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Bitcoin mining and electricity consumption

BITCOIN MINING AND ELECTRICITY CONSUMPTION

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We propose a dynamic industry equilibrium model for Bitcoin electricity consumption in a general framework, including Bitcoin miners' optimal entry and exit with technology innovation. By adopting average operating costs as an approximation to the true operating costs, we overcome the difficulty of strong path-dependency due to the interaction among entry, exit, and technology innovation. The model can capture both the upside and downside co-movements of miners' computing power, electricity consumption, and mining revenue. Our model shows that the Bitcoin electricity consumption will not grow indefinitely, with the ratio of Bitcoin electricity consumption to the miners' revenue fluctuating within a range.

KEYWORDS: Industry Equilibrium, Singular Control, Computing Power.

1. INTRODUCTION

Unlike centralized payment systems such as PayPal and Alipay, Bitcoin is the first decentralized payment system with all transactions recorded in a public blockchain, solving a long-standing open problem in computer science – how to prevent double-spending in decentralized systems. Bitcoin miners compete to validate users' transactions via a proof-of-work mechanism. Thanks to their efforts, miners are compensated with a predetermined number of newly

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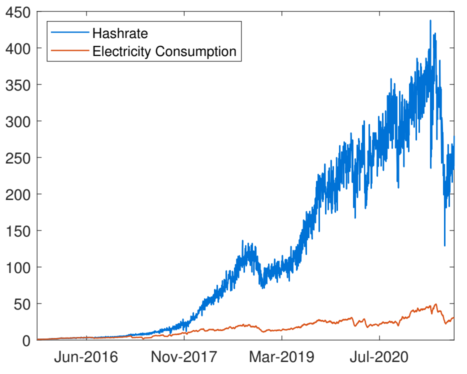
minted Bitcoins plus transaction fees whenever the verified users' transactions are successfully added to the blockchain.

One of the main concerns about Bitcoin mining is its electricity consumption. Rauchs et al. (2021) report that nowadays, the electricity consumed annually by Bitcoin mining has amounted to 0.53% of annual global electricity consumption, even higher than that of medium-sized countries like Sweden, Ukraine, and Argentina. Mora et al. (2018) even argue that Bitcoin alone could produce CO_2 emissions that accelerate global warming by increasing temperature over $2°$C within a few decades. The trend of electricity consumption is even more worrisome. As we have witnessed a tremendous increase in Bitcoin price and mining revenue in terms of the US dollar in the past decade, substantial new miners are lured into joining the mining business. Consequently, both the electricity consumption and the Bitcoin computing power measured by hashrate have grown up rapidly.

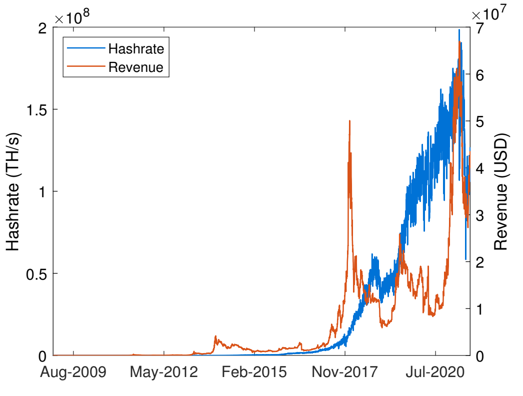
Figure 1(a) depicts both the estimated hashrate and electricity consumption in the Bitcoin network since 2016. Figure 1(b) displays the historical daily mining revenue and hashrate since the inception of Bitcoin. It can be seen that: (i) The hashrate has soared since 2015, though significant dips occurred in 2018 and 2021. (ii) The electricity consumption may fall as well, and its overall growth rate is much slower than that of the computing power, mainly due to technology innovation that results in less electricity consumption per unit of computing power.

The research questions in this paper are: (1) Will the electricity consumption by Bitcoin mining grow indefinitely? This question is important in view of public concerns about Bitcoin's carbon footprint and long-run sustainability. (2) To answer the first question using an economic model, we need to answer the second question: Can we build a dynamic equilibrium model to capture the co-movements of miners' computing power, electricity consumption, and mining revenue by incorporating entry, exit, and technology innovation simultaneously?

Motivation and Our Contribution. There is relatively little literature on the economic modeling of computing power in the Bitcoin network. An exception is Prat and Walter (2021), in which a novel industry equilibrium model is developed by first introducing technology innovation to the model in Caballero and Pindyck (1996) and then calibrating the model to capture the evolution of miners' computing power, as measured by the hashrate. However, two main issues remain:



(a)



(b)

FIGURE 1.— Miners' mining revenue, computing power, and electricity consumption. We retrieve daily mining revenue and the estimated hashrate (computing power) from blockchain.com. The estimated electricity consumption is retrieved from cbeci.org and is available from July 1, 2014. The hashrate and electricity consumption data in panel (a) are normalized such that both of the values on July 1, 2014, are one.

(1) They focus on examining the entry rule of the miners, and the hashrate in their primary model cannot decrease over time.[1] However, there are significant dips in the hashrate after 2018, especially in late 2018 and in 2021, as can be seen from Figures 1(a) and 1(b), motivating us to consider a general model with the exit option.

[1]In Appendix M of their online supplement, they extend the model by including the exit option of mining machines, but under a strong assumption (Assumption 6) that miners ignore machines' historical vintage distribution. This assumption seems reasonable only when there is no significant dip in the hashrate (e.g., before 2018).

(2) How to link hashrate to electricity consumption in a model, as different mining machines have different electricity consumption rates due to technology innovation? Prat and Walter (2021) do not focus on electricity consumption.[2] For an entry-only model, the resulting electricity consumption must be non-decreasing, and the ratio of electricity consumption to revenue may grow to infinity. The intuition is that an entry-only model tends to overestimate future electricity consumption, mainly due to the omission of the exit option.

For the first issue, it is natural to think about a possible extension of their model to a general industry equilibrium with entry, exit, and technology innovation. However, this is generally a difficult problem because the interaction among entry, exit, and technology innovation leads to strong path-dependency, and one needs to keep track of mining machines' vintage distribution due to technology innovation.[3]

To the best of our knowledge, this paper is the first that proposes an industry equilibrium model with entry, exit, and technology innovation simultaneously by tracking the path-dependency. We circumvent the difficulty of path-dependency by introducing average operating costs as an approximation. The idea stems from the literature in portfolio selection with capital gains taxes (cf. Dammon et al. (2001) and Ben Tahar et al. (2010)), where the average tax basis is used to keep track of asset purchasing prices.

For the second issue, introducing average operating costs also helps us keep track of the total operating costs rate (in addition to the hashrate), which is directly linked to the electricity consumption under the assumption of constant electricity price .

In summary, the contribution of this paper is threefold. First, from the real investment viewpoint, we propose a general framework with entry, exit, and technology innovation to study industry equilibrium. Second, from the application viewpoint, we use the model to analyze the upside and downside co-movements of Bitcoin computing power, electricity consumption, and mining revenue. In particular, our model can capture both the dips of hashrate and impetuous entries of miners. Third, calibrating the model's parameters enables us to forecast future electricity consumption and hashrate in terms of miners' revenue. Our model implies that the

[2]They present Equation (14) for electricity consumption, showing that the expectation of Bitcoin mining electricity consumption is proportional to the mining revenue. In contrast, we will give a detailed study of electricity consumption by linking it to miners' optimal strategy. In our model, the electricity consumption corresponds to miners' running costs and is no longer a linear function of the mining revenue due to the path-dependency of machines' purchasing costs.

[3]This is not an issue for an entry-only model with technology innovation.

Bitcoin electricity consumption will not grow indefinitely, with the ratio of Bitcoin electricity consumption to the miners' revenue fluctuating within a range.

Literature Review. This paper is related to four strands of literature. First, industry equilibrium has been widely studied in the field of real investment. Debreu (1954) shows the equivalence between the industry equilibrium and the social planner's problem in a static setting. Lucas and Prescott (1971) and Jovanovic (1982) consider similar problems by introducing uncertainty into the demand or supply function. Leahy (1993) develops a competitive industry equilibrium with entry and exit options and shows that a firm's myopic strategy is also optimal under the equilibrium setting. Dixit and Pindyck (1994) present a systematic analysis of the industry equilibrium under uncertainty by viewing investment opportunities as options. Caballero and Pindyck (1996) study the effects of different sources of uncertainty with entry options only, while Prat and Walter (2021) further extend the model by adding technology innovation to study the computing power of the Bitcoin mining system. Kogan (2001) develops a competitive equilibrium model of a two-sector production economy with irreversible investment and examines the effects of irreversibility on investment; based on this work, Kogan (2004) further studies the stock return behavior of competitive firms. Federico and Pham (2014) solve the industrial investment problem from a social planner's perspective by optimally controlling production capacity and deriving the optimal investment boundaries. However, except Prat and Walter (2021), none of the above works consider the possibility of technology innovation. We extend this strand of literature by proposing a general framework that includes technology innovation, entry, and exit simultaneously.

Second, the paper is related to the literature of portfolio selection with capital gains taxes, which dates back at least to Constantinides (1983, 1984). Dammon et al. (2001) introduce the average tax basis as an approximation into a multi-period model to handle the strong path-dependency incurred by tax basis. Ben Tahar et al. (2010), Dai et al. (2015), and Cai et al. (2018) extend the idea to formulate continuous-time models with the average tax basis by using the singular control technique (see, e.g., Constantinides (1986), Liu and Loewenstein (2002), and Fleming and Soner (2006)). We borrow the idea to propose the concept of average operating costs, which enables us to develop a tractable industry equilibrium model with technology innovation, entry, and exit.

Third, this paper relies on the previous empirical study of the electricity consumption related to Bitcoin mining. It is not easy to estimate the electricity consumption, as one needs to collect the data related to mining machine distribution worldwide and corresponding machine efficiency. Early attempts are made by O'Dwyer and Malone (2014), who conclude that the total electricity consumption of the Bitcoin network as of March 2014 is comparable to Irish national energy consumption, and by Hayes (2017), who uses a top-down approach by ignoring sunk costs in purchasing new mining machines and assuming that the marginal revenue rate equals the marginal electricity costs rate. The first Bitcoin electricity consumption index, Bitcoin Energy Consumption Index (BECI), proposed by De Vries (2018), is updated daily on Digiconomist.net. Although BECI is frequently cited both in industry and academia, Bevand (2017) points out limitations of the index and suggests an alternative bottom-up approach, which is further extended and implemented as the Cambridge Bitcoin Electricity Consumption Index (CBECI) (see Rauchs et al. (2021)). We use CBECI as the benchmark when examining the electricity consumption implied by our model.

Finally, besides Prat and Walter (2021) on Bitcoin's computing power, there is a growing body of literature on the economics of Bitcoin mining. Biais et al. (2019) analyze the robustness of the proof-of-work mining protocol. Easley et al. (2019) develop a Nash equilibrium model to investigate transaction fees' role and explain miners' and users' strategic behaviors. Huberman et al. (2021) use a congestion queuing game to examine how transaction fees and infrastructure levels are determined. Dai et al. (2021) study the co-movements of Bitcoin transaction fees, block sizes, Bitcoin prices, and endogenous inventory holdings. Cong et al. (2018) explore the role of mining pools in Bitcoin and find that mining pools do not necessarily undermine the decentralization of the Bitcoin network. Alsabah and Capponi (2019) study the arms race among miners for endogenous R&D investment and show that higher investments in research translate into a more aggressive mining game. Ma et al. (2018) model the mining protocol as a dynamic R&D racing game between miners and show that free entry is the cause of resource usage by Bitcoin mining. We complement this strand of literature by investigating the co-movements of computing power and electricity consumption in the Bitcoin mining market through a dynamic equilibrium model.

The rest of the paper is organized as follows. Section 2 presents the model formulation and shows that the social planner's optimal strategy is an industry equilibrium. We conduct a theoretical analysis on the social planner's problem in Section 3. In Section 4, we extend the model by incorporating impetuous miners. Section 5 is devoted to calibration for our model. We then examine the electricity consumption in Section 6. Section 7 concludes. Technique details are all relegated to Appendix.

2. THE MODEL

In this section, after presenting the model setup, we formulate the mining problem from a social planner's perspective and show that the social planner's optimal strategy is an industry equilibrium.

2.1. *Model Setup*

The miners' revenue comes from the block rewards and transaction fees associated with newly added blocks. The miners' costs consist of two parts. First, upon entry, new miners need to purchase mining machines, whose costs are referred to as the fixed cost afterward. Second, miners incur operating costs when operating machines. The following assumption about the miners' revenue and costs is taken from Prat and Walter (2021).

ASSUMPTION 1: *Let R_t be the miners' aggregate revenue rate in US dollar at time t, I_t the fixed cost for purchasing a unit of mining hardware at time t, and c_t the operating costs rate for operating a unit of hardware purchased at time t (i.e., with vintage t).*

(i) *The revenue rate R_t satisfies the following geometric Brownian motion,*

$$dR_t = R_t\left(\alpha dt + \sigma dW_t\right), \tag{1}$$

where constants $\alpha \in \mathbb{R}$ and $\sigma > 0$ are the annualized expected growth rate and volatility, respectively, of the revenue, and $\{W_t; t > 0\}$ is a standard Brownian Motion on a filtered probability space $(\Omega, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ with $W_0 = 0$.

