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Some properties of sets in the plane closed under linear extrapolation by a fixed parameter

Fixed-Parameter Extrapolation and Aperiodic Order

Stephen Fenner[*] University of South Carolina fenner@cse.sc.edu

Frederic Green[†] Clark University fgreen@clarku.edu

Steven Homer[‡] Boston University homer@cs.bu.edu

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Abstract

Fix any $\lambda \in \mathbb{C}$. We say that a set $S \subseteq \mathbb{C}$ is λ-*convex* if, whenever a and b are in S, the point $(1-\lambda)a + \lambda b$ is also in S. If S is also (topologically) closed, then we say that S is λ-*clonvex*. We investigate the properties of λ-convex and λ-clonvex sets and prove a number of facts about them. Letting $R_\lambda \subseteq \mathbb{C}$ be the least λ-clonvex superset of $\{0, 1\}$, we show that if R_λ is convex in the usual sense, then R_λ must be either $[0, 1]$ or $\mathbb{R}$ or $\mathbb{C}$, depending on λ. We investigate which λ make R_λ convex, derive a number of conditions equivalent to R_λ being convex, and give several conditions sufficient for R_λ to be convex or not convex; in particular, we show that R_λ is either convex or uniformly discrete. Letting $\mathcal{C} := \{\lambda \in \mathbb{C} \mid R_\lambda \text{ is convex}\}$, we show that $\mathbb{C} \setminus \mathcal{C}$ is closed, discrete and contains only algebraic integers. We also give a sufficient condition on λ for R_λ and some other related λ-convex sets to be discrete by introducing the notion of a strong PV number. These conditions give rise to a number of periodic and aperiodic Meyer sets (the latter sometimes known as "quasicrystals").

The paper is in four parts. Part I describes basic properties of λ-convex and λ-clonvex sets, including convexity versus uniform discreteness. Part II explores the connections between λ-convex sets and quasicrystals and displays a number of such sets, including several with dihedral symmetry. Part III generalizes a result from Part I about the λ-convex closure of a path, and Part IV contains our conclusions and open problems.

Our work combines elementary concepts and techniques from algebra and plane geometry.

Keywords: discrete geometry, point set, convex, Meyer set, cut-and-project scheme, quasicrystal, aperiodic order, idempotent medial groupoid, mode, a-convex, quasiaddition, quasicrystal addition, τ-inflation

[*]Computer Science and Engineering Department, Columbia, SC 29208 USA. Partially supported by NSF grant CCF-0915948.

[†]Department of Mathematics and Computer Science, Clark University, Worcester, MA 01610

[‡]Computer Science Department, Boston University, Boston, MA 02215. Partially supported by NSF grant CCF-1533663.

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Part I: Introduction and Basic Properties

1 Introduction

Definition 1.1. Fix a number $\lambda \in \mathbb{C}$. For any $a, b \in \mathbb{C}$ define $a \star_\lambda b := (1-\lambda)a + \lambda b$.
Then for any set $S \subseteq \mathbb{C}$,

1. we say that S is λ-*convex* iff for every $a, b \in S$, the point $a \star_\lambda b$ is in S, and

2. we say that S is λ-*convex closed* (or λ-*clonvex* for short) iff S is λ-convex and (topologically) closed.

In either case, we say that S is *nontrivial* if S contains at least two distinct elements. We will informally say, "λ-c[l]onvex" when we want to assert analogous things about both notions, respectively.

For fixed λ, we defined $\star := \star_\lambda$ as a two-place operation on $\mathbb{C}$. We call $a \star b$ the λ-*extrapolant* of a and b, and we say that $a \star b$ is obtained from a and b by λ-*extrapolation*. Then the first property in Definition 1.1 just says that S is closed under λ-extrapolation. Of course, if $0 \le \lambda \le 1$, then this might more appropriately be called λ-interpolation, but as we will see, the case where $\lambda \notin [0,1]$ is much more interesting. When we are not explicit about λ, we refer to the operation $\star_\lambda$ as *fixed-parameter extrapolation* or *fixed-parameter affine combination*.

We may drop the subscript and just say $a \star b$ if the value of λ is clear from the context. We may also drop parentheses in an expression involving $\star$ or $\star_\lambda$, assuming that this operator binds more tightly than $+$ or $-$ but less tightly than multiplication or division.

By the definition of convexity, a set $S \subseteq \mathbb{C}$ is convex if and only if, for all $\lambda \in (0,1)$, S is λ-convex. Definition 1.1 above is in part motivated by the following additional observation (Proposition 2.17, below): If S is a *closed* set, then for *any fixed* $\lambda \in (0,1)$, we have that S is convex if and only if S is λ-convex. We are generally interested in λ-c[l]onvexity for $\lambda \notin [0,1]$, and we are particularly interested in minimal nontrivial λ-c[l]onvex sets.

Definition 1.2. For any $\lambda \in \mathbb{C}$ and any set $S \subseteq \mathbb{C}$,

1. We define the λ-*convex closure* of S, denoted $Q_\lambda(S)$, to be the $\subseteq$-minimum λ-convex superset of S. We let Q_λ be shorthand for $Q_\lambda(\{0,1\})$, the λ-convex closure of $\{0,1\}$.

2. We define the λ-*clonvex closure* of S, denoted $R_\lambda(S)$, to be the $\subseteq$-minimum λ-clonvex superset of S. We let R_λ be shorthand for $R_\lambda(\{0,1\})$, the λ-clonvex closure of $\{0,1\}$.

R_λ is a minimal nontrivial λ-clonvex set because it is generated by just two distinct points. We choose the points 0 and 1 for convenience, but since λ-c[l]onvexity is invariant under orientation-preserving similarity transformations (i.e., $\mathbb{C}$-affine transformations, i.e., polynomials of degree 1; see Definition 2.6, below), any two initial points would yield a set with the same essential properties. One of our main goals, then, is to characterize R_λ for as many λ as we can.

We conclude this introduction with some historical background and motivation. The notion of fixed-parameter extrapolation was first investigated for its own intrinsic interest by Calvert [5] and, some years later, by Pinch [30], under the name of "a-convexity." Berman & Moody [4] were the first to notice its application to discrete Q_λ in the context of *quasicrystals*, where they call it "quasicrystal addition," specifically in the case of $\lambda = 1 + \varphi$ (where φ is the golden ratio), which we cover in this paper as well. This value of λ is significant because it gives the simplest example where Q_λ is discrete and aperiodic. Other authors have investigated fixed-parameter extrapolation under various other names, including quasiaddition and τ-inflation [22], [23], [24].

The comparatively young field of *aperiodic order* [3] arose, in part, to explain such mathematical phenomena as aperiodic tilings and natural phenomena such as quasicrystals. Traditionally, aperiodic order has been studied in terms of paradigms such as local substitution, inflation tilings, and model sets (i.e., cut-and-project sets). As Berman and Moody [4] have pointed out, and as investigated further by other authors [22], [23], and [24], fixed-parameter extrapolation offers an alternative approach. Indeed, as is hinted in [4] and stated explicitly in [23], one may view the binary extrapolation operation as a mathematical model of aperiodic crystal growth. The discrete sets generated by fixed-parameter extrapolation share many properties with *Meyer sets* [26], widely regarded as the mathematical counterpart of quasicrystals. Indeed, we can establish in many cases that these sets *are* Meyer sets.

The goals of this paper are threefold. First, we work towards a systematic and unified theory of fixed-parameter extrapolation and the point sets closed under that operation. We characterize convex sets closed under the operation, although many open questions remain in this regard. Secondly, we extend aspects of the theory already accomplished in previous work (notably [4], [5], [30], [22], [23], and [24]) from the case of *real* λ to *complex* λ. To our knowledge, our work is the first to examine $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In this context, we study precise connections between fixed-parameter extrapolation and other constructions such as inflation tilings and model sets. For example, we give a partial characterization of parameters that lead to discrete sets, in terms of a refinement of Pisot-Vijayaraghavan (PV) numbers, which we call *strong PV* numbers. These generalize the subset of real irrationals considered in [30]. All of the sets we find to be discrete are in fact subsets of cut-and-project sets. It remains to be established if they are also Meyers sets, although we have been able to establish this in many cases. Finally, and along somewhat different lines, although the sets we construct are manifestly hierarchical (with the fixed parameter playing a role analogous to the inflation scale), the relationship between our approach and inflation tilings is at present unclear. Ultimately, we would like to understand what added insight our particular approach offers to the theory of quasicrystals. For that purpose, we strive in this paper to classify these sets according to a number of their properties, including but not limited to aperiodicity, uniform discreteness, relative density, finite local complexity, and repetivity.

The techniques of this paper draw on diverse fields, including complex analysis, algebra, algebraic number theory, topology, combinatorics, and computer science. In support of theoretical studies, computation (including symbolic computation) and computer graphics have been used to guide our work and determine future problems and directions.

This paper is divided into four parts. Part I treats the basic definitions and properties of λ-convexity and fixed-parameter extrapolation. Here we include various characterizations of λ-clonvex sets and criteria for determining when Q_λ is or is not discrete. Part II investigates the connection between fixed-parameter extrapolation and aperiodic order. The central result of Part II is a sufficient condition for discreteness of $Q_\lambda(S)$ for certain finite $S \subseteq \mathbb{C}$. We also determine the λ-convex closure of various regular shapes, in both two and three dimensions, and explore particular values of λ that yield discrete sets that are also relatively dense. Part III generalizes one of the characterizations of Part I, Section 3, from differentiable paths to what we call "bent paths," which are not required to be differentiable. Finally, in Part IV we present concluding remarks and open problems.

2 Basics

We start with a few basic facts and definitions. In this paper, we call a theorem a "Fact" when it is either immediately obvious or has a routine, straightforward proof. We omit the proofs of Facts.

For $z \in \mathbb{C}$, we let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of z, respectively, and we let z^* denote the complex conjugate of z.

If $f : X \to Y$ is some function with domain X, and A is any subset of X, then we let $f|A$ denote the function f restricted to domain A.

Any topological references assume the usual topology on $\mathbb{C} \cong \mathbb{R}^2$. For $A \subseteq \mathbb{C}$, we let $\overline{A}$ denote the topological closure of A.

We use the symbol $:=$ to mean, "equals by definition." We set $\tau := 2\pi$ throughout. For any $x \in \mathbb{R}$, let $x \bmod \tau$ denote the unique $y \in [0, \tau)$ such that $(x - y)/\tau$ is an integer.

We let $\mathbb{Z}^+$ denote the set of positive integers.

Whenever a ring is mentioned, it will be assumed to be unital, that is, possessing a multiplicative identity.

Operations on numbers lift to operations on sets of numbers in the usual way. This includes subtraction, and so $S - T = \{x - y \mid x \in S \ \ \& \ \ y \in T\}$, i.e., the Minkowski difference of S and T. We use $S \setminus T$ to denote the relative complement of T in S.

We note the following simple property of the fixed-parameter extrapolation operator.

Fact 2.1. *For all $a, b, c, d, x, y, \lambda \in \mathbb{C}$,*

$$a \star_\lambda a = a \ ,$$
$$(xa + yb) \star_\lambda (xc + yd) = x(a \star_\lambda c) + y(b \star_\lambda d) \ .$$

In particular, setting $x := 1 - \mu$ and $y := \mu$ gives $(a \star_\mu b) \star_\lambda (c \star_\mu d) = (a \star_\lambda c) \star_\mu (b \star_\lambda d)$ for any $\mu \in \mathbb{C}$. If $\mu = \lambda$, then we have the entropic law

$$(a \star_\lambda b) \star_\lambda (c \star_\lambda d) = (a \star_\lambda c) \star_\lambda (b \star_\lambda d) \ .$$

Remark. A groupoid satisfying the identity $(ab)(cd) \approx (ac)(bd)$ is known as a *medial* groupoid. If the operation is also idempotent ($aa \approx a$), then the groupoid is sometimes called a *medial band*, *(groupoid) mode*, or *idempotent medial groupoid*, as well as other names. These structures have been studied extensively in the literature. See, for example, [7, 9, 10, 17]. These two identities are not the only ones universally satisfied by $\star_\lambda$. For example, $(u \star_\lambda (v \star_\lambda w)) \star_\lambda ((x \star_\lambda y) \star_\lambda z) = (u \star_\lambda (x \star_\lambda w)) \star_\lambda ((v \star_\lambda y) \star_\lambda z)$ for all $u, v, w, x, y, z, \lambda \in \mathbb{C}$, and this identity does not follow from idempotence and the entropic law, above. $\square$

Notation 2.2. For the rest of this section, we use $\star$ with no subscript to mean $\star_\lambda$.

We can stratify the set $Q_\lambda(S)$ as follows:

Definition 2.3. For any $\lambda \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, we define $Q_\lambda^{(0)}(S) := S$, and for all integers $n \geq 0$ we inductively define $Q_\lambda^{(n+1)}(S) := \{a \star b \mid a, b \in Q_\lambda^{(n)}(S)\}$. We use $Q_\lambda^{(n)}$ to denote $Q_\lambda^{(n)}(\{0,1\})$.

Fact 2.4. *For any $\lambda \in \mathbb{C}$ and $S \subseteq \mathbb{C}$,*

- *$Q_\lambda^{(n)}(S) \subseteq Q_\lambda^{(n+1)}(S)$ for all integers $n \geq 0$ (noticing that $\star$ is idempotent), and*
- *$\bigcup_{n=0}^\infty Q_\lambda^{(n)}(S) = Q_\lambda(S)$.*
- *If S is countable, then $Q_\lambda(S)$ is countable.*

Definition 2.5. For any $\lambda \in \mathbb{C}$, any $S \subseteq \mathbb{C}$, and any $z \in Q_\lambda(S)$, we define the (λ, S)-*rank* of z to be the least n such that $z \in Q_\lambda^{(n)}(S)$.

Some of our proofs will use induction on the (λ, S)-rank of a point.
In the expression $(1-\lambda)a + \lambda b$, it will sometimes be useful to treat λ as the variable.

Definition 2.6. For any $a, b \in \mathbb{C}$, define the function $\rho_{a,b} : \mathbb{C} \to \mathbb{C}$ by

$$\rho_{a,b}(z) := a \star_z b = (1-z)a + zb$$

for all $z \in \mathbb{C}$.

Fact 2.7. *For all $a, b \in \mathbb{C}$,*

1. *$\rho_{a,b}$ is the unique $\mathbb{C}$-affine map (polynomial of degree ≤ 1 or orientation-preserving similarity transformation) that maps $0 \mapsto a$ and $1 \mapsto b$.*
2. *$\rho_{a,b}$ is continuous.*
3. *If $a \neq b$, then $\rho_{a,b}$ is a bijection (a homeomorphism, in fact), and for all $z \in \mathbb{C}$,*
$$(\rho_{a,b})^{-1}(z) = \frac{z-a}{b-a} .$$
It follows that $(\rho_{a,b})^{-1} = \rho_{x,y}$, where
$$x = \frac{a}{a-b} , \qquad\qquad y = \frac{a-1}{a-b} .$$
4. *For all $x, y \in \mathbb{C}$,*
$$\rho_{a,b} \circ \rho_{x,y} = \rho_{\rho_{a,b}(x), \rho_{a,b}(y)} .$$
Equivalently, we have the following distributive law: for all $z \in \mathbb{C}$,
$$\rho_{a,b}(x \star_z y) = \rho_{a,b}(x) \star_z \rho_{a,b}(y) .$$

Lemma 2.8. *For any $a, b, \lambda \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, if S is λ-convex (respectively, λ-clonvex), then $\rho_{a,b}(S)$ is λ-convex (respectively, λ-clonvex).*

Proof. Suppose S is λ-convex. If $a = b$, then the statement is trivial, so we assume $a \neq b$. Fix any $x, y \in \rho_{a,b}(S)$ and let $u, v \in S$ be such that $x = \rho_{a,b}(u)$ and $y = \rho_{a,b}(v)$. Then

$$x \star y = \rho_{a,b}(u) \star \rho_{a,b}(v) = \rho_{a,b}(u \star v)$$

by the distributive law above. We have $u \star v \in S$ because S is λ-convex; thus $x \star y \in \rho_{a,b}(S)$. This proves that $\rho_{a,b}(S)$ is λ-convex.

If, in addition, S is closed, then so is $\rho_{a,b}(S)$, because $\rho_{a,b}$ is a homeomorphism. This proves that $\rho_{a,b}$ preserves λ-clonvexity as well. $\square$

Fact 2.9. *$Q_\lambda(S) \subseteq R_\lambda(S)$ for all $\lambda \in \mathbb{C}$ and $S \subseteq \mathbb{C}$.*

The next lemma gives a basic relationship between Q_λ and R_λ. Recall that $\overline{A}$ denotes the topological closure of set A.

Lemma 2.10. *For any $\lambda \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, $\overline{Q_\lambda(S)} = R_\lambda(S)$.*

Proof. The $\subseteq$-containment is obvious because $R_\lambda(S)$ is closed and contains $Q_\lambda(S)$. For the $\supseteq$-containment, we just need to show that $\overline{Q_\lambda(S)}$ is λ-convex. This just follows from the continuity of $\star : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$: for all $A, B \subseteq \mathbb{C}$, we have $\overline{A} \star \overline{B} = \star\left(\overline{A} \times \overline{B}\right) = \star\left(\overline{A \times B}\right) \subseteq \overline{\star(A \times B)} = \overline{A \star B}$. Setting $A := B := Q_\lambda(S)$ gives $\overline{Q_\lambda(S)} \star \overline{Q_\lambda(S)} \subseteq \overline{Q_\lambda(S) \star Q_\lambda(S)} \subseteq \overline{Q_\lambda(S)}$. $\square$

Next we give a general lemma from which many of the results of this section follow easily. This lemma will also be used in Part II.

Lemma 2.11. *For all $x, y, \lambda \in \mathbb{C}$ and all $S, T \subseteq \mathbb{C}$,*

$$\begin{aligned} Q_\lambda(xS + yT) &= xQ_\lambda(S) + yQ_\lambda(T) , \\ R_\lambda(xS + yT) &= \overline{xR_\lambda(S) + yR_\lambda(T)} . \end{aligned}$$

Proof. For the first equation of the lemma, we get $\subseteq$ by noticing that the right-hand side includes $xS + yT$ and is λ-convex. To show this latter fact, start with any $a, b \in xQ_\lambda(S) + yQ_\lambda(T)$, let $a_s \in Q_\lambda(S)$ and $a_t \in Q_\lambda(T)$ be such that $a = xa_s + ya_t$, and choose $b_s \in Q_\lambda(S)$ and $b_t \in Q_\lambda(T)$ similarly for b. Then

$$a \star b = (xa_s + ya_t) \star (xb_s + yb_t) = x(a_s \star b_s) + y(a_t \star b_t) \in xQ_\lambda(S) + yQ_\lambda(T) ,$$

the second equation above using Fact 2.1.

To prove $\supseteq$ in the first equation of the lemma, let $a \in Q_\lambda(S)$ and $b \in Q_\lambda(T)$ be arbitrary. Let s be the (λ, S)-rank of a and let t be the (λ, T)-rank of b (cf. Definition 2.5). We show that $xa + yb \in Q_\lambda(xS + yT)$ by induction on $s + t$. If $s + t = 0$, then $a \in S$ and $b \in T$, so $xa + yb \in xS + yT \subseteq Q_\lambda(xS + yT)$. Now suppose $s + t > 0$ and the inclusion holds for all rank sums less than $s + t$. We prove the case where $s > 0$, the case where $t > 0$ being similar. Since $s > 0$, we have $a = a_1 \star a_2$ for some $a_1, a_2 \in Q_\lambda(S)$, both with (λ, S)-rank less than s. Then by Fact 2.1 again,

$$xa + yb = x(a_1 \star a_2) + y(b \star b) = (xa_1 + yb) \star (xa_2 + yb) .$$

By the inductive hypothesis, $xa_1 + yb$ and $xa_2 + yb$ are both in $Q_\lambda(xS + yT)$, and so by λ-convexity, $xa + yb \in Q_\lambda(xS + yT)$.

The second equation follows from the first by taking the closure of both sides and using Lemma 2.10 and the fact that $xC + yD \subseteq x\overline{C} + y\overline{D} \subseteq \overline{xC + yD}$ for any $C, D \subseteq \mathbb{C}$. $\square$

The next lemma helps to justify our arbitrary choice of 0 and 1 in the definitions of Q_λ and R_λ.

Lemma 2.12. *For any $a, b, \lambda \in \mathbb{C}$ and any set $S \subseteq \mathbb{C}$,*

$$\begin{aligned}\rho_{a,b}(Q_\lambda(S)) &= Q_\lambda(\rho_{a,b}(S)) \ ,\\ \rho_{a,b}(R_\lambda(S)) &= R_\lambda(\rho_{a,b}(S)) \ .\end{aligned}$$

In particular, $\rho_{a,b}(R_\lambda)$ is the λ-clonvex closure of $\{a, b\}$.

Proof. Note that $\rho_{a,b}(z) = a + (b-a)z$ for any $z \in \mathbb{C}$. So applying Lemma 2.11, we get

$$\begin{aligned}\rho_{a,b}(Q_\lambda(S)) &= a + (b-a)Q_\lambda(S) = a\{1\} + (b-a)Q_\lambda(S) = aQ_\lambda(\{1\}) + (b-a)Q_\lambda(S)\\ &= Q_\lambda(a\{1\} + (b-a)S) = Q_\lambda(a + (b-a)S) = Q_\lambda(\rho_{a,b}(S)) \ ,\end{aligned}$$

which establishes the first equation. The second equation is obtained by taking the closure of both sides of the first, observing that $\overline{\rho_{a,b}(Q_\lambda(S))} = \rho_{a,b}(\overline{Q_\lambda(S)})$, and using Lemma 2.10. $\square$

The next fact can be seen by noticing that $a \star_\lambda b = b \star_\mu a$ for all $a, b, \lambda \in \mathbb{C}$, where $\mu = 1 - \lambda$.

Fact 2.13. *A set is λ-c[l]onvex if and only if it is $(1-\lambda)$-c[l]onvex. Thus $Q_\lambda(S) = Q_{1-\lambda}(S)$ and $R_\lambda(S) = R_{1-\lambda}(S)$ for any $S \subseteq \mathbb{C}$ and $\lambda \in \mathbb{C}$.*

Fact 2.14. *For any $\lambda \in \mathbb{C}$,*

- *$Q_\lambda(S)^* = Q_{\lambda^*}(S^*)$ and $R_\lambda(S)^* = R_{\lambda^*}(S^*)$ for any $S \subseteq \mathbb{C}$.*
- *In particular, $(Q_\lambda)^* = Q_{\lambda^*}$ and $(R_\lambda)^* = R_{\lambda^*}$.*
- *Thus R_λ is convex if and only if R_{λ^*} is convex.*

The following geometric picture of a, b and $a \star b$ is especially useful for constructions involving $\lambda \in \mathbb{C} \setminus \mathbb{R}$. See Figure 1.

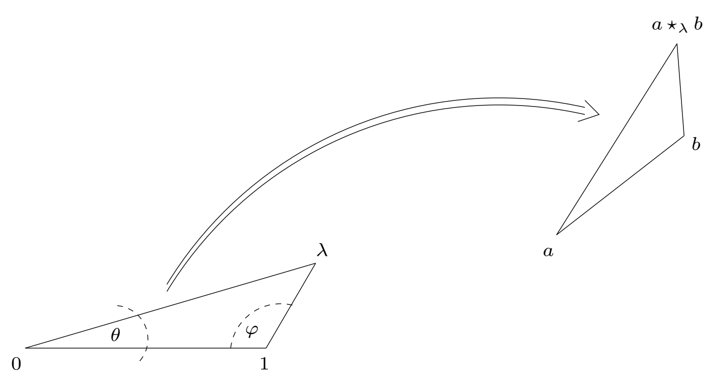


Figure 1: The $\mathbb{C}$-affine transformation that takes $(0, 1, \lambda) \mapsto (a, b, a \star_\lambda b)$ preserves angles, so that the triangles $(0, 1, \lambda)$ and $(a, b, a \star_\lambda b))$ are similar.

Fact 2.15. *By Fact 2.7(1), for any $a, b, \lambda \in \mathbb{C}$, the points a, b, and $a \star b$ form a triangle that is similar to the one formed by 0, 1, and λ.*

Definition 2.16. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we call the angles θ (formed by $(1, 0, \lambda)$) and φ (formed by $(\lambda, 1, 0)$), as indicated in Figure 1, the *characteristic angles* of λ. We assume $0 < \theta, \varphi < \pi$.

Proposition 2.17. *If $S \subseteq \mathbb{C}$ is closed, then for any fixed $\lambda \in (0,1)$, we have that S is convex if and only if S is λ-convex.*

Proof. Clearly, if S is convex, it is λ-convex for any fixed $\lambda \in (0,1)$.

Now suppose S is λ-convex for some fixed $\lambda \in (0,1)$. Let $I := [0,1]$. Then consider two points $a, b \in S$, and the line segment $L := \rho_{a,b}(I) = \{x \mid x = (1-\ell)a + \ell b \text{ for some } \ell \in I\}$, which connects a and b. We now show that S is dense in the set L. This is sufficient for the proposition: Since S is closed by hypothesis, this implies that in fact $L \subseteq S$, from which it follows that S is convex.

Suppose, then, that S is not dense in L. That is, there is some nonempty open[1] subset $O \subseteq L$ that does not intersect with S. O is the unique union of disjoint nonempty open intervals. Let $J := (x, y) \subseteq O$ be one of these intervals, for some $x < y$. Then $x \in S$ and $y \in S$, and by λ-convexity, $x \star y \in S$. But $x < x \star y < y$, and so $x \star y \in S \cap J$, a contradiction. $\square$

Corollary 2.18. *If $T \subseteq \mathbb{C}$ and $0 < \lambda < 1$, then $R_\lambda(T)$ is the closure of the convex hull of T.*

Proof. Let $S := R_\lambda(T)$, and let S' be the (topological) closure of the convex hull of T. We have $T \subseteq S$ and S is closed and λ-convex, whence it follows that S is convex by Proposition 2.17, and thus $S' \subseteq S$. Conversely, the closure of the convex hull of any set is also convex. Thus S' is λ-convex by the same proposition, and this together with the inclusion $T \subseteq S'$ imply $S \subseteq S'$. $\square$

Now we consider the minimal nontrivial λ-clonvex set $R_\lambda = R_\lambda(\{0,1\})$. If R_λ happens to be convex, then characterizing R_λ is easy.

Theorem 2.19. *Suppose R_λ is convex.*

1. *If $\lambda \in [0,1]$, then $R_\lambda = [0,1]$.*
2. *If $\lambda \in \mathbb{R} \setminus [0,1]$, then $R_\lambda = \mathbb{R}$.*
3. *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $R_\lambda = \mathbb{C}$.*

It will be convenient later to define the following:

Definition 2.20. For any $\lambda \in \mathbb{C}$, define

$$F_\lambda = \begin{cases} [0,1] & \text{if } \lambda \in [0,1], \\ \mathbb{R} & \text{if } \lambda \in \mathbb{R} \setminus [0,1], \\ \mathbb{C} & \text{if } \lambda \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

Then Theorem 2.19 states simply that if R_λ is convex, then $R_\lambda = F_\lambda$.

Proof. For (1), we have $[0,1] \subseteq R_\lambda$ by convexity, and if $\lambda \in [0,1]$, then it is obvious that $[0,1]$ is λ-clonvex, since $(1-\lambda)a + \lambda b$ always lies on the line segment connecting a and b. Thus $R_\lambda \subseteq [0,1]$ by the minimality of R_λ.

For (2), we can assume WLOG that $\lambda > 1$ (otherwise consider $1 - \lambda$ and use Fact 2.13). Certainly, $\lambda^0 = 1 \in R_\lambda$, and if $\lambda^n \in R_\lambda$ for some integer $n \geq 0$, then $\lambda^{n+1} = 0 \star \lambda^n \in R_\lambda$ as

[1] with respect to the induced topology on L

well. Thus by induction, $\lambda^n \in R_\lambda$ for all integers $n \geq 0$. Since the sequence $1, \lambda, \lambda^2, \lambda^3, \ldots$ increases without bound, we have $[1, \infty) \subseteq R_\lambda$ by convexity. Similarly, the sequence $1, 1-\lambda, 1-\lambda^2, 1-\lambda^3, \ldots$ lies entirely within R_λ (by induction, if $1 - \lambda^n$ is in R_λ, then so is $1 - \lambda^{n+1} = 1 \star (1 - \lambda^n)$). This latter sequence decreases without bound, and thus $(-\infty, 1] \subseteq R_\lambda$ by convexity.

For (3), we use a trick suggested by George McNulty: we show that R_λ is open, and thus, since R_λ is nonempty and also closed, we must have $R_\lambda = \mathbb{C}$. Since $\lambda \notin \mathbb{R}$, we can represent λ in polar form as $\lambda = re^{i\theta}$, where $r = |\lambda| > 0$, and $\theta = \arg \lambda \in \mathbb{R}$ is not a multiple of π. The value of θ is determined modulo τ, and so we take θ to have the least possible absolute value, giving $0 < |\theta| < \pi$. Now consider any point $a \in R_\lambda$. Since R_λ has at least two points, there is some other point $b \in R_\lambda \setminus \{a\}$. Now define the following sequence of points, all of which are in R_λ:

$$\begin{aligned} b_0 &:= b \ , \\ b_1 &:= a \star b_0 \ , \\ &\ \vdots \\ b_{i+1} &:= a \star b_i \ , \\ &\ \vdots \end{aligned}$$

Set

$$k := \left\lfloor \frac{\pi}{|\theta|} \right\rfloor + 1 \ ,$$

the least integer such that $k|\theta| > \pi$. Then a lies in the interior of the convex hull of $\{b_0, b_1, \ldots, b_k\}$, as illustrated in Figure 2.

Since R_λ is convex, it contains this convex hull, whence a lies in the interior of R_λ. Since $a \in R_\lambda$ was chosen arbitrarily, it follows that R_λ is open. □

In light of Theorem 2.19, most of the rest of the paper concentrates on determining, for various $\lambda \in \mathbb{C}$, whether or not R_λ is convex, and if not, characterizing R_λ. We start with a basic definition followed by a trivial observation.

Definition 2.21. Let $\mathcal{C} := \{\lambda \in \mathbb{C} : R_\lambda \text{ is convex}\}$. Let $\mathcal{D} := \mathbb{C} \setminus \mathcal{C}$.

Fact 2.22.

1. $R_0 = R_1 = Q_0 = Q_1 = \{0, 1\}$, *hence* $0 \notin \mathcal{C}$ *and* $1 \notin \mathcal{C}$.
2. *If* $0 < \lambda < 1$, *then* $R_\lambda = [0, 1]$ *by Proposition 2.17 and Theorem 2.19, hence* $(0, 1) \subseteq \mathcal{C}$.

Lemma 2.23. *For any* $a, b, \lambda \in \mathbb{C}$ *and any* $S \subseteq \mathbb{C}$,

1. *if* $\rho_{a,b}(S) \subseteq Q_\lambda(S)$, *then* $\rho_{a,b}(Q_\lambda(S)) \subseteq Q_\lambda(S)$;
2. *if* $\rho_{a,b}(S) \subseteq R_\lambda(S)$, *then* $\rho_{a,b}(R_\lambda(S)) \subseteq R_\lambda(S)$.

Proof. Using the first assumption and Lemma 2.12, we get

$$\rho_{a,b}(Q_\lambda(S)) = Q_\lambda(\rho_{a,b}(S)) \subseteq Q_\lambda(Q_\lambda(S)) = Q_\lambda(S) \ .$$

Using the second assumption and Lemma 2.12, we get

$$\rho_{a,b}(R_\lambda(S)) = R_\lambda(\rho_{a,b}(S)) \subseteq R_\lambda(R_\lambda(S)) = R_\lambda(S) \ .$$

□

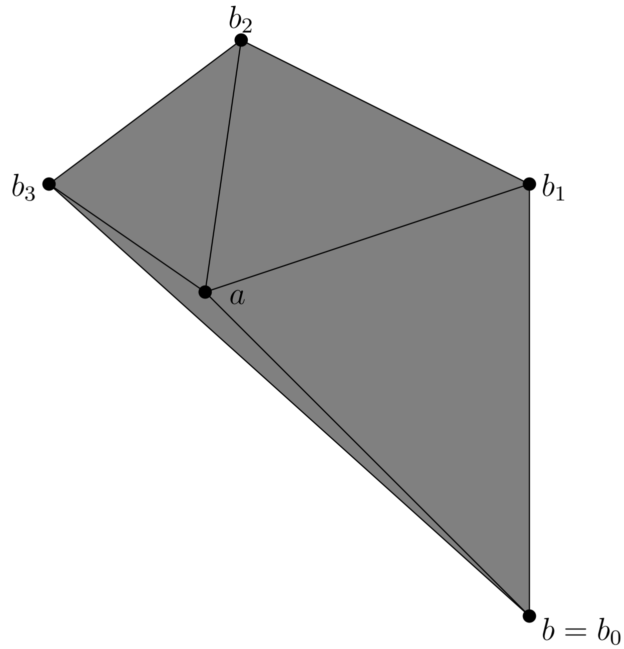


Figure 2: In this example where $\lambda = (1 + 2i)/3$ and $k = 3$, the point a lies in the interior of the convex hull of $\{b_1, b_1, b_2, b_3\}$.

Lemma 2.23 and Lemma 2.25 (below) have some useful corollaries.

Corollary 2.24. *For any $a, b, \lambda \in \mathbb{C}$,*

1. *if $a \in Q_\lambda$ and $b \in Q_\lambda$, then $\rho_{a,b}(Q_\lambda) \subseteq Q_\lambda$;*
2. *if $a \in R_\lambda$ and $b \in R_\lambda$, then $\rho_{a,b}(R_\lambda) \subseteq R_\lambda$.*

Proof. Set $S = \{0, 1\}$ and use Lemma 2.23. □

Part (1.) of the next lemma will be used in Section 16.

Lemma 2.25. *For any $\lambda, \mu \in \mathbb{C}$ and $S \subseteq \mathbb{C}$,*

1. *if $\mu \in Q_\lambda$, then $Q_\lambda(S)$ is μ-convex, and consequently, $Q_\mu(S) \subseteq Q_\lambda(S)$;*
2. *if $\mu \in R_\lambda$, then $R_\lambda(S)$ is μ-clonvex, and consequently, $R_\mu(S) \subseteq R_\lambda(S)$.*

Proof. For part (1.), suppose $\mu \in Q_\lambda$. Then for any $a, b \in Q_\lambda(S)$,

$$a \star_\mu b = \rho_{a,b}(\mu) \in \rho_{a,b}(Q_\lambda) = Q_\lambda(\{a, b\}) \subseteq Q_\lambda(Q_\lambda(S)) = Q_\lambda(S) ,$$

where the second equation follows from Lemma 2.12 with $S = \{0, 1\}$. This shows that $Q_\lambda(S)$ is μ-convex. A similar argument holds for part (2.). □

Corollary 2.26. *For any $\lambda, \mu \in \mathbb{C}$,*

1. *if $\mu \in Q_\lambda$, then Q_λ is μ-convex, and consequently, $Q_\mu \subseteq Q_\lambda$;*

2. *if $\mu \in R_\lambda$, then R_λ is μ-clonvex, and consequently, $R_\mu \subseteq R_\lambda$.*

Corollary 2.27. *For any $\lambda \in \mathbb{C}$, the sets Q_λ and R_λ are both closed under the ternary operation $(\mu, a, b) \mapsto a \star_\mu b$. In particular, Q_λ and R_λ are both closed under multiplication.*

Proof. Given any $\mu, a, b \in Q_\lambda$, we have $a \star_\mu b \in Q_\mu(\{a, b\}) \subseteq Q_\lambda(\{a, b\}) \subseteq Q_\lambda(Q_\lambda) = Q_\lambda$, the first inclusion using part (1.) of Lemma 2.25. Similarly for R_λ, using part (2.) of Lemma 2.25. For closure under multiplication, we notice that $\mu\nu = 0 \star_\mu \nu$ for any $\mu, \nu \in \mathbb{C}$. □

Definition 2.28. For any $S \subseteq \mathbb{C}$, define $1 - S := \{1 - x \mid x \in S\}$ as usual.

Note that $1 - S = \rho_{1,0}(S)$, for any $S \subseteq \mathbb{C}$.

Corollary 2.29. *$R_\lambda = 1 - R_\lambda$ for any $\lambda \in \mathbb{C}$.*

Proof. We have

$$1 - R_\lambda = \rho_{1,0}(R_\lambda) \subseteq R_\lambda = \rho_{1,0}(\rho_{1,0}(R_\lambda)) \subseteq \rho_{1,0}(R_\lambda) = 1 - R_\lambda \ .$$

Both $\subseteq$-steps follow from Corollary 2.24. □

Corollary 2.30. *For any $\lambda, \mu \in \mathbb{C}$, if $\mu \in R_\lambda$ and R_μ is convex, then R_λ is convex.*

Proof. Assume $\mu \in R_\lambda$ and R_μ is convex. To show that R_λ is convex, it suffices to show that for any $a, b \in R_\lambda$ and $x \in [0, 1]$, the point $a \star_x b$ is in R_λ. We have $0, 1 \in R_\mu$, and so by Corollary 2.26 and the convexity of R_μ, we have

$$[0, 1] \subseteq R_\mu \subseteq R_\lambda \ .$$

Thus, for any $x \in [0, 1]$, we have

$$a \star_x b = \rho_{a,b}(x) \in \rho_{a,b}([0, 1]) \subseteq \rho_{a,b}(R_\lambda) \subseteq R_\lambda \ .$$

□

Corollary 2.31. *For any $\lambda \in \mathbb{C}$, $\lambda \in \mathcal{C}$ if and only if $R_\lambda \cap \mathcal{C} \neq \emptyset$.*

Proposition 2.32. *If R_λ is convex, then all λ-clonvex sets are convex.*

Proof. Suppose R_λ is convex, and let A be any λ-clonvex set. For any $a, b \in A$, the line segment connecting a and b is $\rho_{a,b}([0, 1])$. Since R_λ is convex, we have $[0, 1] \subseteq R_\lambda$, and thus

$$\rho_{a,b}([0, 1]) \subseteq \rho_{a,b}(R_\lambda) = R_\lambda(\rho_{a,b}(\{0, 1\})) = R_\lambda(\{a, b\}) \subseteq R_\lambda(A) = A \ .$$

The first equality follows from Lemma 2.12; the last equality holds because A is λ-clonvex. □

3 Equivalent characterizations of convexity for λ-clonvex sets

Throughout this section, we continue to use $\star$ without a subscript to denote $\star_\lambda$.

For any $a, b \in \mathbb{C}$, a *path* from a to b is a continuous function $\sigma : [0,1] \to \mathbb{C}$ such that $\sigma(0) = a$ and $\sigma(1) = b$. If $a = b$, then σ is a *loop*. A set $S \subseteq \mathbb{C}$ is said to be *path-connected* if it contains a path between any two of its points.[2]

In this section we consider five possible properties of a λ-clonvex set and the implications between them. Throughout this section, we will adopt the convention that λ denotes an arbitrary complex number and that A denotes an arbitrary λ-clonvex set containing at least two distinct points. Here are the five properties we will consider:

1. A is convex.
2. A is path-connected.
3. A contains a nontrivial (i.e., nonconstant) path.
4. A has an accumulation point.
5. There exist $a, b \in \mathbb{C}$ such that $0 < |a - b| < 1$ and $\rho_{a,b}(A) \subseteq A$ (i.e., A is self-similar).

In particular, we show (Corollary 3.5, below) that these five properties are all equivalent when $A = R_\lambda$, while some implications do not hold for all λ-clonvex sets. Results similar to some of these below were shown in the case of $\lambda \in \mathbb{R}$ by Pinch [30].

We refer to the above properties by their numbers in parentheses.

Fact 3.1. *For all λ and A subject to this section's convention, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).*

Theorem 3.2. *For all λ and A subject to this section's convention, (1) $\Rightarrow$ (5).*

Proof. Choose any point $x \in A$, and consider the map

$$\psi := \rho_{x,x+1} \circ \rho_{0,1/2} \circ (\rho_{x,x+1})^{-1} .$$

It is easy to check that for any $z \in \mathbb{C}$, $\psi(z)$ is the midpoint $(x+z)/2$ of x and z. Thus $\psi(A) \subseteq A$, because A is assumed to be convex. Using Fact 2.7, we get $\psi = \rho_{a,b}$, where

$$a = \psi(0) = \frac{x}{2} , \qquad\qquad b = \psi(1) = \frac{x}{2} + \frac{1}{2} .$$

We have $|a - b| = 1/2$, which implies (5). $\square$

Theorem 3.3. *For all λ and A subject to this section's convention, (5) $\Rightarrow$ (4).*

Proof. Let a, b be such that $0 < |a - b| < 1$ and $\rho_{a,b}(A) \subseteq A$. Letting $\delta := |a - b|$, we see that for any $x, y \in \mathbb{C}$,

$$\rho_{a,b}(x) - \rho_{a,b}(y) = (1-x)a + xb - (1-y)a - yb = (b-a)(x-y) ,$$

[2] Strictly speaking, as we identify the path with the function σ, it is more accurate to say that S contains all the points in the image of some path connecting the two points. However, we will assume that the meaning will be clear from the context.

and thus

$$|\rho_{a,b}(x) - \rho_{a,b}(y)| = \delta|x - y| \ . \tag{1}$$

It is easy to check that the map $\rho_{a,b}$ on $\mathbb{C}$ has the unique fixed point

$$z := \frac{a}{1 + a - b} \ ,$$

and a routine induction on n using Equation (1) shows that for any $w \in \mathbb{C}$, $\left|z - \rho_{a,b}^{(n)}(w)\right| = \delta^n|z-w|$ for $n = 0, 1, 2, \ldots$. Thus if $z \neq w$, then z is an accumulation point of the sequence

$$w, \ \rho_{a,b}(w), \ \rho_{a,b}(\rho_{a,b}(w)), \ \ldots, \ \rho_{a,b}^{(n)}(w), \ \ldots .$$

If, in addition, $w \in A$ (and there must exist such a w, because A contains at least two points by convention), then all the elements of this sequence are in A by the self-similarity assumption. We then get $z \in A$ by the fact that A is closed. □

Recall the definition of F_λ in Definition 2.20.

Theorem 3.4. *For all λ and A subject to this section's convention and such that $\lambda \notin \{0, 1\}$ and $A \subseteq F_\lambda$, (4) ⇒ (1).*

Proof. We consider the case where $\lambda \in \mathbb{R} \setminus [0, 1]$ first, which was essentially proved by Pinch [30, Proposition 7]. This case is required, but it also gives a simpler version of the proof for when λ is complex. By Proposition 2.17, if $0 < \lambda < 1$, then A is convex, regardless of whether or not $A \subseteq F_\lambda$ (which equals $[0, 1]$ in this case). Therefore—since $\lambda \notin \{0, 1\}$ by assumption—we may assume that $\lambda \notin [0, 1]$, in which case, $A \subseteq F_\lambda = \mathbb{R}$. Since λ-clonvexity is the same as $(1 - \lambda)$-clonvexity by Fact 2.13, we may further assume that $\lambda > 1$.

Now let C be the set of all accumulation points of A, and suppose that $C \neq \emptyset$. Note that $C \subseteq A$, because A is closed. We show for any $x \in \mathbb{R}$ that $x \in C$, from which it follows that $A = \mathbb{R}$. Let $a \in C$ be closest to x among all the elements of C. Such a point a exists, because C is closed and nonempty. If $x = a$, then we are done, so suppose that $x < a$ (there is no essential difference with the case where $a < x$). Then for some sequence $\{a_n\} \subseteq A \setminus \{a\}$, $a = \lim_{n\to\infty} a_n$. Define the sequence $\{b_n\}$ as follows:

$$b_n = \begin{cases} a \star a_n & \text{if } a_n < a, \\ a_n \star a & \text{if } a_n > a. \end{cases}$$

Evidently, $b_n < a$ for all n, and $\lim_{n\to\infty} b_n = a$. Furthermore, for each n, we have $b_n \in C$, because

$$b_n = \begin{cases} \lim_{m\to\infty} a_m \star a_n & \text{if } a_n < a, \\ \lim_{m\to\infty} a_n \star a_m & \text{if } a_n > a, \end{cases}$$

and moreover, for all m, n, $a_m \star a_n \in A$, and $a_m \star a_n \neq b_n$ (respectively, $a_n \star a_m \neq b_n$) if $a_n < a$ (respectively, $a_n > a$). Since b_n converges to a, we find for sufficiently large n that $x < b_n < a$, so b_n is in C and is closer to x than a is, contradicting the hypothesis that a was the closest.

Now suppose that $\lambda \notin \mathbb{R}$, and that A is arbitrary (subject to this section's convention) but contains an accumulation point a. We show that in such a case, $A = \mathbb{C}$. We do this by showing that any point $x \in \mathbb{C}$ is an accumulation point of A, from which the result follows by the fact that

A is closed. The proof is in the same in spirit as the case where $\lambda \in \mathbb{R}$, but here there is no division into cases as to whether x is to the "left" or "right" of a. For ease of illustration, we will assume that $\mathrm{Im}(\lambda) > 0$, as depicted in Fig. 1. The case where $\mathrm{Im}(\lambda) < 0$ is entirely similar.

The proof is by contradiction. Again, let $C \subseteq A$ be the set of all accumulation points of A, and let $b \in C$ be the closest to x of any point in C. Such a point b exists, because the set $\{z \in C : |z - x| \leq |a - x|\}$, as well as being nonempty, is closed and bounded, and hence compact. Now assume for the sake of contradiction that $x \notin A$. Then $x \neq b$. Draw a circle with x as the center and b on the circle. Let D denote the open disk bounded by the circle. We will show now that $C \cap D \neq \emptyset$, which contradicts the hypothesis that b is the closest point in C to x. For ease in visualization, suppose b is at the top of the circle (see Fig. 3).

