

2025-03-15

A Taylor–Leibniz expansion for time domain viscoelastic studies

E. Abercrombie, J.G. McDaniel. 2025. "A Taylor–Leibniz expansion for time domain viscoelastic studies" *Modelling and Simulation in Materials Science and Engineering*, Volume 33, Issue 2, pp.025003-025003. <https://doi.org/10.1088/1361-651x/ada175>

<https://hdl.handle.net/2144/50255>

"Downloaded from OpenBU. Boston University's institutional repository."

PAPER • OPEN ACCESS

A Taylor–Leibniz expansion for time domain viscoelastic studies

To cite this article: Eric Abercrombie and J Gregory McDaniel 2025 *Modelling Simul. Mater. Sci. Eng.* **33** 025003

View the [article online](#) for updates and enhancements.

You may also like

- [A non-isothermal phase-field crystal model with lattice expansion: analysis and benchmarks](#)
Maik Punke, Marco Salvalaglio, Axel Voigt et al.
- [Ab initio computational study of hydration thermodynamics in cubic yttria-stabilized zirconia](#)
A G Marinopoulos
- [Effect of anisotropic Peierls barrier on the evolution of discrete dislocation networks in Ni](#)
John D Shimanek, Darshan Bamney, Laurent Capolungo et al.

A Taylor–Leibniz expansion for time domain viscoelastic studies

Eric Abercrombie*  and J Gregory McDaniel

Department of Mechanical Engineering, Boston University, Boston, MA 02215,
United States of America

E-mail: abere@bu.edu

Received 1 August 2024; revised 10 December 2024

Accepted for publication 19 December 2024

Published 17 January 2025



CrossMark

Abstract

Transient simulations of viscoelastic materials remain common, as does interest in the formulation of such simulations. This paper offers an alternative to the current set of solution schemes, including the Newmark-Beta or space-time Galerkin methods. The solution to an equation of motion for a viscoelastic simulation can be expressed by a Taylor series, when a Leibniz integration rule is applied to the time integral appearing in the viscoelastic stress equation. The result is a remarkably simple scheme to implement that also offers extremely rapid solution times. This effort includes a formulation of the approach, as well as numerical results for a simple mass and viscoelastic spring system, including an assessment of the number of Taylor series terms needed to accurately depict transient behavior. The paper outlines how the approach could be extended to finite element analysis and how complex loading conditions could be accounted for. Care is given in addressing the potential limitations of the approach.

Keywords: viscoelasticity, Taylor series, transient, shock

1. Introduction

Viscoelastic materials are a common class of materials that experience a time decay in stress for a step in strain. This property gives them the capacity to damp energy while vibrating. Viscoelastic behavior plays a wide role in many current areas of research such as contact

* Author to whom any correspondence should be addressed.



Original Content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

mechanics [1, 2], nanoscale design [3], and biomechanics [4, 5]. Mathematically, viscoelastic behavior is often expressed in the time domain as a stress that combines the initial fixed stiffness of the material with a time integral. In the frequency domain, treatment of this integral is simple and it disappears in the wake of using the Fourier transform. However, transient behavior in the time domain remains a popular area of research [6, 7] and of interest in many practical analyses. The common approach of many commercial finite element packages [8, 9] to linear viscoelastic materials remains a time stepping Newmark-Beta scheme to evaluate vibration in discrete steps. These step-wise integration schemes are well documented [10]. The last two decades have produced an alternative in the space-time Galerkin methods [11, 12]. While these are not the common approach in commercial tools, they may offer some advantage in convergence rate [13]. Also noteworthy are data-driven schemes [14], which have recently been applied to viscoelastic and inelastic transient studies [15]. However, that work is new and the literature in regard to its use on viscoelasticity is still very limited. It too does not represent the current approach of commercial tools, but future studies may show it has computational advantage for these same problems.

This paper presents a very different alternative approach to evaluating transient behavior in viscoelastic material studies. It uses a Taylor series approximation of the solution of an ordinary differential equation (ODE) in parallel with the use of the Leibniz rule to collapse the viscoelastic time integral around the Taylor series expansion point. This method may eventually be applied for finite element analysis, but can also be used for solutions of any system that can be expressed as an ODE. The benefit of this approach over others is two fold. First, the approach is far easier to implement in a tool such as MATLAB, which means it is easier to use for various tailored applications. Secondly, the solution time is rapid, for small problems taking only a matter of seconds. In comparison, step-wise integration schemes for these same studies are known to be laborious, which is why recent work has taken place to improve upon them [16].

The notion of using a series to represent the solution to an ODE is not new [17, 18], and power series expressions are also common [19, 20]. Furthermore, Taylor series expansions have recently been used to study other problems in physics [21], chemical physics [22], and electrochemistry [23]. The unique feature of this approach is in the solution of transient viscoelastic studies where the Taylor series, when coupled with a Leibniz rule reduction, avoids a challenging convolution integral over time. Of course, there are limitations to this approach as well. It is well understood that more terms in a Taylor series are needed to expand further around any given point. To that end, the study provides the reader with the number of terms in a Taylor series that were needed to consider longer solution times for an example problem. The study also identifies an approach to reduce calculation time by expanding around additional points in the solution time. Furthermore, when expanded around a single point, it may seem the study would be incapable of handling multiple excitations at different discrete times or functions in time. However, this too can be incorporated by use of linear superposition, which is explored later in this document.

The formulation of a Taylor series expression for an elastic system will be reviewed in section 2. This will set the stage for development of the approach and the introduction of the Leibniz integration rule in section 3. Section 4 contains an example solution for a vibration study, an example of superposition for the same application, as well as discussion and explanation of the use of additional expansion points.

2. Mathematical background, Taylor series of an ODE

To motivate the proposed method, we begin with a simple mass-spring system. The equation of motion can be expressed as

$$m\ddot{x} + kx = 0, \quad (1)$$

where m is mass, k is the stiffness of a spring and x is displacement. The system is assumed to have initial displacement x_0 and initial velocity \dot{x}_0 .

A Taylor series expansion of the displacement solution in time would be

$$x(\tau) = x_t + \frac{\dot{x}_t(\tau - t)}{1!} + \frac{\ddot{x}_t(\tau - t)^2}{2!} + \dots, \quad (2)$$

where τ denotes the point being evaluated and t is the point around which the expansion is being taken. For a Maclaurin series taken around point $t = 0$ this expression becomes

$$x(\tau) = x_0 + \frac{\dot{x}_0(\tau)}{1!} + \frac{\ddot{x}_0(\tau)^2}{2!} + \dots \quad (3)$$

Where x_0 and \dot{x}_0 are already known and the expressions of consecutive derivative of the displacement x are derived from the equation of motion. Each such further derivative in the series is represented as

$$\frac{d^n x}{dt^n} = -\frac{k}{m} \frac{d^{(n-2)} x}{dt^{(n-2)}}. \quad (4)$$

This is the result of differentiating the equation of motion. Some conditions on the use of a Taylor series are worth noting here and will become even more relevant in the next section. In general, Taylor series require numerous derivatives of an analytic function. The resulting series has a convergent radius around an expansion point up to the nearest singularity [24]. In this way the function approximation by Taylor series is best for an easily differentiable form with derivatives that do not produce singularities. In this case, each further derivative of displacement is primarily defined by a multiplication of $\frac{-k}{m}$, so no singularities form. This limitation will be re-examined in the next section. These results motivate the below application to a viscoelastic system.

3. Mathematical development

For linear viscoelastic materials, stress can be given as [25].

$$\sigma(t) = \epsilon(t)Y(0) + \int_{0+}^t \epsilon(\tau) \frac{dY(t-\tau)}{d(t-\tau)} d\tau, \quad (5)$$

where $\sigma(t)$ is stress, $\epsilon(t)$ is strain, and $Y(t)$ is the relaxation function. Other formats of the same expression for stress are common [25] and can be derived using integration by parts [26]. Additionally, strain history can also be used to describe linear viscoelastic materials.

