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Singular solutions of ordinary differential equations of simple degree and order ..

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Boston University

Graduate School

Thesis

Singular Solutions of Ordinary Differential Equations
of Simple Degree and Order.

by

Alice Elizabeth Ward

(B.A., Wellesley, 1929)

submitted in partial fulfillment of the
requirements for the degree of

Master of Arts

1931

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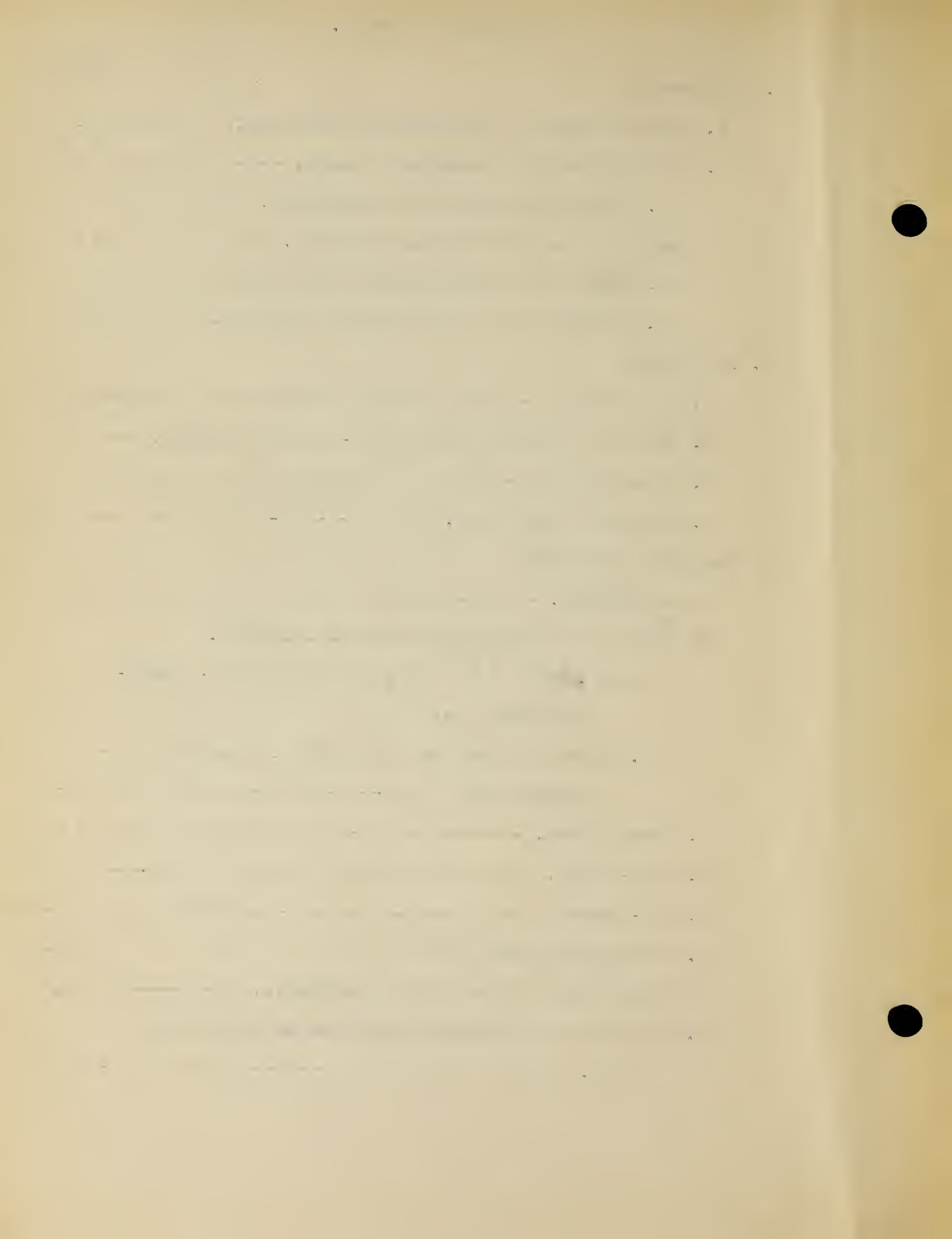
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Introduction.

Professor Piaggio in his book "An Elementary Treatise on Differential Equations and Their Application" remarks in his preface that "the Theory of Differential Equations is the most important branch of modern mathematics. The subject may be considered to occupy a central position from which different lines of development extend in many directions." It seems to play an important part in problems in physics, mechanics, and chemistry. We are going to consider only a small portion of this field, namely, singular solutions of ordinary differential equations.

Definitions.

By an ordinary differential equation we mean an equation in which there is only one independent variable and which involves derivatives. Thus (1) $\frac{d^2y}{dx^2} = \sqrt{1 + y^2} + \frac{dy}{dx}$ and (2) $y - x\frac{dy}{dx} + 3\frac{dx}{dy} = 0$ are ordinary differential equations. The order of the equation is the order of the highest derivative in the differential equation. Thus equation (2) is of the first order and (1) is of the second. To find the degree of a differential equation it is necessary first to rationalize it and clear of fractions. Then the degree of the equation is the exponent of the highest ordered derivative in it. Hence equations (1) and (2) mentioned above are of the second degree. $x^2y^3dx + x^3y^2dy = 0$ is of the first, as is $\frac{d^2y}{dx^2} + a^2y = 0$. We are only going to consider those of low degree and order. A final definition, so that we may know what our topic means, is that of a solution which may be defined as a relation which connects the dependent and independent variables and satisfies the equation.

¹ Piaggio, Differential Equations, p.v.

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Envelope.

As envelopes are so closely connected with the subject of singular solutions we will first examine these.

Definition: An envelope is the locus of the limiting positions of the points of intersection of each curve of a family of curves, $f(x,y,c)=0$, with a nearby curve as the nearby curve approaches the former.

The equation $f(x,y,c)=0$, where c is an arbitrary constant, I.E. we may assign a value to it at pleasure, represents a family of curves, for for each value c we get a curve of the family. Then $f(x,y,c_0)=0$ represents one curve, where c_0 is some value assigned to c . The equation of a nearby curve will be $f(x,y,c_0+\Delta c)=0$, where Δc is a small constant. Solving these two equations

$\begin{cases} f(x,y,c_0)=0 \\ f(x,y,c_0+\Delta c)=0 \end{cases}$ for x and y we would get the points of intersection of the two nearby curves. $f(x,y,c_0+\Delta c) - f(x,y,c_0) = 0$ represents a c curve through the intersection. Dividing by Δc we get

$$\frac{f(x,y,c_0+\Delta c) - f(x,y,c_0)}{\Delta c} = 0.$$

Letting Δc approach 0 we get nothing more than the first derivative or $\frac{\partial f(x,y,c)}{\partial c} = 0$ which represents a curve through the limiting points of intersection of the two nearby curves which approach coincidence as Δc approaches 0. Solving our original equation with this, I.E.

$\begin{cases} f(x,y,c_0)=0 \\ \frac{\partial f(x,y,c_0)}{\partial c} = 0 \end{cases}$ for x and y we would obtain the intersections of the

curve with the consecutive one. According to our definition we want the locus of the intersections. By letting c vary, ^{and eliminating c} $\begin{cases} f(x,y,c)=0 \\ \frac{\partial f(x,y,c)}{\partial c} = 0 \end{cases}$ we get the locus which we defined as the envelope of the family.

In other words, $\begin{cases} f(x,y,c)=0 \\ \frac{\partial f(x,y,c)}{\partial c} = 0 \end{cases}$ is the equation of the envelope in parametric form, where c is the parameter. This is known as finding the c -discriminant.

Received of the Treasurer of the State of New York

the sum of Five Hundred Dollars

for the purchase of the land described in the

order of the Board of Regents of the University of the State of New York

dated the 10th day of June 1879

and the receipt of the Treasurer of the State of New York

dated the 10th day of June 1879

is hereby acknowledged

in witness whereof I have hereunto set my hand and seal

this 10th day of June 1879

at Albany, New York

John W. Alden

Secretary of the Board of Regents of the University of the State of New York

and the receipt of the Treasurer of the State of New York

dated the 10th day of June 1879

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Example: Find the envelope of the family of parabolas $y^2 = a(x - a)$.¹

Differentiating with respect to a :

$$0 = x - 2a, \quad a = \frac{x}{2}$$

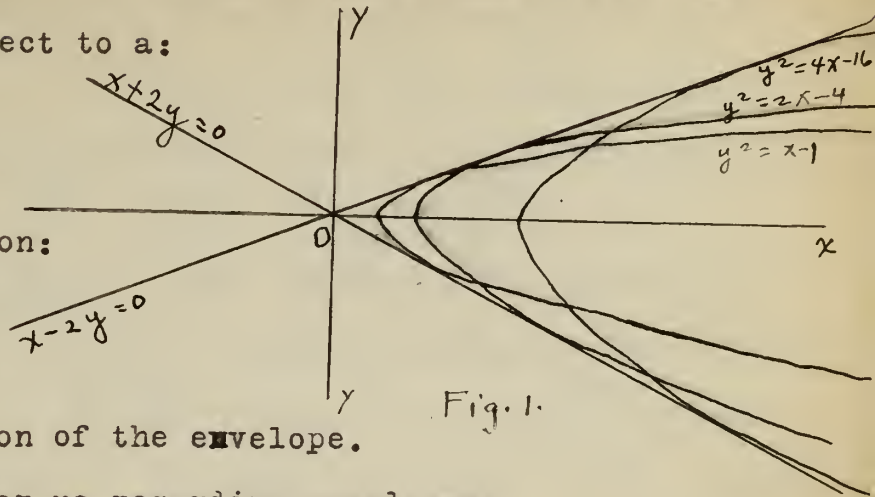
Substituting this value

of a in the given equation:

$$y^2 = \frac{x}{2} \left(x - \frac{x}{2} \right)$$

$$y^2 = \frac{x^2}{2} - \frac{x^2}{4} \quad \text{or} \quad x^2 - 4y^2 = 0$$

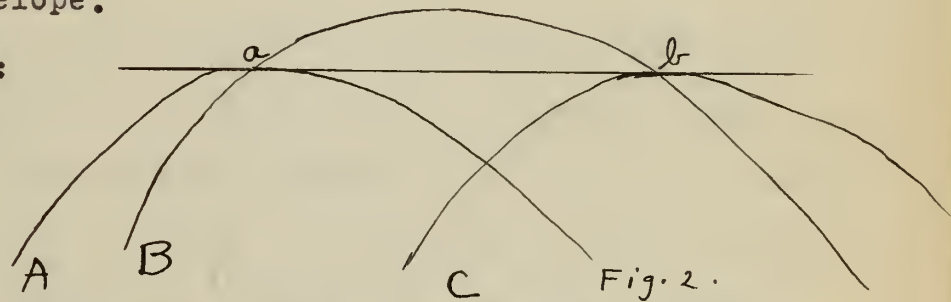
$x^2 - 4y^2 = 0$ is the equation of the envelope.