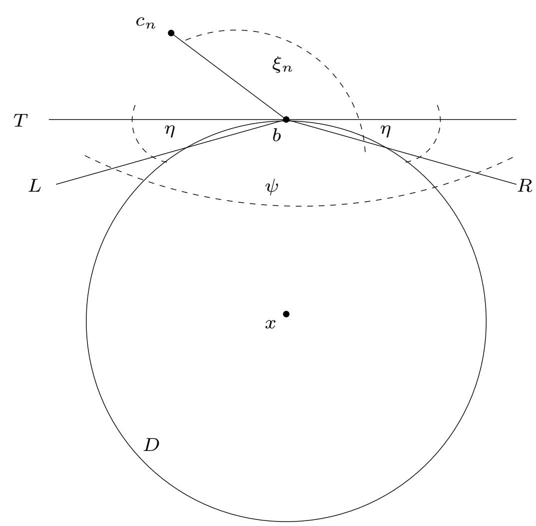


Figure 3: Theorem 3.4: Construction of accumulation point x.

Suppose the sequence $\{b_n\} \subseteq A \setminus \{b\}$ converges to b, that is, $\lim_{n\to\infty} b_n = b$. Let $\{c_n\}$ denote the sequence defined by $c_n = b_n \star b$ for all n. Note that for each n, $c_n \in A \setminus \{b\}$, and moreover, c_n is itself in C, because $c_n \neq b_n \star b_m \in A$ for all m, and

$$\lim_{m\to\infty} b_n \star b_m = \lim_{m\to\infty} ((1-\lambda)b_n + \lambda b_m) = (1-\lambda)b_n + \lambda b = c_n .$$

Thus if any $c_n \in D$, then we are done, but it is possible that $c_n \notin D$ for all n. We therefore show how to "rotate" the sequence $\{c_n\}$ so that it is contained in D for all sufficiently large n. Let T denote the tangent to the circle at b. With θ and φ denoting the characteristic angles of λ (see Definition 2.16 and Fig. 1), let η be any angle obeying $0 < \eta < \min(\theta, \varphi, (\pi - \theta - \varphi)/2)$, and observe that $\eta < \pi/2$. Form the two rays L and R intersecting at b and making an angle η with T as shown in the figure. Let ξ_n denote the counterclockwise angle formed by R and the line segment (b, c_n) connecting c_n with b, also as shown in the figure. Since c_n could be anywhere except b, we have $0 \leq \xi_n < \tau$, where $\xi_n = 0$ corresponds to the ray R. Let ψ denote the angle subtended by L and R; thus $\psi = \pi - 2\eta$. Note that by the choice of η, we have $\varphi < \theta + \varphi < \pi - 2\eta = \psi$. Our goal now is to find an accumulation point below the lines L and R, and inside D.

To do this, let q_n denote the least integer such that $q_n\varphi > \xi_n$. Then $q_n\varphi = (q_n - 1)\varphi + \varphi \leq \xi_n + \varphi < \xi_n + \psi$. Thus $\xi_n < q_n\varphi < \xi_n + \psi$, so that the angle $q_n\varphi$ takes us from the line segment

(b, c_n) *clockwise* to a ray through b, below T, and strictly between L and R. Note that since $\xi_n < \tau$, $0 \le q_n - 1 \le \lfloor \tau/\varphi \rfloor$, so for all n, q_n can only take on a finite number of values, independent of n.

Next, for each n, form the finite sequence $c_n^{(0)}, c_n^{(1)}, \ldots, c_n^{(q_n)}$, where

$$\begin{aligned} c_n^{(0)} &= c_n \\ c_n^{(1)} &= c_n^{(0)} \star b \\ &\vdots \\ c_n^{(i+1)} &= c_n^{(i)} \star b \\ &\vdots \end{aligned}$$

(This is essentially the same construction as in Theorem 2.19. Also note Fig. 2, although it is not necessary here to form a convex hull.) Each $c_n^{(i)}$ is in A, similarly to c_n. By the definition of φ, for each i, the clockwise angle going from the line segment $(b, c_n^{(i)})$ to the line segment $(b, c_n^{(i+1)})$ is φ. Thus the clockwise angle from (b, c_n) to $(b, c_n^{(q_n)})$ is $q_n\varphi$. Hence, the point $c_n^{(q_n)}$ is in the desired wedge-shaped region beneath L and R. However, it may be too far from b to be in the interior of D. Now observe that, by virtue of the fact that $c_n^{(i+1)}$ is always constructed from b and $c_n^{(i)}$ via similar triangles, there is a constant k such that $|b - c_n^{(i+1)}| \le k|b - c_n^{(i)}|$. Thus $|b - c_n^{(q_n)}| \le k^{q_n}|b - c_n|$. But since $\{c_n\}$ converges to b, for any $\epsilon > 0$, there exists an m such that $|b - c_m| \le \epsilon/k^{\lfloor 2\pi/\varphi \rfloor + 1} \le \epsilon/k^{q_m}$. In that case, $|b - c_m^{(q_m)}| \le \epsilon$, so ϵ may be chosen sufficiently small that $c_m^{(q_m)}$ is contained in D. (And indeed, the sequence $\{c_n^{(q_n)}\}$ converges to b.) $\square$

Some kind of constraint on λ and A in Theorem 3.4—beyond this section's convention—is necessary to obtain the implication (4) $\Rightarrow$ (1). For example, if $\lambda \in \{0, 1\}$, then any closed subset of $\mathbb{C}$ is λ-clonvex, and so we may take $A := \{0\} \cup \{1/n : n \in \mathbb{Z}^+\}$, which has 0 as an accumulation point but is not convex. If $\lambda \in \mathbb{R} \setminus [0, 1]$ but $A \not\subseteq \mathbb{R}$, then the implication still holds provided either A lies entirely on a single line or A contains a nonempty open set (cf. Proposition 3.8, below). Otherwise, the implication may not hold: let $\lambda := 2$ and consider the set $A := R_2(\{0, 1, \sqrt{2}, i\})$. Then it is a short exercise to show that

$$A = \{x + yi \mid x \in \mathbb{R} \ \& \ y \in \mathbb{Z}\},$$

which has accumulation points (paths, in fact) but is not convex.

Property (5) of the next corollary provides a useful shortcut for proving that R_λ is convex. Pinch essentially proved for $\lambda \in \mathbb{R}$ that (4) $\Rightarrow$ (1) [30, Propositions 5,7] and that (5) $\Rightarrow$ (4) [30, Proposition 10].

Corollary 3.5. *For any $\lambda \in \mathbb{C}$, the following are equivalent:*

1. *R_λ is convex.*
2. *R_λ is path-connected.*
3. *R_λ contains a path.*
4. *R_λ has an accumulation point.*

5. *There exist $a, b \in R_\lambda$ such that $0 < |a - b| < 1$.*

Proof. If $A = R_\lambda$, then we merely note that the property (5) of Corollary 3.5 is equivalent to the property (5) given earlier in this section: if $a, b \in R_\lambda$, then $\rho_{a,b}(R_\lambda) \subseteq R_\lambda$ by Corollary 2.24. Conversely, if $\rho_{a,b}(R_\lambda) \subseteq R_\lambda$, then $\{a, b\} = \rho_{a,b}(\{0,1\}) \subseteq \rho_{a,b}(R_\lambda) \subseteq R_\lambda$. □

Corollary 3.5 presents a nice dichotomy: R_λ is either convex (hence either $[0,1]$, $\mathbb{R}$, or $\mathbb{C}$) or uniformly discrete (with no two points less than unit distance apart). The former holds when $\lambda \in \mathcal{C}$; the latter when $\lambda \in \mathcal{D}$.

We end this section with some basic facts about $Q_\lambda(S)$ for certain λ and S. First we show that, given any disk $D \subseteq \mathbb{C}$ and any $\lambda \in \mathbb{C} \setminus [0,1]$, we can construct a larger concentric disk $D' \subseteq Q_\lambda(D)$ (analogous to the construction of successively larger intervals in the case $\lambda \in \mathbb{R}$). From this it follows immediately that $Q_\lambda(D) = \mathbb{C}$ (Corollary 3.7, below).

Lemma 3.6. *Fix $\lambda \in \mathbb{C}$ and let $\delta := |\lambda| + |1 - \lambda|$. For any $x \in \mathbb{C}$ and $r \geq 0$, let $D_{x,r} := \{z \in \mathbb{C} : |z - x| \leq r\}$ be the closed disk of radius r centered at x. Then $Q_\lambda^{(1)}(D_{x,r}) = D_{x,\delta r}$.*

Proof. For any $a, b \in D_{x,r}$, we have $|a \star b - x| = |(a - x) \star (b - x)| \leq |1 - \lambda||a - x| + |\lambda||b - x| \leq \delta r$, and thus $Q_\lambda^{(1)}(D_{x,r}) \subseteq D_{x,\delta r}$. For the reverse inclusion, pick any $y \in D_{x,\delta r}$ (and so $|y - x| \leq \delta r$). We can assume $\lambda \notin \{0, 1\}$, for otherwise the result is trivial. Define

$$a := x + (y - x)\frac{|1 - \lambda|}{\delta(1 - \lambda)}\ , \qquad\qquad b := x + (y - x)\frac{|\lambda|}{\delta\lambda}\ .$$

One readily checks that $|a - x| \leq r$ and $|b - x| \leq r$ (so $a, b \in D_{x,r}$) and that $y = a \star b$. □

Corollary 3.7. *Let $D_{x,r}$ be defined as in Lemma 3.6. If $r > 0$ and $\lambda \notin [0,1]$, then $Q_\lambda(D_{x,r}) = \mathbb{C}$.*

Proof. We have $\delta = |1 - \lambda| + |\lambda| > 1$ and $Q_\lambda^{(n)} = D_{x,\delta^n r}$ for all n. □

Thus we have the following:

Proposition 3.8. *Fix any $\lambda \in \mathbb{C} \setminus [0,1]$ and $B \subseteq \mathbb{C}$.*

1. *If B includes a nonempty open subset of $\mathbb{C}$, then $Q_\lambda(B) = R_\lambda(B) = \mathbb{C}$.*

2. *If B includes a nonempty open subset of $\mathbb{R}$, then $F_\lambda \subseteq Q_\lambda(B)$.*

Proof. Part (1.) follows from Corollary 3.7. For Part (2.), we have two cases: (i) $\lambda \in \mathbb{R}$ and $F_\lambda = \mathbb{R}$; and (ii) $\lambda \notin \mathbb{R}$ and $F_\lambda = \mathbb{C}$. In case (i), we apply a one-dimensional version of the disk expansion construction above to expand any interval $[a, b] \subseteq Q_\lambda(B)$ to a larger interval $[c, d] \subseteq Q_\lambda(B)$, where (assuming $\lambda > 1$ without loss of generality) $c = b \star a$ and $d = a \star b$. Note that for every point $z \in [c, d]$ there exist $x, y \in [a, b]$ such that $z = x \star y$. The expansion is by a factor of $2\lambda - 1 > 1$. Applying the expansion repeatedly puts all of $\mathbb{R}$ into $Q_\lambda(B)$. For case (ii), if we start with some interval $[a, b] \subseteq B$, then the entire triangle formed by a, b, and $a \star b$ and its interior lies in $Q_\lambda(B)$. Indeed, for any point z inside this triangle, there exist $x, y \in [a, b]$ such that $z = x \star y$, as shown in Fig. 4. Then applying Part (1.) to $Q_\lambda(B)$ gives $Q_\lambda(Q_\lambda(B)) = Q_\lambda(B) = \mathbb{C}$. □

The proof above can easily be generalized to show that if B includes a differentiable path in $\mathbb{C}$, then $Q_\lambda(B) = \mathbb{C}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In fact, we can prove something much stronger:

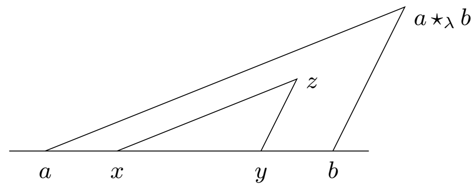


Figure 4: From the interval $[a, b] \subseteq B$ we get a triangle in $Q_\lambda(B)$ with a nonempty interior.

Theorem 3.9. *If d is any path in $\mathbb{C}$ that is not contained in a straight line, then $Q_\lambda(d) = R_\lambda(d) = \mathbb{C}$ for all $\lambda \in \mathbb{C} \setminus [0, 1]$.*

In Theorem 3.9, we do not require d to be differentiable, or even simple; we only require that d be continuous. We defer the proof of this theorem until Section 16.

4 Finding λ such that R_λ is convex

As in previous sections, we continue to use $\star$ without subscript to denote $\star_\lambda$.

Corollary 3.5 itself has two useful corollaries:

Corollary 4.1. *R_λ is convex for any $\lambda \in \mathbb{C}$ such that either $0 < |\lambda| < 1$ or $0 < |1 - \lambda| < 1$.*

Corollary 4.2. *For any $\lambda \in \mathbb{C}$, R_λ is convex if and only if there exists $\mu \in R_\lambda$ such that either $0 < |\mu| < 1$ or $0 < |1 - \mu| < 1$.*

Proof. The forward implication is obvious, since $[0, 1] \subseteq R_\lambda$ if R_λ is convex. For the converse, we have R_μ is convex by Corollary 4.1, whence R_λ is convex by Corollary 2.30. $\square$

The proof of Corollary 3.5 is the last place where we explicitly use the fact that R_λ is closed. In fact, all convexity arguments for R_λ from now on can follow directly or indirectly from Corollaries 2.30 and 4.2, or alternatively from Corollary 3.5. They now allow us to expand our set of λ such that R_λ is known to be convex.

Proposition 4.3. *If $|\lambda| = 1$ and λ is neither a fourth nor a sixth root of unity, then R_λ is convex.*

Proof. Let $\lambda \in \mathbb{C}$ be any point on the unit circle. Write $\lambda = x + iy$ for real x, y such that $x^2 + y^2 = 1$. The following point is evidently in R_λ:

$$\mu := 1 \star \lambda = \lambda^2 - \lambda + 1 = (x^2 - y^2 - x + 1) + (2xy - y)i = (2x^2 - x) + y(2x - 1)i = (2x - 1)\lambda \,.$$

Thus $|\mu| = |2x - 1|$. If $0 < x < 1$, then $|\mu| < 1$, and moreover, $0 < |\mu|$ if $x \neq 1/2$. Corollary 4.1 then implies that R_μ is convex (and thus R_λ is convex) for all $x \in (0, 1) \setminus \{1/2\}$, which proves the current proposition when $x \geq 0$. (The cases where $x \in \{0, 1/2, 1\}$ correspond to λ being a fourth or a sixth root of unity.)

Now assume $x < 0$. It is easy to check (on geometric grounds alone) that either λ^2 or λ^3 has positive real part, and so, provided λ^2 (respectively λ^3) is not a sixth root of unity, we have R_{λ^2} (respectively R_{λ^3}) is convex by the argument in the previous paragraph, and hence R_λ is convex. Thus the only cases left to show are where: (1) λ is neither a fourth nor a sixth root of unity; but

(2) λ^2 has nonpositive real part or is a sixth root of unity, and (3) similarly for λ^3. There are only four such cases: $\lambda = e^{\pm i\tau(5/12)}$ and $\lambda = e^{\pm i\tau(5/18)}$. If $\lambda = e^{\pm i\tau(5/12)}$, then $\lambda^5 = e^{\pm i\tau/12}$, which has positive real part and is not a sixth root of unity. If $\lambda = e^{\pm i\tau(5/18)}$, then the same can be said for $\lambda^4 = e^{\pm i\tau/9}$. So we can apply the first paragraph argument to R_{λ^5} and R_{λ^4}, respectively. $\square$

The converse of Proposition 4.3 for λ on the unit circle follows from the following fact:

Fact 4.4. *If D is any subring of $\mathbb{C}$ that is (topologically) closed, and $\lambda \in D$, then $R_\lambda \subseteq D$.*

So in particular, if $\lambda \in \mathbb{Z}$, then $R_\lambda \subseteq \mathbb{Z}$; if λ is a Gaussian integer, then R_λ consists only of Gaussian integers; if λ is an Eisenstein integer,[3] then R_λ consists only of Eisenstein integers. The fourth roots of unity are all Gaussian integers, and the sixth roots of unity are all Eisenstein integers. None of these choices of λ makes R_λ convex.

R_λ is usually a proper subset of D for the choices of ring D mentioned above. More on this in Section 10.

Corollary 4.5. *If $\lambda \in \mathbb{C} \setminus [0,1]$, then R_λ is unbounded (and thus Q_λ is unbounded).*

Proof. Suppose $\lambda \notin [0,1]$. If $|\lambda| > 1$, then R_λ is unbounded, since $\lambda^n \in R_\lambda$ for all integers $n > 0$. Similarly, if $|1-\lambda| > 1$, then R_λ is unbounded, since $(1-\lambda)^n \in R_{1-\lambda} = R_\lambda$ for all integers $n > 0$. If either $|\lambda| < 1$ or $|1-\lambda| < 1$, then R_λ is convex by Corollary 4.1, and thus $\mathbb{R} \subseteq R_\lambda$ by Theorem 2.19. The only case left is when $|\lambda| = |1-\lambda| = 1$. In this case, $\lambda = (1 \pm i\sqrt{3})/2$. Letting

$$\mu := \lambda \star 1 = 2\lambda - \lambda^2 = \frac{3 \pm i\sqrt{3}}{2} ,$$

we have $|\mu| > 1$, and thus R_μ is unbounded. But since $\mu \in R_\lambda$, we have $R_\mu \subseteq R_\lambda$, which makes R_λ unbounded.

If R_λ is unbounded, then so is Q_λ, because $R_\lambda = \overline{Q_\lambda}$ by Lemma 2.10. $\square$

Lemma 4.6. *R_λ is convex for all $\lambda = x + iy$ where $0 < x \le 1/2$ and $\sqrt{1-x^2} < y \le 1$.*

Proof. We know that $\mu \in R_\lambda$, where $\mu := 1 \star \lambda = 1 - \lambda + \lambda^2 = 1 - x + x^2 - y^2 + y(2x-1)i$. Letting $\alpha := \mathrm{Re}(\mu) = 1 - x + x^2 - y^2$ and $\beta := \mathrm{Im}(\mu) = y(2x-1)$, we have, for the values of x and y in question,

$$\begin{gathered} -x < x(x-1) \le \alpha < 2x^2 - x = x(2x-1) \le 0 , \\ 2x - 1 \le \beta \le 0 . \end{gathered}$$

Then $0 < \alpha^2 + \beta^2 < (-x)^2 + (2x-1)^2 = 5x^2 - 4x + 1 < 1$, giving $0 < |\mu| < 1$. It follows from Corollary 4.2 that R_λ is convex. $\square$

Proposition 4.7. *R_λ is convex for all $\lambda = x + iy$ where $0 < x < 1$ and $-1 \le y \le 1$, except for the two points $e^{i\tau/6}$ and $e^{-i\tau/6}$.*

Proof. We just need to notice that the rectangular region given in the proposition is included in the union of a handful of other known subregions of $\mathcal{C}$ (see Definition 2.21). Let λ be as in the proposition. If $|\lambda| < 1$ or $|1-\lambda| < 1$, then R_λ is convex by Corollary 4.1. If $|\lambda| = 1$ or $|1-\lambda| = 1$, then R_λ is convex by Proposition 4.3 and the fact that $R_\lambda = R_{1-\lambda}$. Let W be the wedge-shaped region of Lemma 4.6. Then the rest of the possible values of λ are covered by either W, $1-W$, W^*, or $1-W^*$, which all yield convex R_λ by Lemma 4.6 and Facts 2.13 and 2.14. $\square$

[3] i.e., a number of the form $a + be^{i\tau/3}$ for some $a, b \in \mathbb{Z}$

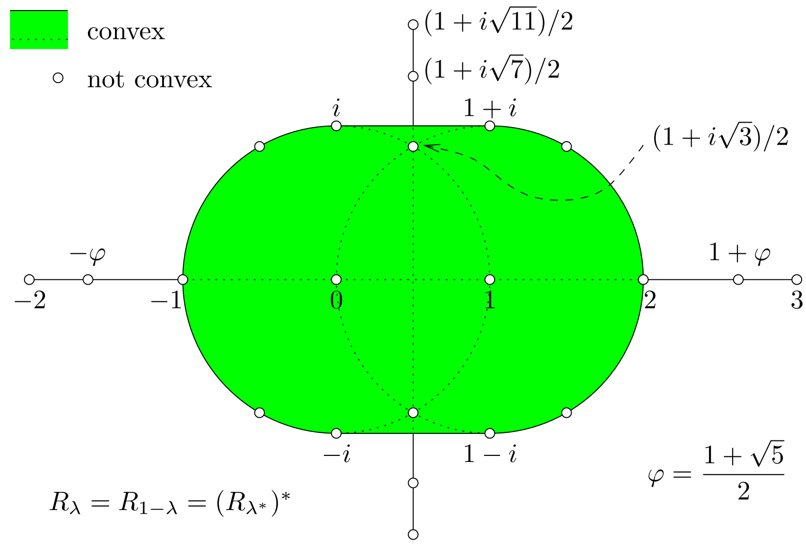


Figure 5: A portion of the complex plane showing points λ such that R_λ is convex by our results (green region and lines) and points λ where we know that R_λ is discrete (white points).

Figure 5 shows in part what points are in $\mathcal{C}$ and in $\mathcal{D}$, based on the results of this and the next section.

5 Some λ such that $R_\lambda = \mathbb{R}$

In this section we establish that R_λ is convex (and thus $R_\lambda = \mathbb{R}$) for various $\lambda \in \mathbb{R} \setminus [0,1]$. We can assume without loss of generality that $\lambda > 1$, since $R_\lambda = R_{1-\lambda}$. If $1 < \lambda < 2$, then we already know that R_λ is convex by Corollary 4.1, and if $\lambda \in \mathbb{Z}$, then $R_\lambda \subseteq \mathbb{Z}$ and thus is not convex. So we investigate the case where $\lambda > 2$ and $\lambda \notin \mathbb{Z}$. At one point in time, we conjectured that R_λ is convex for all λ strictly between 2 and 3, but this turns out not to be the case, and the *unique* counterexample—where $\lambda = 1 + \varphi \approx 2.618\ldots$ where $\varphi := (1+\sqrt{5})/2$ is the Golden Ratio—gives a discrete set R_λ that is aperiodic. In fact, $R_{1+\varphi}$ is an example of an aperiodic Meyer set (see Part II); although unbounded, it has no infinite arithmetic progressions.

Proposition 5.1. *If $2 < \lambda < 3$ and $\lambda \neq 1 + \varphi$, then R_λ is convex.*

Proof. Set $\beta := (\lambda - 1)^2$. Then one checks that $\beta = 1 - \lambda \star 1 \in 1 - R_\lambda = R_\lambda$. Further, if $2 < \lambda < 3$, then $-1 < \beta - \lambda < 1$. One easy way to see this is to note that the function $f(x) := (x-1)^2 - x$ satisfies $f(2) = -1$ and $f(3) = 1$, and $f'(x) = 2x - 3 > 0$ for all $x \in [2,3]$. Thus f is strictly monotone increasing on $[2,3]$, and $f(x) = 0$ only when $x = 1 + \varphi$. Thus for all the λ in question, we have $0 < |\lambda - \beta| < 1$, and so R_λ is convex by Corollary 3.5, since λ and β are both in R_λ. $\square$

6 $R_{1+\varphi}$ is not convex

The next proposition shows that $R_{1+\varphi}$ is not convex. It was originally shown by Berman & Moody [4]. This is a special case of a more general theorem (Theorem 12.14) in Part II.

Proposition 6.1 (Berman & Moody [4])**.**

$$R_{1+\varphi} = \left\{ a + b\varphi : a, b \in \mathbb{Z} \ \ \& \ \ \frac{b}{\varphi} \le a \le \frac{b}{\varphi} + 1 \right\} = \{1\} \cup \left\{ \left\lceil \frac{b}{\varphi} \right\rceil + b\varphi : b \in \mathbb{Z} \right\} .$$

In particular, $R_{1+\varphi}$ is discrete, and except for 0 and 1, any two adjacent points of $R_{1+\varphi}$ differ either by φ or by $1+\varphi$.

The set of pairs (b, a) such that $a + b\varphi \in R_{1+\varphi}$ is illustrated in Figure 6. Although we give a complete, self-contained proof here, the first inclusion we show below—that $R_{1+\varphi}$ is a subset of the right-hand side—is actually a special case of a more general result (Theorem 12.14) we prove in Part II, Section 12. We prove the inclusion here both to make Part I self-contained and to give a foreshadowing of the more general proof in Part II.

Proof of Proposition 6.1. For this proof, set $\lambda := 1 + \varphi$. The second equality is obvious, because φ is irrational. For the first equality, let

$$S := \{a + b\varphi : a, b \in \mathbb{Z} \ \ \& \ \ b/\varphi \le a \le b/\varphi + 1\} \ .$$

We show that $R_\lambda = S$ via two containments.

$R_\lambda \subseteq S$: It suffices to show that S is λ-convex, since $\{0, 1\} \subseteq S$ and S is closed. For any $x = a + b\varphi \in \mathbb{Z}[\varphi] = \mathbb{Z} + \varphi\mathbb{Z}$, define $\delta(x) := a - b/\varphi$. Then $S = \{x \in \mathbb{Z}[\varphi] : \delta(x) \in [0, 1]\}$. For all $x, y \in \mathbb{Z}[\varphi]$, $x \star_\lambda y$ is also in $\mathbb{Z}[\varphi]$, and using the fact that $1/\varphi = \varphi - 1$, a routine calculation shows that

$$\delta(x \star_\lambda y) = \delta(y) \star_\mu \delta(x) \ ,$$

where $\mu := 1/\varphi$. Since $0 < \mu < 1$, we have $\delta(y) \star_\mu \delta(x) \in [0, 1]$ provided $\delta(x), \delta(y) \in [0, 1]$. This just means that $x \star_\lambda y \in S$ provided $x, y \in S$. Thus S is λ-convex, and so $R_\lambda \subseteq S$.

$S \subseteq R_\lambda$: It is enough to show that $\lceil b/\varphi \rceil + b\varphi \in R_\lambda$ for all $b \in \mathbb{Z}$. We show this by induction on $|b|$. For $b \in \{-1, 0, 1\}$ this is easily checked; in particular, $\lambda = 0 \star_\lambda 1$ and $-\varphi = 1 \star_\lambda 0$. Thus we can start the induction with $|b| \ge 2$.

Notice that for all $x \in \mathbb{Z} \setminus \{0\}$,

$$\left\lceil \frac{-x}{\varphi} \right\rceil + (-x)\varphi = 1 - \left(\left\lceil \frac{x}{\varphi} \right\rceil + x\varphi \right) \ ,$$

which implies that the left-hand side is in R_λ if and only if the right-hand side is in R_λ, which in turn is true if and only if $\lceil x/\varphi \rceil + x\varphi \in R_\lambda$. From this fact, we can assume WLOG that $b \ge 2$, the result for $-b$ following immediately.

Assume $b \in \mathbb{Z}$ and $b \ge 2$. Set $a := \lfloor (b+1)/\varphi \rfloor$. We have $1 \le a < b$, and so by the inductive hypothesis, both $\lceil a/\varphi \rceil + a\varphi$ and $\lceil -a/\varphi \rceil - a\varphi$ are in R_λ. Then letting $y := \lceil -a/\varphi \rceil - a\varphi$, the following two values are both elements of R_λ:

$$\begin{aligned} y \star_\lambda 0 &= -\varphi y = -\varphi \left(\left\lceil \frac{-a}{\varphi} \right\rceil - a\varphi \right) = -\varphi(\lceil -a(\varphi - 1) \rceil - a\varphi) = -\varphi(\lceil -a\varphi \rceil + a - a\varphi) \\ &= -\varphi(-\lfloor a\varphi \rfloor + a - a\varphi) = \varphi \lfloor a\varphi \rfloor - a\varphi + a\varphi^2 = \varphi \lfloor a\varphi \rfloor + a = a + (\lceil a\varphi \rceil - 1)\varphi \ , \\ y \star_\lambda 1 &= -\varphi y + 1 + \varphi = a + (\lceil a\varphi \rceil - 1)\varphi + 1 + \varphi = a + 1 + \lceil a\varphi \rceil \varphi \ . \end{aligned}$$

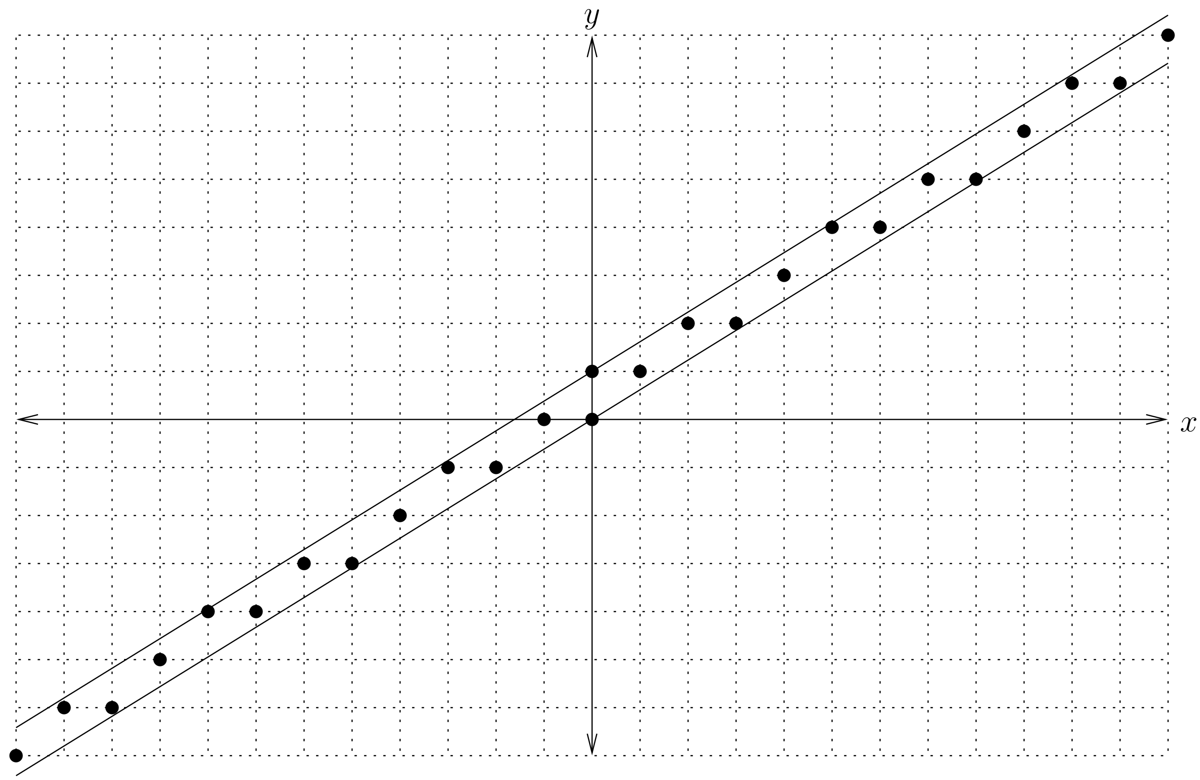


Figure 6: The points $(b, a) \in \mathbb{Z} \times \mathbb{Z}$ such that $a + b\varphi \in R_{1+\varphi}$ are shown. They are all the lattice points lying in the closed strip bounded by the lines $y = x/\varphi$ and $y = x/\varphi + 1$ (also shown). The figure illustrates the fact that $R_{1+\varphi}$ contains no infinite arithmetic progressions, because any two points are connected either by the y-axis or by a line with rational slope, and this line eventually leaves the strip. $R_{1+\varphi}$ is a typical example of an aperiodic model set obtained by a cut-and-project scheme (see Part II).

By the definition of a, we have $b-1 < b+1-\varphi < a\varphi < b+1$, and so the following two cases are exhaustive:

Case 1: $\lceil a\varphi \rceil = b+1$. Then $y \star_\lambda 0 = a+b\varphi \in R_\lambda$. Furthermore, in this case, we have $b < a\varphi < b+1$, and thus

$$\frac{b}{\varphi} < a < \frac{b+1}{\varphi} < \frac{b}{\varphi} + 1 ,$$

and so $a = \lceil b/\varphi \rceil$ as desired.

Case 2: $\lceil a\varphi \rceil = b$. Then $y \star_\lambda 1 = a+1+b\varphi \in R_\lambda$. Furthermore, in this case, we have $b-1 < a\varphi < b$, and thus

$$\frac{b}{\varphi} - 1 < \frac{b-1}{\varphi} < a < \frac{b}{\varphi} .$$

Adding 1 to both sides gives

$$\frac{b}{\varphi} < a+1 < \frac{b}{\varphi} + 1 ,$$

and so $a+1 = \lceil b/\varphi \rceil$ as desired.

The case for $-b$ follows immediately as described above. This finishes the induction. □

The following corollary implies that $R_{1+\varphi}$ is aperiodic, that is, it possesses no translational symmetry, and neither does any nonempty subset of $R_{1+\varphi}$.

Corollary 6.2. *$R_{1+\varphi}$ contains no infinite arithmetic progressions.*

Proof. Suppose $x, x+d, x+2d, x+3d, \ldots \in R_{1+\varphi}$ for some $x \in \mathbb{R}$ and $d \in \mathbb{R} \setminus \{0\}$. Then since $R_{1+\varphi} \subseteq \mathbb{Z}[\varphi]$, we must have $x, d \in \mathbb{Z}[\varphi]$ as well. Defining the function δ as in the proof of Proposition 6.1, one can easily check that $\delta(x+jd) = \delta(x) + j\delta(d)$ for all $j \in \mathbb{Z}$. Since $\delta(d) \neq 0$, we have $\delta(x+jd) \notin [0,1]$—and hence $x+jd \notin R_{1+\varphi}$—for all sufficiently large j, contradicting our assumption. □

7 $\mathcal{C}$ is a big set

In this section, we show that $\mathcal{C}$ is open and contains all transcendental numbers, which implies that its complement is countable. We also show that every element of $\mathcal{D} = \mathbb{C} \setminus \mathcal{C}$ has a deleted neighborhood contained in $\mathcal{C}$. From these two facts it follows immediately that $\mathcal{D}$ is discrete, with no accumulation points in $\mathbb{C}$, and contains only algebraic numbers. In Section 8, below, we prove the stronger result that $\mathcal{D}$ contains only algebraic *integers* (Theorem 8.2), via a much more difficult proof. Beforehand, we introduce some new facts and concepts that will also be useful elsewhere, including the set $Q_{[x]} \subseteq \mathbb{Z}[x]$ (Definition 7.1) and a characterization of it due to Pinch [30] (Lemma 7.8). (We give another useful characterization of $Q_{[x]}$ in Section 11.)

Recall the definition of Q_λ in Definition 1.2.

Definition 7.1. For any polynomials $S, T \in \mathbb{Z}[x]$, define $S \star_{[x]} T := (1-x)S + xT$ (which is clearly also in $\mathbb{Z}[x]$). Let $Q_{[x]}$ denote the least set of polynomials such that

1. The constant polynomials 0 and 1 are both in $Q_{[x]}$, and
2. For every $S, T \in Q_{[x]}$, $S \star_{[x]} T \in Q_{[x]}$.

Note that $Q_{[x]} \subseteq \mathbb{Z}[x]$. The operation $\star_{[x]}$ and set $Q_{[x]}$ are completely analogous to the various $\star_\lambda$ and Q_λ, respectively, for $\lambda \in \mathbb{C}$. For example, the analogue of Fact 2.1 holds for $\star_{[x]}$, and, similarly to Definitions 2.3 and 2.5, we can define $Q^{(0)}_{[x]} := \{0,1\} \subseteq \mathbb{Z}[x]$ and $Q^{(n+1)}_{[x]} := \{S \star_{[x]} T \mid S, T \in Q^{(n)}_{[x]}\}$ for all integers $n \geq 0$. Then the analogue of Fact 2.4 holds for $Q_{[x]}$, which allows us to define the *rank* of a polynomial $P \in Q_{[x]}$ as the least n such that $P \in Q^{(n)}_{[x]}$.

In Section 11, we will obtain some further constraints on the elements of $Q_{[x]}$, including upper bounds on the number of elements of $Q_{[x]}$ of degree $\leq n$, for $n = 0, 1, 2, \ldots$.

Fact 7.2. *For any $\lambda \in \mathbb{C}$, the evaluation map $P \mapsto P(\lambda)$ is a ring homomorphism from $\mathbb{Z}[x]$ into $\mathbb{C}$, and $(S \star_{[x]} T)(\lambda) = S(\lambda) \star_\lambda T(\lambda)$ for all $S, T \in \mathbb{Z}[x]$).*

The next lemma says that this map maps $Q_{[x]}$ onto Q_λ.

Lemma 7.3. *For any $\lambda \in \mathbb{C}$, $Q_\lambda = \{P(\lambda) \mid P \in Q_{[x]}\}$. In fact, $Q^{(n)}_\lambda = \{P(\lambda) \mid P \in Q^{(n)}_{[x]}\}$ for any integer $n \geq 0$.*

Proof. The first statement follows immediately from the second, which is proved by a routine induction on n: We clearly have $Q^{(0)}_\lambda = \{0,1\} = \{P(\lambda) \mid P \in Q^{(0)}_{[x]}\}$. For any $n \geq 0$, if $Q^{(n)}_\lambda = \{P(\lambda) \mid P \in Q^{(n)}_{[x]}\}$, then

$$
\begin{aligned}
Q^{(n+1)}_\lambda &= \left\{a \star_\lambda b \mid a, b \in Q^{(n)}_\lambda\right\} = \left\{S(\lambda) \star_\lambda T(\lambda) \mid S, T \in Q^{(n)}_{[x]}\right\} \\
&= \left\{(S \star_{[x]} T)(\lambda) \mid S, T \in Q^{(n)}_{[x]}\right\} = \{P(\lambda) \mid P \in Q^{(n+1)}_{[x]}\} .
\end{aligned}
$$

□

Now we can prove the first of the two main theorems of this section. Theorem 7.4 was proved in the real case by Pinch [30, Proposition 9]. The complex case is also straightforward.

Theorem 7.4. *$\mathcal{C}$ is open.*

Proof. Let $D := \{x \in \mathbb{C} : 0 < |x| < 1 \ \vee \ 0 < |1 - x| < 1\}$. Note that D is open. For any $\lambda \in \mathbb{C}$, we have

$$
\begin{aligned}
R_\lambda \text{ is convex} &\iff R_\lambda \cap D \neq \emptyset && \text{(Corollary 4.2)} \\
&\iff Q_\lambda \cap D \neq \emptyset && \text{(Lemma 2.10)} \\
&\iff (\exists P \in Q_{[x]})[P(\lambda) \in D] && \text{(Lemma 7.3)} \\
&\iff \lambda \in \bigcup_{P \in Q_{[x]}} P^{-1}(D) .
\end{aligned}
$$

Thus $\mathcal{C} = \bigcup_{P \in Q_{[x]}} P^{-1}(D)$, which is the union of open sets, because each $P \in Q_{[x]}$ corresponds to a continuous map $\mathbb{C} \to \mathbb{C}$. Thus $\mathcal{C}$ is open. □

Corollary 7.5. *$\mathcal{D}$ is closed.*

To prove the second main theorem of this section, that R_λ is convex for all transcendental λ, we first give two lemmas, the first is routine, and the second is a key observation made by Stuart Kurtz.

Lemma 7.6. *For any natural number n, the set $\{x^k(1-x)^{n-k} \mid k \in \mathbb{Z} \ \& \ 0 \le k \le n\}$ is linearly independent over $\mathbb{C}$, and is thus a basis for the space of all polynomials in $\mathbb{C}[x]$ of degree $\le n$.*

Proof. Let $r_0, \dots, r_n$ be any complex numbers, not all zero. Let j be least such that $r_j \ne 0$. Then letting $P(x) := \sum_{k=0}^{n} r_k x^k (1-x)^{n-k}$, we have

$$P(x) = \sum_{k=j}^{n} r_k x^k (1-x)^{n-k} = x^j \left(r_j (1-x)^{n-j} + x \sum_{k=j+1}^{n} r_k x^{k-j-1} (1-x)^{n-k} \right) .$$

Evaluating the expression in the big parentheses at $x = 0$ shows that it is not the zero polynomial, whence P is not the zero polynomial, either. $\square$

Lemma 7.7 (Kurtz)**.** *For all $\lambda \in \mathbb{C}$ and integers $n \ge 0$, let $c_\lambda(n)$ be the cardinality of $Q_\lambda^{(n)}$. Let $c(n)$ be the cardinality of $Q_{[x]}^{(n)}$.*

1. *For any $\lambda \in \mathbb{C}$, if $c_\lambda(n) \notin e^{\mathcal{O}(n)}$ as $n \to \infty$, then R_λ is convex.*
2. *If $c(n) \notin e^{\mathcal{O}(n)}$ as $n \to \infty$, then R_λ is convex for all transcendental $\lambda \in \mathbb{C}$.*

Proof. We first prove Part (1.). For any $r \ge 0$, define $D_r \subseteq \mathbb{C}$ to be the closed disk of radius r centered at the origin. Set $\delta := |1-\lambda| + |\lambda|$. Notice that $Q_\lambda^{(0)} = \{0, 1\} \subseteq D_1$. By Lemma 3.6 (and induction on n), we have $Q_\lambda^{(n)} \subseteq D_{\delta^n}$ for all $n \ge 0$. If R_λ is not convex, then by Corollary 3.5, any two distinct elements of R_λ are at least unit distance apart, and so we can draw an open disk around each element of $Q_\lambda^{(n)}$ of radius $1/2$, and these disks are pairwise disjoint, for a total area of $c_\lambda(n)\pi/4$. These disks must in turn all be included in $D_{\delta^n+1/2}$, which has area $\pi(\delta^n + 1/2)^2$. Thus we get $c_\lambda(n) \le 4(\delta^n + 1/2)^2 \in e^{\mathcal{O}(n)}$ if R_λ is not convex.

For Part (2.), notice that if λ is transcendental, then the evaluation map $\mathbb{Z}[x] \to \mathbb{C}$ sending P to $P(\lambda)$ is one-to-one. By Lemma 7.3, this means that $c(n) = c_\lambda(n)$ for all $n \ge 0$. So we get that R_λ is convex by Part (1.) if λ is transcendental. $\square$

By the second item of Lemma 7.7, we are done if we can get a good lower bound on $c(n)$. To this end, we next characterize the level sets $Q_{[x]}^{(n)}$ so as to determine their cardinalities exactly. The following was proved by Pinch using a straightforward induction [30, Proposition 4, Corollary 4.1]. Here we include an alternate, holistic proof.

Lemma 7.8 (Pinch)**.** *Fix any integer $n \ge 0$. For any polynomial $P \in \mathbb{Z}[x]$, P is in $Q_{[x]}^{(n)}$ if and only if there exist integers $b_0, \dots, b_n$ such that $0 \le b_k \le \binom{n}{k}$ for all $0 \le k \le n$ and*

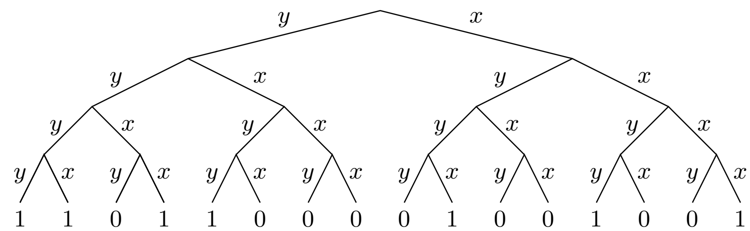
$$P(x) = \sum_{k=0}^{n} b_k x^k (1-x)^{n-k} .$$

Proof. One could prove this formally by induction on n, but it is more illustrative to consider the general case all at once. For convenience, set $y := 1 - x$. Then a typical polynomial $P \in Q_{[x]}^{(n)}$ for $n \ge 2$ is of the form $P_0 \star_{[x]} P_1 = yP_0 + xP_1$, for some $P_0, P_1 \in Q_{[x]}^{(n-1)}$. Then P_0 is of the form $P_{00} \star_{[x]} P_{01} = yP_{00} + xP_{01}$ and similarly P_1 is of the form $P_{10} \star_{[x]} P_{11} = yP_{10} + xP_{11}$ for some

$P_{00}, P_{01}, P_{10}, P_{11} \in Q_{[x]}^{(n-2)}$, making $P = y^2 P_{00} + yx(P_{01} + P_{10}) + x^2 P_{11}$. Similarly, if $n \geq 3$, then there are eight polynomials $P_{000}, \ldots, P_{111} \in Q_{[x]}^{(n-3)}$ such that

$$P = y^3 P_{000} + y^2 x(P_{001} + P_{010} + P_{100}) + yx^2(P_{011} + P_{101} + P_{110}) + x^3 P_{111} \ .$$

This continues until we get polynomials in $Q_{[x]}^{(0)}$, i.e., 0 or 1. Then the completely expanded expression for P resembles a full binary tree with leaves either 0 or 1. Such a tree is shown below for $n = 4$ with leaves chosen arbitrarily:



In the expression tree above, each edge represents multiplication by either y or x, and each internal node is the sum of its children, weighted by y and x, respectively. The root of the tree yields P. Note that each path in the tree from the root to a leaf contributes one term to P of the form $bx^k y^{n-k}$, where k is the number of right jogs in the path and b is the value at the leaf (either 0 or 1). For each possible k, there are exactly $\binom{n}{k}$ many paths with k right jogs, and each of these contributes either 0 or $x^k y^{n-k}$ to P. The lemma follows. (The polynomial given by the tree above is $y^4 + y^3 x + 3y^2 x^2 + 0yx^3 + x^4$.) $\square$

Lemma 7.9 (Pinch [30, Corollary 4.2])**.** *For every $n \geq 0$, the cardinality $c(n)$ of $Q_{[x]}^{(n)}$ is exactly $\prod_{k=0}^{n} \left(1 + \binom{n}{k}\right)$.*

Proof. Lemma 7.8 immediately gives $\prod_{k=0}^{n} \left(1 + \binom{n}{k}\right)$ as an upper bound on $c(n)$. For the lower bound, we have that the set of monomials $\{x^k(1-x)^{n-k} \mid 0 \leq k \leq n\}$ is linearly independent by Lemma 7.6, and thus any two distinct choices of $b_0, \ldots, b_n$ in Lemma 7.8 must give different polynomials. $\square$

Lemma 7.10. *Let $c(n)$ be as in Lemmas 7.7 and 7.9. Then $c(n) \in e^{\Omega(n \log n)}$ as $n \to \infty$.*

Proof. For $n \geq 2$, we have $c(n) \geq \prod_{k=1}^{n-1} \binom{n}{k} \geq n^{n-1} = e^{(n-1)\log n}$. $\square$

Lemmas 7.7 and 7.10 together prove the second main theorem of this section:

Theorem 7.11. *R_λ is convex for all transcendental $\lambda \in \mathbb{C}$.*

In the next section, we strengthen this result by showing (by a very different proof) that if Q_λ is discrete, then λ must be an algebraic *integer*. This fact was proved for real λ by Pinch [30]. The generalization to all complex λ is *not* straightforward.