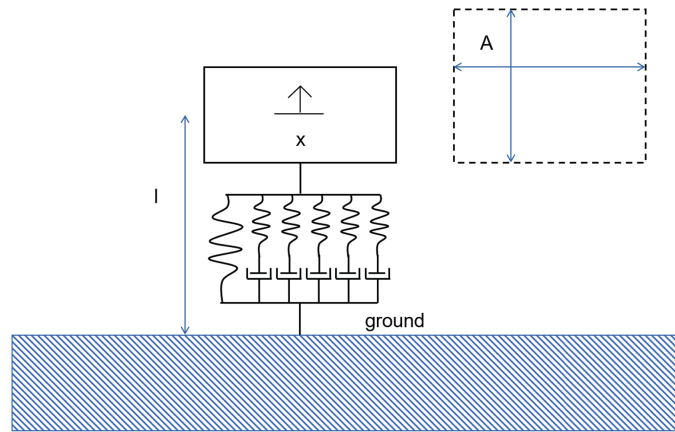


Figure 1. Diagram of a simple mass-spring system with a single viscoelastic support.

3.1. Inclusion of viscoelasticity

Now consider the replacement of the elastic spring with a viscoelastic spring. This system will consist of a mass, m , supported by a network of springs and dashpots equivalent to a generalized Maxwell model of a viscoelastic material, and positioned off the ground by a notional length, l , as depicted below.

In figure 1, a reference area, A , is provided for the mass subject to initial conditions, x_0 and \dot{x}_0 , and will later allow the incorporation of stress as a force. A general form for the equation of motion defined by stress is

$$m\ddot{x} + A\sigma(t) = 0, \quad (6)$$

where A is a notional area that relates a stress to a force and $\sigma(t)$ is stress. If $\sigma(t)$ has the form in equation (5) then the equation of motion can be used to describe the behavior of a viscoelastic material. For the proposed system, strain will be related to a reference length of the viscoelastic spring in preparation for the analysis of the simple system. This is given as

$$\epsilon(t) = \frac{x(t)}{l}. \quad (7)$$

Combining equation (7) with equation (5) yields

$$l\sigma(t) = x(t)Y(0) + \int_{0+}^t x(\tau) \frac{dY(t-\tau)}{d(t-\tau)} d\tau, \quad (8)$$

an expression for stress in terms of displacement.

Moving the stress term to the right hand side of equation (6) and taking additional derivatives of both sides, allows for additional derivatives of displacement having the form of

$$\frac{d^n x}{dt^n} = -\frac{A}{m} \frac{d^{(n-2)} \sigma}{dt^{(n-2)}}. \quad (9)$$

It is critical to recognize that derivatives of stress, which contain the important time integral, are needed. Derivatives of the stress integral $\sigma(t)$, will follow this form:

$$l \frac{d\sigma^n}{dt^n} = \frac{d^n x}{dt^n} Y(0) + \frac{d^n}{dt^n} \int_{0+}^t x(\tau) \frac{dY(t-\tau)}{d(t-\tau)} d\tau. \quad (10)$$

3.2. Leibniz rule for integral reduction

This subsection touches on the heart of the benefit of this approach. Through the use of the Leibniz integral rule and consideration of the Taylor series around the point $t=0$ (Maclaurin Series), the time integral can be collapsed into a simple series expression. The Leibniz rule found in many textbooks on calculus [27] is applied to the right hand side of equation (10) as

$$\frac{d}{dt} \int_0^t f(t, \tau) d\tau = f(t, t) \frac{dt}{dt} - f(t, 0) \frac{d0}{dt} + \int_0^t \frac{d}{dt} f(t, \tau) d\tau. \quad (11)$$

In the below discussion the initial term already appearing in equation (10) will be ignored, since it will not contribute to the treatment of the integral. The first derivative of only the time integral in equation (10) following the well known Leibniz integral rule is

$$\frac{d}{dt} \int_0^t x(\tau) \frac{dY(t-\tau)}{d(t-\tau)} d\tau = x(t) \frac{dY}{dt} \Big|_{t=0} + \int_0^t x(\tau) \frac{d^2 Y(t-\tau)}{d(t-\tau)^2} d\tau, \quad (12)$$

where an initial term taken at time $t=0$ has appeared, which evaluates to a constant. Then, another integral containing a higher order derivative of the relaxation function $Y(t)$ remains. The second derivative taken in this way is

$$\frac{d^2}{dt^2} \int_0^t x(\tau) \frac{dY(t-\tau)}{d(t-\tau)} d\tau = \frac{dx}{dt} \frac{dY}{dt} \Big|_{t=0} + x(t) \frac{d^2 Y}{dt^2} \Big|_{t=0} + \int_0^t x(\tau) \frac{d^3 Y(t-\tau)}{d(t-\tau)^3} d\tau. \quad (13)$$

This continues and the n th order derivative for n number of terms taken in the Taylor Series can be expressed as

$$\frac{d^n}{dt^n} \int_0^t x(\tau) \frac{dY(t-\tau)}{d(t-\tau)} d\tau = \sum_{p=1}^n \frac{d^{(n-p)} x}{dt^{(n-p)}} \frac{d^p Y}{dt^p} \Big|_{t=0} + \int_0^t x(\tau) \frac{d^{(n+1)} Y(t-\tau)}{d(t-\tau)^{(n+1)}} d\tau, \quad (14)$$

where p is introduced to account for the higher order derivatives of the initial condition terms that result from higher order derivatives of the integral. Each successive order derivative of the integral will produce one additional initial condition series term. The derivative order of the displacement will always be shifted by $(n-p)$, while it can be seen that the relaxation function derivative is simply of order p . For any given number of derivatives taken, a form of the integral on the right hand side of equation (14) will still exist and would then require a step-wise marching scheme or equivalent to evaluate in time. However, expanding around the point $t=0$, results in the replacement of t in the hereditary integral with zero. This is advantageous because integrals over a span of zero also converge to zero [28]. This can be expressed as

$$\int_0^0 x(\tau) \frac{d^{(n+1)} Y(t-\tau)}{d(t-\tau)^{(n+1)}} d\tau = 0. \quad (15)$$

In place of the evaluation of this integral, the solution now uses the series of terms resulting from the application of the Leibniz Integral rule, taken at $t = 0$. The n th order derivative expression of these terms can be reduced to

$$l \frac{d\sigma^n}{dt^n}(0) = \left. \frac{d^n x}{dt^n} Y(0) \right|_{t=0} + \sum_{p=0}^n \left. \frac{d^{(n-p)} x}{dt^{(n-p)}} \frac{d^p Y}{dt^p} \right|_{t=0}. \quad (16)$$

Of immediate concern to the reader may be the perceived loss of information in regard to the relaxation function $Y(t)$. However, almost all common relaxation functions map an analytic function to measured relaxation data across the entire time decay of the relaxation. This means that so long as the user is applying a relaxation function fit to the overall curve, such as the Generalized Maxwell Model, Kelvin–Voight Model or even a quadratic function, the higher order derivatives of $Y(t)$ fulfill the same role. The implication of this approach then is to say that all future behavior of a system can be expressed as being a result of what is known about the material, initial conditions and forcing at a single expansion point.

It is worth now examining what conditions apply to the above formulation. In particular, the conditions for the application of a Taylor series. Review of equations (16) and (9) collectively impose the condition that only derivatives of the relaxation function itself put a limit on the use of the Taylor series. The case of a Prony series form of relaxation and its derivatives are a clear example of an infinitely differentiable function with no singularities. This is because each additional derivative only differs from the previous derivative by a multiplication by the negative inverse constant within the exponential function. Other representations of relaxation are more complex, such as fractional derivative models which present gamma functions. Finding values for derivatives of the gamma function may prove challenging [29]. The extension of the formulation to all constitutive relations is worthy of a separate effort.

