

# Monotone Lagrangians in flag varieties

---

Yunhyung Cho, Y Kim. "Monotone Lagrangians in flag varieties."

<https://hdl.handle.net/2144/29007>

*Downloaded from DSpace Repository, DSpace Institution's institutional repository*

# MONOTONE LAGRANGIANS IN FLAG VARIETIES

YUNHYUNG CHO AND YOOSIK KIM

ABSTRACT. In this paper, we give a formula for the Maslov index of a gradient holomorphic disc, which is a relative version of the Chern number formula of a gradient holomorphic sphere for a Hamiltonian  $S^1$ -action. Using the formula, we classify all monotone Lagrangian non-toric fibers of Gelfand-Cetlin systems on partial flag manifolds.

## CONTENTS

1. Introduction	1
2. Gradient $J$ -holomorphic discs	2
3. Maslov index formula	4
4. Gelfand-Cetlin systems	9
5. Monotone Lagrangian Gelfand-Cetlin fibers	16
References	25

## 1. INTRODUCTION

Every Lagrangian submanifold  $L$  in a  $2n$ -dimensional symplectic manifold  $(M, \omega)$  comes up with the so-called Maslov homomorphism  $\mu: \pi_2(M, L) \rightarrow \mathbb{Z}$ . The output  $\mu(\beta)$  of a homotopy class  $\beta$  is called the *Maslov index* of  $\beta$ , which can be thought as a relative version of the Chern number of a spherical class in  $\pi_2(M)$ . The Maslov index is particularly important in the theory of moduli spaces of pseudo-holomorphic curves from bordered Riemann surfaces as it involves the Fredholm index of a linearization of the Cauchy-Riemann operator via the Riemann-Roch theorem. In particular, for the purpose of studying Lagrangian Floer theory on symplectic toric manifolds, Cho [Cho] and Cho-Oh [CO] introduced a formula for the Maslov index of a holomorphic disc bounded by a Lagrangian toric fiber. The Maslov index is twice the intersection number of the disc and the toric anti-canonical divisor. In a more general context, Auroux [Aur] derived a Maslov index formula for a special Lagrangian submanifold in the complement of an anti-canonical divisor of a Kähler manifold. Those formulae are crucially used for classifying the holomorphic discs, which leads to a mirror Landau-Ginzburg model arising from deformations of Floer theory in Fukaya-Oh-Ohta-Ono [FOOO1, FOOO2].

The first goal of this paper is to deduce a formula for the Maslov index of a *gradient disc*, which is an analogue of a *gradient sphere* in Karshon [Ka], bounded by an  $S^1$ -invariant Lagrangian submanifold in a symplectic manifold admitting a Hamiltonian  $S^1$ -action.

**Theorem A** (Theorem 3.5). *Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic manifold equipped with an effective Hamiltonian  $S^1$ -action with a moment map  $H: M \rightarrow \mathbb{R}$ . Suppose that  $L$  is an  $S^1$ -invariant Lagrangian submanifold of  $(M, \omega)$  lying on some level set of  $H$ . Then, for any gradient holomorphic disc  $u: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$ , we have*

$$\mu([u]) = 2(n_z^- - n_z^+)$$

where  $n_z^-$  (resp.  $n_z^+$ ) is the sum of negative (resp. positive) weights at the unique fixed point  $z$  in  $u(\mathbb{D})$ . In particular, if the action is semifree and  $H(z)$  is the maximum, then the Maslov index  $\mu([u])$  equals the codimension of the maximal fixed component of the action.

This formula can be understood as a relative version of the Chern number formula of a gradient sphere in Ahara-Hattori [AH]. Indeed, we use Lerman's symplectic cut [Le] to reduce our the gradient disc to the gradient sphere.

It is worth mentioning that Theorem A can be applied to a symplectic manifold having a locally defined Hamiltonian  $S^1$ -action. More precisely, if a Hamiltonian  $S^1$ -action is defined on an open subset  $U$  of  $(M, \omega)$ , then one can apply Theorem A to a gradient disc as long as the image of the disc is fully contained in  $U$ . It exactly fits into the situation of *Gelfand-Cetlin systems* on partial flag manifolds for instance.

A *Gelfand-Cetlin system*, or shortly a *GC-system*, is a completely integrable system on a partial flag manifold constructed by Guillemin and Sternberg [GS2]. The image is a convex polytope  $\Delta$ , which is called a *Gelfand-Cetlin polytope*, or a *GC-polytope* for short. As the big torus action does not extend to the ambient manifold, non-torus Lagrangian fibers can appear over a lower dimensional face of the polytope  $\Delta$ . For the case of partial flag manifolds of type A, the authors with Oh [CKO] locate the non-toric GC fibers and describe their topology. One consequence of [CKO, Theorem A] is that the fiber over a point in the relative interior of a face  $f$  is Lagrangian if and only if the fiber over any point in the relative interior of  $f$  is also Lagrangian. In this regard, a face  $f$  of  $\Delta$  is said to be *Lagrangian* if one fiber over its relative interior point is Lagrangian (and hence all).

Using the formula, we classify all *monotone* Lagrangian fibers of GC systems. A Lagrangian submanifold is said to be *monotone* if the symplectic area of discs are positively proportional to their Maslov index, that is, for some positive real number  $c > 0$ ,

$$\omega(\beta) = c \cdot \mu(\beta), \quad \beta \in \pi_2(M, L).$$

The notion of monotone Lagrangian submanifolds was introduced by Oh [Oh] as a nice condition for constructing Lagrangian Floer homology. Our second main theorem states that every monotone Lagrangian submanifold of a partial flag manifold is located at the center of each Lagrangian face.

**Theorem B** (Theorem 5.1). *Consider a partial flag manifold equipped with a monotone Kirillov-Kostant-Souriau symplectic form and let  $\Delta$  be the corresponding GC polytope. For a point  $\mathbf{u} \in \Delta$ , the fiber of the GC-system at  $\mathbf{u}$  is monotone Lagrangian if and only if  $\mathbf{u}$  is the center of a Lagrangian face of  $\Delta$ .*

As a special case of Theorem B, it immediately follows that the GC toric fiber at  $\mathbf{u}$  is monotone if and only if  $\mathbf{u}$  is the center of  $\Delta$  since  $\Delta$  is itself the (unique) improper Lagrangian face of  $\Delta$ .

Finally, we address the importance of monotone Lagrangians the aspects of Fukaya category and mirror symmetry. In the light of the work of Nohara-Ueda [NU2] and Evans-Lekili [EL] proving that certain monotone GC fibers (together with deformation data) in certain Grassmannians split-generates the Fukaya category, monotone Lagrangian fibers serve as candidates for non-zero objects of the Fukaya category over a ring (with certain characteristic). Also, a preferred Landau-Ginzburg mirror constructed by Rietsch [Ri] in the setting of closed mirror symmetry is defined on a partial compactification of algebraic torus. Non-toric monotone Lagrangians are presumably in charge of the added part for the partial compactification.

The paper is organized as follows. In Section 2, we define a gradient holomorphic disc generated by a Hamiltonian  $S^1$ -action. Section 3 discusses the Maslov index for gradient discs and prove Theorem A. In Section 4, we review GC systems and recall some results in [CKO]. Section 5 is devoted to classify the monotone Lagrangian GC fibers and prove Theorem B.

**Acknowledgements.** The authors would like to thank Cheol-Hyun Cho, Yong-Geun Oh, Kaoru Ono for helpful discussions. The first author is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP; Ministry of Science, ICT & Future Planning) (NRF-2017R1C1B5018168). This project was initiated when two authors were affiliated to IBS-CGP and were supported by IBS-R003-D1.

## 2. GRADIENT $J$ -HOLOMORPHIC DISCS

In this section, we introduce the notion of a *gradient disc*, an analogue of a gradient sphere (cf. [AH, Aud, Ka]), in a Hamiltonian  $S^1$ -manifold.

Let  $(M, \omega)$  be a symplectic manifold. Assume that the unit circle group  $S^1$  acts effectively on  $M$  and denote by  $\xi$  the vector field on  $M$  generated by the  $S^1$ -action. The action is said to be *Hamiltonian* if

$$(2.1) \quad \iota_\xi \omega = -dH$$

for some smooth function  $H$  on  $M$ . Such a function  $H$  is called a *moment map* for the  $S^1$ -action, or a *periodic Hamiltonian*. Then we may obtain the following from (2.1), see [Aud, Remark II.3.1].

**Lemma 2.1.** *The fixed point set  $M^{S^1}$  of the action coincides with the critical point set  $\text{Crit}H$  of  $H$ .*

Let  $J$  be an  $S^1$ -invariant  $\omega$ -compatible almost complex structure on  $(M, \omega)$ . With respect to the Riemannian metric  $g_J(\cdot, \cdot) := \omega(J\cdot, \cdot)$ , the gradient vector field  $\nabla H$  of  $H$  is characterized by

$$(2.2) \quad g_J(\nabla H, Y) = dH(Y), \quad \text{or equivalently,} \quad \omega(J\nabla H, Y) = -\omega(\xi, Y)$$

for every vector field  $Y$  on  $M$ . Therefore we have  $\nabla H = J\xi$ . Assuming of the completeness of the vector field  $J\xi$  on  $M$ , the one-parameter subgroup action

$$(2.3) \quad \begin{aligned} \gamma: \mathbb{R} \times M &\rightarrow M \\ (s, q) &\mapsto \gamma_s(q) \end{aligned}$$

is well-defined on  $\mathbb{R}$  where  $\gamma_s$  is an integral curve of  $J\xi$  defined by the following differential equation

$$\begin{cases} \frac{d}{ds}\gamma_s(q) = J\xi(\gamma_s(q)) \\ \gamma_0(q) = q \end{cases}$$

for every  $q \in M$ .

Now, pick a point  $p \in M$  whose stabilizer is the trivial subgroup of  $S^1$  and let  $\sigma_p := S^1 \cdot p$  be the free  $S^1$ -orbit containing  $p$ . Define a map

$$(2.4) \quad \begin{aligned} u: \mathbb{R}_{\geq 0} \times S^1 &\rightarrow M \\ (s, t) &\mapsto \gamma_s(t \cdot p), \end{aligned}$$

whose image consists of  $S^1$ -orbits.

**Lemma 2.2.** *Suppose that  $\gamma_s(p)$  converges to some fixed point  $z_p \in M^{S^1}$  as  $s$  goes to infinity. Then every point in  $\sigma_p$  converges to the point  $z_p$  along  $\gamma_s$ . In other words,*

$$\lim_{s \rightarrow \infty} \gamma_s(q) = z_p$$

for every  $q \in \sigma_p$ .

*Proof.* Fix  $q \in \sigma_p$ . Since  $J$  is  $S^1$ -invariant, so  $J\xi$  is and hence the one-parameter group action  $\gamma$  commutes with the  $S^1$ -action. Therefore, we have

$$\lim_{s \rightarrow \infty} t \cdot \gamma_s(q) = \lim_{s \rightarrow \infty} \gamma_s(t \cdot q) = \lim_{s \rightarrow \infty} \gamma_s(p) = z_p.$$

where  $t \in S^1$  such that  $t \cdot q = p$ . □

**Lemma 2.3.** *The map  $u$  in (2.4) is  $(j, J)$ -holomorphic where  $j$  is the almost complex structure on  $\mathbb{R} \times S^1$  given by*

$$j \cdot \frac{\partial}{\partial t} = \frac{\partial}{\partial s}, \quad j \cdot \frac{\partial}{\partial s} = -\frac{\partial}{\partial t}.$$

*Proof.* Comparing

$$J\xi(u(s_0, t_0)) = J\xi(\gamma_{s_0}(t_0 \cdot p)) = \frac{\partial}{\partial s} \Big|_{s=s_0} \gamma_s(t_0 \cdot p) = \frac{\partial u}{\partial s}(s_0, t_0) = du_{(s_0, t_0)} \left( \frac{\partial}{\partial s} \right) = du \left( j \cdot \frac{\partial}{\partial t} \right)$$

with

$$\xi(u(s_0, t_0)) = \xi(\gamma_{s_0}(t_0 \cdot p)) = \frac{\partial}{\partial t} \Big|_{t=t_0} \gamma_{s_0}(t \cdot p) = \frac{\partial u}{\partial t}(s_0, t_0) = du_{(s_0, t_0)} \left( \frac{\partial}{\partial t} \right) = du \left( \frac{\partial}{\partial t} \right),$$

we obtain

$$du \left( j \cdot \frac{\partial}{\partial t} \right) = J \circ du \left( \frac{\partial}{\partial t} \right).$$

Similarly, we see that  $du \left( j \cdot \frac{\partial}{\partial s} \right) = J \circ du \left( \frac{\partial}{\partial s} \right)$  and this completes the proof. □

Recall that the half-infinite cylinder  $\mathbb{R}_{\geq 0} \times S^1$  is conformally equivalent to the punctured unit disc  $\mathbb{D} \setminus \{0\}$  in  $\mathbb{C}$ . If  $\gamma_s(p)$  converges to some fixed point  $z_p$ , by Lemma 2.2, the map  $u$  can be extended to a topological map  $\tilde{u}: \mathbb{D} \rightarrow M$ . Furthermore, the  $L^2$ -norm (called the *energy of  $u$* ) of the derivative of a  $J$ -holomorphic curve satisfies

$$\int_{\mathbb{D}} \|du\|^2 = \int_{\mathbb{D}} u^* \omega = H(z_p) - H(p),$$

see [McS2, Section 4]. In particular, since the energy of  $u$  is bounded (uniformly), we apply the Riemann extension theorem to confirm that  $u$  can be extended to a  $J$ -holomorphic map  $\tilde{u}: \mathbb{D} \rightarrow M$ .

**Definition 2.4.** The extended map  $\tilde{u}: \mathbb{D} \rightarrow M$  is called a *gradient  $J$ -holomorphic disc* of an  $S^1$ -orbit  $\sigma_p$  and is denoted by  $u_p^J$ .

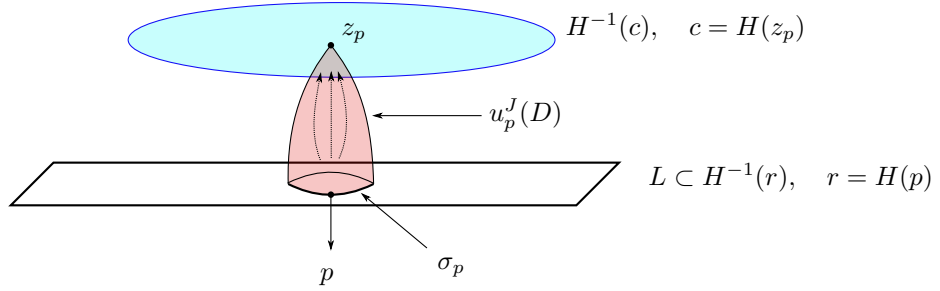


FIGURE 1. Gradient holomorphic disc

*Remark 2.5.* The map  $u$  in (2.4) cannot be only defined on  $\mathbb{R}_{\geq 0} \times S^1$ , but defined on the infinite cylinder  $\mathbb{R} \times S^1$  by considering the gradient flow with respect to  $g_{-J}$ . Under the assumption that  $\gamma_s$  converges to a fixed point as  $s$  goes to  $\pm\infty$ , the map  $u$  can be extended to a smooth map defined on  $S^2$ . It is called a *gradient sphere* in [Aud, AH, Ka]. Note that a gradient sphere containing an  $S^1$ -orbit  $\sigma_p$  can be obtained by gluing two gradient holomorphic discs  $u_p^J$  and  $u_p^{-J}$  along their common boundary  $\sigma_p$ .

### 3. MASLOV INDEX FORMULA

The aim of this section is to derive a formula for the Maslov index of a gradient holomorphic disc. More precisely, we will show that the Maslov index of a gradient  $J$ -holomorphic disc  $u: \mathbb{D} \rightarrow M$  bounded by an  $S^1$ -invariant Lagrangian submanifold having constant momentum is determined by the weights of the tangential  $S^1$ -representation at the unique fixed point contained in  $u(\mathbb{D})$ .

We begin with the following well-known fact on Maslov indices.

**Lemma 3.1.** *Let  $L$  be a Lagrangian submanifold of a  $2n$ -dimensional symplectic manifold  $(M, \omega)$ . Let  $\varphi: [0, 1] \times L \rightarrow M$  be a Lagrangian isotopy, that is, a smooth map such that  $\varphi_s := \varphi(s, \cdot): L \rightarrow M$  is a Lagrangian embedding for each  $s \in [0, 1]$ . Then, the Maslov index is preserved through the isotopy. More precisely, for any disc  $u_0: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, \varphi_0(L))$ , consider the (extended) disc  $u_s: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, \varphi_s(L))$  obtained by gluing the disc  $u_0$  and the cylinder  $v: [0, s] \times \partial\mathbb{D} \rightarrow M$  along the boundary of the disc  $u_0(\mathbb{D})$ . Then, we have*

$$\mu([u_0]) = \mu([u_s])$$

where  $[u_s] \in \pi_2(M, \varphi_s(L))$  is the homotopy class represented by  $u_s$  and  $\mu([u_s])$  denotes its Maslov index.