(ii) *Due to technology innovation, I_t and c_t decline exponentially at a constant rate $a > 0$,*

$$I_t = I_0 e^{-at},\ c_t = c_0 e^{-at},\ \text{for } t \geq 0, \tag{2}$$

where the parameter a represents the speed of technology innovation.

One of the main contributions of this paper is that we add the miners' exit option to Prat and Walter (2021). More precisely, miners can permanently switch off part or all of their mining machines. However, it is difficult to track all existing operating costs, as technology innovation causes the operating costs to be heavily path-dependent. To have a tractable model, we borrow an idea from the tax literature by assuming that the same fraction of different machines are switched off upon exit, which allows us to model the average operating costs. Furthermore, as the mining machines have been devised specifically for Bitcoin mining and have no other use, it seems reasonable to assume that the mining machines become worthless after exiting from the mining market. Thus, we propose the following assumption:

ASSUMPTION 2: *Miners have an exit option. Moreover, upon exit,*

(i) *the same fraction of different machines are retired, and*

(ii) *the retired mining machines have no resale value.*

Although it helps us gain tractability, Assumption 2(i) may introduce some bias in terms of the optimal strategy for miners. Indeed, since newer machines consume less electricity, miners may prefer to retire more older machines instead of retiring the same fraction of different machines upon exit. To reduce the bias, later we shall introduce two features: (1) the impetuous entry in the enhanced model to allow more new machines and other types of entry in the system; and (2) the default intensity β in Assumption 3 to put more penalties on older machines and to compensate for default risk due to government policy changes. Our out-of-sample results suggest that the two features can be effective in the out-of-sample data fitting. Thus, Assumption 2 seems to strike a balance between tractability and reality.[4]

Let Q_t be the hashrate (i.e., the total number of tarahashes per second) at time t, which measures the Bitcoin network computing power. Without loss of generality, we assume that each unit of mining hardware produces exactly one tarahash per second. Since we have both entry and exit, we introduce two processes L_t and M_t and model the evolution process of the hashrate Q_t as follows:

$$dQ_t = dL_t - Q_t dM_t, \tag{3}$$

where $dL_t \geq 0$ is the amount of increase in hashrate, $dM_t \in [0, 1]$ is the proportion of computing units retired, and L_t and M_t are right-continuous with left limit, non-decreasing processes with $L_{0-} = M_{0-} = 0$. We call such a pair $(L, M) := (L_t, M_t)_{t \geq 0}$ an admissible strategy if L_t and M_t are time-adapted. We denote by $\mathcal{A}$ the set of all admissible strategies. The average mining revenue per hashrate at t is denoted by P_t, i.e., $P_t = R_t/Q_t$.

[4]In the tax literature, DeMiguel and Uppal (2005) and Haugh et al. (2016) compare the models with the average tax basis and with the exact tax basis and find that the error is small in terms of the certainty equivalent wealth loss. Dai et al. (2015) point out that using the average tax basis alone may not apply to the US market with asymmetric long-term and short-term capital gains tax rates. However, they show that adding another variable based on the average holding period of assets can significantly improve the model.

To study electricity consumption, we need to compute k_t, the total operating costs rate for all machines under mining at time t. Thanks to Assumption 2(i), the total operating costs rate declines with the same fraction dM_t as the hashrate Q_t. Thus, k_t evolves according to

$$dk_t = c_t dL_t - k_t dM_t, \tag{4}$$

where c_t is the operating costs rate for a unit of hardware with vintage t given in (2), and L_t and M_t are as given in (3). As the miners' costs decline in time, we always have $k_t \geq c_t Q_t$.

Grobys and Sapkota (2020) report that around 60% of cryptocurrencies launched before 2014 had defaulted by the end of 2018. This motivates us to make the following assumption.

ASSUMPTION 3: *The Bitcoin system may default, and the default time τ is the first jump time of a Poisson process with constant intensity* $\beta \geq 0$.

Assumption 3 can provide an additional benefit by giving more penalties to the older machines, helping to reduce the bias from Assumption 2. Through the paper, we shall impose Assumptions 1–3.

2.2. *Industry Equilibrium*

To establish an industry equilibrium, we need to figure out how miners respond to the competitive market by assessing their potential revenue and costs. We first investigate the behavior of an individual miner at t when all other miners follow an arbitrarily chosen admissible strategy (L, M) since t.

Consider a unit of hardware under mining at t. This unit of hardware can receive revenue until ν_t, where $\nu_t := \inf\{s \geq t : \Delta M_s = 1\}$ is the first arrival time all machines are retired since t. Assumption 2(i) indicates that only a proportion,

$$D_{t,s} := e^{-(M_s^c - M_t^c)} \prod_{t<u\leq s, \Delta M_u > 0} (1 - \Delta M_u),$$

of this unit remains running at $s \in [t, \tau \wedge \nu_t)$, where M_s^c and ΔM_s represent the continuous part and the jump part of the process M_s, respectively. Thus, the discounted expectation of total future revenue of a unit of hardware at t is

$$E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)} P_s D_{t,s} ds \middle| \mathcal{F}_t\right]. \tag{5}$$

The individual miner will buy a new unit of hardware if and only if the expected future revenue given by (5) can cover the fixed cost and the discounted expectation of cumulative operating costs, i.e.,

$$E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}P_sD_{t,s}ds\middle|\mathcal{F}_t\right] \geq I_t + c_tE\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}D_{t,s}ds\middle|\mathcal{F}_t\right]. \tag{6}$$

Conversely, the individual miner will retire a unit of hardware if the future revenue given by (5) is less than the operating costs to be waived, i.e.,

$$E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}P_sD_{t,s}ds\middle|\mathcal{F}_t\right] \leq \frac{k_t}{Q_t}E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}D_{t,s}ds\middle|\mathcal{F}_t\right]. \tag{7}$$

Notice that in the above inequality, the operating costs rate to be waived is k_t/Q_t rather than c_t due to Assumption 2(i).

Market free entry and exit impose that the inverse inequalities of inequalities (6) and (7) should hold for an equilibrium strategy (L, M), resulting in

$$E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}P_sD_{t,s}ds\middle|\mathcal{F}_t\right] \leq I_t + c_tE\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}D_{t,s}ds\middle|\mathcal{F}_t\right] \tag{8}$$

and

$$E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}P_sD_{t,s}ds\middle|\mathcal{F}_t\right] \geq \frac{k_t}{Q_t}E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}D_{t,s}ds\middle|\mathcal{F}_t\right]; \tag{9}$$

otherwise, the individual miner can benefit from deviating the mining strategy that all other miners follow, violating the concept of equilibrium. In summary, we have the following definition of industry equilibrium.

DEFINITION 2.1—Industry Equilibrium: *For a given initial state (R_0, Q_0, k_0), an industry equilibrium is an admissible strategy $(L_t, M_t)_{t\geq 0} \in \mathcal{A}$ such that*

(i) *inequalities (8) and (9) hold at any t;*

(ii) *inequality (8) (inequality (9)) holds with equality if L_t (M_t) increases strictly at t.*

Here $P_t = R_t/Q_t$, and $(R_t, Q_t, k_t)_{t\geq 0}$ in (8) and (9) represents a triple of processes governed by (1), (3), and (4) with strategy $(L_t, M_t)_{t\geq 0}$.

When $dM_t \equiv 0$ and $\beta = 0$, i.e., no exit option and no bankruptcy, Definition 2.1 reduces to the definition of industry equilibrium in Prat and Walter (2021). Earlier definitions of industry equilibrium without technology innovation were given in Lucas and Prescott (1971), Jovanovic (1982), and Leahy (1993).

2.3. *A Social Planner's Optimization Problem*

We consider a social planner, who wants to maximize the discounted expectation of societal surplus as given below:

$$\int_0^\tau e^{-rt}\Big(R_t \log Q_t dt - (I_t + C_t)\, dL_t + K_t dM_t\Big) + e^{-r\tau} K_\tau, \tag{10}$$

by choosing $(L_t, M_t)_{t\geq 0}$ over all admissible strategies, where r is the discount rate with $r+\beta > \alpha$,[5] I_t is the fixed cost given in (2), C_t is the cumulative discounted operating costs for a unit of hardware with vintage t since purchase, and K_t is the cumulative discounted total operating costs for all machines under mining. By Assumption 1, we have

$$C_t = \int_t^{+\infty} e^{-r(s-t)} c_t ds = \frac{c_t}{r},\ K_t = \int_t^{+\infty} e^{-r(s-t)} k_t ds = \frac{k_t}{r},\ \text{ for } t \geq 0. \tag{11}$$

The objective function (10) is similar to the formulation in Lucas and Prescott (1971) and Jovanovic (1982). In particular, the first term $R_t \log Q_t$ is related to the consumer surplus $\int_\delta^{Q_t} \frac{R_t}{q} dq = R_t \log Q_t - R_t \log \delta$ for any given small constant $\delta > 0$, where $R_t \log \delta$ is irrelevant to the mining strategy and thus can be dropped. The second term $(I_t + C_t)\, dL_t$ is the potential expenditure upon and after market entry, while the term $K_t dM_t$ and K_τ are the future cumulative operating costs foregone upon market exit and Bitcoin bankruptcy, respectively.

THEOREM 2.1: *The social planner's optimal strategy is an industry equilibrium.*

This theorem is a special case of Theorem A.1, whose proof is in Appendix B.3.

Thanks to Theorem 2.1, we will study the social planner's problem in what follows.

3. THEORETICAL ANALYSIS ON THE SOCIAL PLANNER'S PROBLEM

This section is devoted to theoretical analysis on the social planner's optimization problem. We first present analytical solutions of the optimal strategies in two special cases. For the general case in which an analytical solution to the social planner's problem is unavailable, we study the corresponding HJB equation and numerically find the optimal strategy. All technical details and proofs are relegated to Appendix.

[5] This is to ensure that the maximization problem is well-defined. Qin (2021) discusses how to remove the constraint.

3.1. *Two Special Cases*

THEOREM 3.1: *We have analytical solutions for the optimal strategies in the following two special cases.*

(i) *In the absence of fixed cost, i.e., $I_0 = 0$, the optimal strategy*[6] *is to keep upgrading the mining machines and maintain the optimal hashrate $Q_t^* = \frac{R_t}{c_t}$.*

(ii) *In the absence of technology innovation, i.e., $a = 0$, the optimal strategy is to enter (exit) the mining market if and only if the payoff P_t hits the upper (lower) barrier q (p), where $(p, q) \in (0, c_0) \times (c_0 + (r+\beta)I_0, +\infty)$ is the unique solution to*

$$\begin{cases} \left(\frac{1-\gamma_-}{r+\beta-\alpha}p + \frac{c_0+I_0}{r+\beta}\gamma_-\right)p^{-\gamma_+} = \left(\frac{1-\gamma_-}{r+\beta-\alpha}q + \frac{c_0+(r+\beta)I_0}{r+\beta}\gamma_-\right)q^{-\gamma_+} \\ \left(\frac{1-\gamma_+}{r+\beta-\alpha}p + \frac{c_0+I_0}{r+\beta}\gamma_+\right)p^{-\gamma_-} = \left(\frac{1-\gamma_+}{r+\beta-\alpha}q + \frac{c_0+(r+\beta)I_0}{r+\beta}\gamma_+\right)q^{-\gamma_-} \end{cases} \tag{12}$$

with $\gamma_\pm = \frac{-(\alpha-\frac{\sigma^2}{2})\pm\sqrt{(\alpha-\frac{\sigma^2}{2})^2+2\sigma^2(r+\beta)}}{\sigma^2}$.

Part (i) of Theorem 3.1 indicates that when the fixed cost is absent, miners can always use the latest mining machines, and the optimal hashrate is chosen such that the payoff P_t equals the marginal operating costs rate, i.e., $\frac{R_t}{Q_t} = c_t$. Part (ii) of Theorem 3.1 shows that in the absence of technology innovation, there are two threshold payoffs p and q such that it is optimal to keep the payoff process P_t within $[p, q]$.

3.2. *Value Functions and HJB Equations*

In the presence of fixed cost and technology innovation, analytical strategies are no longer available. We shall use the HJB equation approach.

Introduce

$$x_t := e^{at}R_t, \qquad z_t = e^{at}k_t$$

as the trended revenue rate and trended total operating costs rate, respectively. It is easy to verify that

$$dx_t = x_t\left((\alpha+a)dt + \sigma dW_t\right), \tag{13}$$

$$dz_t = az_t dt + c_0 dL_t - z_t dM_t. \tag{14}$$

[6]Rigorously speaking, the optimal strategy in this case has unbounded variation and is thus not admissible. However, the strategy can be approximated by a sequence of admissible strategies as closely as possible.

Thus, we can reformulate problem (10) and define a time-homogeneous social planner's value function as follows:

$$V(x_0, Q_0, z_0) = \sup_{(L,M)\in\mathcal{A}} E\left[\int_0^\tau e^{-(r+a)t}\left(x_t \log Q_t dt - \bar{c}_0 dL_t + \frac{z_t}{r} dM_t\right) + e^{-(r+a)\tau}\frac{z_\tau}{r}\right] \tag{15}$$

subject to (3), (13), and (14), for $(x_0, Q_0, z_0) \in \bar{\mathcal{D}}$, where $\bar{c}_0 = \frac{c_0}{r} + I_0$, and $\mathcal{D}$ is the solvency region defined as $\mathcal{D} = \{(x, Q, z) : x > 0, z > c_0 Q > 0\}$.