The conditions on the use of a Leibniz integral rule, sometimes noted as ‘differentiation under the integral sign’ are also worth considering. In general two conditions apply [30]. The first is that the function inside the integral and its derivative(s) be continuous within the considered domain. The second is that the bounds of the integration be continuous and have continuous derivatives within the given domain. To the first condition, the continuity of the function inside the integral has already been discussed in regard to the use of a Taylor series. The dependence on the form of the relaxation function has already been noted. To the second condition, the bounds of integration are 0 and t , which are both continuous and differentiable regardless of the form of the relaxation function.

3.3. Solutions with time-dependent forcing functions

The above formulation is done assuming free vibration of a system. However, the inclusion of a forcing function need not be an issue. In the same way a relaxation function can be incorporated still accounting for its behavior over a period of time, a forcing function with an analytical definition can also be incorporated as

$$\frac{d^n x}{dt^n} = \frac{k}{m} \frac{d^{(n-2)} x}{dt^{(n-2)}} - \frac{A}{m} \frac{d^{(n-2)} \sigma}{dt^{(n-2)}} + \frac{1}{m} \frac{d^{(n-2)} f}{dt^{(n-2)}}. \quad (17)$$

Where a term has also been included for the case of a simple spring in parallel, but is not needed to evaluate forcing. In the case of a non-analytically defined forcing function or one that varies by steps, analysts can refer to the superposition section below.

3.4. In three dimensions

Following the formulation for viscoelasticity presented in [31], a 3D framework for the implementation of this computational scheme is presented. $Y(t - \tau)$ is a generalized relaxation function, but when it is taken to be the Youngs relaxation function, it can be further decomposed into

$$\mathbf{E} = G\mathbf{E}_G + B\mathbf{E}_B. \quad (18)$$

Where G is the shear relaxation function, B is the bulk relaxation function and E_G and E_B are modulus matrices. Combining equations (5) and (18) yields a stress function describing a viscoelastic material in terms of bulk and shear relaxation.

$$\boldsymbol{\sigma}(t) = [G(0)\mathbf{E}_G + B(0)\mathbf{E}_B]\boldsymbol{\epsilon}(t) + \mathbf{E}_G \int_{0+}^t G'(t - \tau)\boldsymbol{\epsilon}(\tau) d\tau + \mathbf{E}_B \int_{0+}^t B'(t - \tau)\boldsymbol{\epsilon}(\tau) d\tau. \quad (19)$$

Further development comes from [32], which will allow a transition from strain to displacement and results in a standard equation of motion as

$$\mathbf{A} + \mathbf{M}\ddot{\boldsymbol{\Delta}} = \mathbf{f} + \mathbf{Q}, \quad (20)$$

where \mathbf{A} replaces stress and contains the additional terms needed to re-express the strain equation in terms of displacement, \mathbf{M} is the traditionally understood mass matrix, $\boldsymbol{\Delta}$ is the three dimensional displacement vector, \mathbf{f} is a body forcing function and \mathbf{Q} is a traction forcing function. Stress had already been decomposed into shear and bulk components and similar, \mathbf{A} can be partitioned into the same three components as

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_B + \mathbf{A}_G, \quad (21)$$

where A_0 , A_B , and A_G are defined by

$$\mathbf{A}_0 = \int_{\Omega} \mathbf{B}^T [G(0)\mathbf{E}_G + B(0)\mathbf{E}_B] \mathbf{B} \boldsymbol{\Delta}(t) dx dy dz, \quad (22)$$

$$\mathbf{A}_G = \int_{\Omega} \mathbf{B}^T \mathbf{E}_G \mathbf{B} \left(\int_{0+}^t G'(t - \tau) \boldsymbol{\Delta}(\tau) d\tau \right) dx dy dz, \quad (23)$$

$$\mathbf{A}_B = \int_{\Omega} \mathbf{B}^T \mathbf{E}_B \mathbf{B} \left(\int_{0+}^t B'(t - \tau) \boldsymbol{\Delta}(\tau) d\tau \right) dx dy dz. \quad (24)$$

Here, B is the strain displacement matrix. Creating a lumped integral as

$$\mathbf{J}_G = \int_{\Omega} \mathbf{B}^T \mathbf{E}_G \mathbf{B} dx dy dz, \quad (25)$$

$$\mathbf{J}_B = \int_{\Omega} \mathbf{B}^T \mathbf{E}_B \mathbf{B} dx dy dz, \quad (26)$$

yielding the following reduction

$$\mathbf{A}_0 = [G(0)\mathbf{J}_G \boldsymbol{\Delta}(t) + B(0)\mathbf{J}_B \boldsymbol{\Delta}(t)], \quad (27)$$

$$\mathbf{A}_G = \mathbf{J}_G \int_{0+}^t G'(t-\tau) \Delta(\tau) d\tau, \quad (28)$$

$$\mathbf{A}_B = \mathbf{J}_B \int_{0+}^t B'(t-\tau) \Delta(\tau) d\tau. \quad (29)$$

Applying the Taylor–Leibniz Expansion for a displacement vector around the expansion point $t = 0$ would look like

$$\Delta(\tau) = \Delta_0 + \frac{\dot{\Delta}_0(\tau)}{1!} + \frac{\ddot{\Delta}_0(\tau)^2}{2!} + \dots \quad (30)$$

Which requires definition of the derivatives of Δ , which can be derived from the equation of motion. Formulating equation (20) in terms of the second displacement derivative yields

$$\ddot{\Delta} = -\mathbf{M}^{-1}\mathbf{A} + \mathbf{M}^{-1}\mathbf{f} + \mathbf{M}^{-1}\mathbf{Q}. \quad (31)$$

Equation (31) starts with the second derivative of displacement. Accounting for the zeroth and first order derivatives of displacement introduces the initial conditions of the system, namely initial displacement and initial velocity. Taking further derivatives of displacement creates the following result

$$\frac{d^n \Delta}{dt^n} = -\mathbf{M}^{-1} \left(\frac{d^{(n-2)} \mathbf{A}}{dt^{(n-2)}} - \frac{d^{(n-2)} \mathbf{f}}{dt^{(n-2)}} - \frac{d^{(n-2)} \mathbf{Q}}{dt^{(n-2)}} \right). \quad (32)$$

Assuming that \mathbf{f} and \mathbf{Q} are differentiable functions, then only higher order derivatives of \mathbf{A} still require definition. Recalling equations (22)–(24) and taking further derivatives using the Taylor–Leibniz expansion yields

$$\begin{aligned} \frac{d^{(n)} \mathbf{A}}{dt^{(n)}} = & \mathbf{J}_G \frac{d^n \Delta}{dt^n} G(0) \Big|_{t=0} + \mathbf{J}_B \frac{d^n \Delta}{dt^n} B(0) \Big|_{t=0} + \mathbf{J}_G \sum_{p=1}^n \frac{d^{(n-p)} \Delta}{dt^{(p)}} \frac{d^p G}{dt^p} \Big|_{t=0} \\ & + \mathbf{J}_B \sum_{p=1}^n \frac{d^{(n-p)} \Delta}{dt^{(p)}} \frac{d^p B}{dt^p} \Big|_{t=0}. \end{aligned} \quad (33)$$

4. Application

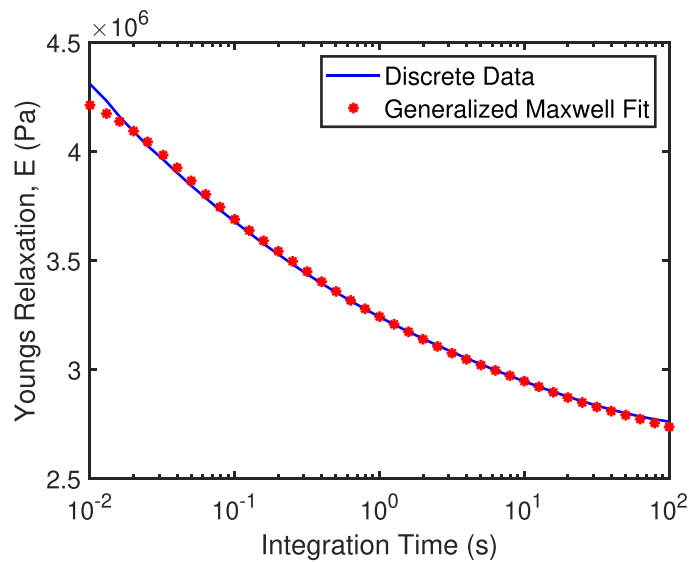
4.1. A mass and viscoelastic spring system

A system was defined in section 3 that can now be used to demonstrate the application of this approach. The viscoelastic spring consists of a 20 °C nitrile rubber support [33]. The Young's relaxation function has a five term Prony fit based on the terms provided in the same source. The function is the stress response of the material for a unit step in axial strain. The material parameters for the rubber are defined in table 1 and depicted in figure 2. The generalized Maxwell model fit presented here follows the format seen in ABAQUS [8], which is

$$E(t) = E_0 \left[1 - \sum_{n=1}^N \left(1 - C_n e^{(-t/s_n)} \right) \right]. \quad (34)$$

Table 1. Generalized Maxwell model curve fit parameters for 20 °C nitrile rubber.