Consider a  $2n$ -dimensional closed symplectic manifold  $(M, \omega)$  endowed with an effective Hamiltonian  $S^1$ -action. Let  $\xi$  be the vector field generated by the action and  $H: M \rightarrow \mathbb{R}$  a moment map. For a point  $p \in M$  with  $H(p) = r$ , consider an  $S^1$ -invariant Lagrangian submanifold  $L$  containing  $p$  and contained in the level set  $H^{-1}(r)$ . Taking an  $\omega$ -compatible  $S^1$ -invariant almost complex structure  $J$ , the gradient vector field  $J\xi$  of  $H$  with respect to the metric  $g_J$  can be defined as in (2.2). For any  $s \in \text{Im } H$  that is not an extreme value of  $H$ , we denote by  $M_s$  the quotient space  $H^{-1}(s)/S^1$  and by  $\omega_s$  the reduced symplectic form on the smooth locus of  $M_s$ . The following

lemma tells us that a neighborhood of a free  $S^1$ -orbit  $\sigma_a$  in  $L$  can be identified with that of another free  $S^1$ -orbit as long as the orbits are related by the gradient flow generated by  $J\xi$ .

**Lemma 3.2.** *Let  $[a, b]$  be the closed interval in  $\text{Im } H$  not containing any extreme values of  $H$  and let  $\sigma_a$  a free  $S^1$ -orbit in  $H^{-1}(a)$ . Assume that the orbit  $\sigma_a$  flows into a free orbit  $\sigma_b$  through free orbits along the gradient vector field. For any  $S^1$ -invariant Lagrangian submanifold  $L \subset H^{-1}(a)$  containing  $\sigma_a$ , there exists an  $n$ -dimensional manifold  $\mathcal{L}$  and an embedding*

$$\varphi: [a, b] \times \mathcal{L} \rightarrow M, \quad (s, z) \mapsto \varphi_s(z)$$

obeying

- for each  $s \in [a, b]$ ,  $\varphi_s(\mathcal{L})$  is an  $S^1$ -invariant Lagrangian submanifold of  $(M, \omega)$ ,
- for each  $s \in [a, b]$ ,  $\varphi_s(\mathcal{L})$  is in the level set  $H^{-1}(s)$ ,
- $\varphi_a(\mathcal{L})$  is a tubular neighborhood of  $\sigma_a$  in  $L$ , and
- $\varphi_b(\mathcal{L})$  is a Lagrangian submanifold of  $M$  containing the orbit  $\sigma_b$ .

In particular, we have a Lagrangian isotopy  $\varphi$  from  $\varphi_a(\mathcal{L})$  to  $\varphi_b(\mathcal{L})$ .

*Proof.* Recall that for any point  $q \in M$ ,  $\gamma_s(q) := \gamma(s, q): \mathbb{R} \rightarrow M$  is a parametrized integral curve of  $J\xi$  such that  $\gamma(0, q) = q$  where  $\gamma$  is defined in (2.3).

Let  $\sigma_b \in H^{-1}(b)$  be the  $S^1$ -orbit into which the given free orbit  $\sigma_a$  flows along the gradient vector field  $J\xi$ . Fix a point  $p \in \sigma_a$ . By reparametrizing the variable  $s$  if necessary, we may assume that  $\gamma_s(p) \in H^{-1}(s)$  for every  $s \in [a, b]$ . Consider an  $S^1$ -equivariant embedding of a closed cylinder

$$\Gamma: [a, b] \times S^1 \rightarrow M \quad (s, t) \mapsto \gamma_s(t \cdot p).$$

Now we take an  $S^1$ -invariant neighborhood  $\mathcal{V}$  of the cylinder  $\text{Im } \Gamma$  in  $H^{-1}([a, b])$  sufficiently small so that the induced  $S^1$ -action is still free on  $\mathcal{V}$ . The quotient  $\underline{\mathcal{V}} := \mathcal{V}/S^1$  can be thought as a parametrized (by  $s$ ) family of open neighborhoods of each point  $[\sigma_s]$  in the quotient space  $M_s$ . Moreover, we may choose  $\mathcal{V}$  such that  $\underline{\mathcal{V}}$  is contractible, and therefore we may assume that  $\mathcal{V} = \underline{\mathcal{V}} \times S^1$  where the  $S^1$ -action on  $\mathcal{V}$  is just the rotation of  $S^1$  on the second factor of  $\underline{\mathcal{V}} \times S^1$ .

Since the level  $[a, b]$  is compact, we may choose a sufficiently small open ball  $\mathcal{W} \subset \mathbb{R}^{2n-2}$  centered at the origin  $O$  of  $\mathbb{R}^{2n-2}$  such that there is a symplectic embedding

$$\iota_s: (\mathcal{W}, \omega_{\mathbb{R}^{2n-2}}) \hookrightarrow (M_s, \omega_s),$$

satisfying  $O \mapsto [\sigma_s]$  for each  $s \in [a, b]$ . Here,  $\omega_{\mathbb{R}^{2n-2}}$  is the standard symplectic form on  $\mathcal{W} \subset \mathbb{R}^{2n-2}$ . We then have a smooth embedding

$$\psi: [a, b] \times (\mathcal{W}, \omega_{\mathbb{R}^{2n-2}}) \rightarrow \underline{\mathcal{V}} \subset \bigcup_{a \leq s \leq b} (M_s, \omega_s)$$

such that

- $\psi(s, \cdot) = \iota_s$ , and
- $\psi_s(O) = [\sigma_s]$  for each  $s \in [a, b]$ .

Consider any  $S^1$ -invariant Lagrangian submanifold  $L$  of  $(M, \omega)$  in the level set  $H^{-1}(a)$  containing the orbit  $\sigma_a$ . Setting  $\underline{L}$  to be the quotient of  $L$  by the  $S^1$ -action, take a tubular neighborhood  $\underline{U}$  of  $[\sigma_a]$  in  $\underline{L}$ , which is a Lagrangian submanifold of  $(M_a, \omega_a)$ . By taking the neighborhood  $\underline{U}$  small enough, we may assume that its closure is contained in  $\psi_a(\mathcal{W})$ . Denote by  $\underline{\mathcal{L}}$  the Lagrangian submanifold  $\psi_a^{-1}(\underline{U})$  of  $(\mathcal{W}, \omega_{\mathbb{R}^{2n-2}})$ . We then have a family  $\{\psi_s(\underline{\mathcal{L}}) \mid s \in [a, b]\}$  such that each  $\psi_s(\underline{\mathcal{L}})$  is Lagrangian and contains  $[\sigma_s]$  of  $(M_s, \omega_s)$ .

Finally, letting  $\mathcal{L} := \underline{\mathcal{L}} \times S^1$ , we define

$$\begin{aligned} \varphi: [a, b] \times \mathcal{L} &\rightarrow \mathcal{V} = \underline{\mathcal{V}} \times S^1, & (\underline{\ell}, t) \in \underline{\mathcal{L}} \times S^1 = \mathcal{L}. \\ (s, (\underline{\ell}, t)) &\mapsto (\psi(s, \underline{\ell}), t) \end{aligned}$$

Then, the following diagram commutes:

$$(3.1) \quad \begin{array}{ccc} [a, b] \times \mathcal{L} & \xrightarrow{\varphi} & \mathcal{V} \\ \downarrow /S^1 & & \downarrow /S^1 \\ [a, b] \times \underline{\mathcal{L}} & \xrightarrow{\psi} & \underline{\mathcal{V}}. \end{array}$$

The map  $\varphi$  is our desired embedding.  $\square$

For a given  $r \in \text{Im } H$ , let  $L \subset (M, \omega)$  be an  $S^1$ -invariant Lagrangian submanifold lying on a level set  $H^{-1}(r)$  and let  $\sigma_p$  be the free  $S^1$ -orbit containing a point  $p \in L$ . Our goal is to compute the Maslov index of the gradient  $J$ -holomorphic disc  $u_p^J: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$  in Definition 2.4. We begin with the following special case.

**Lemma 3.3.** *Suppose that a gradient  $J$ -holomorphic disc  $u_p^J$  maps the center of  $\mathbb{D}$  to some  $S^1$ -fixed point  $z_p$ . If the connected component  $Z_p$  of  $M^{S^1}$  containing  $z_p$  is of codimension two in  $M$ , then the Maslov index of the gradient disc  $u_p^J$  is two, that is  $\mu([u_p^J]) = 2$ .*

*Proof.* Since it has codimension two, the component  $Z_p$  is the maximal fixed component of the  $S^1$ -action. Let  $c := H(Z_p) = H(z_p)$ , the maximum of  $H$ . By the equivariant Darboux theorem, there exists an open neighborhood  $\mathcal{U}_p$  of  $z_p$  that is  $S^1$ -equivariantly symplectomorphic to an open neighborhood  $\mathcal{V}_p$  of the origin in  $\mathbb{C}^n$  equipped with the standard symplectic form and the linear  $S^1$ -action on  $\mathbb{C}^n$  given by

$$t \cdot (z_1, \dots, z_n) = (t^{-1}z_1, z_2, \dots, z_n).$$

Note that  $(0, z_2, \dots, z_n)$  serves as a local coordinate for  $Z_p$  near  $z_p$ . A moment map of the action on  $\mathbb{C}^n$  is taken as

$$H(z) = c - \frac{1}{2}|z_1|^2, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Applying Lemma 3.2 to  $\sigma_r := \sigma_p$ ,  $L$ , and  $[r, c - \epsilon]$  for any positive number  $\epsilon (< (c - r))$ , we obtain an  $n$ -dimensional manifold  $\mathcal{L}$  and an embedding

$$\varphi: [r, c - \epsilon] \times \mathcal{L} \rightarrow M$$

such that  $L_s := \varphi(s, \mathcal{L})$  is an  $S^1$ -invariant Lagrangian submanifold of  $(M, \omega)$  lying on  $H^{-1}(s)$ . Let  $p_0 \in \mathcal{L}$  such that  $\varphi(r, p_0) = p$ .

By taking  $\epsilon$  sufficiently small, we can make  $\sigma_{c-\epsilon}$  contained in the Darboux neighborhood  $\mathcal{U}_p$ . Since the Lagrangian  $L_r = \varphi(r, \mathcal{L})$  still bounds the disc  $u_p^J$  and  $L_r \subset L$ , the Maslov index of  $u_p^J$  bounded by  $L_r$  is equal to that by  $L$ . Moreover, it suffices to calculate the Maslov index of  $u_{\varphi(c-\epsilon, p_0)}^J$  (bounded by  $L_{c-\epsilon}$ ) since the Maslov index is preserved through the Lagrangian isotopy by Lemma 3.1. Therefore, we may assume without any loss of generality that  $r = H(p) = c - \epsilon$ .

In the neighborhood  $\mathcal{V}_p$  of the origin (corresponding to the Darboux neighborhood  $\mathcal{U}_p$ ), every  $S^1$ -orbit containing  $z = (z_1, \dots, z_n)$  can be written as  $\{(w, z_2, \dots, z_n) \mid |w| = |z_1|\}$ . In particular,  $L_{c-\epsilon} \cap \mathcal{U}_p$  corresponds to  $S^1 \times L' \subset \mathbb{C} \times \mathbb{C}^{n-1}$  in  $\mathcal{V}_p$  for some Lagrangian  $L'$  in  $\mathbb{C}^{n-1}$  where  $S^1 = \{w \in \mathbb{C} \mid |w|^2 = 2\epsilon\}$ . Passing  $\mathcal{U}_p$  to  $\mathcal{V}_p$ , the map  $u_{\varphi(c-\epsilon, p_0)}^J$  is precisely  $z \mapsto (\sqrt{2\epsilon} \cdot z, 0, \dots, 0)$  from  $\mathbb{D}$  to  $\mathcal{V}_p \subset \mathbb{C}^n$ . Thus, the Maslov index of  $u_{c-\epsilon}^J$  is two. This finishes the proof.  $\square$

**Lemma 3.4** (Page 75 in [McS]). *Consider a continuous map  $u: S^2 \rightarrow (M, \omega)$  and suppose that  $S^2$  splits into two discs  $\mathbb{D}_1$  and  $\mathbb{D}_2$  such that  $u$  splits into two maps  $u_+: (\mathbb{D}_1, \partial\mathbb{D}_1) \rightarrow (M, L)$  and  $u_-: (\mathbb{D}_2, \partial\mathbb{D}_2) \rightarrow (M, L)$  for some Lagrangian submanifold  $L \subset (M, \omega)$ . Then,*

$$2c_1([u]) = \mu([u_+]) + \mu([u_-]).$$

Now, we are ready to state and prove our main theorem in this section.

**Theorem 3.5.** *Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic manifold equipped with an effective Hamiltonian  $S^1$ -action. Let  $L$  be an  $S^1$ -invariant Lagrangian submanifold of  $(M, \omega)$  lying on some level set of a moment map  $H$ .*

Suppose that a class  $\beta \in \pi_2(M, L)$  is represented by a gradient holomorphic disc of a free  $S^1$ -orbit  $\sigma_p$  containing a point  $p \in L$ . Then

$$\mu(\beta) = \mu([u_p^J]) = 2(n_p^- - n_p^+)$$

where  $n_p^-$  (resp.  $n_p^+$ ) is the sum of negative (resp. positive) weights at  $z_p$ , the image of the center of  $\mathbb{D}$  under  $u_p^J$ .

*Proof.* Put  $r = H(p)$  and choose a sufficiently small positive number  $\epsilon > 0$  such that  $r - \epsilon$  is a regular value of  $H$ . We denote by

$$\overline{M}_{r-\epsilon} := M_{\geq r-\epsilon} / \sim$$

the symplectic cut **[Le]** of  $M$  along  $H^{-1}(r - \epsilon)$  where  $p_1 \sim p_2$  if  $p_1 = t \cdot p_2$  for some  $t \in S^1$ . In general, the reduced space  $\overline{M}_{r-\epsilon}$  is a symplectic orbifold that inherits the reduced symplectic form  $\overline{\omega}_{r-\epsilon}$  from  $M$ . Moreover,  $\overline{M}_{r-\epsilon}$  admits the induced  $S^1$ -action, which is also Hamiltonian. Let  $\overline{H}_{r-\epsilon}$  be a moment map of the induced  $S^1$ -action. Note that the minimal fixed component denoted by  $M_{r-\epsilon}$  is of codimension two. We also denote by  $\overline{J}$  the induced  $S^1$ -invariant almost complex structure on  $\overline{M}_{r-\epsilon}$ , which is compatible with  $\overline{\omega}_{r-\epsilon}$ .

Since the Lagrangian  $L$  remains in  $\overline{M}_{r-\epsilon}$  and nothing has changed in a small neighborhood of  $L$ , flowing the orbit  $\sigma_p$  along  $\overline{J}\xi$  and  $-\overline{J}\xi$ , respectively, we have two gradient  $\overline{J}$ -holomorphic discs

- (1)  $u_{p,+}^{\overline{J}} : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$  intersecting  $Z_p$  at  $z_p$  and
- (2)  $u_{p,-}^{\overline{J}} : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$  intersecting  $M_{r-\epsilon}$  at  $[\sigma_{r-\epsilon}]$

where  $\sigma_{r-\epsilon}$  denotes a free  $S^1$ -orbit in  $H^{-1}(r - \epsilon)$  obtained by flowing  $\sigma_p$  down along the negative gradient flow  $-J\xi$  and  $[\sigma_{r-\epsilon}]$  denotes the corresponding point in  $M_{r-\epsilon}$ . Note that  $[\sigma_{r-\epsilon}]$  is a smooth point in  $M_{r-\epsilon}$ . As their boundaries match up, by decomposing  $S^2$  into the upper hemisphere and the lower hemisphere, we may consider a map  $\widehat{u}_p^{\overline{J}} := u_{p,+}^{\overline{J}} \# u_{p,-}^{\overline{J}} : S^2 \rightarrow \overline{M}_{r-\epsilon}$ .

Since  $(\overline{M}_{r-\epsilon} \setminus M_{r-\epsilon}, \overline{\omega}_{r-\epsilon})$  and  $(M_{>r-\epsilon}, \omega)$  are symplectomorphic, we have

$$(3.2) \quad \mu([u_{p,+}^{\overline{J}}]) = \mu([u_p^J]).$$

By Lemma 3.4, we have

$$(3.3) \quad \mu([u_{p,+}^{\overline{J}}]) + \mu([u_{p,-}^{\overline{J}}]) = 2 \cdot c_1([\widehat{u}^{\overline{J}}]).$$

Since the minimal fixed component is diffeomorphic to the quotient space  $M_{r-\epsilon}$ , see Figure 2, Lemma 3.3 yields

$$(3.4) \quad \mu([u_{p,-}^{\overline{J}}]) = 2$$

Also, by **[AH, Lemma 4.3]** or **[Go, Lemma 3.1]**, we have

$$\langle c_1(M), [\widehat{u}^{\overline{J}}] \rangle = 2 - n_p^+ + n_p^- - 1 = -n_p^+ + n_p^- + 1.$$

Consequently, combining (3.2), (3.3), and (3.4), we have

$$\mu([u_p^J]) = \mu([u_{p,+}^{\overline{J}}]) = 2 \cdot c_1([\widehat{u}^{\overline{J}}]) - \mu([u_{p,-}^{\overline{J}}]) = 2(n_p^- - n_p^+).$$

□

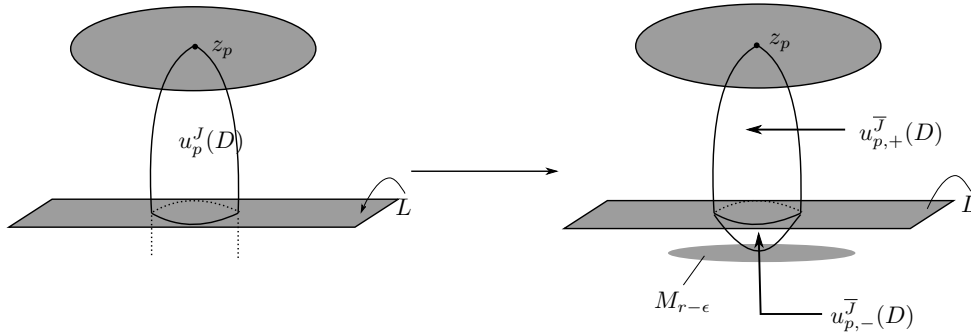


FIGURE 2. Symplectic cut along  $H^{-1}(r - \epsilon)$



If the action is semifree near a fixed point  $z \in M^{S^1}$ , every non-zero weight on  $T_z M$  is either  $+1$  or  $-1$ . In particular, twice the sum of negative (resp. positive) weights at  $z$  is the Morse-Bott index of  $H$  (resp. of  $-H$ ) at  $z$ . Therefore, we obtain the following corollary.