One theorem is important for us regarding envelopes:

One curve of the family is tangent to the envelope at each point of the envelope.²

Geometric Proof:



Let A, B, and C be three curves of a family which coincide ultimately and a be the point of intersection of A and B and b the intersection of B and C. As B approaches A, a becomes the limiting position of the point of intersection of A and B and as C approaches B, b becomes the limiting position of the point of intersection of B and C. Therefore a and b are points on the envelope and the line through a and b becomes a tangent to the envelope but they are also on B, so that the line ab also becomes a tangent to B. Thus B is tangent to the envelope at the limiting position of a and b . Similarly, it can be proved for the other curves of the family.

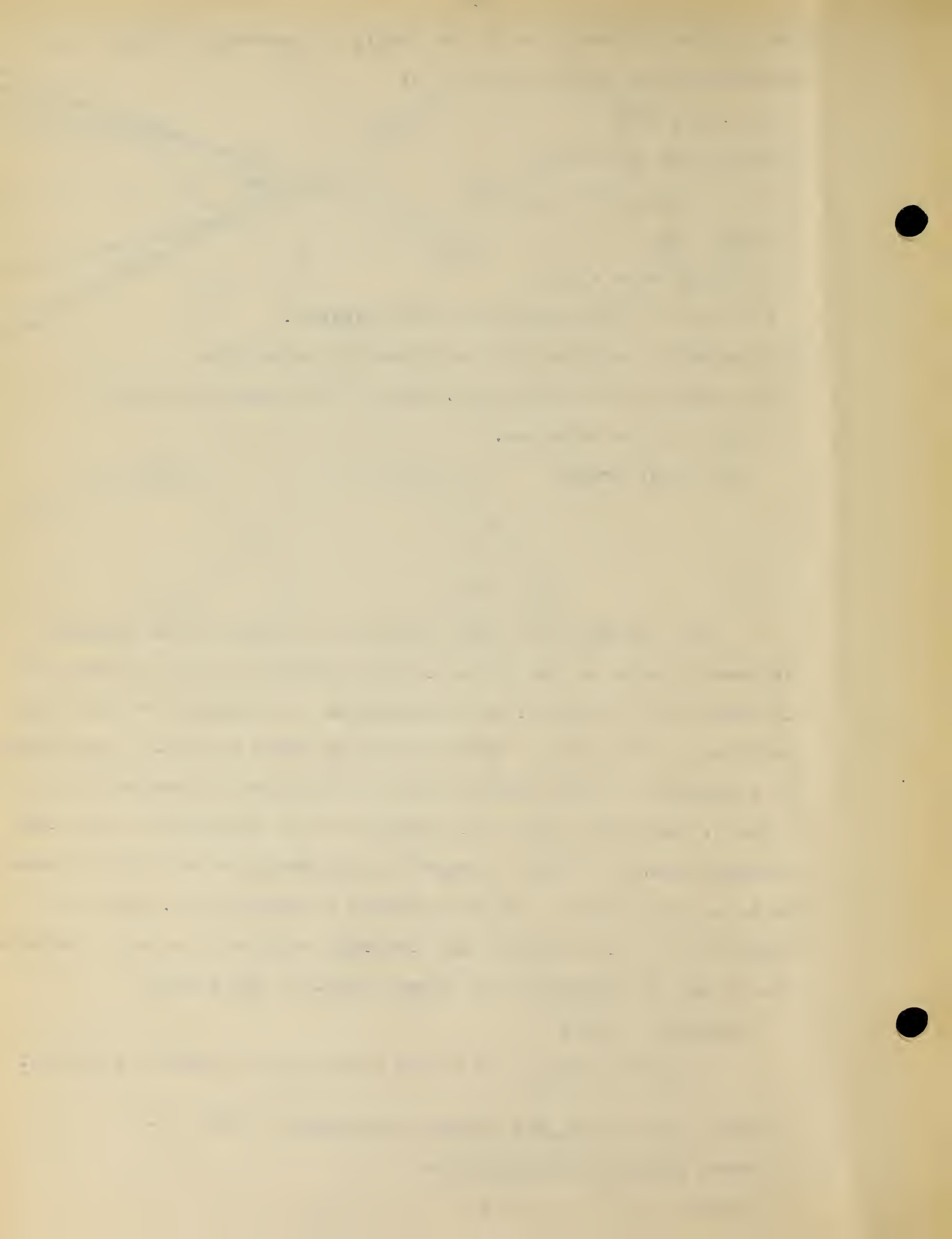
Analytic Proof:³

Suppose $f(x, y, c) = 0$ is the equation of a family of curves.

¹ Love, Differential and Integral Calculus, p. 254. Ex. 3.

² Cohen, Differential Equations, p. 62.

³ Snyder and Hutchinson, Calculus, p. 202-203.



Let us solve $\frac{\partial f(x,y,c)}{\partial c} = 0$ for c in the form $c = \psi(x,y)$. Substituting this for c in the equation we have $f[x, \psi(x,y)] = 0$.

Taking the total derivative with respect to x and setting it equal to zero, we get $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial \psi} \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} \right) = 0$.

Since $\frac{\partial f}{\partial \psi} = \frac{\partial f}{\partial c} = 0$, the slope of the tangent to the envelope at the point (x,y) is $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0$ which also defines the direction of the tangent to the curve $f(x,y,c) = 0$ at the ^{same} point ~~and~~ and therefore one curve of the family is tangent to the envelope at each point of the envelope.

Singular Solutions.

Suppose $f(x,y,c) = 0$ is the solution of $\phi(p,x,y) = 0$ where $p = \frac{dy}{dx}$. $\phi(p,x,y) = 0$ defines the slope of the tangent of that curve of the family which passes through any point (x,y) , which we choose, which is the value for p for that particular point. In other words the tangent and the curve of the family have a common value of x,y,p . We have just noticed above that the tangent of any point of the envelope of the family of curves coincides with that of the curve through that point. Therefore, the equation of the envelope will satisfy the differential equation and is a solution which is different from the general solution, for we can not obtain its equation by assigning a certain value to the parameter. This solution is called the singular solution.

C-Discriminant.

We have already seen how we find the equation of the envelope, which is known as the c-discriminant method. To use this method we have to have the complete primitive or solution or be able to find it.

Example: Find the singular solution of $y = px + \frac{1}{p}$ by the c-discriminant method.¹

¹ Cohen, Differential Equations, p. 63. Ex. 1.

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We recall that if an equation is of the form

$y = px + f(p)$, $p = c$. Hence to get the general solution we may substitute c for p .

Therefore the solution of our equation is $y = cx + \frac{1}{c}$ or $c^2x - cy + 1 = 0$.

Differentiating with respect to c we get:

$$2cx - y = 0 \quad \text{or} \quad c = \frac{y}{2x}$$

Eliminating c , $\frac{y^2}{4x^2}x - \frac{y^2}{2x} + 1 = 0$

Thus $y^2 = 4x$ is the singular solution.

Example: Find the singular solution of $xp^2 - 2yp - x = 0$ ¹ by the c -discriminant method.

Here we need to recall another method of finding the general solution, namely, solving for p .

$$p = \frac{y \pm \sqrt{x^2 + y^2}}{x} \quad \text{or} \quad \begin{cases} xp - y = \sqrt{x^2 + y^2} & (1) \\ xp - y = -\sqrt{x^2 + y^2} & (2) \end{cases}$$

Since $p = \frac{dy}{dx}$, our equation (1) may be written $\frac{xdy - ydx}{\sqrt{x^2 + y^2}} = dx$ or

$$\frac{xdy - ydx}{x^2 \sqrt{1 + \left(\frac{y}{x}\right)^2}} = \frac{dx}{x}$$

Integrating, we get

$$\log \left[\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} \right] = \log x + k, \quad \text{or} \quad \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} - cx = 0.$$

Similarly, on integrating $xp - y = -\sqrt{x^2 + y^2}$, we get

$$\frac{y}{x} - \sqrt{1 + \left(\frac{y}{x}\right)^2} - cx = 0.$$

Therefore, the general solution of $xp^2 - 2yp - x = 0$ is

$$\left[\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} - cx \right] \left[\frac{y}{x} - \sqrt{1 + \left(\frac{y}{x}\right)^2} - cx \right] = 0 \quad \text{or} \\ c^2x^2 - 2cy - 1 = 0.$$

Differentiating with respect to c , we get $2cx^2 - 2y = 0$

$$c = \frac{y}{x}$$

Eliminating c , we obtain $\frac{y^2}{x} - \frac{2y^2}{x} - x = 0$ or $y^2 - 2y^2 - x^2 = 0$.

Thus we might expect $x^2 + y^2 = 0$ to be the singular solution

¹ Cohen, Differential Equations, p. 64. Ex. 2.

² Ibid., p. 50. Ex. 2.

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but it is not, for a point, the origin, has no derivative.

Example. Find the singular solution by the same method of

$$(1 + x^2)p^2 = 1.$$

We note that the equation is solvable for p , $p = \pm \sqrt{\frac{1}{1+x^2}}$ or

$$\frac{dy}{dx} = \pm \frac{\sqrt{1+x^2}}{1+x^2} \quad dy = \frac{dx}{\sqrt{1+x^2}} \quad (1)$$

$$dy = -\frac{dx}{\sqrt{1+x^2}} \quad (2)$$

Integrating (1), we get $y + k = \log(x + \sqrt{x^2+1})$ or

$$x + \sqrt{x^2+1} - ce^y = 0.$$

Similarly, integrating (2), $y + k = -\log(x + \sqrt{x^2+1})$ or

$$x - \sqrt{x^2+1} - ce^y = 0.$$

Therefore the general solution of $(1+x^2)p^2=1$ is

$$\left[x + \sqrt{x^2+1} - ce^y \right] \left[x - \sqrt{x^2+1} - ce^y \right] = 0 \quad \text{or}$$

$$c^2 e^{2y} - 2cxe^y - 1 = 0.$$

Differentiating with respect to c , $2xe^y = 2ce^{2y}$ or $c = \frac{x}{e^y}$

Substituting this value for c , $2x = x - \frac{1}{x}$

$$2x^2 = x^2 - 1 \quad x^2 + 1 = 0 \text{ is the singular solution. But this}$$

also is not really a singular solution
for it is a point.