Property	Description	Value
E_0	Initial relaxation value	4.7 MPa
E_0^s	Adjusted relaxation value	5.3 MPa
C_1	First Prony multiplier	0.0526
s_1	First Prony time constant	27.374 s
C_2	Second Prony multiplier	0.0570
s_2	Second Prony time constant	3.3234 s
C_3	Third Prony multiplier	0.07589
s_3	Third Prony time constant	0.44 676 s
C_4	Fourth Prony multiplier	0.09849
s_4	Fourth Prony time constant	0.07 532 s
C_5	Fifth Prony multiplier	0.1325
s_5	Fifth Prony time constant	0.0121 s

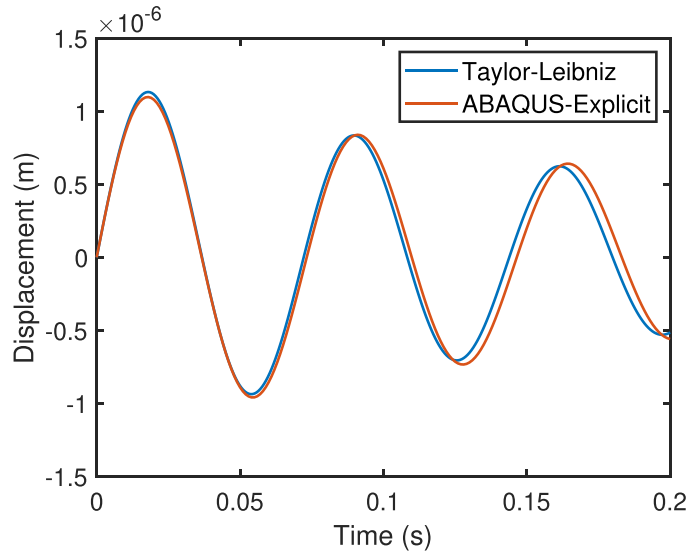
**Figure 2.** Young's relaxation function for 20 °C nitrile rubber derived from data and Prony terms in [33]. Reproduced from [33]. © IOP Publishing Ltd. [CC BY 3.0](#)

The choice here of a generalized Maxwell model to define the relaxation is done for convenience. The formulation works for any expression of the relaxation function analytically across the entire time decay. This fit could just have easily been geometric, cubic spline, or any other fit. Other properties of the system including the dimensions used, initial conditions and basic rubber material properties are provided in table 2 alongside boundary conditions and solver settings for an ABAQUS benchmark study. Non-linear effects have not been included here and will not be included in any other part of this analysis.

The displacement of the system is depicted in figure 3. The results of the function are consistent with a damped single frequency system. To validate the results, the same system was modeled in ABAQUS as a simply supported one degree of freedom bar and evaluated using an iterative time-stepping scheme. While the mass input is direct in units of kilograms for the

Table 2. Model parameter and solver setting comparison for figure 3 with the Taylor–Leibniz expander left and the ABAQUS Explicit solver right.

Parameter	Taylor–Leibniz	Bar parameter	ABAQUS
Spring length	1 m	Bar length	1 m
Area	1 m ²	Bar area	1 m ²
Density	—	Density	1300 kg m ⁻³
Mass	650 kg	Nodal mass	650 kg
Displacement (x_0)	0	Displacement (x_0)	0
Velocity (\dot{x}_0)	0.0001 m s ⁻¹	Velocity (\dot{x}_0)	0.0001 m s ⁻¹
Poisson ratio	0.47	Poisson ratio	0.47
Solver	MATLAB	Solver	Explicit
Evaluation points	200	Time increment	0.001 s
Boundary	Grounded	Boundary	Fixed Node
Solution time	0.2 s	Step end time	0.2 s
Elements	1	Elements	1

**Figure 3.** Displacement of a viscoelastic system with comparison to an explicit ABAQUS solution.

spring code, a density needs to be used in ABAQUS that creates an equivalent nodal mass. This can be resolved based on the form of the mass matrix for a bar element and verified by examination of matrix output from within the solver. It was desirable to use unity values for area and length, so the parameter parallel was the closest between ABAQUS and MATLAB. However, the resulting dimensions of the object are cube like and so have an intermediate modulus between Young's and Bulk. We have corrected the effective modulus of the viscoelastic spring in MATLAB based on the formula and parameters for a rubber like material in Snow, 1968 [34]. The change in effective modulus is small and shown in table 1. The ABAQUS explicit solver uses numerical dissipation schemes for stability [8] and The parameters used were the default. The ABAQUS solution is susceptible to small changes based on various solver and element assumptions, including small Poisson effects in the bar. However, the results still

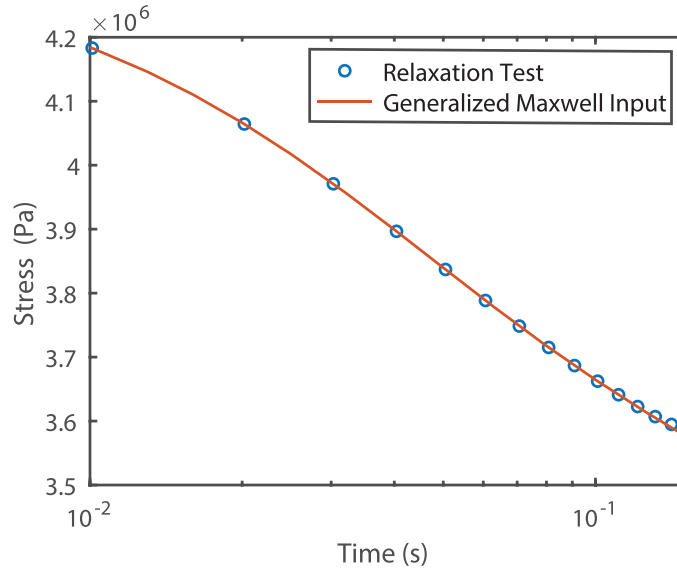


Figure 4. Relaxation test versus prescribed generalized Maxwell model values.

indicate a high degree of accuracy. Both calculations produced run times that were at or below the minimum threshold for reference in MATLAB or ABAQUS. The system used to produce this figure is a 14 core system with Intel i9 processors.

To further ensure the validity of the approach, a basic relaxation test was formulated. In the study the mass of the system will be set to zero and strain will be set to a unit step, such that the expansion actually solves for the stress required to maintain the unit strain. The material is viscoelastic, so the stress should decrease over time. Furthermore, the resulting stress should be exactly equal to the generalized Maxwell model used to define the material. Without mass the equation of motion becomes

$$A\sigma(t) = f(t), \quad (35)$$

where $f(t)$ is a force representing stress related by the unit area A . $f(0)$ is easily calculated as

$$f(0) = AY(0)x(0), \quad (36)$$

when substituting the stress equation from equation (10). Furthermore, displacement, x , is constant because strain is constant so the higher order derivatives of $f(t)$ reduce to

$$f^n(0) = x(0) \left. \frac{d^n Y}{dt^n} \right|_{t=0}. \quad (37)$$

Evaluating the Taylor series around $t = 0$ as was done with the oscillating system then yields figure 4.