**Corollary 3.6.** *Let  $(M, \omega)$ ,  $L, p \in L$ ,  $z_p \in M^{S^1}$ , and  $\beta \in \pi_2(M, L)$  be given in Theorem 3.5. Additionally, assume that the action is semifree<sup>1</sup> near  $z_p$ . Then the Maslov index  $\mu(\beta)$  is the signature of  $-H$  at  $z_p$ , that is, the difference between the number of negative and positive eigenvalues of the Hessian of  $H$  at  $z_p$ .*

*In particular, if  $H$  attains the maximum at  $z_p$ , then  $\mu(\beta)$  is the codimension of the maximal fixed component  $Z_p$ .*

In the remaining part of the section, we apply Theorem 3.5 and Corollary 3.6 to calculate the Maslov indices of some classes in some well-known examples in [Cho, CO, Aur].

**Example 3.7.** Consider  $\mathbb{C}^n$  with the standard symplectic form  $\omega_0$ . Let  $S_a^1$  denote the circle centered at the origin of radius  $a \in \mathbb{R}_+$  in  $\mathbb{C}$ . For  $(a_1, \dots, a_n) \in (\mathbb{R}_+)^n$ , the torus  $T^n := S_{a_1}^1 \times \dots \times S_{a_n}^1$  is a Lagrangian submanifold of  $\mathbb{C}^n$ . Let  $\mathbb{D}$  be the unit disc in  $\mathbb{C}$ . It is straightforward to see that the action on  $(\mathbb{C}^n, \omega_0)$  given by

$$t \cdot (z_1, \dots, z_k, z_{k+1}, \dots, z_n) = (t^{-1}z_1, \dots, t^{-1}z_k, z_{k+1}, \dots, z_n)$$

is Hamiltonian with a moment map

$$\begin{aligned} H: \mathbb{C}^n &\rightarrow \mathbb{R} \\ z &\mapsto -\frac{1}{2}(|z_1|^2 + \dots + |z_k|^2). \end{aligned}$$

For  $p = (a_1, \dots, a_k, \dots, a_n) \in T^n$ , consider the gradient holomorphic disc of the  $S^1$ -orbit containing  $p$  with respect to the standard complex structure, which is  $S^1$ -invariant. It is explicitly written as

$$\begin{aligned} u: (\mathbb{D}, \partial\mathbb{D}) &\rightarrow (\mathbb{C}^n, T^n) \\ z &\mapsto (a_1 z, \dots, a_k z_k, a_{k+1}, \dots, a_{n-k}). \end{aligned}$$

Note that the fixed point set

$$F = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1 = \dots = z_k = 0 \}$$

of the action occurs at the maximum of  $H$  so that there is no positive weight at  $F$ . Since the action is semifree, the sum of negative weights at  $F$  is exactly the Morse-Bott index of  $H$  at  $F$ , which is the codimension of  $F$ . Therefore, the Maslov index  $\mu([u])$  is equal to  $2(n - k)$  by Corollary 3.6. According to [CO, Theorem 5.1], the Maslov index of  $[u]$  is exactly twice the intersection number between the disc and the toric divisor, which also gives us  $\mu([u]) = 2(n - k)$ .

**Example 3.8.** Let  $\mathbb{C}P^2$  be the projective space equipped with the standard Fubini-Study form  $\omega_0$  and the standard complex structure  $J_0$ . To generalize the Strominger-Yau-Zaslow conjecture to Fano manifolds, Auroux came up with a conic fibration in order to construct a special Lagrangian torus fibration on  $\mathbb{C}P^2$ . He explicitly exhibited the wall-crossing phenomenon using the fibration, which leads to the quantum correction for complex variables on mirror Landau-Ginzburg models, see [Aur] for more details.

To recall his construction, we restrict ourselves to an affine chart of  $\mathbb{C}P^2$  by taking  $x = X/Z, y = Y/Z$  where  $[X : Y : Z]$  is the homogeneous coordinate for  $\mathbb{C}P^2$ . Then, the conic fibration  $f: \mathbb{C}^2 \rightarrow \mathbb{C}^1$  is given by  $f(x, y) = xy$ . We consider the fiberwise  $S^1$ -action determined by

$$t \cdot (x, y) = (t^1 x, t^{-1} y).$$

A periodic Hamiltonian function is chosen as

$$\lambda(x, y) = \frac{1}{2} \cdot \frac{|x|^2 - |y|^2}{1 + |x|^2 + |y|^2}$$

with respect to  $\omega_0$ . For a positive number  $\varepsilon > 0$ , we then have a family of Lagrangian tori

$$T_{r, \lambda} = \{ (x, y) \in \mathbb{C}^2 \mid |xy - \varepsilon| < r, \lambda(x, y) = \lambda_0 \}.$$

parametrized by  $\{(r, \lambda) \mid r > 0, \lambda \in \mathbb{R}\}$  as in Figure 3.

<sup>1</sup>For a Lie group  $G$  acting on a manifold  $M$ , we call the action is *semifree* if the action is free on the complement of the fixed point set.

We focus on a point  $r := \varepsilon$  and  $\lambda := \lambda_0 < 0$  as an example. First, choose any point  $p_1 \in T_{\varepsilon, \lambda_0}$  over the origin, any point of the green circle in Figure 3. Putting  $p_1 = (0, y_1) \in T_{r, \lambda_0}$ , the gradient disc bounded by  $T_{\varepsilon, \lambda_0}$  is

$$u_{p_1}^{J_0}: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (\mathbb{C}^2, T_{\varepsilon, \lambda_0})$$

$$z \mapsto (0, y_1 z).$$

The connected component containing  $u_{p_1}^{J_0}(0) = (0, 0)$  consists of the origin. Because the weights are  $n_{p_1}^- = 1$  and  $n_{p_1}^+ = 1$ , the Maslov index of  $[u_{p_1}^{J_0}]$  is zero by Theorem 3.5.

Next, choose any point  $p_2 \in T_{r, \lambda_0}$  not over the origin. For instance, let us choose a point in the red circle in Figure 3. Putting  $p_2 = (x_2, y_2)$ , the gradient disc bounded by  $T_{r, \lambda_0}$  is written as

$$u_{p_2}^{J_0}: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (\mathbb{C}^2, T_{\varepsilon, \lambda_0})$$

$$z \mapsto (x_2 z^{-1}, y_2 z).$$

The origin maps into the point  $[X : Y : Z] = [1 : 0 : 0]$  via  $u_{p_2}^{J_0}$  in terms of the homogeneous coordinate. Because the weights are  $n_{p_2}^- = \langle (1, -1), (1, -1) \rangle + \langle (1, -1), (0, -1) \rangle = 3$  and  $n_{p_2}^+ = 0$ . Therefore, the Maslov index of the  $J_0$ -holomorphic disc is six.

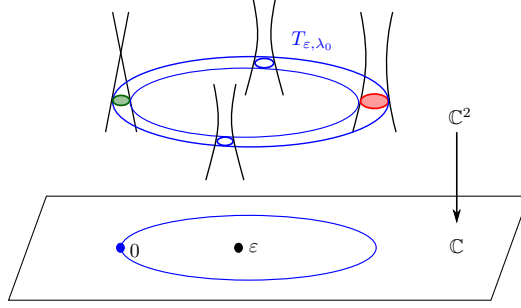


FIGURE 3. A family of Lagrangian tori.

*Remark 3.9.* As one can see in Example 3.8, depending on a choice of a point  $p$ , we can obtain gradient holomorphic discs having different Maslov indices.

#### 4. GELFAND-CETILN SYSTEMS

A *Gelfand-Cetlin system*, or simply a *GC-system*, is a completely integrable system on a partial flag manifold constructed by Guillemin-Sternberg [GS2] as a symplectic geometric analogue of the Gelfand-Cetlin basis [GC]. As in the toric case, the image of a GC system is a convex polytope, called a *Gelfand-Cetlin polytope* (*GC polytope*). In this section, we briefly review various notions and results on GC polytopes and GC systems in [GS2, CKO] that will be used in Section 5.

**4.1. Monotone Lagrangians.** A symplectic manifold  $(M, \omega)$  is called *monotone* if  $[\omega] = \delta \cdot c_1(TM)$  for some positive real number  $\delta$ , called a *monotonicity constant*. A Lagrangian submanifold  $L$  of  $(M, \omega)$  comes with two group homomorphisms: the symplectic energy  $I_\omega$  and the Maslov index  $\mu$  on  $\pi_2(M, L)$ . For any class  $\beta \in \pi_2(M, L)$ , after choosing any disc  $u: (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$  in  $\beta$ , we trivialize the pull-backed tangent bundle  $u^*TM$  over  $\mathbb{D}$ . Restricting the trivialization to the boundary of the disc, the family of tangent spaces of the Lagrangian submanifold  $L$  over the boundary of the disc gives rise to a loop  $\ell$  in the Lagrangian Grassmannian  $\Lambda(n) \simeq U(n)/O(n)$  of  $\mathbb{R}^{2n}$ . We compose the square of the determinant of unitary matrices and then obtain a map from  $S^1$  to  $S^1$ . The *Maslov index* of  $\beta$  is defined by the degree of the map. We call  $L$  *monotone* if the symplectic area of discs bounded by  $L$  is proportional to their Maslov index, that is,

$$(4.1) \quad I_\omega(\beta) = c \cdot \mu(\beta) \quad \text{for every } \beta \in \pi_2(M, L)$$

for some positive number  $c > 0$ . A monotone Lagrangian submanifold can exist only in a monotone symplectic manifold  $(M, \omega)$ . The monotonicity constant  $\delta$  of  $(M, \omega)$  is equal to  $2c$  if  $(M, \omega)$  is not *symplectically aspherical*<sup>2</sup>, see Remark 2.3 in [Oh]. In particular, we have the following.

**Lemma 4.1.** *Let  $(M, \omega)$  be a monotone symplectic manifold such that  $c_1(TM) = [\omega]$  and  $I_\omega|_{\pi_2(M)} \neq 0$ . Then a Lagrangian submanifold  $L \subset (M, \omega)$  is monotone if and only if*

$$2 \cdot \omega(\beta) = \mu(\beta) \quad \text{for every } \beta \in \pi_2(M, L).$$

**4.2. Partial flag manifolds.** For a positive integer  $n$ , consider a sequence of integers

$$(4.2) \quad 0 = n_0 < n_1 < n_2 < \cdots < n_r < n_{r+1} = n.$$

The *partial flag manifold*  $\mathcal{F}(n_1, \dots, n_r; n)$  is the space of nested sequences of complex vector subspaces of  $\mathbb{C}^n$  defined by

$$\mathcal{F}(n_1, \dots, n_r; n) = \{V_\bullet := (0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = n_i\}.$$

Since  $U(n)$  acts on  $\mathcal{F}(n_1, \dots, n_r; n)$  transitively, it becomes a homogeneous manifold diffeomorphic to

$$U(n)/(U(k_1) \times \cdots \times U(k_{r+1})), \quad k_i = n_i - n_{i-1}.$$

One well-known fact is that any partial flag manifold is a complete Fano variety and carries a canonical projective embedding called the *Plücker embedding*.

To construct a GC system on  $\mathcal{F}(n_1, \dots, n_r; n)$ , it is convenient to think of  $\mathcal{F}(n_1, \dots, n_r; n)$  as a co-adjoint orbit of  $U(n)$ , which is defined as follows. Let  $U(n)$  be the unitary group and denote its Lie algebra by  $\mathfrak{u}(n)$ . Choosing a suitable Killing form, the dual  $\mathfrak{u}(n)^*$  of  $\mathfrak{u}(n)$  with the co-adjoint action can be identified with the set  $\mathcal{H}_n$  of  $(n \times n)$  Hermitian matrices with the conjugate action as a  $U(n)$ -space, see [Aud, page 51] for details. We choose any non-decreasing sequence  $\lambda := (\lambda_1, \dots, \lambda_n)$  of real numbers such that

$$(4.3) \quad \lambda_1 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \cdots = \lambda_{n_2} > \cdots > \lambda_{n_r+1} = \cdots = \lambda_{n_{r+1}} (= \lambda_n)$$

where  $\{n_0, \dots, n_r\}$  is given in (4.2). Then, the corresponding co-adjoint orbit is given by

$$\mathcal{O}_\lambda := \{U \cdot \text{diag}(\lambda_1, \dots, \lambda_n) \cdot \overline{U}^T : U \in U(n)\} = \{H \in \mathcal{H}_n : \text{Spec}(H) = \{\lambda_1, \dots, \lambda_n\}\},$$

which is diffeomorphic to  $U(n)/(U(k_1) \times \cdots \times U(k_{r+1}))$ . Therefore,  $\mathcal{O}_\lambda \cong \mathcal{F}(n_1, \dots, n_r; n)$ .

Any co-adjoint orbit  $\mathcal{O}_\lambda$  admits a  $U(n)$ -invariant Kähler form  $\omega_\lambda$ , called the *Kirillov-Kostant-Souriau form*, and  $(\mathcal{O}_\lambda, \omega_\lambda)$  becomes a Hamiltonian  $U(n)$ -manifold whose moment map is the inclusion map  $\mu: \mathcal{O}_\lambda \hookrightarrow \mathfrak{u}(n)^* \cong \mathcal{H}_n$ . The maximal torus  $T^n$  of  $U(n)$ , consisting of diagonal matrices in  $U(n)$ , acts on  $(\mathcal{O}_\lambda, \omega_\lambda)$  in a Hamiltonian fashion with a moment map

$$(a_{ij}) \mapsto \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mapsto (a_{11}, \dots, a_{nn}) \in \mathbb{R}^n \cong \mathfrak{t}^*.$$

The following tells us how one should choose  $\lambda$  so that  $\omega_\lambda$  becomes a monotone symplectic form on  $\mathcal{O}_\lambda$ .

**Proposition 4.2** (See p.653-654 in [NNU] for example). *The symplectic form  $\omega_\lambda$  on  $\mathcal{O}_\lambda$  satisfies*

$$c_1(T\mathcal{O}_\lambda) = [\omega_\lambda]$$

*if and only if*

$$\lambda = \underbrace{(n - n_1, \dots, n - n_1 - n_2, \dots)}_{k_1}, \underbrace{\dots, n - n_1 - n_2, \dots}_{k_2}, \dots, \underbrace{\dots, n - n_{r-1} - n_r, \dots}_{k_r}, \underbrace{\dots, -n_r, \dots, -n_r}_{k_{r+1}} + \underbrace{(m, \dots, m)}_n,$$

*for some  $m \in \mathbb{R}$ .*

<sup>2</sup>A symplectic manifold  $(M, \omega)$  is called *symplectically aspherical* if  $\int_\alpha \omega = 0$  for every  $\alpha \in \pi_2(M)$ . Note that any partial flag manifold is not symplectically aspherical.

**4.3. Gelfand-Cetlin systems.** Let  $\lambda$  be given in (4.3). In [GS2], Guillemin and Sternberg constructed a completely integrable system

$$(4.4) \quad \Phi_\lambda := (\Phi_\lambda^{i,j})_{2 \leq i+j \leq n} : \mathcal{O}_\lambda \rightarrow \mathbb{R}^{\frac{n(n-1)}{2}}$$

on  $(\mathcal{O}_\lambda, \omega_\lambda)$  where  $\Phi_\lambda^{i,j} : \mathcal{O}_\lambda \rightarrow \mathbb{R}$  assigns the  $i$ -th largest eigenvalue of the leading principal minor of size  $(i+j-1)$  for each element of  $\mathcal{O}_\lambda$ . It is called a *Gelfand-Cetlin system*, a *GC system* for short.