One may speculate that the value function satisfies the following HJB equation, which we will study rigorously later:

$$\min\left\{-\mathcal{L}V - \frac{\beta}{r}z - x\log Q, -V_Q - c_0 V_z + \bar{c}_0, QV_Q + zV_z - \frac{z}{r}\right\} = 0 \ \text{ in } \ \bar{\mathcal{D}}, \tag{16}$$

where $\mathcal{L}V = \frac{1}{2}\sigma^2 x^2 V_{xx} + (\alpha + a)xV_x + azV_z - (r + a + \beta)V$. The value function has a nice homogeneity property:

$$V(\lambda x, \lambda Q, \lambda z) = \lambda V(x, Q, z) + \frac{\lambda \log \lambda}{r + \beta - \alpha} x \tag{17}$$

for all $(x, Q, z) \in \bar{\mathcal{D}}$ and $\lambda > 0$. Thanks to the homogeneity, we can make the following change of variables:

$$g(b, y) = \frac{1}{Q}V(x, Q, z) - \frac{1}{r + \beta - \alpha}\frac{x}{Q}\log Q, \ \ b = \frac{z}{Q}, \ \ y = \frac{x}{Q},$$

which allows us to study the optimal mining strategy in the $b - y$ plane. Here b and y stand for the detrended average operating costs rate and the detrended payoff, respectively. The solvency region in $b - y$ plane is defined as $\mathcal{S} = \{(b, y) : b > c_0, \ y > 0\}$.

After the change of variables, we can verify that $g(b, y)$ should satisfy the following HJB equation:

$$\min\left\{-\bar{\mathcal{L}}g, \mathcal{I}g, \mathcal{O}g\right\} = 0, \ (b, y) \in \bar{\mathcal{S}}, \tag{18}$$

where

$$\begin{aligned}
\bar{\mathcal{L}}g &= \frac{1}{2}\sigma^2 y^2 g_{yy} + (\alpha + a)yg_y - (r + a + \beta)g + abg_b + \frac{\beta b}{r}, \\
\mathcal{I}g &= -\left(g - yg_y + \frac{y}{r + \beta - \alpha}\right) - (c_0 - b)\, g_b + \bar{c}_0, \\
\mathcal{O}g &= g - yg_y + \frac{y}{r + \beta - \alpha} - \frac{b}{r}.
\end{aligned}$$

3.3. *Optimal Mining Strategy*

From (18) we can define the exit region $\mathbf{ER}$, the entry region $\mathbf{IR}$, and the no-action region $\mathbf{NR}$ as

$$\begin{aligned} &\mathbf{ER} = \{(b,y) \in \bar{\mathcal{S}} : \mathcal{O}g(b,y) = 0\}, \quad \mathbf{IR} = \{(\mathbf{b},\mathbf{y}) \in \bar{\mathcal{S}} : \mathcal{I}\mathbf{g}(\mathbf{b},\mathbf{y}) = \mathbf{0}\}, \\ &\mathbf{NR} = \{(b,y) \in \bar{\mathcal{S}} : \mathcal{O}g(b,y) > 0 \ \& \ \mathcal{I}g(b,y) > 0\}, \end{aligned} \tag{19}$$

respectively. As the names suggest, it is optimal to exit the market, enter the market, and take no action in $\mathbf{ER}$, $\mathbf{IR}$, and $\mathbf{NR}$, respectively. We also define a sub-region of $\mathbf{ER}$, called the upgrading region $\mathbf{UR}$, as

$$\mathbf{UR} = \{(b,y) \in \mathbf{ER} : \mathcal{O}g(b,\bar{y}) = 0 \text{ for any } \bar{y} \geq y\}. \tag{20}$$

In $\mathbf{UR}$, it is optimal to liquidate all existing machines and re-enter the mining market. Once the (reduced) value function g is available, we can characterize the optimal strategy through (19)-(20).

It is worth pointing out that (18) has infinitely many constrained viscosity solutions.[7] However, the following can be proved using a similar approach developed by Bian et al. (2021): (i) The value function g is the minimum constrained viscosity solution of (18). (ii) The penalty method developed in Dai and Zhong (2010) converges to the value function. For details, see Qin (2021).

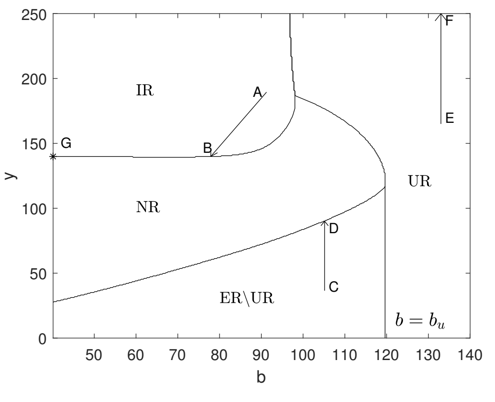
Applying the penalty method, we numerically obtain the value function g and the action and no-action regions as shown in Figure 2(a). (i) When the detrended average operating costs rate b is greater than a threshold value b_u, it is optimal to upgrade hardware by liquidating all the mining machines and re-entering the mining market so that the future operating costs can be effectively reduced. Indeed, upon re-entry, the current detrended average operating costs rate is adjusted to c_0, which suggests that the optimal strategy immediately after liquidation is to instantly enter the market with a certain number of mining machines such that the after-adjusted state in the $b-y$ plane is the point G in Figure 2(a). (ii) The social planner chooses an optimal exit and entry strategy when b is less than b_u: if the detrended payoff y is sufficiently low, it

[7]In fact, we can show that

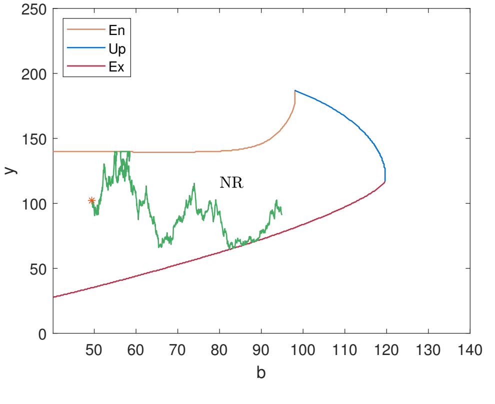
$$\hat{g}(b,y;M) := \left(\log \frac{y}{c_0} + \frac{\frac{1}{2}\sigma^2 + \alpha + a}{r+\beta-\alpha} - 1 + M \right) \frac{y}{r+\beta-\alpha} + \frac{b}{r}$$

is always a constrained viscosity solution of (18) if $M \geq 0$.

corresponds to a state (b, y) lying in $\mathbf{ER}\backslash\mathbf{UR}$, indicating the optimality of market exit (see the arrow from C to D in Figure 2(a)); if y is sufficiently high and b is close to b_u, upgrading mining machines is still optimal as the forgone operating costs can compensate for the fixed cost incurred; and if y is sufficiently high while b is relatively low compared to b_u, it is optimal to enter the market along the line[8] $\frac{y_t}{b_t - c_0} = \frac{y_{t-}}{b_{t-} - c_0}$ at t until the entry boundary is reached (see the arrow from A to B in Figure 2(a)).



(a)



(b)

FIGURE 2.— Optimal mining strategy. Panel (a) shows the action and no-action regions and optimal mining directions in action regions. Panel (b) highlights the upper and lower boundaries En, Up, and Ex of the no-action region and draws a typical path of the state variables which follow the optimal mining strategy. Default parameter values: $\alpha = 0.2$, $\sigma = 0.4$, $r = 0.35$, $\beta = 0.65$, $a = 0.3$, $c_0 = 40.15$, and $I_0 = 54.75$.

[8]The line is nothing but the characteristic line of the equation $\mathcal{I}g = 0$ in $\mathbf{IR}$.

The optimal strategy ensures that (b_t, y_t) must lie in $\overline{\mathbf{NR}}$ almost surely. $\overline{\mathbf{NR}}$ has two upper barriers (entry barrier En and upgrading barrier Up) and one lower barrier (exit barrier Ex), as shown in Figure 2(b), which also depicts a typical path of the state variables following the optimal mining strategy with the initial position marked by an asterisk.

4. AN ENHANCED MODEL WITH IMPETUOUS ENTRY

We further introduce impetuous entries as an extension of the model. Impetuous entry can enhance our model in two aspects. First, as mentioned before, additional entry can help to reduce the bias from Assumption 2(i). Second, there are indeed possibilities for exogenous entry. For example, people may use Bitcoin mining as a rational way to store extra electricity produced by solar, hydro, wind generators, which is difficult to store by other means; in addition, with the prosperity of blockchains in recent years, some miners may enter the mining market regardless of market fluctuations. More precisely, we assume that the entry rate of the impetuous entries is proportional to the current hashrate level, as in the following assumption.

ASSUMPTION 4: *Impetuous miners keep entering the market at a constant rate* $\rho \in [0, r + \beta + a)$ *proportional to the current hashrate level.*

The hashrate Q_t and the total operating costs rate k_t evolve according to

$$dQ_t = dL_t - Q_t dM_t + \rho Q_t dt, \tag{21}$$

$$dk_t = c_t dL_t - k_t dM_t + \rho c_t Q_t dt, \tag{22}$$

where the term $\rho Q_t dt$ models impetuous entries at a constant rate proportional to the current hashrate level, and $(L, M) \in \mathcal{A}$.

We can incorporate impetuous entries into the industry equilibrium problem and solve the social planner's optimization problem. The details are relegated to Appendix A.

5. CALIBRATION

In what follows, we calibrate our equilibrium model to historical hashrate data by using the data of revenue rate as input. Prat and Walter (2021) also use the revenue data as input. Later we shall use the calibrated parameters to study the electricity consumption.

5.1. *Data*

To calibrate the parameters in the model and make predictions on the hashrate, we need the following data: (1) the revenue rate R_t and (2) the network hashrate Q_t. The mining revenue consists of block rewards and transaction fees. The exchange rate of Bitcoin against USD, all block rewards, and all transaction fees in each block are directly observable from the Bitcoin blockchain. We use the data of Bitcoin revenue retrieved from blockchain.com as R_t. The estimated hashrate Q_t is also reported on the same web as well. To obtain more stable data, rather than directly use the daily hashrate/revenue, we take the average over the previous ten days of the corresponding data. We use the period 11/Sep/2018 - 10/May/2021, whose hashrate and revenue rate processes are shown in Figure 3.

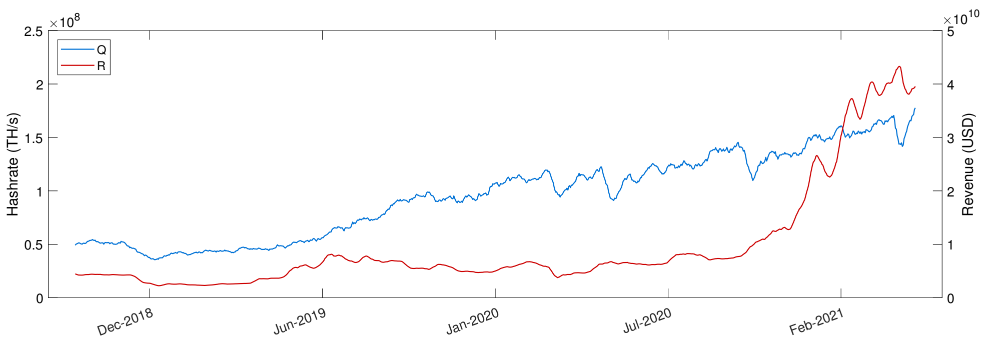


FIGURE 3.— Historical hashrate and revenue rate processes in the period of Sep/11/2018 - May/10/2021. Both data presented here are ten-day averages of the original data. During this period, several dips occurred in the hashrate process, such as in late 2018 and Mar/2020. Note that there is an upsurge in mining revenue in early 2021.

5.2. *Calibration Method*

Instead of calibrating the model using the entire period, we first conduct an in-sample calibration with 20 months of data and roll over the model every three months to update the model parameters. More specifically, we calibrate the model within the following five subperiods: Sep/11/2018–May/10/2020, Dec/11/2018–Aug/10/2020, Mar/11/2019–Nov/10/2020, Jun/11/2019–Feb/10/2021, and Sep/11/2019–May/10/2021.

For the drift term α and diffusion term σ in the revenue dynamic, we directly estimate them via the maximum likelihood approach and present them in Table I.

TABLE I

CALIBRATED PARAMETERS IN THE REVENUE DYNAMIC

Parameter	Interpretation	1^{st} subperiod Sep/11/2018 -May/10/2020	2^{nd} subperiod Dec/11/2018 -Aug/10/2020	3^{rd} subperiod Mar/11/2019 -Nov/10/2020	4^{th} subperiod Jun/11/2019 -Feb/10/2021	5^{th} subperiod Sep/11/2019 -May/10/2021
α	Trend R_t	0.24	0.74	0.88	0.98	1.08
σ	Volatility R_t	0.26	0.24	0.24	0.25	0.25

The most time-consuming part of the in-sample calibration is the simultaneous calibration of the operating costs rate c_0, the fixed cost I_0, default intensity β, discount rate r, technology improvement rate a, and impetuous entry rate ρ. Note that by making a further transformation, $\tilde{g}(b,y) = g(b,y) - \frac{b}{r}$, we can see that r and β appear only in the form of $r+\beta$ in the HJB equation related to $\tilde{g}$. Thus, we need to estimate c_0, I_0, $\Lambda := r+\beta$, a, and ρ simultaneously. The calibration algorithm is given below.