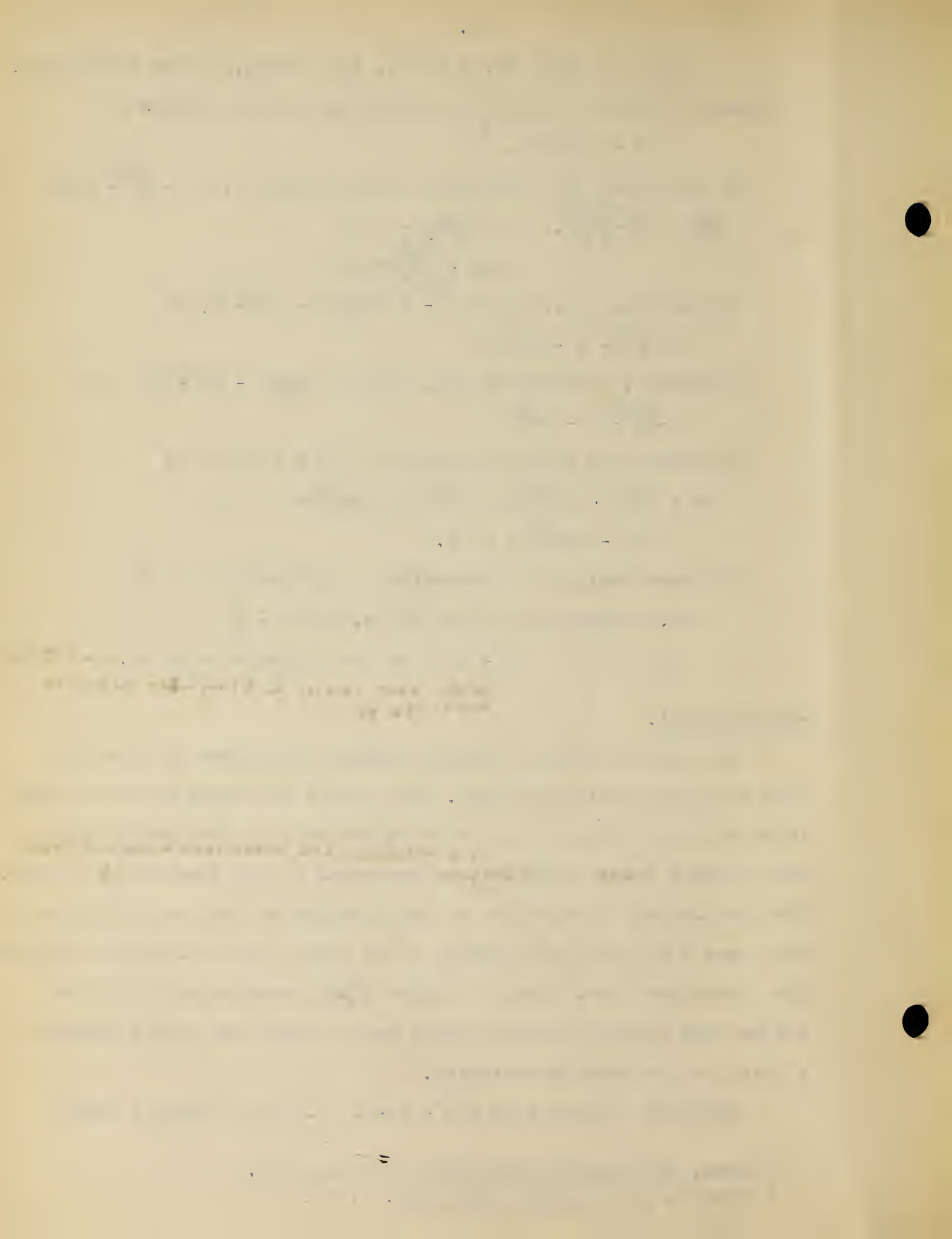
P-Discriminant.

The other method of finding singular solutions is directly from the differential equation. This method is called the p-discriminant method. A discriminant of an equation is a function of the coefficients whose vanishing ^{is a necessary and sufficient condition that} when two roots of the equation are equal. Thus in finding the equation of the envelope of the family $f(x,y,c)=0$ which was a solution of $\phi(p,x,y)=0$ we found the discriminant relation. Thus there are fewer integral curves through each point of the envelope than through general points and at least two curves through a point of the envelope coincide.

Consider the equation $x^2p^2 - yp + 1 = 0$.² If we assign x and y

¹ Cohen, Differential Equations, p. 52. Ex. 5.

² Piaggio, Differential Equations, p. 71.



any definite values, we obtain a quadratic in p . E.g. if $\begin{cases} x = \sqrt{2} \\ y = 3 \end{cases}$, $2p^2 - 3p + 1 = 0$, $p = \frac{1}{2}$ or 1 . Thus we find that, in general, there are two curves of the family through every point satisfying this equation. These two curves will have the same tangent at all points where the equation has equal roots in p , i.e. where the discriminant $y^2 - 4x^2 = 0$. We might note that since there must be two roots to be equal, we cannot expect to find any singular solution in equations of the first order and degree. Thus to find the p -discriminant we eliminate p from $\begin{cases} \phi(p, x, y) = 0 \\ \frac{\partial \phi}{\partial p} = 0 \end{cases}$ where $\phi(p, x, y) = 0$ represents a family of curves. This is the condition for a repeated root.

Example: Find the singular solution of $py = p^2x + 1$, if there is one, directly from the differential equation.¹

Differentiating with respect to p : $y = 2px$

$p = \frac{y}{2x}$. Substituting this value for p in the given equation,

we get $y = \frac{y}{2} + \frac{2x}{y}$, $2y^2 = y^2 + 4x$

$y^2 = 4x$ is the singular solution.

Geometrically, let us consider the curve ABC as it approaches CDF (Fig. 3) always remaining

tangent to the envelope BDE.

C will approach D and the tangents

to the two curves at C will

finally coincide with each other

and the tangent at D. Thus D is the point at which equal p 's

occur and the p -discriminant vanishes.

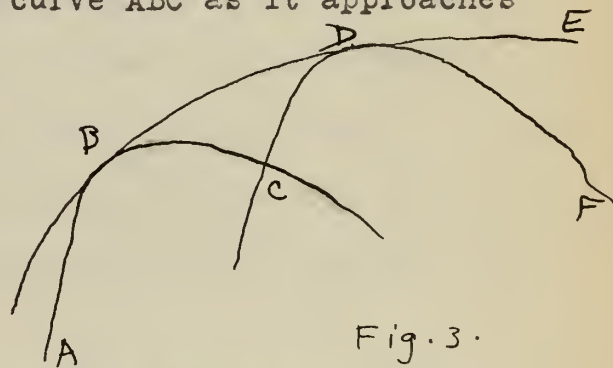


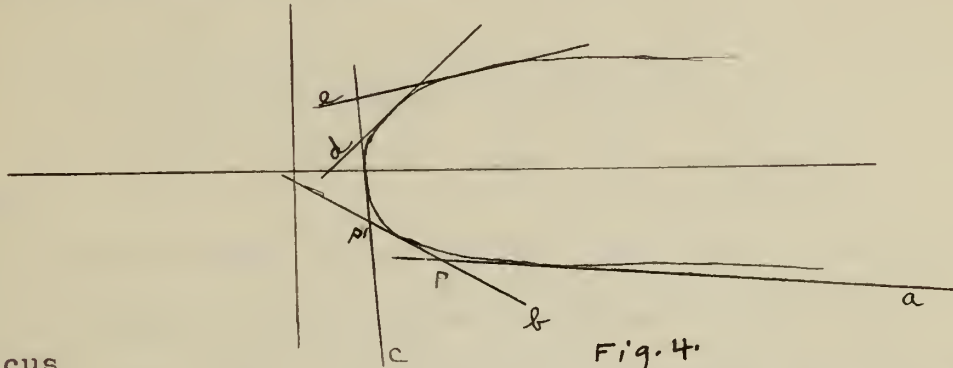
Fig. 3.

We have discovered how to find the c and p -discriminants and our next task is to examine these closely to see if there are any singularities or whether we get only the equation of the envelope which is the singular solution in the discriminants.

¹ Cohen, Differential Equations, p. 63. Ex. 1.

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We may have an envelope with no singularities where every two members of the family intersect in one point only, as in Fig. 4. Here we have a family of straight lines, a, b, c, \dots . a and b intersect in a point P , b and c in P' ; etc.



Nodal-locus.

Fig. 4.

On the other hand, every two members of a family may intersect twice, as in Fig. 5, and we may have double points. Suppose we have several equal nodes as in Fig. 5. Each node of the family intersects with the neighboring node in two points and EE' and AA' are the loci of the points of intersection. NN' is the locus of the double points, i.e. where the curve crosses itself, and there are equal c 's. This type is known as the nodal locus. Each curve of the family has a node and two consecutive curves intersect in three points. The c -discriminant might contain the square of the equation of the node-locus as well as the equation of the envelope, for as we let the consecutive curves become closer and closer AA' and BB' will coincide with NN' . (Figure 5.)

