|}{|\Delta V|} \right]$$

hence the charge deposit has increased to $Q = \kappa Q_0$, for the same value of $V_0 = \mathcal{E}$, so that the energy stored in this final configuration is

$$\mathcal{U}_f = \frac{1}{2} C \mathcal{E}^2 = \frac{1}{2} K C_0 \mathcal{E}^2 > \mathcal{U}_i \quad .$$

The change in the stored energy is

$$\Delta \mathcal{U}_c = \mathcal{U}_f - \mathcal{U}_i = \frac{1}{2} C_0 \mathcal{E}^2 (K-1) \quad .$$

But the battery does work in charging the capacitor from Q_0 to $K Q_0$. This work is

$$-\Delta W_b = \int_{Q_0}^{KQ_0} V_c \, dq = \mathcal{E} Q_0 (K-1) = C_0 \mathcal{E}^2 (K-1) .$$

Therefore, there is a decrease in the battery's energy which goes into energy stored in the capacitor. But

$$-\Delta W_b = 2(\Delta \mathcal{U}_c)$$

which means that the work done by the battery is not wholly stored in the capacitor field. The other one-half should be found in the mechanical work done on the dielectric. This work may be used in one of two ways: it may be used to accelerate the slab, in the absence of friction, such that the rate of this dissipation is equal to the rate of storage of energy in the capacitor; or the work may be dissipated in such a way that the dielectric always has negligible acceleration, in which case the frictional force is at all times just equal to the electrical force. It is the latter case which we wish to consider here, for which, the increase in potential energy of the field and the work done by the field in moving the dielectric should be equal, and their rates of change should also be equal.

In order to find the work done on the dielectric, it is necessary to determine the force exerted on the dielectric. This force should be derivable from the negative gradient of the total potential energy of the system, U_T. If the system is defined to include the battery, then the total potential energy is that stored in the battery plus that stored in the field, i.e.

$$\mathcal{U}_T = \mathcal{U}_c + W_b \qquad .$$

Hence,

$$\Delta \mathcal{U}_T = \Delta W_b + \Delta \mathcal{U}_c$$

but

$$-\Delta W_b = 2(\Delta \mathcal{U}_c)$$

so that

$$\Delta \mathcal{U}_T = -2(\Delta \mathcal{U}_c) + (\Delta \mathcal{U}_c) = -(\Delta \mathcal{U}_c).$$

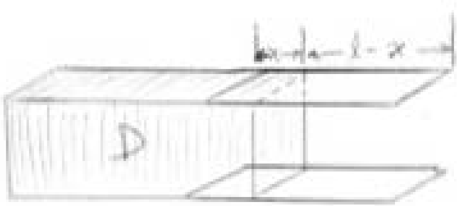
Thus,

$$F_x^V = -\left(\frac{\partial \mathcal{U}_T}{\partial x}\right)_V = -\left[\frac{\partial}{\partial x}\left(-\mathcal{U}_c\right)\right] = \left(\frac{\partial \mathcal{U}_c}{\partial x}\right)_V$$

or

$$F_x^V = \left(\frac{\partial \mathcal{U}_c}{\partial x}\right)_V$$

where x is the axis of the capacitor. The potential energy stored in the capacitor's field may be written as a function of x since the capacitor acts as two in parallel, one air-filled and one filled with the dielectric. That is



$$; \quad \mathcal{U}_x = \mathcal{U}_{dielectric} + \mathcal{U}_{air} \, .$$

Now,

$$\mathcal{U}_x = \frac{1}{2} C \mathcal{E}^2$$

where

$$C = KC_o \frac{x}{\ell} + C_o \frac{\ell - x}{\ell}$$

or

$$\mathcal{U}_x^v = \frac{1}{2} \frac{C_o \mathcal{E}^2}{\ell} \left[x(K-1) + \ell \right]$$

so that when $x = 0$, $\mathcal{U}_x = \frac{1}{2} C_o \mathcal{E}^2$ and when $x = 1$, $\mathcal{U}_x = \frac{1}{2} K C_o \mathcal{E}^2$.
Therefore,

$$\Delta \mathcal{U}_x^v = \mathcal{U}_x - \mathcal{U}_i = \frac{1}{2} \frac{C_o \mathcal{E}^2}{\ell} \left[x(K-1) + \ell \right] - \frac{1}{2} C_o \mathcal{E}^2$$

$$= \frac{1}{2} C_o \mathcal{E}^2 \frac{x}{\ell} (K-1) .$$

Since $K > 1$ always, $\mathcal{U}_c$ increases as x increases. Therefore, the force

$$F_x^v = \left(\frac{\partial \mathcal{U}_c}{\partial x} \right)_V = \frac{1}{2} \frac{C_o \mathcal{E}^2}{\ell} (K-1)$$

is a constant positive force independent of x which indicates that we must postulate a constant dissipative force if we wish to avoid acceleration. The work done by the force is

$$\Delta W_{F_x}^v = \int_0^x F_x dx = \frac{1}{2} C_o \mathcal{E}^2 \frac{(K-1)}{\ell} \int_0^x dx$$

$$= \frac{1}{2} C_o \mathcal{E}^2 \left(\frac{K-1}{\ell} \right) x = \Delta \mathcal{U}_c^v .$$