The image of  $\Phi_\lambda$  is a convex polytope, called a *Gelfand-Cetlin polytope* and denoted by  $\Delta_\lambda$ , of dimension  $\dim_{\mathbb{C}} \mathcal{O}_\lambda$  and it is precisely the set  $\{(u_{i,j})_{2 \leq i+j \leq n}\} \subset \mathbb{R}^{\frac{n(n-1)}{2}}$  given by

$$(4.5) \quad \begin{array}{cccccccc} & \lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_{n-1} & & \lambda_n \\ & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow \\ & & u_{1,n-1} & & u_{2,n-2} & & & & & u_{n-1,1} \\ & & & \searrow & & \swarrow & & & & \searrow \\ & & & & u_{1,n-2} & & & & & u_{n-2,1} \\ & & & & & \searrow & & \swarrow & & \\ & & & & & & \ddots & & \ddots & \\ & & & & & & & \searrow & & \swarrow \\ & & & & & & & & & u_{1,1} \end{array}$$

This pattern comes from the min-max principle. We give a list of properties of GC systems.

**Theorem 4.3.** Let  $\Phi_\lambda$  be the Gelfand-Cetlin system on  $(\mathcal{O}_\lambda, \omega_\lambda)$  given in (4.4). Let  $\mathring{\mathcal{O}}_\lambda := \Phi_\lambda^{-1}(\mathring{\Delta}_\lambda)$  where  $\mathring{\Delta}_\lambda$  denotes the set of interior points of  $\Delta_\lambda$ .

(1) [GS2, page 113]  $\Phi_\lambda$  is a completely integrable system in the sense that  $\Phi_\lambda$  is smooth on an open dense subset  $\mathcal{U}$  of  $\mathcal{O}_\lambda$  satisfying the followings.

- the components  $\{\Phi_\lambda^{i,j}\}_{2 \leq i+j \leq n}$  Poisson commute with each other on  $\mathcal{U}$ .
- the differentials are linearly independent in each cotangent space at  $x \in \mathcal{U}$ .

Moreover,  $\mathring{\mathcal{O}}_\lambda$  is contained in  $\mathcal{U}$ .

(2) [GS2, Proposition 5.3] The Hamiltonian vector field of each component  $\Phi_\lambda^{i,j}$  generates a Hamiltonian circle action on the subset of  $\mathcal{O}_\lambda$  on which  $\Phi_\lambda^{i,j}$  is smooth.

(3) [CKO, Theorem 5.12] For any  $\mathbf{u} \in \Delta_\lambda$ , the fiber  $\Phi_\lambda^{-1}(\mathbf{u})$  is an isotropic submanifold of  $(\mathcal{O}_\lambda, \omega_\lambda)$ .

(4) A fiber over a point in  $\mathring{\Delta}_\lambda$  is a Lagrangian torus by the Arnold-Liouville theorem.

(5) [CKO, Theorem 7.9] For a  $k$ -dimensional face  $F$  of  $\Delta_\lambda$  and a point  $\mathbf{u}$  in the relative interior  $\mathring{F}$  of  $F$ , the fiber  $\Phi_\lambda^{-1}(\mathbf{u})$  is diffeomorphic to  $(S^1)^k \times Y_F$  for some closed manifold  $Y_F$  such that

$$\pi_1(Y_F) = \pi_2(Y_F) = 0.$$

Now, let us investigate when components and their linear combinations of a GC system are smooth. Since  $\mathcal{O}_\lambda \subset \mathcal{H}_n$  and each component of  $\Phi_\lambda$  is the restriction of an eigenvalue function on  $\mathcal{H}_n$  to  $\mathcal{O}_\lambda$ , it is enough to study the smoothness of eigenvalue functions on  $\mathcal{H}_n$ . For each  $i, j \geq 1$  with  $i + j - 1 = n$ , let

$$(4.6) \quad \begin{aligned} \mathcal{U}_-^{i,j} &:= \{A \in \mathcal{H}_n : \text{the } i\text{-th largest eigenvalue is strictly bigger than the } (i+1)\text{-th largest eigenvalue.}\}, \\ \mathcal{U}_+^{i,j} &:= \{A \in \mathcal{H}_n : \text{the } (i-1)\text{-th largest eigenvalue is strictly bigger than the } i\text{-th largest eigenvalue.}\}. \end{aligned}$$

Denote by  $\Phi^{i,j} : \mathcal{H}_n \rightarrow \mathbb{R}$  the function which assigns the  $i$ -th largest eigenvalue of each element of  $\mathcal{H}_n$ . Note that  $\Phi^{1,n} + \cdots + \Phi^{n,1}$  is a trace function and hence it is smooth on  $\mathcal{H}_n$ . For each  $(i, j)$ , define a *partial trace* by

$$(4.7) \quad \Psi^{i,j} := \Phi^{1,n} + \cdots + \Phi^{i,j} : \mathcal{H}_n \rightarrow \mathbb{R}.$$

**Proposition 4.4.** A partial trace  $\Psi^{i,j}$  is smooth on  $\mathcal{U}_-^{i,j}$ .

*Proof.* For any integer  $k$  with  $1 \leq k \leq n$ , consider the  $k$ -th elementary symmetric polynomial

$$s_k(\Phi^{1,n}, \Phi^{2,n-1}, \dots, \Phi^{n,1}) := \sum_{\{i_1 < \dots < i_k\} \subset [n]} \Phi^{i_1, n+1-i_1} \dots \Phi^{i_k, n+1-i_k},$$

in terms of  $n$  variables  $\Phi^{1,n}, \dots, \Phi^{n,1}$ . Note that each  $s_k(\Phi^{1,n}, \Phi^{2,n-1}, \dots, \Phi^{n,1})$  gives us the  $k$ -th coefficient (up to sign) of the characteristic polynomial of each element in  $\mathcal{H}_n$  and hence it is smooth on  $\mathcal{H}_n$ . Since any symmetric polynomial in the variables  $\Phi^{1,n}, \dots, \Phi^{n,1}$  can be expressed as a polynomial in the  $s_1, \dots, s_n$  variables, every symmetric polynomial in the variables  $\Phi^{1,n}, \dots, \Phi^{n,1}$  is smooth on  $\mathcal{H}_n$ .

Define

$$F: \mathbb{R} \times \mathcal{H}_n \rightarrow \mathbb{R} \\ (x, A) \mapsto \prod_{\{i_1 < \dots < i_k\} \subset [n]} (x - (\Phi^{i_1, n+1-i_1}(A) + \dots + \Phi^{i_k, n+1-i_k}(A))),$$

which is smooth because each coefficient is a symmetric polynomial in the variables  $\Phi^{1,n}, \dots, \Phi^{n,1}$ . Observe that  $x = \Psi^{i,j}(A) = \Phi^{1,n}(A) + \dots + \Phi^{i,j}(A)$  is a zero of  $F$ . For any  $A \in \mathcal{U}_-^{i,j}$ ,  $x = \Psi^{i,j}(A)$  is a simple root since it is the unique maximal root of  $F$  so that  $\frac{\partial F}{\partial x}$  does not vanish at  $(x, A)$ . By the implicit function theorem,  $\Psi^{i,j}$  is smooth at  $A$ . Since this argument holds for every  $A \in \mathcal{U}_-^{i,j}$ ,  $\Psi^{i,j}$  is smooth on  $\mathcal{U}_-^{i,j}$ .  $\square$

**Corollary 4.5.** For each  $i, j \geq 1$  with  $i + j - 1 = n$ ,  $\Phi^{i,j}$  is smooth on  $\mathcal{U}_-^{i,j} \cap \mathcal{U}_+^{i,j}$  where  $\mathcal{U}_-^{i,j}$  and  $\mathcal{U}_+^{i,j}$  are given in (4.6).

*Proof.* A symmetric argument of the proof of Proposition 4.4 shows that  $\Phi^{i,j} + \dots + \Phi^{n,1}$  is also smooth on  $\mathcal{U}_+^{i,j}$ . Since

$$\Phi^{i,j} = \Psi^{i,j} + (\Phi^{i,j} + \dots + \Phi^{n,1}) - (\Phi^{1,n} + \dots + \Phi^{n,1}),$$

the component  $\Phi^{i,j}$  is smooth on  $\mathcal{U}^{i,j} := \mathcal{U}_+^{i,j} \cap \mathcal{U}_-^{i,j}$ .  $\square$

Let us restrict our attention to the co-adjoint orbit  $\mathcal{O}_\lambda$ . Note that for each  $(i, j) \in (\mathbb{Z}_+)^2$  with  $2 \leq i + j \leq n$ , we have

$$\Phi_\lambda^{i,j} := \Phi^{i,j} \circ \pi_\lambda^{i,j}: \mathcal{O}_\lambda \rightarrow \mathcal{H}_{i+j-1} \rightarrow \mathbb{R}$$

where  $\pi_\lambda^{i,j}: \mathcal{O}_\lambda \rightarrow \mathcal{H}_{i+j-1}$  is the projection map whose output is the leading principal minor matrix of size  $(i + j - 1) \times (i + j - 1)$  for each element in  $\mathcal{O}_\lambda$ . Set a partial trace

$$(4.8) \quad \Psi_\lambda^{i,j} := \Phi_\lambda^{1, n-1} + \dots + \Phi_\lambda^{i,j}$$

on  $\mathcal{O}_\lambda$ . Also we denote by

$$\mathcal{U}_{\lambda,-}^{i,j} := \mathcal{U}_-^{i,j} \cap \mathcal{O}_\lambda = \{A \in \mathcal{O}_\lambda \mid \Phi_\lambda^{i+1, j-1}(A) < \Phi_\lambda^{i,j}(A)\}, \\ \mathcal{U}_{\lambda,+}^{i,j} := \mathcal{U}_+^{i,j} \cap \mathcal{O}_\lambda = \{A \in \mathcal{O}_\lambda \mid \Phi_\lambda^{i,j}(A) < \Phi_\lambda^{i-1, j+1}(A)\}.$$

Then each component  $\Phi_\lambda^{i,j}$  is smooth on

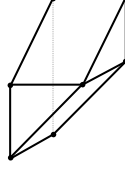
$$\mathcal{U}_\lambda^{i,j} := \mathcal{U}_{\lambda,+}^{i,j} \cap \mathcal{U}_{\lambda,-}^{i,j} = \mathcal{U}^{i,j} \cap \mathcal{O}_\lambda$$

by Corollary 4.5 which is open and dense in  $\mathcal{O}_\lambda$ .

Recall that each component  $\Phi_\lambda^{i,j}$  generates a Hamiltonian circle action on  $\mathcal{U}_\lambda^{i,j}$  by Theorem 4.3 (2). In other words, each  $(\mathcal{U}_\lambda^{i,j}, \omega_\lambda|_{\mathcal{U}^{i,j}})$  is a Hamiltonian  $S^1$ -manifold with a moment map  $\Phi_\lambda^{i,j}$ . Moreover, the following lemma tells us that  $\Psi_\lambda^{i,j}$  is a moment map for a Hamiltonian circle action with respect to the symplectic form on  $\mathcal{U}_{\lambda,-}^{i,j}$  induced from  $\omega_\lambda$ .

**Lemma 4.6.** Any partial trace  $\Psi_\lambda^{i,j}$  generates a Hamiltonian circle action on  $(\mathcal{U}_{\lambda,-}^{i,j}, \omega_\lambda|_{\mathcal{U}_{\lambda,-}^{i,j}})$ .

*Proof.* By Proposition 4.4,  $\Psi_\lambda^{i,j}$  is smooth on  $\mathcal{U}_{\lambda,-}^{i,j}$  so that we only need to prove that every orbit generated by the Hamiltonian vector field  $\xi$  of  $\Psi_\lambda^{i,j}$  is periodic on  $\mathcal{U}_{\lambda,-}^{i,j}$ . Since  $\Psi_\lambda^{i,j} = \Phi_\lambda^{1, i+j-1} + \dots + \Phi_\lambda^{i,j}$  and each summand  $\Phi_\lambda^{\ell, i+j-\ell}$  generates a Hamiltonian circle action on  $\mathcal{U} := \bigcap_{\ell=1}^i \mathcal{U}_\lambda^{\ell, i+j-\ell}$  of  $\mathcal{O}_\lambda$  by [GS2], so does  $\Psi_\lambda^{i,j}$ . Also, the set  $\mathcal{U}$  is also open dense in  $\mathcal{U}_{\lambda,-}^{i,j}$  and every orbit of the Hamiltonian vector field  $\xi$  of  $\Psi_\lambda^{i,j}$  is periodic on  $\mathcal{U}$ . Any orbit of  $\xi$  is periodic on  $\mathcal{U}_{\lambda,-}^{i,j}$  since the *periodicity* is a closed condition.  $\square$


 FIGURE 4. Gelfand-Cetlin polytope  $\Delta_\lambda$  for  $\lambda = \{2, 0, -2\}$ 

It is not guaranteed that the Hamiltonian  $S^1$ -action generated by  $\Psi_\lambda^{i,j}$  may not have an extremal fixed point set in  $\mathcal{U}_{\lambda-}^{i,j}$  since  $\mathcal{U}_{\lambda-}^{i,j}$  is open. Nevertheless, we will see in Section 5 that

- $\mathcal{U}_{\lambda-}^{i,j}$  contains the maximal fixed point set of the action generated by  $\Psi_\lambda^{i,j}$ , and
- the action is semifree near the maximal fixed component.

that will be crucially used for our purpose.

**Example 4.7.** Let  $\lambda = \{2, 0, -2\}$ . The co-adjoint orbit  $\mathcal{O}_\lambda$  is the complete flag variety diffeomorphic to  $U(3)/(U(1))^3$ . The corresponding GC polytope is the intersection of six half planes such that

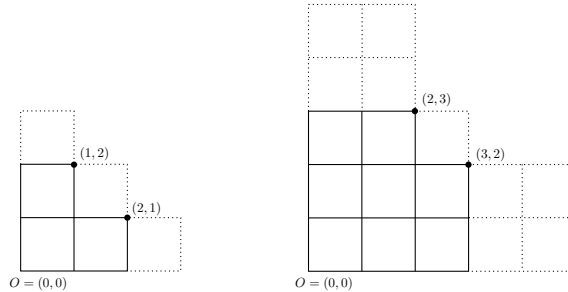
$$u_{2,1} \leq u_{1,1} \leq u_{1,2}, \quad 0 \leq u_{1,2} \leq 2, \quad -2 \leq u_{2,1} \leq 0$$

where  $(u_{1,1}, u_{1,2}, u_{2,1})$  is a coordinate system of  $\mathbb{R}^3$ , see Figure 4. For each Hermitian matrix  $(a_{ij})_{1 \leq i,j \leq 3} \in \mathcal{O}_\lambda$ , we have  $\Phi_\lambda^{1,1}((a_{ij})) = a_{11}$  and  $\Phi_\lambda^{1,1}$  is smooth on the whole  $\mathcal{O}_\lambda$ , that is,  $U_\lambda^{1,1} = \mathcal{O}_\lambda$ . Also,

$$\mathcal{U}_\lambda^{1,2} = \mathcal{U}_\lambda^{2,1} = \{A \in \mathcal{O}_\lambda \mid \Phi_\lambda^{1,2}(A) > \Phi_\lambda^{2,1}(A)\}.$$

Indeed, the set of points at which both  $\Phi_\lambda^{1,2}$  and  $\Phi_\lambda^{2,1}$  are not smooth is contained in  $\Phi_\lambda^{-1}(0, 0, 0)$ , while the sum  $(\Phi_\lambda^{1,2} + \Phi_\lambda^{2,1})((a_{ij})) = a_{11} + a_{22}$  is a smooth function on  $\mathcal{O}_\lambda$ .

**4.4. Ladder diagrams.** Let  $\lambda$  be given in (4.3). According to the work [CKO] of the first author with An and Kim, the face structure of a GC polytope  $\Delta_\lambda$  can be understood in terms of certain subgraphs of a *ladder diagram*.


 FIGURE 5. Ladder diagrams for  $\mathbf{n} = (1, 2; 3)$  and  $\mathbf{n} = (2, 3; 5)$ .

**Definition 4.8.** Consider  $\Gamma_{\mathbb{Z}^2} \subset \mathbb{R}^2$ , the square grid graph such that

- its vertex set is  $\mathbb{Z}^2 \subset \mathbb{R}^2$  and
- each vertex  $(a, b) \in \mathbb{Z}^2$  connects to exactly four vertices  $(a, b \pm 1)$  and  $(a \pm 1, b)$ .

The *ladder diagram*  $\Gamma_\lambda$  is the induced subgraph whose vertex set is

$$\bigcup_{j=0}^r \{(i, j) \in \mathbb{Z}^2 \mid (i, j) \in [n_j, n_{j+1}] \times [0, n - n_{j+1}]\}.$$

Here, we denote the ladder diagram by  $\Gamma_\lambda$  because  $\lambda$  determines  $\mathbf{n}$ .

We call the *origin* the vertex  $O = (0, 0)$  and a *top vertex* a farthest vertex from the origin with respect to the taxi-cap metric. In other words,  $(i, j)$  is a top vertex of  $\Gamma_\lambda$  if  $(i, j)$  is a vertex of  $\Gamma_\lambda$  and  $i + j = n$ . (See Figure 5, the big dots denote the top vertices.) Also, a shortest path from the origin to a top vertex of  $\Gamma_\lambda$  is called a *positive path*. Now, we equip a face structure on  $\Gamma_\lambda$  as follows.

**Definition 4.9** (Definition 1.5 in [ACK]). A subgraph  $\gamma$  of  $\Gamma_\lambda$  is called a *face* of  $\Gamma_\lambda$  if

- $\gamma$  contains all top vertices of  $\Gamma_\lambda$ , and
- $\gamma$  can be represented as a union of positive paths.