It can be seen that the simple relaxation test successfully approximated the expected Generalized Maxwell Model behavior, up until the 100 term expansion reached its limit.

Table 3. Single Prony term fit and system parameters for a parametric sweep of s_1 .

Property	Description	Value
E_0	Initial relaxation value	4.7 MPa
C_1	First Prony multiplier	0.5
s_1	First Prony time constant	x

Table 4. Number of Taylor series terms needed for calculation based on material relaxation time and total number of periods.

Number of periods	Number of periods				
	1	2	3	4	5
s_1					
.01	20	45	65	—	—
.025	18	38	58	74	90
.05	17	35	53	70	88
.1	17	34	51	68	85
.25	16	34	51	68	85
.5	16	34	51	67	85
1	16	34	51	67	84

4.2. Evaluation of required series terms and numerical demand

One concern of the approach may be the number of Taylor series terms required to represent a signal response of meaningful duration and what limitations may exist in expanding the series around a single point. A similar system was simplified to allow for a parametric study. The generalized Maxwell model of the system was reduced to a single term, with the intent of varying the Prony time constant. Table 3 defines the new viscoelastic parameters of the generalized Maxwell model fit used for these variations.

The Prony time constant s_1 was varied in small increments along with the number of Taylor series terms to produce table 4. For every given time constant evaluated, the number of Taylor series terms required to accurately depict the specified number of periods of oscillation is provided. This assessment was based on visual convergence of the displacement plot. The change in calculated displacement in early periods was found to be small for changes in τ . As expected, greater time ranges required a greater number of Taylor series terms. The number of terms required to calculate a period was found to be higher for small time constants, but ultimately converged to a steady number as the constant value increased. A select number of these time constants are depicted in figure 5.

As expected the lowest time constants create the greatest short term damping, as the relaxation decays rapidly. In table 4, the number of required terms for any given Prony relaxation constant, s_1 , appears to continue linearly, approximately following the trend of 18 additional Taylor terms per additional period of oscillation. The relationship is depicted in figure 6. However, these results and projection should be viewed only as a notional result for two primary reasons. The first, is that the residual which determines convergence in Taylor series expansions will change depending on the physical properties of the system. The second is that the relationship was seen to break down sharply past 100 terms. More analysis is needed to test the limits of a single expansion point and to understand where further expansion points would be advantageous for extending solution time. This would best be done with a more complex and physically meaningful system in mind.

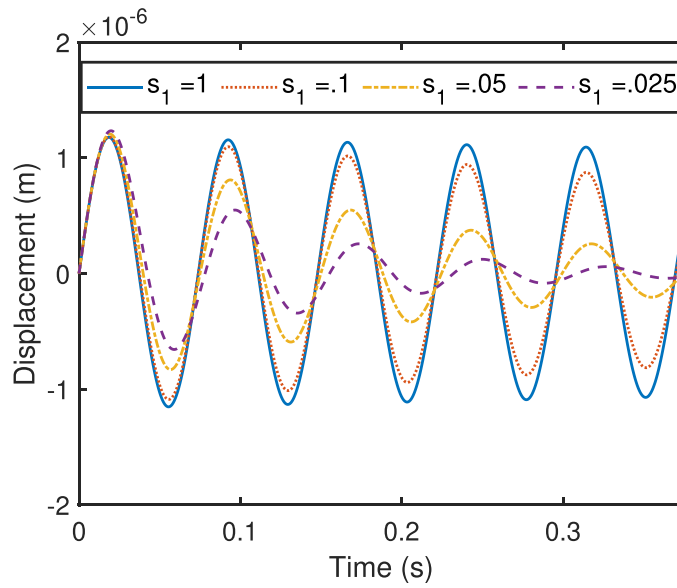


Figure 5. Displacement results for selected single Prony term constants.

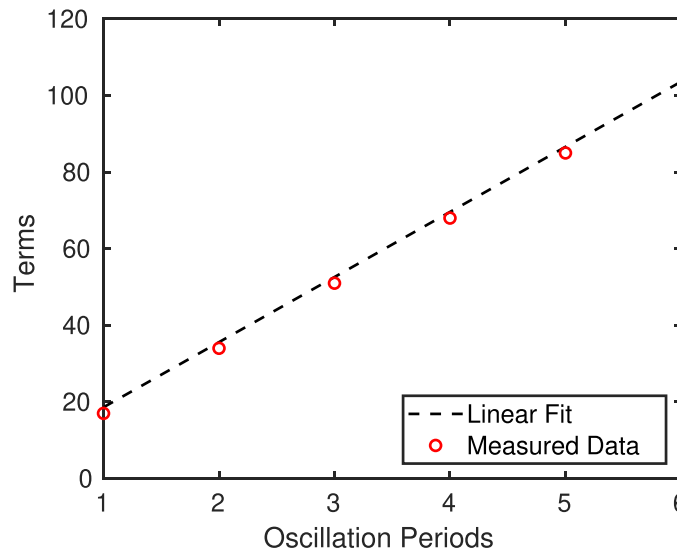


Figure 6. Projections of required terms for increased periods of oscillation.

While the number of terms required per oscillations in a more complex study may be dependent on the physical system, the numerical floating point operation (flop) count for a given number of terms is easier to evaluate. To assess the numerical demand in flop count of the Taylor series expansion versus a traditional Newmark-Beta scheme, three conditions are considered. In all cases flop count will be assessed for a simple one degree of freedom system similar to the one developed above. The first case will be the Taylor series expansion implemented here. The second will be the Newmark-Beta scheme with only simple damping, which

is computationally simpler. The third case will be the Newmark-Beta scheme with the full viscoelastic terms developed in Abercrombie and et al [31]. This is a useful reference point because it provides the complete expressions for viscoelastic forcing at each time step, which is otherwise unavailable. Implementation of the Newmark-Beta method is based on widely published literature [10]. Flop calculations are based on a combination of sources [35, 36].

From Chang [10] we have a description of the iterative calculations for displacement, velocity and acceleration as

$$\begin{aligned} \Delta_{i+1} = & \left[\frac{1}{\beta(\Delta t)^2} m + \frac{\gamma}{\beta(\Delta t)} c + k \right]^{-1} [f_{i+1} \\ & + \left(\frac{m}{\beta(\Delta t)^2} \left(\Delta_i + (\Delta t) \dot{\Delta}_i + \left(\frac{1}{2} - \beta \right) (\Delta t)^2 \ddot{\Delta}_i \right) \right) \\ & - c \left(\dot{\Delta}_i + (1 - \gamma) (\Delta t) \ddot{\Delta}_i \right) + c \left(\frac{\gamma}{\beta(\Delta t)} \right) \left(\Delta_i + (\Delta t) \dot{\Delta}_i + \left(\frac{1}{2} - \beta \right) (\Delta t)^2 \ddot{\Delta}_i \right)], \quad (38) \end{aligned}$$

$$\dot{\Delta}_{i+1} = \left(\frac{1}{\gamma(\Delta t)} m + c \right)^{-1} \left(f_{i+1} - k \Delta_{i+1} + \left(\frac{m}{\gamma(\Delta t)} \right) \left(\dot{\Delta}_i + (1 - \gamma) (\Delta t) \ddot{\Delta}_i \right) \right), \quad (39)$$

and

$$\ddot{\Delta}_{i+1} = \frac{1}{m} \left(f_{i+1} - c \dot{\Delta}_{i+1} - k \Delta_{i+1} \right), \quad (40)$$

respectively. Here, β and γ are coefficients that inform the specific behavior of the numerical method. Practically, γ is included with a value of $\frac{1}{2}$, while β varies based on assumption as further outlined in [10]. The flop count associated with the calculation of displacement is 40, velocity is 15, and acceleration is 6, when m , c , k and Δ are scalar. This means that each Newmark-Beta iteration has a flop count cost of 61 for the case of simple damping.