If the energy delivered by the battery is only that used in charging the capacitor from Q to $Q' = C\mathcal{E}$, then

$$-\Delta W_b = \int_{Q_o}^{Q'} \mathcal{E}\,dq = \mathcal{E}(Q' - Q_o)$$

$$= \mathcal{E}^2\left[\left(KC_o\frac{x}{\ell} - C_o\frac{\ell - x}{\ell}\right) - C_o\frac{\ell}{\ell}\right]$$

or

$$-\Delta W_b = \frac{C_o\mathcal{E}^2}{\ell}(K-1)x \quad .$$

Hence, the energy supplied to the system is equally divided between energy stored and energy lost as work done on the dielectric. But

$$-\Delta W_b = -\int_0^t \frac{dW_b}{dt}\,dt \quad ,$$

$$\Delta U_c = \int_0^t \frac{dU_c}{dt}\,dt \quad ,$$

and

$$\Delta W_{F_x} = \int_0^t \frac{dW_{F_x}}{dt}\,dt \quad .$$

And furthermore, $\Delta U_c = \Delta W_{F_x}$ for any interval δx and δt, thus not only is

$$\int_0^t \frac{dU_c}{dt}\,dt = \int_0^t \frac{dW_{F_x}}{dt}\,dt$$

but since this is true for any time interval, then

$$\frac{dU_c}{dt} = \frac{dW_{F_x}}{dt} \quad .$$

That is, just as in the case of the satellite in the circular orbit, the rates of change of energy are also equal. The argument may be extended to give

$$-\frac{dW_b}{dt} = 2\frac{dU_c}{dt} = 2\frac{dW_{F_x}}{dt} \quad .$$

These energy relations are true for any interval δx and δt. The results are thus true independently of the initial and final conditions of the problem, as was the case for the satellite problem. Therefore, only when the change is independent of the initial and final conditions do the rates of change of energies seem to be equal.

If the dielectric were drawn into the capacitor plates at constant charge rather than constant voltage, the change in electrostatic energy equals the mechanical work done in moving the dielectric. The force drawing the dielectric in is F_x^Q and can be shown to be equal to

$$F_x^Q = \frac{1}{2}\frac{Q_o^2}{C_o}\ell\left[\frac{K-1}{\ell + x(K-1)^2}\right]$$

which is not constant but decreases as x increases. This is not equal to F_x^V, a constant force, in general, although the following statements erroneously indicate otherwise:

> "The force in both cases must be the same, since it depends on the initial state of the system which may be expressed in terms of either charge or potential."[1]

> "For a given configuration of the system the forces are the same, or course, whether charges or potentials are kept constant."[2]

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1. Scott, W. T., The Physics of Electricity and Magnetism, John Wiley and Sons, Inc., New York, 1960, pp.39-40.

2. Page and Adams, Principles of Electricity, D. Van Nostrand Co., Inc., Princeton, N.J., 1958, p.62.

Contrary to the first quotation, the force does not depend only on the "initial state of the system" but on both the initial and final conditions, and since the final energy states differ in the two methods, i.e. constant potential and constant charge, then there will be a difference in the magnitudes of the forces acting on the dielectric. For constant potential, the force is constant; and for constant charge, the force falls off from the same initial value.

Chapter IV
Problems in Which Energy Equipartition Occurs

The following problems, found in the literature or suggested by Dr. Michael Rice, are statements of processes in which an equipartition of energy takes place. Although all were not included in detail in this text, the problems chosen were considered representative of those listed here, had dissipative forces which were known explicitly and therefore the energy dissipated therein could easily be calculated.

1. Satellite[1]: A satellite of mass m moves about the earth of mass M in a near circular orbit which continually decreases in radius due to the presence of a resistive medium. For any change in the orbits, the resultant potential energy change is found to be equally divided between the increase in the energy dissipated as the satellite moves through the resistive medium and the energy stored in the system.

2. Sand on Conveyer Belt[2]: Sand pours at a constant rate onto a conveyer belt moving horimontally. Calculate the force required to keep the belt moving with a constant velocity. Calculation of the force from the time rate of change of momentum yields a horizontal force twice as large as that calculated to give the sand its kinetic energy.