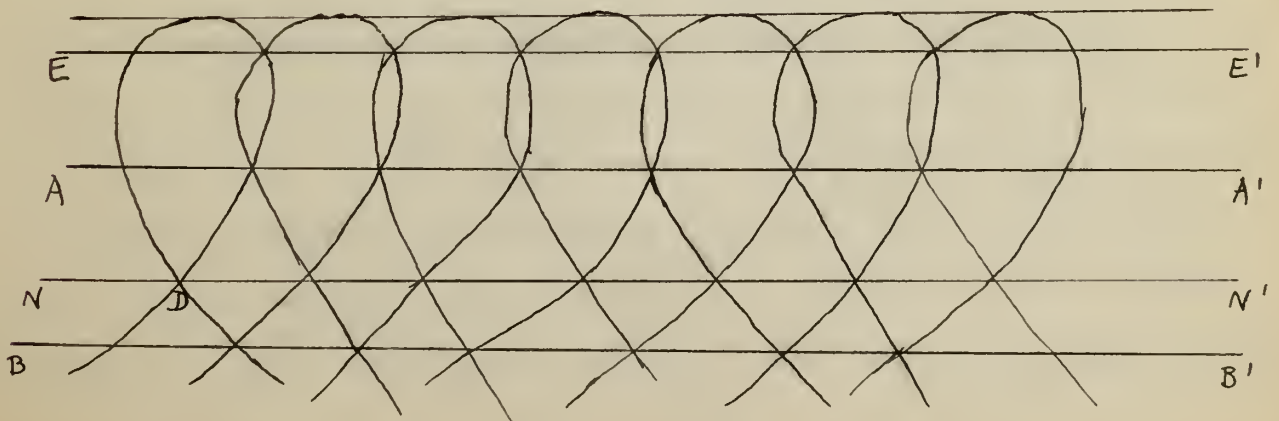
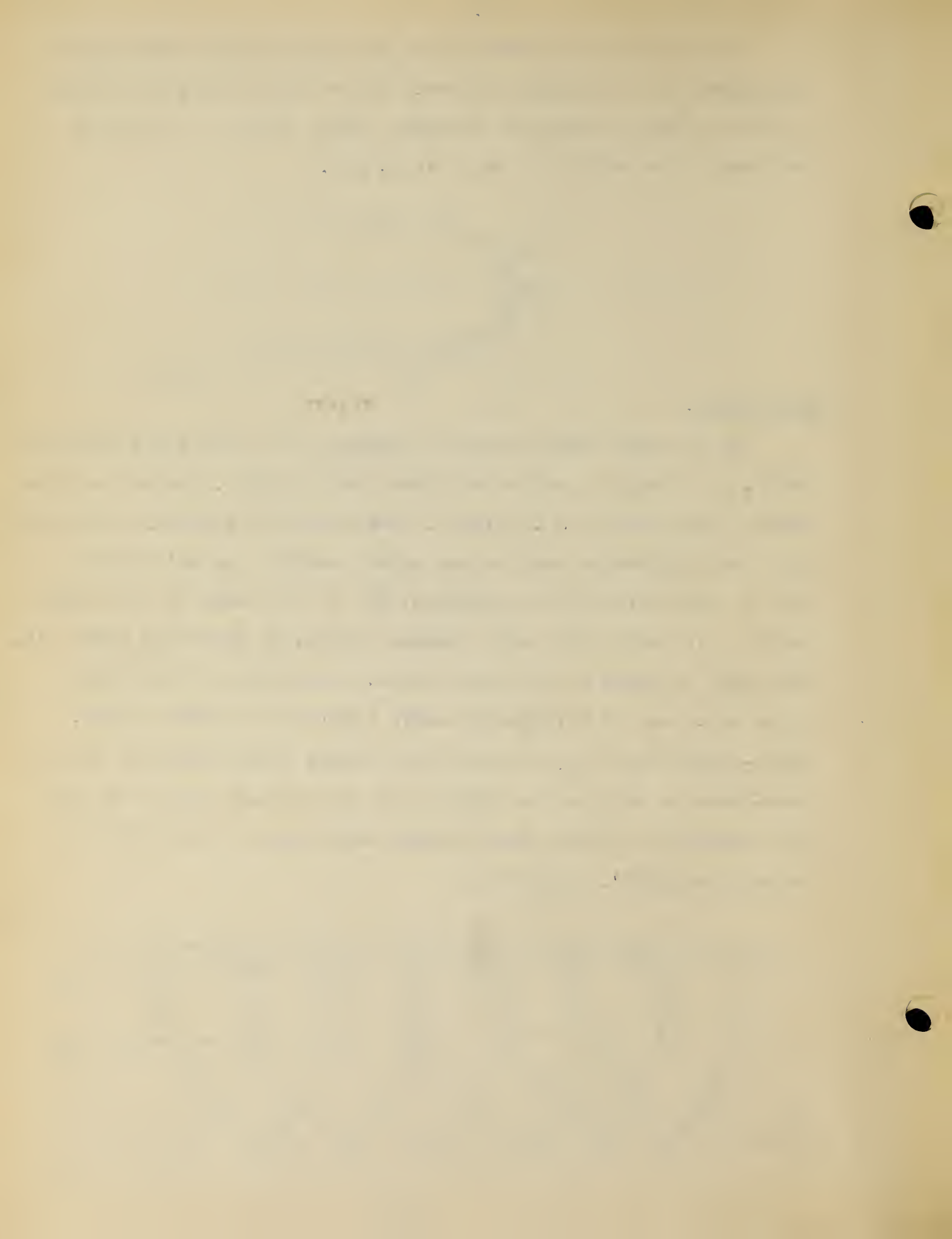


Fig. 5.



The node-locus is not a solution of the differential equation of the family of curves, as it has x and y in common with the curve at D and not p , except in the rare case when it is also an envelope. (Fig. 6.)

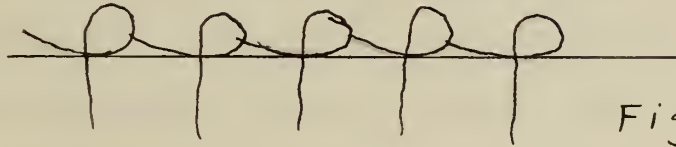


Fig. 6.

Cusp-locus.

When these nodes become closer and closer together and as the area of each node approaches zero, AA' and BB' (Fig. 5) approach NN' while EE' will become an envelope, and we get a cusp-locus.

Consider the equation of the c -discriminant of a family of curves, $f(x,y,c) = 0$, $\begin{cases} f(x,y,c) = 0 \\ \frac{\partial f}{\partial c}(x,y,c) = 0 \end{cases}$. Solving for x and y and putting it in the parametric form we get an equation of the type $\begin{cases} x = \phi(c) \\ y = \psi(c) \end{cases}$.

The direction of the tangent at any point of the c -discriminant is

$\frac{\psi'(c)}{\phi'(c)}$. Let (x_0, y_0) be the coordinates of the point of intersection of the two curves $\begin{cases} f(x,y,c_0) = 0 \\ \frac{\partial f}{\partial c}(x,y,c_0) = 0 \end{cases}$. If ϕ and ψ above are many-valued,

let us determine them, so that ~~in this case~~ $\begin{cases} \phi(c_0) = x_0 \\ \psi(c_0) = y_0 \end{cases}$, so that

in this case $\begin{cases} x_0 = \phi(c_0) \\ y_0 = \psi(c_0) \end{cases}$ would represent a branch of the c -discriminant through (x_0, y_0) . Taking the total derivative of the family of curves,

$f(x,y,c) = 0$, with respect to c we get $\frac{\partial f}{\partial x} \frac{\partial x}{\partial c} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial c} = 0$ which is true

at any point of the c -discriminant. Therefore, at (x_0, y_0) , sub-

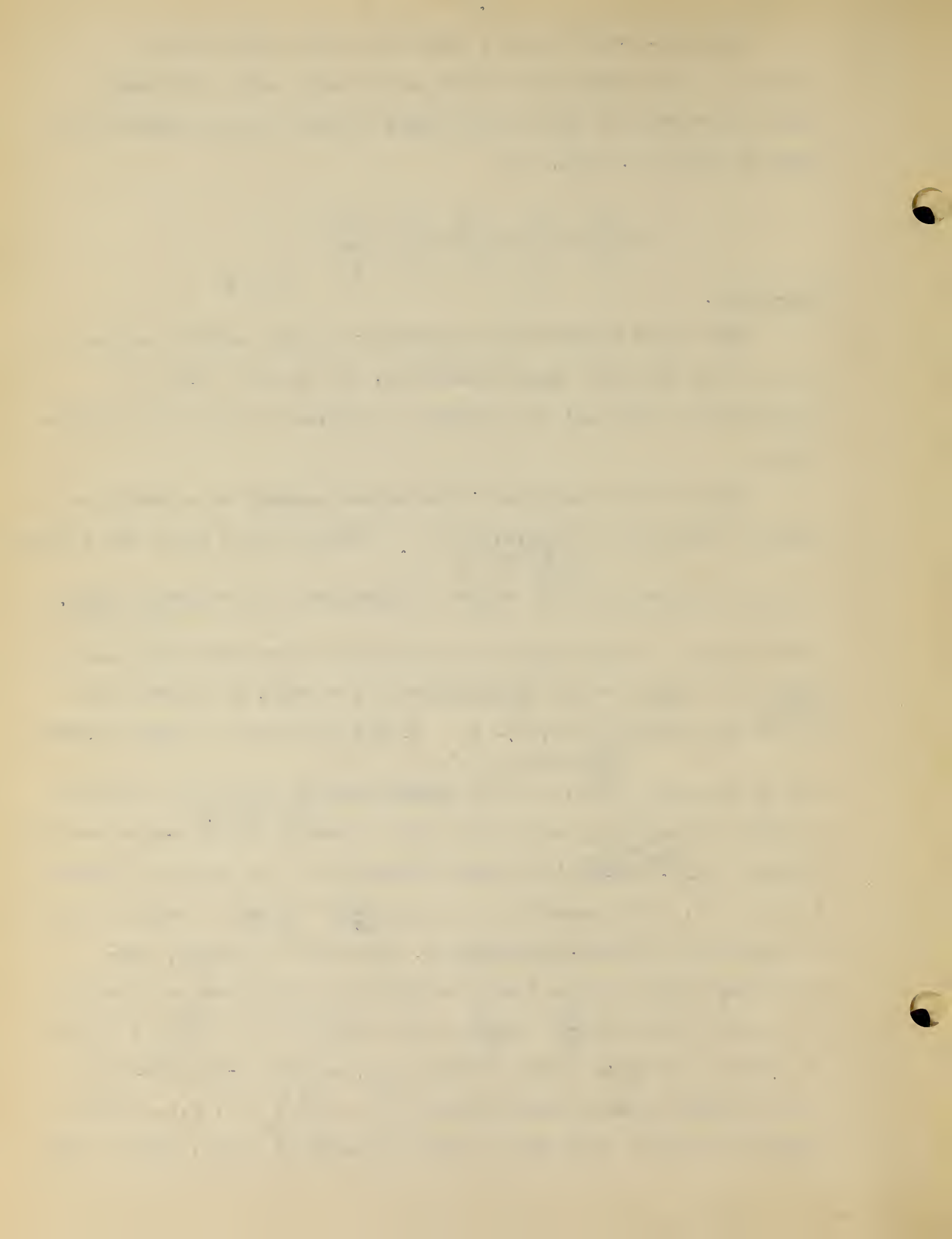
stituting (x_0, y_0) for (x, y) and the partial derivatives of x_0 and y_0 ,

$\phi'(c_0)$ and $\psi'(c_0)$, for $\frac{\partial x}{\partial c}$ and $\frac{\partial y}{\partial c}$ we get $\left(\frac{\partial f}{\partial x}\right)_0 \phi'(c_0) + \left(\frac{\partial f}{\partial y}\right)_0 \psi'(c_0) = 0$,

so that the integral curve through (x_0, y_0) and the c -discriminant

have a common tangent except when $\left(\frac{\partial f}{\partial x}\right)_0 = 0$, $\left(\frac{\partial f}{\partial y}\right)_0 = 0$, i.e. except

when the integral curve has a singular point at (x_0, y_0) . Thus we see



that the branch of the c -discriminant, $\begin{cases} x = \phi(c) \\ y = \psi(c) \end{cases}$, through (x_0, y_0) is either an envelope or a locus of singular points. The c -discriminant usually breaks up into two parts, one which gives us the envelope, the other the locus or loci of singular points, which are cusps and nodes in the most general case.¹

We may expect to find the equation of the cusp-locus cubed in the c -discriminant as three loci, AA' , BB' , and NN' coincide.

(Fig. 5.)

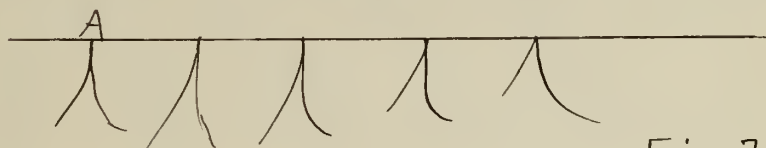


Fig. 7.

The cusp-locus also, in general, is not a solution of the differential equation, except in the rare case when it is an envelope as in figure 8.

the cusp-locus at a
the same as that



Fig. 8.

The direction of the
point A (Fig. 7) is not
of the tangent. Hence

the cusp-locus is not a solution of the differential equation. The c -discriminant vanishes at this point, the cusp, and also two values of p are equal there, as the tangent to the two branches of the curve coincide, i.e. the p -discriminant also vanishes. Therefore, we would expect to find the equation of the cuspidal locus in the p -discriminant as well as in the c -discriminant but only of the first degree here as there is only one locus as we approach the limiting case, i.e. as the nodes become smaller and smaller.