For now, we prove next that $\mathcal{C}$ "surrounds" all elements of $\mathcal{D}$. The restriction of this theorem to $\mathbb{R}$ was shown by Pinch [30, Proposition 11]. We give an independent proof for the general case.

Theorem 7.12. *The set $\mathcal{D}$ has no accumulation points in $\mathbb{C}$. That is, for any $\lambda \in \mathbb{C}$, there exists an open neighborhood N of λ such that $N \setminus \{\lambda\} \subseteq \mathcal{C}$.*

Proof. If $\lambda \in \mathcal{C}$, then the result is immediate by Theorem 7.4, so suppose $\lambda \notin \mathcal{C}$.

We can find distinct polynomials $p, q \in Q_{[x]}$ such that $p(\lambda) = q(\lambda)$. This can be seen as follows: Let $c(n)$ and $c_\lambda(n)$ be the functions defined in Lemma 7.7. By part (1.) of that lemma, we have $c_\lambda(n) \in e^{\mathcal{O}(n)}$ as $n \to \infty$ (because R_λ is not convex), but by Lemma 7.10, we have $c(n) \in e^{\Omega(n \log n)}$ as $n \to \infty$. Therefore, we can choose n such that $c_\lambda(n) < c(n)$, and it follows by the pigeonhole principle that there exist distinct polynomials $p, q \in Q_{[x]}^{(n)}$ such that $p(\lambda) = q(\lambda)$, that is, λ is a root of the nonzero polynomial $r := p - q$.

By the continuity of r, there exists a neighborhood N' of λ such that $|r(z)| < 1$ for all $z \in N'$. Since r has only finitely many roots, there exists an $\varepsilon > 0$ such that $0 < |r(z)| < 1$ for all z such that $0 < |z - \lambda| < \varepsilon$. Letting $N := \{z \in \mathbb{C} : |z - \lambda| < \varepsilon\}$, we have $0 < |r(z)| = |p(z) - q(z)| < 1$ for all $z \in N \setminus \{\lambda\}$. For these z, since $p(z)$ and $q(z)$ are distinct members of R_z (Lemma 7.3) that are less than unit distance apart, we know that R_z is convex by Corollary 3.5. Thus $z \in \mathcal{C}$ for all $z \in N \setminus \{\lambda\}$. ☐

7.1 Threshold polynomials

The next lemma does not apply to $\mathcal{C}$, but it is an easy consequence of Lemma 7.8 and it will be used in Part II, so we include it in this section. First, a definition.

Definition 7.13. For every $n \in \mathbb{Z}^+$ and $\gamma \in (0, 1)$, define the polynomial

$$t_\gamma^{(n)}(x) := \sum_{i=0}^{\lfloor \gamma n \rfloor} \binom{n}{i} x^i (1-x)^{n-i} .$$

For large n, the polynomial $t_\gamma^{(n)}$ approximates a "threshold" function on $[0, 1]$.

Lemma 7.14. *For any $0 < \gamma < 1$ and $\varepsilon > 0$, there exists a polynomial $T_{\gamma,\varepsilon} \in Q_{[x]}$ such that $1 - \varepsilon \le T_{\gamma,\varepsilon}(x) \le 1$ for all $x \in [0, \gamma - \varepsilon]$ and $0 \le T_{\gamma,\varepsilon}(x) \le \varepsilon$ for all $x \in [\gamma + \varepsilon, 1]$.*

Proof. All the $t_\gamma^{(n)}$ are in $Q_{[x]}$ by Lemma 7.8. Also, for all $x \in [0, 1]$,

$$0 \le t_\gamma^{(n)}(x) \le \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = 1 .$$

Taking $T_{\gamma,\varepsilon} := t_\gamma^{(n)}$ for sufficiently large n will satisfy the lemma. This follows from Hoeffding's inequality [15], which in the current context states that for all x such that $\gamma + \varepsilon \le x \le 1$,

$$t_\gamma^{(n)}(x) \le \exp\left(-2n(x-\gamma)^2\right) \le \exp\left(-2n\varepsilon^2\right) .$$

The right-hand side is $\le \varepsilon$ provided $n \ge -(\log \varepsilon)/(2\varepsilon^2)$.

By symmetry, we have for all $0 \le x \le \gamma - \varepsilon$,

$$\begin{aligned} 1 - t_\gamma^{(n)}(x) &= \sum_{i=\lfloor \gamma n \rfloor + 1}^{n} \binom{n}{i} x^i (1-x)^{n-i} = \sum_{j=0}^{n - \lfloor \gamma n \rfloor - 1} \binom{n}{n-j} x^{n-j} (1-x)^j \\ &= \sum_{j=0}^{n - \lfloor \gamma n \rfloor - 1} \binom{n}{j} x^{n-j} (1-x)^j \le \sum_{j=0}^{\lfloor (1-\gamma) n \rfloor} \binom{n}{j} x^{n-j} (1-x)^j = t_{1-\gamma}^{(n)}(1-x) . \end{aligned}$$

Since $(1-\gamma)+\varepsilon \le 1-x \le 1$, we apply Hoeffding's inequality again to get

$$t_\gamma^{(n)}(x) \ge 1 - t_{1-\gamma}^{(n)}(1-x) \ge 1 - \exp(-2n\varepsilon^2) \ge 1-\varepsilon$$

provided $n \ge -(\log \varepsilon)/(2\varepsilon^2)$ as above.

Therefore we can choose $T_{\gamma,\varepsilon} := t_\gamma^{(n)}$, where $n := \lceil -(\log\varepsilon)/(2\varepsilon^2)\rceil$. (We can assume $\varepsilon < 1$ without loss of generality.) $\square$

8 If Q_λ is Discrete, Then λ is an Algebraic Integer

Pinch proved that for $\lambda \in \mathbb{R}$, if Q_λ is discrete, then λ is an algebraic integer [30].

Theorem 8.1 (Pinch [30, Theorem 8])**.** *For any $\lambda \in \mathbb{R}$, if Q_λ is discrete, then λ is an algebraic integer.*

We have the same result for arbitrary complex λ.

Theorem 8.2. *For any $\lambda \in \mathbb{C}$, if Q_λ is discrete, then λ is an algebraic integer.*

The rest of this section is devoted to the proof of this theorem. It adapts Pinch's overall technique to the complex case but is considerably more intricate. Along the way, we prove a weak relative density result for Q_λ. We do this in stages, obtaining stronger and stronger density results for Q_λ.

Notation 8.3. For any nonzero $z \in \mathbb{C}$, we define $\arg z$ to be the unique θ such that $-\pi \le \theta < \pi$ and $z = |z|e^{i\theta}$.

The following technical lemma will make our later proofs easier. We defer the proof until the end of this section.

Lemma 8.4. *For all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, there exists $\nu \in Q_\lambda$ such that $|\nu| > 1$ and $0 < \arg\nu < \pi/6$.*

Now fix $\lambda \in \mathbb{C}$ such that Q_λ is discrete. If $\lambda \in \mathbb{R}$, then λ is an algebraic integer by Theorem 8.1, so we assume $\lambda \notin \mathbb{R}$. We then fix some $\nu \in Q_\lambda$ satisfying Lemma 8.4, above.

Notation 8.5. We define ℓ to be the least positive integer such that, setting $\mu := \nu^\ell$:

- $|\mu| > (1+\sqrt{3})/2$,
- $0 < \arg(\mu) < \pi/6$, and
- for all integers m such that $-6 \le m \le 5$, there exists an integer $0 < j < \ell$ such that $\pi m/6 < \arg(\nu^j) < \pi(m+1)/6$.

Set $B := |\mu|$ and $q := \mu/(\mu-1)$. Define

$$P := \{z \in \mathbb{C} : 1 < |z| < B \ \ \& \ \ 0 < \arg z < \pi/6\}\ .$$

The first two items guarantee that $\mathrm{Re}(\mu) > 1/2$, and thus $|\mu - 1| < |\mu|$, which in turn implies $|q| > 1$. The idea of the third item is that we have a power of ν (and thus an element of Q_λ) in each of the twelve 30° "pie slices" of $\mathbb{C}$ centered at the origin 0, that is, $Q_\lambda \cap e^{i\pi m/6}P \neq \emptyset$ for all $m \in \mathbb{Z}$. Obviously, $\ell > 12$, and $1, \nu, \nu^2, \ldots, \nu^\ell$ are all contained in the closed ball of radius B centered at the origin.

Note that, since $\mu \in Q_\lambda$, Q_λ is closed under $\star_\mu$. Also, for all $x, y, z \in \mathbb{C}$, we have $x \star_\mu y = z$ if and only if $z \star_q y = x$.

Definition 8.6. Define the open region

$$W := \bigcup_{0 \le \theta \le \pi/6} e^{i\theta} P = \{w \in \mathbb{C} : 1 < |w| < B \ \& \ 0 < \arg w < \pi/3\} .$$

For $z \in \mathbb{C}$ where $|z| \ge 1$, we will call regions of the form zW *wedges*.

The four "corners" of a wedge zW are z, $e^{i\pi/3}z$, Bz, and $Be^{i\pi/3}z$.

Lemma 8.7. *Every wedge intersects Q_λ.*

Proof. Given z such that $|z| \ge 1$, let $p \in \mathbb{Z}$ be largest such that $|\nu|^p \le |z|$. We must have $p \ge 0$ by our condition on $|z|$. Letting $T := \{\nu, \nu^2, \ldots, \nu^{\ell-1}\}$, by our choice of ℓ, we have that T intersects $e^{i\pi m/6}P$ for all $m \in \mathbb{Z}$ (this is established explicitly for $-6 \le m \le 5$ and extends to all $m \in \mathbb{Z}$ by periodicity). It follows from our choice of p that $|z| < |w| < B|z|$ for all $w \in \nu^p T$, and thus $\nu^p T$ intersects $|z|e^{i(\arg(\nu^p)+\pi m/6)}P$ for all $m \in \mathbb{Z}$. Choose $m \in \mathbb{Z}$ such that $0 \le \theta_m < \pi/6$, where $\theta_m := \arg(\nu^p) + \pi m/6 - \arg z$. Then

$$|z|e^{i(\arg(\nu^p)+\pi m/6)}P = ze^{i\theta_m}P \subseteq zW .$$

Thus zW intersects $\nu^p T$, the latter being a subset of Q_λ, and we are done. □

Notation 8.8. For $z \in \mathbb{C}$ and real $k > 0$, we define $D(z; k) := \{w \in \mathbb{C} : |w - z| < k|z|\}$, that is, the open disk centered at z with radius $k|z|$.

Definition 8.9. Let $k > 0$ be given. For $x, y \in \mathbb{C}$, we say that *y is k-close to x* iff $y \in D(x; k)$. If $S \subseteq \mathbb{C}$ is some point set and $R \subseteq \mathbb{C}$ is some open region, we say that *S is k-dense in R* if every point in R is k-close to a point in $S \cap R$.

Notice that k-closeness is not a symmetric relation.

By definition, $\{x\}$ is k-dense in $D(x; k)$ for all $k > 0$ and nonzero $x \in \mathbb{C}$. The next lemma says that we can increase the radius a bit for certain elements of Q_λ.

Lemma 8.10. *For every k such that $0 < k < 1 - B^{-1}\sqrt{B^2 - \sqrt{3}B + 1}$, there exists $u > k$ such that, for all $x \in Q_\lambda$ with $|x| \ge \frac{B}{k|q|}$, Q_λ is k-dense in $D(\mu x; u)$.*

Proof. Given k, let

$$u := \frac{k}{2B}\left(\sqrt{3} + \sqrt{4B^2(1-k)^2 - 1}\right) ,$$

that is, the larger of the two solutions to the quadratic equation

$$u^2 - \frac{k\sqrt{3}}{B}u + \frac{k^2}{B^2} - k^2(1-k)^2 = 0 .$$

The upper bound on k guarantees that $u > k$, which can be seen as follows: The inequality $k < u$ is clearly equivalent to

$$2B - \sqrt{3} < \sqrt{4B^2(1-k)^2 - 1}\ .$$

Since $B = |\mu| > (1+\sqrt{3})/2$, both sides are nonnegative, so squaring both sides yields an equivalent inequality:

$$4B^2 - 4B\sqrt{3} + 3 < 4B^2(1-k)^2 - 1\ ,$$

or equivalently,

$$(1-k)^2 > 1 - B^{-1}\sqrt{3} + B^{-2}\ .$$

The lower bound on B above makes both sides nonnegative, so we can take the square root of both sides to get the equivalent statement,

$$|1-k| > B^{-1}\sqrt{B^2 - \sqrt{3}B + 1}\ ,$$

which is implied by our constraint on k.

Now let z be any point in $D(\mu x; u)$. We show that z is k-close to an element of $Q_\lambda \cap D(\mu x; u)$. In fact, we show that z is k-close to an element of $Q_\lambda \cap D(\mu x; k)$, from which k-density follows, because $D(\mu x; k) \subseteq D(\mu x; u)$. If $z \in D(\mu x; k)$, we are done, so assume otherwise. Let z' be the point on the line segment connecting z with μx that is distance $k|\mu x|/B = k|x|$ away from μx, as in Figure 7, and let $y := z' \star_q x$. (Note then that $z' = y \star_\mu x$.) Using the lower bound on $|x|$ and noting that $|q| = |1-\mu|^{-1}B$, we have

$$|y| = |z' \star_q x| = |(1-q)z' + qx| = |1-\mu|^{-1}|z' - \mu x| = |1-\mu|^{-1}k|x| = |q|k|x|/B \geq 1\ .$$

Set $Y := y\, e^{-i\pi/6} W$. Note that Y is a wedge, because $|y| \geq 1$, and thus $Y \cap Q_\lambda \neq \emptyset$ by Lemma 8.7. Letting $Z := Y \star_\mu x = (1-\mu)Y + \mu x$, we have $Z \cap Q_\lambda \neq \emptyset$ as well. (Note that Y and Z are similar.) To finish the proof, we will show that z is k-close to every point in Z and that $Z \subseteq D(\mu x; k)$.

Let $w \in Z$ be arbitrary. Then Y contains the point $w' := w \star_q x = (1-\mu)^{-1}(w - \mu x)$. We can thus write $w' = r\, e^{i(\theta + \arg y)}$, where $|y| < r < B|y|$ and $-\pi/6 < \theta < \pi/6$. Translating back, we have

$$w - \mu x = (1-\mu)\, r\, e^{i(\theta+\arg y)} = (1-\mu)\, \frac{r}{|y|}\, y\, e^{i\theta} = \frac{r}{|y|}\, e^{i\theta}(z' - \mu x) = s\, e^{i\theta}(z' - \mu x)\ ,$$

where we set $s := r/|y|$ and thus $1 < s < B$. It follows that

$$|w - \mu x| = s|z' - \mu x| = sk|x| < Bk|x| = k|\mu x|\ ,$$

which shows that $Z \subseteq D(\mu x; k)$, as $w \in Z$ was chosen arbitrarily. Let z'' be the point on the line connecting μx with z such that z is between μx and z'' and $|z'' - \mu x| = u|\mu x| = uB|x|$ (z'' is on the boundary of $D(\mu x; u)$). Figure 7 shows what is going on. We also identify two corners of Z, namely, $b := \mu x + e^{-i\pi/6}(z' - \mu x)$ and $a := \mu x + B(b - \mu x)$. By definition, $|b - \mu x| = |z' - \mu x| = k|x|$ and $|a - \mu x| = B|b - \mu x| = Bk|x|$. Evidently, $|w - \mu x| < |a - \mu x|$, whence by the triangle inequality, $|w| > |\mu x| - Bk|x| = B(1-k)|x|$. It is also evident from the diagram that the point b is farther away from z'' than any point in Z is from z, and so $|z - w| < |z'' - b|$. (The point a is closer to z than b is to z; this follows from the fact that $B > (1+\sqrt{3})/2$). Using, say, the Law of Cosines with the triangle $(z'', b, \mu x)$, we can find $|z'' - b|$:

$$\begin{aligned} |z'' - b|^2 &= |z'' - \mu x|^2 + |b - \mu x|^2 - 2|z'' - \mu x|\,|b - \mu x| \cos(\pi/6) \\ &= B^2|x|^2 \left(u^2 - \frac{k\sqrt{3}}{B}\, u + \frac{k^2}{B^2} \right) = B^2|x|^2k^2(1-k)^2 \end{aligned}$$

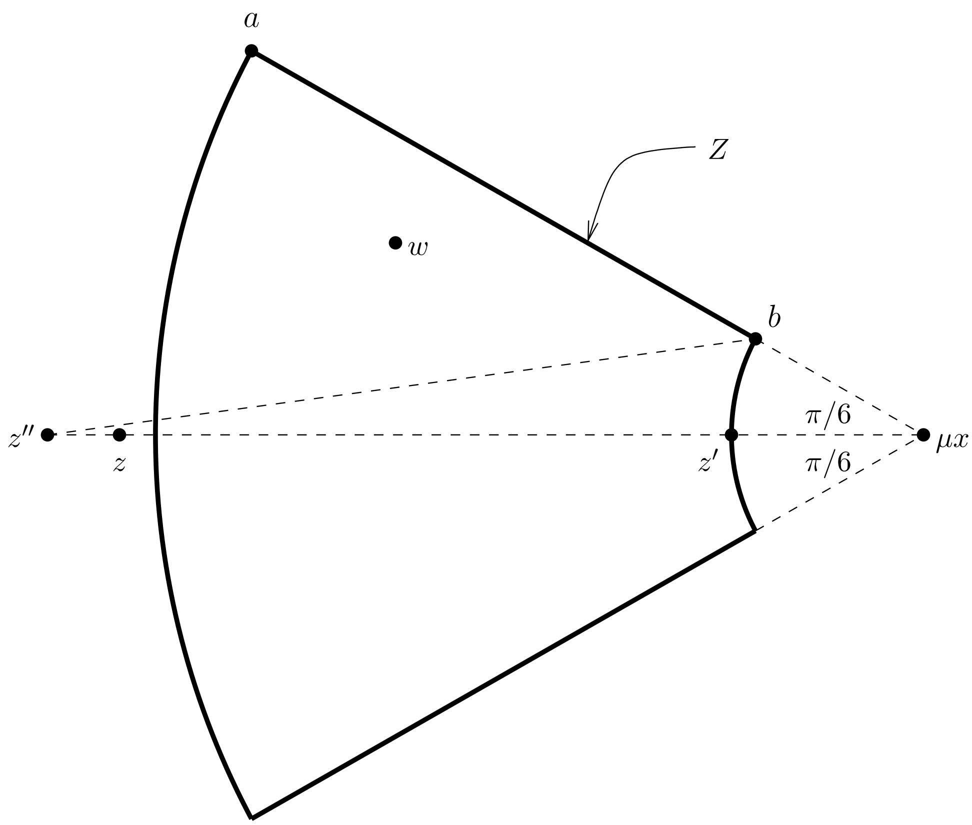


Figure 7: The Z region is bounded by the thick lines and arcs. The arcs are concentric about the point μx through an angle of $\pi/3$. The dotted line connecting μx with z bisects Z and intersects the inner arc at the point z'. The point z lies outside the outer arc, and z'' lies to its left. Two corners a and b and an arbitrary element w of Z are also labeled.

by our choice of u. Thus $|z-w| < |z''-b| = kB(1-k)|x| < k|w|$, making z k-close to w. $\square$

Suppose $x \in \mathbb{C} \setminus \{0\}$ and $r > 0$ are such that Q_λ is k-dense in $D(x;r)$ (for some $k > 0$). It immediately follows, just by multiplying everything by μ, that μQ_λ is k-dense in $D(\mu x; r)$. As with Lemma 8.10, the next lemma increases the diameter a little bit going from x to μx. It is actually a generalization of Lemma 8.10

Lemma 8.11. *Let k and u be as in Lemma 8.10. Suppose* $0 < r < 1$*, and let* $v := r + (1-r)(u-k)$*. Then Q_λ is k-dense in $D(\mu x; v)$ for all $x \in \mathbb{C}$ such that Q_λ is k-dense in $D(x;r)$ and* $|x| \geq \frac{B}{(1-r)k|q|}$.

Proof. Let $z \in D(\mu x; v)$ be arbitrary. Then z is at most $(v-r)|\mu x|$ distance away from some element $y \in D(\mu x; r)$. Since Q_λ is k-dense in $D(x;r)$ by assumption, μQ_λ is k-dense in $D(\mu x; r)$, and thus y is k-close to $\mu x'$ for some $x' \in Q_\lambda \cap D(x;r)$. By the triangle inequality, $|x'| > (1-r)|x| \geq \frac{B}{k|q|}$, and so we can apply Lemma 8.10 to x' to get that Q_λ is k-dense in $D(\mu x'; u)$. It remains to show that $z \in D(\mu x'; u)$, thus making z k-close to some element of Q_λ. By the triangle inequality using the triangle $(\mu x', y, z)$, and noting that $|\mu x'| = B|x'| > B(1-r)|x| = (1-r)|\mu x|$, we have

$$\begin{aligned}|z - \mu x'| &\leq |y - \mu x'| + |z - y| < k|\mu x'| + (v-r)|\mu x| \\ &< (k + (v-r)/(1-r))|\mu x'| = (k + (u-k))|\mu x'| = u|\mu x'|\,,\end{aligned}$$

and thus $z \in D(\mu x'; u)$ as required. $\square$

Remark. In the last lemma, we needed $|x| \geq \frac{B}{(1-r)k|q|}$ so that every point $x' \in D(x;r)$ satisfies $|x'| \geq \frac{B}{k|q|}$, allowing us to apply Lemma 8.10 to x'. The same is true for every point $y \in D(\mu x; v)$, that is, $|y| \geq \frac{B}{k|q|}$. The latter condition is equivalent to the inequality $1 - u + k \geq 1/B$, which can be verified via a rather tedious calculation.[4] We also have $0 < v < 1$, because $1-v = (1-r)(1-u+k) > 0$. These facts will be important for the proof of Theorem 8.16, because they allow us to iterate the passage from $D(x;r)$ to $D(\mu x; v)$ while maintaining k-density of Q_λ throughout. $\square$

Theorem 8.16 below, the first main result of this section, asserts that Q_λ is close to being relatively dense, at least asympototically. Before giving it, we present a few technical lemmas.

Recall that μ was chosen such that $0 < \arg\mu < \pi/6$.

Definition 8.12. Define $n := \lceil 2\pi/\arg\mu \rceil$, noting that n is the least positive integer such that $0 \leq \arg(\mu^n) < \arg\mu$. Define the closed region $V := \{r\,e^{i\varphi} \mid 1 \leq r \leq B^n \;\;\&\;\; 0 \leq \varphi \leq \pi/6\}$.

Define the closed annulus $A := \{z \in \mathbb{C} : B^{n-1} \leq |z| \leq B^n\}$.

Note that n is chosen so that every closed pie slice $S_\theta := \{r\,e^{i\varphi} \mid r \geq 0 \;\;\&\;\; \theta - \pi/6 \leq \varphi \leq \theta\}$ for $\theta \in \mathbb{R}$ contains μ^j for some integer $0 \leq j < n$. V resembles the region P, but extends out much farther away from the origin and is closed. The next lemma is routine and stated without proof.

Lemma 8.13. *V is included within the open disk $C := D(B^n; 1 - B^{-n}/2)$ centered at B^n with radius $B^n - 1/2$.*

Lemma 8.14. *$A \subseteq U$, where $U := \bigcup_{j=0}^{n-1} \mu^j V$.*

[4] Using the fact that $B > 1$, this inequality can be converted into the equivalent form $p(B) \geq 0$, where p is a real quadratic polynomial with leading term $(1+1/k)^2 - (1-k)^2 > 0$ and discriminant $4(1-k)^2 - (1+1/k)^2 < 0$.

Proof. Given $z \in A$, let $\theta := \arg z$. Evidently, $z \in S_\theta$, the pie slice defined above. Let $0 \le j < n$ be such that S_θ contains μ^j. Then

$$\mu^j V = \{r\, e^{i\varphi} \mid B^j \le r \le B^{j+n} \;\;\&\;\; \arg(\mu^j) \le \varphi \le \arg(\mu^j) + \pi/6\} \,,$$

which contains z. $\square$

Lemma 8.15. *$\{z \in \mathbb{C} : |z| \ge B^{n-1}\} \subseteq T$, where $T := \bigcup_{j=0}^{\infty} \mu^j C$ and C is as in Lemma 8.13.*

Proof. Every z such that $|z| \ge B^{n-1}$ is contained in $\mu^p A$ for some integer $p \ge 0$, and by Lemmas 8.14 and 8.13 the latter region is included in $\mu^p U = \bigcup_{j=p}^{p+n-1} \mu^j V \subseteq \bigcup_{j=p}^{p+n-1} \mu^j C \subseteq T$. $\square$

Theorem 8.16. *Given any $k > 0$, there exists $R > 0$ such that Q_λ is k-dense in $\{z \in \mathbb{C} : |z| > R\}$.*

Proof. The idea is that we can increase the sizes of disks in which Q_λ is k-dense until one of them includes zC for some z. Without loss of generality, we can take k to be as small as we want, so we assume it satisfies the conditions in Lemma 8.10, and we also define u as in that lemma. Let n be as in Definition 8.12 and C be as in Lemma 8.13. Fix some $x \in Q_\lambda$ such that $|x| \ge \frac{B}{(1-k)k|q|}$. For example, we can take x to be the lowest power of μ satisfying this norm bound, whence $|x| < \frac{B^2}{(1-k)k|q|}$. Set $r_0 := k$, and for all integers $j \ge 0$, inductively define $r_{j+1} := r_j + (1 - r_j)(u - k)$. Then by induction, for all $j \ge 0$, we have that Q_λ is k-dense in $D(\mu^j x; r_j)$. Also by induction we have $r_j = 1 - (1 - u + k)^j (1 - k)$ for all j. We know that $0 < 1 - u + k < 1$ (see the Remark following Lemma 8.11), so we can choose an $m \ge n$ large enough so that $1 - B^{-n}/2 < r_m < r_{m+1} < \cdots < 1$. Then for all $p \ge m$, $D(\mu^p x; r_p)$ is big enough to include zC for some z. In fact,

$$\frac{\mu^p x}{B^n}\, C = D(\mu^p x; 1 - B^{-n}/2) \subseteq D(\mu^p x; r_m) \subseteq D(\mu^p x; r_p)$$

by our choice of m. Finally, letting $E := \{z \in \mathbb{C} : |z| \ge B^{n-1}\}$ and using Lemma 8.15,

$$\{z \in \mathbb{C} : |z| \ge B^{m-1}|x|\} = \frac{\mu^m x}{B^n}\, E \subseteq \frac{\mu^m x}{B^n}\, T = \frac{x}{B^n} \bigcup_{p=m}^{\infty} \mu^p C \subseteq \bigcup_{p \ge m} D(\mu^p x; r_p) \,.$$

Q_λ is k-dense in the right-hand side, so we can take $R := \frac{B^{m+1}}{(1-k)k|q|} > B^{m-1}|x|$. $\square$

Remark. Although we were assuming all along that Q_λ is discrete, Theorem 8.16 actually holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, for if Q_λ is not discrete, then it is *dense* in $\mathbb{C}$ and hence trivially k-dense in $\mathbb{C}$ for all $k > 0$. For real λ, we have the following situation: if Q_λ is not discrete, then we know that Q_λ is dense in either $[0, 1]$ or $\mathbb{R}$ (depending on λ being in $(0, 1)$ or $\mathbb{R} \setminus [0, 1]$, respectively). In this case, Q_λ is again obviously k-dense in these respective sets, for all $k > 0$. If Q_λ is discrete and $\lambda \notin \{0, 1\}$, then given $k > 0$, Pinch implicitly proves k-density of Q_λ in $\mathbb{R} \setminus [-A, A]$ for some $A > 0$ (depending on k) [30]. $\square$

We now turn to the second main result of this section, showing that λ is an algebraic integer. Pinch's proof for real $\lambda > 1$ works by showing that every sufficiently large $x \in Q_\lambda$ is a $\mathbb{Z}$-linear combination of elements of $Q_\lambda \cap (0, R)$ for some fixed $R > 0$. In this case, given x, he finds elements $u, v \in Q_\lambda$ such that $u < x < v$ and that are "close enough" to x so that the three points $r := x \star_\lambda u$, $s := v \star_\lambda u$, and $t := v \star_\lambda x$ are all strictly between 0 and x. One has $x = r - s + t$, and then he can argue by induction using the discreteness of Q_λ.

Here, given $x \in Q_\lambda$ (for nonreal λ) such that $|x|$ is sufficiently large, we follow roughly the same outline as Pinch, using the k-density of Q_λ to find $u, v \in Q_\lambda$ such that the points $r := x \star_\lambda u$, $s := v \star_\lambda u$, and $t := v \star_\lambda x$ are all smaller than x in norm, allowing a similar inductive argument. Our situation is complicated by the fact that, not only must u and v be close enough to x, they must also be oriented in suitable directions relative to x and to each other.

Lemma 8.17. *For all $x \in Q_\lambda$ with $|x|$ sufficiently large, there exist $u, v \in Q_\lambda$ such that the three points $r := x \star_\mu u$, $s := v \star_\mu u$, and $t := v \star_\mu x$ all have norms strictly smaller than $|x|$.*

Proof. Recall that $q = \mu/(\mu - 1)$, and it follows that $\mu = q/(q-1)$. Also, $|q| > 1$. Let a be the square root of q with positive real part, i.e, $a^2 = q$ and $\mathrm{Re}(a) > 0$ (we know that $q \not< 0$). Let $c := |a| = |q|^{1/2}$ and let $d := |a + 1|$, noting that $1 < c < d$. Choose k such that

$$0 < k < \min\left(\frac{d-c}{d-c+Bdc}\,,\ \frac{1}{B(c+c^{-1})+1}\right)\ , \tag{2}$$

observing that $k < 1$. From (2) it follows that

$$0 < \frac{k}{1-k} < \frac{1}{B}\min\left(c^{-1} - d^{-1}\,,\ (c+c^{-1})^{-1}\right)\ . \tag{3}$$

Given k as above, let R be as in Theorem 8.16, and let x be any element of Q_λ such that $|x| > cR$. Let $y := x/a$ and $z := ax$. We have $R < |y| < |z|$, so by k-density we can choose $u, v \in Q_\lambda$ such that y is k-close to u and z is k-close to v. We have

$$|u| \le |y| + |u - y| < |x|/c + k|u| \qquad \Longrightarrow\ |u| < \frac{c^{-1}|x|}{1-k}\ ,$$
$$|v| \le |z| + |v - z| < c|x| + k|v| \qquad \Longrightarrow\ |v| < \frac{c|x|}{1-k}\ .$$

Define r, s, and t as in the lemma. Observe that $z \star_\mu y = (1-\mu)ax + \mu x/a = (x/a)((1-\mu)q + \mu) = 0$, and thus, using (3) for the last step,

$$\begin{aligned}|s| = |v \star_\mu u| &\le |z \star_\mu u| + |v \star_\mu u - z \star_\mu u| \le |z \star_\mu y| + |z \star_\mu u - z \star_\mu y| + |v \star_\mu u - z \star_\mu u| \\ &= |z \star_\mu u - z \star_\mu y| + |v \star_\mu u - z \star_\mu u| = |\mu(u-y)| + |(1-\mu)(v-z)| < Bk(|u| + |v|) \\ &< B\left(\frac{k}{1-k}\right)(c + c^{-1})|x| < |x|\ .\end{aligned}$$

Using the fact that $\mu = q/(q-1) = a^2/(a^2-1)$, we get

$$a \star_\mu 1 = \left(1 - \frac{a^2}{a^2-1}\right)a + \frac{a^2}{a^2-1} = \frac{a^2 - a}{a^2-1} = \frac{a}{a+1}\ ,$$

and we plug this into the following calculation:

$$\begin{aligned}|t| = |v \star_\mu x| &\le |z \star_\mu x| + |v \star_\mu x - z \star_\mu x| = |ax \star_\mu x| + |(1-\mu)(v-z)| \le |x||a \star_\mu 1| + B|v - z| \\ &< |x||a \star_\mu 1| + Bk|v| < |x|\left(|a \star_\mu 1| + B\frac{k}{1-k}c\right) = |x|\left(\left|\frac{a}{a+1}\right| + B\frac{k}{k-1}c\right) \\ &< |x|\left(\frac{c}{d} + (c^{-1} - d^{-1})c\right) = |x|\ .\end{aligned}$$

$$\begin{aligned}|r| = |x \star_\mu u| &\le |x \star_\mu y| + |x \star_\mu u - x \star_\mu y| = |x \star_\mu a^{-1}x| + |\mu(u-y)| = c^{-1}|x||a \star_\mu 1| + B|u - y| \\ &< c^{-1}|x||a \star_\mu 1| + Bk|u| < c^{-1}|x|\left(|a \star_\mu 1| + B\frac{k}{1-k}\right) < |x|\left(|a \star_\mu 1| + B\frac{k}{1-k}c\right) < |x|\ .\end{aligned}$$

(We reused some of the calculation for $|t|$ for the bound on $|r|$.) $\square$

Proof of Theorem 8.2. The case where $\lambda \in \mathbb{R}$ was proved by Pinch [30], so we assume (as we have throughout this section) that $\lambda \notin \mathbb{R}$. Let R and c be as in the proof of Lemma 8.17, and let $D := \{z \in Q_\lambda : |z| \le cR\}$. We show first that every $x \in Q_\lambda$ is a $\mathbb{Z}$-linear combination of elements of D. This is done by induction on $|x|$, which is possible because Q_λ is discrete: If $|x| \le cR$, then already $x \in D$ and we are done. Otherwise, by Lemma 8.17 we have $u, v \in Q_\lambda$ such that r, s, and t all have norm less than $|x|$, where r, s, and t are as in Lemma 8.17. Obviously, $r, s, t \in Q_\lambda$, so applying the inductive hypothesis to r, s, and t, each is a $\mathbb{Z}$-linear combination of elements of D. It is straightforward to check that $x = r - s + t$, and thus x is a $\mathbb{Z}$-linear combination of elements of D as well. This ends the inductive argument.

Note that D is finite, because Q_λ is discrete. Every element of Q_λ can be expressed as $p(\lambda)$, where $p \in Q_{[x]}$ is a polynomial with integer coefficients. Choose some positive integer N large enough so that every $z \in D$ can be written as $p(\lambda)$ where $p \in \mathbb{Z}[x]$ and $\deg(p) < N$. We have $\lambda^N \in Q_\lambda$. By our inductive argument, λ^N is a $\mathbb{Z}$-linear combination of elements of D, each of which is a $\mathbb{Z}$-linear combination of lower powers of λ. Thus λ is the root of an integer polynomial, and this polynomial is monic, having leading term λ^N. $\square$

8.1 Proof of Lemma 8.4

Proof of Lemma 8.4. If Q_λ is not discrete, then it is dense in $\mathbb{C}$ (by Theorem 2.19 and Corollary 3.5) and we are done, so we can assume that Q_λ is discrete. We may also assume that $\mathrm{Im}(\lambda) > 0$, for otherwise, we can argue the following with $1-\lambda$ in place of λ (recall that $Q_\lambda = Q_{1-\lambda}$). If $|\lambda| < 1$, then Q_λ is not discrete, so we can assume $|\lambda| \ge 1$. We know from Proposition 4.3 that there are exactly three values of λ on the unit circle where $\mathrm{Im}(\lambda) > 0$ and Q_λ is discrete:

$$e^{\pi i/3} = \frac{1+i\sqrt{3}}{2}\,, \qquad e^{\pi i/2} = i\,, \qquad e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}\,.$$

For the first value, $\lambda = (1+i\sqrt{3})/2$, one can see that $Q_\lambda = \mathbb{Z}[\lambda]$, the set of Eisenstein integers, and so ν exists. (More explicitly, we can set

$$\nu := (5+i\sqrt{3})/2 = 2+\lambda = (\lambda \star_\lambda 1) \star_\lambda 2 = (\lambda \star_\lambda 1) \star_\lambda ((\lambda \star_\lambda 1) \star_\lambda 1) \in Q_\lambda\,.$$

For the second value, $\lambda = i$, one can see that $Q_\lambda = \mathbb{Z}[i]$, the Gaussian integers, so ν clearly exists. Explicitly, we can take

$$\nu := 2+i = (i \star_\lambda 0) \star_\lambda 1 \in Q_\lambda\,.$$

For the third value, $\lambda = (-1+i\sqrt{3})/2$, we have $1-\lambda = (3-i\sqrt{3})/2$, and we can take

$$\nu := (5+i\sqrt{3})/2 = (1 \star_\lambda (1-\lambda)) \star_\lambda 1 \in Q_\lambda\,.$$

Thus from now on, we can assume that $|\lambda| > 1$.

If $\frac{\arg\lambda}{\pi}$ is irrational, then we can take ν to be some appropriate positive power of λ, so we can henceforth assume that $\frac{\arg\lambda}{\pi}$ is rational. Let $n \in \mathbb{Z}$ be least such that $n > 1$ and $\lambda^n > 0$ (in fact, we must have $n > 2$ since $\lambda \notin \mathbb{R}$). Then for any nonnegative $k \in \mathbb{Z}$, we have $\arg(\lambda^{kn}) = 0$, and thus $\arg(\lambda^{kn+1}) = \arg\lambda$. Set $\gamma_k := 1 - \lambda^{kn+1}$. We have $\gamma_k \in Q_\lambda$ for all integers $k \ge 0$, and $|\gamma_k| \ge |\lambda|^{kn+1} - 1 > 1$ if k is sufficiently large.

Now consider $\arg \gamma_k$ for any k such that $|\gamma_k| > 1$. If $\frac{\arg \gamma_k}{\pi}$ is irrational, then we can let ν be some appropriate positive power of γ_k, as we did above with λ. Otherwise, let $m \in \mathbb{Z}$ be least such that $m > 1$ and $(\gamma_k)^m > 0$. Thus $\arg \gamma = 2\pi a/m$ for some $a \in \mathbb{Z}$ such that $-m/2 \leq a < m/2$ and a is coprime with m. Then there exists $b \in \mathbb{Z}$ such that $b > 0$ and $m \mid ab - 1$ (b is a modular reciprocal of a modulo m). This gives $\arg((\gamma_k)^b) = 2\pi/m$. If $m > 12$, then we can set $\nu := (\gamma_k)^b$, giving $0 < \arg \nu = 2\pi/m < \pi/6$. Thus the only unresolved case is where $m \leq 12$. Note that there are only finitely many possible values of $\arg \gamma_k$ with $m \leq 12$. Hence we finish by showing that there exists k such that this case does not happen.

It is evident on geometrical grounds that

$$(\arg \lambda) - \pi = (\arg(\lambda^{kn+1})) - \pi = \arg(-\lambda^{kn+1}) < \arg(1 - \lambda^{kn+1}) = \arg \gamma_k < 0 \,.$$

Letting $\theta_k := (\arg \gamma_k) - (\arg \lambda) + \pi$, we see that θ_k is one of the interior angles of the triangle $(0, -\lambda^{kn+1}, \gamma_k)$, namely, the angle at the origin. The interior angle at $-\lambda^{kn+1}$ is $\arg \lambda$. We have $0 < \theta_k < \pi/2$, and by the Law of Sines,

$$\sin \theta_k = \frac{\sin(\arg \lambda)}{|\gamma_k|} \,.$$

Since $|\gamma_k| \to \infty$ as $k \to \infty$, it follows that $\theta_k \to 0$ as $k \to \infty$. From this we see that there are infinitely many values of θ_k, and thus of $\arg(\gamma_k)$, for different k, and so for some positive $k \in \mathbb{Z}$ we have $|\gamma_k| > 1$ and $\arg((\gamma_k)^m) \not> 0$ for all integers $1 \leq m \leq 12$. □

9 R_λ when $\mathrm{Re}(\lambda) = 1/2$

As in previous sections, we use $\star$ without subscript to mean $\star_\lambda$.

Here we look at R_λ for some λ with real part $1/2$. For these λ, we have $1 - \lambda = \lambda^*$, and so R_λ is closed under complex conjugate. Furthermore, R_λ is convex iff R_{λ^*} is convex, and so we can assume throughout this section that $\mathrm{Im}(\lambda) \geq 0$. We also have in particular, $\lambda \star 0 = (1-\lambda)\lambda = |\lambda|^2$, and so $|\lambda|^2 \in R_\lambda$.

Proposition 9.1. *Suppose* $\mathrm{Re}(\lambda) = 1/2$ *and* $|\lambda| \leq \sqrt{3}$*. Then* R_λ *is convex if and only if* $|\lambda| \notin \{1, \sqrt{2}, \sqrt{3}\}$*.*

Proof. If $|\lambda| = \sqrt{n}$ for any $n \in \mathbb{Z}^+$, then $\lambda = (1 + i\sqrt{4n-1})/2$. In this case, λ is a nonreal quadratic integer. (In fact, λ is a root of the monic, quadratic polynomial $x^2 - x + n \in \mathbb{Z}[x]$, which is irreducible over $\mathbb{Q}$.) Thus $R_\lambda \subseteq \mathbb{Z}[\lambda] = \mathbb{Z} + \lambda\mathbb{Z}$, which is a discrete subring of $\mathbb{C}$.

Now suppose $|\lambda| \notin \{1, \sqrt{2}, \sqrt{3}\}$. We have $1/4 \leq |\lambda|^2 < 3$ but $|\lambda|^2 \notin \{1, 2\}$. If, in addition, $|\lambda|^2 \neq 1 + \varphi$, then $R_{|\lambda|^2}$ is convex by previous results. Since $|\lambda|^2 \in R_\lambda$, we have that R_λ is convex for these λ by Corollary 2.30.

Finally, we consider the case where $|\lambda|^2 = 1 + \varphi$, or equivalently, $\lambda = (1 + i\sqrt{5 + 2\sqrt{5}})/2$. We have $|\lambda| = \varphi$ in this case, and in fact, $\lambda = \varphi e^{i\tau/5}$. The points $0, 1, \lambda$ form the vertices of an acute Robinson triangle, i.e., a triangle with side lengths $(1, \varphi, \varphi)$. Given what we know about $R_{|\lambda|^2} = R_{1+\varphi}$, it may come as a surprise that R_λ is indeed convex. We show below via an explicit derivation that the point $\mu := e^{i\tau(7/10)}$ is in R_λ. (The derivation below was found by a computer-assisted search.) It then follows from Corollary 2.30 and Proposition 4.3 that R_λ is convex.

We first note that λ is an algebraic integer of degree 4 with minimum polynomial $x^4 - 2x^3 + 4x^2 - 3x + 1$. Thus

$$\lambda^4 = -1 + \lambda - 4\lambda^2 + 2\lambda^3 \; . \tag{4}$$

It can also be readily checked (on purely geometric grounds, even) that $\mu = 2\lambda - \lambda^2 + \lambda^3$. The derivation of μ follows:

$$\begin{aligned}
x_0 &:= 0 \\
x_1 &:= 1 \\
x_2 &:= x_0 \star x_1 = \lambda \\
x_3 &:= x_1 \star x_0 = 1 - \lambda \\
x_4 &:= x_0 \star x_2 = \lambda^2 \\
x_5 &:= x_2 \star x_1 = \lambda(2-\lambda) = 2\lambda - \lambda^2 \\
x_6 &:= x_1 \star x_3 = (1+\lambda)(1-\lambda) = 1 - \lambda^2 \\
x_7 &:= x_4 \star x_0 = \lambda^2(1-\lambda) = \lambda^2 - \lambda^3 \\
x_8 &:= x_1 \star x_6 = (1+\lambda+\lambda^2)(1-\lambda) = 1 - \lambda^3 \\
x_9 &:= x_7 \star x_5 = \lambda^2(3 - 3\lambda + \lambda^2) = -1 + 3\lambda - \lambda^2 - \lambda^3 && \text{(using (4))} \\
x_{10} &:= x_2 \star x_8 = \lambda(1-\lambda)(2+\lambda+\lambda^2) = 1 - \lambda + 3\lambda^2 - 2\lambda^3 && \text{(using (4))} \\
\mu = x_{11} &:= x_9 \star x_{10} = \lambda^2(1-\lambda)(5 - 2\lambda + 2\lambda^2) = 2\lambda - \lambda^2 + \lambda^3 && \text{(using (4))}
\end{aligned}$$

□

Corollary 9.2. *Suppose* $\mathrm{Re}(\lambda) = 1/2$*, and let* $y := |\mathrm{Im}(\lambda)|$*. If* $y < \sqrt{11}/2$ *and* $y \notin \left\{\sqrt{3}/2, \sqrt{7}/2\right\}$*, then* R_λ *is convex.*

10 R_λ for λ contained in a discrete subring of $\mathbb{C}$

In this section, consider the case (hinted at in the previous section) where λ belongs to a discrete subring of $\mathbb{C}$. We start with two standard lemmas that characterize the discrete subrings of $\mathbb{C}$.