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1. Mills, B. D., Jr., Am. J. Phys. 27, 115 (1959).
   Bacon, R.H., Am. J. Phys. 27, 69 (1959).
2. This problem was suggested by Dr. Rice. Hereafter, all problems which were suggested by Dr. Rice will not be foot-noted, all others will be noted appropriately.

3. Rope-pull[1]: A rope is pulled with constant velocity across a smooth floor. Of the energy supplied, only one-half goes into increasing the kinetic energy of the moving rope. Account for the distribution of energy.

4. Train in trough[2]: A locomotive scoops up water from a trough of given length while moving at a constant velocity. Find the additional force of traction which must be applied by the locomotive in order to keep its velocity constant. The work done by this force is equally divided between the captured water's kinetic energy of mass motion with the locomotive, and all other forms of energy, such as energy of agitation within the tank, potential energy due to the elevation, and heat energy due to friction and partial elasticity of impact.

5. Condenser-charging: A battery, switch, resistor and capacitor are connected in series. When the switch is closed, the battery delivers an amount of energy which is equally divided between energy stored in the electric field between the plate, and the energy dissipated as joule heat in the resistor.

6. Coupled-capacitors[3]: A capacitor of capacitance C_1 and charge q_o is connected in parallel to an equal (i.e. C_1=C_2) but uncharged capacitor C_2 . Comparison of the initial and final configurations of the system indicates that

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1. Sleator, W.W., Am. J. Phys. 15, 474 (1947).; Hull, G.F., Am. J. Phys. 16,447 (1948).; Stephenson, R. J., Am. J. Phys. 17, 224 (1949).; Chapin, E. K., Am. J. Phys. 19, 6 (1951).
2. Dadourian, H.M., Am. J. Phys. 16, 344 (1948).
3. Zucker, C., Am. J. Phys. 23, 469 (1955).

of the energy in C_1 initially available to the system only one-half is found stored in the parallel configuration of C_1 plus C_2.

7. Putty-balls[1]: Two particles of masses m_1 and m_2, chosen such that $m_1 = m_2$, are allowed to collide inelastically. The kinetic energy of the system after collision is one-half the kinetic energy of the system before the collision takes place.

8. Coupled fly-wheels: If two fly-wheels, of moments of inertia I_1 and I_2 such that $I_1 = I_2$, one initially stationary, the other turning about an axis through its center, are coupled in a frictional process, the energy of the final configuration is one-half that of the initially moving fly-wheel.

9. Spring (extension): An extension spring of constant κ, is initially at rest in a position of slight extension under the force of the weight of a hook, attached to the free end of the spring. If an additional mass is added to the hook, the spring undergoes simple harmonic motion resulting in a change in gravitational potential. This results in a decrease in the potential energy of the mass, and this decrease is found to be equally divided between potential energy stored in the spring and kinetic energy of the spring. There continues a constant interchange of kinetic and potential energies in the entire system until the imperfect spring causes the motion to die out.

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1. Bacon, R. H., Am. J. Phys. _8_, 154 (1940). This was also suggested by Dr. Rice.

10. Capacitor-dielectric: A capacitor is connected to a battery which maintains a potential difference $\mathcal{E}$ between the conducting surfaces. A slab of dielectric, of dielectric constant κ, is inserted between the surfaces, completely filling the space between them. The work done by the battery during the insertion process is that in charging up the capacitor, $q_o\mathcal{E}(\kappa-1)$ where q_o is the charge on the capacitor surfaces before the slab is inserted. When the slab completely fills the space between the surfaces, the energy stored in the capacitor has been increased by $\frac{1}{2}q_o\mathcal{E}(\kappa-1)$.

11. Flywheel-increasing in moment of inertia [1]: A constant torque is applied to a body rotating about a fixed axis with a constant angular velocity ω. The angular acceleration which would normally result is prevented by increasing the moment of inertia of the system at the rate dI/dt. In terms of ω and $\frac{dI}{dt}$, express the power required and account for the distribution of energy which results. Neglect friction.

12. Conductor of increasing inductance [2]: A source of constant emf is connected to a pair of long parallel conductors. A conductor, free to move at the rate dr/dt, is placed in the circuit. When it moves the inductance of the circuit increases, maintaining I constant. Find the energy required of the source; account for the distribution of the energy supplied.

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1. Chapin, E.K., Am. J. Phys. 19, 6 (1951).
2. Chapin, E.K., Op. Cit.

13. Inductor-paramagnetic substance: A long solenoid of N turns, and length l, carries a current I, the resulting magnetic field is not quite uniform. A rigid body of permeability μ, and volume v is brought into the field at a point where the magnetic induction was $\vec{B}$. Suppose the volume v of the body is small enough that $\vec{B}$ does not vary appreciably throughout this volume and that the magnetic susceptibility of the body is extremely small compared to unity (the case of ordinary paramagnetic or diamagnetic materials) so that we may consider the induction $\vec{B}$ unaltered by the presence of the body. The work done by the battery is $v(\mu-\mu_0)H^2$, but the energy stored in the magnetic field when the material is inserted is $\frac{v}{2}(\mu-\mu_0)H^2$. Account for the other half of the energy.