1. Given c_0, I_0, Λ, a, and ρ, solve the problem to obtain the optimal mining strategy.
2. Set the initial in-sample calibrated hashrate Q_0^{cal} as the historical value, i.e., Q_0. For a given initial detrended average operating costs rate b_0, we generate a path of hashrate Q_i^{cal} at time $i = 1, 2, ..., n$ using the strategy in Step 1.
3. Find the optimal b_0 to minimize the relative ℓ^2 distance between Q^{cal} generated in Step 2 and the historical hashrate Q. The optimal relative ℓ^2 distance is given by

$$\ell(c_0, I_0, \Lambda, a, \rho) := \min_{b_0} \sum_{i=0}^{n} \left(\frac{Q_i^{cal} - Q_i}{Q_i} \right)^2. \tag{23}$$

4. Repeat the procedure to find the optimal $(\hat{C}_0, \hat{I}_0, \hat{\Lambda}, \hat{a}, \hat{\rho})$ to minimize $\ell(c_0, I_0, \Lambda, a, \rho)$.

5.3. *In-Sample Calibration Results*

The calibrated parameters from Prat and Walter's baseline model and our models are reported in Table II. We also present the relative ℓ^2 distance between the in-sample calibrated hashrate and the historical counterpart of these three models in the table. One can observe from Table II that our enhanced model performs better than other models, with much less relative ℓ^2 distances.

To further evaluate different models, we shall investigate whether the calibrated parameters make sense. First, we shall look at a, which is the exponential decay rate of the fixed cost

TABLE II

CALIBRATED PARAMETERS[a]

Parameter	Interpretation	1^{st} subperiod			2^{nd} subperiod			3^{rd} subperiod		
	Calibration from Hashrate	P & W	Exit	E & I	P & W	Exit	E & I	P & W	Exit	E & I
$\bar{P}_0$	Entry Barrier	221.12			141.19			110.19		
a	Rate of Technology Innovation	0.92	0.71	0.47	0.61	0.60	0.33	0.40	0.40	0.25
I_0	Fixed Cost		46.21	49.17		42.34	46.36		37.67	42.01
c_0	Operating Costs Rate		41.68	36.24		39.30	31.75		34.09	30.55
Λ	$r+\beta$		1.91	1.44		1.22	1.58		1.30	1.75
ρ	Impetuous Entry rate			0.61			0.77			0.81
b_0	Average Operating Costs Rate		62.88	47.56		54.03	47.03		37.70	40.98
ℓ	Relative ℓ^2 distance	10.52	9.97	3.51	13.94	12.92	2.52	16.74	16.62	3.48

Parameter	Interpretation	4^{th} subperiod			5^{th} subperiod		
	Calibration from Hashrate	P & W	Exit	E & I	P & W	Exit	E & I
$\bar{P}_0$	Entry Barrier	98.06			233.30		
a	Rate of Technology Innovation	0	0	0.32	0	0	0.35
I_0	Fixed Cost		35.88	38.69		34.75	36.72
c_0	Operating Costs Rate		32.08	29.57		30.88	27.01
Λ	$r+\beta$		1.76	11.41		5.66	11.20
ρ	Impetuous Entry Rate			0.87			0.54
b_0	Average Operating Costs Rate		32.08	37.14		30.88	35.56
ℓ	Relative ℓ^2 distance	52.74	52.74	12.34	33.62	33.62	3.22

[a]This table shows the parameters calibrated from Prat and Walter's baseline model and our models, and the corresponding relative ℓ^2 distance between the in-sample calibrated hashrate and the historical hashrate of these three models are also presented. Here 'P & W' represents Prat and Walter's baseline model, 'Exit' represents our model with the exit option $(\rho = 0)$, and 'E & I' represents our enhanced model with impetuous entries $(\rho \geq 0)$. In Prat and Walter's model, miners will enter the market when the current detrended payoff $e^{-at}P_t$ is higher than the barrier $\bar{P}_0$. Therefore, only two parameters a and $\bar{P}_0$ need to be calibrated.

(the hardware's original price and the delivery fees) due to technology innovation. To do so, we collect historical machine prices on Bitmain, the world's largest designer of ASIC chips for Bitcoin mining by 2018, to get a rough estimate of the decay rate using regression. Based on this data set of machine prices, during 11/Sep/2018-10/May/2021 the estimator is $\bar{a} = 0.26$ with a 95% confidence interval of $[0.10, 0.41]$.[9] Compared to Prat and Walter's baseline model, the parameter a calibrated from our enhanced model is much closer to $\bar{a}$.

[9]We run a linear regression to fit $\{\log p_{t_j}\}_{j=1,2,\cdots,N}$ to $\{t_j\}_{j=1,2,\cdots,N}$, where p_{t_j} is the historical unit price of mining hardware at time t_j, and N is the size of historical price data collected.

Second, we check the monotonicity of the calibrated parameters. In Prat and Walter's baseline model, the entry barrier of the payoff should be exponentially decreasing; however, the calibrated entry barrier $\bar{P}_0$ of the fifth subperiod is even higher than that of the first subperiod, resulting in model inconsistency. As a comparison, the calibrated fixed cost I_0 and the operating costs rate c_0 from our models all decrease over time.

5.4. *Out-of-Sample Prediction of Hashrate*

To make an out-of-sample prediction of hashrate for all five subperiods, we use the calibrated parameters of each subperiod as given in Table II and the observed revenue process R_t, to predict the hashrate level for the next three months. This out-of-sample prediction procedure gives us a total of 15 months of out-of-sample prediction outcome, which is drawn in Figure 4.

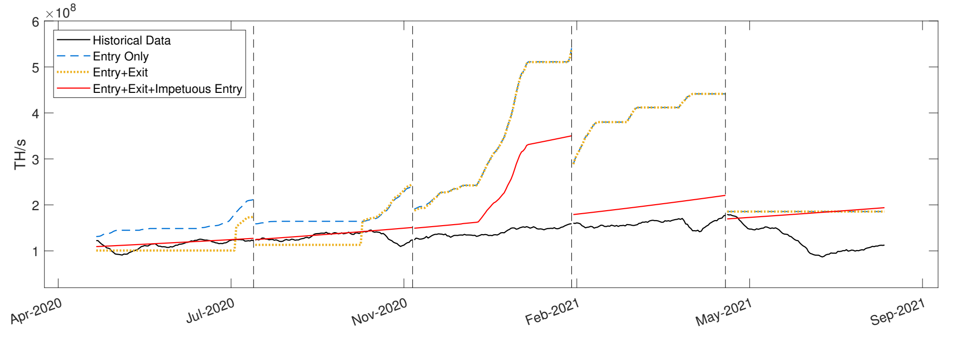


FIGURE 4.—Hashrate's out-of-sample prediction. We use the calibrated parameters in Table II to make a 3-month prediction of the hashrate for each subperiods, yielding a total of 15 months of predictions, as shown in the figure. The black solid line represents the historical hashrate data; the blue dashed line, the yellow dotted line, and the red solid line are the predicted hashrate processes from Prat and Walter's baseline model (entry only), the model with the exit option (entry+exit), and our enhanced model (entry+exit+impetuous entry), respectively.

The model with the exit option seems to perform better at the beginning than Prat and Walter's model and to be almost identical to that model in the last nine months. However, the enhanced model with impetuous entry seems to outperform the other two models in all five out-of-sample periods. This out-of-sample performance result is also supported by Table III, which reports the corresponding relative ℓ^2 distance between the out-of-sample predicted hashrate from different models and the observed hashrate.

To further illustrate the out-of-sample performance, Figure 5 depicts the historical hashrate data as well as the in-sample calibrated hashrate Q^{cal} of different models for five subperiods

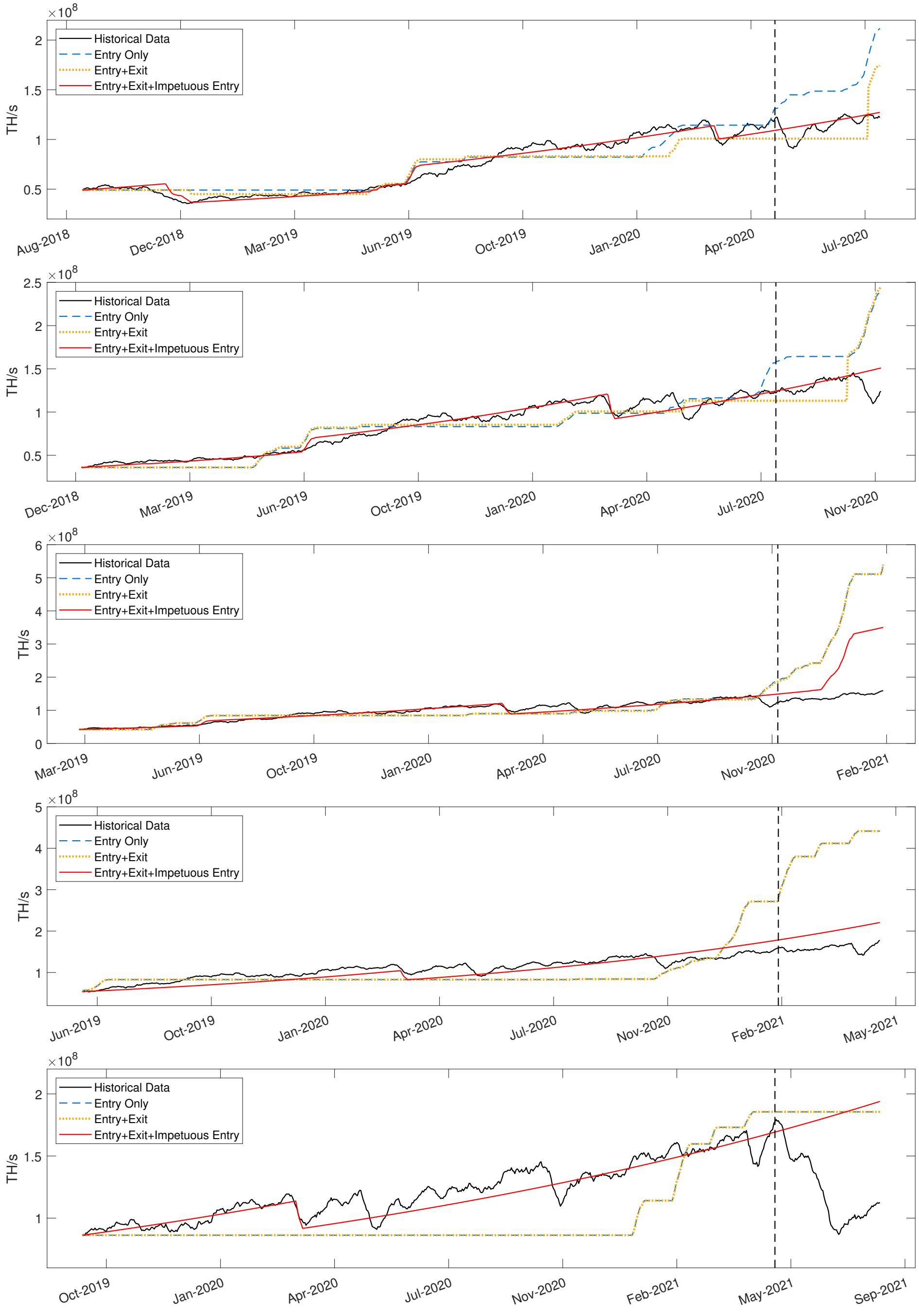


FIGURE 5.—Historical hashrate Q_t and the in-sample calibrated/out-of-sample predicted hashrate Q_t^{cal}. This figure depicts the historical hashrate data as well as the in-sample calibrated hashrate Q^{cal} of different models for five subperiods together with their 3-months out-of-sample prediction. Vertical dashed lines identify the end of in-sample periods. The black solid lines represent the historical hashrate data; the blue dashed lines, the yellow dotted lines, and the red solid lines are the in-sample calibrated/out-of-sample predicted hashrate processes from Prat and Walter's baseline model (entry only), the model with the exit option (entry+exit), and our enhanced model (entry+exit+impetuous entry), respectively.

TABLE III

OUT-OF-SAMPLE RELATIVE ℓ^2 DISTANCE OF HASHRATE IN DIFFERENT MODELS[a]

1^{st} subperiod			2^{nd} subperiod			3^{rd} subperiod			4^{th} subperiod			5^{th} subperiod		
P & W	Exit	E & I	P & W	Exit	E & I	P & W	Exit	E & I	P & W	Exit	E & I	P & W	Exit	E & I
14.76	2.66	0.66	17.04	15.01	1.19	241.98	240.78	57.22	214.90	214.90	6.56	40.17	40.17	40.03

[a]This table shows the relative ℓ^2 distances between the out-of-sample predicted hashrate and the historical data in three models, where "P & W", "Exit", and "E & I" correspond to Prat and Walter's baseline model (entry only), our model with the exit option (entry+exit), and our enhanced model (entry+exit+impetuous entry), respectively.

together with their 3-months out-of-sample predictions. Note that in the last three subperiods, the model with the exit option and Prat and Walter's model have nearly identical out-of-sample prediction paths. A different situation occurs in the first subperiod that the model with the exit option partially catches the hashrate dip in late 2018; even if only one dip in hashrate is partially captured, it allows the model to significantly improve the prediction power. This can also be seen from Table III that the relative ℓ^2 distance of the out-of-sample prediction data in the first subperiod is reduced from 14.76 to 2.66 by incorporating the exit option. When impetuous entries are further incorporated in the model, the prediction power of the enhanced model is much better than the other two models.