The *dimension* of a face  $\gamma$  of  $\Gamma_\lambda$  is defined by

$$\dim \gamma := \text{rank}_{\mathbb{Z}} H_1(\gamma; \mathbb{Z}),$$

that is,  $\dim \gamma$  is the number of bounded regions in  $\gamma$ .

The first author together with An and Kim proved the following correspondence.

**Theorem 4.10** (Theorem 1.9 in [ACK]). *There exists a bijective map*

$$\{\text{faces of } \Gamma_\lambda\} \xrightarrow{\Psi} \{\text{faces of } \Delta_\lambda\}$$

such that for any faces  $\gamma$  and  $\gamma'$  of  $\Gamma_\lambda$ ,

- $\dim \Psi(\gamma) = \dim \gamma$
- $\gamma \subset \gamma'$  if and only if  $\Psi(\gamma) \subset \Psi(\gamma')$ .

Denote the index set of the GC system  $\Phi_\lambda$  by  $I_\lambda \subset (\mathbb{Z}_+)^2$  so that  $(i, j) \in I_\lambda$  if and only if  $\Phi_\lambda^{i,j}$  is a non-constant function on  $\mathcal{O}_\lambda$ . For each  $(i, j) \in I_\lambda$ , let  $\square^{(i,j)}$  be the closed region bounded by the unit square in  $\mathbb{R}^2$  such that the vertices of  $\square^{(i,j)}$  are lying on the lattice  $\mathbb{Z}^2$  and the upper-right vertex is located at  $(i, j)$ . Then we have a natural one-to-one correspondence between the GC components  $\{\Phi_\lambda^{i,j}\}_{(i,j) \in I_\lambda}$  and the unit boxes  $\{\square^{(i,j)}\}_{(i,j) \in I_\lambda}$ . Each point in  $\Delta_\lambda$  corresponds to a “filling each  $\square^{(i,j)}$  by a real number” obeying the inequalities in (4.5). For instance, the vertex  $(0, 0, 0) \in \Delta_\lambda$  in Example 4.7 corresponds to the following tableau in Figure 6.

$$\begin{array}{c} 2 \\ \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \quad -2 \end{array}$$

FIGURE 6. Filling by real numbers.

With this understanding, we briefly describe the bijective map  $\Psi$  in Theorem 4.10. For each face  $\gamma$  in  $\Gamma_\lambda$ , we take the intersection of all facets defined by  $u_{i+1,j} = u_{i+1,j+1}$  (resp.  $u_{i,j+1} = u_{i+1,j+1}$ ) if  $\gamma$  does *not* contain the line segment  $\overline{(i,j)(i+1,j)}$  (resp.  $\overline{(i,j)(i,j+1)}$ ). The face supported by the intersection is the image  $\Psi(\gamma)$ . Conversely, for each point  $\mathbf{u} = (u_{i,j}) \in \Delta_\lambda$ , one can find a unique face  $\gamma$  of  $\Gamma_\lambda$  determined by the rule

- $\gamma$  does *not* contain  $\overline{(i,j)(i+1,j)}$  (resp.  $\overline{(i,j)(i,j+1)}$ ) if  $u_{i+1,j} = u_{i+1,j+1}$  (resp.  $u_{i,j+1} = u_{i+1,j+1}$ ).

**4.5. The topology of Gelfand-Cetlin fibers.** Let  $\lambda$  be given in (4.3) and let  $f$  be any face of the GC-polytope  $\Delta_\lambda$  of dimension  $k$ . For any point  $\mathbf{u}$  in its relative interior  $\overset{\circ}{f}$ , the fiber  $\Phi_\lambda^{-1}(\mathbf{u})$  is an isotropic submanifold of  $(\mathcal{O}_\lambda, \omega_\lambda)$  by Theorem 4.3. In [CKO, Theorem 5.12], the authors and Oh developed a way of “reading off” the topology of  $\Phi_\lambda(\mathbf{u})$  from the face  $\gamma_f$  corresponding to  $f$  in  $\Gamma_\lambda$ . In this section, we review the algorithm and compute some homotopy groups of each fiber of  $\Phi_\lambda$ .

We first consider a closed region in  $\mathbb{R}^2$  bounded by  $\Gamma_\lambda$ , called the *board*. Then we think of the set of edges in  $\gamma_f$  as *walls*. We still denote the “board with the walls” by  $\gamma_f$  if there is no danger of confusion. See Figure 7 for example.

Now, we define a notion “filling of  $\gamma_f$  with  $L$ -blocks” which captures the topology of a fiber over a point contained in the relative interior of  $f$ , where  $L$ -blocks are defined as we see below.

**Definition 4.11** ([CKO], Definition 5.16). Let  $\square^{(i,j)}$  be defined in Section 4.4. For each positive integer  $k \in \mathbb{Z}_{\geq 1}$ , an  $L_k$ -block at  $(a, b)$  is defined by

$$L_k(a, b) := \bigcup_{0 \leq p \leq k-1} \square^{(a,b+p)} \cup \bigcup_{0 \leq p \leq k-1} \square^{(a+p,b)}.$$

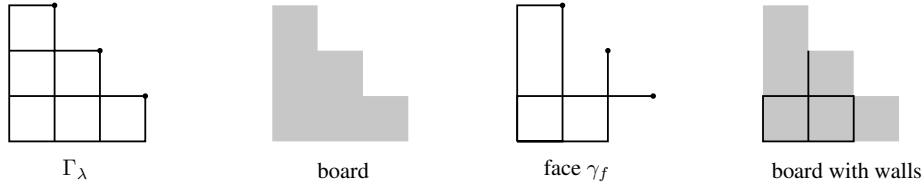


FIGURE 7. Board with walls

An  $L$ -block is meant to be a  $L_k(a, b)$ -block for some  $k \geq 1$  and some  $(a, b) \in (\mathbb{Z}_+)^2$ .

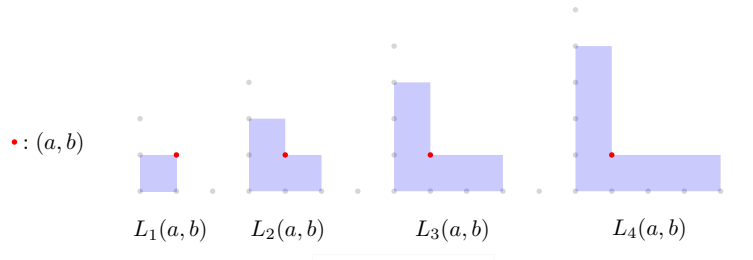


FIGURE 8.  $L$ -blocks

Regarding a face  $\gamma_f$  of  $\Gamma_\lambda$  as the board with walls, we will fill  $\gamma_f$  with  $L$ -blocks satisfying

- (4.9)
  - (1) The interior of a  $L$ -block  $L_k(a, b)$  does *not* contain a wall of  $\gamma_f$ , and
  - (2) Both the rightmost edge and the top edge of  $L_k(a, b)$  are walls of  $\gamma_f$ .

**Definition 4.12.** A *filling* of  $\gamma_f$  with  $L$ -blocks is defined as a collection of all  $L$ -blocks satisfying the conditions (4.9). We say that  $\gamma_f$  is *fillable* by  $L$ -blocks if the filling of  $\gamma_f$  covers the whole board  $\gamma_f$ . If  $\gamma_f$  is fillable by  $L$ -blocks, then we call a face  $\gamma_f$  (resp.  $f$ ) a *Lagrangian face* of  $\Gamma_\lambda$  (resp.  $\Delta_\lambda$ ).

The following theorem characterizes which fiber of  $\Phi_\lambda$  is Lagrangian.

**Theorem 4.13** ([CKO], Corollary 5.23). *Let  $f$  be a face of  $\Delta_\lambda$  and let  $\mathbf{u}$  be any point in its relative interior. The fiber  $\Phi_\lambda^{-1}(\mathbf{u})$  is Lagrangian if and only if  $\gamma_f$  is fillable by  $L$ -blocks.*

*Remark 4.14.* It is worthwhile to note that the diagram  $\Gamma_\lambda$  is cut into the symmetric blocks along the paths in  $\gamma_f$  if  $\gamma_f$  is Lagrangian.

**Example 4.15.** Let  $\lambda = \{6, 4, 2, 0, -2, -4, -6\}$ . Then the co-adjoint orbit  $\mathcal{O}_\lambda$  is diffeomorphic to a complete flag variety. Consider two faces  $\gamma_f$  and  $\gamma_g$  of  $\Gamma_\lambda$  and their fillings with  $L$ -blocks as in Figure 9. Since the filling of  $\gamma_f$  covers whole  $\gamma_f$ , any fiber over a point in the relative interior of  $f$  is Lagrangian. On the other hand, one cannot fill the regions  $\square^{(1,5)}$  and  $\square^{(5,1)}$  in  $\gamma_g$  (white area in the most right one in Figure 9) using  $L_1$ -blocks since the second condition of (4.9) is violated. Thus  $\gamma_g$  is *not* fillable by  $L$ -blocks and any fiber over a point in  $g$  is not Lagrangian.

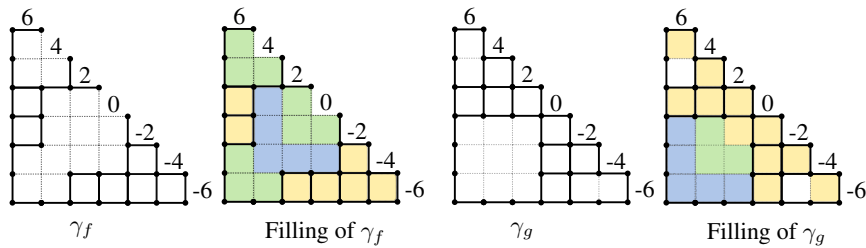


FIGURE 9. Filling with  $L$ -blocks

Now we explain the algorithm provided in [CKO, Theorem 5.12] that decodes the topology of  $\Phi_\lambda^{-1}(\mathbf{u})$  from the fillings of  $\gamma_f$  by  $L$ -blocks.



- **Step 1.** For each  $L_k$ -block located at  $(i, j) \in (\mathbb{Z}_+)^2$ , we associate  $S_{(i,j)}^{2k-1}$ , the  $(2k-1)$ -dimensional sphere labeled by  $(i, j)$ .
- **Step 2.** For each integer  $\ell > 1$ , we denote by

$$F_\ell := \prod_{i+j+k=\ell+1} S_{(i,j)}^{2k-1},$$

which is a product of odd spheres.

- **Step 3.** Then  $\Phi_\lambda^{-1}(\mathbf{u})$  is the total space of an iterated bundle

$$E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_2 \rightarrow E_1 = \text{point}$$

where  $E_\ell$  is an  $F_\ell$ -bundle over  $E_{\ell-1}$  for every  $\ell = 2, \dots, n$ .

Moreover, [CKO, Theorem 7.9] tells us that the filling of  $\gamma_f$  has exactly  $(\dim f)$ -number of  $L_1$ -blocks and every  $S^1$ -factor corresponding to each  $L_1$ -block is a trivial factor so that

$$(4.10) \quad \Phi_\lambda^{-1}(\mathbf{u}) \cong (S^1)^{\dim f} \times Y_f$$

where  $Y_f$  is the total space of an iterated bundle of products of odd spheres. As a corollary, we obtain the following.

**Corollary 4.16.** *Let  $\gamma_f$  be the corresponding face of a face  $f$  in  $\Delta_\lambda$ .*

- *If the filling of  $\gamma_f$  with  $L$ -blocks does not contain any  $L$ -blocks, then  $f$  is a vertex and moreover the fiber over the vertex is a point.*
- *If the filling of  $\gamma_f$  with  $L$ -blocks does not contain any  $L_k$ -blocks for  $k \geq 2$ , then the fiber over any point in the relative interior of  $f$  is a torus.*

Consequently, we have the following.

**Proposition 4.17** (Proposition 7.11 in [CKO]). *Let  $f$  be a  $k$ -dimensional face of  $\Delta_\lambda$  and  $\mathbf{u} \in \overset{\circ}{f}$  be a point in its relative interior. Then the followings hold.*

- (1)  $\pi_1(\Phi_\lambda^{-1}(\mathbf{u})) = \mathbb{Z}^k$ .
- (2)  $\pi_2(\Phi_\lambda^{-1}(\mathbf{u})) = 0$ .
- (3)  $\pi_2(\mathcal{O}_\lambda, \Phi_\lambda^{-1}(\mathbf{u})) = \pi_2(\mathcal{O}_\lambda) \oplus \pi_1(\Phi_\lambda^{-1}(\mathbf{u})) = \pi_2(\mathcal{O}_\lambda) \oplus \mathbb{Z}^k$ .

## 5. MONOTONE LAGRANGIAN GELFAND-CETLIN FIBERS

In this section, using the Maslov index formula derived in Section 3, we classify all monotone Lagrangian fibers of  $\Phi_\lambda$  for a monotone partial flag manifold  $(\mathcal{O}_\lambda, \omega_\lambda)$ .

By scaling and translating  $\lambda$ , without loss of generality, we may assume

$$(5.1) \quad \lambda = \left( \underbrace{n - n_1, \dots, n - n_1 - n_2, \dots}_{k_1}, \dots, \underbrace{n - n_{r-1} - n_r, \dots}_{k_r}, \underbrace{-n_r, \dots, -n_r}_{k_{r+1}} \right)$$

so that  $c_1(\mathcal{O}_\lambda) = [\omega_\lambda]$  by Proposition 4.2 ( $m = 0$ ). The GC polytope  $\Delta_\lambda$  is then a reflexive<sup>3</sup> polytope, which has a unique interior integral lattice point  $\mathbf{u}_{\Delta_\lambda} \in \overset{\circ}{\Delta}_\lambda$  such that the affine distances from  $\mathbf{u}_{\Delta_\lambda}$  to each facet are all equal. In this sense, we call  $\mathbf{u}_{\Delta_\lambda}$  the *center* of  $\Delta_\lambda$ . Choosing  $\lambda$  in (5.1), the center is explicitly written as

$$(5.2) \quad \mathbf{u}_{\Delta_\lambda} = (u_{i,j} := j - i) \in \Delta_\lambda.$$

Let  $f$  be a face of  $\Delta_\lambda$  of dimension  $k$  for some  $k \geq 0$  and  $\gamma_f$  be the corresponding face of the ladder diagram  $\Gamma_\lambda$ . Recall that the dimension of the face  $\gamma_f$  coincides with the number of bounded regions in  $\gamma_f$  by Theorem 4.10. Note that there exists a unique rightmost vertex, a vertex farthest from the origin, of each bounded region by the

<sup>3</sup>A convex polytope  $\Delta$  is called *reflexive* if  $\Delta^* = \Delta$ .

min-max principle. We label the upper rightmost vertex of each bounded region of  $\gamma_f$  by  $\{(a_1, b_1), \dots, (a_k, b_k)\}$  respecting the following order

$$(5.3) \quad i < j \text{ if and only if either } \begin{cases} a_i + b_i = a_j + b_j \text{ and } a_i < a_j, \text{ or} \\ a_i + b_i > a_j + b_j. \end{cases}$$

For instance, in Figure 10, it has five bounded regions and the upper rightmost vertices are ordered as in the figure.

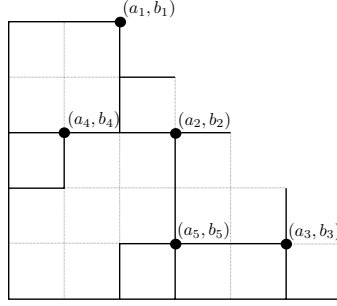


FIGURE 10. The ordering of the upper rightmost vertices

As a face  $f$  is also a polytope, one can consider its center. The *center* of a face  $f$  is a point  $\mathbf{u}_f = (u_{i,j})$  in the interior of  $f$  uniquely determined by

$$(5.4) \quad u_{a_i, b_i} = b_i - a_i, \quad \text{for } i = 1, \dots, k.$$

Note that the positive paths in the face  $\gamma_f$  cut the ladder diagram  $\Gamma_\lambda$  into several regions. All components in any bounded region have same values (determined by  $u_{a_i, b_i}$ 's) and all components in any unbounded regions are determined by the sequence  $\lambda$  in (5.1). Therefore, (5.4) locates a unique point  $\mathbf{u}_f$  in the face  $f$ .

The fiber at the center of  $f$  is a monotone Lagrangian fiber provided  $f$  is Lagrangian.

**Theorem 5.1** (Classification of monotone Lagrangian fibers). *Let  $f$  be a Lagrangian face of the Gelfand-Cetlin polytope  $\Delta_\lambda$  for  $\lambda$  in (5.1). For  $\mathbf{u} \in \mathring{f}$ , the fiber  $\Phi_\lambda^{-1}(\mathbf{u})$  is monotone Lagrangian if and only if  $\mathbf{u}$  is the center of  $f$ .*

**Example 5.2.** Let  $\lambda = (3, 3, 3, -3, -3, -3)$ . The co-adjoint orbit  $\mathcal{O}_\lambda$  is  $\text{Gr}(3, 6)$ . All monotone Lagrangian fibers of  $\Phi_\lambda$  are listed in Figure 11. It admits one monotone  $T^9$ -fiber, one monotone  $U(3)$ -fiber, one monotone  $(S^3)^2 \times T^3$ -fiber, and four monotone  $S^3 \times T^6$ -fibers.