For the case of viscoelasticity, a damping value is still formed and assumed to be fixed, given a constant Δt , so evaluation of the Newmark-Beta method will still cost 61 flops as in the baseline. However, an additional and expensive treatment of the viscoelastic time integral is needed with each iteration in the form of the operators g_i and b_i , which form the viscoelastic forcing function. For one operator this was defined in Abercrombie and et al [31] for the form of a Prony series as

$$\begin{aligned} \mathbf{g}_i = & hJ_G \int_0^{t_i} \frac{C_{G_1}}{S_{G_1}} e^{-\frac{\tau-t_{i+1}}{S_{G_1}}} \Delta(\tau) d\tau + \frac{C_{G_1} J_G h}{2} \left(2\Delta_i - 2\dot{\Delta}_i t_i + 2\dot{\Delta}_i t_{i+1} - 2\Delta_i e^{-\frac{t_i-t_{i+1}}{S_{G_1}}} \right. \\ & \left. - 2\dot{\Delta}_i S_{G_1} + 2\dot{\Delta}_i S_{G_1} e^{-\frac{t_i-t_{i+1}}{S_{G_1}}} \right) - \frac{C_{G_1} \dot{\Delta}_i J_G h}{2\Delta t} \left(2S_{G_1} t_i - 2S_{G_1} t_{i+1} - 2S_{G_1}^2 e^{-\frac{t_i-t_{i+1}}{S_{G_1}}} \right. \\ & \left. - 2t_i t_{i+1} + 2S_{G_1}^2 + t_i^2 + t_{i+1}^2 \right). \end{aligned}$$

Evaluation of g_i , shear viscoelastic forcing, for one Prony term has a flop count of 100, five times that will be required for five Prony terms and then the same will be needed to calculate b_i , bulk viscoelastic forcing. For execution of an exponential, 20 flops was assumed [36]. This means the additional cost of incorporating viscoelastic forcing in the Newmark-Beta method

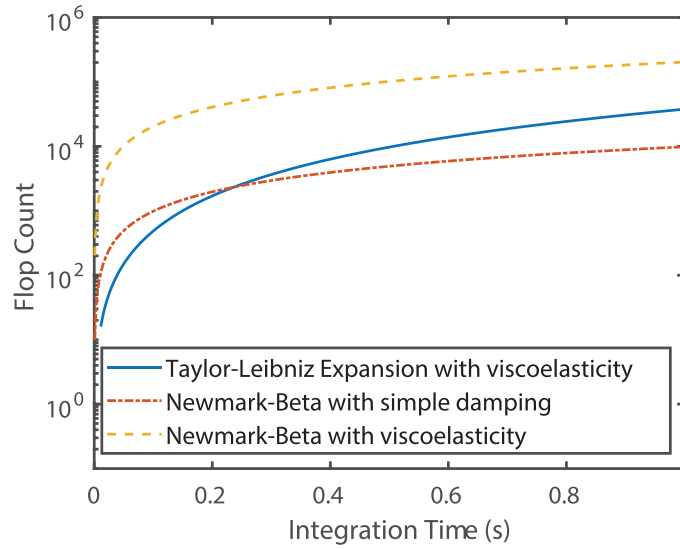


Figure 7. Flop count required for a given integration time using a Taylor–Leibniz expansion with viscoelasticity included, a Newmark-Beta scheme with simple damping, and a Newmark-Beta scheme with viscoelasticity.

with the Maxwell Model is 1200 per timestep. This additional cost is substantial due to the number of operators in g_i , and the high cost of evaluating numerous exponentials.

The Taylor–Leibniz expansion demonstrated here has two primary sources of flop counts during evaluation. The first is the expanding sum in equation (16) and the second is evaluation of the Taylor series in line with equation (30). The computational cost in flops can then be assessed as

$$Flops_{total} = Flops_{es} + Flops_{TS} \quad (42)$$

$$Flops_{es} = 4N - 1 \quad (43)$$

$$Flops_{TS} = Flops_{TS(N-1)} + N \quad (44)$$

where es is used to denote the expanding sum, TS is used to denote the Taylor series, and N is the number of Taylor series terms used.

Unlike the flop count associated with the Newmark-Beta method which grew linearly with each step, the cost of the Taylor–Leibniz grows geometrically with the number of terms used. The cost of the factorial is assumed to be drawn from a stored table and so taken to be free in flop count. With these calculations in mind, it is possible to compare the cost of each approach in flop count as shown in figure 7.

Relating the number of Taylor series terms to integration time was done by consideration of table 4. Examination of figure 7 should be viewed as exemplary of the results of this specific study and not a general study. The time stepping scheme of one tenth of the period of oscillation was adequate in this case, but more refinement is likely needed in more complex systems. Similarly, the cost of the Taylor–Leibniz Expansion does not include the cost of producing a dense number of evaluation points, so the cost likely increases if high resolution is desired, although the same would be true of the Newmark-Beta method. Additionally, the role of the number of degrees of freedom is uncertain and requires further study. Any advantage one

method has over the other in that regard may dominate the calculation as systems of interest become larger and larger. With those conditions acknowledged, the Taylor–Leibniz expansion does appear cheaper than the Newmark–Beta method when considering viscoelasticity in the time domain. Simple damping for the Newmark–Beta scheme is included only as a reference to highlight the costs associated with viscoelasticity in any solver. It is notable that viscoelasticity is numerically expensive compared to simple damping. The geometric increase in flop count observed in the Taylor–Leibniz expansion would eventually equal the Newmark–Beta method for viscoelasticity, but it should be possible to extend integration time by coupling expansion points together. This is discussed later in this work, but would have the added benefit of resetting the number of Taylor terms and thus restarting the geometric increase in flop count. Any benefit in computational cost reduction would then come from successfully coupling expansion points around the inflection point between solver types. In this way, the solver continually operates in the time range after each expansion point for which it is computationally faster than the Newmark–Beta method.

4.3. Comparison with other solver schemes and extension to finite elements

Use of this approach with a finite element formulation may be ideal for increasing the calculation time of short transient responses and even of very large systems. For a finite element calculation, the spring stiffness, k and mass, m , can be replaced by a stiffness matrix, \mathbf{K} and a mass matrix, \mathbf{M} , as was previously shown. The rapid solution time seen in the mass and viscoelastic spring system would of course be slowed down by incorporation of matrix calculations, which could be of any size. However, the improvement for any number of degrees of freedom may still be substantial.

It may seem desirable now to simply compare calculation time in MATLAB to a commercially available tool. However, there are multiple complications in considering relative performance to a commercial solver. In a commercial finite element GUI, it is less clear what contributions in CPU time are based on and some effort may be spent on non-solver operations. Additionally, these two codes would then be programmed in two different languages, which may have preferable or lower speed. Furthermore, the effort is deserving of its own independent consideration on a physical problem to apply real world constraints to the needed solution time. Such a study could even provide guidelines for what type of systems and relaxation behavior is best suited for improved study time with this scheme.