Although our enhanced model outperforms the other two models, the out-of-sample predicted hashrate from the enhanced model in the two subperiods, Jan/2021-Feb/2021 and May/2021-Aug/2021, are much higher than the historical data. There could be some historical events that contribute to this. First, in early 2021, the excessive growth of the Bitcoin price led to high demand for new mining hardware. Therefore, the short supply of mining hardware and the delay in machine deliveries can cause the hashrate to increase at a slower pace than expected for the period Jan/2021-Feb/2021. Second, for the period May/2021-Aug/2021, China, which owns an average monthly hashrate market share of 46.04% on 21/Apr/2021,[10] had begun an effective crackdown on Bitcoin mining and trading since May/2021. After only three months, China's hashrate market share became virtually zero. This unexpectedly implemented policy could be the main reason why our model does not predict the hashrate decline in the period of May/2021-Aug/2021. Furthermore, as can be seen from Table II, the summation of discount rate and default intensity $r + \beta$ calibrated from our models is large in the whole pe-

[10]The market share data can be found on Cambridge Centre for Alternative Finance website.

riod and are very large in the last two subperiods. This seems to suggest the significant default risk in Bitcoin mining. Although this default risk aggregates different sources of risk, such as systemic risk, hacker attacks, and government regulations, the sudden increase of default risk in the last two subperiods may be due to the potential risk from government regulation.

6. ELECTRICITY CONSUMPTION

In this section, we investigate the electricity consumption of the Bitcoin network. Denote by EC_t the annualized electricity consumption, measured by TWh. Note that we shall compute the implied electricity consumption from the calibrated hashrate, rather than doing the calibration of electricity consumption and hashrate simultaneously.

6.1. *Electricity Consumption Estimation*

The total operating costs $(k_t)_{t\geq 0}$ mainly consist of electricity and the overheads of mining farms, such as machine storage fees and cooling fees. As the electricity costs account for the vast majority of operating expenses, we assume that operating costs are fully spent on electricity. In addition, the electricity price p^e is assumed to be constant over time, i.e., $p^e = 0.05$ USD/kWh, consistent with Prat and Walter (2021) and Rauchs et al. (2021). Thus, we can replicate the electricity consumption path with $EC_t = k_t/p^e$ and make prediction on electricity consumption.

For the enhanced model and the model with the exit option, we can compute the implied k_t directly using the calibrated parameters[11] $(c_0, I_0, r+\beta, a, \rho)$ and b_0 obtained in the in-sample calibration procedureof hashrate. Indeed, the path of k_t in both in-sample and out-of-sample periods is given by the dynamic $dk_t = c_t dL_t - k_t dM_t + \rho c_t Q_t^{cal} dt$ with initial value $k_0 = b_0 Q_0$, where Q_t^{cal} is the in-sample calibrated (or out-of-sample predicted) hashrate, and (L, M) is the optimal mining strategy corresponding to the calibrated parameters and the observed revenue process R_t.

For the entry-only model, the in-sample calibration of hashrate only yields $(a, \bar{P}_0)$ as well as α and σ. Thus, a further calibration is needed to estimate electricity consumption. Indeed, let the total operating costs rate be $\bar{k}_t$. To get the implied path of $\bar{k}_t$, which has the dynamic $d\bar{k}_t = c_t dQ_t^{cal}$, we need the initial total operating costs rate $\bar{k}_0$ and the initial operating costs rate c_0. Note that c_0 is given by $c_0 = r\left(\frac{\bar{P}_0(\bar{\gamma}-1)}{(r-\alpha)\bar{\gamma}} - I_0\right)$, where $\bar{\gamma}$ is the positive solution to

[11] For the model with the exit option, we have $\rho \equiv 0$.

$\frac{1}{2}\sigma^2\bar{\gamma}^2 + \left(\alpha + a - \frac{1}{2}\sigma^2\right)\bar{\gamma} - (r + a) = 0$.[12] Since α, σ, $\bar{P}_0$, and a are already calibrated, the electricity consumption path is available once we have additional parameters: discount rate r, initial fixed cost I_0, and initial total operating costs rate $\bar{k}_0$. We calibrate r, I_0, and $\bar{k}_0$ using CBECI data in the in-sample periods to obtain implied electricity consumption paths and make future predictions in the out-of-sample periods.

Figure 6 depicts the Cambridge Bitcoin Electricity Consumption Index (CBECI) and the electricity consumption path implied by different models for five in-sample subperiods, along with the out-of-sample prediction for 3 months. It appears that the enhanced model has the best performance compared to the other two models, and can catch the significant drops of electricity consumption in the five subperiods. We also summarize the relative ℓ^2 distance between the implied electricity consumption and CBECI in Table IV, confirming that the enhanced model is significantly better compared to the other two models.

From Figure 6, the model with the exit option performs the worst, which may be due to the following reasons: (1) Adopting average operating costs without any correction on entry underestimates the portion of efficient new machines and thus overestimates electricity consumption. (2) While the second round calibration has to be made for the entry-only model to match the CBECI electricity data, we do not use the electricity data to calibrate our model with the exit option and our enhanced model. Calibrating our models with the electricity and hashrate data simultaneously would enhance our performance.

TABLE IV

RELATIVE ℓ^2 DISTANCE OF IMPLIED ELECTRICITY CONSUMPTION IN DIFFERENT MODELS[a]

	1^{st} subperiod			2^{nd} subperiod			3^{rd} subperiod			4^{th} subperiod			5^{th} subperiod		
	P & W	Exit	E & I	P & W	Exit	E & I	P & W	Exit	E & I	P & W	Exit	E & I	P & W	Exit	E & I
In Sample	15.87	82.60	5.55	12.49	22.10	7.41	14.75	22.72	13.38	30.59	31.69	20.60	12.92	30.19	16.96
Out of Sample	8.47	15.79	5.27	6.19	4.26	3.66	15.62	41.69	7.90	163.94	140.26	2.49	53.68	22.07	7.29

[a]This table shows the relative ℓ^2 distances between the implied electricity consumption and CBECI for both in-sample and out-of-sample periods under three models. "P & W", "Exit", and "E & I" correspond to Prat and Walter's baseline model (entry only), our model with the exit option (entry+exit), and our enhanced model (entry+exit+impetuous entry), respectively.

[12]This relationship among parameters follows from the optimal mining strategy of the entry-only model.

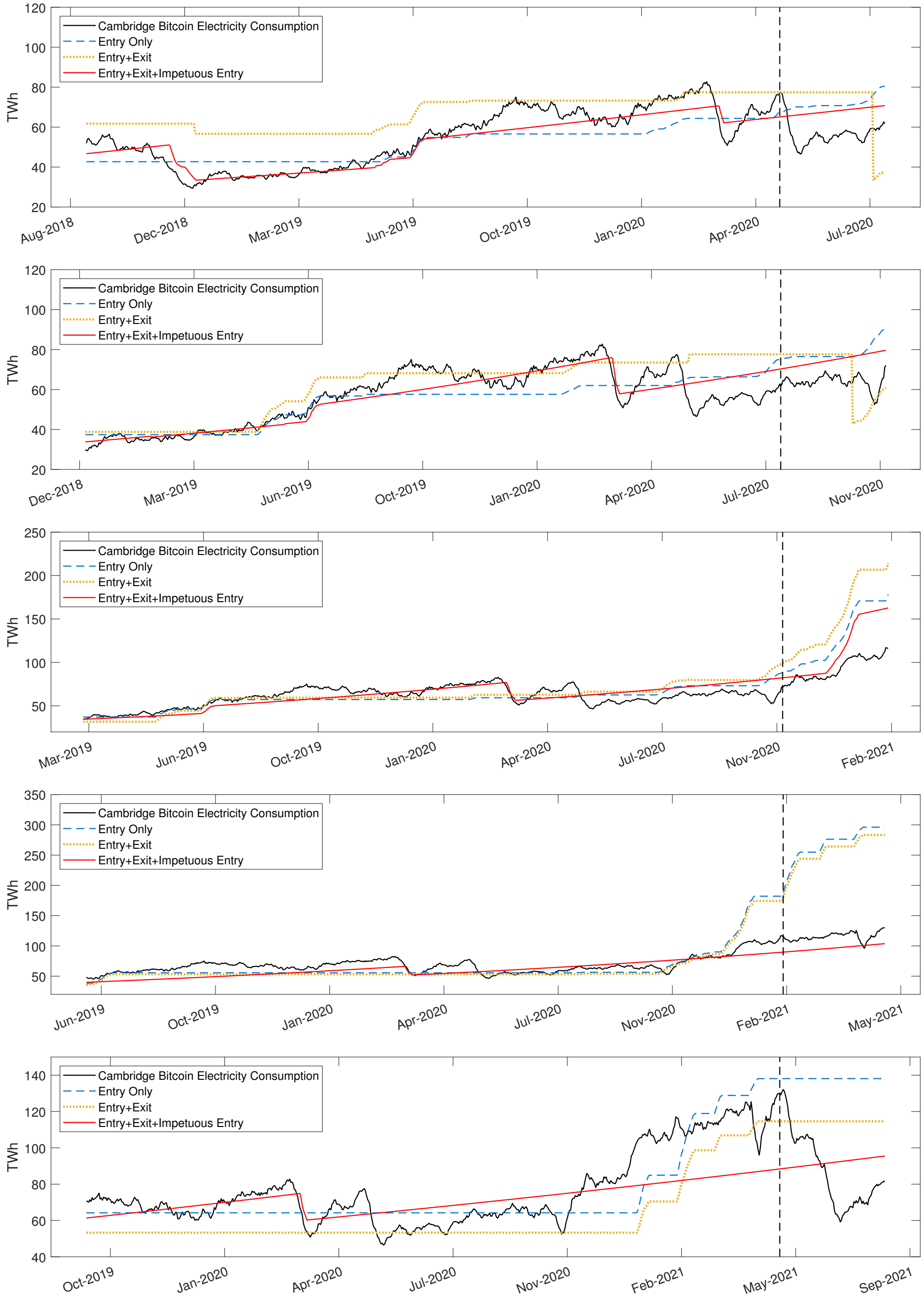


FIGURE 6.—Cambridge Bitcoin Electricity Consumption Index and the implied electricity consumption of different models. This figure shows the CBECI and the implied electricity consumption of three models for five in-sample subperiods together with their 3-months out-of-sample prediction. Vertical dashed lines identify the end of in-sample periods. The black solid lines represent the CBECI; the blue dashed lines, the yellow dotted lines, and the solid red lines are the implied electricity consumption processes of Prat and Walter's baseline model (entry only), our model with the exit option (entry+exit), and our enhanced model (entry+exit+impetuous entry), respectively.

6.2. *The Ratio of Electricity Consumption to Mining Revenue*

We shall show in this subsection that (1) for our enhanced model, the ratio of electricity consumption to mining revenue fluctuates between two finite constants independent of time; and (2) for the entry-only model, along a sample path, the ratio of electricity consumption to mining revenue might go to infinity.[13] The intuition behind this key difference is that, compared to the enhanced model, the entry-only model tends to overestimate future electricity consumption, mainly due to the omission of the exit option.

The ratio can shed some light on the future of electricity consumption in the Bitcoin mining market. Due to the uniform upper and lower bounds of the ratio, the magnitude of electricity consumption is directly linked to the Bitcoin mining revenue. Thus, the electricity consumed by Bitcoin mining cannot grow indefinitely, as it is bounded by the Bitcoin mining revenue.

Figure 7 draws the daily electricity costs as a percentage of miners' daily revenue in the first subperiod. Our enhanced model suggests that the ratio of electricity costs to mining revenue is among $33\% - 83\%$ in this specified subperiod.

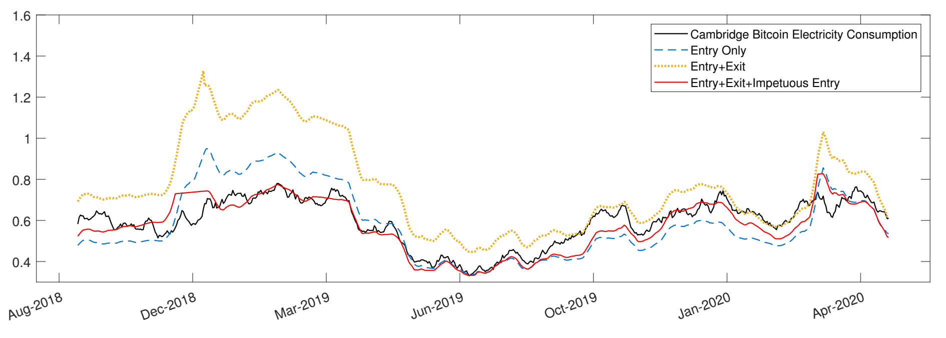


FIGURE 7.—Electricity costs as a percentage of miners' revenue. This figure shows the ratio of daily electricity costs to miners' daily mining revenue implied by different models. The black solid line represents the electricity costs-revenue ratio implied by CBECI; the blue dashed line, the yellow dotted line, and the red solid line are the counterparts derived from Prat and Walter's baseline model (entry only), our model with the exit option (entry+exit), and our enhanced model (entry+exit+impetuous entry), respectively.