Lemma 10.1. *Suppose D is a subring of $\mathbb{C}$ that is discrete in the induced topology. Then no two distinct elements of D are less than unit distance apart. Consequently, D is (topologically) closed, and $D \cap \mathbb{R} = \mathbb{Z}$.*

Proof. Suppose for the sake of contradiction that $a, b \in D$ are such that $0 < |a - b| < 1$. Then $(a-b)^n \in D \setminus \{0\}$ for all integers $n > 0$, and $\lim_{n\to\infty}(a-b)^n = 0$. This means that $0 \in D$ is an accumulation point of D, and hence D is not discrete. The other two consequences follow immediately. □

Lemma 10.2. *Suppose D is a subring of $\mathbb{C}$ that is discrete in the induced topology. Then either $D = \mathbb{Z}$ or $D = \mathbb{Z}[\alpha] = \mathbb{Z} + \alpha\mathbb{Z}$, where α is a nonreal quadratic integer. Equivalently, either $D = \mathbb{Z}$ or there exists $n \in \mathbb{Z}^+$ such that $D = \mathbb{Z} + \alpha\mathbb{Z}$, where α is either $i\sqrt{n}$ or $(1 + i\sqrt{4n-1})/2$.*

Proof. $\mathbb{Z}$ is the smallest subring of $\mathbb{C}$ and is discrete. If $D \neq \mathbb{Z}$, then choose some $\alpha \in D \setminus \mathbb{Z}$. Then α cannot be real by Lemma 10.1. Since $D \setminus \mathbb{Z}$ is closed, we can choose α to have minimum norm.

This implies that $-1/2 \le \mathrm{Re}(\alpha) \le 1/2$, for otherwise, we could add some appropriate integer to α to reduce its norm. We also have $|\alpha| \ge 1$ by Lemma 10.1.

Since $\alpha \in D$, we have $\mathbb{Z} + \alpha\mathbb{Z} \subseteq D$. We now show that $D \subseteq \mathbb{Z} + \alpha\mathbb{Z}$, and thus equality holds. Suppose otherwise, and let β be some element of $D \setminus (\mathbb{Z}+\alpha\mathbb{Z})$. By adding some appropriate member of $\mathbb{Z}+\alpha\mathbb{Z}$ to β, we can assume that β lies somewhere in the parallelogram P with corners $(1+\alpha)/2$, $(1-\alpha)/2)$, $(-1+\alpha)/2$, and $(-1-\alpha)/2$, but not the origin. P is included in the larger parallelogram P' with corners ± 1 and $\pm\alpha$. The norm of any non-corner point in P' is strictly bounded by the norm of one of the corners of P', which is $|\alpha|$, since $|\alpha| \ge 1$. The corners of P' are not included in P, and so we must have $|\beta| < |\alpha|$, contradicting the minimality of $|\alpha|$.

Thus $D = \mathbb{Z}[\alpha] = \mathbb{Z}+\alpha\mathbb{Z}$, and it follows that α is a quadratic integer. By the quadratic formula, all quadratic integers are of the form $(m \pm \sqrt{m^2 - 4n})/2$ for some $m, n \in \mathbb{Z}$. Since $\alpha \notin \mathbb{R}$, we must have $n > 0$, and without loss of generality, we can assume $-1/2 < \mathrm{Re}(\alpha) \le 1/2$ and so $m \in \{0, 1\}$, which gives the result. □

From Fact 4.4 and Lemma 10.2 it follows that if D is discrete and $\lambda \in D$, then $R_\lambda = Q_\lambda \subseteq D$ and is discrete as well. Sometimes equality holds in the inclusion above. For example,

Fact 10.3. *$R_2 = R_{-1} = \mathbb{Z}$.*

Usually, equality does not hold; R_2 is the only case where equality holds for $D := \mathbb{Z}$. R_λ is a proper subset of $\mathbb{Z}$ for all $\lambda \ge 3$, as the next general lemma implies.

Lemma 10.4. *Let D be any subring of $\mathbb{C}$. For any $\lambda \in D$, let $I_\lambda := \lambda(1-\lambda)D = \{a\lambda(1-\lambda) \mid a \in D\}$ be the ideal of D generated by $\lambda(1-\lambda)$. Then*

$$Q_\lambda \subseteq I_\lambda + \{0, 1, \lambda, 1-\lambda\} .$$

If D is discrete, then the same inclusion holds for R_λ.

Proof sketch. One merely checks that the right-hand side is λ-convex. □

The next corollary follows from Lemma 10.4 by the Chinese Remainder Theorem.

Corollary 10.5. *Let D and λ be as in Lemma 10.4, above. Then*

$$Q_\lambda \subseteq (\lambda D + \{0, 1\}) \cap ((1-\lambda)D + \{0, 1\}) .$$

If D is discrete, then the same inclusion holds for R_λ.

When applying Lemma 10.4 with $D := \mathbb{Z}$, it suffices to consider $\lambda > 1$, so in this case, we will assume $\lambda \ge 2$.

Corollary 10.6. *For all $\lambda \in \mathbb{Z}$ such that $\lambda \ge 2$ and all $n \in \mathbb{Z}$, if $n \in R_\lambda$, then $n \equiv d \pmod{\lambda(\lambda-1)}$ for some $d \in \{0, 1, \lambda, 1-\lambda\}$. Equivalently, if n is in R_λ then n is congruent to either 0 or 1 modulo both λ and $\lambda - 1$. In particular, if $n \in R_\lambda$, then $n \equiv n^2 \pmod{\lambda(\lambda-1)}$.*

One would generally like to know when equality holds in Lemma 10.4 for discrete D. We show that it holds at least for $D := \mathbb{Z}$ (Theorem 10.12, below), but we currently have no general proof for all discrete D.

We can at least prove a sufficient condition for equality (Theorem 10.10, below). First, a definition, which is justified by the lemma that follows it.

Definition 10.7. We will say that a point $x \in \mathbb{C}$ is a *translation point* of R_λ iff $\{x, x+1\} \subseteq R_\lambda$.

Lemma 10.8. *If x is a translation point for R_λ, then so is $-x$, and furthermore, $\rho_{x,x+1}(R_\lambda) = \rho_{-x,-x+1}(R_\lambda) = R_\lambda$.*

Proof. $R_\lambda = 1 - R_\lambda$ by Corollary 2.29, so if x is a translation point of R_λ, then so is $-x$. We have $\rho_{x,x+1}(R_\lambda) \subseteq R_\lambda$ and $\rho_{-x,-x+1}(R_\lambda) \subseteq R_\lambda$ by Corollary 2.24. To get the reverse containments, we observe that $\rho_{x,x+1}$ and $\rho_{-x,-x+1}$ are inverses of each other, and so, applying $\rho_{-x,-x+1}$ to both sides of the first containment, we get

$$R_\lambda = \rho_{-x,-x+1}(\rho_{x,x+1}(R_\lambda)) \subseteq \rho_{-x,-x+1}(R_\lambda) ,$$

and applying $\rho_{x,x+1}$ to the second containment similarly yields $R_\lambda \subseteq \rho_{x,x+1}(R_\lambda)$. $\square$

Corollary 10.9. *For any $\lambda \in \mathbb{C}$, the translation points of R_λ form a subgroup of $\mathbb{C}$ under addition.*

Theorem 10.10. *Let D be a discrete subring of $\mathbb{C}$, and let $S \subseteq D$ generate the additive group of D. Suppose $\lambda \in D$ is such that $a\lambda(1-\lambda)$ is a translation point of R_λ for every $a \in S$. Then*

$$R_\lambda = I_\lambda + \{0, 1, \lambda, 1-\lambda\} , \tag{5}$$

where $I_\lambda := \lambda(1-\lambda)D \subseteq D$ is the ideal generated by $\lambda(1-\lambda)$.

Proof. Since D is discrete, we have $R_\lambda = Q_\lambda$, and so the $\subseteq$-inclusion holds by Lemma 10.4. For the reverse inclusion, we have by assumption and Corollary 10.9 that every element of I_λ is a translation point of R_λ. Now suppose $x \in I_\lambda + b$ for some $b \in \{0, 1, \lambda, 1-\lambda\}$. Note that $b \in R_\lambda$. Writing $x := y + b$ for $y \in I_\lambda$, we have

$$x = \rho_{y,y+1}(b) \in \rho_{y,y+1}(R_\lambda) = R_\lambda$$

by Lemma 10.8, because y is a translation point of R_λ. This proves the $\supseteq$-inclusion. $\square$

Theorem 10.10 is useful because the rings in question are finitely generated $\mathbb{Z}$-modules, and so Equation (5) can be verified by testing a finite number of points. For example, Figure 8 shows R_{2i}. Equation (5) holds for $\lambda := 2i$, because $\mathbb{Z}[2i]$ is spanned by $\{1, 2i\}$, and it is evident from the picture that both $4 + 2i = \lambda(1-\lambda)$ and $-4 + 8i = 2i\lambda(1-\lambda)$ are both translation points of R_{2i}.

We end this section by showing (Lemma 10.11 and Theorem 10.12) that Equation (5) holds for all integer $\lambda \geq 2$ (that is, when $D = \mathbb{Z}$ and $I_\lambda = \lambda(1-\lambda)\mathbb{Z}$ in Theorem 10.10). It follows immediately that R_λ is periodic for all $\lambda \in \mathbb{Z} \setminus \{0, 1\}$ and that the period is $\lambda(\lambda-1)$ if $\lambda \geq 3$ (Corollary 10.13). Most of the technical difficulty is in proving Lemma 10.11, so we defer that proof until after Theorem 10.12.

Lemma 10.11. *For every $\lambda \in \mathbb{Z}$ with $\lambda \geq 3$, the value $\lambda(\lambda-1)$ is contained in R_λ; in fact, it is in $Q_\lambda^{(\lambda-1)}$ unless $\lambda = 3$ or $\lambda = 5$, in which case, it is in $Q_\lambda^{(\lambda)}$.*

Theorem 10.12. *For all integers $\lambda \notin \{0, 1\}$,*

$$R_\lambda = \lambda(\lambda-1)\mathbb{Z} + \{0, 1, \lambda, 1-\lambda\} .$$

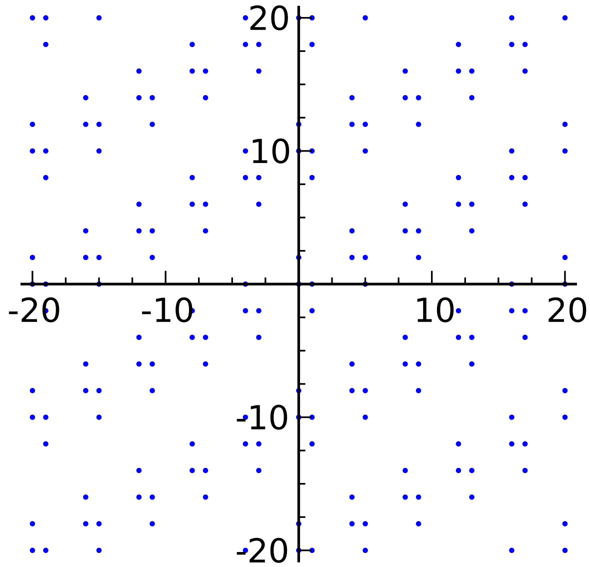


Figure 8: A plot of R_{2i}.

Proof. We can assume WLOG that $\lambda \geq 2$, since both sides of the equation are unchanged by substituting $1-\lambda$ for λ everywhere (q.v. Fact 2.13). The $\lambda = 2$ case is obvious, so assume $\lambda \geq 3$.

Along with 0 and 1, the following are all elements of R_λ:

$$\begin{aligned} 0 \star 1 &= \lambda \ , \\ 1 \star 0 &= 1 - \lambda \ , \\ \lambda \star 0 &= \lambda(1-\lambda) \ , \\ 1 - \lambda(1-\lambda) &= \lambda(\lambda - 1) + 1 \ . \end{aligned}$$

Thus by Lemma 10.11, $\lambda(\lambda-1)$ is a translation point of R_λ. Since the additive group of $\mathbb{Z}$ is generated by $\{1\}$, the theorem follows by Theorem 10.10. $\square$

Proof of Lemma 10.11. Let $n > 1$ be an integer. By Lemma 7.8, $\lambda(\lambda-1) \in Q_\lambda^{(n)}$ if and only if there exist integers $b_0, \ldots, b_n$ such that $0 \leq b_i \leq \binom{n}{i}$ for all $0 \leq i \leq n$ and

$$\lambda(\lambda-1) = \sum_{i=0}^{n} b_i \lambda^i (1-\lambda)^{n-i} \ .$$

Letting b_0 and b_n both be zero[5] and dividing both sides by $\lambda(1-\lambda)$, we get the equivalent equation

$$-1 = \sum_{i=1}^{n-1} b_i \lambda^{i-1}(1-\lambda)^{n-i-1} = \sum_{j=0}^{m} c_j \lambda^j \mu^{m-j} \ , \tag{6}$$

[5]This is necessary, as can be seen by reducing the above equation modulo λ and $\lambda - 1$, respectively, and noting that $b_0, b_n \in \{0, 1\}$.

where we define $m := n-2$ and $c_j := b_{j+1}$ for all $0 \le j \le m$, and we set $\mu := 1-\lambda$ for convenience. The range requirement of each c_j is then

$$0 \le c_j \le \binom{m+2}{j+1} \qquad (\forall j,\ 0 \le j \le m)\,. \tag{7}$$

Thus the theorem is proved for λ if we can find an appropriate m and integer values for $c_0, \dots, c_m$ satisfying both (6) and (7).

Here are explicit values for m and $c_0, \dots, c_m$ satisfying (6) and (7) for $3 \le \lambda \le 8$:

λ	m	c_0	c_1	c_2	c_3	c_4	c_5
3	1	2	1				
4	1	3	2				
5	3	4	1	2	3		
6	3	5	2	3	4		
7	4	6	9	0	3	5	
8	5	7	2	6	4	3	6

Thus from now on, we can assume that $\lambda \ge 9$. We then must set $n := \lambda - 1$, whence $m = \lambda - 3 \ge 6$.

To find $c_0, \dots, c_m$ satisfying (6) and (7), we first find values for the c_j that satisfy (6) but ignore the range requirements (7). We then make a series of adjustments to the c_j in a way that leaves the right-hand side of (6) unchanged, until all coefficients are in their required ranges.

Initially, we set $c_j := -\binom{m}{j}$ for all $0 \le j \le m$.[6] These values satisfy (6) by the Binomial Theorem, using the fact that $\lambda + \mu = 1$. Now what kind of adjustment to the c_j preserves (6)? Choose some j with $0 \le j < m$. If we simultaneously add λ to c_j and subtract μ from c_{j+1} (equivalently, add $\lambda - 1$ to c_{j+1}), then these two changes clearly cancel, and the right-hand side of (6) is unchanged. We call this a *j-adjustment*:

j-adjust:
$c_j \leftarrow c_j + \lambda$
$c_{j+1} \leftarrow c_{j+1} + \lambda - 1$
end

Note that this adjustment increases the values of both c_j and c_{j+1}. Our strategy is then to choose different values of j in some order and, for each j, make just enough j-adjustments so that either c_j is positive or c_{j+1} is positive, depending on the value of j.

The order of our choices of j is important. The strictest range requirements are at the "ends," i.e., for j close to 0 or m. We make those adjustments first, working our way inward, finishing somewhere in the middle. Recalling that $m \ge 6$ and $\lambda = m+3$, we define the middle index to be

$$p := \left\lceil \frac{\lambda m - \lambda + 1}{2\lambda - 1} \right\rceil = \left\lceil \frac{m^2 + 2m - 2}{2m+5} \right\rceil = \left\lceil \frac{m}{2} \right\rceil\,. \tag{8}$$

Note that $3 \le p \le m-3$. We choose this value because setting $j := p$ maximizes the function $t(j)$, defined as

$$t(j) := \binom{m}{j} \left(\frac{\lambda}{\lambda - 1} \right)^j\,,$$

[6] It will be convenient in this proof to treat the c_j as variables whose values can change, as in a computer algorithm, rather than choosing new symbols to denote changed values. Which values of the c_j we are referring to will always be clear from the context.

giving us the most leeway where we need it. One can readily check that the sequence $t(0), \ldots, t(m)$ is bitonic, ascending monotonically from $t(0)$ to $t(p)$, then descending monotonically from $t(p)$ to $t(m)$.

Here is the algorithm to obtain $c_0, \ldots, c_m$ satisfying (6) and all range requirements (7), with explanation afterwards:

```
// Initialization
for j ← 0 to m do
    c_j ← −(m choose j)
endfor
// Adjusting to the left of index p
for j ← 0 to p − 1 do
    while c_j < 0 do
        j-adjust
    endwhile
    // 0 ≤ c_j < λ, satisfying (7), and c_j is not changed subsequently.
endfor
// c_p < λ
// Adjusting to the right of index p
for j ← m downto p + 2 do
    while c_j < 0 do
        (j − 1)-adjust
    endwhile
    // 0 ≤ c_j < λ, satisfying (7), and c_j is not changed subsequently.
endfor
// c_{p+1} < λ
// Adjusting at index p
while c_p < 0 or c_{p+1} < 0 do
    p-adjust
endwhile
```

To show that this algorithm is correct, we first justify the assertions made in comments after the first two inner while-loops. First, c_0 is changed by a single 0-adjustment from -1 to $\lambda - 1$. For $k = 1, 2, \ldots, p-1$ in that order, c_k is first increased by a sequence of $(k-1)$-adjustments followed by zero or more k-adjustments. Since $c_{k-1} \geq -\binom{m}{k-1}$ immediately before the $(k-1)$-adjustments, and each $(k-1)$-adjustment increases c_{k-1} by λ, the number a of $(k-1)$-adjustments satisfies $a \leq \left\lceil \lambda^{-1}\binom{m}{k-1} \right\rceil$. Thus after all $(k-1)$-adjustments, we have (recalling that $t(k-1) \leq t(k)$)

$$
\begin{aligned}
c_k &= -\binom{m}{k} + (\lambda - 1)a < -\binom{m}{k} + (\lambda - 1)\left(\lambda^{-1}\binom{m}{k-1} + 1\right) \\
&= \binom{m}{k}\left(-1 + \frac{t(k-1)}{t(k)}\right) + \lambda - 1 \leq \lambda - 1 \, .
\end{aligned}
$$

Then the k-adjustments (if any) leave $0 \leq c_k < \lambda$ as required, and c_k is not changed subsequently. This justifies the comment after the first inner while-loop. Note that this reasoning also applies to c_p, showing that $c_p < \lambda$ after the second for-loop (although c_p may still be negative at that point).

Justifying the comment after the second inner while-loop is similar. First, c_m is changed by a single $(m-1)$-adjustment, going from -1 to $\lambda-2$. Then for $k = m-1, m-2, \ldots, p+2$ in that order, c_k is first increased by a sequence of k-adjustments followed by zero or more $(k-1)$-adjustments. Since $c_{k+1} \geq -\binom{m}{k+1}$ immediately before the k-adjustments, and each k-adjustment increases c_{k+1} by $\lambda-1$, the number b of k-adjustments satisfies $b \leq \left\lceil (\lambda-1)^{-1}\binom{m}{k+1} \right\rceil$. Thus after all k-adjustments, we have (recalling that $t(k+1) \leq t(k)$)

$$\begin{aligned} c_k &= -\binom{m}{k} + \lambda b < -\binom{m}{k} + \lambda\left((\lambda-1)^{-1}\binom{m}{k+1} + 1\right) \\ &= \binom{m}{k}\left(-1 + \frac{t(k+1)}{t(k)}\right) + \lambda \leq \lambda\ . \end{aligned}$$

Then the $(k-1)$-adjustments (if any) leave $0 \leq c_k < \lambda$ as required, and c_k is not changed subsequently. This justifies the comment after the second inner while-loop. Like before, this reasoning also applies to c_{p+1}, showing that $c_{p+1} < \lambda$ after the third for-loop (although c_{p+1} may still be negative at that point).

It remains to show that after the last while-loop, c_p and c_{p+1} satisfy (7). That loop results in both c_p and c_{p+1} being nonnegative. Thus we are done if $c_p \leq \binom{m+2}{p+1}$ and $c_{p+1} \leq \binom{m+2}{p+2}$ in the end. Let r and s be the number of p-adjustments needed to get $c_p \geq 0$ and $c_{p+1} \geq 0$, respectively. Since $c_p \geq -\binom{m}{p}$ and $c_{p+1} \geq -\binom{m}{p+1}$ just before the final while-loop, we have

$$r \leq \left\lceil \lambda^{-1}\binom{m}{p} \right\rceil = \left\lceil t(p)\frac{(\lambda-1)^p}{\lambda^{p+1}} \right\rceil\ , \qquad s \leq \left\lceil (\lambda-1)^{-1}\binom{m}{p+1} \right\rceil = \left\lceil t(p+1)\frac{(\lambda-1)^p}{\lambda^{p+1}} \right\rceil\ .$$

Since $t(p) \geq t(p+1)$, the last while-loop runs at most $\left\lceil \lambda^{-1}\binom{m}{p} \right\rceil$ times, and since $c_p < \lambda$ and $c_{p+1} < \lambda$ just before this loop, the final values of c_p and c_{p+1} then satisfy

$$\begin{aligned} c_p &< \lambda + \lambda\left\lceil \lambda^{-1}\binom{m}{p} \right\rceil < \lambda + \lambda\left(\lambda^{-1}\binom{m}{p} + 1\right) = \binom{m}{p} + 2\lambda\ , \\ c_{p+1} &< \lambda + (\lambda-1)\left\lceil \lambda^{-1}\binom{m}{p} \right\rceil \leq \lambda + \lambda\left\lceil \lambda^{-1}\binom{m}{p} \right\rceil < \lambda + \lambda\left(\lambda^{-1}\binom{m}{p} + 1\right) = \binom{m}{p} + 2\lambda\ . \end{aligned}$$

Since each quantity in the inequalities above is an integer, and there are two strict inequalities in each chain, we have both c_p and c_{p+1} ending up less than or equal to $f := \binom{m}{p} + 2(\lambda-1) = \binom{m}{p} + 2(m+2)$. To finish the proof, we show that $f \leq \binom{m+2}{p+1}$ and $f \leq \binom{m+2}{p+2}$.

By a straightforward calculation,

$$\binom{m+2}{p+2} = \frac{m-p}{p+1}\binom{m+2}{p+1} = \frac{(m+2)(m+1)}{(p+2)(p+1)}\binom{m}{p}\ .$$

By (8), we have $m-p \leq p+1$, whence $\binom{m+2}{p+2} \leq \binom{m+2}{p+1}$, and so it suffices to show that $f \leq \binom{m+2}{p+2}$, or equivalently,

$$\binom{m}{p} + 2(m+2) \leq \frac{(m+2)(m+1)}{(p+2)(p+1)}\binom{m}{p}\ ,$$

i.e.,

$$2 \leq \binom{m}{p}\left(\frac{m+1}{(p+2)(p+1)} - \frac{1}{m+2}\right) .$$

Finally, to show this last inequality, recalling that $m \geq 6$ and $3 \leq p \leq m-3$, we have

$$\begin{aligned}
\binom{m}{p}\left(\frac{m+1}{(p+2)(p+1)} - \frac{1}{m+2}\right) &= \binom{m}{p}\frac{(m+2)(m+1)-(p+2)(p+1)}{(m+2)(p+2)(p+1)} \\
&\geq \binom{m}{3}\frac{(m+2)(m+1)-(m-1)(m-2)}{(m+2)(m-1)(m-2)} \\
&= \binom{m}{3}\frac{6m}{(m+2)(m-1)(m-2)} = \frac{m^2}{m+2} \geq \frac{9}{2} \geq 2 .
\end{aligned}$$

□

Remark. The ranks given in Theorem 10.12 are tight, i.e., $6 \notin Q_3^{(2)}$ and $20 \notin Q_5^{(4)}$ (which can be checked by exhaustive search), and $\lambda(\lambda-1) \notin Q_\lambda^{(\lambda-2)}$ for any $\lambda \geq 3$. The latter can be seen by reducing (6) modulo λ, which gives $c_0 \equiv -1 \pmod{\lambda}$. Since $0 \leq c_0 \leq \binom{m+2}{1} = m+2$, we must have $\lambda - 1 \leq c_0 \leq m+2$. In particular, $m \geq \lambda - 3$, which implies $n \geq \lambda - 1$. This also shows that no single choice of m can satisfy (6) for all λ. □

Corollary 10.13. *R_λ is periodic with period $\lambda(\lambda-1)$ for all $\lambda \in \mathbb{Z}$ such that $\lambda \geq 3$.*

11 A characterization of $Q_{[x]}$ with some applications

In this section we prove a simple characterization of $Q_{[x]}$ (see Definition 7.1) beyond the characterization given in Lemma 7.8 (and by extension, a new characterization of Q_λ). This lets us, among other things, list all the polynomials in $Q_{[x]}$ of degree ≤ 2 and get a finite upper bound on the number of polynomials in $Q_{[x]}$ of any given degree bound.

Recall that $Q_{[x]} \subseteq \mathbb{Z}[x]$.

Theorem 11.1. *Let f be any polynomial in $\mathbb{Z}[x]$. Then $f \in Q_{[x]}$ if and only if either $f = 0$ or $f = 1$ or $0 < f(\lambda) < 1$ for all $0 < \lambda < 1$.*

Corollary 11.2. *$Q_{[x]}$ is closed under multiplication and the operator $p \mapsto 1-p$.*

Corollary 11.3. *For any $\lambda \in \mathbb{C}$,*

$$Q_\lambda = \{0,1\} \cup \{p(\lambda) \mid p \in \mathbb{Z}[x] \ \& \ 0 < p(\mu) < 1 \text{ for all } 0 < \mu < 1\} .$$

Corollary 11.4. *For every $\lambda \in \mathbb{C}$, Q_λ contains arbitrarily long finite arithmetic progressions. In particular, for every integer $n \geq 0$,*

$$\left\{k\lambda^n(1-\lambda)^n \mid k \in \mathbb{Z} \ \& \ 0 \leq k < 2^{2n}\right\} \subseteq Q_\lambda .$$

Before proving Theorem 11.1, we need a definition and a few lemmas. We extend the definition of the binomial coefficient $\binom{n}{k}$ in the usual way for all $n, k \in \mathbb{Z}$ with $n \geq 0$, namely, by defining $\binom{n}{k} := 0$ if $k < 0$ or $k > n$. Then the recurrence $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ holds for all such n and k.

Definition 11.5. Let $f \in \mathbb{C}[x]$ be any polynomial, and let n be any nonnegative integer such that $\deg(f) \le n$. We let $f_0^{(n)}, f_1^{(n)}, \dots, f_n^{(n)} \in \mathbb{C}$ denote the unique coefficients such that $f(x) = \sum_{k=0}^{n} f_k^{(n)} x^k (1-x)^{n-k}$ (cf. Lemma 7.6). We define $f_k^{(n)} := 0$ for all $k < 0$ and $k > n$.

The next two lemmas relate the $f^{(n)}$-coefficients for different n. The first lemma says that the $f^{(n)}$-coefficients satisfy the same "Pascal's triangle" recurrence as the binomial coefficients.

Lemma 11.6. *Let f and n be as in Definition 11.5. Then for any $k \in \mathbb{Z}$, $f_k^{(n+1)} = f_k^{(n)} + f_{k-1}^{(n)}$.*

Proof. This is clearly true for $k < 0$ and $k > n+1$, since both sides are 0. Moreover, we have

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} f_k^{(n)} x^k (1-x)^{n-k} = (x + (1-x)) \sum_{k=0}^{n} f_k^{(n)} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} f_k^{(n)} x^{k+1} (1-x)^{n-k} + \sum_{k=0}^{n} f_k^{(n)} x^k (1-x)^{n+1-k} \\ &= \sum_{k=1}^{n+1} f_{k-1}^{(n)} x^k (1-x)^{n+1-k} + \sum_{k=0}^{n} f_k^{(n)} x^k (1-x)^{n+1-k} \\ &= \sum_{k=0}^{n+1} \left(f_{k-1}^{(n)} + f_k^{(n)} \right) x^k (1-x)^{n+1-k} . \end{aligned}$$

Comparing coefficients with the equation $f(x) = \sum_{k=0}^{n+1} f_k^{(n+1)} x^k (1-x)^{n+1-k}$, we see that $f_k^{(n+1)} = f_{k-1}^{(n)} + f_k^{(n)}$ for all $0 \le k \le n+1$. □

The next lemma extends the previous one in a natural way.

Lemma 11.7. *Let $f \in \mathbb{C}[x]$ be any polynomial, and let m be any natural number such that $m \ge \deg(f)$. For any integers n and k such that $n \ge m$,*

$$f_k^{(n)} = \sum_{i=0}^{m} f_i^{(m)} \binom{n-m}{k-i} .$$

Proof. We proceed by induction on n. If $n = m$, then for $0 \le k \le m$ we have $\sum_{i=0}^{m} f_i^{(m)} \binom{n-m}{k-i} = f_k^{(m)} \binom{0}{0} = f_k^{(m)}$. Now suppose the lemma holds for some $n \ge m$. We have $f_k^{(n+1)} = f_k^{(n)} + f_{k-1}^{(n)}$ for any $k \in \mathbb{Z}$ by Lemma 11.6, and so by the inductive hypothesis,

$$\begin{aligned} f_k^{(n+1)} &= f_{k-1}^{(n)} + f_k^{(n)} = \sum_{i=0}^{m} f_i^{(m)} \binom{n-m}{k-1-i} + \sum_{i=0}^{m} f_i^{(m)} \binom{n-m}{k-i} \\ &= \sum_i f_i^{(m)} \left[\binom{n-d}{k-i} + \binom{n-d}{k-i-1} \right] = \sum_i f_i^{(m)} \binom{n+1-d}{k-i} . \end{aligned}$$

Thus the lemma holds for $n+1$. □

Lemma 11.8. *Let $f \in \mathbb{C}[x]$ be any polynomial, and let $d := \deg(f)$. For every $\varepsilon > 0$ there exists a $t \ge 0$ such that, for all natural numbers $n \ge d$ and all integers k such that $t < k < n - t$,*

$$\left| f(k/n) - \frac{f_k^{(n)} (1-k/n)^d}{\binom{n-d}{k}} \right| \le \varepsilon .$$

Proof. We can assume WLOG that $f \neq 0$ (and thus $d \geq 0$). Set $\lambda := k/n$. From Lemma 11.7, we have that if $d < k < n - d$, then $0 < \lambda < 1$ and

$$\begin{aligned} f_k^{(n)} &= \sum_{i=0}^{d} f_i^{(d)} \binom{n-d}{k-i} \\ &= \binom{n-d}{k} \left(f_0^{(d)} + f_1^{(d)} \frac{k}{n-d-k+1} + f_2^{(d)} \frac{k(k-1)}{(n-d-k+1)(n-d-k+2)} + \cdots \right) \\ &= \binom{n-d}{k} \sum_{i=0}^{d} f_i^{(d)} \prod_{j=1}^{i} \frac{k-j+1}{n-d-k+j} = \binom{n-d}{k} \sum_{i=0}^{d} f_i^{(d)} \prod_{j=1}^{i} \frac{\lambda - \frac{j-1}{n}}{1 - \lambda - \frac{d-j}{n}} . \end{aligned}$$

We then have

$$\left| f(\lambda) - \frac{f_k^{(n)}(1-\lambda)^d}{\binom{n-d}{k}} \right| = \left| \sum_{i=0}^{d} f_i^{(d)} \lambda^i (1-\lambda)^{d-i} - (1-\lambda)^d \sum_{i=0}^{d} f_i^{(d)} \prod_{j=1}^{i} \frac{\lambda - \frac{j-1}{n}}{1 - \lambda - \frac{d-j}{n}} \right| \tag{9}$$

$$\leq \sum_{i=0}^{d} \left| f_i^{(d)} \right| \cdot \left| \lambda^i (1-\lambda)^{d-i} - (1-\lambda)^d \prod_{j=1}^{i} \frac{\lambda - \frac{j-1}{n}}{1 - \lambda - \frac{d-j}{n}} \right| \tag{10}$$

$$= \sum_{i=0}^{d} \left| f_i^{(d)} \right| \cdot \lambda^i (1-\lambda)^{d-i} \cdot \left| 1 - \prod_{j=1}^{i} \left(\frac{\lambda - \frac{j-1}{n}}{\lambda} \right) \left(\frac{1-\lambda}{1 - \lambda - \frac{d-j}{n}} \right) \right| \tag{11}$$

$$\leq \sum_{i=0}^{d} \left| f_i^{(d)} \right| \cdot \left| 1 - \prod_{j=1}^{i} \left(\frac{\lambda - \frac{j-1}{n}}{\lambda} \right) \left(\frac{1-\lambda}{1 - \lambda - \frac{d-j}{n}} \right) \right| . \tag{12}$$

The product $\prod_{j=1}^{i} \cdots$ above is positive. If k and $n - k$ are both large compared to d, then it is also close to 1, but it may be less than or greater than 1, depending on λ. We will bound it from above and below. Note that, for $1 \leq j \leq d$,

$$1 - \frac{d}{k} = 1 - \frac{d}{\lambda n} < \frac{\lambda - \frac{j-1}{n}}{\lambda} < 1$$

and

$$1 < \frac{1-\lambda}{1 - \lambda - \frac{d-j}{n}} < 1 + \frac{d/n}{1 - \lambda - d/n} = 1 + \frac{d}{n-k-d} .$$

Letting $M := \sum_{i=0}^{d} \left| f_i^{(d)} \right|$, we see that (12) above is then less than or equal to

$$\begin{aligned} M \cdot \max & \left\{ 1 - \left(1 - \frac{d}{k}\right)^d , \left(1 + \frac{d}{n-k-d}\right)^d - 1 \right\} \\ & \leq M \cdot \max \left\{ \left(1 + \frac{d}{k}\right)^d - 1, \left(1 + \frac{d}{n-k-d}\right)^d - 1 \right\} \\ & \leq M \cdot \left(\max \left\{ \exp(d^2/k), \exp(d^2/(n-k-d)) \right\} - 1 \right) \end{aligned}$$

Now we just need to let $t \geq d$ be large enough so that this quantity is at most ε when $t < k < n-t$. Letting

$$t := \left\lceil \frac{d^2}{\log\left(\frac{\varepsilon}{M}+1\right)} + d \right\rceil$$

suffices. (Note that $M > 0$ (because $f \neq 0$) and that t only depends on f and ε and not on n.) $\square$

Lemma 11.9. *Let $f \in \mathbb{R}[x]$ be a polynomial such that $f(\lambda) > 0$ for all $0 < \lambda < 1$. Then for all sufficiently large n, $f_0^{(n)}, \ldots, f_n^{(n)} \geq 0$.*

Proof. Notice that if $f_0^{(n)}, \ldots, f_n^{(n)} \geq 0$ for *some* natural number $n \geq \deg(f)$, then by Lemma 11.7, we have $f_0^{(n')}, \ldots, f_n^{(n')} \geq 0$ for all $n' \geq n$ as well. Thus it suffices to find some n such that $f_0^{(n)}, \ldots, f_n^{(n)} \geq 0$.

We use induction on $d := \deg(f)$. If $d = 0$, then $f(x)$ is some constant $c > 0$. We then have $f_0^{(0)} = c > 0$, and so taking $n := 0$ suffices. Now suppose $d > 0$ and the lemma holds for all polynomials of degree less than d. If $f(0) = 0$, then $f(x) = xg(x)$ for some polynomial $g \in \mathbb{R}[x]$ of degree $d-1$. By the inductive hypothesis, there is an $n \geq d-1$ such that $g_0^{(n)}, \ldots, g_n^{(n)} \geq 0$. Then $f_0^{(n+1)} = f(0) = 0$, and for all $1 \leq k \leq n+1$, we see that $f_k^{(n+1)} = g_{k-1}^{(n)} \geq 0$. Thus the lemma holds for f witnessed by $n+1$.

A similar argument applies if $f(1) = 0$: we get $f(x) = (1-x)h(x)$ for some h of degree $d-1$. Letting n be such that $h_0^{(n)}, \ldots, h_n^{(n)} \geq 0$, we have $f_k^{(n+1)} = h_k^{(n)} \geq 0$ for all $0 \leq k \leq n$, and furthermore, $f_{n+1}^{(n+1)} = f(1) = 0$.

We can now assume (by the continuity of f) that $f(0) > 0$ and $f(1) > 0$. By compactness, there exists $\varepsilon > 0$ such that $f(\lambda) \geq \varepsilon$ for all $0 \leq \lambda \leq 1$. Now given f and ε, let t be the number obtained from Lemma 11.8. Then for all $n \geq d$ and all k such that $t < k < n-t$, that lemma implies $f_k^{(n)} \geq 0$, because $f_k^{(n)}(1-k/n)^d/\binom{n-d}{k} \geq 0$. (The latter quantity is within ε of $f(k/n)$, which itself is at least ε.)

It remains to show that if n is sufficiently large, then $f_k^{(n)} \geq 0$ and $f_{n-k}^{(n)} \geq 0$ for all $0 \leq k \leq t$. To this end, it suffices to prove the following statement for all k, which we do by induction on k:

There exists an integer $n_k \geq d$ such that, for all integers $n \geq n_k$, $f_k^{(n)} \geq \varepsilon$ and $f_{n-k}^{(n)} \geq \varepsilon$.

For $k = 0$, we have $f_0^{(n)} = f(0) \geq \varepsilon$ and $f_n^{(n)} = f(1) \geq \varepsilon$ for all $n \geq d$, and so we can set $n_0 := d$. Now let $k > 0$, and assume the statement holds for $k-1$. Let $y := f_k^{(n_{k-1})}$. (It could be that $y < 0$.) By Lemma 11.6,

$$f_k^{(n_{k-1}+1)} = f_k^{(n_{k-1})} + f_{k-1}^{(n_{k-1})} \geq y + \varepsilon\ .$$

Similarly,

$$f_k^{(n_{k-1}+2)} = f_k^{(n_{k-1}+1)} + f_{k-1}^{(n_{k-1}+1)} \geq y + 2\varepsilon\ ,$$

and so on, yielding, for all $\ell \geq n_{k-1}$,

$$f_k^{(\ell)} \geq y + (\ell - n_{k-1})\varepsilon\ .$$

Thus $f_k^{(\ell)} \geq \varepsilon$ for all ℓ large enough. By a similar argument, letting $z := f_{n_{k-1}-k}^{(n_{k-1})}$, we get

$$f_{m-k}^{(m)} \geq z + (m - n_{k-1})\varepsilon$$

for all $m \geq n_{k-1}$. Now setting

$$n_k := \left\lceil \max\left\{ n_{k-1},\ n_{k-1} + 1 - \frac{y}{\varepsilon},\ n_{k-1} + 1 - \frac{z}{\varepsilon} \right\} \right\rceil ,$$

the statement holds for k. $\square$

Proof of Theorem 11.1. First we show the "only if" part. If $f \in Q_{[x]}$, then $f \in Q_{[x]}^{(n)}$ for some n. By Lemma 7.8, there exist integers $b_0, \ldots, b_n$ such that $0 \leq b_k \leq \binom{n}{k}$ for all $0 \leq k \leq n$ and $f(x) = \sum_{k=0}^{n} b_k x^k (1-x)^{n-k}$. (In fact, $b_k = f_k^{(n)}$ by Lemma 7.6.) If $b_k = 0$ for all k, then $f = 0$. If $b_k = \binom{n}{k}$ for all k, then $f = (x + (1-x))^n = 1$ by the Binomial Theorem. Otherwise, some $b_i > 0$, and this clearly implies $f(\lambda) > 0$ for all $0 < \lambda < 1$; also, some $b_j < \binom{n}{j}$, which similarly implies $f(\lambda) < 1$ for all $0 < \lambda < 1$.

Now we show the "if" part. If $f = 0$ or $f = 1$, then $f \in Q_{[x]}^{(0)}$, and we are done. Otherwise, if $0 < f(\lambda) < 1$ for all $0 < \lambda < 1$, then by Lemma 11.9, there exists n' such that for all $n \geq n'$ and $0 \leq k \leq n$, we have $f_k^{(n)} \geq 0$. Letting $g := 1 - f$, we have $0 < g(\lambda)$ for all $0 < \lambda < 1$, and so also by Lemma 11.9, there exists n'' such that $g_k^{(n)} \geq 0$ for all $n \geq n''$ and $0 \leq k \leq n$. Now let $n := \max\{n', n''\}$. From the Binomial Theorem,

$$1 = (x + (1-x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} ,$$

viewed as a polynomial in x. Then

$$g(x) = 1 - f(x) = \sum_{k=0}^{n} \left[\binom{n}{k} - f_k^{(n)} \right] x^k (1-x)^{n-k} .$$

Comparing coefficients, we have $0 \leq g_k^{(n)} = \binom{n}{k} - f_k^{(n)}$, and thus $0 \leq f_k^{(n)} \leq \binom{n}{k}$, for all $0 \leq k \leq n$.

We can now apply Lemma 7.8 again (letting $b_k := f_k^{(n)}$) to put f into $Q_{[x]}^{(n)}$ and be done, provided $f_0^{(n)}, \ldots, f_n^{(n)}$ are all integers. This is true, and one way to see it is as follows: Consider the $\mathbb{C}$-linear map $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ that maps any vector $(b_0, \ldots, b_n)$ to the unique vector $(c_0, \ldots, c_n)$ such that

$$\sum_{k=0}^{n} b_k x^k (1-x)^{n-k} = \sum_{k=0}^{n} c_k x^k .$$

(The left-hand side is a polynomial in x of degree $\leq n$, so this map is well-defined and easily seen to be linear.) Let M be the $(n+1) \times (n+1)$ matrix representing φ. By expanding the left-hand side above for various choices of $(b_0, \ldots, b_n)$, one sees that M is an integer matrix. Note that if $b_0 = \cdots = b_{i-1} = 0$ for some $i \leq n$, then x^i divides the left-hand side, and thus $c_0 = \ldots c_{i-1} = 0$. This means that M is triangular. If, in addition, $b_i = 1$, then $c_i = 1$ as well, and this means that all diagonal entries of M are 1. Therefore, $\det M = 1$, which implies M^{-1} is an integer matrix. Now since $f \in \mathbb{Z}[x]$, we have $f(x) = \sum_{k=0}^{n} c_k x^k$ for integers $c_0, \ldots, c_n$. If follows that $f_0^{(n)}, \ldots, f_n^{(n)}$ are all integers, since $(f_0^{(n)}, \ldots, f_n^{(n)}) = \varphi^{-1}(c_0, \ldots, c_n)$. $\square$

Proposition 11.10.

1. *There are exactly four elements of $Q_{[x]}$ of degree ≤ 1, namely,*

$$P_0 := 0 \,, \qquad P_1 := 1 \,, \qquad P_2 := x \,, \qquad P_3 := 1 - x \,.$$

2. *There are exactly ten elements of $Q_{[x]}$ of degree 2, namely,*

$$\begin{aligned} P_4 &:= x^2 \,, & P_5 &:= -x^2 + 1 \,, \\ P_6 &:= -x^2 + 2x \,, & P_7 &:= x^2 - 2x + 1 \,, \\ P_8 &:= -x^2 + x \,, & P_9 &:= x^2 - x + 1 \,, \\ P_{10} &:= -2x^2 + 2x \,, & P_{11} &:= 2x^2 - 2x + 1 \,, \\ P_{12} &:= -3x^2 + 3x \,, & P_{13} &:= 3x^2 - 3x + 1 \,. \end{aligned}$$

Proof. For (1.), we note that these are the only four polynomials $P \in \mathbb{Z}[x]$ of degree ≤ 1 satisfying the conclusion of Theorem 11.1.

Any polynomial P of degree ≤ 2 is uniquely determined by its values on three distinct inputs. We consider $P(0)$, $P(1/2)$, and $P(1)$. If, in addition, $P \in Q_{[x]}$ and is nonconstant, then by Theorem 11.1, we have: (i) $P(0) \in \{0, 1\}$; (ii) $P(1) \in \{0, 1\}$; and (iii) $0 < P(1/2) < 1$. Since $P \in \mathbb{Z}[x]$, $P(1/2)$ is a multiple of $1/4$, and thus (iii) implies $P(1/2) \in \{1/4, 1/2, 3/4\}$. Taking all possible combinations, there are then at most $2 \cdot 2 \cdot 3 = 12$ many $P \in Q_{[x]}$ with degree 1 or 2 satisfying (i), (ii), and (iii). Two of these have degree 1 (P_2 and P_3, above). The other ten have degree 2 and are listed above as $P_4, \ldots, P_{13}$. We verify that they are all in $Q_{[x]}$ by giving explicit derivations:

$$\begin{aligned} P_4 &= (1-x)P_0 + xP_2 \,, & P_5 &= (1-x)P_1 + xP_3 \,, \\ P_6 &= (1-x)P_2 + xP_1 \,, & P_7 &= (1-x)P_3 + xP_0 \,, \\ P_8 &= (1-x)P_0 + xP_3 \,, & P_9 &= (1-x)P_1 + xP_2 \,, \\ P_{10} &= (1-x)P_2 + xP_3 \,, & P_{11} &= (1-x)P_3 + xP_2 \,, \\ P_{12} &= (1-x)P_6 + xP_5 \,, & P_{13} &= (1-x)P_7 + xP_4 \,. \end{aligned}$$

□

We can use the same technique to get finite upper bounds on the number of elements of $Q_{[x]}$ with any given degree. If the degree is at least four, then slightly better bounds can be obtained by using a classic theorem of Chebyshev [6] (see [1, Chapter 21]) to bound the leading coefficient by 4^{n-1} in absolute value. We also can eliminate some polynomials from $Q_{[x]}$ using the following fact:

Fact 11.11. *Let $P \in \mathbb{R}[x]$ be any real polynomial such that $\{P(0), P(1)\} \subseteq \{0, 1\}$. Then $0 < P(\lambda) < 1$ for all $0 < \lambda < 1$ if and only if $0 < P(r) < 1$ for every root r of P' (the derivative of P) such that $0 < r < 1$.*

Part II: Aperiodic Order

12 R_λ for some algebraic integers λ

In this section, we prove a result (Theorem 12.14, below) that gives some sufficient conditions for R_λ to be discrete for certain algebraic integers λ, including some (e.g., $1+\varphi$) not belonging to any discrete subring of $\mathbb{C}$. In fact, all cases we currently know of where R_λ is discrete follow from Theorem 12.14.