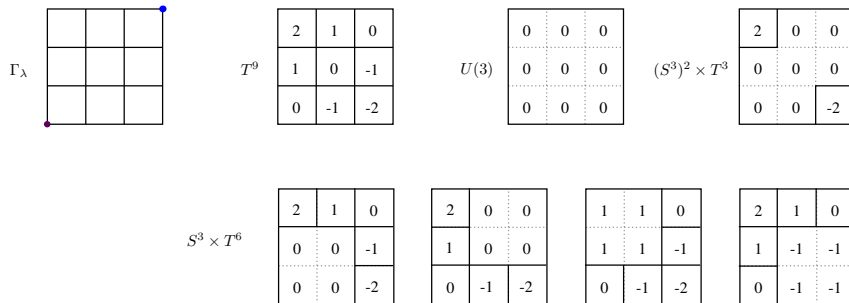
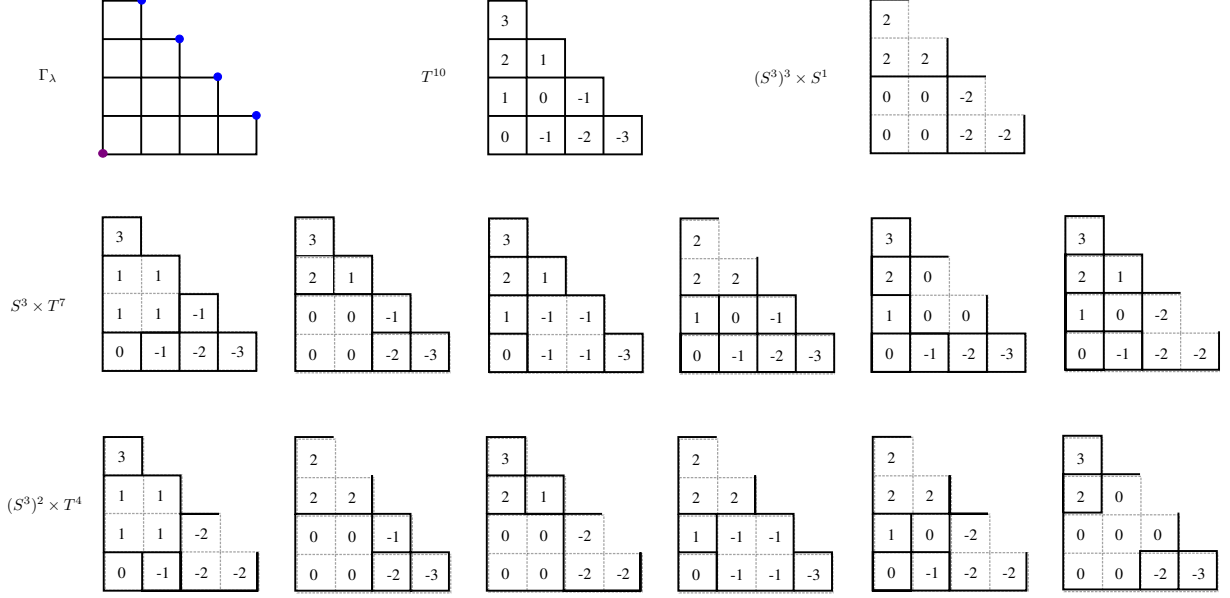


FIGURE 11. Monotone Lagrangian fibers in  $\text{Gr}(3, 6)$

**Example 5.3.** Let  $\lambda = (4, 2, 0, -2, -4)$ . Then the co-adjoint orbit  $\mathcal{O}_\lambda$  is a complete flag manifold  $\mathcal{F}(5)$ . The following figure 12 illustrate all monotone Lagrangian GC fibers in  $\mathcal{F}(5)$ .

FIGURE 12. Monotone Lagrangian fibers in  $\mathcal{F}(5)$ 

The remaining part of this section is reserved for the proof of Theorem 5.1. We are going to use circle actions generated by partial traces (4.7) to obtain gradient holomorphic discs and then apply Corollary 3.6 to compute their Maslov indices. The action is *not* globally defined because a partial trace is *not* a global smooth function in general. So, we need to confirm that a constructed disc generated by the partial trace is contained in its smooth locus. Moreover, to apply Corollary 3.6, we will check the semifreeness of the action.

Fix an index  $i \in \{1, \dots, k\}$  for the upper rightmost vertices in the corresponding graph  $\gamma_f$  to a face  $f$ . Consider the Hamiltonian  $S^1$ -space  $(\mathcal{U}_{\lambda, -}^{a_i, b_i}, \omega_\lambda)$  whose periodic Hamiltonian is the partial trace in Lemma 4.6

$$(5.5) \quad \Psi_\lambda^{a_i, b_i} = \Phi_\lambda^{1, a_i + b_i - 1} + \dots + \Phi_\lambda^{a_i, b_i}.$$

The following lemma tells us that the fiber over any point  $\mathbf{u}$  in the relative interior of  $f$  is contained in the smooth locus of  $\Psi_\lambda^{a_i, b_i}$ .

**Lemma 5.4.** *The inverse image  $\Phi_\lambda^{-1}(\mathring{f})$  is contained in  $\mathcal{U}_{\lambda, -}^{a_i, b_i}$ .*

*Proof.* By our choice of the point  $(a_i, b_i)$ , the edge connecting  $(a_i, b_i)$  and  $(a_i, b_i - 1)$  is contained in the face  $\gamma_f$ . Thus any point  $\mathbf{u} \in \mathring{f}$  satisfies  $u_{a_i, b_i} > u_{a_i+1, b_i} \geq u_{a_i+1, b_i-1}$  and hence  $\Phi_\lambda^{-1}(\mathbf{u})$  is in  $\mathcal{U}_{\lambda, -}^{a_i, b_i}$ .  $\square$

A free orbit generated by  $\Psi_\lambda^{a_i, b_i}$  and contained in  $\Phi_\lambda^{-1}(\mathbf{u})$  can be always taken due to the following. Namely, if  $f$  is Lagrangian, then we may choose a free  $S^1$ -orbit in  $\Phi_\lambda^{-1}(\mathbf{u})$  for any  $\mathbf{u} \in \mathring{f}$ .

**Lemma 5.5.** *Consider an effective symplectic  $S^1$ -action on a  $2n$ -dimensional, possibly open, symplectic manifold  $(M, \omega)$  and suppose that  $L \subset M$  is an  $S^1$ -invariant closed Lagrangian submanifold of  $(M, \omega)$ . Then there exists at least one free  $S^1$ -orbit in  $L$ .*

*Proof.* Since  $L$  is closed, there are only finitely many orbit types, namely types of  $\mathbb{Z}_{p_1}, \dots$ , and  $\mathbb{Z}_{p_r}$  for some positive integers  $p_1, \dots, p_r$  greater than one. Also, since the action is effective and symplectic, each  $\mathbb{Z}_{p_i}$ -fixed point set denoted by  $Z_i$  is a symplectic submanifold of  $(M, \omega)$  and has dimension less than  $2n$ . Thus  $Z_i \cap L$  has positive codimension in  $L$ . Therefore, we have  $L - \bigcup_{i=1}^r Z_i \neq \emptyset$  and this finishes the proof.  $\square$

Now, we figure out the maximal fixed component of the circle action generated by  $\Psi_\lambda^{a_i, b_i}$  and compute its dimension for the purpose of applying Corollary 3.6.

**Lemma 5.6.** *Let  $\lambda = \{\lambda_1, \dots, \lambda_n\}$  be given in (5.1). For any vertex  $(a, b)$  in  $\Gamma_\lambda$ , consider the circle action on  $\mathcal{U}_{\lambda, -}^{a, b}$  generated by  $\Psi_\lambda^{a, b}$  in (4.8). Then, there exists a maximal fixed component, denoted by  $Z_\lambda^{a, b}$ , of the action contained in  $\mathcal{U}_{\lambda, -}^{a, b}$ . Moreover,  $Z_\lambda^{a, b}$  is connected and*

$$(5.6) \quad \text{codim}_{\mathbb{R}} Z_\lambda^{a, b} = 2 \left( \sum_{i=1}^a \lambda_i - \Psi_\lambda^{a, b}(\mathbf{u}_{\Delta_\lambda}) \right)$$

where  $\mathbf{u}_{\Delta_\lambda}$  is the center of  $\Delta_\lambda$ .

*Proof.* Since each  $\Phi_\lambda^{i, j}$  has the maximal value  $\lambda_i$ , the maximal value of  $\Psi_\lambda^{a, b}$  is  $\lambda_1 + \dots + \lambda_a$  on  $\mathcal{O}_\lambda$ . Therefore, the maximal component can be expressed as

$$Z_\lambda^{a, b} = \mathcal{U}_{\lambda, -}^{a, b} \cap \bigcap_{j=1}^a \left( \Phi_\lambda^{j, a+b-j} \right)^{-1} (\lambda_j).$$

Let  $F$  be the face of  $\Delta_\lambda$  given by

$$(5.7) \quad F = \bigcap_{i=1}^a \{ \mathbf{u} \in \Delta_\lambda \mid u_{i, a+b-i} = \lambda_i \}.$$

Then,  $Z_\lambda^{a, b}$  can be expressed as

$$(5.8) \quad Z_\lambda^{a, b} = \mathcal{U}_{\lambda, -}^{a, b} \cap \Phi_\lambda^{-1}(F)$$

By Theorem 4.10, the face  $F$  is non-empty. Because  $u_{a, b} > u_{a+1, b-1}$  for every point  $\mathbf{u} = (u_{i, j}) \in \mathring{F}$ ,  $\Phi_\lambda^{-1}(\mathring{F})$  is contained in  $\mathcal{U}_{\lambda, -}^{a, b}$  so that  $\Phi_\lambda^{-1}(\mathring{F}) \subset Z_\lambda^{a, b}$ . It implies that  $Z_\lambda^{a, b}$  is non-empty. The connectedness of  $Z_\lambda^{a, b}$  follows from [Aud, Corollary IV.3.2].

Let  $\gamma_F$  be the face of  $\Gamma_\lambda$  corresponding to  $F$  in Theorem 4.10, see the second picture of Figure 13 as an example. Observe that  $\gamma_F$  does not contain  $L_k$ -blocks ( $k \geq 2$ ) satisfying (4.9), that is, the filling of  $\gamma_F$  with  $L$ -blocks consists of only  $L_1$ -blocks. By Corollary 4.16, the fiber over any point in  $\mathring{F}$  is a  $(\dim F)$ -dimensional torus. We then obtain the following dimension formula:

$$\text{codim}_{\mathbb{R}} Z_\lambda^{a, b} = 2(\dim \Delta_\lambda - \dim F).$$

On the other hand, we have

$$\sum_{i=1}^a \lambda_i - \Psi_\lambda^{a, b}(\mathbf{u}_{\Delta_\lambda}) = \sum_{j=1}^a \left( \lambda_j - \Phi_\lambda^{j, a+b-j}(\mathbf{u}_{\Delta_\lambda}) \right) = \sum_{j=1}^a (\lambda_j - (a + b - j - j)),$$

where the equalities come from (5.2) and (5.5). From (5.1), it follows that

$$\begin{aligned} \sum_{i=1}^a \lambda_i - \Psi_\lambda^{a, b}(\mathbf{u}_{\Delta_\lambda}) &= \sum_{i=1}^s \sum_{j=n_{i-1}+1}^{\min(n_i, a)} (n - n_{i-1} - n_i - (a + b) + 2j) \\ &= \sum_{i=1}^{s+1} k_i ((n - n_i) - (a + b - n_i - 1)) + (a - n_s) ((n - n_{s+1}) - (a + b - n_s - 1)) \end{aligned}$$

where  $s$  is the integer satisfying  $n_s < a \leq n_{s+1}$ . Observe that it is exactly the number of unit boxes of  $\Gamma_\lambda$  not in  $\gamma_F$ . Thus, by Theorem 4.10, we obtain

$$\sum_{i=1}^a \lambda_i - \Psi_\lambda^{a, b}(\mathbf{u}_{\Delta_\lambda}) = \dim \Delta_\lambda - \dim F.$$

Hence, (5.6) is established.  $\square$

**Example 5.7.** Let  $\lambda = \{8, 8, 3, 3, 3, -2, -2, -5, -8, -8\}$ . We choose a vertex  $(4, 3)$  of  $\Gamma_\lambda$  and consider

$$\Psi_\lambda^{4, 3} = \Phi_\lambda^{1, 6} + \Phi_\lambda^{2, 5} + \Phi_\lambda^{3, 4} + \Phi_\lambda^{4, 3}$$

as an example. Any point  $z$  in the maximal component  $Z_\lambda^{4, 3}$  satisfies

$$\Phi_\lambda^{1, 6}(z) = \Phi_\lambda^{2, 5}(z) = 8, \quad \Phi_\lambda^{3, 4}(z) = \Phi_\lambda^{4, 3}(z) = 3.$$

Then, the image  $\Phi_\lambda(Z_\lambda^{4,3})$  is contained in the face  $F$  where the face  $\gamma_F$  of  $\Gamma_\lambda$  corresponding to  $F$  is given in the second diagram of Figure 13 by the min-max principle. By Theorem 4.10, we have

$$\text{codim}_{\mathbb{R}} \Phi_\lambda^{-1}(\mathring{F}) = \text{codim}_{\mathbb{R}} Z_\lambda^{3,4} = 28.$$

On the other hand, we have

- $\sum_{i=1}^4 \lambda_i = 8 + 8 + 3 + 3 = 22$ ,
- $\Psi_\lambda^{3,4}(\mathbf{u}_{\Delta_\lambda}) = 5 + 3 + 1 + (-1) = 8$ .

Therefore (5.6) holds for  $(a, b) = (3, 4)$ .

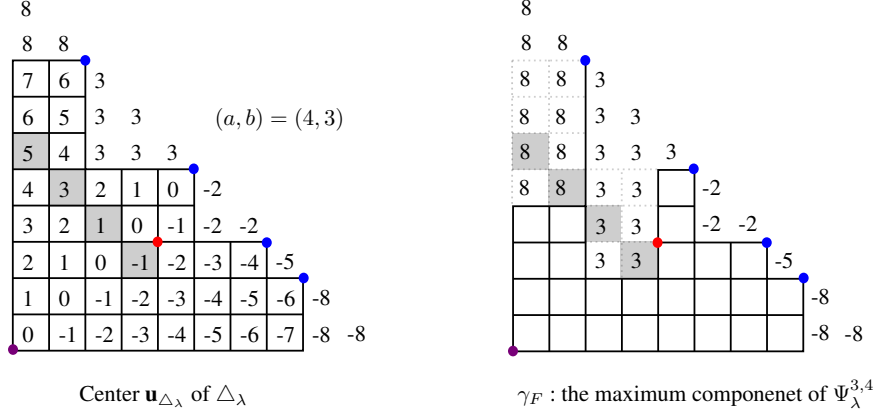


FIGURE 13. Maximal component for  $\Psi_\lambda^{4,3}$

**Proposition 5.8.** *The  $S^1$ -action on  $\mathcal{U}_{\lambda,-}^{a,b}$  generated by  $\Psi_\lambda^{a,b}$  is semifree near  $Z_\lambda^{a,b}$ .*

Before proving Proposition 5.8, we recall some well-known facts needed for the proof. For any (complex)  $n$ -dimensional projective toric variety  $X$  with a projective embedding  $\iota: X \rightarrow \mathbb{P}^N$ , we denote by  $T$  and  $\mathfrak{t}$  the compact torus in  $(\mathbb{C}^*)^n$  acting on  $X$  holomorphically and its Lie algebra, respectively. Then a moment map  $\mu_X: X \rightarrow \mathfrak{t}^* \cong \mathbb{R}^n$  with respect to the Kähler form induced from the Fubini-Study form on  $\mathbb{P}^N$  is the restriction of a moment map  $\mu_{\mathbb{P}^N}$  for the linearly extended Hamiltonian  $T$ -action on  $\mathbb{P}^N$  to  $\iota(X)$ . Moreover, the image  $\mu_X(X)$  is a convex polytope  $\Delta_X$  by the Atiyah-Guillemin-Sternberg convexity theorem [At, GS1]. Let  $\ell \in \mathfrak{t}$  be any primitive integral vector that generates a circle subgroup of  $T$  and let  $e$  be an edge of  $\Delta_X$ . Then the inverse image  $\mu_X^{-1}(e)$  is a  $T$ -invariant 2-sphere and the order of the isotropy subgroup of a point in  $\mu_X^{-1}(e)$  for the  $S^1$ -action is equal to  $|\langle \ell, \vec{e} \rangle|$  where  $\vec{e}$  denotes a primitive edge vector of  $e$ .

Also, we employ the following toric degeneration constructed by Nishinou-Nohara-Ueda [NNU] in order for us to pass our case into the toric case.

**Theorem 5.9** ([NNU]). *There exists a toric degeneration  $\pi: \mathcal{X} \rightarrow \mathbb{C}$  such that*

- *the central fiber  $X_0 = \pi^{-1}(0)$  is the toric variety associated to  $\Delta_\lambda$ , and*
- *For  $X_1 = \pi^{-1}(1)$  is isomorphic to  $\mathcal{O}_\lambda$  as a complex manifold.*

Furthermore, there is a continuous map  $\phi: X_1 \rightarrow X_0$  making the diagram

$$(5.9) \quad \begin{array}{ccc} X_1 & \xrightarrow{\phi} & X_0 \\ & \searrow \Phi_\lambda & \swarrow \mu \\ & & \Delta_\lambda \end{array}$$

commute where  $\mu = (\mu^{i,j})$  is a moment map on  $X_0$ .