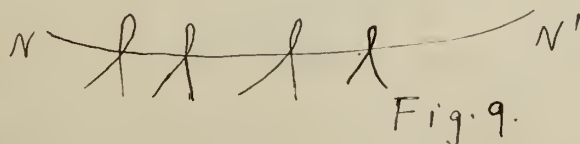


Fig. 9.

¹ Ince, Ordinary Differential Equations, p. 86.

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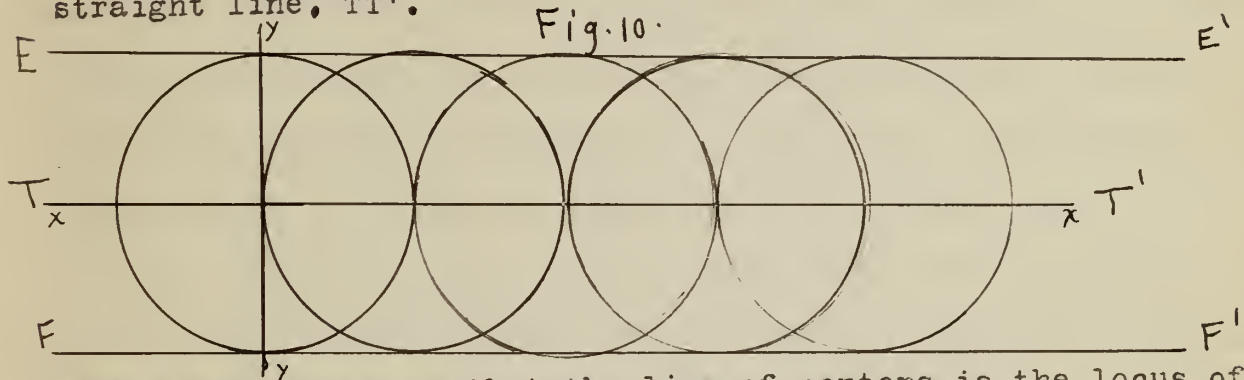
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Tac-locus.

There is another extraneous locus which we must note and this occurs when two integral curves are tangent to each other, for then the number of distinct values of p is decreased and the p -discriminant vanishes at this point. This is called the tac-locus. Consider a family of circles of equal radii, with centers on a straight line, TT' .



This figure shows us that the line of centers is the locus of the points of contact of pairs of circles and it is a tac-locus. An envelope we defined as the locus of the points of intersection of two consecutive curves of a family, having equal values of p 's. In the tac-locus the points of the locus are not intersections of consecutive curves. We know from geometry that two circles intersect in two points.

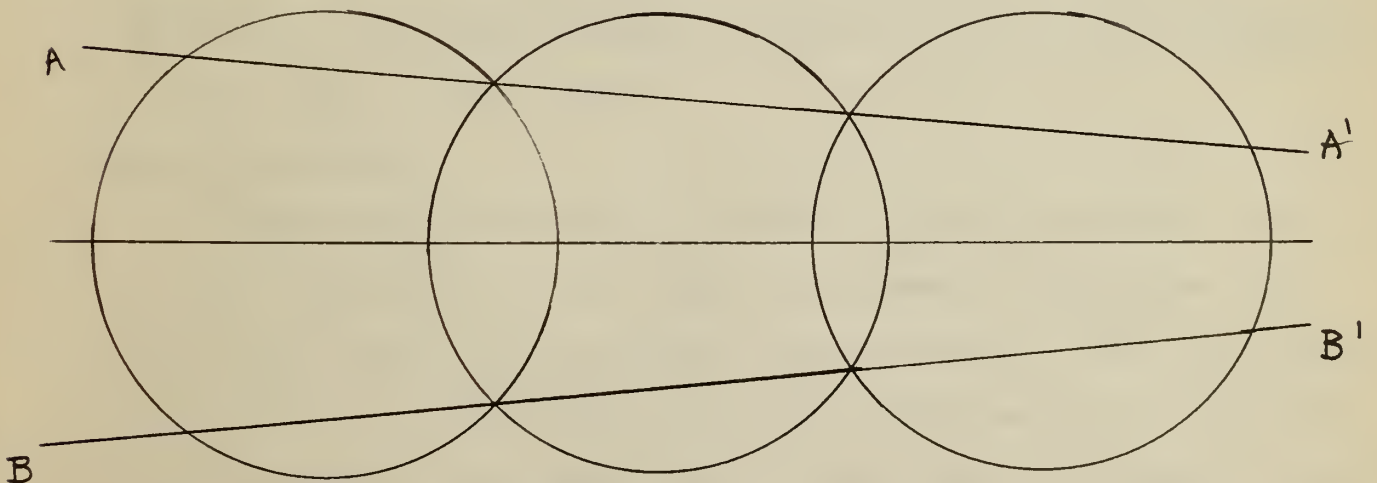


Fig. 11.

Suppose we have nearby equal circles intersecting with centers on a straight line and AA' and BB' are the loci of these

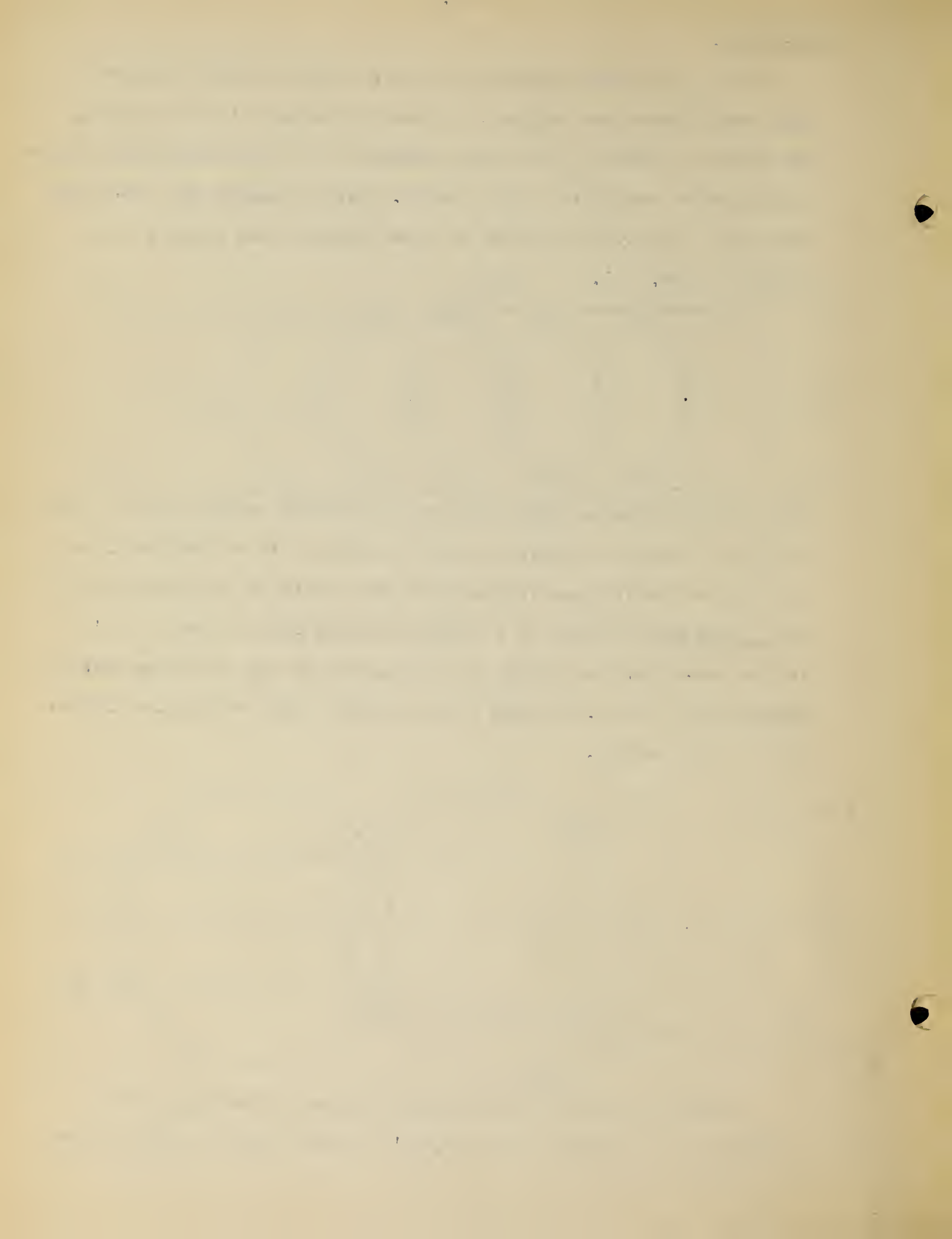


Fig. 11.

intersections. When we proceed to the limiting case, i.e. when the circles are tangent to each other, the two loci, AA' and BB', coincide. Therefore we might expect to find the equation of the tac-locus squared in the p-discriminant. Also, in general, the equation of the tac-locus is not a solution of the differential equation, for at the point C in Fig. 11 the direction of the tac-locus is not that of the two circles. Thus a relation between x, y, and p which will satisfy the circles will not satisfy the tac-locus as they have different values of p at this point. For example, suppose we have a family of circles of equal radii, Fig. 10, where TT' the line of centers is the x-axis. The equation of the family is $(x - c)^2 + y^2 = r^2$. To find the differential equation of the circles, take the square root of each side of the equation, $x - c = \pm \sqrt{r^2 - y^2}$, and differentiate with respect to x: $1 = \pm \frac{yp}{\sqrt{r^2 - y^2}}$. Therefore, $y^2 p^2 + y^2 - r^2 = 0$ is the differential equation of the family of circles. To find the p-discriminant we differentiate this equation with respect to p: $2py^2 = 0$, $p = 0$ or $y^2 = 0$. Therefore the p-discriminant is $y^2(y^2 - r^2) = 0$. Thus $y = \pm r$ is the envelope and $y = 0$ is the tac-locus as it occurs twice in the p-discriminant. $y = 0$ does not satisfy the equation and $y = \pm r$ does, for on substituting $\begin{cases} y = \pm r \\ y' = 0 \end{cases}$ we get $0 + r^2 - r^2 = 0$, $0 = 0$.

Particular Curve.

Sometimes there may also occur in the discriminant what is called a particular curve. When the parameter approaches some value, the curves of the family often approach a limiting one which is different in shape from the others and this particular curve may be tangent to all the others at one point. An infinite number of curves pass through this point but through the other points of the curve fewer curves than usual pass and the equation of the

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particular curve is found in the p and c-discriminants.

Comparison of c and p-discriminants.