A *Pisot-Vijayaraghavan number* (or *PV number* for short) is an algebraic integer $\alpha > 1$ whose Galois conjugates α' (other than α) all lie inside the unit disk in $\mathbb{C}$, i.e., satisfy $|\alpha'| < 1$. This notion can be relaxed to allow for non-real α by excluding both α and its complex conjugate α^* from the norm requirement. We need a stronger definition.

Definition 12.1. We call an algebraic integer $\alpha \in \mathbb{C}$ a *strong PV number* iff its Galois conjugates, other than α and α^*, all lie in the unit interval $(0,1)$. In this case, we also say that α *is sPV*. We say that a strong PV number α is *trivial* if it has no conjugates other than α and α^* (i.e., no conjugates in $(0,1)$). Otherwise, α is *nontrivial*.

Nontrivial strong PV numbers include $1+\varphi$ and $2+\sqrt{2}$, and there are infinitely many real, irrational—hence nontrivial—strong PV numbers (Corollary 13.5, below). Every strong PV number greater than 1 is a PV number, but not conversely; for example, φ and $1+\sqrt{2}$ are PV numbers but not sPV. Theorem 12.14 implies that R_λ is discrete for all strong PV numbers λ. This result does not extend to all PV numbers; for example, $R_\varphi = R_{1+\sqrt{2}} = \mathbb{R}$.

Fact 12.2. *If α is sPV, then so are α^* and $1-\alpha$; furthermore, $\alpha \notin (0,1)$.*

The trivial strong PV numbers come in two types:

Fact 12.3. *By Lemma 10.2, every trivial sPV number is of one of the following two types:*

1. *The strong PV numbers of degree 1 coincide with the integers.*
2. *All non-real algebraic integers of degree 2 are (trivial) sPV. These numbers coincide with the non-real members of discrete subrings of $\mathbb{C}$ and are of the form either $a + ib\sqrt{n}$ or $a + b(1 + i\sqrt{4n-1})/2$, for integers a, b, n with $b \neq 0$ and $n > 0$.*

Pinch proved that if λ is a real sPV number, then R_λ is discrete [30, Proposition 12].

Proposition 12.4 (Pinch)**.** *If λ is a real sPV number, then R_λ is discrete.*

His proof idea generalizes to complex λ is a straightforward way.

Proposition 12.5. *If λ is a nonreal sPV number, then R_λ is discrete.*

Proof. Suppose R_λ is not discrete. Then $R_\lambda = \mathbb{C}$ by Corollary 3.5 and Theorem 2.19. Since Q_λ is dense in R_λ, there is a point $z \in Q_\lambda$ such that $0 < |z| < 1$. Let polynomial $p \in Q_{[x]}$ be such that $p(\lambda) = z$ (Lemma 7.3). Since λ is an algebraic integer, so is z, and it follows that z has integer norm in the algebraic sense—that is, $N(z) \in \mathbb{Z}$, where $N(z)$ is the product of the conjugates of z. By Corollary A.3, there is a positive integer m such that $\prod_\nu p(\nu) = N(z)^m$, where ν runs over the conjugates of λ. These conjugates include λ itself and its complex conjugate λ^*, and the rest of the conjugates (if any) all satisfy $0 < \nu < 1$. We thus have

$$N(z)^m = \prod_\nu p(\nu) = p(\lambda)p(\lambda^*) \prod_{\nu\,:\,0<\nu<1} p(\nu) = |z|^2 \prod_{\nu\,:\,0<\nu<1} p(\nu) \ .$$

We have $0 < p(\nu) < 1$ if $0 < \nu < 1$ (Theorem 11.1), and thus $0 < N(z)^m < 1$. But $N(z)^m$ is an integer. Contradiction. □

Theorem 12.14, below, substantially strengthens Proposition 12.5, and leads to sharper results about the nature of R_λ, particularly, the cut-and-project schemes of Meyer (see Sections 12.1 and 12.3 below). Its proof was found independently of Pinch's paper [30] as were the connections to cut-and-project schemes explored in [4] for one specific λ. Propositions 12.4 and 12.5, however, can be combined with results on affine embedding to give a short proof of Theorem 12.14. (See Theorems 15.36 and 15.37 in Section 15.3.3.) The short proof has the disadvantage of not connecting with cut-and-project schemes at all, and for this reason we include the longer, more informative proof of Theorem 12.14 in Section 12.2, below, which includes concepts used elsewhere in the paper.

12.1 Meyer sets and cut-and-project schemes

Before stating and proving Theorem 12.14, we recall some concepts from discrete geometry, particularly concepts relating to ordered but aperiodic point sets in Euclidean space. The theory of such sets has gained intense interest recently, following the discovery of so-called "quasicrystals"—materials whose atomic arrangement shares many properties of crystals (e.g., sharp spikes in X-ray diffraction patterns), but—unlike with true crystals—lacks translational symmetry. Much of the mathematical theory of the corresponding point sets is due to Meyer [26, 27], and it also relates closely to aperiodic tilings of the plane [8, 13, 29]. This section draws somewhat from the recent exposition of Baake & Grimm [3] as well as papers by Moody [28]. Also consulted are some related papers by Lagarias [19, 20]. We do not need to present the concepts in their full generality.

Definition 12.6. Let X be any metric space, and let S be any subset of X with at least two elements.

- S is *uniformly discrete* iff there exists an $r > 0$ such that $B_r(x) \cap B_r(y) = \emptyset$ for all distinct $x, y \in S$. The supremum of the set of such r is the *packing radius* of S, denoted $\mu_\mathrm{p}(S)$.
- S is *relatively dense* (in X) iff there exists an $R > 0$ such that $\bigcup_{x\in S} B_R(x) = X$. The infimum of the set of such R is the *covering radius* of S (in X), denoted $\mu_\mathrm{c}(S)$.
- S is a *Delone set* (in X) iff it is both uniformly discrete and relatively dense (in X).

The packing radius is also $1/2$ times the infimum of interpoint distances in S. Thus if R_λ is discrete, then it has packing radius $1/2$ by Corollary 3.5. We will use the following lemma a couple of times in the proof of Theorem 12.14. Stronger statements are possible, but this is all we need.

Lemma 12.7. *If $A \subseteq \mathbb{R}^n$ is uniformly discrete and $B \subseteq \mathbb{R}^n$ is bounded, then $A \cap B$ is finite.*

Proof. Let $r > 0$ be the packing radius of A, and let $R > 0$ be such that B is included in some closed ball of radius R. Then

$$|A \cap B| \le \left(1 + \frac{R}{r}\right)^n \ ,$$

because the open balls of radius r centered at the points of $A \cap B$ are pairwise disjoint subsets of some fixed ball of radius $r + R$. $\square$

The next definition is not Meyer's original definition, but was shown equivalent to it by Jeffrey Lagarias [19].

Definition 12.8. A subset $S \subseteq \mathbb{R}^d$ is a *Meyer set* iff S is relatively dense in $\mathbb{R}^d$ and $S - S$ is uniformly discrete.

There are many equivalent characterizations of Meyer sets; see, for example, Moody [28, Theorem 9.1]. All Meyer sets are Delone sets, but Meyer sets have additional properties, such as finite local complexity, not shared by all Delone sets. Other properties of interest include repetivity, diffractivity, and aperiodicity. A Meyer set may or may not possess any of these additional properties. For definitions, see [3, 28].

One way of producing aperiodic Meyer sets is through a *cut-and-project scheme*. Our definition here follows Moody [28] with some minor alterations. Recall that if $f : X \to Y$ is some function with domain X, and A is any subset of X, then we let $f|A$ denote the function f restricted to domain A.

Definition 12.9. A *cut-and-project scheme* is a tuple $(G, \mathbb{R}^n, L)$, where

- G is a locally compact abelian (topological) group,
- n is a positive integer,
- L is a discrete subgroup of $G \times \mathbb{R}^n$ such that the quotient group $(G \times \mathbb{R}^n)/L$ is compact,
- letting $\pi_1 : G \times \mathbb{R}^n \to G$ and $\pi_2 : G \times \mathbb{R}^n \to \mathbb{R}^n$ be the two canonical projection maps,
    1. $\pi_1(L)$ is dense in G, and
    2. $\pi_2|L$ is one-to-one.

G is called the *internal space* and $\mathbb{R}^n$ the *physical space* of the cut-and-project scheme.

Definition 12.10. Fix a cut-and-project scheme $\mathcal{M} = (G, \mathbb{R}^n, L)$. A *window* of $\mathcal{M}$ is any relatively compact[7] subset $W \subseteq G$ with nonempty interior. A set $M \subseteq \mathbb{R}^n$ is a *model set* (of $\mathcal{M}$) iff $M = \pi_2(L \cap \pi_1^{-1}(W))$ for some window W.

All model sets are Delone sets. Model sets can also be used to characterize Meyer sets. Citing work of Meyer [26] and Lagarias [19] as well as his own work, Moody [28] gives various characterizations of Meyer sets, two of which are the following:

Fact 12.11 (Meyer, Moody [28])**.** *Let S be a relatively dense subset of $\mathbb{R}^n$. Then S is a Meyer set if and only if one (or both) of the following equivalent conditions holds:*

1. *S is a subset of a model set.*
2. *$S - S$ is uniformly discrete.*

In this paper, G will always be $\mathbb{R}^m$ (for some positive m) with the usual vector addition, and L will always be an integer lattice spanning $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$.

[7] A subspace Y of a topological space X is *relatively compact* iff the closure of Y is compact.

Remark. Another means of generating Delone and Meyer sets is via substitution and inflation (see Lagarias & Wang [21]). □

We end this subsection with a standard result about cut-and-project schemes used to show aperiodicity of the corresponding model sets.

Lemma 12.12. *Let $\mathcal{M} = (\mathbb{R}^m, \mathbb{R}^n, L)$ be a cut-and-project scheme with internal space $\mathbb{R}^m$ and associated projectors π_1 and π_2 as in Definition 12.9. If $\pi_1|L$ is one-to-one, then no model set of $\mathcal{M}$ intersects any arithmetic progression at infinitely many points.*

Proof. Let M be a model set with corresponding window $W \subseteq \mathbb{R}^m$ as in Definition 12.10. Let $A := \langle a_0, a_1, a_2, \ldots \rangle$ be an infinite arithmetic progression in $\mathbb{R}^n$ that intersects $\pi_2(L)$ in at least two distinct points, say a_i and a_j for some least $i < j$. Since $\pi_2(L)$ is an additive subgroup of $\mathbb{R}^n$, we get that $A \cap \pi_2(L)$ is an arithmetic subprogression of A. Letting $B := \langle b_0, b_1, b_2, \ldots \rangle$ be this subprogression, we see that $b_k = a_{i+k(j-i)} = a_i + k(a_j - a_i)$ for $k = 0, 1, 2, \ldots$. Furthermore, $M \cap A = M \cap B$, because $M \subseteq \pi_2(L)$.

Let $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots \in L$ be the unique vectors such that $\pi_2(\mathbf{b}_k) = b_k$ for $k = 0, 1, 2, \ldots$. For any such k we have by linearity

$$\pi_2(\mathbf{b}_0 + k(\mathbf{b}_1 - \mathbf{b}_0)) = b_0 + k(b_1 - b_0) = b_k = \pi_2(\mathbf{b}_k) ,$$

and so $\mathbf{b}_k = \mathbf{b}_0 + k(\mathbf{b}_1 - \mathbf{b}_0)$ for all k, making $\langle \mathbf{b}_0, \mathbf{b}_1, \ldots \rangle$ a proper arithmetic progression in L. Define $c_k := \pi_1(\mathbf{b}_k) \in \mathbb{R}^m$ for $k = 0, 1, 2, \ldots$. Again, this time by linearity of π_1, we have $c_k = c_0 + k(c_1 - c_0)$ for all k. Moreover, $c_0 \neq c_1$ because $\pi_1|L$ is one-to-one, making $\langle c_0, c_1, c_2, \ldots \rangle$ a proper arithmetic progression. Since W is bounded, for all but finitely many k we have $c_k \notin W$, which puts $\mathbf{b}_k$ out of $\pi_1^{-1}(W)$, and so puts b_k out of M. Thus M intersects B (hence A) at only finitely many points. □

We explore the connection between cut-and-project schemes and λ-convex sets in Section 12.3, below. As a warm-up, we have the following proposition, which is an easy corollary to Corollary 11.3:

Proposition 12.13. *For any algebraic integer λ of degree $d > 0$,*

$$Q_\lambda \subseteq \{0,1\} \cup \{p(\lambda) \mid p \in \mathbb{Z}[x] \ \ \& \ \ \deg(p) < d \ \ \& \ \ 0 < p(\mu) < 1 \textit{ for all } 0 < \mu < 1 \textit{ conjugate to } \lambda\} .$$

Proof. Let $m \in \mathbb{Z}[x]$ be the minimal (monic) polynomial of λ of degree d. Corollary 11.3 says that $Q_\lambda = \{0,1\} \cup \{p(\lambda) \mid p \in \mathbb{Z}[x] \ \ \& \ \ 0 < p(\mu) < 1 \text{ for all } 0 < \mu < 1\}$. Since m is monic, we can write any $p \in \mathbb{Z}[x]$ as $p = qm + r$ for some $q, r \in \mathbb{Z}[x]$ with $\deg(r) < d$. Moreover, if μ is conjugate to λ, then $p(\mu) = r(\mu)$, which means that we can restrict the degree of the polynomial in the set-former to be less than d by substituting r for p. □

We will see below that if λ is an sPV number, then the right-hand side of Proposition 12.13 is discrete. In fact, it is the model set of a cut-and-project scheme.

12.2 Main results

Proposition 12.4 [30] is a special case of the following theorem, the main theorem of this section:

Theorem 12.14. *If λ is sPV, then $R_\lambda(S)$ is uniformly discrete for any finite set $S \subseteq \mathbb{Q}(\lambda)$.*

Let D be a discrete subring of $\mathbb{C}$. D is closed by Lemma 10.1. If λ belongs to D, then R_λ is discrete for an "easy" reason: $R_\lambda \subseteq D$. Proposition 6.1 gave us our first example of a discrete R_λ that is *not* contained in any discrete subring of $\mathbb{C}$. Theorem 12.14 will give us infinitely many other examples, i.e., values of λ such that R_λ is discrete but $\mathbb{Z}[\lambda]$ is not.

To prove Theorem 12.14, we build up some more machinery using some results of linear algebra—especially the spectral decomposition of an operator.

Some of our proof technique resembles work on generalized Fibonacci sequences done by Kalman [18], who found a closed form for the n^{th} element a_n of a sequence satisfying the k^{th} order linear recurrence $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$, where the c_i are fixed integers. Both proofs use the spectral decomposition of the companion matrix of an irreducible polynomial.

Throughout this section, for convenience, we start the indexing of vectors and matrices with 0 instead of with 1. If E is some expression of matrix type, we let $[E]_{ij}$ denote the $(i,j)^{th}$ entry of E, for any appropriate integers $i,j \geq 0$. We let an expression of the form $(x_0, x_1, x_2, \ldots)$ denote either a row vector or a column vector, depending on the context.

We first extend our definition of $\star_\lambda$ and $\rho_{a,b}$ to a more general setting. As in previous sections, we use $\star$ without a subscript to denote $\star_\lambda$.

Definition 12.15. Let R be a commutative ring and let M be an R-module. For any $\lambda \in R$ and $u, v \in M$, we define $u \star v$ and $\rho_{u,v}(\lambda)$ to be the point $(1-\lambda)u + \lambda v \in M$.[8]

A set $S \subseteq M$ is *λ-convex* iff S is closed under the $\star$ operation. For any $T \subseteq M$, we define $Q_\lambda(T)$ to be the least λ-convex superset of T.

Part I of this paper is mostly concerned with the case where $R = M = \mathbb{C}$, except in Sections 7 and 11, where $R = M = \mathbb{Z}[x]$. Many of the basic results of that part carry over to the more general setting. For example, Lemma 12.17 below is analogous to Lemma 2.12.

Fact 12.16. *Let R be a commutative ring, let M and N be R-modules, and let $f : M \to N$ be a homomorphism of R-modules. Then for any $\lambda \in R$ and $u, v \in M$,*

$$f(u \star v) = f(u) \star f(v) .$$

Lemma 12.17. *Let R, M, N, and f be as in Fact 12.16. Then for any $\lambda \in R$ and $S \subseteq M$,*

$$f(Q_\lambda(S)) = Q_\lambda(f(S)) .$$

Proof. Given any $x, y \in f(Q_\lambda(S))$, let $u, v \in Q_\lambda(S)$ be such that $x = f(u)$ and $y = f(v)$. Then $x \star y = f(u) \star f(v) = f(u \star v) \in f(Q_\lambda(S))$. This shows that $f(Q_\lambda(S))$ is λ-convex, and, since $f(S) \subseteq f(Q_\lambda(S))$, this proves the $\supseteq$-inclusion of the lemma. For the $\subseteq$-inclusion, we first show that $T := f^{-1}(Q_\lambda(f(S)))$ is a λ-convex superset of S. We have $S \subseteq f^{-1}(f(S)) \subseteq T$. Now let $u, v \in T$ be arbitrary. Then because $f(u)$ and $f(v)$ are both in $Q_\lambda(f(S))$, we have $f(u \star v) = f(u) \star f(v) \in Q_\lambda(Q_\lambda(f(S))) = Q_\lambda(f(S))$. This then puts $u \star v$ into T, and so T is λ-convex. By minimality, $Q_\lambda(S) \subseteq T$. Applying f to both sides gives $f(Q_\lambda(S)) \subseteq f(T) \subseteq Q_\lambda(f(S))$. □

We will need to consider another special case of Definition 12.15: Let V be any vector space over some field k, and fix a k-linear map $\Lambda : V \to V$. Then Λ naturally turns V into a $k[x]$-module, where scalar multiplication is defined for all $g \in k[x]$ and $v \in V$ as $gv := (g(\Lambda))(v)$ (see, for example, Jacobson [16, Chapter 3]). We denote this $k[x]$-module by V_Λ. We will only use the case where $\lambda = x \in k[x]$, and thus we can identify λ with Λ.

[8]Since $u \star v$ is an affine combination of u and v, the operation $\star$ is well-defined on any R-affine space. We will not need this additional generality here, however.

Definition 12.18. Let V be a vector space over a field k, and let $\Lambda : V \to V$ be a k-linear map.

1. For any vectors $u, v \in V$, define $u \star_\Lambda v := (I - \Lambda)u + \Lambda v = u + \Lambda(v - u)$.
2. A set $S \subseteq V$ is Λ-*convex* iff $u \star_\Lambda v \in S$ whenever $u, v \in S$.
3. For any set of vectors $S \subseteq V$, we define $Q_\Lambda(S) \subseteq V$ as the least Λ-convex superset of S.

We could easily define Λ-clonvexity and $R_\Lambda(S)$ for vector spaces over $\mathbb{R}$ or $\mathbb{C}$, but we will not need this notion here.

Fact 12.19. *Let V, k, and Λ be as in Definition 12.18. For any $u, v, w, x \in V$ and any $a \in k$, we have*

$$(u + v) \star_\Lambda (w + x) = u \star_\Lambda w + v \star_\Lambda x \qquad \textit{and} \qquad au \star_\Lambda av = a(u \star_\Lambda v)\ .$$

Lemma 12.20. *Let U and V be vector spaces over some field k, and let $A : U \to U$, $B : V \to V$, and $t : U \to V$ be k-linear maps such that $B \circ t = t \circ A$. For any $S \subseteq U$, we have*

$$t(Q_A(S)) = Q_B(t(S))\ .$$

Proof. This is a special case of Lemma 12.17. The combined condition that t is linear and $B \circ t = t \circ A$ is equivalent to t being a homomorphism of $k[x]$-modules $U_A \to V_B$. ☐

We use Lemma 12.17 one more time to get a generalization of Lemma 7.3.

Lemma 12.21. *Let R be a commutative ring, $\lambda \in R$, and M an R-module. For any $u \in M$,*

$$Q_\lambda(\{0, u\}) = \left\{ P(\lambda)u \mid P \in Q_{[x]} \right\}\ ,$$

where $P(\lambda)$ is interpreted in R for each $P \in Q_{[x]}$ (see Definition 7.1).

Proof. The lemma follows from Lemma 12.17 provided we set things up the right way. We can view M as a $\mathbb{Z}[x]$-module (which we denote by M_λ) by defining scalar multiplication of a vector $w \in M$ with a scalar $P \in \mathbb{Z}[x]$ by $Pw := P(\lambda)w$, where $P(\lambda)$ is interpreted in R as usual and $P(\lambda)w$ is scalar multiplication in M. Then $\star_\lambda$, interpreted in M, is the same map as $\star_x$ interpreted in M_λ. Consequently, for any $S \subseteq M$, $Q_\lambda(S)$ interpreted in M is the same set as $Q_x(S)$ interpreted in M_λ. Now given $u \in M$, consider the map $g : \mathbb{Z}[x] \to M_\lambda$ that sends $P \in \mathbb{Z}[x]$ to $Pu \in M_\lambda$. One checks that g is a homomorphism of $\mathbb{Z}[x]$-modules. Using Lemma 12.17, we now have, working in M_λ,

$$Q_x(\{0, u\}) = Q_x(g(\{0, 1\})) = g(Q_x(\{0, 1\})) = g(Q_{[x]}) = \{g(P) \mid P \in Q_{[x]}\} = \{Pu \mid P \in Q_{[x]}\}\ .$$

The lemma follows by interpreting both sides in M. ☐

The first extended proof of this section considers the case where $k = \mathbb{R}$, $V = \mathbb{R}^d$ for some finite $d > 0$, and $\Lambda : V \to V$ is an $\mathbb{R}$-linear map given by some *diagonal* matrix. It gives a good illustration of the general concepts above. For $0 \le i < d$, let $\mu_i := [\Lambda]_{ii}$ be the diagonal entries of Λ. So on the i^{th} coordinate, Λ acts as scalar multiplication by μ_i. One can consider Lemma 12.22 below to be a d-dimensional generalization of Theorem 2.19(1).

Lemma 12.22. *Let d, Λ, and $\mu_0, \ldots, \mu_{d-1}$ be as in the last paragraph, and suppose that all of the μ_i are in $[0,1]$. Let $\mathbf{1} = (1,1,\ldots,1)$ be the vector of d ones, and let $S := \{0, \mathbf{1}\}$. Then*

1. *$Q_\Lambda(S) \subseteq [0,1]^{\times d}$, and*

2. *$Q_\Lambda(S)$ is dense in $[0,1]^{\times d}$ if and only if all the μ_i are pairwise distinct and lie strictly between 0 and 1.*

Proof. Let $\pi_i : \mathbb{R}^d \to \mathbb{R}$ be the projection onto the i^{th} coordinate, for all $0 \le i < d$. For part (1.): by Lemma 12.20, for all i, we have $\pi_i(Q_\Lambda(S)) = Q_{\mu_i}(\pi_i(S)) = Q_{\mu_i}$, which is contained in $[0,1]$ by Theorem 2.19(1).

For part (2.): By permuting coordinates if necessary, we can assume without loss of generality that $0 \le \mu_0 \le \mu_1 \le \cdots \le \mu_{d-1} \le 1$. We show the "only if" part first. If $\mu_0 = 0$, then $\pi_0(Q_\Lambda(S)) = Q_0 = \{0,1\}$, and so every point in $Q_\Lambda(S)$ has zeroth coordinate 0 or 1. Similarly for the $(d-1)^{st}$ coordinate if $\mu_{d-1} = 1$. If $\mu_i = \mu_{i+1}$ for some i, then every point in $Q_\Lambda(S)$ has equal i^{th} and $(i+1)^{st}$ coordinates. This follows from the observation that both points in S have equal i^{th} and $(i+1)^{st}$ coordinates and that this property is preserved by $\star_\Lambda$. In none of these cases is $Q_\Lambda(S)$ dense in $[0,1]^{\times d}$.

For the "if" part, now assume that $0 < \mu_0 < \mu_1 < \cdots < \mu_{d-1} < 1$, and let $R := \overline{Q_\Lambda(S)}$. We show that $R = [0,1]^{\times d}$. By a proof similar to that of Lemma 2.10 in Part I, we have that R is Λ-convex. Our proof now is in two steps: we first show that R is convex and then show that R contains all the corners of $[0,1]^{\times d}$, i.e., all points in $\{0,1\}^{\times d}$. This suffices for the "if" part.

For convexity, fix any two points $a, b \in R$. Define $a_0 := a$ and $b_0 := b$, and for every integer $n \ge 0$, inductively define

$$a_{n+1} := b_n \star_\Lambda a_n\ , \qquad\qquad b_{n+1} := a_n \star_\Lambda b_n\ .$$

All these points are in R, because R is Λ-convex. Set $m = (a+b)/2$, the midpoint of a and b, and set $\sigma := \max_{0 \le i < d} |2\mu_i - 1|$, noting that $0 \le \sigma < 1$. We then check by induction that for all $n \ge 0$,

$$a_n + b_n = 2m\ , \qquad\qquad \|a_n - b_n\|_\infty \le \sigma^n \|a - b\|_\infty\ ,$$

where for all $x \in \mathbb{R}^d$, $\|x\|_\infty := \max_{0 \le i < d} |\pi_i(x)|$ denotes the ℓ_∞-norm of x. This certainly holds for $n = 0$. For the inductive case, given $n \ge 0$, we have

$$a_{n+1} + b_{n+1} = b_n \star_\Lambda a_n + a_n \star_\Lambda b_n = (b_n + a_n) \star_\Lambda (a_n + b_n) = a_n + b_n = 2m\ .$$

Also,

$$a_{n+1} - b_{n+1} = b_n \star_\Lambda a_n - a_n \star_\Lambda b_n = (b_n - a_n) \star_\Lambda (a_n - b_n)\ ,$$

and thus for all $0 \le i < d$, letting $\delta := a_n - b_n$,

$$\begin{aligned}|\pi_i(a_{n+1} - b_{n+1})| &= |\pi_i((-\delta) \star_\Lambda \delta)| = |(-\pi_i(\delta)) \star_{\mu_i} \pi_i(\delta))| = |(2\mu_i - 1)\pi_i(\delta)| \le \sigma |\pi_i(\delta)| \le \sigma \|\delta\|_\infty \\ &\le \sigma^{n+1} \|a - b\|_\infty\ ,\end{aligned}$$

and from this it follows that $\|a_{n+1} - b_{n+1}\|_\infty \le \sigma^{n+1} \|a - b\|_\infty$, which finishes the induction. Now for all n,

$$2\|a_n - m\|_\infty = \|2a_n - 2m\|_\infty = \|a_n - b_n\|_\infty \le \sigma^n \|a - b\|_\infty\ ,$$

and since $|\sigma| < 1$, we then have $m = \lim_{n\to\infty} a_n \in R$, because R is closed. All of this shows that R is closed under midpoints, and it follows from the closedness of R that R is convex.

To finish, we show that every corner $\vec{b} = (b_0, b_1, \ldots, b_{d-1})$—where each $b_i \in \{0,1\}$—is in R. The idea is to use the polynomials $T_{\gamma,\varepsilon}$ of Lemma 7.14 to construct a polynomial $P \in Q_{[x]}$ such that for all i, $P(\mu_i)$ approximates b_i as closely as we want. More precisely, for every $\delta > 0$, we find some $P_\delta \in Q_{[x]}$ such that for all $0 \le i < d$, $|P_\delta(\mu_i) - b_i| \le \delta$. Then we invoke Lemma 12.21 to get a point in $Q_\Lambda(S)$ close to $\vec{b}$. We set $\mu_{-1} := 0$ and let $\mu_d := 1$ for convenience. For $0 \le i \le d$, we let $\gamma_i := (\mu_{i-1} + \mu_i)/2$ be the midpoint of μ_{i-1} and μ_i. Notice that $0 < \gamma_0 < \mu_0 < \gamma_1 < \mu_1 < \cdots < \gamma_{d-1} < \mu_{d-1} < \gamma_d < 1$. Now let $\delta > 0$ be arbitrary. For a certain $\varepsilon > 0$ that we choose later, we define

$$P_\delta := \prod_{0\le i<d\,:\,b_i=0} (1 - T_{\gamma_{i+1},\varepsilon}(1 - T_{\gamma_i,\varepsilon}))\ ,$$

where the $T_{\cdot,\cdot}$ polynomials are given by Lemma 7.14. P_δ is in $Q_{[x]}$ by Corollary 11.2. If we choose ε not to exceed $(\mu_i - \mu_{i-1})/2$ for any $0 \le i \le d$, then no interval $(\gamma_i - \varepsilon, \gamma_i + \varepsilon)$ contains any of the μ_j. This means that we can apply Lemma 7.14 to μ_i for each $0 \le i < d$: Since $\gamma_i \le \mu_i - \varepsilon$ and $\mu_i + \varepsilon \le \gamma_{i+1}$, we have $T_{\gamma_j,\varepsilon}(\mu_i) \le \varepsilon$ for all $j \le i$ and $T_{\gamma_j,\varepsilon}(\mu_i) \ge 1 - \varepsilon$ for all $j > i$. Hence, $1 - T_{\gamma_{i+1},\varepsilon}(\mu_i)(1 - T_{\gamma_i,\varepsilon}(\mu_i)) \le 1 - (1-\varepsilon)^2 \le 2\varepsilon$, but $1 - T_{\gamma_{j+1},\varepsilon}(\mu_i)(1 - T_{\gamma_j,\varepsilon}(\mu_i)) \ge 1 - \varepsilon$ for all $j \ne i$. Now considering $P_\delta(\mu_i)$:

- If $b_i = 1$, then all terms in the product are $\ge 1 - \varepsilon$, and so $P_\delta(\mu_i) \ge (1-\varepsilon)^d \ge 1 - d\varepsilon$.
- If $b_i = 0$, then one term in the product is $\le 2\varepsilon$, and the rest are ≤ 1, and so $P_\delta(\mu_i) \le 2\varepsilon$.

So choosing ε to be the smaller of $\delta/(d+2)$ and $\min_{0\le i<d}((\mu_i - \mu_{i-1})/2)$, we obtain $|P_\delta(\mu_i) - b_i| \le \delta$ for all i.

Finally, we apply Lemma 12.21 with the ring being $\mathbb{Z}[x]$, with $\lambda := x \in \mathbb{Z}[x]$ as always, with $M := V_\Lambda = (\mathbb{R}^d)_\Lambda$ (restricted to being a $\mathbb{Z}[x]$-module), and with $u := \mathbf{1}$. That lemma says that $Q_\Lambda(S) = \{P(\Lambda)\mathbf{1} \mid P \in Q_{[x]}\}$, and so $Q_\Lambda(S)$ contains the points $P_\delta(\Lambda)\mathbf{1}$ for all $\delta > 0$. Now, $P_\delta(\Lambda)$ is the $d \times d$ diagonal matrix with diagonal entries $P_\delta(\mu_0), \ldots, P_\delta(\mu_{d-1})$, and so $P_\delta(\Lambda)\mathbf{1} = (P_\delta(\mu_0), \ldots, P_\delta(\mu_{d-1}))$, which is within distance δ of $\vec{b}$ (ℓ_∞ distance) by the previous paragraph. So we have that $\vec{b}$ is arbitarily close to points in $Q_\Lambda(S)$, which puts $\vec{b} \in R$ as required. $\square$

Corollary 12.23. *Let d, Λ, and $\mu_0, \ldots, \mu_{d-1}$ be as in Lemma 12.22. Let $\mathbf{x} := (x_0, \ldots, x_{d-1}) \in \mathbb{R}^d$ be a vector. If $\mu_0, \ldots, \mu_{d-1}$ are pairwise distinct elements of $(0,1)$, then $Q_\Lambda(\{0, \mathbf{x}\})$ is a dense subset of $[0, x_0] \times \cdots \times [0, x_{d-1}]$.*

Proof. Let D be the $d \times d$ diagonal matrix with diagonal elements $x_0, \ldots, x_{d-1}$. Then $\mathbf{x} = D\mathbf{1}$, and so by Lemma 12.20 and the fact that Λ and D commute, we get

$$Q_\Lambda(\{0, \mathbf{x}\}) = Q_\Lambda(D\{0, \mathbf{1}\}) = D(Q_\Lambda\{0, \mathbf{1}\})\ ,$$

which is dense in $D\left([0,1]^d\right) = [0, x_0] \times \cdots \times [0, x_{d-1}]$ by Lemma 12.22. $\square$

Definition 12.24. Let $p(x) = x^d + \sum_{j=0}^{d-1} c_j x^j$ be some monic polynomial in $\mathbb{C}[x]$ of degree $d > 0$ with coefficients $c_0, \ldots, c_{d-1} \in \mathbb{C}$. We let Λ_p denote the companion matrix of p, that is, the $d \times d$

matrix

$$\Lambda_p := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ 0 & 0 & 1 & \cdots & 0 & -c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{d-1} \end{bmatrix} . \tag{13}$$

We can view Λ_p as representing a linear map $\mathbb{C}^d \to \mathbb{C}^d$ relative to the standard basis $\{\mathbf{e}_0, \ldots, \mathbf{e}_{d-1}\}$ of $\mathbb{C}^d$. It is well-known that Λ_p has characteristic polynomial $\pm p$, and so its eigenvalues are the roots of p.

The next lemma is standard.

Lemma 12.25. *Let $p(x)$ and Λ_p be as in Definition 12.24 where p has degree $d > 0$. Let $\mu_0, \ldots, \mu_{d-1} \in \mathbb{C}$ be the (not necessarily distinct) roots of p. Let*

$$V := V(\vec{\mu}) := \begin{bmatrix} 1 & \mu_0 & (\mu_0)^2 & \cdots & (\mu_0)^{d-1} \\ 1 & \mu_1 & (\mu_1)^2 & \cdots & (\mu_1)^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mu_{d-1} & (\mu_{d-1})^2 & \cdots & (\mu_{d-1})^{d-1} \end{bmatrix} \tag{14}$$

be the $d \times d$ Vandermonde matrix with respect to $\vec{\mu} := (\mu_0, \ldots, \mu_{d-1})$ (that is, $[V]_{ij} = (\mu_i)^j$ for all $i, j \in \{0, \ldots, d-1\}$), and let D be the $d \times d$ diagonal matrix with diagonal entries $[D]_{ii} := \mu_i$ for all $i \in \{0, \ldots, d-1\}$. Then

$$V\Lambda_p = DV .$$

Proof. For any $i, k \in \{0, \ldots, d-1\}$, the $(i,k)^{th}$ entry of $V\Lambda_p$ is given by

$$[V\Lambda_p]_{ik} = \sum_{j=0}^{d-1} [V]_{ij}[\Lambda_p]_{jk} = \sum_{j=0}^{d-1} (\mu_i)^j [\Lambda_p]_{jk} = \begin{cases} (\mu_i)^{k+1} & \text{if } k < d-1, \\ -\sum_{j=0}^{d-1} c_j(\mu_i)^j & \text{if } k = d-1. \end{cases}$$

Since μ_i is a root of p, we have $-\sum_{j=0}^{d-1} c_j(\mu_i)^j = (\mu_i)^d$, and so in either case we get $[V\Lambda_p]_{ik} = (\mu_i)^{k+1} = \mu_i(\mu_i)^k = [D]_{ii}[V]_{ik} = [DV]_{ik}$. $\square$

Remark. If V is invertible (which is true when all the μ_i are pairwise distinct), then Lemma 12.25 says that the columns of V^{-1} are eigenvectors of Λ_p with respective eigenvalues $\mu_0, \ldots, \mu_{d-1}$. $\square$

We now turn to the proof of Theorem 12.14, the main result of this section. We essentially prove that $R_\lambda(S)$ is a subset of a certain model set (this is the content of Eq. (16) below). Although it would then follow directly from previous results of Meyer [26] (also see [28]) that $R_\lambda(S)$ is uniformly discrete, we give a self-contained proof for the sake of completeness. We will give more details about the connection with model sets after the proof. The case where λ is not real (and $S = \{0, 1\}$) was proved by Rohit Gurjar [14].

To denote the model set containing $R_\lambda(S)$, we use the following notation generalizing that of Berman & Moody [4]:

Definition 12.26. Let $\lambda \in \mathbb{C} \setminus (0,1)$ be an algebraic integer and let $\mu_0, \ldots, \mu_{k-1}$ be the conjugates of λ that are in $(0,1)$. For any $x \in \mathbb{Z}[\lambda]$, let $\mathbf{x}'$ denote the vector of conjugates $(x'_0, \ldots, x'_{k-1}) \in \mathbb{R}^k$, where x'_i is the i^{th} conjugate of x, i.e., the image of x under the ring isomorphism $\mathbb{Z}[\lambda] \to \mathbb{Z}[\mu_i]$ sending λ to μ_i, for all $0 \leq i < k$. Let P be any interval of $\mathbb{R}^k$, defined as $P := [\ell_0, h_0] \times \cdots \times [\ell_{k-1}, h_{k-1}]$, where for $0 \leq i < k$, we have $\ell_i, h_i \in \mathbb{R}$ and $\ell_i < h_i$. We define

$$\Sigma_P^{(\lambda)} := \{x \in \mathbb{Z}[\lambda] \mid \mathbf{x}' \in P\} . \tag{15}$$

If $k = 0$, then we take $\mathbf{x}'$ to be the empty tuple for every $x \in \mathbb{Z}[\lambda]$, and we take P to be the singleton set containing this tuple, whence $\Sigma_P^{(\lambda)} = \mathbb{Z}[\lambda]$. (We leave off the superscript if λ is clear from the context.)

For any set $S \subseteq \mathbb{Z}[\lambda]$, let $S' := \{\mathbf{s}' \mid s \in S\}$ denote the image of S under the mapping $\cdot' : \mathbb{Z}[\lambda] \to \mathbb{Z}[\mu_0] \times \cdots \times \mathbb{Z}[\mu_{k-1}]$ defined above.

When we use Definition 12.26, we generally assume P contains some set S' of conjugates of a finite set S. Furthermore, if $S = \{0,1\}$, it turns out that $P = [0,1]^{\times k}$ is the appropriate choice. The reason for this should become clear as we use the definition and the result below. Also, although here we are assuming $S \subseteq \mathbb{Z}[\lambda]$, we will see in the proof of Theorem 12.14 that some choice for P is sufficient for proving that $R_\lambda(S)$ is uniformly discrete for any finite $S \subseteq \mathbb{Q}[\lambda]$.

A key result is that $Q_\lambda(S)$ is always contained in a set of the form $\Sigma_P^{(\lambda)}$.

Theorem 12.27. *For any algebraic integer $\lambda \in \mathbb{C} \setminus [0,1]$ and any finite $S \subseteq \mathbb{Z}[\lambda]$, let P be any interval of $\mathbb{R}^k$—as defined in Definition 12.26—that includes S'. Then*

$$Q_\lambda(S) \subseteq \Sigma_P^{(\lambda)}. \tag{16}$$

Proof. For convenience, we fix λ and write Σ_P for $\Sigma_P^{(\lambda)}$. First we observe that $S \subseteq \Sigma_P$, since by hypothesis $S' \subseteq P$, and hence $s \in \Sigma_P$ for each $s \in S$. We now show that Σ_P is closed under $\star_\lambda$. Let $x, y \in \Sigma_P$. Then the respective vectors of conjugates $\mathbf{x}'$ and $\mathbf{y}'$ are contained in P. Now let $z := x \star_\lambda y$. The i^{th} coordinate of $\mathbf{z}'$ is $z'_i = (1-\mu_i)x'_i + \mu_i y'_i$. But by definition, $\mu_i \in [0,1]$ for each i between 0 and $k-1$. Thus due to the fact that $[\ell_i, h_i]$ is convex, we conclude that $\ell_i \leq (1-\mu_i)x'_i + \mu_i y'_i \leq h_i$, so that $z'_i \in [\ell_i, h_i]$ for all i. Clearly, $z \in \mathbb{Z}[\lambda]$, and hence $z \in \Sigma_P$, so Σ_P is closed under $\star_\lambda$ as claimed. It therefore contains the closure of S under $\star_\lambda$, i.e., $\Sigma_P \supseteq Q_\lambda(S)$. ☐

The proof of Theorem 12.14 now reduces to proving that $\Sigma_P^{(\lambda)}$ is uniformly discrete for sPV λ.

Proof of Theorem 12.14. We are given a finite set $S \subseteq \mathbb{Q}(\lambda)$, which we can assume contains 0 and 1 (so that $Q_\lambda \subseteq Q_\lambda(S)$). We can also assume without loss of generality that $S \subseteq \mathbb{Z}[\lambda]$. This can be seen as follows: since S is finite, there exists a positive $\delta \in \mathbb{Z}$ such that $\delta S \subseteq \mathbb{Z}[\lambda]$, but by Lemma 2.12, $R_\lambda(\delta S) = R_\lambda(\rho_{0,\delta}(S)) = \rho_{0,\delta}(R_\lambda(S)) = \delta R_\lambda(S)$, so $R_\lambda(S)$ is uniformly discrete if and only if $R_\lambda(\delta S)$ is uniformly discrete. We can thus substitute δS for S in the theorem.

Now it suffices to show that $Q_\lambda(S)$ is uniformly discrete; in that case, $R_\lambda(S) = Q_\lambda(S)$ by Lemma 2.10, because $Q_\lambda(S)$ is closed. We can assume without loss of generality that λ is nontrivial (cf. Definition 12.1), for if λ is trivial, then $\mathbb{Z}[\lambda]$ is a (uniformly) discrete subring of $\mathbb{C}$ including $Q_\lambda(S)$ (by Fact 12.3), and we are done.

Let $p(x) := x^d + \sum_{i=0}^{d-1} c_i x^i$ be the minimal polynomial of λ, where $d \geq 2$ and each c_i is an integer. By Fact 12.2 and λ being nontrivial, we have $\lambda \notin [0,1]$. Let $\mu_0, \ldots, \mu_{k-1}$ be the conjugates of λ that are in $[0,1]$, as in Definition 12.26. In fact, $\mu_i \in (0,1)$ for all $0 \le i < k$. Since λ is sPV, we have two cases: (1) $\lambda \in \mathbb{R}$, or (2) $\lambda \notin \mathbb{R}$. Since λ is sPV, we have $k = d-1$ in case 1 and $k = d-2$ in case 2. In either case, define $\mu_k := \lambda$. In case 2, also set $\mu_{k+1} := \lambda^*$. Thus $\mu_0, \dots, \mu_{d-1}$ are all the conjugates of λ in either case. Since $\lambda \notin [0,1]$, $Q_\lambda(S)$ is unbounded (Corollary 4.5), a fact we will need toward the end of this proof.

Each $x \in \mathbb{Z}[\lambda]$ is a polynomial $x = \sum_{i=0}^{d-1} a_i \lambda^i$, where $a_i \in \mathbb{Z}$ for each i, and d is the degree of λ. We may represent the set of coefficients $a_i \in \mathbb{Z}$ as a column vector $\mathbf{a} := (a_0, \dots, a_{d-1}) \in \mathbb{Z}^d$. If we regard the powers of λ as a row vector $\boldsymbol{\lambda} := (1, \lambda, \dots, \lambda^{d-1})$, then $x = \boldsymbol{\lambda}\mathbf{a}$. Similarly, for the conjugates μ_i $(0 \le i < d-1)$ of λ, we define the row vector $\boldsymbol{\mu}_i := (1, \mu_i, \dots, \mu_i^{d-1})$, and thus $(\boldsymbol{\lambda}\mathbf{a})' = (\boldsymbol{\mu}_0\mathbf{a}, \dots, \boldsymbol{\mu}_{k-1}\mathbf{a})$, where we understand the last vector to be a column vector in $\mathbb{R}^k$. As $S \subseteq \mathbb{Z}[\lambda]$, any element of S can be written in the same form $\boldsymbol{\lambda}\mathbf{a}$ for some $\mathbf{a} \in \mathbb{Z}^d$. Let $\pi_i : \mathbb{C}^d \to \mathbb{C}$ denote the linear map projecting an element of $\mathbb{C}^d$ onto the i^{th} coordinate. Thus, for example, $\pi_i((\boldsymbol{\lambda}\mathbf{a})') = \boldsymbol{\mu}_i\mathbf{a}$. Let S'_i denote the set of projections of elements of S' onto the i^{th} coordinate: $S'_i := \{\pi_i(\mathbf{s}') \mid s \in S\}$. Put $\ell_i := \min(S'_i)$ and $h_i := \max(S'_i)$ and $P := [\ell_0, h_0] \times \cdots \times [\ell_{k-1}, h_{k-1}]$ as in Definition 12.26 (note that if $S = \{0,1\}$, then $P = [0,1]^{\times k}$). Then clearly $P \supseteq S'$, and hence by Theorem 12.27, $Q_\lambda(S) \subseteq \Sigma_P$. (This inclusion also holds when λ is trivial, i.e., when $k = 0$.)

Thus it now suffices to show that Σ_P is uniformly discrete. By the containment $Q_\lambda(S) \subseteq \Sigma_P$ it then follows that $Q_\lambda(S)$ is uniformly discrete.