This numerical result motivates us to derive theoretical bounds between electricity costs and mining revenue. Recall that the optimal mining strategy of our enhanced model is only

[13] Prat and Walter (2021) show that in the sense of expectation, the ratio of electricity consumption to mining revenue is a fixed constant in their entry-only model (see Equation (14) in their paper).

determined by the detrended average operating costs rate b_t and the detrended payoff y_t. There is a bounded no-action region $\mathbf{NR}$ in $b-y$ plane such that (b_t, y_t) lies in it all the time.[14] As a result, the ratio of the electricity costs to miners' revenue, $\frac{k_t}{R_t}$, which equals $\frac{b_t}{y_t}$, is bounded as well. Theoretically, we have Theorem 6.1, whose proof is given in Appendix C.

THEOREM 6.1: *Assume the following regularity conditions: The free boundaries En, Ex, and Up of the no-action region exist; there exist constants $b_u \geq b_m \geq c_0$ and two continuous functions $y^+(b) \geq y^-(b) \geq 0$ such that $En := \{(b, y^+(b)) : c_0 \leq b \leq b_m\}$, $Up := \{(b, y^+(b)) : b_m < b \leq b_u\}$, $Ex := \{(b, y^-(b)) : c_0 \leq b < b_u\}$; g, g_b, g_y and g_{yy} exist and are continuous in $\mathbf{NR} \cup (\mathbf{ER} \backslash \mathbf{UR})$. Then there are two positive constants A and B, independent of time t, such that*

$$A \leq \frac{k_t}{R_t} \leq B.$$

Based on Theorem 6.1 and the relationship $EC_t = k_t/p^e$, we conclude that the ratio of electricity consumption to miners' revenue fluctuates within a certain range under our enhanced model with exit.

In contrast, for the entry-only model the ratio of the electricity costs to the miners' revenue in the future can go to infinity along some sample paths. To show this, note that for the entry-only model the mining strategy is fully characterized by the detrended payoff $\bar{y}_t := \frac{e^{at} R_t}{Q_t}$. In the entry-only model, $\bar{y}_t$ evolves as a geometric Brownian motion with a reflective barrier $\bar{P}_0$. Let $\bar{b}_t := \frac{e^{at} \bar{k}_t}{Q_t}$ be the detrended average operating costs rate at t. The action boundary is a horizontal line in the $(\bar{b}, \bar{y})$ plane as shown in Figure 8, where $\mathbf{IR_0}$ and $\mathbf{NR_0}$ represent the entry region and the no-action region, respectively. Thus, for the entry-only model there is no upper bound on sample paths of $\frac{\bar{b}_t}{\bar{y}_t}$ or $\frac{\bar{k}_t}{R_t}$.

7. CONCLUSION

Now we are ready to answer the two questions raised in the beginning of the introduction section. (1) To answer the first question, we show that the ratio of the electricity costs to the miners' revenue is bounded uniformly, demonstrating how miners' electricity consumption is affected by the Bitcoin mining revenue. Thus, the electricity consumed by Bitcoin mining cannot grow indefinitely. (2) To answer the second question, we develop a dynamic equilibrium model for Bitcoin mining with entry and exit options, technology innovation, and impetuous entries. Our

[14]The boundedness of region $\mathbf{NR}$ can be theoretically proved under certain conditions (see Appendix C).

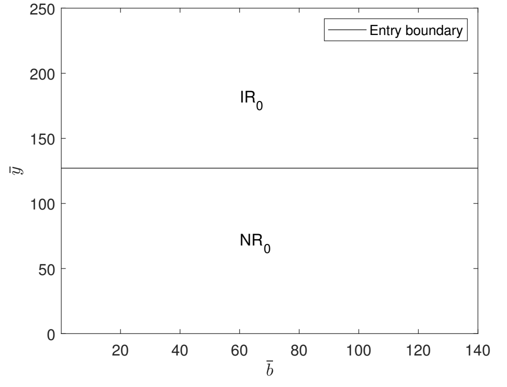


FIGURE 8.— Optimal mining strategy of the entry-only model. Default parameter values: $\alpha = 0.2$, $\sigma = 0.4$, $r = 0.35$, $a = 0.3$, $c_0 = 40.15$, and $I_0 = 54.75$.

model can be calibrated to explain the co-movements of Bitcoin's computing power, electricity consumption, and mining revenue. In particular, we can catch drops in computing power and electricity consumption.

We should point out the key idea here is to adopt the average operating costs to track the path-dependency, along with two corrections, impetuous entry and default risk. The approximation appears to work well. How to improve the approximation further, such as using neural network to track the path-dependency, remains an interesting research problem.

APPENDIX A: ENHANCED MODEL WITH IMPETUOUS ENTRY

In this section, we suppose Assumption 4 holds and list technical results related to the model with impetuous entry. All theoretical proofs are postponed to Appendix B.

A.1. *Industry Equilibrium*

We examine an individual miner's behavior when all other miners follow an arbitrarily chosen admissible strategy (L, M). Consider a unit of hardware under mining at t. The discounted future revenue of this unit of hardware at t is $NP_t := E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}P_sD_{t,s}ds\middle|\mathcal{F}_t\right]$. This unit of hardware results in some future impetuous entries, and thus we also take those impetuous entries into consideration. At $s \in [t, \tau \wedge \nu_t)$, the impetuous entry rate resulting from this unit of hardware is $\rho_{t,s}$ where $\rho_{t,s} := \rho e^{\rho(s-t)}D_{t,s}$. Thus, the total discounted future revenue due to the existence of a unit of hardware at t is $NP_t + E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)}\rho_{t,s} NP_s ds \middle| \mathcal{F}_t\right]$, i.e.,

$$E\left[\int_t^{\tau\wedge\nu_t} e^{(\rho-r)(s-t)} D_{t,s} P_s \, ds \middle| \mathcal{F}_t\right]. \tag{24}$$

The individual will set up a new unit of hardware if and only if the total discounted future revenue given by (24) can cover the total fixed cost and the discounted cumulative operating costs, including the part caused by impetuous entries. He will retire a unit of hardware if the future revenue is less than the fixed cost and operating costs to be waived. Thus, for an equilibrium strategy (L, M), we should have

$$\begin{aligned} E\left[\int_t^{\tau\wedge\nu_t} e^{(\rho-r)(s-t)} D_{t,s} P_s ds \middle| \mathcal{F}_t\right] &\le I_t + E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)} \rho_{t,s} I_s ds \middle| \mathcal{F}_t\right] \\ &+ c_t E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)} D_{t,s} ds \middle| \mathcal{F}_t\right] + E\left[\int_t^{\tau\wedge\nu_t} \rho_{t,s} c_s \int_s^{\tau\wedge\nu_t} e^{-r(u-t)} D_{s,u} du ds \middle| \mathcal{F}_t\right] \end{aligned} \tag{25}$$

and

$$\begin{aligned} E\left[\int_t^{\tau\wedge\nu_t} e^{(\rho-r)(s-t)} D_{t,s} P_s ds \middle| \mathcal{F}_t\right] &\ge E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)} \rho_{t,s} I_s ds \middle| \mathcal{F}_t\right] \\ &+ \frac{k_t}{Q_t} E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)} D_{t,s} ds \middle| \mathcal{F}_t\right] + E\left[\int_t^{\tau\wedge\nu_t} \rho_{t,s} c_s \int_s^{\tau\wedge\nu_t} e^{-r(u-t)} D_{s,u} du ds \middle| \mathcal{F}_t\right]. \end{aligned} \tag{26}$$

DEFINITION A.1—Industry Equilibrium with Impetuous Entry: *Suppose Assumption 4 holds. For a given initial state* (R_0, Q_0, k_0)*, an industry equilibrium with impetuous entry is an admissible strategy* $(L_t, M_t)_{t\ge 0} \in \mathcal{A}$*, such that*

(i) inequalities (25) *and* (26) *hold at any* t*;*

(ii) inequality (25) *(inequality* (26)*) holds with equality if* L_t *(*M_t*) increases strictly at* t*.*

Here $P_t = R_t/Q_t$*, and* $(R_t, Q_t, k_t)_{t\ge 0}$ *in* (25) *and* (26) *represents a triple of processes governed by* (1)*,* (21)*, and* (22) *with strategy* $(L_t, M_t)_{t\ge 0}$*.*

A.2. *The Social Planner's Problem*

The social planner's problem is to maximize the discounted expectation of societal surplus in the form of

$$\int_0^\tau e^{-rt}\Big(R_t \log Q_t dt - (I_t + C_t)(dL_t + \rho Q_t dt) + K_t dM_t\Big) + e^{-r\tau} K_\tau$$

subject to (1), (21), and (22) by choosing $(L_t, M_t)_{t\geq 0}$ over all admissible strategies, where I_t, C_t and K_t are given by (2) and (11).

THEOREM A.1: *The social planner's optimal strategy with impetuous entry is an industry equilibrium with impetuous entry.*

A.3. *Theoretical Analysis*

A.3.1. *Two Special Cases.*

We extend Theorem 3.1 to the case with impetuous entries.

THEOREM A.2: *Suppose Assumption 4 holds.*

(1) In the absence of fixed cost, i.e., $I_0 = 0$, the optimal mining strategy is to keep upgrading the mining machines and maintain the optimal hashrate $Q_t^* = \frac{R_t}{c_t}$.

(2) In the absence of technology innovation, i.e., $a = 0$, the optimal strategy is to enter (exit) the mining market if and only if the payoff P_t hits the upper (lower) barrier q (p), where $(p,q) \in (0, c_0 + \rho I_0) \times (c_0 + (r+\beta)I_0, +\infty)$ is the unique solution to

$$\begin{cases} \left(\dfrac{1-\gamma_-}{r+\beta-\alpha}p + \dfrac{c_0+\rho I_0}{r+\beta-\rho}\gamma_-\right)p^{-\gamma_+} = \left(\dfrac{1-\gamma_-}{r+\beta-\alpha}q + \dfrac{c_0+(r+\beta)I_0}{r+\beta-\rho}\gamma_-\right)q^{-\gamma_+} \\ \left(\dfrac{1-\gamma_+}{r+\beta-\alpha}p + \dfrac{c_0+\rho I_0}{r+\beta-\rho}\gamma_+\right)p^{-\gamma_-} = \left(\dfrac{1-\gamma_+}{r+\beta-\alpha}q + \dfrac{c_0+(r+\beta)I_0}{r+\beta-\rho}\gamma_+\right)q^{-\gamma_-} \end{cases} \tag{27}$$

with $\gamma_\pm = \frac{-(\alpha-\rho-\frac{\sigma^2}{2})\pm\sqrt{(\alpha-\rho-\frac{\sigma^2}{2})^2+2\sigma^2(r+\beta-\rho)}}{\sigma^2}$.

A.3.2. *Value Functions and HJB Equations*

Denote by $x_t := e^{at}R_t$ and $z_t := e^{at}k_t$ the trended revenue rate and trended total operating costs rate, respectively. Substitution of (2) into (22) leads to

$$dz_t = az_t dt + c_0(dL_t + \rho Q_t dt) - z_t dM_t. \tag{28}$$

Define the value function as

$$\begin{aligned} & V(x_0, Q_0, z_0) \\ = & \sup_{(L,M)\in\mathcal{A}} E\Big[\int_0^\tau e^{-(r+a)t}\left(x_t \log Q_t dt - \bar{c}_0\,(dL_t + \rho Q_t dt) + \frac{z_t}{r}dM_t\right) + e^{-(r+a)\tau}\frac{z_\tau}{r}\Big] \end{aligned}$$

subject to (13), (21), and (28), for $(x_0, Q_0, z_0) \in \bar{\mathcal{D}}$.

The HJB equation associated with our value function is

$$\min\Big\{ -\mathcal{L}V - \frac{\beta}{r}z - x\log Q + \bar{c}_0\rho Q, -V_Q - c_0V_z + \bar{c}_0, QV_Q + zV_z - \frac{z}{r}\Big\} = 0 \text{ in } \bar{\mathcal{D}},$$

where $\mathcal{L}V = \frac{1}{2}\sigma^2x^2V_{xx} + (\alpha+a)xV_x + \rho QV_Q + (az + \rho c_0 Q)V_z - (r+a+\beta)V$.

We can reduce the problem dimension even when impetuous entry is incorporated:

$$g(b,y) = \frac{1}{Q}V(x,Q,z) - \frac{1}{r+\beta-\alpha}\frac{x}{Q}\log Q, \tag{29}$$

for all $(x,Q,z) \in \bar{\mathcal{D}}$. The HJB equation related to g is

$$\min\big\{-\bar{\mathcal{L}}g + \rho\cdot\mathcal{I}g, \mathcal{I}g, \mathcal{O}g\big\} = 0,\ (b,y) \in \bar{\mathcal{S}},$$

where operators are defined in Equation (18). As in (19) and (20), action and no-action regions can be defined.