We have examined both the c and p-discriminants for extraneous loci and we might tabulate our results thus:

Curves we may expect in the c-discriminant	Number of times we may expect them to appear in c-discriminant	Curves we may expect in the p-discriminant	Number of times we may expect them to appear in p-discriminant
1. Envelope	1	1. Envelope	1
2. Node-locus	2	2. Tac-locus	2
3. Cusp-locus	3	3. Cusp-locus	1
4. Particular Curve	1	4. Particular Curve	3

Existence of Singular Solutions and Extraneous Loci.

Thus our method of finding singular solutions is to find the c and p-discriminants and then, setting the factors equal to zero, see whether they satisfy the equation or not. If they, or some of the factors, do satisfy the equation, then they are singular solutions. If, on the other hand, they do not satisfy the equation, they are extraneous loci. Then we may use the above table as a check on our work. There are a few cases where we need not be so careful about extraneous roots, as in the case where the integral curves are straight lines, for there are no extraneous loci then. When the integral curves are conics, there are no nodal or cuspidal loci.

We have already noted that we need not look for singular solutions unless the equation is of at least the second degree. Also if the differential equation is linear or decomposable into a set of rational linear factors, it has no singular solution, for

¹ Cohen, Differential Equations, p.72.

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any solution of it is only a particular solution derived from the general solution by assigning a special value to the constant. However, this does not mean that every irreducible differential equation has a singular solution.

Theorem: An irreducible differential equation does not always have a singular solution.¹

Proof:

Suppose we have a differential equation $\phi(x, y, p) = 0$ where $p = \frac{dy}{dx}$.

Let U equal the discriminant of this with regard to p . U is a function of the coefficients of p by our definition of discriminant and assume that it cannot be decomposed into linear factors.

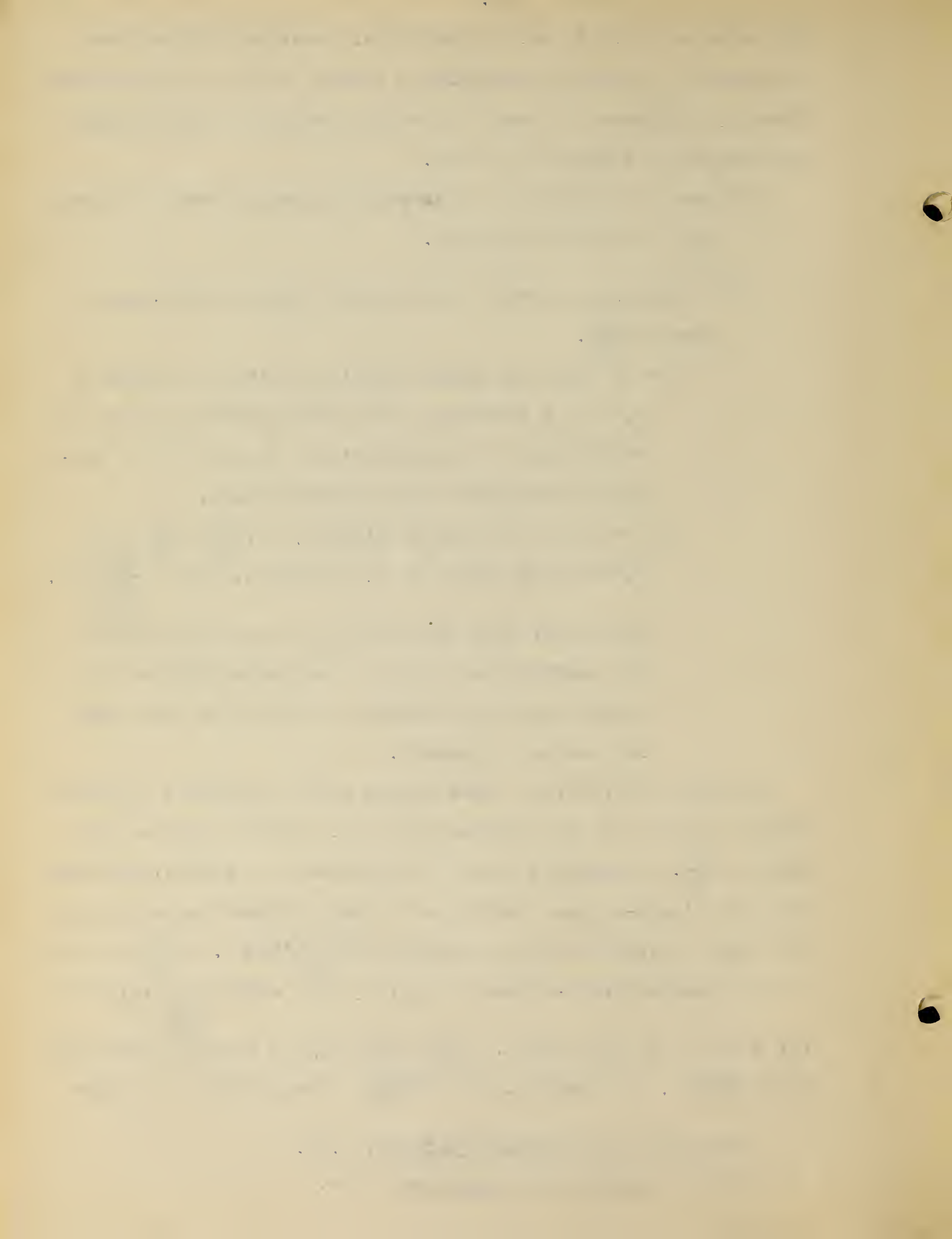
If $U=0$ is a solution of $\phi(x, y, p) = 0$, $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} p = 0$ gives us the value of p . Therefore, $\phi\left(x, y, \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}}\right) = 0$.

This means that there must be a relation between the coefficients of p in ϕ and their differential coefficients with regard to x and y but this does not happen in general.

In fact, as Professor Cohen points out,² in general, a differential equation has no singular solution. Consider $f(x, y, p) = 0$ where $p = \frac{dy}{dx}$. In general, p may be expressed as a finite, continuous, and single-valued function of x and y in some region (x_0, y_0) and takes a finite value p_0 as long as $\frac{\partial f(x, y, p)}{\partial p} \neq 0$. If $\frac{\partial f}{\partial p} = 0$, then p is no longer single-valued in (x_0, y_0) but, solving $\begin{cases} f(x, y, p) = 0 \\ \frac{\partial f}{\partial p} = 0 \end{cases}$ for y and p , we get $y = \phi(x)$, $p = \phi_1(x)$ and only in special cases will $\phi_1(x) = \frac{d\phi(x)}{dx}$. If, however, $\phi_1(x) = \frac{d\phi(x)}{dx}$, then $y = \phi(x)$ is a solu-

¹ Forsyth, Differential Equations, p.35.

² Cohen, Differential Equations, p.75.



tion of the equation and as it is usually separate from the general solution it is a singular solution.

We might note that differential equations of higher order than the first may have singular solutions also. If there is a solution of $f(x, y, p_1, p_2, \dots, p_n) = 0$ for which $\frac{\partial f}{\partial p_n} = 0$, this solution is a singular solution. ¹ p_1, p_2 , etc. represent $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc.

Geometrical Interpretation.

We have considered the analytic side of singular solutions, so let us turn now for a moment to the geometrical. From elementary work we know that to represent $y = f(x)$ graphically all we need to do is to assign values to x and then y is fixed. Consider the differential equation $\frac{dy}{dx} = f(x, y)$. If we assign a value to x , this does not determine the value of y but if we assign values to both x and y then $\frac{dy}{dx}$, that is, the direction, is determined for these values if the curve passes through that point. Suppose we have determined the directions of many points and represented them by arrows (Figure 12).²



Fig. 12.

¹ Cohen, Differential Equations, p.169.

² Fry, Differential Equations, p.23.

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Each arrow seems to carry us along to the next one, so that we think of them in groups. If we connect the arrows beginning at some point A or B, we get a curve, which has the required slope at the points denoted by arrows. Theoretically, if we increased the number of arrows, we would approach the limiting curve, which is the solution of the differential equation. Thus "A differential equation defines a curve by telling the direction in which it passes through each of its points." Thus every particular solution corresponds to a curve, and the general solution to the family as a whole. Suppose the family has an envelope, that is, a singular solution. Then the slope and coordinates for a point of the envelope are the same as those for the curve to which it is tangent there. therefore the envelope satisfies the differential equation, as we have already seen, but cannot be found by giving a certain value to the parameter.

Illustrations.

In conclusion, let us consider several examples and examine them for singular solutions and extraneous loci.

Example 1. $4p^2 = 9x$.¹

This equation is solvable for p , $p = \pm \frac{3}{2}\sqrt{x}$. Integrating, we get $(y + c)^2 = x^3$ as the general solution. Differentiating with respect to c , $2y + 2c = 0$, $c = -y$. Hence the c -discriminant is $x^3 = 0$.

Differentiating our given equation with respect to p , we get $8p = 0$, $p = 0$ and hence the p -discriminant is $x = 0$. By looking at our equation $4p^2 = 9x$, we can readily see that $x = 0$

¹ Cohen, Differential Equations, p.74. Ex. 3.

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does not satisfy it and hence is not a singular solution but a cusp-locus, appearing once in the p-discriminant and three times in the c-discriminant as we would expect.

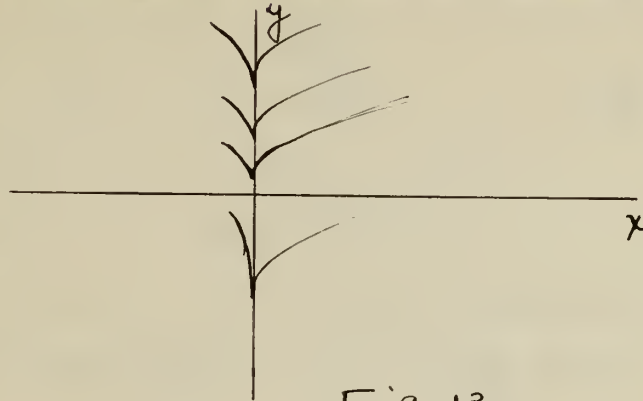


Fig. 13.

Example 2. $4e^{2y} p^2 + 2xp - 1 = 0.$

To find the general solution let $e^{2y} = t.$

$p = \frac{dy}{dx} = \frac{1}{2t} \frac{dt}{dx}$ Therefore, $t = x \frac{dt}{dx} + \left(\frac{dt}{dx}\right)^2$ and the general solution is $t = cx + c^2$ or $e^{2y} = cx + c^2.$

Differentiating with respect to c , $0 = x + 2c$, $c = -\frac{x}{2}.$

Substituting, this value for c , $e^{2y} = -\frac{x^2}{2} + \frac{x^2}{4}$ or $e^{2y} + \frac{x^2}{4} = 0$ is the c-discriminant.