Our first goal is to put the set $\Sigma_P = \{x \in \mathbb{Z}[\lambda] \mid \mathbf{x}' \in P\}$ in a form (Eq. (22) below) convenient for proving its uniform discreteness. By the notation introduced above, $\Sigma_P = \{\boldsymbol{\lambda}\mathbf{a} \mid \mathbf{a} \in \mathbb{Z}^d \ \& \ (\boldsymbol{\lambda}\mathbf{a})' \in P\}$. Now recall the Vandermonde matrix $V = V(\mu_0, \dots, \mu_{d-1})$ defined in Eq. (14). For any vector $\mathbf{v} \in \mathbb{C}^d$, let $\mathbf{v}_F$ denote the vector consisting of the First k elements of $\mathbf{v}$. Using this notation, $(\boldsymbol{\lambda}\mathbf{a})' = (V\mathbf{a})_F$. Then we have

$$\Sigma_P = \{\boldsymbol{\lambda}\mathbf{a} \mid \mathbf{a} \in \mathbb{Z}^d \ \& \ (\boldsymbol{\lambda}\mathbf{a})' \in P\} = \{\boldsymbol{\lambda}\mathbf{a} \mid \mathbf{a} \in \mathbb{Z}^d \ \& \ (V\mathbf{a})_F \in P\}\,. \tag{17}$$

The first k rows of V can be chopped up into two matrices: a $k \times k$ matrix W consisting of the first k rows and columns of V, followed by a $k \times (d-k)$ matrix X consisting of the first k rows and remaining columns of V. (Both W and X are real matrices.) Thus V looks as follows when $\lambda \in \mathbb{R}$ (case 1 where $k = d-1$) and $\lambda \notin \mathbb{R}$ (case 2 where $k = d-2$), respectively:

$$V = \left[\begin{array}{cccc|c} & & \vdots & & \vdots \\ \cdots & & W & \cdots & X \\ & & \vdots & & \vdots \\ \hline 1 & \lambda & \lambda^2 & \cdots & \lambda^{d-1} \end{array}\right], \qquad V = \left[\begin{array}{cccc|cc} & & \vdots & & \vdots & \\ \cdots & & W & \cdots & X & \\ & & \vdots & & \vdots & \\ \hline 1 & \lambda & \lambda^2 & \cdots & \lambda^{d-2} & \lambda^{d-1} \\ 1 & \lambda^* & (\lambda^*)^2 & \cdots & (\lambda^*)^{d-2} & (\lambda^*)^{d-1} \end{array}\right].$$

Decompose the vector $\mathbf{a}$ as $(\mathbf{a}_F; \mathbf{a}_L)$, where as above $\mathbf{a}_F$ denotes the first k coordinates of $\mathbf{a}$, and $\mathbf{a}_L$ denotes the Last $d-k$ coordinates of $\mathbf{a}$. Thus $(V\mathbf{a})_F = W\mathbf{a}_F + X\mathbf{a}_L$, and Eq. (17) becomes

$$\begin{aligned} \Sigma_P &= \{\boldsymbol{\lambda}\mathbf{a} \mid \mathbf{a}_F \in \mathbb{Z}^k \ \& \ \mathbf{a}_L \in \mathbb{Z}^{d-k} \ \& \ W\mathbf{a}_F + X\mathbf{a}_L \in P\} \\ &= \{\boldsymbol{\lambda}\mathbf{a} \mid \mathbf{a}_F \in \mathbb{Z}^k \ \& \ \mathbf{a}_L \in \mathbb{Z}^{d-k} \ \& \ W\mathbf{a}_F \in P - X\mathbf{a}_L\}\,. \end{aligned}$$

Note that $W = V(\mu_0, \ldots, \mu_{k-1})$ is invertible and W^{-1} is a real matrix. Hence, defining $\Omega = W^{-1}P$ and $Y = W^{-1}X$, we have,

$$\Sigma_P = \{\boldsymbol{\lambda}\mathbf{a} \mid \mathbf{a}_F \in \mathbb{Z}^k \cap (\Omega - Y\mathbf{a}_L) \;\;\&\;\; \mathbf{a}_L \in \mathbb{Z}^{d-k}\} \; . \tag{18}$$

Clearly, $\Omega \subseteq \mathbb{R}^k$ is a bounded, convex parallelepiped depending only on S and $\mu_0, \ldots, \mu_{k-1}$.

Letting $\boldsymbol{\lambda}_F := (1, \lambda, \ldots, \lambda^{k-1})$ and $\boldsymbol{\lambda}_L := (\lambda^k, \ldots, \lambda^{d-1})$ (both row vectors which concatenate to $\boldsymbol{\lambda}$, so that $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_F; \boldsymbol{\lambda}_L)$ is the k^{th} row of V), Equation (18) establishes that

$$\Sigma_P = \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} \Big\{ (\boldsymbol{\lambda}_F; \boldsymbol{\lambda}_L)(\mathbf{a}_F; \mathbf{a}_L) \mid \mathbf{a}_F \in \mathbb{Z}^k \cap (\Omega - Y\mathbf{a}_L) \Big\} \; . \tag{19}$$

We will use the F- and L-subscripts for new, stand-alone vectors we introduce to indicate their types: F-subscripts for k-dimensional vectors and L-subscripts for $(d-k)$-dimensional vectors. For any $\mathbf{c}_L \in \mathbb{Z}^{d-k}$, define

$$\Omega_{\mathbf{c}_L} := \Big(\mathbb{Z}^k + Y\mathbf{c}_L\Big) \cap \Omega \; . \tag{20}$$

For any $\mathbf{c} = (\mathbf{c}_F; \mathbf{c}_L) \in \mathbb{Z}^d$, let $\hat{\mathbf{c}}_F$ denote $\mathbf{c}_F + Y\mathbf{c}_L$ (column vector in $\mathbb{C}^k$). Then (19) becomes

$$\begin{aligned}
\Sigma_P &= \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} \Big\{ \boldsymbol{\lambda}_F\mathbf{a}_F + \boldsymbol{\lambda}_L\mathbf{a}_L \mid \mathbf{a}_F \in \mathbb{Z}^k \cap (\Omega - Y\mathbf{a}_L) \Big\} \\
&= \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} \Big\{ \boldsymbol{\lambda}_F\mathbf{a}_F + \boldsymbol{\lambda}_L\mathbf{a}_L \mid \hat{\mathbf{a}}_F \in \Big(\mathbb{Z}^k + Y\mathbf{a}_L\Big) \cap \Omega \Big\} \\
&= \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} \{ \boldsymbol{\lambda}_L\mathbf{a}_L + \boldsymbol{\lambda}_F(\hat{\mathbf{a}}_F - Y\mathbf{a}_L) \mid \hat{\mathbf{a}}_F \in \Omega_{\mathbf{a}_L} \} \\
&= \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} \{ (\boldsymbol{\lambda}_L - \boldsymbol{\lambda}_F Y)\,\mathbf{a}_L + \boldsymbol{\lambda}_F\hat{\mathbf{a}}_F \mid \hat{\mathbf{a}}_F \in \Omega_{\mathbf{a}_L} \} \\
&= \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} [(\boldsymbol{\lambda}_L - \boldsymbol{\lambda}_F Y)\,\mathbf{a}_L + \boldsymbol{\lambda}_F\Omega_{\mathbf{a}_L}] \, .
\end{aligned}$$

The set $\boldsymbol{\lambda}_F\Omega_{\mathbf{a}_L}$ in the right-hand side is bounded, independent of $\mathbf{a}_L$, because $\Omega_{\mathbf{a}_L} \subseteq \Omega$. Set

$$\Delta := \boldsymbol{\lambda}_L - \boldsymbol{\lambda}_F Y \; , \tag{21}$$

noting that Δ is a $(d-k)$-dimensional row vector. With these definitions, we have

$$\Sigma_P = \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} (\Delta\mathbf{a}_L + \boldsymbol{\lambda}_F\Omega_{\mathbf{a}_L}) \; . \tag{22}$$

Consider the set $A := \Delta\mathbb{Z}^{d-k} = \big\{\Delta\mathbf{a}_L \mid \mathbf{a}_L \in \mathbb{Z}^{d-k}\big\}$. A is a subgroup of the additive group of $\mathbb{C}$, being the image of $\mathbb{Z}^{d-k}$ under the homomorphism given by Δ.

Claim 12.28. *Suppose that (1) A has no accumulation points in $\mathbb{C}$ (whence A is uniformly discrete), and (2) the map given by Δ is one-to-one*[9] *restricted to $\mathbb{Z}^{d-k}$, i.e., for any point $p \in A$, there is only one $\mathbf{a}_L \in \mathbb{Z}^{d-k}$ such that $p = \Delta\mathbf{a}_L$. Then Σ_P is uniformly discrete.*

[9]This restriction can be relaxed to "finite-to-one," and the proof of the claim still goes through. It does not matter here, because every nontrivial subgroup of $\mathbb{Z}^{d-k}$ is infinite, but this relaxation may be useful if the claim is ever generalized.

Proof of the claim. We show Σ_P as expressed in Eq. (22) is uniformly discrete. We must find an $r > 0$ such that any two distinct points in Σ_P are at least r distance apart. Let x and y be any two distinct elements of Σ_P, and assume that $|x - y| \leq 1$, i.e., $x - y \in \overline{D}$, where $\overline{D}$ is the closed unit disk in $\mathbb{C}$. By Eq. (22), we can write

$$x = \Delta \mathbf{a}_L + \boldsymbol{\lambda}_F \mathbf{x}_F \ ,$$
$$y = \Delta \mathbf{b}_L + \boldsymbol{\lambda}_F \mathbf{y}_F \ ,$$

for some $\mathbf{a}_L, \mathbf{b}_L \in \mathbb{Z}^{d-k}$ and some $\mathbf{x}_F \in \Omega_{\mathbf{a}_L}$ and $\mathbf{y}_F \in \Omega_{\mathbf{b}_L}$. By the definition of $\Omega_{\mathbf{c}_L}$ (Eq. (20)), we can write $\mathbf{x}_F = \mathbf{a}_F + Y\mathbf{a}_L$ and $\mathbf{y}_F = \mathbf{b}_F + Y\mathbf{b}_L$ for some $\mathbf{a}_F, \mathbf{b}_F \in \mathbb{Z}^k$. Thus we find,

$$\begin{aligned}
x - y &= \Delta \mathbf{a}_L - \Delta \mathbf{b}_L + \boldsymbol{\lambda}_F(\mathbf{x}_F - \mathbf{y}_F) && (23)\\
&= \Delta \mathbf{a}_L - \Delta \mathbf{b}_L + \boldsymbol{\lambda}_F \mathbf{a}_F - \boldsymbol{\lambda}_F \mathbf{b}_F + \boldsymbol{\lambda}_F Y \mathbf{a}_L - \boldsymbol{\lambda}_F Y \mathbf{b}_L && (24)\\
&= \boldsymbol{\lambda}_L \mathbf{a}_L - \boldsymbol{\lambda}_L \mathbf{b}_L - \boldsymbol{\lambda}_F Y \mathbf{a}_L + \boldsymbol{\lambda}_F Y \mathbf{b}_L + \boldsymbol{\lambda}_F \mathbf{a}_F - \boldsymbol{\lambda}_F \mathbf{b}_F + \boldsymbol{\lambda}_F Y \mathbf{a}_L - \boldsymbol{\lambda}_F Y \mathbf{b}_L && (25)\\
&= \boldsymbol{\lambda}_L(\mathbf{a}_L - \mathbf{b}_L) + \boldsymbol{\lambda}_F(\mathbf{a}_F - \mathbf{b}_F), && (26)
\end{aligned}$$

where in the third equality we used the definition of Δ. The remainder of the proof is essentially to show that $\boldsymbol{\lambda}_L(\mathbf{a}_L - \mathbf{b}_L)$ and $\boldsymbol{\lambda}_F(\mathbf{a}_F - \mathbf{b}_F)$ take on finitely many possible values, from which we can conclude the same for $x - y$.

To see this, note that by rearranging Eq. (23), we find, $x - y - \boldsymbol{\lambda}_F(\mathbf{x}_F - \mathbf{y}_F) = \Delta(\mathbf{a}_L - \mathbf{b}_L)$. From $\mathbf{x}_F - \mathbf{y}_F \in \Omega_{\mathbf{a}_L} - \Omega_{\mathbf{b}_L} \subseteq \Omega - \Omega$, and since $x - y \in \overline{D}$, we conclude that $\Delta(\mathbf{a}_L - \mathbf{b}_L) \in \overline{D} - \boldsymbol{\lambda}_F(\Omega - \Omega)$. We also have $\Delta(\mathbf{a}_L - \mathbf{b}_L) \in A$, so $\Delta(\mathbf{a}_L - \mathbf{b}_L) \in A \cap (\overline{D} - \boldsymbol{\lambda}_F(\Omega - \Omega))$. But A is discrete and $\overline{D} - \boldsymbol{\lambda}_F(\Omega - \Omega)$ is bounded, so by Lemma 12.7, $A \cap (\overline{D} - \boldsymbol{\lambda}_F(\Omega - \Omega))$ is finite. For this reason, and because Δ is 1-1, the set G defined as

$$G := \left\{ \mathbf{a}_L \in \mathbb{Z}^{d-k} \mid \Delta \mathbf{a}_L \in \overline{D} - \boldsymbol{\lambda}_F(\Omega - \Omega) \right\}$$

is finite. Because $\Delta(\mathbf{a}_L - \mathbf{b}_L) \in \overline{D} - \boldsymbol{\lambda}_F(\Omega - \Omega)$, we have that $\mathbf{a}_L - \mathbf{b}_L \in G$, and hence $\boldsymbol{\lambda}_L(\mathbf{a}_L - \mathbf{b}_L)$ is contained in $\boldsymbol{\lambda}_L G$, which is finite as well. Now we know $\mathbf{x}_F - \mathbf{y}_F = \mathbf{a}_F - \mathbf{b}_F + Y(\mathbf{a}_L - \mathbf{b}_L)$, so that $\mathbf{a}_F - \mathbf{b}_F = \mathbf{x}_F - \mathbf{y}_F - Y(\mathbf{a}_L - \mathbf{b}_L) \in \Omega - \Omega - YG$. However, because G is finite and $\Omega - \Omega$ is bounded, the set H defined by $H := \mathbb{Z}^k \cap (\Omega - \Omega - YG)$ is finite, again by Lemma 12.7. We thus have $\mathbf{a}_F - \mathbf{b}_F \in H$, and hence $\boldsymbol{\lambda}_F(\mathbf{a}_F - \mathbf{b}_F) \in \boldsymbol{\lambda}_F H$, where $\boldsymbol{\lambda}_F H$ is finite. Then by Eq. (26), $x - y \in \boldsymbol{\lambda}_L G + \boldsymbol{\lambda}_F H$, where, by the foregoing, the sets G and H are independent of x and y. Since $\boldsymbol{\lambda}_L G + \boldsymbol{\lambda}_F H$ is finite, its nonzero elements are bounded away from 0 in absolute value, and so we may set $r := \min(\{1\} \cup \{|z| : z \in \boldsymbol{\lambda}_L G + \boldsymbol{\lambda}_F H \text{ and } z \neq 0\})$. This establishes the Claim. □

Continuing with the proof of Theorem 12.14, we just need to see when the vector Δ and the set $A = \Delta \mathbb{Z}^{d-k}$ satisfy the assumptions of the Claim. Start with the observation that, by Eq. (22), we have

$$\begin{aligned}
\Sigma_P &= \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} (\Delta \mathbf{a}_L + \boldsymbol{\lambda}_F \Omega_{\mathbf{a}_L}) && (27)\\
&\subseteq \bigcup_{\mathbf{a}_L \in \mathbb{Z}^{d-k}} (\Delta \mathbf{a}_L + \boldsymbol{\lambda}_F \Omega) && (28)\\
&= \{\Delta \mathbf{a}_L \mid \mathbf{a}_L \in \mathbb{Z}^{d-k}\} + \boldsymbol{\lambda}_F \Omega && (29)\\
&= A + \boldsymbol{\lambda}_F \Omega \ . && (30)
\end{aligned}$$

We now show that the assumptions (1) and (2) of the Claim are satisfied in both of the two cases given at the start of this proof.

1. When $k = d-1$ (whence $\lambda \in \mathbb{R}$), X and Y are both $(d-1)$-dimensional column vectors, $\boldsymbol{\lambda}_F$ is a $(d-1)$-dimensional row vector, $\boldsymbol{\lambda}_L$ is the scalar λ^{d-1}, and Δ (cf. Eq. (21)) is the scalar $\lambda^{d-1} - \boldsymbol{\lambda}_F Y$. The set $A = \Delta\mathbb{Z}$ is clearly discrete. Also, we must have $\Delta \neq 0$, for otherwise, $A = \{0\}$, and so by Theorem 12.27 and Equations (16) and (27–30), $Q_\lambda(S) \subseteq \Sigma_P \subseteq \boldsymbol{\lambda}_F \Omega$, which is bounded, but we know that $Q_\lambda(S)$ is unbounded. Thus, for any point $p \in A$ there is exactly one $a \in \mathbb{Z}$ such that $p = \Delta a$.

2. When $k = d-2$ (and $\lambda \notin \mathbb{R}$), W is a $k \times k$ matrix, X and Y are $k \times 2$ matrices, $\boldsymbol{\lambda}_F = (1, \ldots, \lambda^{d-3})$, and $\boldsymbol{\lambda}_L = (\lambda^{d-2}, \lambda^{d-1})$ (both row vectors). Let $\mathbf{x}_{d-2} := ((\mu_0)^{d-2}, \ldots, (\mu_{d-3})^{d-2})$ and $\mathbf{x}_{d-1} := ((\mu_0)^{d-1}, \ldots, (\mu_{d-3})^{d-1})$ be the columns of X, and let $\mathbf{y}_{d-2} := W^{-1}\mathbf{x}_{d-2}$ and $\mathbf{y}_{d-1} := W^{-1}\mathbf{x}_{d-1}$ be the columns of Y. Then $\Delta = (\alpha, \beta) \in \mathbb{C}^2$, where $\alpha := \lambda^{d-2} - \boldsymbol{\lambda}_F \mathbf{y}_{d-2}$ and $\beta := \lambda^{d-1} - \boldsymbol{\lambda}_F \mathbf{y}_{d-1}$, and this gives $A = \Delta\mathbb{Z}^2 = \alpha\mathbb{Z} + \beta\mathbb{Z}$. It is easy to see that if $\alpha \neq 0$ and $\beta/\alpha \notin \mathbb{R}$ (i.e., α and β are in different directions), then the set A is uniformly discrete, and also, for any $p \in A$ there is exactly one pair $(a_{d-2}, a_{d-1}) \in \mathbb{Z}^2$ such that $p = \alpha a_{d-2} + \beta a_{d-1}$, satisfying the hypotheses of Claim 12.28. Now, we show that $\alpha \neq 0$ and $\beta/\alpha \notin \mathbb{R}$, which finishes the proof.

   By replacing $\mathbf{y}_{d-2}$ and $\mathbf{y}_{d-1}$ by their definitions, we can write

   $$\alpha = \lambda^{d-2} - (\lambda^0, \ldots, \lambda^{d-3})W^{-1}\mathbf{x}_{d-2} \quad \text{and} \quad \beta = \lambda^{d-1} - (\lambda^0, \ldots, \lambda^{d-3})W^{-1}\mathbf{x}_{d-1} \,. \tag{31}$$

   We know that $WW^{-1} = I$, so for any $0 \leq i \leq d-3$, $((\mu_i)^0, (\mu_i)^1, \ldots, (\mu_i)^{d-3})W^{-1} = \mathbf{e}_i^\mathsf{T}$. And hence,

   $$((\mu_i)^0, (\mu_i)^1, \ldots, (\mu_i)^{d-3})W^{-1}\mathbf{x}_{d-2} = \mathbf{e}_i^\mathsf{T}\mathbf{x}_{d-2} = (\mu_i)^{d-2} \,, \tag{32}$$
   $$((\mu_i)^0, (\mu_i)^1, \ldots, (\mu_i)^{d-3})W^{-1}\mathbf{x}_{d-1} = \mathbf{e}_i^\mathsf{T}\mathbf{x}_{d-1} = (\mu_i)^{d-1} \,. \tag{33}$$

   Let us now look at the α and β of (31) as polynomials in λ. With some abuse of notation let us define the monic polynomial $\alpha(z) := z^{d-2} - (z^0, z^1, \ldots, z^{d-3})W^{-1}\mathbf{x}_{d-2}$. Equation (32) tells us that $\alpha(z)$ has $\mu_0, \mu_1, \ldots, \mu_{d-3}$ as its roots. As its degree is $d-2$, we can write

   $$\alpha(z) = (z-\mu_0)(z-\mu_1)\cdots(z-\mu_{d-3}) \,.$$

   Similarly, by Equation (33), $\mu_0, \ldots, \mu_{d-3}$ are also roots of the monic polynomial $\beta(z) := z^{d-1} - (z^0, \ldots, z^{d-3})W^{-1}\mathbf{x}_{d-1}$. But $\beta(z)$ has degree $d-1$, so it has one more root. The sum of all its roots is equal to minus the coefficient on z^{d-2} in $\beta(z)$, which is zero, and so the other root is $(-\mu_0 - \mu_1 - \cdots - \mu_{d-3})$. Hence we can write the following:

   $$\alpha = \alpha(\lambda) = (\lambda-\mu_0)(\lambda-\mu_1)\cdots(\lambda-\mu_{d-3}) \,,$$
   $$\beta = \beta(\lambda) = (\lambda-\mu_0)(\lambda-\mu_1)\cdots(\lambda-\mu_{d-3})(\lambda+\mu_0+\ldots+\mu_{d-3}) \,.$$

   Now it is clear that $\alpha \neq 0$ and $\beta/\alpha = \lambda + \mu_0 + \ldots + \mu_{d-3}$, which is non-real because λ is non-real.

Thus in both cases, the assumptions of Claim 12.28 are satisfied, and the proof of Theorem 12.14 is complete. $\square$

Remark. Case 1 of the proof of Theorem 12.14 (the case where $k = d-1$) would still go through if we allowed $p \in D[x]$ for any discrete subring $D \subseteq \mathbb{C}$ (instead of insisting that $p \in \mathbb{Z}[x]$), that is, we could relax the definition of strong PV number to allow algebraic D-integers λ rather than $\mathbb{Z}$-integers, and we would still get discrete Q_λ in Case 1. This generalization does not yield any new values of λ that are not already sPV by our original definition, however. See Proposition A.1 in the Appendix. □

We summarize the most important findings of Theorem 12.27 and the proof of Theorem 12.14—Equation (16) in particular—in the following fact (cf. Equation (18)) for any finite $S \subseteq \mathbb{Z}[\lambda]$.

Fact 12.29. *Let $\lambda \in \mathbb{C} \setminus [0,1]$ be an algebraic integer of degree $d > 0$. Let $\mu_0, \ldots, \mu_{d-1} \in \mathbb{C}$ be the conjugates of λ, and assume that $\mu_0, \ldots, \mu_{k-1}$ are elements of $[0,1]$, for some $k \geq 1$, and that $\lambda = \mu_k$. Let $S \subseteq \mathbb{Z}[\lambda]$ be finite. Then letting*

- $W := V(\mu_0, \ldots, \mu_{k-1})$,
- $\mathbf{x}_j := \big((\mu_0)^j, \ldots, (\mu_{k-1})^j\big)$ *(column vector) for all $k \leq j < d$,*
- $\mathbf{y}_j := W^{-1}\mathbf{x}_j \in \mathbb{R}^k$ *for all $k \leq j < d$,*
- *for all $0 \leq j < k$, $\ell_j := \min S_j$ and $h_j := \max S_j$, where*

$$S_j := \left\{ \sum_{i=0}^{d-1} a_i(\mu_j)^i \ : \ a_0, \ldots, a_{d-1} \in \mathbb{Z} \ \ \& \ \sum_{i=0}^{d-1} a_i\lambda^i \in S \right\} ,$$

  and
- $\Omega := W^{-1}P$, *where* $P := [\ell_0, h_0] \times [\ell_1, h_1] \times \cdots \times [\ell_{k-1}, h_{k-1}]$,

we have

$$Q_\lambda(S) \subseteq \Sigma_P = \Big\{ (\boldsymbol{\lambda}\mathbf{a}) \mid \mathbf{a} \in \mathbb{Z}^d \ \ \& \ \ \mathbf{a}_F \in \mathbb{Z}^k \cap (\Omega - Y\mathbf{a}_L) \Big\} \tag{34}$$

$$= \left\{ \sum_{j=0}^{d-1} a_j\lambda^j \ \middle|\ (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \ \ \& \ \ (a_0, \ldots, a_{k-1}) \in \Omega - \sum_{j=k}^{d-1} a_j\mathbf{y}_j \right\} \tag{35}$$

$$= \left\{ \sum_{j=0}^{d-1} a_j\lambda^j \ \middle|\ (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \ \ \& \ \ W(a_0, \ldots, a_{k-1}) + \sum_{j=k}^{d-1} a_j\mathbf{x}_j \in P \right\} \tag{36}$$

$$= \left\{ \sum_{j=0}^{d-1} a_j\lambda^j \ \middle|\ (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \ \ \& \ \ \ell_i \leq \sum_{j=0}^{d-1} a_j(\mu_i)^j \leq h_i \textit{ for all } 0 \leq i < k \right\} . \tag{37}$$

In the special case where $S = \{0,1\}$, we have $\ell_0 = \cdots = \ell_{k-1} = 0$ and $h_0 = \cdots = h_{k-1} = 1$, whence

$$Q_\lambda \subseteq \Sigma_P = \left\{ \sum_{j=0}^{d-1} a_j\lambda^j \ \middle|\ (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \ \ \& \ \ 0 \leq \sum_{j=0}^{d-1} a_j(\mu_i)^j \leq 1 \textit{ for all } 0 \leq i < k \right\} . \tag{38}$$

12.3 Connection to cut-and-project schemes

Here we show that the set $\Sigma_P^{(\lambda)}$ of Definition 12.26, used explicitly in Theorem 12.14 and Fact 12.29 and which includes $Q_\lambda(S)$ $(= R_\lambda(S))$ as a subset by Theorem 12.27, is always the model set of a particular cut-and-project scheme (see Section 12.1). Let λ be sPV with minimal polynomial $p(x)$ of degree d, and let S be some finite subset of $\mathbb{Z}[\lambda]$, as in the proof of Theorem 12.14. As in that proof, we can assume that λ is nontrivial with $k \geq 1$ conjugates $\mu_0, \ldots, \mu_{k-1}$ in $(0, 1)$. If $\lambda \in \mathbb{R}$, then $k = d - 1$, and otherwise, $k = d - 2$ and we set $\mu_{d-1} := \lambda^*$. In either case, we set $\mu_k := \lambda$. We define P and $\Sigma_P = \Sigma_P^{(\lambda)}$ accordingly.

In what follows, V will be the $(k + 1) \times d$ matrix consisting of the first $k + 1$ rows of the Vandermonde matrix $V' := V(\mu_0, \ldots, \mu_{d-1})$. Our conventions dictate that if $\lambda \in \mathbb{R}$, then $V = V'$, and otherwise, V is missing the last row of V'—the row corresponding to λ^*. In either case, the last row of V contains powers of $\mu_k = \lambda$, and all the other entries of V are real. We also let F be $\mathbb{R}$ if $\lambda \in \mathbb{R}$, and $F := \mathbb{C}$, otherwise. We identify $\mathbb{C}$ with $\mathbb{R}^2$ via real and imaginary parts, and so the columns of V can naturally be viewed as vectors in d real dimensions.

Definition 12.30. Given λ, etc. as above, define the tuple

$$\mathcal{M} := (\mathbb{R}^k, F, L) \ ,$$

where L is the d-(real)-dimensional integer lattice spanned by the columns of V:

$$L := V\mathbb{Z}^d = \left\{ \sum_{j=0}^{d-1} a_j \mathbf{u}_j : a_0, \ldots, a_{d-1} \in \mathbb{Z} \right\} \ ,$$

where, for all $0 \leq j < d$, $\mathbf{u}_j := ((\mu_0)^j, \ldots, (\mu_k)^j)$ is the j^{th} column of V.

We also have the corresponding projection maps π_1 and π_2, where $\pi_1 : \mathbb{R}^k \times F \to \mathbb{R}^k$ takes a vector $\mathbf{v} \in \mathbb{R}^k \times F$ to the vector of its first k components (all real), and $\pi_2 : \mathbb{R}^k \times F \to F$ takes $\mathbf{v}$ to its last component (either real or complex, depending on λ).

We can give a short, self-contained proof that the tuple $\mathcal{M}$ is a cut-and-project scheme (see Definition 12.9), based on some of our previous results.

Proposition 12.31. *Let $\lambda \in \mathbb{C}$ be sPV of degree d, and let $\mathcal{M}$ be as in Definition 12.30. Then $\mathcal{M}$ is a cut-and-project scheme. Further, if λ is nontrivial, then $\pi_1|L$ is also one-to-one.*

Proof. The vectors $\mathbf{u}_j$ can be seen to be $\mathbb{R}$-linearly independent. (This is clearly true when $\lambda \in \mathbb{R}$ and so $V = V'$, since the conjugates of λ are pairwise distinct. If $\lambda \notin \mathbb{R}$, then V is missing the last row (powers of λ^*) of V'. But any $\mathbb{R}$-linear combination of the columns of V' makes the $(k+1)^{st}$ (i.e., the last) component the complex conjugate of the k^{th} component, and so the former is nonzero whenever the latter is.) Since the $\mathbf{u}_j$ are linearly independent, L spans $\mathbb{R}^d \cong \mathbb{R}^k \times F$, and thus $\mathbb{R}^d/L$ is compact (the d-dimensional torus).

The only other nonobvious things to prove are that: (a) $\pi_1(L)$ is dense in $\mathbb{R}^k$, (b) $\pi_2|L$ is one-to-one, and (c) $\pi_1|L$ is one-to-one if λ is nontrivial.

For part (a), let $p(x) = x^d + \sum_{i=0}^{d-1} c_i x^i \in \mathbb{Z}[x]$ be the minimal polynomial of λ, and let $\Lambda = \Lambda_p$ be the $d \times d$ companion matrix of p as in Definition 12.24. Let D be the $(k+1) \times (k+1)$ diagonal matrix where $[D]_{ii} = \mu_i$ for all $0 \le i \le k$. By adapting the proof of Lemma 12.25, we see that $V\Lambda = DV$, which implies

$$DL = DV\mathbb{Z}^d = V\Lambda\mathbb{Z}^d \subseteq V\mathbb{Z}^d = L\ ,$$

the inclusion following from the fact that Λ is an integer matrix. Thus D maps L into L, and hence the same is true for $I - D$. It follows that L is closed[10] under the binary operation $\star_D$ (cf. Definition 12.18), and so $Q_D(S) \subseteq L$ for any $S \subseteq L$.

Define $\mathbf{w} := V\mathbf{1}$, where $\mathbf{1}$ is the d-dimensional vector of all ones. Note that $\mathbf{w}$ is in L. Set $\mathbf{x} := \pi_1(\mathbf{w})$. The entries $x_0, \dots, x_{k-1}$ of $\mathbf{x}$ (i.e., all but the last entry of $\mathbf{w}$) are all positive reals; indeed, $x_i = (1 - \mu_i^d)/(1 - \mu_i)$ for all $0 \le i < k$, and each such μ_i is in $(0, 1)$. Also note that $\pi_1 \circ D = D' \circ \pi_1$, where D' is the $k \times k$ diagonal matrix whose diagonal elements are $\mu_0, \dots, \mu_{k-1}$, i.e., the first k elements of the diagonal of D, which are pairwise distinct elements of $(0, 1)$ (both compositions above are maps $\mathbb{R}^k \times F \to \mathbb{R}^k$). Now by Lemma 12.20, we have $\pi_1(Q_D(\{0, \mathbf{w}\})) = Q_{D'}(\pi_1(\{0, \mathbf{w}\})) = Q_{D'}(\{0, \mathbf{x}\})$, which is dense in the k-dimensional box $B := [0, x_0] \times \cdots \times [0, x_{k-1}]$ by Corollary 12.23. But $\pi_1(Q_D(\{0, \mathbf{w}\})) \subseteq \pi_1 L$ by the previous paragraph, and thus $\pi_1 L \cap B$ is dense in B as well. By this fact and the translation invariance of L, $\pi_1 L$ must be dense in all of $\mathbb{R}^k$, which proves part (a).

For part (b), consider two distinct points $\mathbf{x}, \mathbf{y} \in L$. Then $\mathbf{x} - \mathbf{y} \in L \setminus \{0\}$, and we can write $\mathbf{x} - \mathbf{y} = V\mathbf{a}$ for some nonzero $\mathbf{a} = (a_0, \dots, a_{d-1}) \in \mathbb{Z}^d$.

Not only is $V\mathbf{a}$ nonzero, but all of its components are nonzero as well. Consider any component of $V\mathbf{a}$, say, the j^{th} component (for some j). This component is $\sum_{i=0}^{d-1} a_i \mu_j^i$, which cannot be 0, for that would contradict the fact that μ_j has degree d (being a conjugate of λ).

We have $\pi_2(\mathbf{x}) - \pi_2(\mathbf{y}) = \pi_2(\mathbf{x} - \mathbf{y}) = \pi_2(V\mathbf{a})$. The k^{th} component of $V\mathbf{a}$ is $\pi_2(V\mathbf{a})$, which is nonzero, and thus $\pi_2|L$ is one-to-one.

For part (c), suppose λ is nontrivial. For $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{a}$ as above, $\pi_1(\mathbf{x}) - \pi_1(\mathbf{y}) = \pi_1(\mathbf{x} - \mathbf{y}) = \pi_1(V\mathbf{a})$. Since λ is nontrivial, $k \ge 1$. Thus the vector $\pi_1(V\mathbf{a})$ has positive dimension, i.e., the 0^{th} component of $V\mathbf{a}$ is included in $\pi_1(V\mathbf{a})$ and is nonzero. From this we conclude that $\pi_1|L$ is one-to-one. ☐

Proposition 12.32. *Let λ and $\mathcal{M}$ be as in Proposition 12.31. For any finite $S \subseteq \mathbb{Z}[\lambda]$, the set $Q_\lambda(S)$ is a subset of some model set of $\mathcal{M}$ (cf. Definition 12.10).*

Proof. We show that the set Σ_P (Definition 12.26) corresponding to λ and S is a model set of $\mathcal{M}$. To see how Σ_P fits into Definition 12.30, we essentially perform a change of basis. First, we recall the notation used in the proof of Theorem 12.14 and in Definition 12.30:

- V is the $(k+1) \times d$ matrix of the first $k+1$ rows of the Vandermonde matrix $V(\mu_0, \dots, \mu_{d-1})$, indexed by 0 through k. We have no use for the $(k+1)^{st}$ row of the Vandermonde matrix (if it exists).

[10] Here is an alternative proof that does not use the companion matrix: Let $\mathbf{x}, \mathbf{y} \in L = V\mathbb{Z}^d$. Then for some $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$, we have

$$\mathbf{x} = \left(\sum_{j=0}^{d-1} a_j \mu_0^j, \dots, \sum_{j=0}^{d-1} a_j \mu_{d-1}^j \right) , \qquad \mathbf{y} = \left(\sum_{j=0}^{d-1} b_j \mu_0^j, \dots, \sum_{j=0}^{d-1} b_j \mu_{d-1}^j \right) .$$

The i^{th} component of $\mathbf{x} \star_D \mathbf{y}$ is then $(1 - \mu_i) \sum_{j=1}^{d-1} a_j \mu_i^j + \mu_i \sum_{j=1}^{d-1} b_j \mu_i^j = \sum_{j=1}^{d-1} \left(a_j \mu_i^j + (b_j - a_j) \mu_i^{j+1} \right)$. But μ_i^d is an integer combination of $\mu_i^0, \dots, \mu_i^{d-1}$, so this is of the form $z_i = \sum_{j=0}^{d-1} c_j \mu_i^j$, where each $c_j \in \mathbb{Z}$. Thus the resulting vector $\mathbf{z} = \mathbf{x} \star_D \mathbf{y}$ is contained in L.

- The last (i.e., the k^{th}) row of V is the row vector $\mathbf{e}_k{}^{\mathsf{T}}V = (1, \lambda, \ldots, \lambda^{d-1})$ of powers of $\lambda = \mu_k$. (Here, $\mathbf{e}_k$ is the $(k+1)$-dimensional column vector $(0, 0, \ldots, 0, 1)$.)

- The projection maps π_1 and π_2 and the lattice $L = V\mathbb{Z}^d$ are as in Definition 12.30.

Now we can start with Equation (36), using the notation of Fact 12.29. Let V' be the first k rows of V (i.e., all but the last row). Noting that V' is formed by appending the column vectors $\mathbf{x}_k, \ldots, \mathbf{x}_{d-1}$ onto the right end of W, we have

$$\begin{aligned}\Sigma_P &= \left\{ \sum_{j=0}^{d-1} a_j \lambda^j \;\middle|\; (a_0, \ldots, a_{d-1}) \in \mathbb{Z}^d \;\;\&\;\; W(a_0, \ldots, a_{k-1}) + \sum_{j=k}^{d-1} a_j \mathbf{x}_j \in P \right\} \\ &= \left\{ \mathbf{e}_k{}^{\mathsf{T}} V\mathbf{a} \mid \mathbf{a} \in \mathbb{Z}^d \;\;\&\;\; V'\mathbf{a} \in P \right\} .\end{aligned}$$

Letting $\mathbf{b} := V\mathbf{a}$, we finally get

$$\Sigma_P = \left\{ \mathbf{e}_k{}^{\mathsf{T}}\mathbf{b} \mid \mathbf{b} \in V\mathbb{Z}^d \;\;\&\;\; \pi_1(\mathbf{b}) \in P \right\} = \{\pi_2(\mathbf{b}) \mid \mathbf{b} \in L \;\;\&\;\; \pi_1(\mathbf{b}) \in P\} = \pi_2\left(L \cap \pi_1^{-1}(P)\right) .$$

The right-hand side is evidently the model set with window P in the cut-and-project scheme of Definition 12.30. □

Remark. Unfortunately, Proposition 12.32 does not show that the various discrete $Q_\lambda(S)$ are Meyer sets, only because we do not know in general whether any particular $Q_\lambda(S)$ is relatively dense. We address the issue of relative density in Section 15, below, where we prove relative density in many cases. □

The next proposition speaks to the aperiodicity of various $Q_\lambda(S)$. It follows immediately from Lemma 12.12 and Proposition 12.31.

Proposition 12.33. *Let λ be sPV and $S \subseteq \mathbb{Q}(\lambda)$ be finite. If λ is nontrivial (by Definition 12.1), then $Q_\lambda(S)$ intersects any arithmetic progression in only finitely many points (and is therefore aperiodic).*

Proof. By scaling everything up by some appropriate integer as in the proof of Theorem 12.14, we may assume that $S \subseteq \mathbb{Z}[\lambda]$.

By Proposition 12.32, $Q_\lambda(S)$ is a subset of Σ_P, the model set of a cut-and-project scheme $\mathcal{M}$ as in Proposition 12.31, and since λ is nontrivial, the map $\pi_1|L$ is one-to-one. Then Σ_P intersects any arithmetic progression in only finitely many points by Lemma 12.12. □

13 Further properties of strong PV numbers

In this section, we collect a some additional facts about strong PV numbers, and we characterize all strong PV numbers of degree ≤ 3.

13.1 The topology of sPV

Recall from Part I that $\mathcal{C}$ is the set of all λ such that R_λ is convex. Theorem 12.14 says that no strong PV number is in $\mathcal{C}$. Thus Theorems 7.12 and 12.14 immediately imply the following topological fact about the set of sPV numbers, which contrasts with the set of PV numbers, which is known to have infinitely many accumulation points [33, 34].

Corollary 13.1. *The strong PV numbers form a discrete subset of* $\mathbb{C}$.

13.2 Abundance of strong PV numbers

In this section, we show how an sPV number gives rise to infinitely many more sPV numbers.

Proposition 13.2. *Let λ be a strong PV number, and let $k \geq 0$ be the number of conjugates of λ that are in $(0,1)$. Every element of $\Sigma_P^{(\lambda)}$ is a strong PV number, where $P := [0,1]^{\times k}$.*

Proof. Let z be any element of $\Sigma_P := \Sigma_P^{(\lambda)}$. Then $z \in \mathbb{Z}[\lambda]$. By standard results in algebra (see Corollary A.5 in the Appendix), z is an algebraic integer, and for every conjugate c of z there exists a conjugate μ of λ such that c is the image $h_\mu(z)$ of z under the ring isomorphism $h_\mu : \mathbb{Z}[\lambda] \to \mathbb{Z}[\mu]$ that maps λ to μ. If $\mu = \lambda$ or $\mu = \lambda^*$, then $c = z$ or $c = z^*$ accordingly. Equivalently, if $c \notin \{z, z^*\}$, then $c = h_\mu(z)$ for some $\mu \notin \{\lambda, \lambda^*\}$ conjugate to λ. Since λ is sPV, this μ is in $(0,1)$. This implies that c is one of the coordinates of $\mathbf{z}'$ (see Definition 12.26). Since $z \in \Sigma_P$, we have $\mathbf{z}' \in P$, whence $c \in [0,1]$. It follows that z is sPV. □

Corollary 13.3. *If λ is a strong PV number, then so are all elements of R_λ.*

Proof. By Theorem 12.27 with $S := \{0,1\}$, Proposition 13.2, and the fact that $R_\lambda = Q_\lambda$. □

One of our main conjectures is the converse of Theorem 12.14, that is, if R_λ is discrete, then λ is sPV. All examples of discrete R_λ that we currently know of are where λ is sPV. We have a natural way of generating new elements of $\mathbb{C} \setminus \mathcal{C}$ from old ones: by Corollary 2.26, if R_λ is discrete, then R_μ is discrete for all $\mu \in R_\lambda$. Corollary 13.3 says that this process cannot be used to disprove the conjecture.

Suppose λ is nontrivial sPV and Σ_P is as in Proposition 13.2. Although every element of Σ_P is sPV by that proposition, it is not the case that every sPV element of $\mathbb{Z}[\lambda]$ is in Σ_P. For example, $2 \in \mathbb{Z}[\lambda]$ and is sPV, but $\mathbf{2}' = (2, 2, \ldots, 2) \notin P$, and so $2 \notin \Sigma_P$. We do have the following, however:

Proposition 13.4. *Suppose λ is a strong PV number of prime degree, with k many conjugates in $(0,1)$. Then every sPV number in $\mathbb{Z}[\lambda] \setminus \mathbb{Z}$ is in $\Sigma_P^{(\lambda)}$, where $P := [0,1]^{\times k}$.*

Proof. Let d be the degree of λ, let z be any strong PV number in $\mathbb{Z}[\lambda] \setminus \mathbb{Z}$, and let n be the degree of z. We have $n > 1$ and $n | d$, hence $n = d$. If $\lambda \in \mathbb{R}$, then so is z, and thus z has $k = d - 1$ many conjugates in $(0,1)$. In any case, z has at least k many conjugates in $(0,1)$.[11] Since all of these conjugates are coordinates of $\mathbf{z}'$, it follows that all coordinates of $\mathbf{z}'$ are in $(0,1)$, whence $\mathbf{z}' \in P$, which puts z into $\Sigma_P^{(\lambda)}$. □

[11] In fact, it has exactly k many such conjugates. This can be seen as follows: The only way z can have more conjugates in $(0,1)$ than λ is if $z \in \mathbb{R}$ but $\lambda \notin \mathbb{R}$. But then $\mathbb{Q}(z) \subseteq \mathbb{R}$ and $\mathbb{Q}(\lambda) \not\subseteq \mathbb{R}$. Since $\mathbb{Q}(z)$ is properly included in $\mathbb{Q}(\lambda)$, we would thus have $n = [\mathbb{Q}(z) : \mathbb{Q}] < [\mathbb{Q}(\lambda) : \mathbb{Q}] = d$. Contradiction.

13.3 Characterizing nontrivial strong PV numbers of low degree

The next corollary characterizes the strong PV numbers $\lambda \in \mathbb{R}$ of degree 2 and gives bounds on the corresponding R_λ sets. It uses case 1 of Theorem 12.14 with $d = 2$. It only applies to negative λ; for positive numbers, we can use the fact that $R_{1-\lambda} = R_\lambda$ and the fact (Fact 12.2) that λ is sPV if and only if $1 - \lambda$ is.

Corollary 13.5. *Let λ be any negative (real) number. Then λ is sPV of degree 2 if and only if $\lambda = (-m - \sqrt{m^2 + 4n})/2$ for some integers m and n with $0 < n \leq m$. If this is the case, then*

$$R_\lambda \subseteq \Sigma_P^{(\lambda)} = \{a - b\lambda \mid a, b \in \mathbb{Z} \;\&\; b\mu \leq a \leq b\mu + 1\} = \{1\} \cup \{\lceil b\mu \rceil - b\lambda \mid b \in \mathbb{Z}\} \; , \tag{39}$$

where $\mu := (-m + \sqrt{m^2 + 4n})/2$ is the conjugate of λ and $\Sigma_P^{(\lambda)}$ is as in Definition 12.26 with $P := [0, 1]$. Furthermore, $\lambda < -1$, and except for 0 and 1, any two adjacent elements of R_λ differ by either $-\lambda$ or at least $1 - \lambda$.

Proof. The quadratic polynomial $p(x) := x^2 + mx - n \in \mathbb{Z}[x]$ has roots $(-m \pm \sqrt{m^2 + 4n})/2$. If λ and μ are the roots of $p(x)$ as above, then the inequality $0 < n \leq m$ is equivalent to $0 < \mu < 1$, and it guarantees that $p(x)$ is irreducible (and so λ and μ are conjugates). We can thus apply case 1 of Theorem 12.14 with $(\mu_0, \mu_1) := (\mu, \lambda)$, which says that R_λ is uniformly discrete (and thus $R_\lambda = Q_\lambda$). Applying Equation (38) in Fact 12.29 with $d := 2$, $k := 1$, and $(\mu_0, \mu_1) := (\mu, \lambda)$, we have (using (a, b) instead of (a_0, a_1) as the index)

$$Q_\lambda \subseteq \{a + b\lambda \mid \; a, b \in \mathbb{Z} \;\&\; 0 \leq a + b\mu \leq 1\} \; .$$

By switching b with $-b$, this set inclusion is seen to be equivalent to the first inclusion of (39). The subsequent set equality in (39) follows from the fact that μ is irrational.