There are substantial differences between the proposed solution scheme and other traditional schemes, such as the Newmark–Beta method and the Runge–Kutta approximation. Of primary interest are accuracy and efficiency. For the purpose of considering the differences, it should for now be assumed that in either solver the same constitutive relation is used, so that fitting error is uniform among solver types. The accuracy of a step-wise integration scheme is then dependent primarily on the number of time points used, with an increasing number of time points yielding a more accurate interpretation. Such step-wise integration schemes are known to exhibit numerical overshoot [10], particularly in early time. The error arising from each step compounds over time as each point influences the next. In a Taylor series expansion, error is expressed differently. Each time point is evaluated entirely independently of the previous or next, so the error is not based on how many time points are evaluated. While it is true that each Taylor term is associated with an order of the approximation, the order does not present as a level of global inaccuracy for all time points. Instead, early time points are highly accurate, as can be seen in the relaxation test in figure 4, while all time points past a certain time are completely inaccurate. To summarize, a Newmark–Beta scheme with a fixed time step can be used to represent solutions over any length of time with a degree of accuracy based on

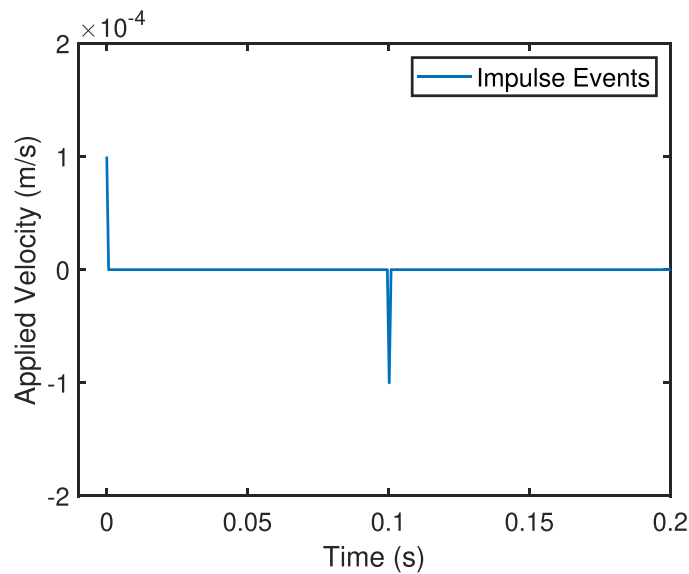


Figure 8. Imposed velocities on a viscoelastic mass-spring system.

the acceleration assumption and time step duration. Each Taylor–Leibniz expansion provides a very accurate approximation over a short time range defined by the number of Taylor terms.

Due to these differences in evaluating time, comparison of efficiency is less clear than may be expected.

The number of time points solved for has a minimum in the case of the Newmark-Beta scheme, while the Taylor–Leibniz expansion can be calculated for only those times of interest to the analyst. Additionally, each Taylor–Leibniz expansion as elucidated here presents the displacement result. It takes additional effort to produce final results for other variables. Ultimately, the relative efficiency is defined in part by the needs of the user.

The introduction highlighted the broad range of studies that incorporate viscoelasticity. The fact that step-wise integration schemes are so common means that the number of potential applications that have incorporated such a scheme is likely extensive. This approach similarly should be just as applicable. Further work and utilization will be the more definitive measure of where the benefits of this approach are felt most.

4.4. Accommodating multiple time events with linear superposition

The authors envision the approach may be particularly beneficial when considering short transient events, such as shock. However, such studies may have multiple events effecting motion in the system and the formulation so far has revolved around a single expansion point. If a study had two discrete shock events in time as depicted in figure 8, the second event would not appear in the initial conditions at time $t = 0$ and be invisible to the study.

This condition can be accommodated in the approach by a superposition of solutions at two expansion points, in this case at $t = 0$ and $t = 0.1$ seconds. Employing this for the system described above with all five Prony terms, the superposition system displacement can still be evaluated as shown in figure 9. In this way two discrete Dirac impulses have been included in the solution of the system.

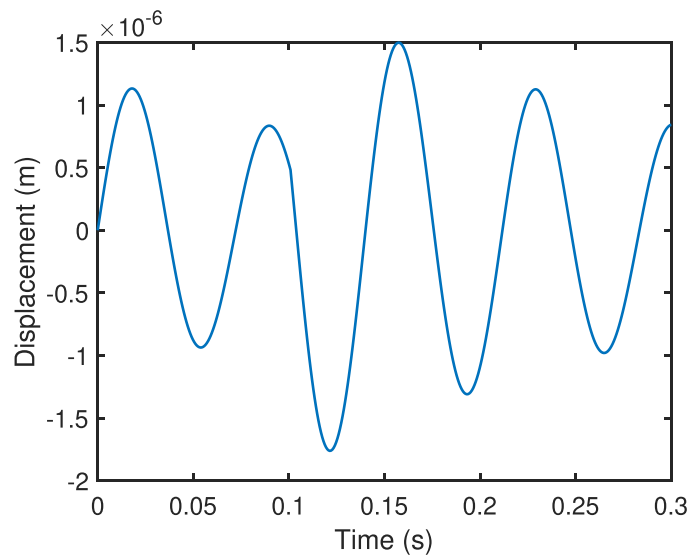


Figure 9. Superposition of a viscoelastic mass-spring system subject to two shock events.

4.5. Extension of solution time with further expansion points

The use of additional expansion points has already been discussed for the purpose of handling multiple shock events. However, it may also be desirable to incorporate multiple expansion points to create longer time studies while minimizing computational costs. A Maclaurin series is limited by the number of terms required to represent further and further oscillations. However, the result can be extended by incorporating an expansion of the Taylor series around a new time, t . In effect, this new expansion point may be best incorporated by still treating it as a time $t = 0$, but using the solution in the first expansion as initial conditions in the next. Optimization of expansion point selection in reference to time convergence may prove to be the key to implementing this approach as a generalized analysis tool. However, such work is extensive and worthy of a future effort. Figure 10 is an example of the extension of solution time using an additional expansion point. The Taylor–Leibniz solution in figure (10) was again implemented in MATLAB, while benchmarking was done using the ABAQUS explicit solver and a single bar element. The mass matrix relationship with density was used again to create an equivalent system out of a bar with the equivalent mass established by matrix output from ABAQUS. The ABAQUS explicit solver was once again implemented with the default numerical dissipation. The parameters of the study are listed in table 5. The material values are arbitrarily selected and not necessarily indicative of a rubber, so the Snowdon formula has not been used.

5. Conclusion

A formulation for a Taylor series expression of the solution of a viscoelastic system has been provided that was enabled by incorporation of the Leibniz integration rule and expansion around a single point in time. An explanation of how a Taylor series can be used to express the solution of an ODE was provided and then refined for the case of a viscoelastic material.

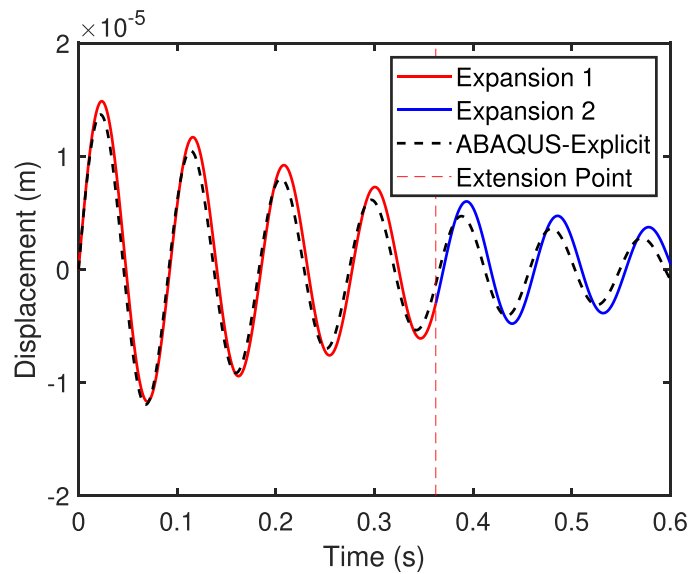


Figure 10. Extension of solution time using an additional expansion point with a numerical accuracy benchmark using an ABAQUS explicit solver.

Table 5. Model parameter and solver setting comparison for figure 10 with the Taylor–Leibniz expander left and the ABAQUS explicit solver [8] right.