APPENDIX B: TECHNICAL PROOFS RELATED TO APPENDIX A

In this section, we provide technical proofs related to our singular control model with the exit option and impetuous entries. First, we show that the value function is well-defined. Next, we present the proof of Theorem A.2 for degenerate cases. At last, we show that the optimal mining strategy of the social planner's problem is an industry equilibrium.

B.1. *Elementary Properties of the Value Function*

We first present the value on the boundary $\partial^x\mathcal{D} := \{(x,Q,z) \in \bar{\mathcal{D}} : x = 0\}$. Given an initial state $(x,Q,z) \in \partial^x\mathcal{D}$, it follows from the dynamic of the revenue process that $x_t \equiv 0$ for $t \geq 0$, indicating there is no future revenue anymore. Thus, intuitively, the social planner will terminate the mining business by retiring all machines (i.e., $Q_t \equiv 0$ for $t \geq 0$) in order to stop paying operating costs, z/r. However, technically, $x_t \log Q_t$ in the objective function is not well defined when $x_t \equiv Q_t \equiv 0$. This issue can be avoided by a sequence of admissible strategies approximating the liquidating strategy.

PROPOSITION B.1: *$V(x,Q,z) = z/r$, on $\partial^x\mathcal{D}$.*

PROOF: Let $x = 0$ and fix $\epsilon > 0$ to be sufficiently small. Take the strategy $(\bar{L},\bar{M})$ where $\bar{L}_t \equiv \epsilon, \bar{M}_t \equiv 1$, for $t \geq 0$. We obtain $V(0,Q,z) \geq -\epsilon\bar{c}_0(1 + \frac{\rho}{r+a+\beta-\rho}) + \frac{z}{r}$. Let $\epsilon \to 0$, we have $V(0,Q,z) \geq \frac{z}{r}$. Since $x=0$ implies $x_t=0$, $\forall t \geq 0$, it is sufficient to show

$$\sup_{(L,M)\in\mathcal{A}} E\left[\int_0^\tau e^{-(r+a)t}\big(-c_0(dL_t+\rho Q_t dt)+z_t dM_t\big)+e^{-(r+a)\tau}z_\tau\right]\leq z.$$

Fixing an arbitrary (L,M) and applying $It\hat{o}$'s lemma gives

$$\begin{aligned} d\left(e^{-(r+a)t}z_t\right)=&-rz_t e^{-(r+a)t}dt+e^{-(r+a)t}\left(c_0(dL_t+\rho Q_t dt)-z_t dM_t\right)\\ \leq& e^{-(r+a)t}\left(c_0(dL_t+\rho Q_t dt)-z_t dM_t\right). \end{aligned}$$

Thus,

$$\begin{aligned} &E\left[\int_0^\tau e^{-(r+a)t}\left(-c_0(dL_t+\rho Q_t dt)+z_t dM_t\right)+e^{-(r+a)\tau}z_\tau\right]\\ \leq& -E\left[\int_0^\tau d\left(e^{-(r+a)t}z_t\right)+e^{-(r+a)\tau}z_\tau\right]=z. \end{aligned}$$

This completes the proof. *Q.E.D.*

By providing upper and lower bounds, we show that the value function is well-defined.

PROPOSITION B.2: *The value function $V(x,Q,z)$ is finite for any $(x,Q,z)\in\bar{\mathcal{D}}$ when $r+\beta>\alpha$. More precisely,*

$$\frac{x\log x}{r+\beta-\alpha}-\frac{r+a+\beta}{r+a+\beta-\rho}\bar{c}_0 x+\frac{z}{r}\leq V(x,Q,z)\leq V^u(x,Q,z), \tag{30}$$

where $V^u(x,Q,z)$ is the value function when the fixed cost is absent and is given in Appendix B.2.

PROOF: Proposition B.1 implies that (30) holds with equality on $\partial^x\mathcal{D}$. Now we assume $x>0$. We obtain a lower bound

$$\begin{aligned} V(x,Q,z)\geq& E\left[\int_0^\tau e^{-(r+a)t}x_t\log x dt\right]-\bar{c}_0 x\left(1+\frac{\rho}{r+a+\beta-\rho}\right)+\frac{z}{r}\\ =&\frac{x\log x}{r+\beta-\alpha}-\frac{r+a+\beta}{r+a+\beta-\rho}\bar{c}_0 x+\frac{z}{r} \end{aligned}$$

by first switching off all the machines and then immediately adjusting the hashrate to x. The other side is obtained by the definition of the value function. *Q.E.D.*

B.2. *Outline of the Proof of Theorem A.2*

(i) The Case without Fixed Cost.

We establish the link between our optimization problem and the following control problem

$$\bar{V}(x) := \sup_{\{Q_t \geq 0\}_{t\geq 0}} E\left[\int_0^\tau e^{-(r+a)t}\Big(x_t \log Q_t dt - c_0 Q_t dt\Big)\right]. \tag{31}$$

In fact, we can prove $V(x,Q,z) = \frac{z}{r} + \bar{V}(x)$. Then a verification theorem can be given to show that the optimal strategy of the control problem (31) is to keep upgrading the mining machines and maintain the optimal hashrate $Q_t^* = \frac{R_t}{c_t}$. Though this strategy implies unbounded variation in the corresponding (L,M) process, it can be approximated by a sequence of admissible strategies $\{(L^m, M^m)\}_{m\in\mathbb{N}}$ as follows.

For $m \in \mathbb{N}$, let $N = \lfloor m\tau \rfloor$, $t_n = \frac{n}{m}$ for $n \in \{0,1,2,...,N\}$ and $t_{N+1} = \tau$. Consider such a strategy: switch off all machines and immediately buy $\frac{x_{t_n}}{c_0}$ units of hardware at t_n for $n \in \{0,1,2,...,N\}$, i.e., $(L_t^m, M_t^m) = \Big(\frac{\sum_{i=0}^n x_{t_i}}{c_0}, n+1\Big)$ if $t \in [t_n, t_{n+1})$.

(ii) The Case without Technology Innovation.

Applying the intermediate value theorem and the implicit function theorem, we can show the existence of the pair (p,q). Smooth pasting condition leads to the explicit expression of the value function:[15]

$$\begin{cases} \left(\dfrac{\alpha+\sigma^2/2}{m} - \dfrac{c_0+\rho I_0}{p} + \log\dfrac{x}{p}\right)\dfrac{x}{m} + C_0 Q, & x \leq pQ \\ \left(d_1\left(\dfrac{x}{Q}\right)^{\gamma_+} + d_2\left(\dfrac{x}{Q}\right)^{\gamma_-} + \dfrac{\beta c_0 - r\rho\bar{c}_0}{r(r+\beta-\rho)}\right)Q + \dfrac{\rho x}{m^2} + \dfrac{x\log Q}{m}, & pQ < x < qQ \\ \left(\dfrac{\alpha+\sigma^2/2}{m} - \dfrac{c_0+(r+\beta)I_0}{q} + \log\dfrac{x}{q}\right)\dfrac{x}{m} + \bar{c}_0 Q, & x \geq qQ, \end{cases} \tag{32}$$

where $m = r+\beta-\alpha$, $d_1 = \frac{\frac{1-\gamma_-}{r+\beta-\alpha}q + \frac{c_0+(r+\beta)I_0}{r+\beta-\rho}\gamma_-}{q^{\gamma_+}(\gamma_- - \gamma_+)(1-\gamma_+)}$, $d_2 = \frac{\frac{1-\gamma_+}{r+\beta-\alpha}p + \frac{c_0+\rho I_0}{r+\beta-\rho}\gamma_+}{p^{\gamma_-}(\gamma_+ - \gamma_-)(1-\gamma_-)}$.

The corresponding HJB equation is

$$\min\{-\mathcal{L}_0 V - \beta C_0 Q - x\log Q + \rho\bar{c}_0 Q, -V_Q + \bar{c}_0, V_Q - C_0\} = 0 \text{ in } \bar{\mathcal{D}}_0, \tag{33}$$

where $\mathcal{D}_0 = \{(x,Q) : x > 0, Q > 0\}$ and $\mathcal{L}_0 V = \frac{1}{2}\sigma^2 x^2 V_{xx} + \alpha x V_x + \rho Q V_Q - (r+\beta)V$. We can prove that the value function and (32) are both the minimum constrained viscosity solution to (33) with boundary condition $V(0,Q) = C_0 Q$, and the value function implies that the optimal strategy is as stated in Theorem A.2(ii).

[15] If we ignore technology improvements (i.e., $a = 0$), then the social planner's optimization problem depends only on the revenue rate $R = x$ and the hashrate Q. Consequently, we can rewrite the value function as $V(x,Q)$.

B.3. *Relation between Industry Equilibrium and Social Planner's Problem*

PROOF OF THEOREM A.1: (i) Assuming that $\pi := (L_t, M_t)_{t\geq 0}$ is an optimal strategy of the social planner, we prove (25) by contradiction.

If inequality (25) does not hold, then

$$E\left[\int_t^{\tau\wedge\nu_t} e^{(\rho-r)(s-t)} D_{t,s} P_s \, ds \middle| \mathcal{F}_t\right] > I_t + E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)} \rho_{t,s} I_s \, ds \middle| \mathcal{F}_t\right]$$
$$+ c_t E\left[\int_t^{\tau\wedge\nu_t} e^{-r(s-t)} D_{t,s} \, ds \middle| \mathcal{F}_t\right] + E\left[\int_t^{\tau\wedge\nu_t} \rho_{t,s} c_s \int_s^{\tau\wedge\nu_t} e^{-r(u-t)} D_{s,u} du ds \middle| \mathcal{F}_t\right]$$

for some t. Without loss of generality, we may assume $t = 0$. Then for some $N > 0$,

$$E\left[\int_0^{\tau\wedge\nu_0\wedge N} e^{(\rho-r)s} D_{0,s} P_s \, ds\right] > I_0 + E\left[\int_0^{\tau\wedge\nu_0} e^{-rs} \rho_{0,s} I_s \, ds\right]$$
$$+ c_0 E\left[\int_0^{\tau\wedge\nu_0} e^{-rs} D_{0,s} \, ds\right] + E\left[\int_0^{\tau\wedge\nu_0} \rho_{0,s} c_s \int_s^{\tau\wedge\nu_0} e^{-ru} D_{s,u} du ds\right] \tag{34}$$

with initial position $(R, Q, k) = (R_0, Q_0, k_0)$.

We replace the strategy at time 0 by increasing δQ additional units of hardware for some small positive δ and keeping the strategy unchanged afterward. Denoting this strategy as $\hat{\pi}$, we will show that

$$V^{\hat{\pi}}(yQ, Q, bQ, T) - V^{\pi}(yQ, Q, bQ, T) \geq \epsilon Q \tag{35}$$

for some positive constant ϵ when T is sufficiently large, where

$$V^i(x, Q, z, T) := E\left[\int_0^{T\wedge\tau} e^{-(r+a)t} \Big(x_t \log Q_t^i dt \right.$$
$$\left. - \bar{c}_0 (dL_t^i + \rho Q_t^i dt) + \frac{z_t^i}{r} dM_t^i \Big) + e^{-(r+a)\tau} \frac{z_\tau^i}{r} \mathbf{1}_{\tau\leq T}\right], \; i = \pi, \hat{\pi}.$$

That contradicts the optimality of strategy π.

Starting from $(x, Q, z) = (yQ, Q, bQ)$, under strategy $\hat{\pi}$, the social planner first adjusts to the state $(yQ, Q + \delta Q, bQ)$. Since the strategies are the same after ν_0, these δQ units of hardware result in an additional expected total discounted fixed cost during $[0, T]$:

$$\delta Q \left(I_0 + E\left[\int_0^{T\wedge\nu_0\wedge\tau} e^{-rs} \rho_{0,s} I_s \, ds\right]\right). \tag{36}$$

Similarly, since $\rho < r+\beta+a$, when $T>N$, these δQ units of hardware influence the term $\int_0^{T\wedge\nu_0\wedge\tau} e^{-rt}R_t\log Q_t dt$ by

$$
\begin{aligned}
&\int_0^{T\wedge\nu_0\wedge\tau} e^{-rt}R_t(\log Q_t^{\hat\pi}-\log Q_t^{\pi})dt\\
\geq&\int_0^{N\wedge\nu_0\wedge\tau} e^{-rt}R_t(\log Q_t^{\hat\pi}-\log Q_t^{\pi})dt=\int_0^{N\wedge\nu_0\wedge\tau} e^{-rt}\frac{R_t}{Q_t^{\pi}}(Q_t^{\hat\pi}-Q_t^{\pi})dt+o(\delta)\\
=&Q\delta\int_0^{N\wedge\nu_0\wedge\tau} e^{-rt}\frac{R_t}{Q_t^{\pi}}\left(1+\int_0^t \rho e^{\rho s}ds\right)e^{-M_t^c}\prod_{0<u\leq t,\Delta M_u>0}(1-\Delta M_u)dt+o(\delta)\\
=&Q\delta\int_0^{N\wedge\nu_0\wedge\tau} e^{(\rho-r)t}P_t^{\pi}D_{0,t}dt+o(\delta).
\end{aligned}
\tag{37}
$$

The difference of discounted electricity costs from both endogenous and impetuous entries is

$$
Q\delta\Bigg(c_0E\left[\int_0^{T\wedge\nu_0\wedge\tau} e^{-rs}D_{0,s}ds\right]+E\left[\int_0^{T\wedge\nu_0\wedge\tau}\rho_{0,s}c_s\int_s^{T\wedge\nu_0\wedge\tau} e^{-ru}D_{s,u}duds\right]\Bigg). \tag{38}
$$

Combining (36), (37) and (38), the difference between the revenue and the total costs of these additional δQ units of hardware is no less than

$$
\begin{aligned}
&Q\delta\Bigg(E\left[\int_0^{N\wedge\nu_0\wedge\tau} e^{(\rho-r)s}D_{0,s}P_sds\right]-I_0-E\left[\int_0^{T\wedge\nu_0\wedge\tau} e^{-rs}\rho_{0,s}I_sds\right]\\
&\quad-c_0E\left[\int_0^{T\wedge\nu_0\wedge\tau} e^{-rs}D_{0,s}ds\right]-E\left[\int_0^{T\wedge\nu_0\wedge\tau}\rho_{0,s}c_s\int_s^{T\wedge\nu_0\wedge\tau} e^{-ru}D_{s,u}duds\right]\Bigg)+o(\delta).
\end{aligned}
$$

We then obtain (35) by noticing that inequality (34) holds and δ is independent of T. Similarly, we can show inequality (26).