Differentiating the given equation with respect to p :

$$8e^{2y} p + 2x = 0, \quad p = -\frac{x}{4e^{2y}}$$

$$\text{Substituting this, } \frac{4e^{2y} x^2}{16e^{4y}} - \frac{2x^2}{4e^{2y}} - 1 = 0$$

$$\frac{x^2}{4e^{2y}} - \frac{x^2}{2e^{2y}} - 1 = 0 \quad \text{or } x^2 - 2x^2 - 4e^{2y} = 0. \quad x^2 + 4e^{2y} = 0 \text{ is the}$$

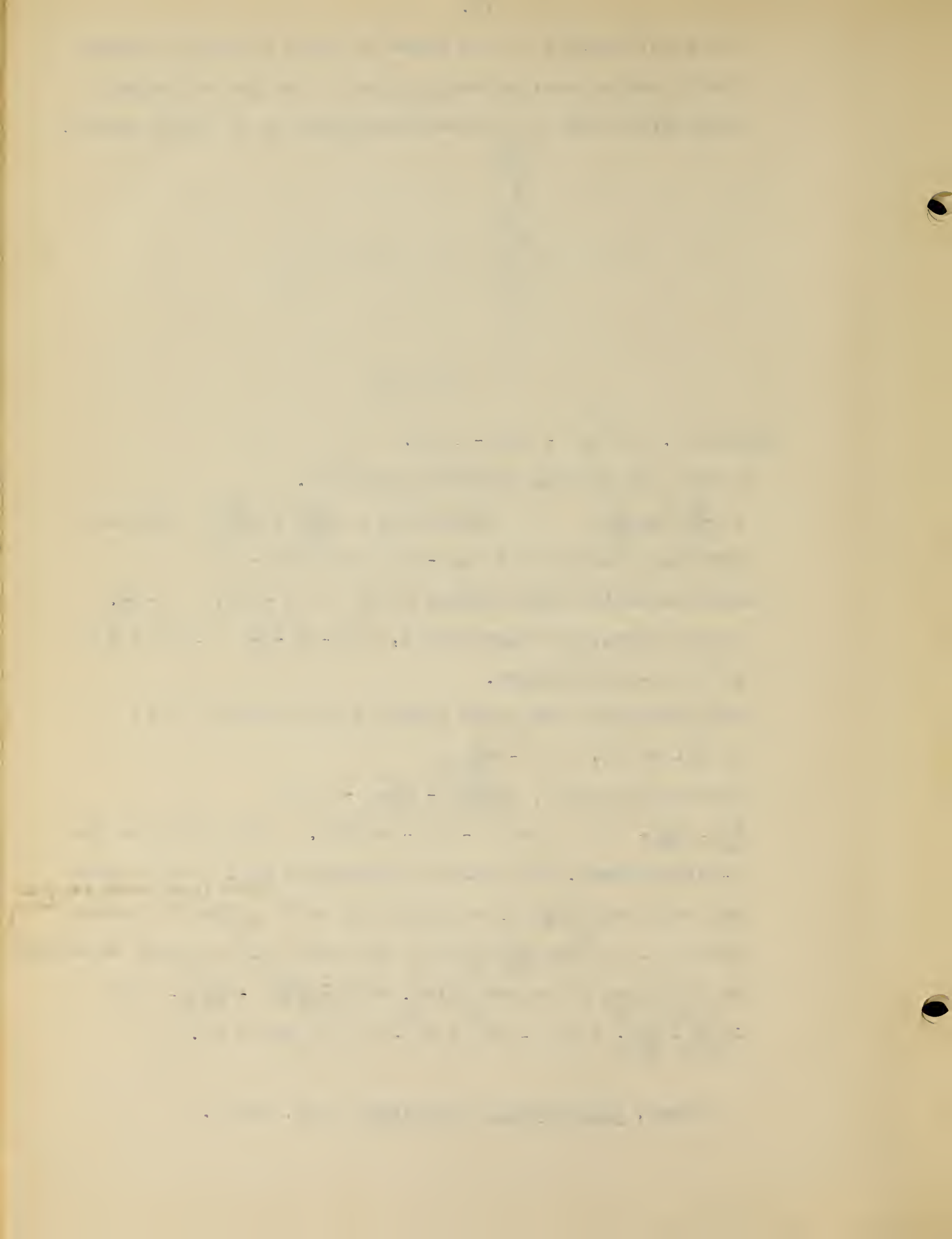
p-discriminant. There are no extraneous loci in this case

but only the singular solution $x^2 + 4e^{2y} = 0$, which is not really a singular solution When we substi-

tute $x^2 = -4e^{2y}$ and $\frac{dy}{dx} = \frac{-x}{4e^{2y}}$ in our given differential equation,

$$\text{we find that it is satisfied. } 4e^{2y} \left(\frac{-4e^{2y}}{16e^{4y}}\right) + \frac{8e^{2y}}{4e^{2y}} - 1 = 0$$

$$\frac{-4e^{2y}}{4e^{2y}} + \frac{8e^{2y}}{4e^{2y}} - 1 = 0, \quad -4e^{2y} + 8e^{2y} - 4e^{2y} = 0 \text{ and } 0 = 0.$$



Example 3. $3y = 2px - 2\frac{p^2}{x}$.

Differentiating to find the general solution:

$$3p = 2p + 2\frac{p^2}{x^2} + (2x - 4\frac{p}{x})\frac{dp}{dx}$$

$$\text{i.e. } px^2 - 2p^2 = (2x^3 - 4px)\frac{dp}{dx}$$

$$x^2 - 2p = 0 \quad \text{or } p = 2x\frac{dp}{dx} \quad \frac{dx}{x} = 2\frac{dp}{p}$$

Integrating: $\log x = 2\log p - \log c$

$$cx = p^2 \quad 3y = 2c^{\frac{1}{2}}x^{\frac{3}{2}} - 2c \quad \text{or}$$

$(3y + 2c)^2 = 4cx^3$ is the general solution and is a family of semi-cubical parabolas with their cusps on the y-axis.

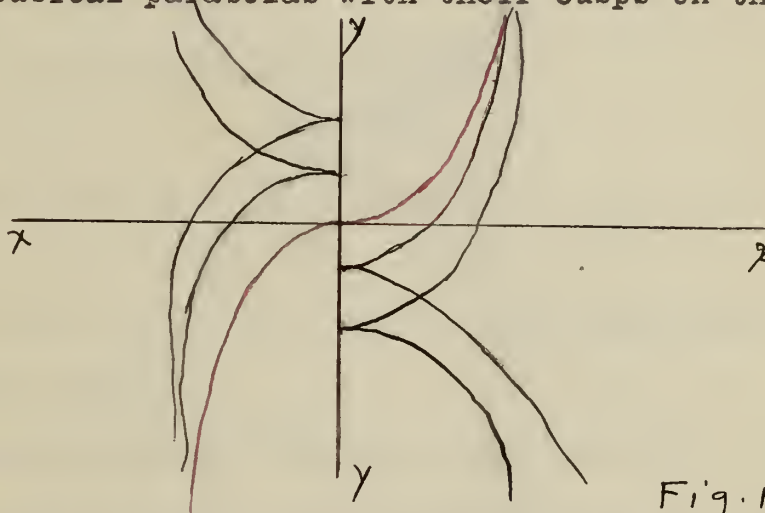


Fig. 14.

To find the c-discriminant differentiate the general solution with respect to c : $12y + 8c = 4x^3$

$$8c = 4x^3 - 12y \quad 2c = x^3 - 3y \quad c = \frac{x^3 - 3y}{2}$$

Substituting this value for c : $(3y + x^3 - 3y)^2 = 2x^3(x^3 - 3y)$

$$x^6 = 2x^6 - 6x^3y \quad x^6 - 2x^6 + 6x^3y = 0$$

$x^3(6y - x^3) = 0$ is the c-discriminant.

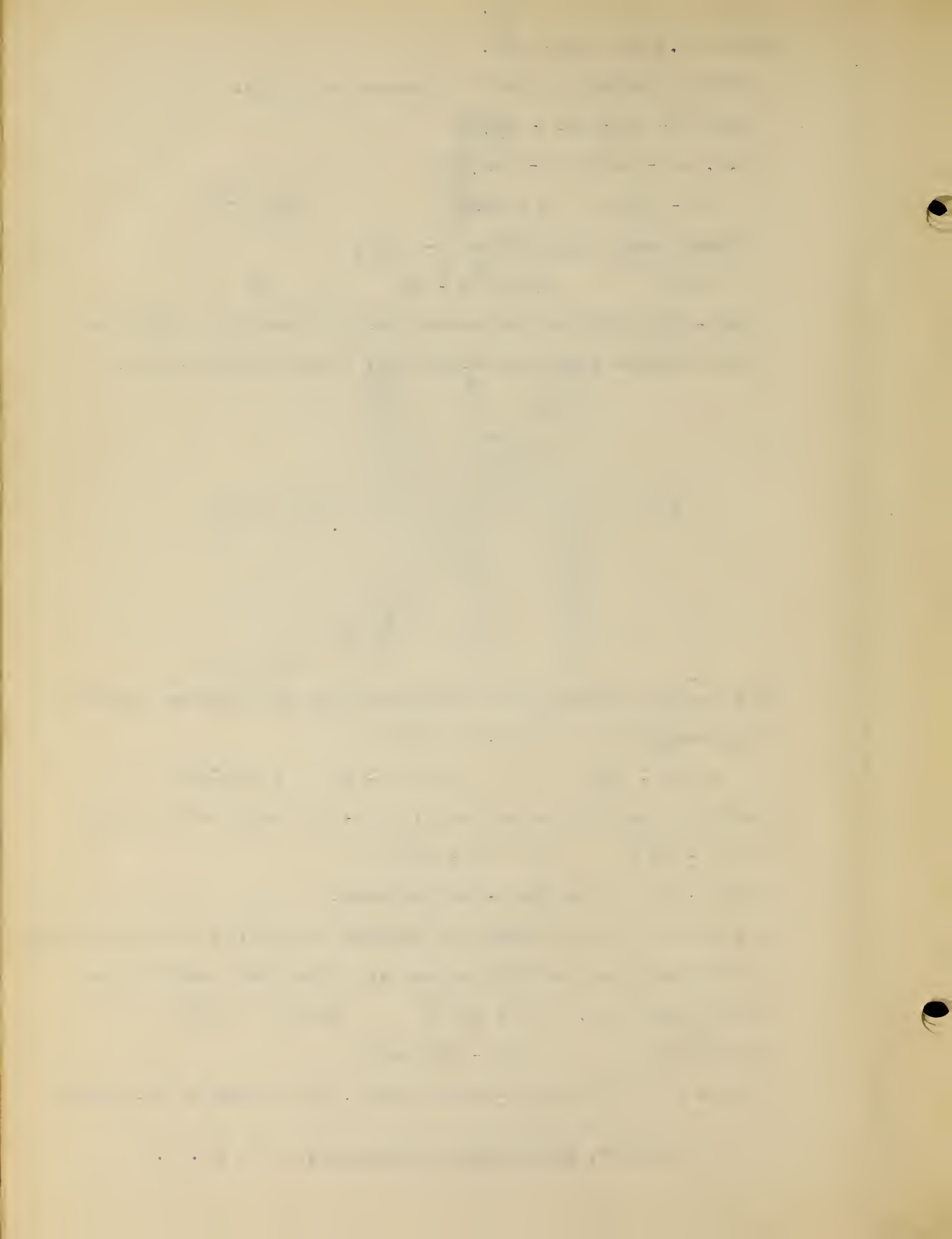
To find the p-discriminant we proceed as before, differentiating

our given equation with respect to p and then substituting

$$\text{this value of } p. \quad 2x^2 - 4p = 0 \quad 2p = x^2 \quad p = \frac{x^2}{2}$$

$$3yx = x^4 - \frac{x^4}{2} \quad 6yx - 2x^4 + x^4 = 0$$

$x(6y - x^3) = 0$ is the p-discriminant. By looking at our given



equation we can see that $x=0$ does not satisfy it and is an extraneous locus, while $6y - x^3=0$ does satisfy it. $y = \frac{x^3}{6}$, $\frac{dy}{dx} = \frac{x^2}{2}$. Substituting these in the equation: $\frac{x^4}{2} = x^4 - \frac{x^4}{2}$ or $x^4 = 2x^4 - x^4$, $x^4 = x^4$. Hence $6y - x^3=0$ is the singular solution. and $x=0$ is a cusp locus as it occurs once in the p-discriminant and three times in the c-discriminant.