It is clear that $\lambda < -1$. The quantity $\lceil b\mu \rceil - b\lambda$ increases strictly monotonically in b, so adjacent points of $R_\lambda \setminus \{1\}$ correspond (at least) to adjacent integer values of b. When b increases by 1, $\lceil b\mu \rceil$ increases by 0 or 1, giving a difference of either $-\lambda$ or $1 - \lambda$. Finally, the closest point to 1 in R_λ, other than 0, is $\lceil \mu \rceil - \lambda = 1 - \lambda$, which is $-\lambda$ away from 1. $\square$

The case where $m = n = 1$ was already shown in Proposition 6.1. In that case, $\lambda = -\varphi$ and $R_\lambda = R_{-\varphi} = R_{1+\varphi}$. Here are a few other cases:

$m = 2$ and $n = 1$: $\lambda = -1 - \sqrt{2}$ and $R_\lambda = R_{1-\lambda} = R_{2+\sqrt{2}}$.

$m = 2$ and $n = 2$: $\lambda = -1 - \sqrt{3}$ and $R_\lambda = R_{1-\lambda} = R_{2+\sqrt{3}}$.

$m = 3$ and $n = 1$: $\lambda = -(3 + \sqrt{13})/2$ and $R_\lambda = R_{1-\lambda} = R_{(5+\sqrt{13})/2}$.

The next corollary characterizes the non-real strong PV numbers of degree 3. It applies case 2 of Theorem 12.14 with $d = 3$.

Corollary 13.6. *A non-real number λ is sPV of degree 3 if and only if λ is a root of a polynomial $p(x) := x^3 + ax^2 + bx + c$ for some $a, b, c \in \mathbb{Z}$ such that*

1. *$c < 0$,*

2. *$a + b + c \geq 0$, and*

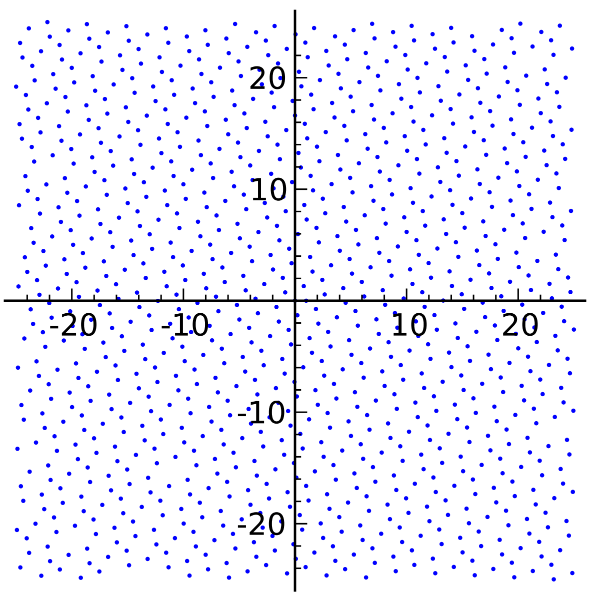


Figure 9: Two plots illustrating Corollary 13.6. The left plot is of R_λ, where λ is the root of the polynomial $x^3 + x^2 - 1$ closest to $-0.877 + 0.745i$. The right plot is of R_μ, where μ is the root of the polynomial $x^3 + x - 1$ closest to $-0.341 + 1.162i$. One can easily show that $R_\lambda = R_{\lambda^2} = R_{\lambda^3}$.

3. the discriminant $\Delta < 0$, where $\Delta := a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$.

If this is the case, then let μ be the (unique) root of p in $(0, 1)$, and let λ be one of the non-real roots of p. Then

$$\begin{aligned} R_\lambda &\subseteq \left\{a_0 + a_1\lambda + a_2\lambda^2 \mid a_0, a_1, a_2 \in \mathbb{Z} \;\;\&\;\; a_0 + a_1\mu + a_2\mu^2 \in [0,1]\right\} \\ &= \left\{a_0 + a_1\lambda + a_2\lambda^2 \mid a_0, a_1, a_2 \in \mathbb{Z} \;\;\&\;\; -a_1\mu - a_2\mu^2 \le a_0 \le -a_1\mu - a_2\mu^2 + 1\right\} \\ &= \{1\} \cup \left\{m\lambda + n\lambda^2 - \lfloor m\mu + n\mu^2 \rfloor \mid m, n \in \mathbb{Z}\right\} \ . \end{aligned}$$

Proof sketch. The first two conditions give $p(0) = c < 0$ and $p(1) = 1 + a + b + c > 0$, respectively, and together they force p to have a root in $(0, 1)$. The negative discriminant implies that the other two roots are non-real. (The only other way to force a single real root in $(0, 1)$ is for $p(0) > 0$ and $p(1) < 0$, but this makes p have three real roots.) Thus the conditions of case 2 of Theorem 12.14 are satisfied if and only if the conditions of the corollary are satisfied. If this is the case, we apply Fact 12.29 (Equation (38)) with $d := 3$, with $k := 1$, and with $\mu_0 := \mu$, with $k := 1$ to get the first inclusion of the corollary. The next equality is immediate. The final equality follows from the fact that p must be irreducible, and thus the set $\{1, \mu, \mu^2\}$ is linearly independent over $\mathbb{Q}$, whence $m\mu + n\mu^2 \notin \mathbb{Z}$ unless $m = n = 0$. $\square$

Figure 9 shows two applications of Corollary 13.6. Let $r := \lambda - \mu$, let $s := \lambda^2 - \mu^2 = r(\lambda + \mu)$, and consider the lattice $r\mathbb{Z} + s\mathbb{Z}$. This lattice is discrete, since r and s are $\mathbb{R}$-linearly independent, and it is interesting to notice that each point in R_λ differs from a point in this lattice by some real displacement between 0 and 1. In Propositions 15.16 and 15.17 we show that equality holds in

Corollary 13.6 for the values of λ and μ indicated in Figure 9. It would be nice to know in general whether or when equality holds.

The next corollary covers the case of strong PV numbers of degree 3 which are real. It considers only negative λ; the positive numbers are then all of the form $1-\lambda$.

Corollary 13.7. *A negative real number λ is sPV of degree 3 if and only if λ is the least root of a polynomial $p(x) := x^3 + ax^2 + bx + c$ for some $a, b, c \in \mathbb{Z}$ such that*

1. $c > 0$,
2. $a + b + c \geq 0$,
3. $-2a - 3 < b < 0$, *and*
4. *the discriminant $\Delta > 0$, where $\Delta := a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$.*

If this is the case, then $a > 0$ as well, and letting $\mu < \nu$ be the two roots of p in $(0, 1)$ and letting $v_{\max}(a_1, a_2)$ and $v_{\min}(a_1, a_2)$ be $\max\{-a_1\mu-a_2\mu^2, -a_1\nu-a_2\nu^2\}$ and $\min\{-a_1\mu-a_2\mu^2, -a_1\nu-a_2\nu^2\}$, respectively, we have

$$\begin{aligned} R_\lambda &\subseteq \left\{a_0 + a_1\lambda + a_2\lambda^2 \mid a_0, a_1, a_2 \in \mathbb{Z} \ \& \ a_0 + a_1\mu + a_2\mu^2 \in [0, 1] \ \& \ a_0 + a_1\nu + a_2\nu^2 \in [0, 1]\right\} \\ &= \left\{a_0 + a_1\lambda + a_2\lambda^2 \mid a_0, a_1, a_2 \in \mathbb{Z} \ \& \ v_{\max}(a_1, a_2) \leq a_0 \leq v_{\min}(a_1, a_2) + 1\right\} \\ &= \{1\} \cup \left\{m\lambda + n\lambda^2 + \lceil v_{\max}(m, n)\rceil \mid m, n \in \mathbb{Z} \ \& \ \lceil v_{\max}(m, n)\rceil = \lceil v_{\min}(m, n)\rceil\right\} \\ &= \{1\} \cup \left\{m\lambda + n\lambda^2 - \lfloor m\mu + n\mu^2\rfloor \mid m, n \in \mathbb{Z} \ \& \ \lfloor m\mu + n\mu^2\rfloor = \lfloor m\nu + n\nu^2\rfloor\right\} \ . \end{aligned}$$

Proof sketch. The discriminant Δ is positive if and only if p has three distinct real roots: $\lambda < 0$ and its conjugates μ and ν such that $\mu < \nu$. As in Corollary 13.6, conditions 1 and 2 are equivalent to $p(0), p(1) > 0$. The derivative of $p(x)$ is $p'(x) = 3x^2+2ax+b$ and has roots $x_\pm = (-a\pm\sqrt{a^2 - 3b})/3$. Condition 4 implies $x_+, x_- \in \mathbb{R}$ and $p(x_+) < 0$.

($\Longrightarrow$): Suppose λ is sPV of degree 3 with minimal polynomial p as above (and thus $\lambda < 0 < \mu < \nu < 1$). Then $\Delta > 0$ (Condition 4), and λ is the least root of p. Additionally, $0 < \mu < x_+ < \nu < 1$ and $p(0), p(1) > 0$, the latter implying Conditions 1 and 2. From $0 < x_+ < 1$ we get $a < \sqrt{a^2 - 3b} < a + 3$ (and in particular, $a > -3$ and $b \leq a^2/3$). Squaring both sides of the inequality $\sqrt{a^2 - 3b} < a + 3$ yields $-2a - 3 < b$. If $a \geq 0$, then squaring both sides of the inequality $a < \sqrt{a^2 - 3b}$ gives $b < 0$, which implies Condition 3. We have left to consider when $a < 0$, and here there are only two possible cases: when $a = -1$ (whence $b = 0$, because $-2a - 3 < b \leq a^2/3$) or when $a = -2$. We cannot have $a = -2$, because this violates $-2a - 3 < b \leq a^2/3$ for any integer b. If $a = -1$ and $b = 0$, however, then $\Delta = 4c - 27c^2 < 0$ for all integers $c > 0$ (cf. Condition 1), so this case cannot happen either. This establishes the forward direction. Finally, we also have $a > 0$, for if $a = 0$, then $b \in \{-2, -1\}$ by Condition 3, and in either case, $\Delta = -4b^3 - 27c^2 < 0$ for any integer $c \geq -b$ (cf. Condition 2).

($\Longleftarrow$): Suppose λ is the least root of p as above satisfying Conditions 1–4, with the other real roots $\mu < \nu$ (by Condition 4). We have $p(0), p(1) > 0$ by Conditions 1 and 2, and $0 < x_+ < 1$ by Condition 3. By Condition 4, $p(x_+) < 0$, and thus there are two roots of p strictly between 0 and 1. These are μ and ν, because $\lambda < 0$. Finally, p is irreducible (over $\mathbb{Q}$), for otherwise, p has an integral root (which must be λ), which makes μ and ν conjugate roots of some monic quadratic polynomial in $\mathbb{Z}[x]$, but this is impossible. Thus λ is sPV of degree 3.

The rest of the corollary follows from Equation (38) of Fact 12.29 where $d = 3$ and $k = 2$. $\square$

Notice that the condition $\lfloor m\mu + n\mu^2 \rfloor = \lfloor m\nu + n\nu^2 \rfloor$ in the corollary implies $|m\nu + n\nu^2 - m\mu - n\mu^2| < 1$, which implies $|m + n(\nu + \mu)| < (\nu - \mu)^{-1}$. Thus given $m \in \mathbb{Z}$, there are only finitely many $n \in \mathbb{Z}$ such that the condition is satisfied, and vice versa.

14 λ-convex closures of some regular shapes

In this section, we consider the λ-convex closures of some point sets, specifically, regular polygons and the regular polyhedra (Platonic solids). We apply Theorem 12.14 to find combinations of finite sets S and values λ that make $Q_\lambda(S)$ uniformly discrete. In all cases in this section, λ will be a real number. A challenge for future research is to find familiar point sets in $\mathbb{C}$ whose λ-convex closures are discrete for nonreal λ.

Recall that if R_λ is convex, then so is $R_\lambda(S)$ for any S, by Proposition 2.32. Thus to find discrete, nontrivially generated λ-convex sets, we must have $R_\lambda = Q_\lambda$ discrete. We thus confine ourselves to considering sPV λ.

14.1 Regular polygons

Here we will prove facts about discrete sets of the form $Q_\lambda(P_n)$, where P_n is the set of vertices of a regular n-gon for $n \geq 3$, and λ is chosen appropriately, depending on n. We will also show plots of some of these sets.

We first review the facts about cyclotomic field extensions of $\mathbb{Q}$ that we will need. Fix an integer $n > 1$, and let

$$\zeta := e^{i\tau/n} \tag{40}$$

be the principal n^{th} root of unity. Then ζ is an algebraic integer of degree $\phi(n)$ (where ϕ is Euler's totient function) whose minimum polynomial is the n^{th} cyclotomic polynomial Φ_n and whose conjugates are the primitive n^{th} roots of 1. The field extension $\mathbb{Q}(\zeta)$ of $\mathbb{Q}$ is a Galois extension of degree $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \phi(n)$ whose Galois group—isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$—consists of those automorphisms η_a that map ζ to ζ^a (and thus map ζ^j to ζ^{ja} for any $j \in \mathbb{Z}$) for all $0 < a < n$ such that $\gcd(a, n) = 1$. Thus the η_a transitively permute the primitive n^{th} roots of 1.

We will use the following lemma repeatedly to lift discreteness results on the real line up to the complex plane.

Lemma 14.1. *Let $\lambda \in \mathbb{R}$ and $S \subseteq \mathbb{C}$. Suppose that: (1) $Q_\lambda(\mathrm{Re}(S))$ is uniformly discrete; and (2) there exist $a, b \in \mathbb{C}$ such that $b - a \notin \mathbb{R}$ and $S = \rho_{a,b}(S)$. Then $Q_\lambda(S)$ is uniformly discrete.*

Proof. Suppose otherwise. Let $d := \max(|b-a|, 1)$. For any $\varepsilon > 0$, we show that there exist distinct $u, v \in Q_\lambda(\mathrm{Re}(S))$ such that $|u - v| < \varepsilon$, contradicting the assumption that $Q_\lambda(\mathrm{Re}(S))$ is uniformly discrete. By assumption, there exist distinct $x, y \in Q_\lambda(S)$ such that $|x - y| < \varepsilon/d$. We have

$$\{\mathrm{Re}(x), \mathrm{Re}(y)\} \subseteq \mathrm{Re}(Q_\lambda(S)) = Q_\lambda(\mathrm{Re}(S)) ,$$

the last equality by Lemma 12.20.[12] If $\mathrm{Re}(x) \neq \mathrm{Re}(y)$, then letting $u := \mathrm{Re}(x)$ and $v := \mathrm{Re}(y)$ gives the contradiction: $0 < |u - v| \leq |x - y| < \varepsilon/d \leq \varepsilon$. If $\mathrm{Re}(x) = \mathrm{Re}(y)$, then $x - y = ir$ for

[12] We use Lemma 12.20 with $U = \mathbb{C}$, $V = k = \mathbb{R}$, $t = \mathrm{Re}$, and the $\mathbb{R}$-linear maps A and B being scalar multiplication by λ on $\mathbb{C}$ and $\mathbb{R}$, respectively.

some nonzero $r \in \mathbb{R}$ with $|r| < \varepsilon/d$. Letting $u := \mathrm{Re}(\rho_{a,b}(x))$ and $v := \mathrm{Re}(\rho_{a,b}(y))$, we have

$$u - v = \mathrm{Re}(\rho_{a,b}(x) - \rho_{a,b}(y)) = \mathrm{Re}((x-y)(b-a)) = \mathrm{Re}(ir(b-a)) = -r\mathrm{Im}(b-a) \neq 0 \ ,$$

and so $0 < |u-v| = |r||\mathrm{Im}(b-a)| \le |r||b-a| < (\varepsilon/d)|b-a| \le \varepsilon$, and

$$\{u, v\} \subseteq \mathrm{Re}(\rho_{a,b}(Q_\lambda(S))) = \mathrm{Re}(Q_\lambda(\rho_{a,b}(S))) = \mathrm{Re}(Q_\lambda(S)) = Q_\lambda(\mathrm{Re}(S)) \ .$$

Contradiction. The first equality is by Lemma 2.12; the last is by Lemma 12.20. □

The primary question we ask in this section is: For which n and which λ is $Q_\lambda(P_n)$ discrete, and if so, what does this set look like? We also wish to show that, if $Q_\lambda(P_n)$ is discrete, then it is a Meyer set. For this it suffices to show uniform discreteness of $Q_\lambda(P_n) - Q_\lambda(P_n)$ and relative density of $Q_\lambda(P_n)$ in $\mathbb{C}$ (see Fact 12.11). We show the former in this section and the latter in Section 15.[13]

We will use the next lemma in conjunction with Lemma 14.1:

Lemma 14.2. *For any $a, b \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, if $S = \rho_{a,b}(S)$, then $S - S = \rho_{0,b-a}(S - S)$.*

Proof. For all $x, y \in \mathbb{C}$, we have $\rho_{0,b-a}(x-y) = (x-y)(b-a) = \rho_{a,b}(x) - \rho_{a,b}(y)$. Therefore,

$$\rho_{0,b-a}(S-S) = \{\rho_{0,b-a}(x-y) \mid x, y \in S\} = \{\rho_{a,b}(x) - \rho_{a,b}(y) \mid x, y \in S\} = \rho_{a,b}(S) - \rho_{a,b}(S) = S - S \ .$$

□

It is most natural to define $P_n \subseteq \mathbb{C}$ formally as the set of all n^{th} roots of 1, which orients the polygon on the unit circle. Purely for reasons of technical convenience, however (including some relating to the computer code generating the plots shown below), we instead orient P_n so that one of its sides coincides with the unit interval and the rest of the polygon lies in the upper halfplane:

$$P_n := \left\{ \sum_{j=0}^{\ell-1} \zeta^j : 0 \le \ell < n \right\} = \left\{ \frac{\zeta^\ell - 1}{\zeta - 1} : 0 \le \ell < n \right\} \ , \tag{41}$$

for ζ as in (40). This definition also has the advantage that $\{0, 1\} \subseteq P_n$.

We first prove a straightforward result for the equilateral triangle, square, and regular hexagon.

Proposition 14.3. *If $\lambda \in \mathbb{R}$ is sPV, then $Q_\lambda(S) - Q_\lambda(S)$ is uniformly discrete for all $S \in \{P_3, P_4, P_6\}$.*

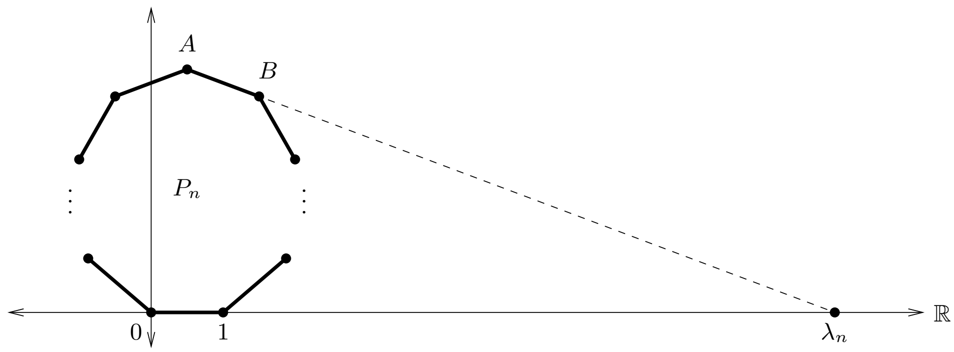
Proof. For $S \in \{P_3, P_4, P_6\}$, we note that $\rho_{a,b}(S) = S$, where $a = 1$, and either $b = (1 + i\sqrt{3})/2$ for $S = P_3$, $b = 1 + i$ for $S = P_4$, or $b = (3 + i\sqrt{3})/2$ for $S = P_6$. It follows by Lemma 14.2 that $\rho_{0,c}(S - S) = S - S$, where $c = (-1 + i\sqrt{3})/2$ or $c = i$ or $c = (1 + i\sqrt{3})/2$, respectively. In all cases, $\mathrm{Re}(S - S) = \mathrm{Re}(S) - \mathrm{Re}(S) \subseteq \left\{-2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\lambda)$, and so by Theorem 12.14, $Q_\lambda(\mathrm{Re}(S - S))$ is uniformly discrete. It follows that $Q_\lambda(S - S)$ is uniformly discrete by Lemma 14.1, and in addition, $Q_\lambda(S - S) = Q_\lambda(S) - Q_\lambda(S)$ by Lemma 2.11. □

We obtain most of the other values of λ that we consider here, corresponding to the various P_n, by perhaps a somewhat arbitrary type of geometric construction: choose two nonparallel lines L_1 and L_2 passing through pairs of points of P_n, then let λ be a certain ratio of distances on L_1 between the two points on the polygon and the point of intersection of L_1 and L_2. For the moment we will choose pairs of adjacent points on the polygon, and so we can assume without loss of generality that L_1 is the real axis.

[13] We cannot apply Proposition 12.32 directly here, because P_n is not a subset of $\mathbb{Z}[\lambda]$.

14.2 Constructions with odd n

Let $n \geq 3$ be odd, and consider the following construction using P_n:



The line through the base of P_n is the real axis, and the other line passes through the apex A of P_n and the point B adjacent to it to the right. The two lines intersect at the point

$$\lambda_n := \frac{1}{2(1-\cos(\pi/n))} = \frac{1}{2+\zeta^k+\zeta^{-k}}\ , \tag{42}$$

where $k := (n-1)/2$ and ζ is as in Equation (40). Note that $\gcd(n,k) = 1$. (One way to see that this formula for λ_n is correct is to drop a vertical line segment from A down to the base at the point $1/2$, forming a right triangle T with the point λ_n. The hypotenuse of T has length λ_n by symmetry, and the base of T has length $\lambda_n - 1/2$, and finally, the acute angle at λ_n is π/n. To see the second equation, note that $(\zeta^k + \zeta^{-k})/2 = \cos(2k\pi/n) = \cos(\pi - \pi/n) = -\cos(\pi/n)$.) Of course, the first equation of (42) makes sense for all $n \in \mathbb{Z}^+$, not necessarily odd, and we will on occasion refer to λ_n for even n, but for the time being, we assume that n is odd.

Now consider the set $Q_{\lambda_n}(P_n)$. Since $0, 1 \in P_n$, this set includes Q_{λ_n} as a subset, and so it can be discrete only if Q_{λ_n} is discrete. The next proposition comes close to a converse.

Proposition 14.4. *For any odd $n \geq 3$, let λ_n be given by Equation (42). Suppose λ_n is sPV. Then $Q_{\lambda_n}(P_n) - Q_{\lambda_n}(P_n)$ is uniformly discrete; in fact, $Q_{\lambda_n}(P_{2n}) - Q_{\lambda_n}(P_{2n})$ is uniformly discrete.*

Remark. Note that if $Q_{\lambda_n}(P_{2n}) - Q_{\lambda_n}(P_{2n})$ is uniformly discrete, then so is $Q_{\lambda_n}(P_n) - Q_{\lambda_n}(P_n)$: since P_n can be embedded into P_{2n} by a $\mathbb{C}$-affine transformation, the same holds for $P_n - P_n$ into $P_{2n} - P_{2n}$, and thus for $Q_{\lambda_n}(P_n) - Q_{\lambda_n}(P_n)$ into $Q_{\lambda_n}(P_{2n}) - Q_{\lambda_n}(P_{2n})$ by Lemma 2.11. $\square$

Proof of Proposition 14.4. Assume that $n \geq 3$ is odd and that λ_n is sPV. By the previous remark, it suffices to show that $Q_{\lambda_n}(P_{2n}) - Q_{\lambda_n}(P_{2n})$ is uniformly discrete. By symmetry, we have $\rho_{1,1+\eta}(P_{2n}) = P_{2n}$, where $\eta := e^{i\tau/(2n)}$ is the principal $(2n)$-th root of 1. Then by Lemmas 2.11, 14.1, and 14.2, it suffices to show that $Q_{\lambda_n}(\mathrm{Re}(P_{2n} - P_{2n}))$ is uniformly discrete.

By Equation (41),

$$\mathrm{Re}(P_{2n}) = \left\{ \sum_{j=0}^{\ell-1} \cos(j\pi/n) : 0 \leq \ell < n \right\} .$$

All the points in this finite set are elements of $\mathbb{Q}(\lambda_n)$. This is because $\cos(\pi/n) = 1 - 1/(2\lambda_n)$ is clearly in $\mathbb{Q}(\lambda_n)$, and all the terms in the sum above are of the form $\cos(j\pi/n)$ for integer j, and these can in turn be expressed as (Chebychev) polynomials $T_j(\cos(\pi/n))$ of $\cos(\pi/n)$, where $T_j(x) \in \mathbb{Z}[x]$. It follows that $\mathrm{Re}(P_{2n} - P_{2n}) = \mathrm{Re}(P_{2n}) - \mathrm{Re}(P_{2n}) \subseteq \mathbb{Q}(\lambda_n)$ and is finite. Theorem 12.14 then says that $Q_{\lambda_n}(\mathrm{Re}(P_{2n} - P_{2n}))$ is uniformly discrete. $\square$

The "good" news regarding Proposition 14.4 is that there exist sVP λ_n, which yield interesting sets $Q_{\lambda_n}(P_n)$ and $Q_{\lambda_n}(P_{2n})$. The "bad" news is that there are only finitely many such n.

Proposition 14.5. *Let $n \geq 3$ be odd. Then λ_n is sPV if and only if $n \in \{3, 5, 7, 9, 15\}$.*

We prove Proposition 14.5 via the following three lemmas. Define $\mathbb{Z}_n^* := \{j \in \mathbb{Z} \mid 0 < j < n \ \& \ \gcd(j, n) = 1\}$ as is customary.

Lemma 14.6. *Let $n \geq 3$ be odd. Then λ_n (see Eq. (42)) is an algebraic integer of degree $\phi(n)/2$, and its Galois conjugates are of the form*

$$\frac{1}{2\,(1 + \cos(2j\pi/n))}$$

for all integers $0 < j < n/2$ such that $\gcd(j, n) = 1$.

Proof. From Equation (42), letting $k := (n-1)/2$, we see that $1/\lambda_n = 2 + \zeta^k + \zeta^{-k}$ is an algebraic integer, since $\zeta^{-k} = \zeta^{n-k}$ and the algebraic integers form a ring. Let m be the degree of $1/\lambda_n$, and let $p(x) := c_0 + c_1 x + \cdots + x^m \in \mathbb{Z}[x]$ be its (monic) minimal polynomial. All numbers of the form $\eta_j(1/\lambda_n)$ for $j \in \mathbb{Z}_n^*$ must be conjugates of $1/\lambda_n$, since the η_j are all field automorphisms. It is more convenient to work with one such conjugate, $\mu := \eta_2(1/\lambda_n) = 2 + \zeta^{2k} + \zeta^{-2k} = 2 + \zeta + \zeta^{-1} = 2(1 + \cos(2\pi/n))$, rather than $1/\lambda_n$ itself. Then, for $j \in \mathbb{Z}_n^*$, the numbers

$$\eta_j(\mu) = 2 + \zeta^j + \zeta^{-j} = 2(1 + \cos(2j\pi/n))$$

are all conjugates of μ (and hence of $1/\lambda_n$), and these numbers are pairwise distinct for $0 < j < n/2$, since $\cos(x)$ is decreasing on $[0, \pi]$. There are exactly $\phi(n)/2$ such j, and so μ has at least this many conjugates, and these are also conjugates of $1/\lambda_n$. It follows that $m \geq \phi(n)/2$. On the other hand, since $\mu \in \mathbb{Q}(\zeta) \cap \mathbb{R}$, we have that $\mathbb{Q}(\mu)$ is a proper subfield of $\mathbb{Q}(\zeta)$, and so $[\mathbb{Q}(\zeta) : \mathbb{Q}(\mu)] \geq 2$, and because $[\mathbb{Q}(\zeta) : \mathbb{Q}(\mu)][\mathbb{Q}(\mu) : \mathbb{Q}] = [\mathbb{Q}(\zeta) : \mathbb{Q}] = \phi(n)$, it follows that $[\mathbb{Q}(\mu) : \mathbb{Q}] \leq \phi(n)/2$, which implies $m \leq \phi(n)/2$. Thus $m = \phi(n)/2$ is the degree of μ, which is also the degree of $1/\lambda_n$ and of λ_n. Furthermore, μ and $1/\lambda_n$ share the same set of conjugates $\{\eta_j(\mu) : j \in \mathbb{Z}_n^* \ \& \ j < n/2\} = \{2(1 + \cos(2j\pi/n)) : j \in \mathbb{Z}_n^* \ \& \ j < n/2\}$, and hence the conjugates of λ_n are exactly the reciprocals of these, being the roots of the polynomial $q(x) := c_0 x^m + c_1 x^{m-1} + \cdots + 1$.

It remains to show that the constant term c_0 of $p(x)$ is ± 1, which implies that λ_n is an algebraic integer, since c_0 is also the leading coefficient of $q(x)$. Using some trigonometric identities, we have

$$\frac{1}{\lambda_n} = 2(1 - \cos(\pi/n)) = \frac{2\sin^2(\pi/n)}{1 + \cos(\pi/n)} = \frac{1 - \cos(2\pi/n)}{1 + \cos(\pi/n)} = \frac{2 - \zeta - \zeta^{-1}}{2 - \zeta^k - \zeta^{-k}} = \frac{\nu}{\eta_k(\nu)} \ ,$$

where we have set $\nu := 2 - \zeta - \zeta^{-1}$. Then (recalling that $k := (n-1)/2$),

$$\mu = \eta_2\left(\frac{1}{\lambda_n}\right) = \frac{\eta_2(\nu)}{\eta_{2k}(\nu)} = \frac{\eta_2(\nu)}{\eta_{n-1}(\nu)} = \frac{\eta_2(\nu)}{\eta_{-1}(\nu)} = \frac{\eta_2(\nu)}{\nu} \ .$$

Now c_0 is, up to a change of sign, the product of all the roots of $p(x)$, i.e., the conjugates of $1/\lambda_n$ (or of μ). We then have, noticing that $\eta_j(\mu) = \eta_{n-j}(\mu)$ for all $j \in \mathbb{Z}_n^*$,

$$c_0^2 = \left(\prod_{j \in \mathbb{Z}_n^* \ \&\ j<n/2} \eta_j(\mu) \right)^2 = \prod_{j \in \mathbb{Z}_n^* \ \&\ j<n/2} \eta_j(\mu) \prod_{j \in \mathbb{Z}_n^* \ \&\ j<n/2} \eta_{n-j}(\mu) = \prod_{j \in \mathbb{Z}_n^*} \eta_j(\mu) = \prod_j \frac{\eta_{2j}(\nu)}{\eta_j(\nu)} ,$$

where the index $2j$ is assumed to be reduced modulo n. But the right-hand side is 1, because the numerators and denominators both run through the same values. Thus $c_0 = \pm 1$, and we are done. □

Lemma 14.7. *Let $n \geq 3$ be odd. Then λ_n is sPV if and only if there are no $j \in \mathbb{Z}_n^*$ such that $n/3 \leq j \leq 2n/3$, except for $(n \pm 1)/2$.*

Proof. Set $k := (n-1)/2$. By the previous lemma, λ_n is an algebraic integer with conjugates $(2 + 2\cos(2j\pi/n))^{-1} = (2 + \zeta^j + \zeta^{-j})^{-1}$ for $j \in \mathbb{Z}_n^*$ with $j < n/2$. We can drop the requirement that $j < n/2$, because $\cos(2j\pi/n) = \cos(2(n-j)\pi/n)$ for all j. Since $\lambda_n = (2 + \zeta^k + \zeta^{-k})^{-1} = (2 + \zeta^{k+1} + \zeta^{-(k+1)})^{-1}$, the conjugates of λ_n other than λ_n itself are of the form $(2 + \zeta^j + \zeta^{-j})^{-1} = (2 + 2\cos(2j\pi/n))^{-1}$ for all $j \in \mathbb{Z}_n^* \setminus \{k, k+1\}$. It follows by definition that λ_n is sPV if and only if $0 < (2 + 2\cos(2j\pi/n))^{-1} < 1$, or equivalently, $\cos(2j\pi/n) > -1/2$ for all such j. This latter inequality is equivalent to $j < n/3$ or $j > 2n/3$, for all $j \in \mathbb{Z}_n^*$ other than k or $k+1$. □

Lemma 14.8. *If $n > 21$ and n is odd, then there exists a $j \in \mathbb{Z}_n^*$ such that $n/3 < j < 2n/3$ and $j \notin \{(n-1)/2, (n+1)/2\}$.*

Proof. We give an elementary proof using only the Bertrand-Chebyshev theorem, which states that for all integers $m > 1$, there exists a prime p with $m < p < 2m$. It follows that this must also be true for all *real* $m > 1$, by applying the theorem to $\lfloor m \rfloor$ if $m \geq 2$. Set $k := (n-1)/2$ as usual.

Case 1: $n = qp$ where p is an odd prime and $q \in \{5, 7, 9, 11\}$. Choose two numbers $j_1 < j_2$ depending on p and q according to the following table:

q	j_1	j_2
5	$2p-1$	$2p+1$
7	$3p-1$	$3p+1$
9	$4p-2$	$4p-1$
11	$5p-2$	$5p-1$

Since n is sufficiently large, one can readily check that: (i) $n/3 < j_1 < j_2 < k$; (ii) neither j_1 nor j_2 is a multiple of p; and (iii) at least one of j_1 and j_2 is coprime with q and hence coprime with n, satisfying the lemma.

Case 2: not Case 1 and neither k nor $k+1$ is prime. By Bertrand-Chebyshev, there exists a prime j such that $n/3 < j < 2n/3$. Then $j \nmid n$ and j is neither k nor $k+1$, so j satisfies the lemma.

Case 3: not Case 1, one of k and $k+1$ is prime, and the other is not of the form $2p$ for any prime p. By Bertrand-Chebyshev, there exists a prime r such that $n/6 < r < n/3$. We must have $r \nmid n$, because we are not in Case 1 (and n is odd). Then setting $j := 2r$ satisfies the lemma.

Case 4: not Case 1, one of k and $k+1$ is prime, and the other is of the form $2p$ for some prime p. By Bertrand-Chebyshev, choose a prime s such that $n/12 < s < n/6$. Again, $s \nmid n$ since we are not in Case 1 (and n is odd). Then setting $j := 4s$ satisfies the lemma.

Cases 1–4 are clearly exhaustive. $\square$

Proof of Proposition 14.5. Lemmas 14.7 and 14.8 imply that λ_n cannot be sPV if $n > 21$. For $3 \leq n \leq 21$, one can check case by case that Lemma 14.7 is satisfied if and only if $n \in \{3, 5, 7, 9, 15\}$. $\square$

The following table gives basic information about λ_n for the values of n we are interested in:

n	minimal polynomial of λ_n	approximate value of λ_n
3	$x - 1$	1.000
4	$2x^2 - 4x + 1$	1.707
5	$x^2 - 3x + 1$	2.618
6	$x^2 - 4x + 1$	3.732
7	$x^3 - 6x^2 + 5x - 1$	5.049
9	$x^3 - 9x^2 + 6x - 1$	8.291
15	$x^4 - 24x^3 + 26x^2 - 9x + 1$	22.881

All are sPV except λ_4, which is not an algebraic integer. $2\lambda_4 = 2 + \sqrt{2}$ is sPV, however.

Figures 10–21 on the following pages show plots of the uniformly discrete sets $Q_{\lambda_n}(P_m)$ for $n = 5, 7, 9, 15$ as in Proposition 14.5 and for selected $m \geq 3$ dividing $2n$ (the $n = 3$ case is trivial, as $\lambda_3 = 1$). We order them by increasing n then increasing m for each n. Each displayed set has dihedral D_m symmetry, but none has any translational symmetry. Each displayed set is a Meyer set by Corollary 15.42.

14.3 Constructions with even n

Although the first part of Equation (42) makes sense when n is even, λ_n may or may not be an algebraic integer in this case. For example, $\lambda_4 = 1 + \sqrt{2}/2$ is not an algebraic integer, but $2\lambda_4 = 2 + \sqrt{2}$ and $\lambda_6 = 2 + \sqrt{3}$ both are, and are two of the strong PV numbers given after Corollary 13.5.

Proposition 14.9. *The sets $Q_{2+\sqrt{2}}(P_8) - Q_{2+\sqrt{2}}(P_8)$ and $Q_{2+\sqrt{3}}(P_{12}) - Q_{2+\sqrt{3}}(P_{12})$ are both uniformly discrete.*

Proof. By symmetry, we have $P_8 = \rho_{1,1+c}(P_8)$ for $c := e^{i\tau/8} = \frac{\sqrt{2}}{2}(1+i)$, and $P_{12} = \rho_{1,1+d}(P_{12})$ for $d := e^{i\tau/12} = \frac{1}{2}(\sqrt{3} + i)$. We have $\mathrm{Re}(P_8) = \left\{-\frac{\sqrt{2}}{2}, 0, 1, 1 + \frac{\sqrt{2}}{2}\right\} \subseteq \mathbb{Q}\left(2 + \sqrt{2}\right)$, and thus $\mathrm{Re}(P_8 - P_8) \subseteq \mathbb{Q}\left(2 + \sqrt{2}\right)$ as well. Similarly, $\mathrm{Re}(P_{12}) = \left\{-\frac{1+\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, 1, 1 + \frac{\sqrt{3}}{2}, 1 + \frac{1+\sqrt{3}}{2}\right\} \subseteq \mathbb{Q}\left(2 + \sqrt{3}\right)$, and thus $\mathrm{Re}(P_{12} - P_{12}) \subseteq \mathbb{Q}\left(2 + \sqrt{3}\right)$ also.

As in the proofs of Propositions 14.3 and 14.4, the statement now follows from Theorem 12.14 and Lemmas 2.11, 14.1, and 14.2. $\square$

Figure 22 shows three point sets that do not fit the "odd n" pattern—the 2-convex closure of P_6 (upper left), the $(2+\sqrt{2})$-convex closure of P_8 (upper right), and the $(2+\sqrt{3})$-convex closure of P_{12}. Note that $2 + \sqrt{2} = 2\lambda_4$ and $2 + \sqrt{3} = \lambda_6$. These sets are all Meyer sets by Corollary 15.42.

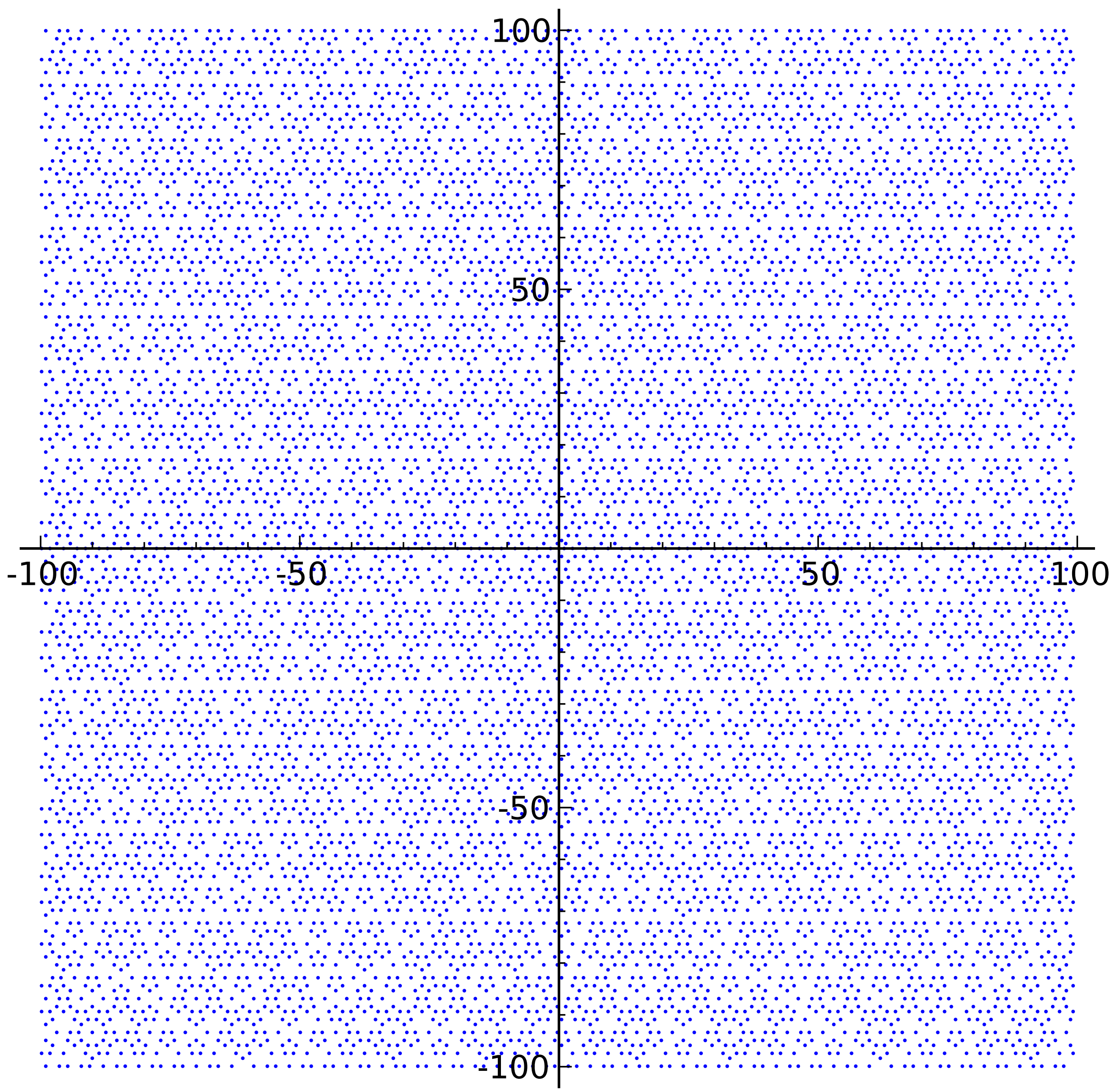


Figure 10: $Q_{\lambda_5}(P_5)$, the λ_5-convex closure of a regular pentagon. $\lambda_5 = 1 + \varphi$.

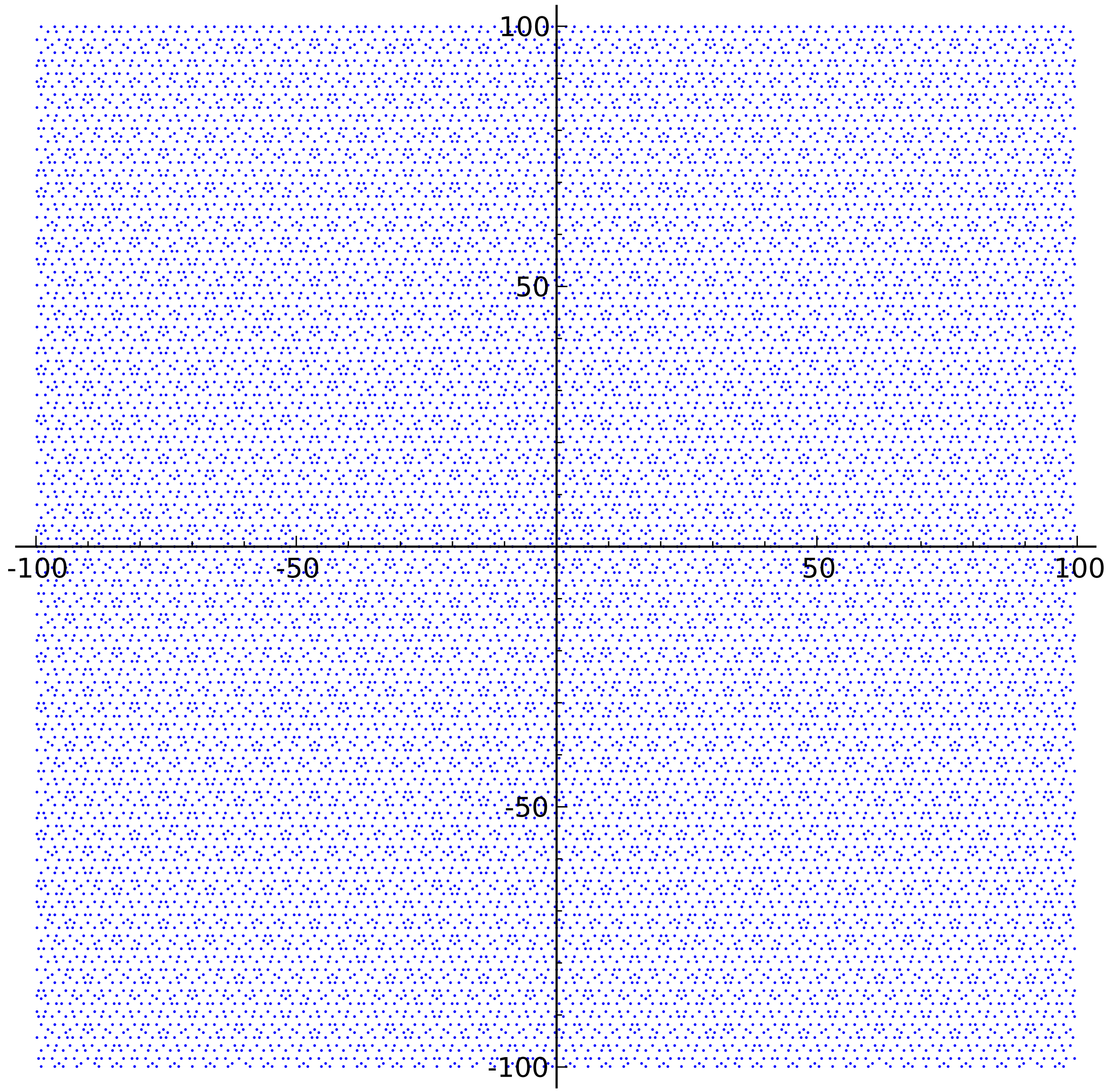


Figure 11: $Q_{\lambda_5}(P_{10})$, the λ_5-convex closure of a regular decagon.