(ii) When L_t (M_t) increases strictly, inequality (25) (inequality (26)) holds with equality, since otherwise, we can similarly adjust the strategy to generate higher net revenue for the social planner.

Q.E.D.

APPENDIX C: BOUNDS OF THE NO-ACTION REGION

In this section, we provide some bounds on the no-action region when $\alpha < r+\beta$.

LEMMA C.1: *For any* $\delta>0$,

$$
V(x,Q,z)+\frac{\beta\delta}{r(r+\beta)}\leq V(x,Q,z+\delta)\leq V(x,Q,z)+\frac{\delta}{r}.
$$

PROOF: It follows from the definition of the value function and the fact that $\mathcal{A}(x,Q,z+\delta) = \mathcal{A}(x,Q,z)$ for any $\delta > 0$. *Q.E.D.*

We make a transformation:

$$g(b,y) = \frac{b}{r} + y\left(f(b,y) + \frac{\log y}{r+\beta-\alpha} + \frac{\frac{1}{2}\sigma^2 + a + \alpha}{(r+\beta-\alpha)^2}\right), \tag{39}$$

where g is given by (29). A direct calculation shows f satisfies

$$\min\left\{-\tilde{\mathcal{L}}f, \tilde{\mathcal{I}}f, \tilde{\mathcal{O}}f\right\} = 0,\ (b,y) \in \bar{\mathcal{S}}, \tag{40}$$

where

$$\begin{aligned}\tilde{\mathcal{L}}f &= \frac{1}{2}\sigma^2 y^3 f_{yy} + (\sigma^2 + \alpha + a - \rho)y^2 f_y + (ab + \rho(c_0 - b))y f_b \\ &\qquad - (r+\beta-\alpha)yf - y\log y - b - \rho I_0, \\ \tilde{\mathcal{I}}f &= y^2 f_y - (c_0 - b)y f_b + I_0, \quad \tilde{\mathcal{O}}f = -f_y.\end{aligned}$$

A straight corollary of Lemma C.1 is

$$-\frac{1}{(r+\beta)y} \le f_b(b,y) \le 0. \tag{41}$$

THEOREM C.1: *Assume the same regularity conditions as in Theorem 6.1. Let* $C_1 := \frac{\frac{\sigma^2}{2}+a+\alpha}{r+\beta-\alpha} + \frac{(r+a+\beta)(r+\beta-\alpha)}{r+a+\beta-\rho}\bar{c}_0$, $C_l := \inf\{y > 0 | \log y + \frac{c_0+\rho I_0}{y} \le C_1\}$, $D_l := \sup\{y > 0 | \log y + \frac{c_0+\rho I_0}{y} \le C_1\}$, $C_r := \inf\left\{b \ge c_0 : b + \rho I_0 + \min\left\{\frac{(a-\rho)b+\rho c_0}{r+\beta}, 0\right\} > e^{C_1 - 1}\right\}$, $C_2 := \left(\rho - \left(\frac{\sigma^2}{2} + \alpha + a\right)\right)^+ \frac{I_0}{C_l} + \frac{\rho C_r}{(r+\beta)C_l} + C_1$, $C_u := \frac{e^{2C_2}}{C_l}$.

Then, C_l, D_l, C_r, and C_u are positive finite constants, and we have the following results of the no-action region:

1. *the lower boundary satisfies $C_l \le y^-(b) \le D_l$, for $b \in [c_0, b_u]$;*
2. *the no-action region is on the left of $b = C_r$, i.e., $b_u \le C_r$;*
3. *the upper boundary satisfies $y^+(b) \le C_u$, for $b \in [c_0, b_u]$.*

PROOF: According to (30),

$$f(b,y) \ge f^l(b,y) := -\frac{\frac{1}{2}\sigma^2 + a + \alpha}{(r+\beta-\alpha)^2} - \frac{r+a+\beta}{r+a+\beta-\rho}\bar{c}_0 = -\frac{C_1}{r+\beta-\alpha}. \tag{42}$$

Owing to (41) and the constraint $\rho < r+\beta+a$,

$$(ab+\rho(c_0-b))yf_b \le -\min\left\{\frac{(a-\rho)b+\rho c_0}{r+\beta},0\right\} \le b-c_0. \tag{43}$$

1. In the exit region, we have $f_y=0$ and thus

$$f_{yy}=f_y=0. \tag{44}$$

It follows from (42), (43), and (44) that in $\mathbf{ER}\backslash\mathbf{UR}$

$$\tilde{\mathcal{L}}f \le C_1y - y\log y - b - \rho I_0 - \min\left\{\frac{(a-\rho)b+\rho c_0}{r+\beta},0\right\} \le C_1y-y\log y - c_0 - \rho I_0. \tag{45}$$

Therefore, $\tilde{\mathcal{L}}f<0$ if $C_1 < \log y + \frac{c_0+\rho I_0}{y}$. According to the continuity of partial derivatives, on the exit boundary we have $\tilde{\mathcal{L}}f=0$. As a result, on the exit boundary,

$$C_1 \ge \log y^-(b) + \frac{c_0+\rho I_0}{y^-(b)},\ \text{for } b\in[c_0,b_u].$$

Since $\lim\limits_{y\to 0}\log y+\frac{c_0+\rho I_0}{y}=+\infty$ and $\lim\limits_{y\to+\infty}\log y+\frac{c_0+\rho I_0}{y}=+\infty$, constants C_l and D_l exist and are positive and finite. Thus, $C_l\le y^-(b)\le D_l,$ for $b\in[c_0,b_u]$.

2. First, $C_r\ge c_0$ is finite due to $\rho<r+\beta+a$, and

$$\left\{b\ge c_0 : b+\rho I_0+\min\left\{\frac{(a-\rho)b+\rho c_0}{r+\beta},0\right\} > e^{C_1-1}\right\} = (C_r,+\infty).$$

We notice $\max_{y>0}\{C_1y-y\log y\}=e^{C_1-1}$. Then due to (45), $\tilde{\mathcal{L}}f<0$ when $b+\rho I_0+\min\left\{\frac{(a-\rho)b+\rho c_0}{r+\beta},0\right\}>e^{C_1-1}$. Since $\tilde{\mathcal{L}}f=0$ on the exit boundary, we have $b_u\le C_r$.

3. It follows from $C_l,C_r\in(0,+\infty)$ and the definition of C_u that C_u is positive and finite. According to (41), in $\mathbf{NR}$,

$$(ab+\rho(c_0-b))f_b \le \frac{(\rho b-(ab+c_0\rho))^+}{(r+\beta)y} \le \frac{\rho C_r}{(r+\beta)C_l}. \tag{46}$$

Combining (42) and (46), we have in the no-action region,

$$\begin{aligned}0=&\frac{1}{2}\sigma^2y^2f_{yy}+(\sigma^2+\alpha+a-\rho)yf_y+(ab+\rho(c_0-b))f_b\\&-(r+\beta-\alpha)f-\log y-\frac{b+\rho I_0}{y}\\\le&\frac{1}{2}\sigma^2y^2f_{yy}+(\sigma^2+\alpha+a-\rho)yf_y+\frac{\rho C_r}{(r+\beta)C_l}+C_1-\log y-\frac{b+\rho I_0}{y}.\end{aligned}$$

Set $j=\log y$ and $h(b,j)=f(b,y)$, we have

$$0\le \frac{1}{2}\sigma^2 h_{jj}+\left(\frac{1}{2}\sigma^2+\alpha+a-\rho\right)h_j+\frac{\rho C_r}{(r+\beta)C_l}+C_1-j-(b+\rho I_0)e^{-j}. \tag{47}$$

Due to $\tilde{\mathcal{I}}f=y^2f_y+(b-c_0)yf_b+I_0\ge 0$, $\tilde{\mathcal{O}}f=-f_y\ge 0$, and $f_b\le 0$, we have $h_j=yf_y\in[-I_0e^{-j},0]$. Therefore,

$$\left(\frac{1}{2}\sigma^2+\alpha+a-\rho\right)h_j\le\left(\rho-\left(\frac{1}{2}\sigma^2+\alpha+a\right)\right)^+ I_0e^{-j}\le\left(\rho-\left(\frac{1}{2}\sigma^2+\alpha+a\right)\right)^+\frac{I_0}{C_l}.$$

Inequality (47) leads to

$$\begin{aligned}
h_{jj}\ge&-\frac{2}{\sigma^2}\left(\left(\rho-\left(\frac{1}{2}\sigma^2+\alpha+a\right)\right)^+\frac{I_0}{C_l}+\frac{\rho C_r}{(r+\beta)C_l}+C_1-(b+\rho I_0)e^{-j}\right)+\frac{2j}{\sigma^2}\\
\ge&-\frac{2}{\sigma^2}\left(\left(\rho-\left(\frac{1}{2}\sigma^2+\alpha+a\right)\right)^+\frac{I_0}{C_l}+\frac{\rho C_r}{(r+\beta)C_l}+C_1\right)+\frac{2j}{\sigma^2}\\
=&-\frac{2C_2}{\sigma^2}+\frac{2j}{\sigma^2}.
\end{aligned}$$

Therefore, for (b,j) in the no-action region,

$$\begin{aligned}
0\ge h_j(b,j)\ge& h_j(b,\log y^-(b))+\int_{y^-(b)}^{j}\left(-\frac{2C_2}{\sigma^2}+\frac{2s}{\sigma^2}\right)ds\\
=&-\frac{2C_2}{\sigma^2}(j-\log y^-(b))+\frac{1}{\sigma^2}(j^2-\log^2 y^-(b)),
\end{aligned}$$

which results in $j\le 2C_2-\log y^-(b)\le 2C_2-\log C_l$. Consequently, we have $y\le\frac{e^{2C_2}}{C_l}$, and thus, $y^+(b)\le C_u$, for $b\in[c_0,b_u]$. *Q.E.D.*

PROOF OF THEOREM 6.1: This follows from Theorem C.1 by choosing $A:=\frac{c_0}{C_u}$ and $B:=\frac{C_r}{C_l}$. *Q.E.D.*

APPENDIX D: DECOMPOSITION OF THE HASHRATE PROCESS

Based on our enhanced model, the variation in hashrate arises from endogenous market entry and exit, as well as impetuous entries. We can derive the following decomposition of the in-sample calibrated hashrate process in terms of their individual contributions

$$Q_t^{cal}=Q_0+L_t^A-M_t^A+\rho_t^A,$$

$$L_t^A:=\int_0^t dL_s^{cal},\ M_t^A:=\int_0^t Q_s^{cal}dM_s^{cal},\ \rho_t^A:=\int_0^t \rho Q_s^{cal}ds,$$

where (L^{cal}, M^{cal}) is the corresponding optimal mining strategy. Note that L_t^A and M_t^A in the above expressions come from endogenous market entry and exit, respectively, and ρ_t^A measures the aggregate amount of impetuous entries. Table D.I shows the proportion of each component to the total increment of the in-sample calibrated hashrate. In the calibrated period, the aggregate amount of market exit turns out to be higher than the amount of endogenous entries, and significant impetuous entries occur. This demonstrates the significant contribution of the exit and impetuous entry to our enhanced model.

TABLE D.I

DECOMPOSITION OF THE HASHRATE PROCESS[a]

	1^{st} subperiod	2^{nd} subperiod	3^{rd} subperiod	4^{th} subperiod	5^{th} subperiod
ρ	0.61	0.77	0.81	0.87	0.54
entry: $L_T^A/(Q_T^{cal} - Q_0)$	0.37	0.18	0.13	0	0
exit: $M_T^A/(Q_T^{cal} - Q_0)$	0.58	0.35	0.32	0.19	0.28
impetuous entry: $\rho_T^A/(Q_T^{cal} - Q_0)$	1.21	1.17	1.18	1.19	1.28

[a]This table shows the proportion of endogenous entries, exit, and impetuous entries to the total increment of the in-sample calibrated hashrate, where T refers to the end of each subperiods.

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