Parameter	Taylor–Leibniz	Bar parameter	ABAQUS
Spring length	1 m	Bar length	1 m
Area	1 m ²	Bar area	1 m ²
Density	—	Density	1500 kg m ⁻³
Mass	750 kg	Nodal mass	750 kg
Displacement (x_0)	0	Displacement (x_0)	0
Velocity (\dot{x}_0)	.001	Velocity (\dot{x}_0)	.001
Time constant (s_1)	.1	Time constant (s_1)	.1
Prony constant (C_1)	.4947	Bulk constant (C_1)	.4947
Shear constant (C_1)	—	Shear constant (C_1)	0
Modulus (E_0)	3.5 MPa	Modulus (E_0)	3.5 MPa
Poisson	—	Poisson	.47
Solver	MATLAB	Solver	Explicit
Time increment	—	Time increment	.001 s
Boundary	Grounded	Boundary	Fixed Node
Step end time	—	Step end time	1 s
Elements	1	Elements	1

Limitations of the approach were considered and a parametric sweep was performed to define the number of Taylor series terms needed to calculate oscillation periods for different material profiles. The capability to expand the formulation to allow finite elements analysis, longer time ranges, and multiple discrete impulses was developed.

The long term impact of the approach may be dominated by the solution speed up time it could provide to transient finite element calculations. A substantial additional effort is needed

to understand that impact and an effort is already underway. However, the methodology here can be used to consider any number of systems that can be defined by a differential equation and so constitutes a potentially valuable tool to any number of the many viscoelastic studies done every day. It may also be valuable in the development of specialized tools for individuals and institutions regularly working with these materials.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

Work supported by ONR under Award Number N00014-22-1-2785.

ORCID iD

Eric Abercrombie  <https://orcid.org/0009-0003-5884-0037>

References

- [1] Ding S, Yiqun H, Jian B, Zhang Y, Xia R and Guoming H 2024 A semi-analytical solution to incipient sliding contact on viscoelastic layer-elastic substrate with imperfectly bonded interface *Appl. Math. Modelling* **125** 35–58
- [2] Ding S, Jian B, Zhang Y, Xia R and Guoming H 2023 A normal contact force model for viscoelastic bodies and its finite element modeling verification *Mech. Mach. Theory* **181** 105202
- [3] Van Thom D, Minh Chinh V, Van Minh P and Dinh Anh Vu N 2024 Mechanical responses of nanoplates resting on viscoelastic foundations in multi-physical environments *Eur. J. Mech. A* **106** 105309
- [4] Hadzipasic M, Zhang S, Huang Z, Passaro R, Sten M S, Shankar G M and Nia H T 2024 Emergence of nanoscale viscoelasticity from single cancer cells to established tumors *Biomaterials* **305** 122431
- [5] Morteza Seyedpour S, Azhdari M, Lambers L, Ricken T and Rezazadeh G 2024 One-dimensional thermomechanical bio-heating analysis of viscoelastic tissue to laser radiation shapes *Int. J. Heat Mass Transfer* **218** 124747
- [6] Belloch Paul and Eric Austin 2020 *Special Topics in Structural Dynamics and Experimental Techniques, Volume 5 of Proc. 38th IMAC, A Conf. and Exposition on Structural Dynamics 2020 (Cham, Switzerland)*, ed S E David (The Society for Experimental Mechanics, Springer Nature Switzerland AG) pp 89–97
- [7] Cortés F, Brun M and Jesús Elejabarrieta M 2023 A finite element formulation for the transient response of free layer damping plates including fractional derivatives *Compos. Struct.* **282** 107039
- [8] Dassault Systèmes Abaqus Theory Manual. Dassault Systèmes, France, 6.11 edn 2011 ([www.3ds.com/+/](http://www.3ds.com/))
- [9] Sandia. Sierra/SD - Theory Manual Sandia National Laboratories Albuquerque, New Mexico 4.58 edn 2003 (www.sandia.gov)
- [10] Chang S-Y 2004 Studies of Newmark method for solving nonlinear systems: (1) basic analysis *J. Chin. Inst. Eng.* **27** 651–62
- [11] Rivière B 2003 Discontinuous Galerkin finite element methods for linear elasticity and quasistatic linear viscoelasticity *Numer. Math.* **95** 347–76
- [12] Idesman A, Niekamp R and Stein E 2000 Continuous and discontinuous Galerkin methods with finite elements in space and time for parallel computing of viscoelastic deformation *Comput. Methods Appl. Mech. Eng.* **190** 1049–63

- [13] Saffarian M A, Ahmadi A R and Bagheripour M H 2022 Convergence study of variational space-time coupled least-squares frameworks in simulation of wave propagation in viscoelastic medium *Adv. Civ. Eng.* **1764236**
- [14] Kirchdoerfer T and Ortiz M 2016 Data-driven computational mechanics *Comput. Methods Appl. Mech. Eng.* **304** 81–101
- [15] Eggersmann R, Kirchdoerfer T, Reese S, Stainier L and Ortiz M 2019 Model-free data-driven inelasticity *Comput. Methods Appl. Mech. Eng.* **350** 81–99
- [16] Cunha-Filho A G, Briend Y, de Lima A M G and Donadon M V 2020 A new and efficient constitutive model based on fractional time derivatives for transient analyses of viscoelastic systems *Mech. Syst. Signal Process.* **146** 107042
- [17] Abbasbandy S and Bervillier C 2011 Analytic continuation of Taylor series and the boundary value problems of some nonlinear ordinary differential equations *Appl. Math. Comput.* **218** 2178–99
- [18] Bervillier C, Boisseau B and Giacomini H 2008 Analytical approximation schemes for solving exact renormalization group equations in the local potential approximation *Nucl. Phys. B* **789** 525–51
- [19] Bervillier C 2009 Conformal mappings versus other power series methods for solving ordinary differential equations: illustration on anharmonic oscillators *J. Phys. A: Math. Theor.* **42** 485202
- [20] Leonard D and Mansfield P 2007 Solving the anharmonic oscillator: tuning the boundary condition *J. Phys. A: Math. Theor.* **40** 10291
- [21] Berry D W, Childs A M, Cleve R, Kothari R and Somma R D 2015 Simulating hamiltonian dynamics with a truncated Taylor series *Phys. Rev. Lett.* **114** 090502
- [22] Usha Rani R and Rajendran L 2020 Taylor's series method for solving the nonlinear reaction-diffusion equation in the electroactive polymer film *Chem. Phys. Lett.* **754** 137573
- [23] Clarence Mary L, Usha Rani R, Meena A and Rajendran L 2021 Nonlinear mass transfer at the electrodes with reversible homogeneous; reactions: Taylor's series and hyperbolic function method *Int. J. Electrochem. Sci.* **16** 151037
- [24] La Mon K 1989 Removal of singularities from Taylor series
- [25] Wilhelm Flügge 1967 *Viscoelasticity* (Blaisdell Publishing Company) pp 22–29
- [26] Wineman A 2009 Nonlinear viscoelastic solids-a review *Math. Mech. Solids* **14** 300–66
- [27] Amazigo J C and Rubinfeld L A 1980 *Advanced Calculus and its Applications to the Engineering and Physical Sciences* (Wiley)
- [28] Strang G 1991 *Calculus* vol 1 (SIAM) p 260
- [29] Rivoal T 2009 Rational approximations for values of derivatives of the gamma function *Trans. Am. Math. Soc.* **361** 6115–49
- [30] Wrede R and Spiegel M R 2002 Theory and problems of advanced calculus
- [31] Eric Abercrombie J G M, McDaniel J G and Walsh T 2024 A generalized time-domain constitutive finite element approach for viscoelastic materials *Modelling Simul. Mater. Sci. Eng.* **32** 035028
- [32] Reddy J N 1993 *An Introduction to the Finite Element Method* 2nd edn (McGraw-Hill) chapter 10
- [33] Chu H, Hong T, Chen Q and Wang R 2019 Establishment of rubber thermo-viscoelastic constitutive model and analysis of temperature field *Mater. Sci. Eng.* **531** 012042
- [34] Snowdon J C 1968 *Vibration and Shock in Damped Mechanical Systems* (Wiley)
- [35] Liem A 2021 Analyzing damping in large models of complex dynamic systems *PhD Thesis* Boston University, 110 Cummington Mall Boston, MA
- [36] Moler C and Van Loan C 1978 Nineteen dubious ways to compute the exponential of a matrix *Soc. Indus. Appl. Math.* **20** 4