In particular, we have  $\mu^{i,j} \circ \phi = \Phi_\lambda^{i,j}$  so that the restriction

$$\phi: \mathcal{U}_{\lambda,-}^{i,j} \rightarrow X_0$$

is equivariant under the  $S^1$ -action on  $\mathcal{U}_{\lambda,-}^{i,j}$  generated by  $\Phi_\lambda^{i,j}$  and the  $S^1$ -action on  $X_0$  generated by  $\mu^{i,j}$ .

To verify Proposition 5.8, it is sufficient to show that the  $S^1$ -action generated by  $\Psi_\lambda^{a,b}$  is semifree at some fixed point in the maximal fixed component  $Z_\lambda^{a,b}$  since it is connected by Lemma 5.6. For the purpose of showing that the action is semifree, we will take one particular vertex.

**Definition 5.10.** For a given  $\lambda$  in (5.1) and any lattice point  $(a, b)$ , let  $s$  be the largest integer satisfying  $n_s < n+1-b$  where  $n_\bullet$ 's are given in (4.2). Consider the subgraph  $\gamma_\lambda^{a,b}$  associated with  $(a, b)$  of the ladder diagram  $\Gamma_\lambda$  such that the vertical segments in  $\gamma_\lambda^{a,b}$  are exactly

$$(5.10) \quad \left\{ \overline{(0, 0), (0, b-1)}, \overline{(n_1, b-1)(n_1, n-n_1)}, \dots, \overline{(n_s, b-1)(n_s, n-n_s)} \right\}.$$

Note that such a face always exists since one can draw horizontal line segments including  $\overline{(0, b-1), (n_s, b-1)}$  to make a face in Definition 4.9. The graph is comb-shaped and it does not bound any closed regions so that the corresponding face is a vertex. The corresponding vertex is called a *comb-shaped vertex* and denoted by  $v_\lambda^{a,b}$ . See Figure 14 for examples.

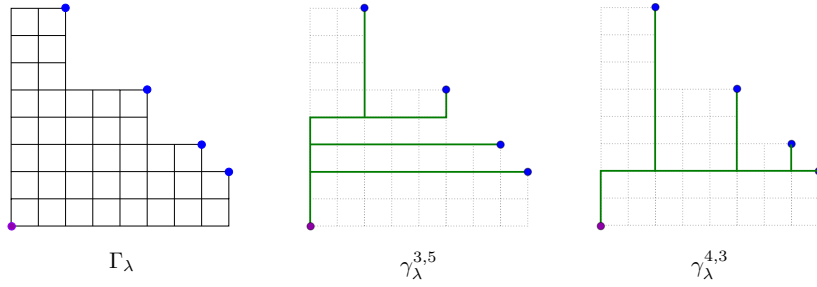


FIGURE 14. Comb-shaped vertices

The comb-shaped vertices enjoy the following properties.

**Lemma 5.11.** For each lattice point  $(a, b) \in \Gamma_\lambda$ , the comb-shaped vertex  $v_\lambda^{a,b}$  satisfies

- (1) The inverse image of  $v_\lambda^{a,b}$  is a point contained in  $Z_\lambda^{a,b}$ .
- (2) The vertex  $v_\lambda^{a,b}$  is incident on edges  $e_1, \dots, e_{\dim \Delta_\lambda}$  such that the fiber over any point in  $\hat{e}_i$  is  $S^1$

*Proof.* Note that the filling of  $\gamma_\lambda^{a,b}$  with  $L$ -blocks is empty, that is, there does not exist any  $L$ -block satisfying the condition (4.9). By Corollary 4.16, the fiber over  $v_\lambda^{a,b}$  is a point. Because of the segment  $\overline{(0, b-1), (n_s, b-1)}$ , the comb-shaped vertex must be contained in  $\mathcal{U}_{\lambda, -}^{a,b}$ . Since  $\gamma_\lambda^{a,b}$  is contained in  $\gamma_F$  corresponding to (5.7),  $v_\lambda^{a,b}$  is contained in  $Z_\lambda^{a,b}$  by (5.8) so that (1) is confirmed.

For (2), we define

$$\begin{aligned} \bullet A_{p,q}(i, j) &:= \bigcup_{1 \leq t \leq p} \square^{(i-t+2, j)} \cup \bigcup_{1 \leq s \leq q} \square^{(i+1, j+s-1)} \\ \bullet B_{p,q}(i, j) &:= \bigcup_{1 \leq t \leq p} \square^{(i+t-1, j+1)} \cup \bigcup_{1 \leq s \leq q} \square^{(i, j-s+2)}, \end{aligned}$$

see Figure 15.

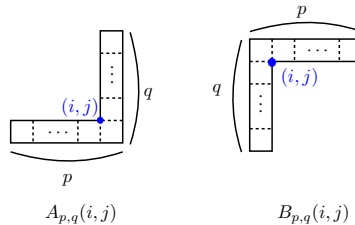


FIGURE 15.  $A_{p,q}(i, j)$  and  $B_{p,q}(i, j)$ .

Note that the ladder diagram  $\Gamma_\lambda$  is divided by  $\gamma_\lambda^{a,b}$  into several regions. Observe that there are two types of regions: the first type contains both the lower edge and the right edge (say  $A$ -type) and the second type contains both the upper edge and the left edge (say  $B$ -type). Consider the faces in  $\Gamma_\lambda$  which is given by the union of

- The comb-shaped face  $\gamma_\lambda^{a,b}$  associated with  $(a, b)$  and
- The boundary of one *single* block  $A_{\bullet, \bullet}(\bullet, \bullet)$  (resp.  $B_{\bullet, \bullet}(\bullet, \bullet)$ ) such that
  - (1) the rightmost edge and the bottom edge (resp. the leftmost edge and the upper edge) contained in  $\gamma_\lambda^{a,b}$
  - (2) it is contained in a single divided region of *A*-type (resp. *B*-type).

Since every graph has one bounded region, the corresponding face is one-dimensional. Note that there are exactly  $\dim \Delta_\lambda$  many such faces. Let  $\{e_1, \dots, e_{\dim \Delta_\lambda}\}$  be the corresponding edges of  $\Delta_\lambda$ . Also, that the generic fiber on each  $e_i$  is  $S^1$  by Corollary 4.16.  $\square$

**Example 5.12.** Let  $\lambda$  and  $(a, b) = (3, 5)$  be given in Figure 14. The graph  $\gamma_\lambda^{3,5}$  divides  $\Gamma_\lambda$  into five regions. There are two *A*-type regions and three *B*-type regions. Figure 16 illustrates the faces corresponding to edges  $\{e_1, \dots, e_{\Delta_\lambda}\}$  in Lemma 5.11.

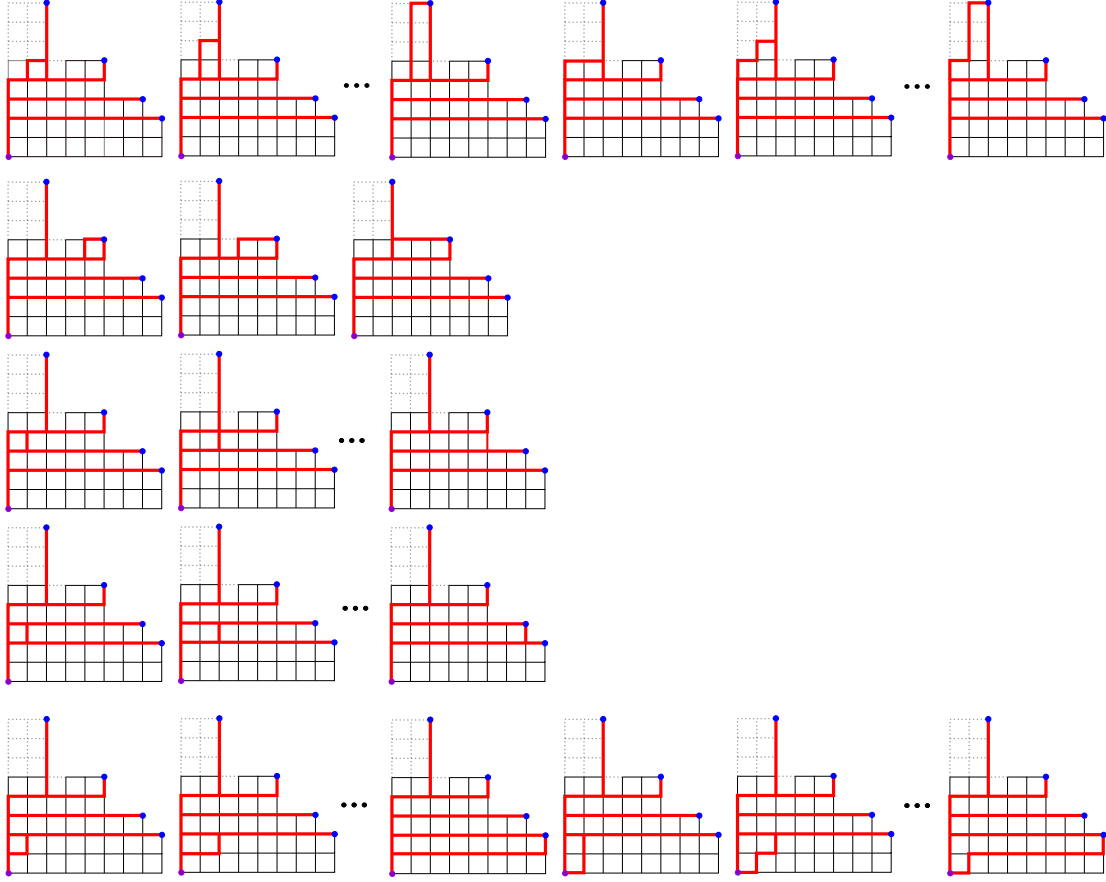


FIGURE 16. Edges from Lemma 5.11.

**Lemma 5.13.** Let  $\{e_1, \dots, e_{\dim \Delta_\lambda}\}$  be the set of edges chosen in Lemma 5.11. Set  $\vec{e}_i$  to be the primitive edge vector of  $e_i$  starting from the comb-shaped vertex  $v_\lambda^{a,b}$ . Let  $\ell = (\ell^{i,j}) \in \mathfrak{t} \cong \mathbb{R}^{\dim \Delta_\lambda}$  be given by

$$(5.11) \quad \begin{cases} \ell^{i,j} = 1 & 1 \leq i \leq a, \quad j = a + b - i \\ \ell^{i,j} = 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$(5.12) \quad \langle \ell, \vec{e}_i \rangle = 0 \text{ or } -1$$

for every  $i = 1, \dots, \dim \Delta_\lambda$ .

*Proof.* Observe that

- (1) The  $(i, j)$ -th component of  $\vec{e}_i$  is  $-1$  if and only if  $\square^{(i,j)}$  is contained in the bounded region of  $e_i$ .

(2) The  $(i, j)$ -th component of  $\vec{e}_i$  is 0 if and only if  $\square^{(i, j)}$  is not contained in the bounded region of  $e_i$ .

Since any added block  $A_{\bullet, \bullet}(\bullet, \bullet)$  in the proof of Lemma 5.11 includes at most one of the boxes  $\{\square^{i, j} \mid i + j = a + b, 1 \leq i \leq a\}$  and any added block  $B_{\bullet, \bullet}(\bullet, \bullet)$  does not, we deduce (5.12).  $\square$

Now we start the proof of Proposition 5.8.

*Proof of Proposition 5.8.* By Lemma 5.11, the fiber over the vertex  $v_\lambda^{a, b}$  consists of a single point  $z$ . Note that each Hamiltonian  $\Phi_\lambda^{i, j}$  is smooth at the point  $z$ . The point is a zero of its Hamiltonian vector field. In particular,  $z$  is a fixed point of the Hamiltonian  $T^n$ -action<sup>4</sup> where  $T^n$  is the maximal torus of  $U(n)$  acting on  $(\mathcal{O}_\lambda, \omega_\lambda)$  in a Hamiltonian fashion. The tangent space  $T_z \mathcal{O}_\lambda$  is decomposed into one-dimensional  $T^n$ -representations  $\xi_1, \dots, \xi_{\dim \Delta_\lambda}$  so that

$$T_z \mathcal{O}_\lambda \cong \bigoplus_{i=1}^{\dim \Delta_\lambda} \xi_i.$$

Moreover, each representation space  $\xi_i$  is equal to the tangent space  $T_z \Phi_\lambda^{-1}(e_i) \cong \mathbb{C}$ . Therefore, it is enough to show that the  $S^1$ -action generated by  $\Psi_\lambda^{a, b}$  is semifree on each  $T_z \Phi_\lambda^{-1}(e_i)$ . Note that by Theorem 5.9, the restriction of the map  $\phi: X_1 \rightarrow X_0$  to  $\mathcal{U}_{\lambda, b}^{a, b}$  is  $S^1$ -invariant under the  $S^1$ -action generated by  $\mu^{1, a+b-1} + \dots + \mu^{a, b}$ , a partial trace of the moment map  $\mu = (\mu^{i, j}): X_0 \rightarrow \Delta_\lambda$ . Therefore, we only need to check

$$\langle \ell, \vec{e}_i \rangle = 0 \text{ or } -1, \quad i = 1, \dots, \Delta_\lambda$$

where  $\ell \in \mathfrak{t}$  that generates the circle action generated by  $\mu^{1, a+b-1} + \dots + \mu^{a, b}$ . Since  $\ell$  is exactly given in (5.11), Lemma 5.13 completes the proof.  $\square$

**Lemma 5.14.** *Let  $f$  be a face of  $\Delta_\lambda$  of dimension  $k$ . For any  $\mathbf{u} \in \overset{\circ}{f}$ , the fundamental group of  $\Phi_\lambda^{-1}(\mathbf{u})$  is generated by  $\{\sigma_1, \dots, \sigma_k\}$  where the vertices  $(a_1, b_1), \dots, (a_k, b_k)$  of  $\Gamma_\lambda$  are defined in (5.3) and  $\sigma_i$  is any free orbit of the  $S^1$ -action generated by  $\Psi_\lambda^{a_i, b_i}$  in  $\Phi_\lambda^{-1}(\mathbf{u})$  for  $i = 1, \dots, k$ .*

*Proof.* Let  $f$  be a  $k$ -dimensional face of  $\Delta_\lambda$  and let  $\mathbf{u} = (u_{i, j}) \in \overset{\circ}{f}$ . As we have seen in Section 4.5,  $\Phi_\lambda^{-1}(\mathbf{u})$  is the total space of an  $n$ -stage iterated bundle

$$E_n \xrightarrow{\pi_n} E_{n-1} \xrightarrow{\pi_{n-1}} \dots \longrightarrow E_2 \xrightarrow{\pi_2} E_1 = \text{point}$$

where each  $E_l$  is a subset of the set  $\mathcal{H}_l$  of  $(l \times l)$  Hermitian matrices and  $\pi_l: E_l \rightarrow E_{l-1}$  is the projection from a  $(l \times l)$  Hermitian matrix to its  $((l-1) \times (l-1))$  leading principal minor.

For each  $i = 1, \dots, k$ , we first describe the  $S^1$ -action on  $\Phi_\lambda^{-1}(\mathbf{u})$  generated by  $\Phi_\lambda^{a_i, b_i}$  more explicitly. Suppose that  $(a_i, b_i)$  is located at the  $\ell$ -th anti-diagonal, i.e.,  $a_i + b_i = \ell$ , and consider the set

$$(5.13) \quad \mathcal{I}_\ell := \{(a_j, b_j) \mid 1 \leq j \leq i, a_j + b_j = \ell\}$$

consisting of  $m$  points where the ordering of  $(a_j, b_j)$ 's is given in (5.3). That is, the set  $\mathcal{I}_\ell$  is equal to

$$\{(a_{i-m+1}, b_{i-m+1}), (a_{i-m+2}, b_{i-m+2}), \dots, (a_i, b_i)\}.$$

Now, we consider the filling of  $\gamma_f$  with  $L$ -blocks. Then the  $L$ -blocks in the filling, whose right vertex of its top edge is located at the  $\ell$ -th anti-diagonal, correspond to odd dimensional sphere factors of the fiber  $F_\ell$  of the projection  $\pi_\ell: E_\ell \rightarrow E_{\ell-1}$ . We denote by  $L_{t_1}, \dots, L_{t_r}$  the  $L$ -blocks in the filling of  $\gamma_f$  each of which is an  $L$ -block of size  $t_i$  and is located at  $(s_j, \ell + 1 - s_j - t_j)$  for some  $s_1 < \dots < s_r$ .