Diagramatically, these problems may be grouped together as shown on page 50. Note that it is also possible to list problems 5-7 in Group I under Group III depending on the definition of the system. This listing is not to be considered as the only possible listing of the problems, others may exist. This list was formulated through considerations resulting from end-point arguments.

In Group I, the equipartition of energy which occurs in each of the problems can be extended to give equal differential energy relations. The same is true of the problems in Group III. For the problems in Group II, however, only equal integral relations involving the energies may be deduced.

Dissipative Problems
(all yield equipartition of energy)

Internal Dissipative Forces

Group I
(rates of energy change are equal and independent of initial and final conditions)

1. capacitor-dielectric
2. inductor-paramagnetic substance
3. fly-wheel of increasing moment of inertia
4. conductor of increasing inductance
5. train in trough
6. sand on conveyer belt
7. rope increasing in mass

Group II
(rates of energy change are not equal and depend on initial and final conditions)

1. condenser-charging
2. coupled-capacitors
3. coupled fly-wheels
4. putty balls
5. spring

External Dissipative Forces

Group III
(rates of energy change are equal and independent of initial and final conditions)

1. satellite

Conclusion

There are many processes which take place (see Chapter IV) such that the energy delivered to a system is divided between the energy stored in the system, and later available for work, and the energy dissipated by the system. The division further yields that some processes are characterized by equal time-differential energy relations, while others are characterized by equal time-integral energy relations. Thus, the equipartition of energy is sometimes dependent on any or all of the following; the initial and final conditions, on the time intervals required for the changes to take place, and on the form of the force in the energy source. It is definitely independent of the form of the dissipative force, and thus negates the possibility of any extension of the classical equipartition of energy law, although this force may be useful in relating rates of energy changes and in developing other parametric relations.

Only three of the problems listed in all the groups in Chapter IV were considered in the main text of the thesis, in detail, one from each group listed. In general, these were chosen because the characteristics were the same as those of the other problems in the same group; because the form of their dissipative force was immediately known, and thus allowed more easily the detailed calculations for energy partitions than did other problems in the same group; and because they were more interesting than the others as a result of being extended to include other results than just the details of the energy partitions. For example, a differential equation was derived for the kinetic energy as a function of time in the satellite problem of Chapter I, and this equation was used to derive the path equation of the satellite. In Chapter II, we demonstrated the nature of the charging process with a hypothetical "cubic" Ohm's law.

Furthermore, the satellite was chosen for Group III since it allowed use of the Virial Theorem whereas the other impact problems (listed as 5-7 in Group I here) did not. (It is also considered an impact problem since the dissipative force is considered to be a series of impulses which serve to retard the satellite.) By use of the Virial Theorem, a more general result, including elliptical as well as circular orbits, was obtained. That is, the time averages of the energies were related. As an added attraction, it introduced the idea of the dissipation function rather nicely into the discussion of the problem of equipartition of energy, especially by allowing extension of the argument using this function to give the path equation of the satellite as it moved through the dissipative medium.

In Group I and Group II, the capacitor problems were chosen because of the known form of the dissipative force. The problems in Group II are such that the systems involved are coupled systems. The capacitor charging may be considered a coupled system by considering the battery as coupled to the capacitor, thus the physical set-up is analagous in all problems. The systems in Group I each have something added to the system, thus resulting in the energy transfer and in the equipartition of energy. The capacitor-dielectric, because of the erroneous force discussions found in some texts on Electricity, was chosen to represent this group. Dr. Rice suggested that an interesting extension of this problem would be the calcul^a^tion of the equations of motion, assuming little or no friction, subject to a driving force which is a periodic step function.

The problems listed here as "equipartition" problems should not be considered as all-inclusive, but it is hoped that should other problems arise, the brief discussions above and the diagram on page 50, will prove helpful in catagorizing the new problems and thus immediately give the energy relations to be expected for the new problems, without going through detailed calculations, and perhaps indicate possible extensions to other characteristics of these problems, as was the case of the satellite-path equation, and the capacitor charging independence of the form of the dissipative force.

The characteristics and results of the problems considered in detail and of the others mentioned above were not sufficiently inter-related to enable at present the desired development of a theorem to govern the problems considered. Although few of the articles listed in the bibliography developed to such an extent the detailed problems or those listed in Chapter IV, the possibility should not be excluded that physical laws and/or principles other than those listed in the introduction, may be found or extended which may satisfy the requirement of equipartition of energy.

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