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## Abstract

Shear wave motion of a soft, electrically-conducting solid in the presence of a strong magnetic field excites eddy currents in the solid. These, in turn, give rise to Lorentz forces that resist the wave motion. We derive a mathematical model for linear elastic wave propagation in a soft electrically conducting solid in the presence of a strong magnetic field. The model reduces to an effective anisotropic dissipation term resembling an anisotropic viscous foundation. The application to magnetic resonance elastography, which uses strong magnetic fields to measure shear wave speed in soft tissues for diagnostic purposes, is considered.

**Keywords:** magnetoelasticity, eddy current, magnetic resonance elastography

## 1 Introduction

Elastic shear wave propagation in soft tissues is of current interest in the field of elastography. Shear wave speed serves as a surrogate for tissue modulus, and is used for diagnosis of various pathologies. Therefore, understanding those parameters which can lead to variations in shear wave speed *in vivo* is necessary. Here the focus is on dependence of shear wave speed and attenuation in the presence of a strong magnetic field. We note that Lorentz force has been used to generate shear waves in soft tissues [1].

Equations of motion for stressed and unstressed magnetoelastic solids have been studied extensively, c.f. [2]. Equations for incremental waves are presented in [3], though the focus there is largely on materials that are magnetoelastic and non-electrically conducting. Electrically conducting materials are considered in [4]. The focus of the present work is on small amplitude waves in a non-magnetoelastic but electrically conducting solid.

## 2 Electromagnetic modeling

We begin by recalling Maxwell's equations [5]:

$$\nabla \cdot \mathbf{E} = \frac{q_f}{\varepsilon_0} \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (4)$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density (current per unit area), $q_f$ is the free charge density, $\mu_0$ is the permeability of free space and $\varepsilon_0$ is the permiattivity of free space. In Ampere's law, (4), we neglect the term $\mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \approx 0$, consistent with an electromagnetic quasistatic approximation.

Assuming Ohm's law holds within the material gives us

$$\mathbf{J} = \sigma\frac{\mathbf{F}}{q} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (5)$$

Here, $\sigma$ is the conductivity. Taking the curl of both sides of (5) and using Maxwell's Equations yields

$$\nabla^2\mathbf{B} - \sigma\mu_0\left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B})\right) = 0. \quad (6)$$

Equation (6) is our main field equation for the magnetic field within the conducting solid.

## 3 Magneto-elasticity model

We start with the linear momentum equation in a continuum:

$$\nabla \cdot \boldsymbol{\tau} - \rho\ddot{\boldsymbol{u}} + \mathbf{f} = 0. \quad (7)$$

Here $\tau$ is the stress, $u$ is the displacement, and $f$ is the body force per unit volume. We take $f$ to be the Lorentz body force, given by

$$\mathbf{f} = \mathbf{J} \times \mathbf{B} \quad (8)$$

$$\text{by (4):} \quad = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \quad (9)$$

Substituting (9) into (7) gives:

$$\nabla \cdot (\mathbf{C} : \nabla \boldsymbol{u}) - \rho\ddot{\boldsymbol{u}} + (\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}) = 0 \quad (10)$$

## 4 Linearization and scaling

We consider shear wave propagation at frequency $\omega_0$, which determines our time scale. We choose to nondimensionalize space with respect to $L_{mech} = \frac{c_{mech}}{\omega_0}$. Two non-dimensional ratios arise in the analysis. These are:

$$\beta = \frac{L_{mech}}{L_{mag}} = c_{mech}\sqrt{\frac{\sigma\mu_0}{\omega_0}} \ll 1 \quad (11)$$

$$\gamma = \frac{B_0^2}{\mu_0 \rho c_{mech}^2} \gg 1 \quad (12)$$

We now assume that the magnetic field is a small perturbation of a strong static field, $\mathbf{B}_0 = B_0\boldsymbol{m}$:

$$\mathbf{B} = B_0(\boldsymbol{m} + \beta^2\boldsymbol{h}), \qquad \beta^2\boldsymbol{h} \ll 1. \quad (13)$$

We thus obtain to leading order:

$$\nabla^2\mathbf{h} + \nabla\times(\frac{\partial \boldsymbol{u}}{\partial t}\times\boldsymbol{m}) = 0 \quad (14)$$

$$\nabla\cdot(\tilde{\mathbf{C}}:\nabla\boldsymbol{u}) - \frac{\partial^2\boldsymbol{u}}{\partial t^2} + \gamma\beta^2((\nabla\times\mathbf{h})\times\boldsymbol{m}) = 0 \quad (15)$$

Here, $\tilde{\mathbf{C}} = \frac{1}{\rho c_{mech}^2}\mathbf{C}$ is the nondimensional elasticity tensor.

## 5 Time-harmonic waves

We assume traveling wave solutions for displacement and magnetic field:

$$\boldsymbol{u} = \boldsymbol{A}\exp(i\kappa\hat{\mathbf{n}}\cdot\boldsymbol{x} - it), \quad (16)$$

$$\boldsymbol{h} = \boldsymbol{d}\exp(i\kappa\hat{\mathbf{n}}\cdot\boldsymbol{x} - it). \quad (17)$$

Substituting into (14) and (15), and eliminating $\boldsymbol{d}$ leads to the following effective dispersion equation for the mechanical wavefield amplitude, $\boldsymbol{A}$:

$$\Big(\boldsymbol{n}\cdot[(\kappa^2\tilde{\mathbf{C}} + i\gamma\beta^2\boldsymbol{D}]\cdot\boldsymbol{n} - \mathbf{I}\Big)\mathbf{A} = 0. \quad (18)$$

Here, $\boldsymbol{D}(\boldsymbol{m})$ is the non-dimensional tensor:

$$\boldsymbol{D}_{ijkl} = m_i m_l\delta_{jk} - m_j m_l\delta_{ik} - m_i m_k\delta_{jl} + \delta_{ij}\delta_{kl}. \quad (19)$$

This tensor, $\gamma\beta^2\boldsymbol{D}(\boldsymbol{m})$, may be interpreted as a non-dimensional damping tensor.

## 6 Discussion

Equations (18) and (19) are our main results, and have several implications regarding the behavior of elastic waves in the presence of a strong magnetic field.

- The dominant effect of the strong magnetic field and the medium conductivity is wave damping through eddy currents.
- The magnitude of the eddy current damping effect is proportional to

$$\gamma\beta^2 = \frac{\sigma B_0^2}{\rho\omega_0}. \quad (20)$$

- The magnitude of the effect is the same for both P-Waves and S-Waves.
- For soft tissues [6,7] the effect is negligible at all practical frequencies.

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