By Corollary 6.10 in [CKO], we know that  $F_\ell$  is diffeomorphic to

$$(5.14) \quad F_\ell \cong \{(z_1, \dots, z_{\ell-1}) \in \mathbb{C}^{\ell-1} \mid |z_{s_j}|^2 + \dots + |z_{s_j+t_j-1}|^2 = C_{s_j} \text{ for some } C_{s_j} > 0 \quad j = 1, \dots, r\}.$$

In particular, each  $(a_j, b_j)$  indicates a location of an  $L_1$ -block in the filling of  $\gamma_f$  and hence we have

$$\{a_{i-m+1}, a_{i-m+2}, \dots, a_i\} \subset \{s_1, \dots, s_r\}.$$

The  $S^1$ -action on the fiber  $F_l$  of  $\pi_l$  generated by  $\Phi_\lambda^{a_i, b_i}$  is expressed by

$$t \cdot (z_1, \dots, z_{l-1}) = (z_1, \dots, z_{a_i-1}, tz_{a_i}, z_{a_i+1}, \dots, z_{l-1}).$$

<sup>4</sup>The  $T^n$ -action is generated by the traces of leading principal minors  $\{\Psi_\lambda^{1,1}, \Psi_\lambda^{2,1}, \dots, \Psi_\lambda^{n,1}\}$ , which are smooth on  $\mathcal{O}_\lambda$ .



Note that the action extends to the total space  $E_n = \Phi_\lambda^{-1}(\mathbf{u})$  such that the projection map  $\pi_q$  ( $q = 2, \dots, n$ ) on each stage is  $S^1$ -equivariant. Moreover, since the action on  $E_l$  is fiberwise, the induced action on  $E_{l'}$  for  $l' < l$  is trivial. Furthermore, as the action is free on  $E_l$ , so is free on  $E_n$ .

Let  $\tau_i$  be any free  $S^1$ -orbit in  $\Phi_\lambda^{-1}(\mathbf{u})$  for the  $S^1$ -action generated by  $\Phi_\lambda^{a_i, b_i}$ . Note that the homotopy class of  $\tau_i$  does not depend on the choice of orbits. By (4.10) and Proposition 4.17,  $\Phi_\lambda^{-1}(\mathbf{u}) \cong (S^1)^k \times Y_f$  for some simply connected manifold  $Y_f$  and its fundamental group  $\pi_1(\Phi_\lambda^{-1}(\mathbf{u})) \cong \mathbb{Z}^k$  is generated by  $\{\tau_1, \dots, \tau_k\}$ . Identifying  $\tau_i$  with  $(\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0) \in \mathbb{Z}^k$ , in order to confirm that the fundamental group generated by  $\sigma_i$ 's, it is enough to show that

$$\sigma_i = (\underbrace{*, \dots, *}_{i-1}, 1, 0, \dots, 0) \in \mathbb{Z}^k \cong \pi_1(\Phi_\lambda^{-1}(\mathbf{u})), \quad i = 1, \dots, k$$

where  $\sigma_i$  is a free  $S^1$ -orbit of the  $S^1$ -action generated by the partial trace  $\Psi_\lambda^{a_i, b_i}$ .

The partial trace  $\Psi_\lambda^{a_i, b_i}$  in (5.5) can be grouped into two parts

$$\begin{aligned} \Psi_\lambda^{a_i, b_i} &= \sum_{j=1}^{a_i} \Phi_\lambda^{j, \ell-j} = \sum_{(j, \ell-j) \in \mathcal{I}_\ell} \Phi_\lambda^{j, \ell-j} + \sum_{(j, \ell-j) \notin \mathcal{I}_\ell} \Phi_\lambda^{j, \ell-j} \\ &= \left( \sum_{j=1}^m \Phi_\lambda^{a_i-j+1, b_i-j+1} \right) + \sum_{(j, \ell-j) \notin \mathcal{I}_\ell} \Phi_\lambda^{j, \ell-j} \end{aligned}$$

where the second summation is over all  $j$  such that  $1 \leq j \leq a_i$  and  $(j, \ell-j) \notin \mathcal{I}_\ell$ . Note the first summand  $(\sum_{j=1}^m \Phi_\lambda^{a_i-j+1, b_i-j+1})$  generates the homotopy class  $\sum_{j=1}^m \tau_{i-j+1}$ . Moreover, the second summand generates an  $S^1$ -action, which is trivial on  $E_l$  (since the portion of  $F_l$  on which the  $S^1$  acts in (5.14) is a point) but may be extended to a non-trivial action on the total space  $E_n$ . Consequently, we have

$$\sigma_i = (\underbrace{*, \dots, *}_\ell, \underbrace{1, \dots, 1}_j, 0, \dots, 0) \in \mathbb{Z}^k \cong \pi_1(\Phi_\lambda^{-1}(\mathbf{u})).$$

This completes the proof.  $\square$

Finally, we prove our main theorem.

*Proof of Theorem 5.1.* Assume that a Lagrangian GC fiber  $\Phi_\lambda^{-1}(\mathbf{u})$  does not contain any circle factors. Then, (4.10) and Proposition 4.17 imply that  $\pi_1(\Phi_\lambda^{-1}(\mathbf{u})) = 0$ . Thus, the fiber is monotone since  $(\mathcal{O}_\lambda, \omega_\lambda)$  is monotone.

It remains to deal with Lagrangian GC fiber having circle factors. Let  $f$  be a Lagrangian face of dimension  $k (> 0)$  and let  $\gamma_f$  be the face of  $\Gamma_\lambda$  corresponding to  $f$ . Note that  $\dim f$  is the number of bounded regions in  $\gamma_f$  and we denote by  $(a_1, b_1), \dots, (a_k, b_k)$  the upper rightmost vertices of the bounded regions of  $\gamma_f$  respecting the ordering given in (5.3).

We denote by  $S^1(a_i, b_i)$  the circle group generated by  $\Psi_\lambda^{a_i, b_i}$  for  $i = 1, \dots, k$ . Let  $\mathbf{u} = (u_{i,j}) \in \mathring{f}$ . For the sake of convenience, we denote  $\Phi_\lambda^{-1}(\mathbf{u})$  by  $L_{\mathbf{u}}$ . Fix  $i \in [k]$ . By Lemma 5.5 and Lemma 5.14, there exists a free  $S^1(a_i, b_i)$ -orbit  $\sigma_i$  such that  $\{\sigma_1, \dots, \sigma_k\}$  forms a basis of  $\pi_1(L_{\mathbf{u}}) \cong \mathbb{Z}^k$ .

To construct a disc bounded by  $\sigma_i$ , we first choose an  $\omega_\lambda$ -compatible  $S^1(a_i, b_i)$ -invariant almost complex structure  $J$  on  $\mathcal{U}_{\lambda, -}^{a_i, b_i}$ . Let  $c_i := \Psi_\lambda^{a_i, b_i}(Z_\lambda^{a_i, b_i})$  be the maximum of  $\Psi_\lambda^{a_i, b_i}$ . With respect to the metric  $\omega_\lambda(J, \cdot)$ , the gradient flow of  $\Psi_\lambda^{a_i, b_i}$  is defined on  $\mathcal{U}_{\lambda, -}^{a_i, b_i}$ . Let  $W_i^u$  be the set of unstable points converging to  $Z_\lambda^{a_i, b_i}$ .

Second, consider the reduced space  $(R_{\mathbf{u}(i)}, \omega_{\mathbf{u}(i)})$  at level  $\mathbf{u}(i)$  where

$$R_{\mathbf{u}(i)} := \left( \Psi_\lambda^{a_i, b_i} \right)^{-1}(\mathbf{u}(i)) / S^1(a_i, b_i), \quad \mathbf{u}(i) := \Psi_\lambda^{a_i, b_i}(\mathbf{u}) = u_{1, a_i + b_i - 1} + \dots + u_{a_i, b_i}$$

equipped with the induced symplectic form which we denote by  $\omega_{\mathbf{u}(i)}$ . Since  $\sigma_i$  is a free orbit,  $[\sigma_i]$  is a smooth point in  $R_{\mathbf{u}(i)}$  so that there exists a smooth Darboux neighborhood  $U \subset R_{\mathbf{u}(i)}$  of  $[\sigma_i]$ . Because  $W_i^u$  is open dense in  $\mathcal{U}_{\lambda, -}^{a_i, b_i}$ ,  $U \cap W_i^u \neq \emptyset$  and hence we may choose an element  $[\sigma'_i] \in U \cap W_i^u$  sufficiently close to  $[\sigma_i]$  such that

- there is a smooth path  $\gamma$  in  $U$  from  $[\sigma_i]$  to  $[\sigma'_i]$ , and

- there is a symplectic isotopy  $\{\phi_t\}_{0 \leq t \leq 1}$  on  $(R_{\mathbf{u}(i)}, \omega_{\mathbf{u}(i)})$  such that the support is in  $U$ ,  $\phi_0 = \text{id}$ , and  $\phi_1$  sends  $[\sigma_i]$  to  $[\sigma'_i]$

If we denote by  $\bar{L}_{\mathbf{u}}$  the quotient image of  $L_{\mathbf{u}}$  in  $R_{\mathbf{u}(i)}$ , then  $\phi_1$  maps  $\bar{L}_{\mathbf{u}}$  to some Lagrangian submanifold  $\bar{L}'_{\mathbf{u}}$  containing  $[\sigma'_i]$ . Moreover,  $\{\phi_t\}_{0 \leq t \leq 1}$  lifts to a Lagrangian isotopy from a neighborhood of  $\sigma_i$  in  $L_{\mathbf{u}}$  to a neighborhood of  $\sigma'_i$  in  $L'_{\mathbf{u}}$ , the preimage of  $\bar{L}'_{\mathbf{u}}$  under the quotient map.

Finally, consider a disc  $u_{\sigma_i} : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (\mathcal{O}_{\lambda}, L_{\mathbf{u}})$  obtained by gluing

- the gradient  $J$ -holomorphic disc  $u_{\sigma'_i}^J : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (\mathcal{O}_{\lambda}, L'_{\mathbf{u}})$  and
- the cylinder  $q^{-1}(\gamma)$  where  $q : \left(\Phi_{\lambda}^{a_i, b_i}\right)^{-1}(\mathbf{u}(i)) \rightarrow R(\mathbf{u}(i))$  is the quotient map.

Because of our choice of  $\lambda$  in (5.1),  $c_1(\mathcal{O}_{\lambda}) = [\omega_{\lambda}]$ . It remains to show that  $\mu([u_{\sigma_i}]) = 2 \int_{\mathbb{D}} (u_{\sigma_i})^* \omega_{\lambda}$  for every  $i = 1, \dots, k$ . We compute the symplectic area and the Maslov index of  $u_{\sigma_i}$ .

- (Symplectic area)

$$\int_{\mathbb{D}} (u_{\sigma_i})^* \omega_{\lambda} = \int_{\mathbb{D}} (u_{\sigma'_i})^* \omega_{\lambda} = c_i - \mathbf{u}(i) = c_i - \Psi_{\lambda}^{a_i, b_i}(\mathbf{u})$$

since the symplectic area of the cylinder  $q^{-1}(\gamma)$  is zero.

- (Maslov index) By Lemma 3.1, Corollary 3.6, and Lemma 5.6,

$$\mu([u_{\sigma_i}]) = \mu([u_{\sigma'_i}]) = 2 \text{codim } F_i = 2 \left( c_i - \Psi_{\lambda}^{a_i, b_i}(\mathbf{u}_{\Delta_{\lambda}}) \right).$$

Observe that the ladder diagram  $\Gamma_{\lambda}$  is cut into several regions  $\{\mathcal{R}_1, \dots, \mathcal{R}_{\kappa}\}$  along  $\gamma_f$ . Because of our choice of  $(a_i, b_i)$ , for each  $j$  ( $1 \leq j \leq \kappa$ ),  $\{\square^{(s,t)} \mid 1 \leq s \leq a_i, a_i + b_i = s + t\} \cap \mathcal{R}_j$  is empty or coincides with  $\{\square^{(s,t)} \mid a_i + b_i = s + t\} \cap \mathcal{R}_j$ . Using the symmetricity of the divided regions (Remark 4.14), by the induction on  $i$  ( $1 \leq i \leq k$ ), one sees that a point  $\mathbf{u} \in \mathring{f}$  is the center of  $f$  if and only if

$$\Psi_{\lambda}^{a_i, b_i}(\mathbf{u}_{\Delta_{\lambda}}) = \Psi_{\lambda}^{a_i, b_i}(\mathbf{u})$$

for  $i = 1, \dots, k$ . In this case, we have

$$\mu([u_{\sigma_i}]) = 2 \left( c_i - \Psi_{\lambda}^{a_i, b_i}(\mathbf{u}_{\Delta_{\lambda}}) \right) = 2 \left( c_i - \Psi_{\lambda}^{a_i, b_i}(\mathbf{u}) \right) = \int_{\mathbb{D}} (u_{\sigma_i})^* \omega_{\lambda}$$

for every  $i = 1, \dots, k$ . It completes the proof.  $\square$

## REFERENCES

- [AH] Kazushi Ahara, Akio Hattori, *4-dimensional symplectic  $S^1$ -manifolds admitting moment map*. J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **38** (1991), 251–298. [1](#), [2](#), [4](#), [7](#)
- [ACK] Byung Hee An, Yunhyung Cho, Jang Soo Kim, *On the  $f$ -vectors of Gelfand-Cetlin polytopes*. European J. Combin. **67** (2018), 61–77. [14](#)
- [At] Michael Atiyah, *Convexity and commuting Hamiltonians*. Bull. London Math. Soc. **14** (1982), 1–15. [20](#)
- [Aud] Michéle Audin, *Topology of Torus actions on symplectic manifolds Second revised edition*. Progress in Mathematics **93**, Birkhäuser Verlag, Basel (2004). [2](#), [3](#), [4](#), [10](#), [19](#)
- [Aur] Denis Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*. J. Gökova Geom. Topol. **1** (2007), 51–91. [1](#), [8](#)
- [Cho] Cheol-Hyun Cho, *Holomorphic discs, spin structures and the Floer cohomology of the Clifford torus*. Int. Math. Res. Not. **35** (2004), 1803–1843. [1](#), [8](#)
- [CO] Cheol-Hyun Cho, Yong-Geun Oh *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*. Asian J. Math. **10** (2006), no. 4, 773–814. [1](#), [8](#)
- [CKO] Yunhyung Cho, Yoosik Kim, Yong-Geun Oh, *Lagrangian fibers of Gelfand-Cetlin systems*. arXiv:1704.07213v2. [2](#), [9](#), [11](#), [13](#), [14](#), [15](#), [16](#), [23](#)
- [EL] Jonathan Evans, Yanki Lekili, *Generating the Fukaya categories of Hamiltonian  $G$ -manifolds*. arXiv:1507.05842. [2](#)
- [FOOO1] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Lagrangian intersection Floer theory: anomaly and obstruction, Part I & Part II*. AMS/IP Studies in Advanced Math. **46**, Amer. Math. Soc., Providence, RI; International Press, Somerville, MA, (2009). [1](#)
- [FOOO2] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Lagrangian Floer theory on compact toric manifolds. I*. Duke Math. J. **151** (2010), no. 1, 23–174. [1](#)
- [GC] Israel M. Gelfand, Michael L. Cetlin, *Finite-dimensional representations of the group of unimodular matrices*. (Russian) Doklady Akad. Nauk SSSR (N.S.) **71**, (1950). 825–828. [9](#)
- [Go] Leonor Godinho, *On certain symplectic circle actions*, J. Symplectic Geom. **3** (2005), no. 3, 357–383. [7](#)

- [GS1] Victor Guillemin, Shlomo Sternberg, *Convexity properties of the moment mapping*. Invent. Math. **67** (1982), 491–513. [20](#)
- [GS2] Victor Guillemin, Shlomo Sternberg, *The Gel'fand-Cetlin system and quantization of the complex flag manifolds*. J. Funct. Anal. **52** (1983), no. 1, 106–128. [2](#), [9](#), [11](#), [12](#)
- [Ka] Yael Karshon, *Periodic Hamiltonian flows on four-dimensional manifolds*. Mem. Amer. Math. Soc. **141** (1999), no. 672. [1](#), [2](#), [4](#)
- [Le] Eugene Lerman, *Symplectic Cuts*, Math. Res. Lett. **2** (1995), 247–258. [1](#), [7](#)
- [McS] Dusa McDuff, Dietmar Salamon, *Introduction to Symplectic Topology*. Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, (1995). [6](#)
- [McS2] Dusa McDuff, Dietmar Salamon, *J-holomorphic curves and quantum cohomology*. University Lecture Series **6**, American Mathematical Society, Providence, RI, (1994). [4](#)
- [NNU] Takeo Nishinou, Yuichi Nohara, Kazushi Ueda, *Toric degenerations of Gelfand-Cetlin systems and potential functions*. Adv. Math. **224** (2) (2010), 648–706. [10](#), [20](#)
- [NU2] Yuichi Nohara, Kazushi Ueda, *Floer cohomologies of non-torus fibers of the Gelfand-Cetlin system*. J. Symplectic. Geom. **14** (2016), no. 4, 1251–1293. [2](#)
- [Oh] Yong-Geun Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks I*. Comm. Pure Appl. Math. **46** (1993), no. 7, 949–993. [2](#), [10](#)
- [Ri] Konstanze Rietsch, *A mirror symmetric construction of  $qH *_{\mathbb{T}} (G/P)_{(q)}$* . Adv. Math. **217** (2008), no. 6, 2401–2442. [2](#)

DEPARTMENT OF MATHEMATICS EDUCATION, SUNGKYUNKWAN UNIVERSITY, SEOUL, REPUBLIC OF KOREA.

*E-mail address*: yunhyung@skku.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, BOSTON UNIVERSITY, BOSTON, MA, USA

*E-mail address*: kimyoosik27@gmail.com, yoosik@bu.edu