Example 4. Given a family of curves $(y + c)^2 = x(x - a)^2$, where c is the parameter and $a > 0$.¹

To find the differential equation of this family we differentiate with respect to x to eliminate the parameter c .

$$y + c = \sqrt{x}(x - a)$$

$$\text{Differentiating, } p = \frac{1}{2\sqrt{x}} + \frac{x - a}{2\sqrt{x}}$$

$$p^2 = x + x - a + \frac{(x - a)^2}{4x}$$

$$4p^2x = 8x^2 - 4ax + x^2 - 2ax + a^2$$

$$4p^2x = 9x^2 - 6ax + a^2$$

Therefore $4p^2x = (3x - a)^2$ is the differential equation of the family.

Differentiating this ^{with} respect to p : $8px = 0$, $p = 0$ or $x = 0$ and the p -discriminant is $x(3x - a)^2 = 0$.

Differentiating our original equation $(y + c)^2 = x(x - a)^2$ with respect to c . we get $2y + 2c = 0$ or $c = -y$.

Therefore the c -discriminant is $x(x - a)^2 = 0$

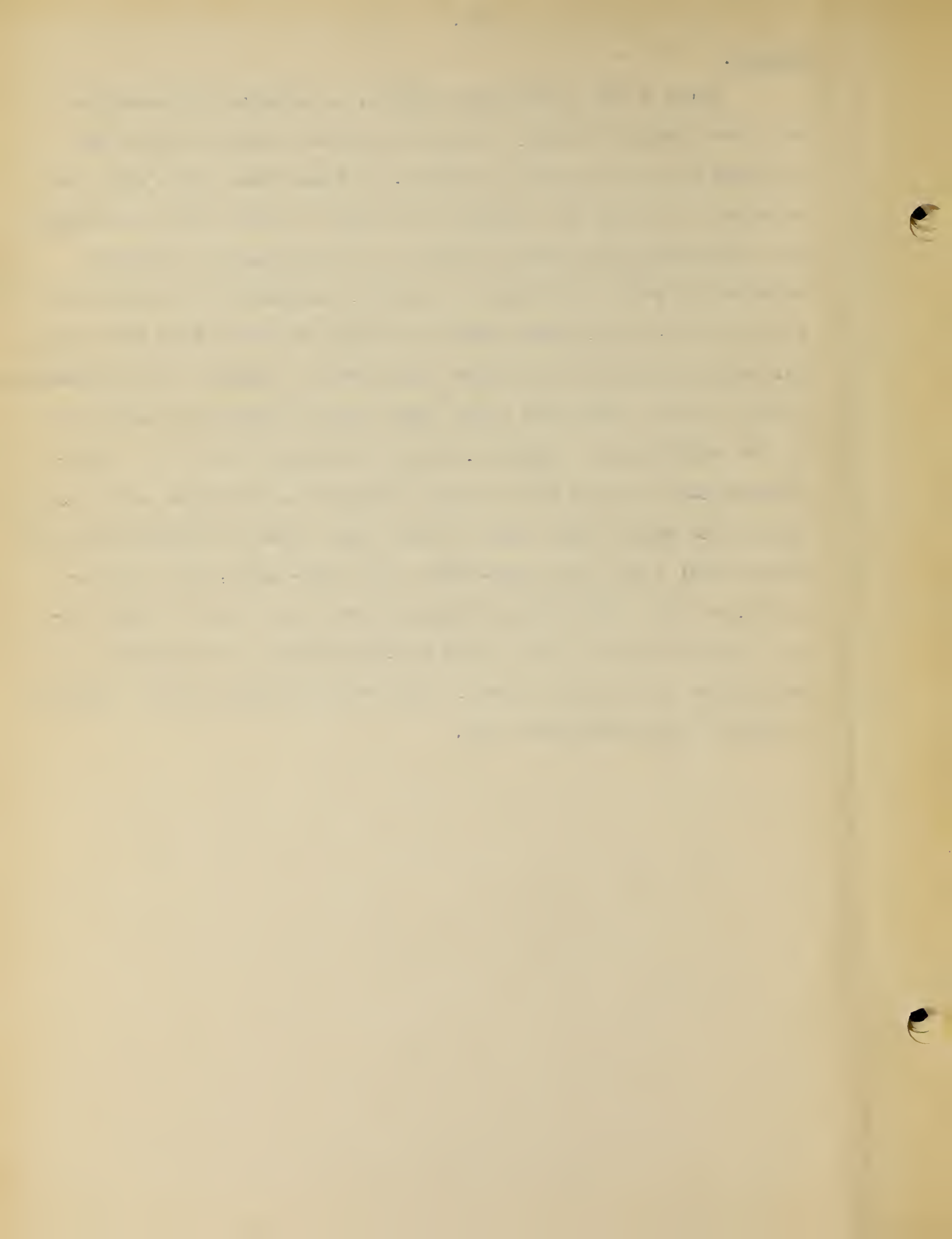
$x = 0$ occurs once in both discriminants and is the singular ^{apparently} solution, but the derivative of the y -axis is indeterminate. The factor $3x - a$ occurs twice in the p -discriminant and therefore $3x - a = 0$ is a tac-locus. $x - a$ appears twice in the c -discriminant and hence is a node-locus.

¹ Ince, Ordinary Differential Equations, p. 86.

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Summary.

After a few preliminary remarks, we studied the envelope of a one parameter family of curves and we discovered that the envelope was the singular solution. We found that there were two methods of finding the singular solution, if there was one, namely the c -discriminant method by which we could find the singular solution by differentiating the general solution with respect to a , and the p -discriminant method by which we could find the singular solution directly from the differential equation by differentiating it with respect to p and then substituting this value of p in the differential equation. Then we examined both the c and p -discriminants to see if we always obtained an envelope and if anything else might occur and we found three kinds of extraneous loci which might occur: the cusp-locus, the node-locus, and the tac-locus. We then proceeded to find out how many times we might expect them to occur in the c and p -discriminants respectively. In conclusion we examined several differential equations for singular solutions and extraneous loci.



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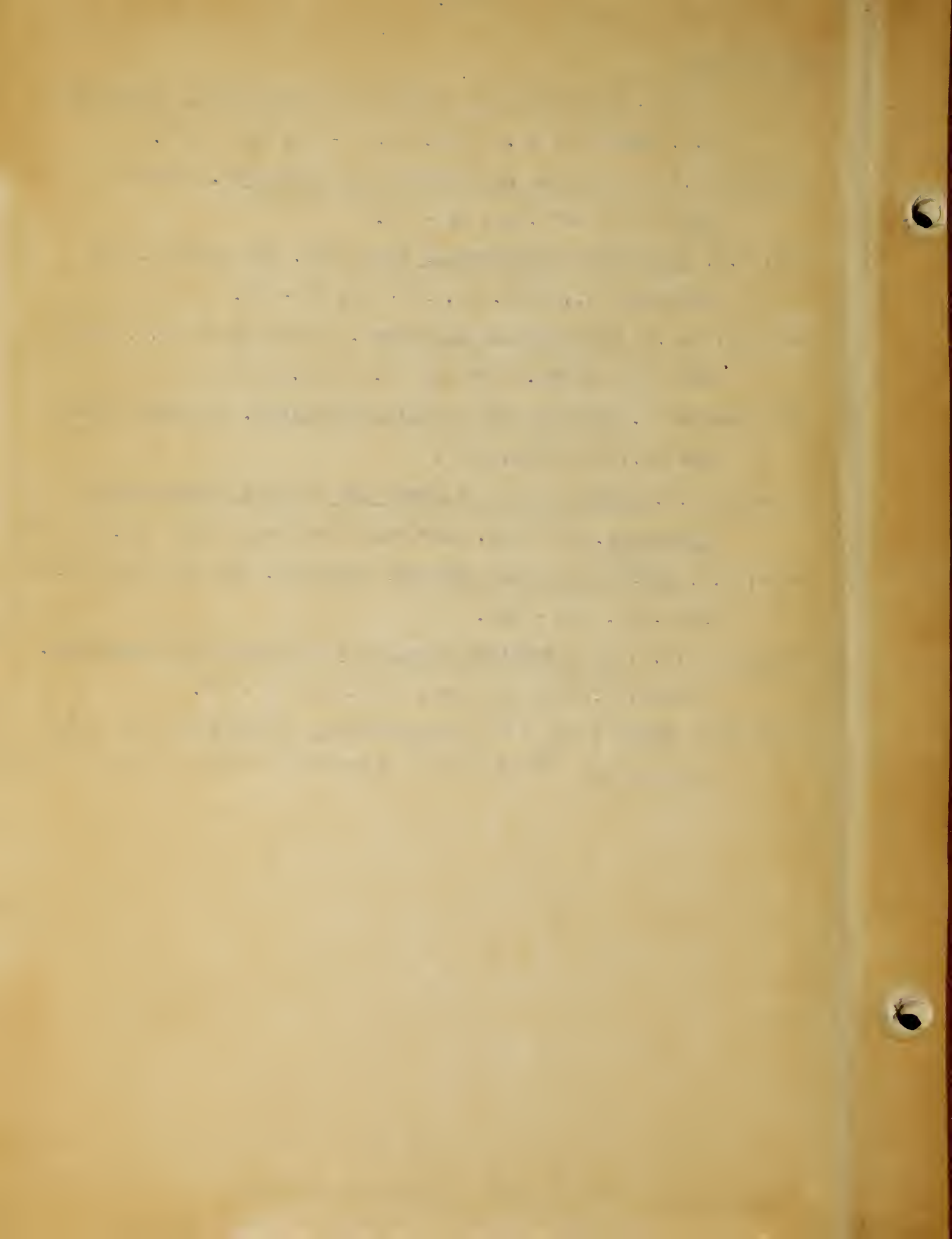
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