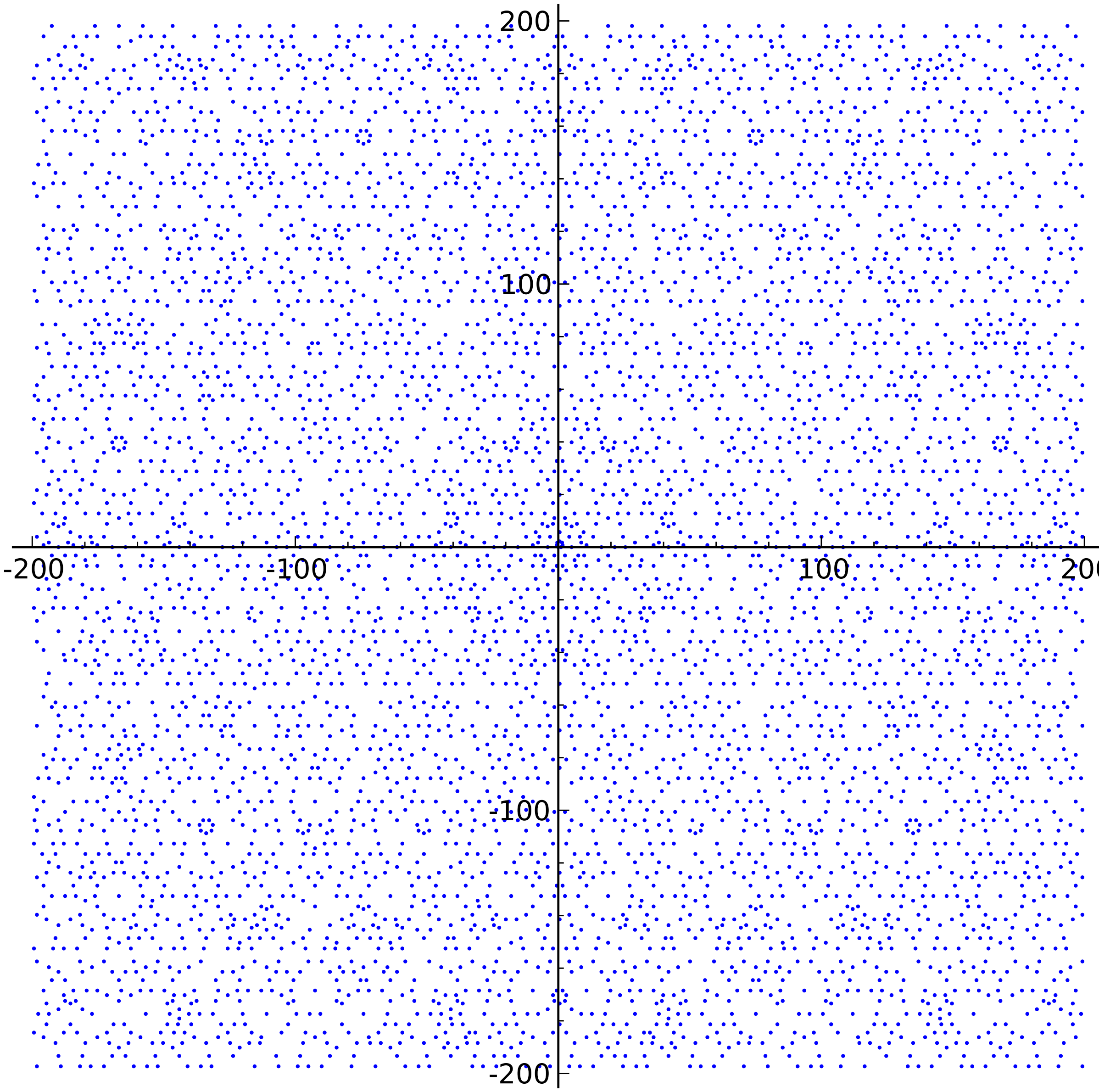


Figure 12: $Q_{\lambda_7}(P_7)$, the λ_7-convex closure of a regular heptagon. $\lambda_7 \approx 5.049$ and has minimal polynomial $x^3 - 6x^2 + 5x - 1$.

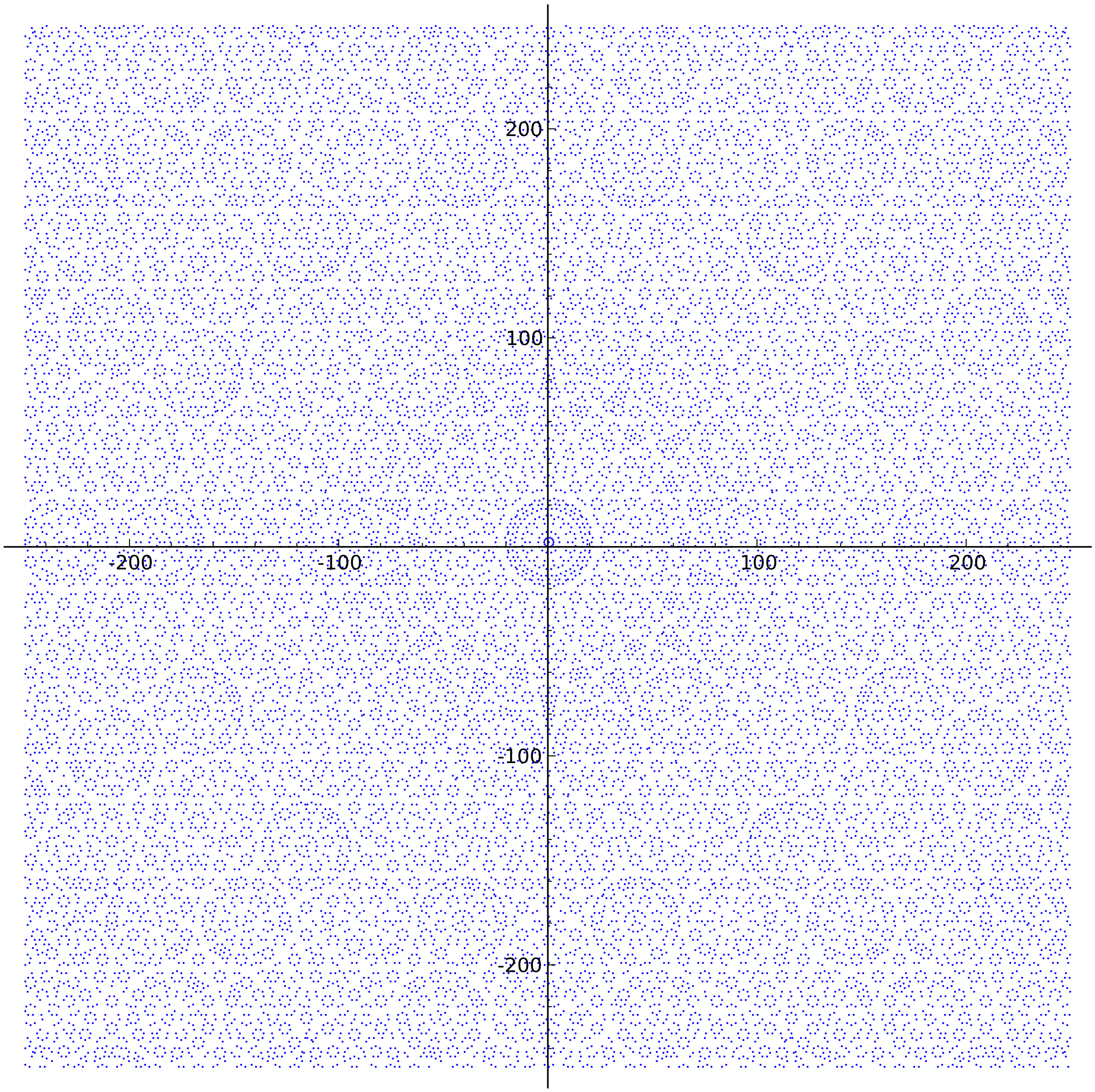


Figure 13: $Q_{\lambda_7}(P_{14})$, the λ_7-convex closure of a regular 14-gon.

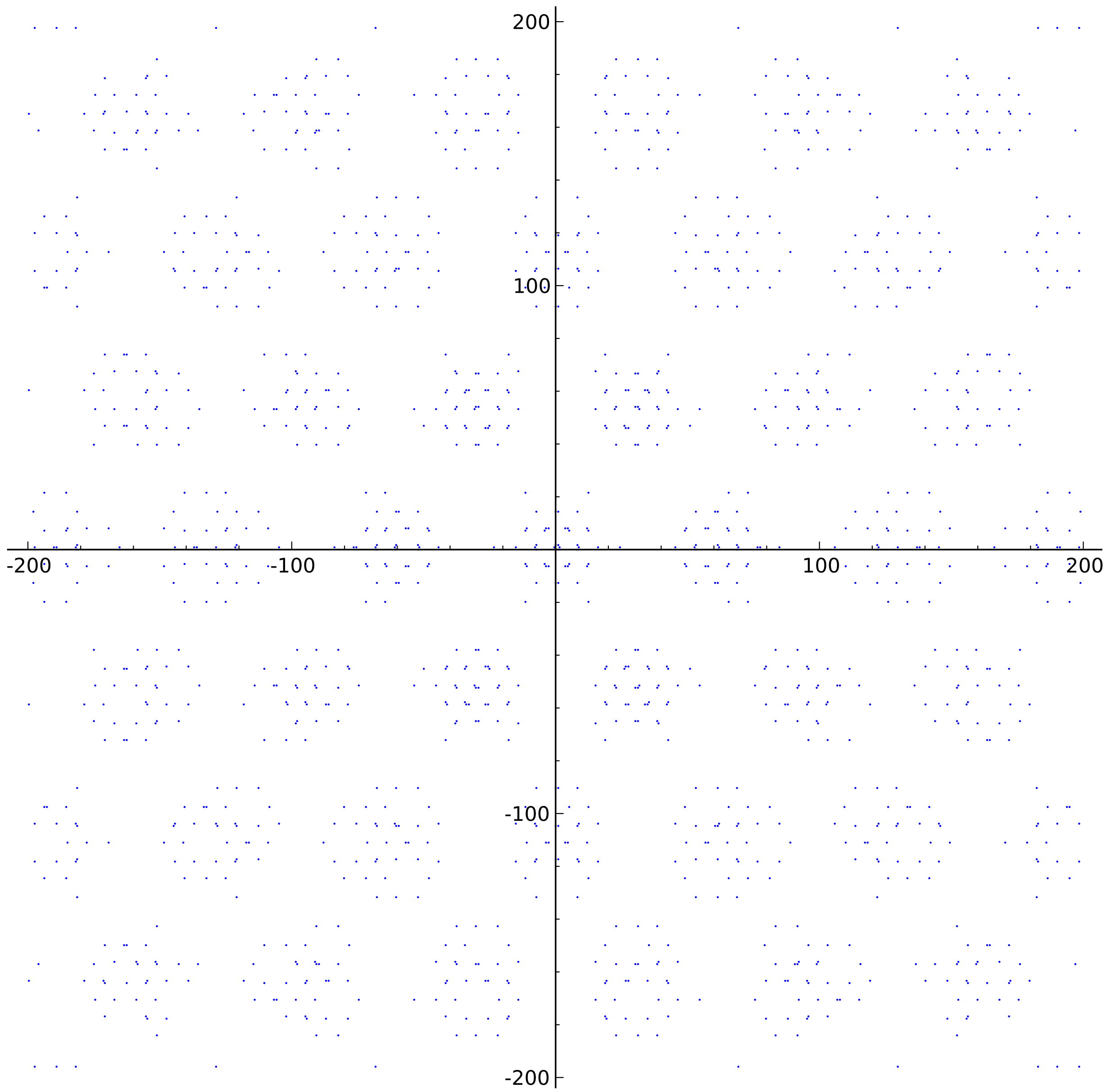


Figure 14: $Q_{\lambda_9}(P_6)$, the λ_9-convex closure of a regular hexagon.

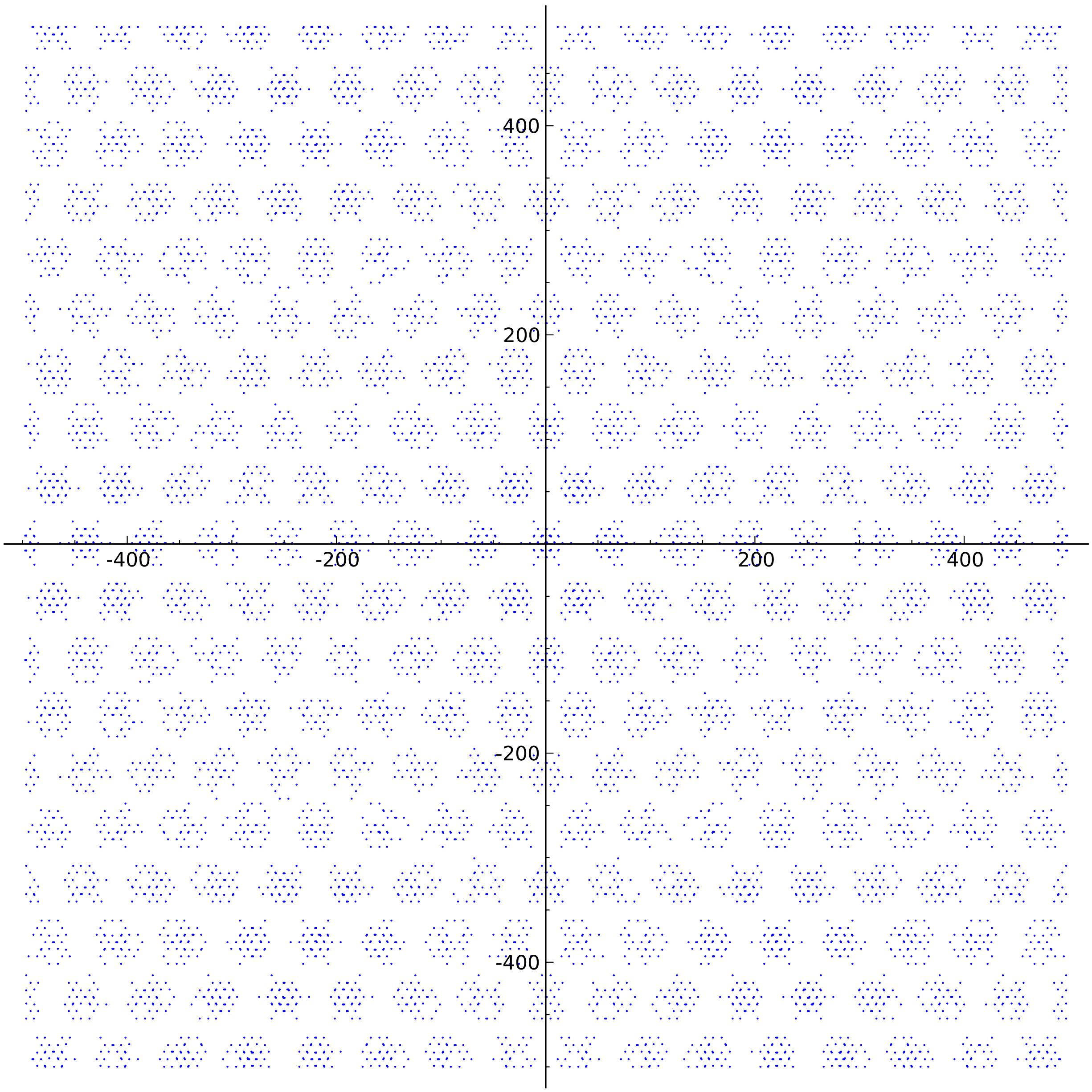


Figure 15: A wider view of $Q_{\lambda_9}(P_6)$.

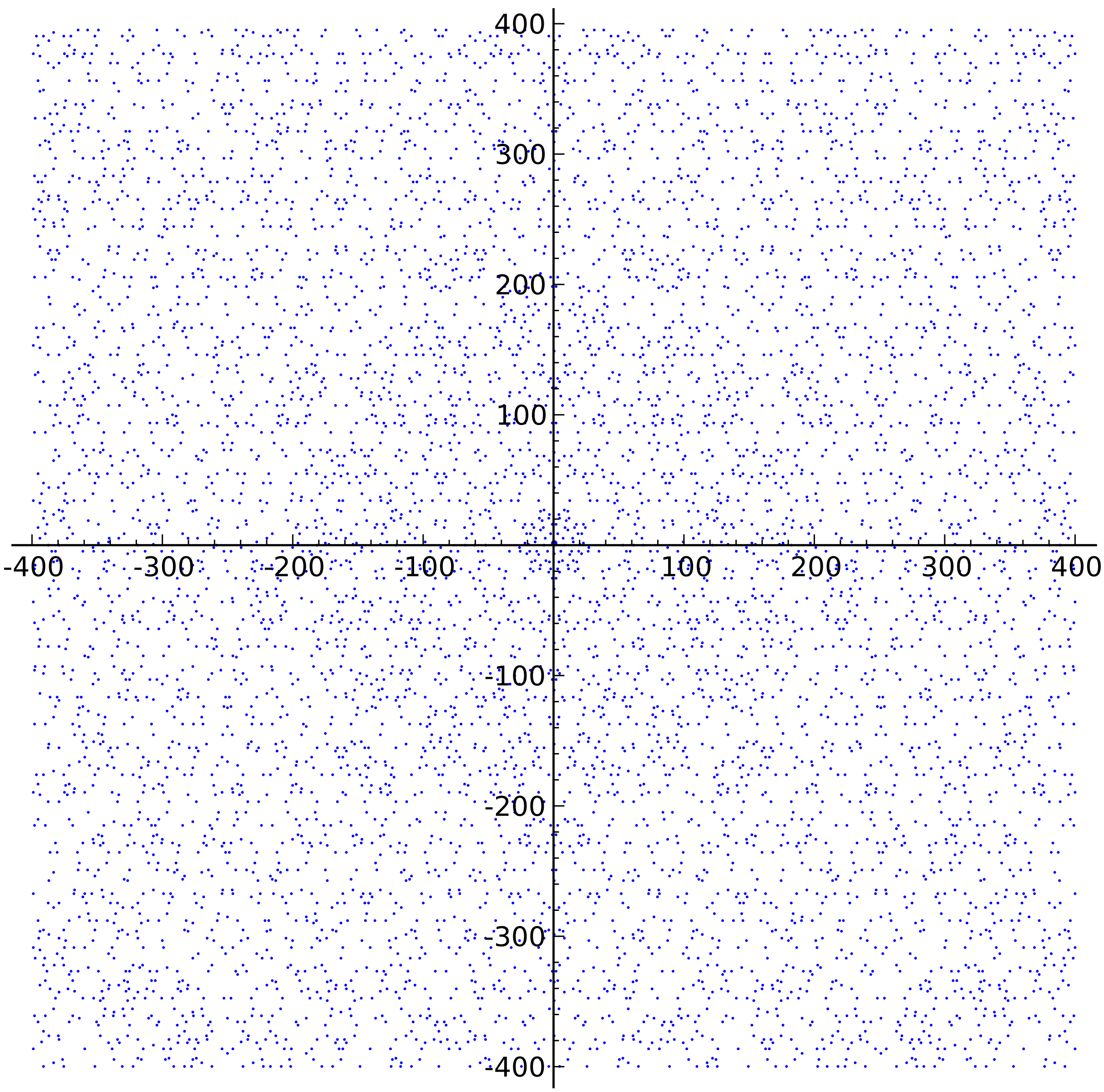


Figure 16: $Q_{\lambda_9}(P_9)$, the λ_9-convex closure of a regular enneagon (nonagon).

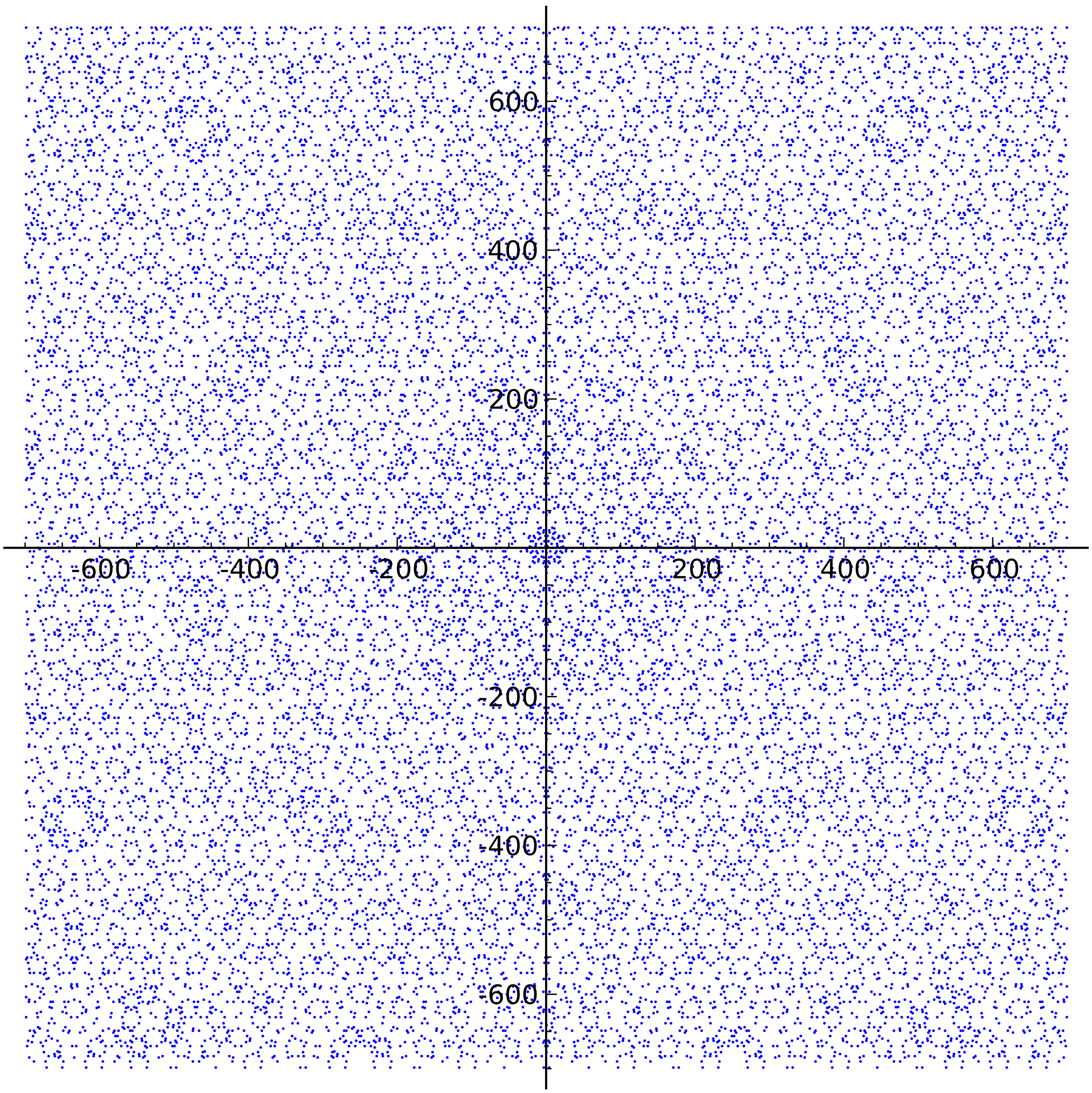


Figure 17: A wider view of $Q_{\lambda_9}(P_9)$.

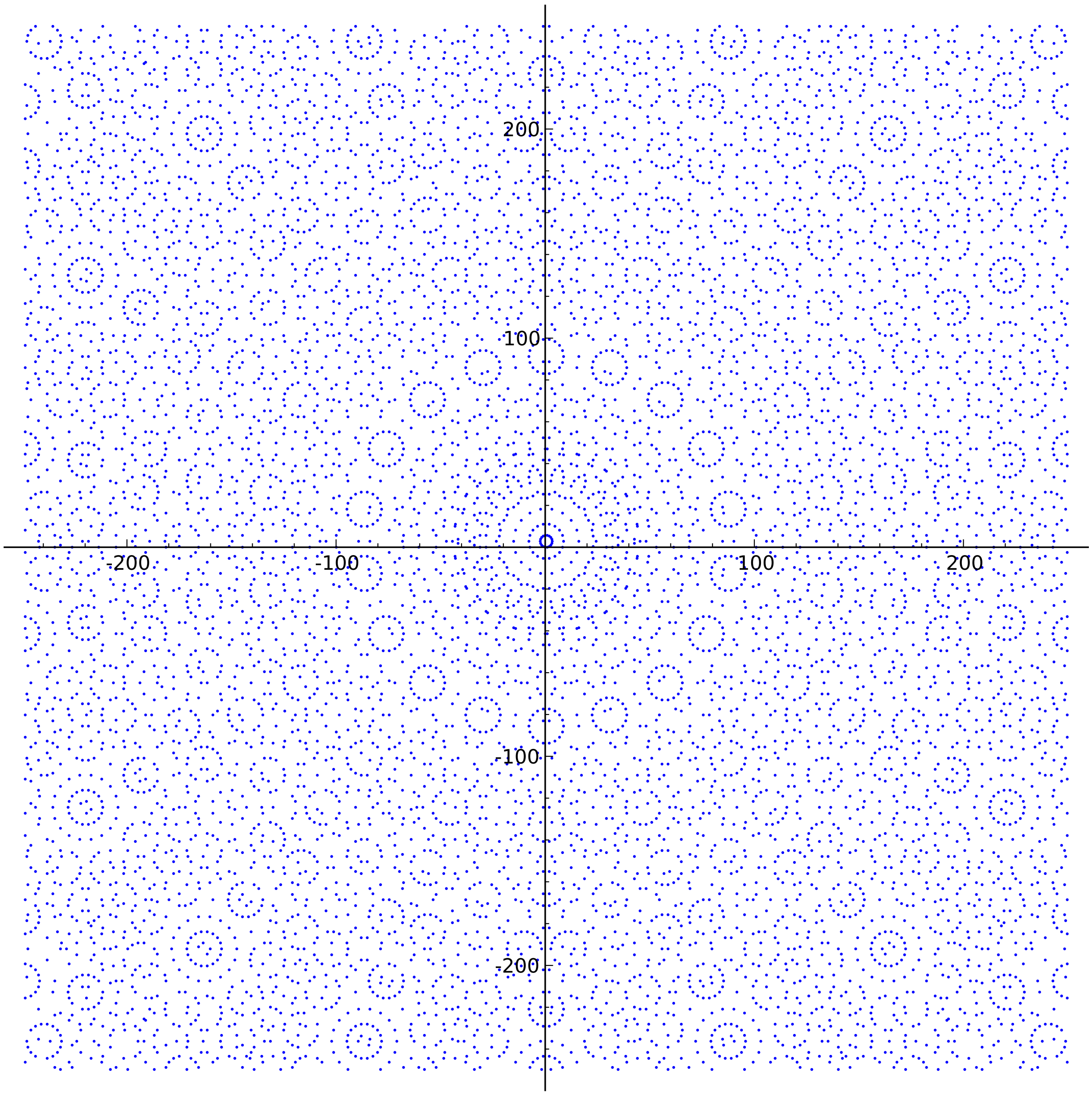


Figure 18: $Q_{\lambda_9}(P_{18})$, the λ_9-convex closure of a regular 18-gon.

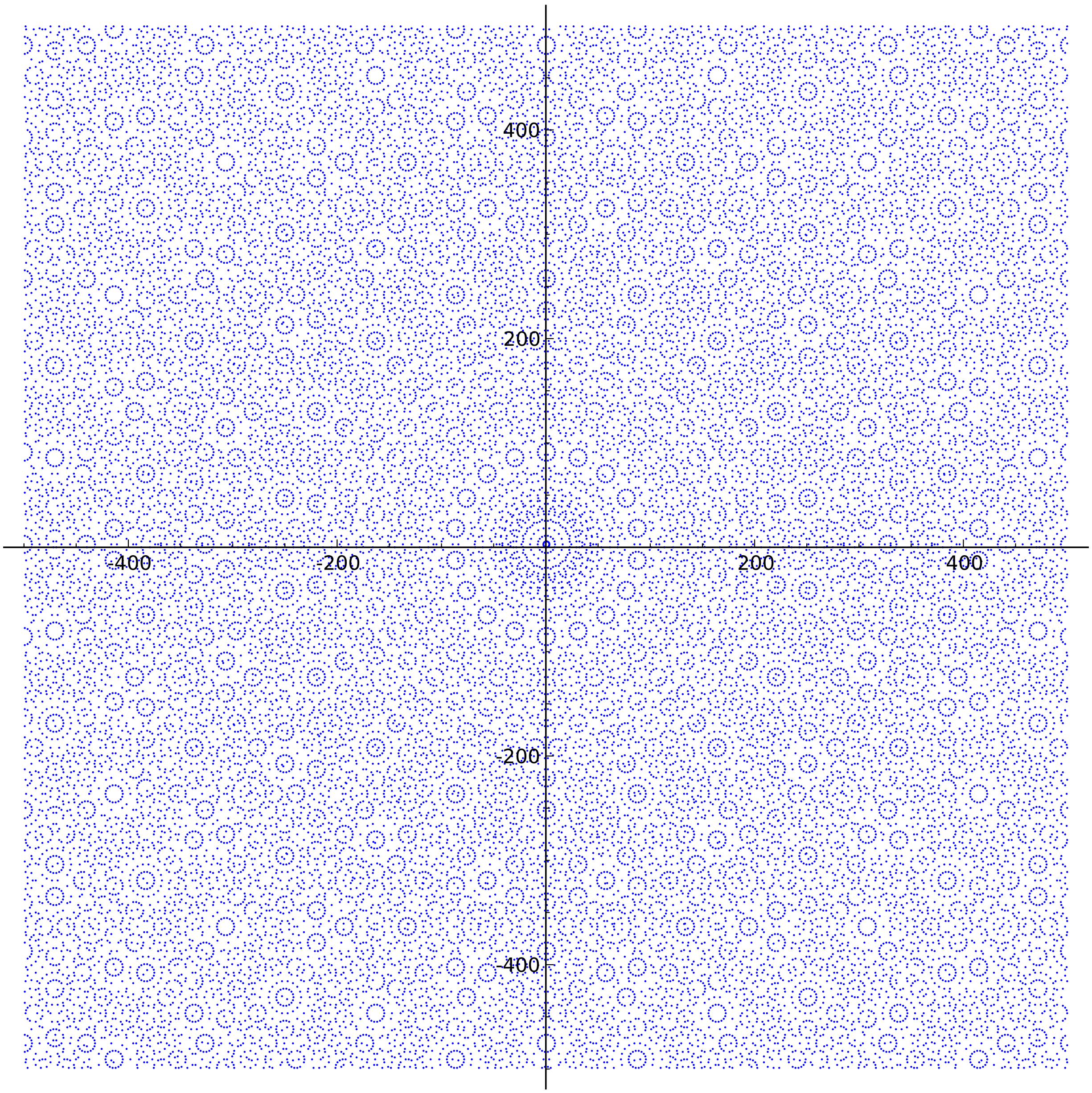


Figure 19: A wider view of $Q_{\lambda_9}(P_{18})$.

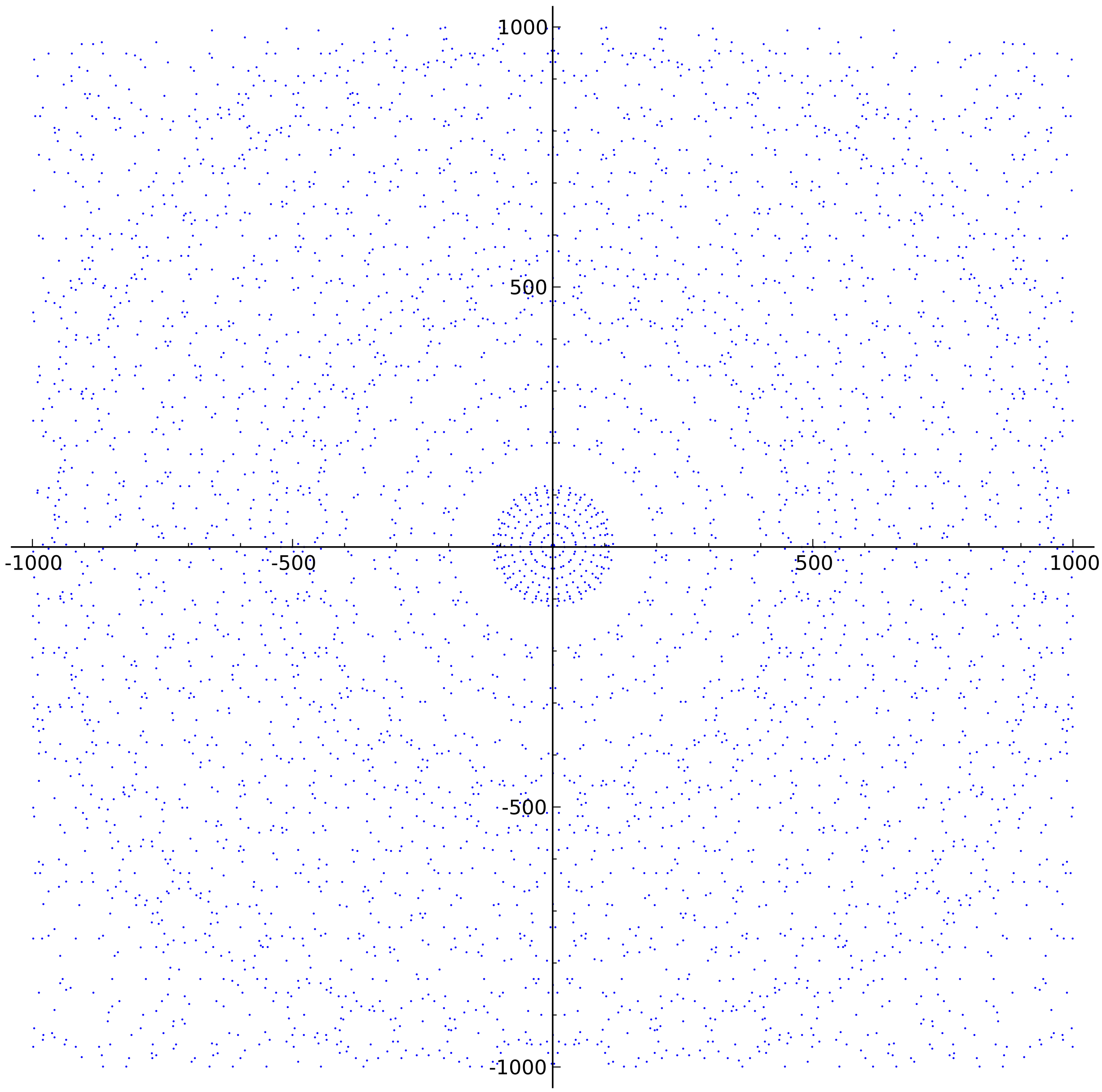


Figure 20: $Q_{\lambda_{15}}(P_{15})$, the λ_{15}-convex closure of a regular 15-gon.

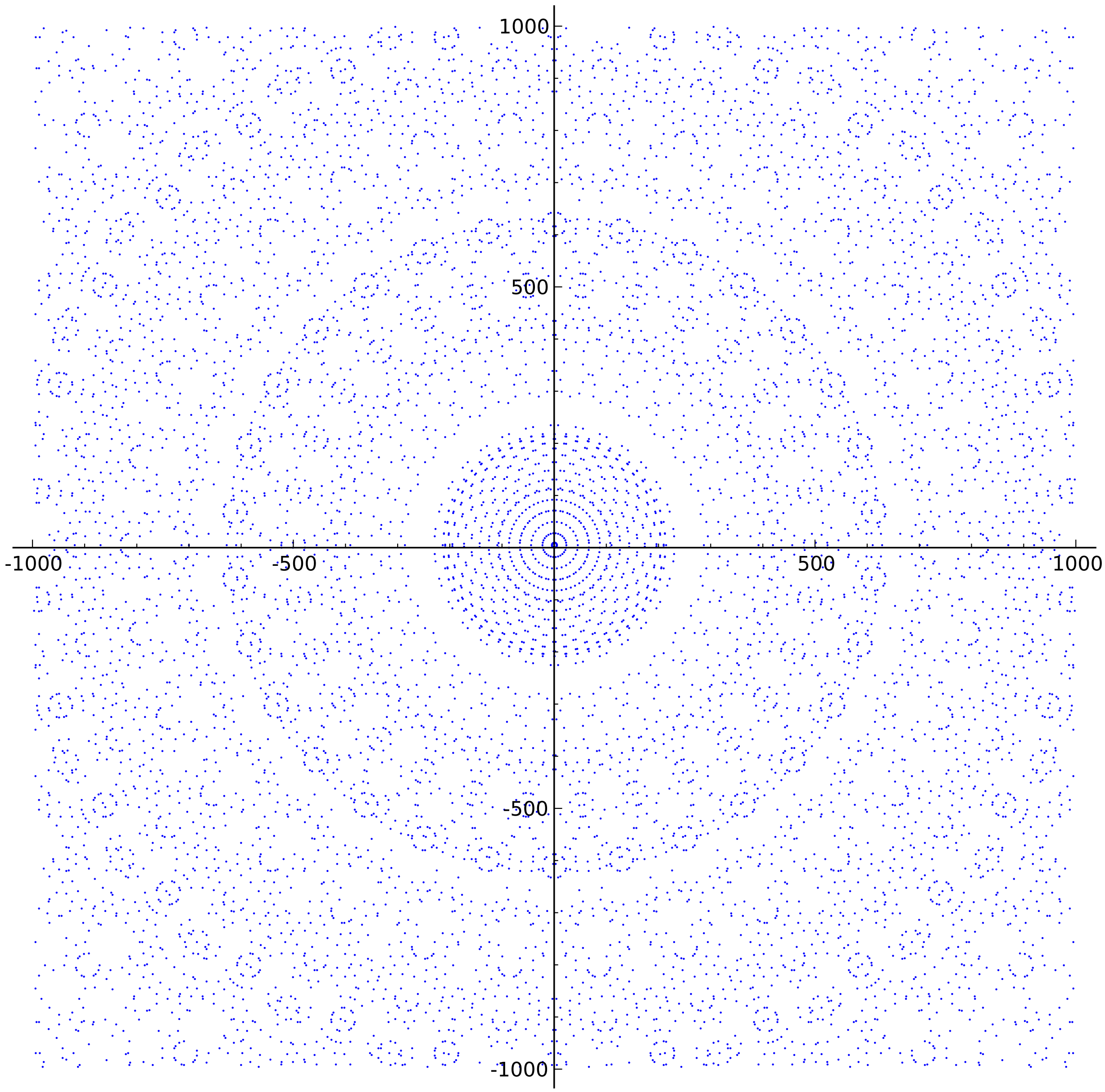


Figure 21: $Q_{\lambda_{15}}(P_{30})$, the λ_{15}-convex closure of a regular 30-gon.

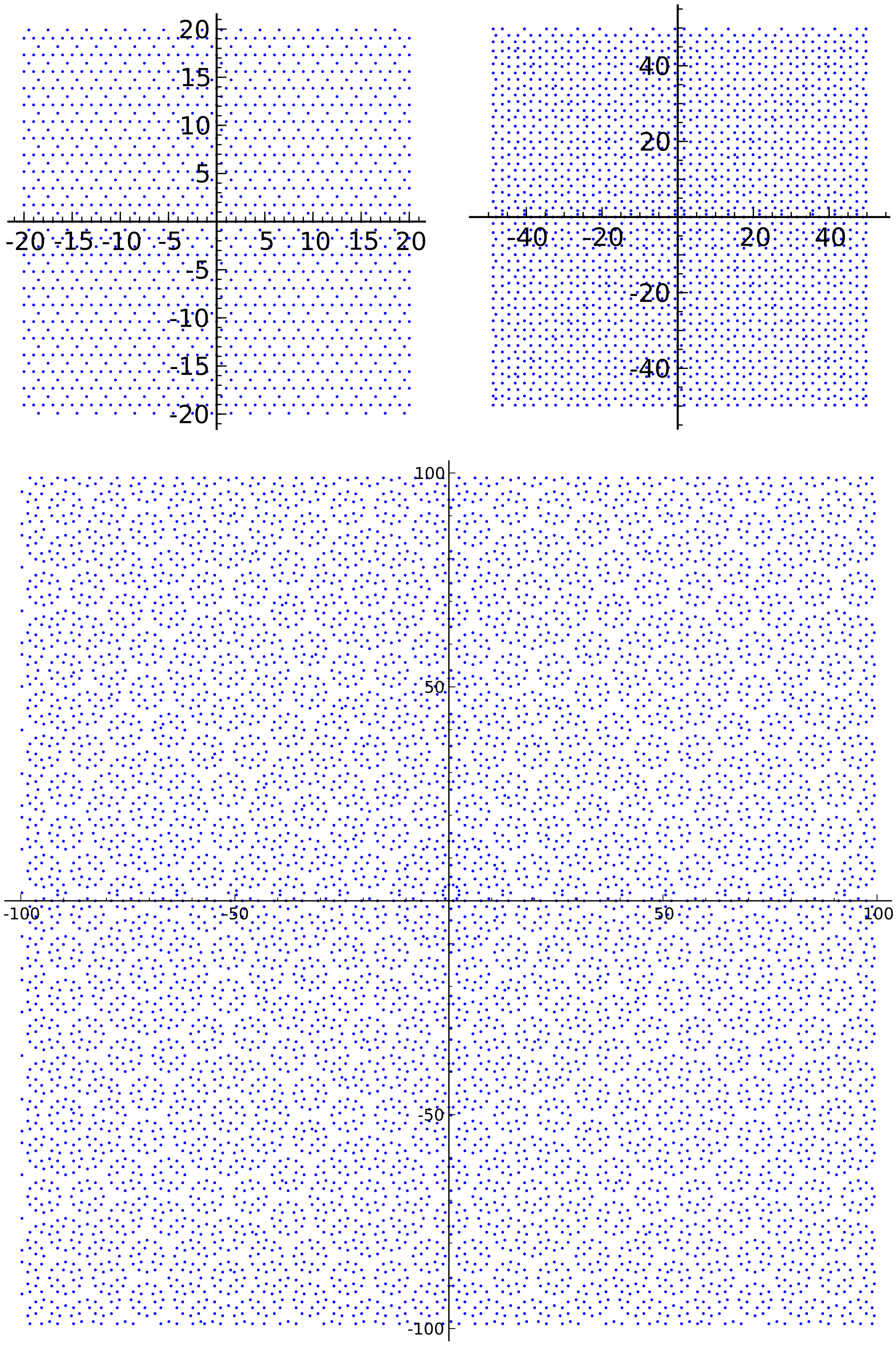


Figure 22: $Q_2(P_6)$ (top left), $Q_{2+\sqrt{2}}(P_8)$ (top right), and $Q_{2+\sqrt{3}}(P_{12})$ (bottom). $Q_2(P_6) = \{a + b\eta \mid a, b \in \mathbb{Z} \;\;\&\;\; ab \equiv 0 \pmod 2\}$, where $\eta := e^{i\tau/3}$ is the principal third root of unity. All three sets are uniformly discrete, with any two points of any set at least unit distance apart. Of these three, only $Q_2(P_6)$ is periodic.

14.4 Regular convex polyhedra

In this section, we investigate cases where λ-convex closures of the five regular convex polyhedra (Platonic solids) in $\mathbb{R}^3$ are uniformly discrete, for real λ. We show that for any real sPV λ, the λ-convex closure of the tetrahedron, cube, and octahedron are uniformly discrete. Berman & Moody showed that the $(1+\varphi)$-convex closures of the dodecahedron and icosahedron are both uniformly discrete using icosians (a finite subgroup of the multiplicative group of the quaterions) [4]. We give an independent, geometry-based proof. We prove all these results using the projection technique we first used in the proof of Lemma 14.1. In fact, one can think of the next proposition as a three-dimensional version of Lemma 14.1.

Proposition 14.10. *For any sPV $\lambda \in \mathbb{R}$, the λ-convex closures of the sets of corners of the regular tetrahedron, cube, and octahedron are all uniformly discrete point sets in $\mathbb{R}^3$.*

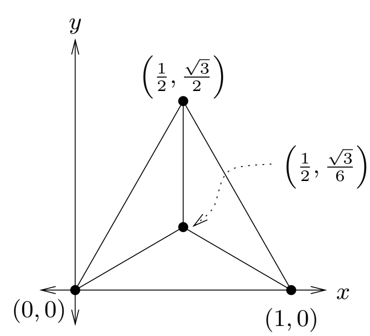
Proof. We prove the result for the regular tetrahedron; the other two have similar proofs. For concreteness—and consistency with how we handled the regular polygons above—we orient the tetrahedron in $\mathbb{R}^3$ so that

- one of its edges coincides with the set $\{(x,0,0) \mid 0 \le x \le 1\}$,
- one of the faces incident to this edge lies in the half-plane $\{(x,y,0) \mid y \ge 0\}$, and
- the tetrahedron itself lies in the half-space $\{(x,y,z) \mid z \ge 0\}$.

We let $T \subseteq \mathbb{R}^3$ be the four corners of this tetrahedron. The perpendicular projection of T into the x,y-plane is

$$T' := \{(0,0),(1,0),(1/2,\sqrt{3}/2),(1/2,\sqrt{3}/6)\} \subseteq \mathbb{R}^2 ,$$

shown below, with edges included:



Projecting T' perpendicularly onto the x-axis yields the set $T'' = \{0, \frac{1}{2}, 1\}$. Since $T'' \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\lambda)$, it follows from Theorem 12.14 that $Q_\lambda(T'')$ is uniformly discrete, and from this it follows (using Lemma 12.20) that $Q_\lambda(T')$ is included in the union U of a discrete set of vertical lines with a uniform positive lower bound on interline spacing. Projecting T' perpendicularly into a line through one of the other outer edges—the line through the origin and $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, say—shows that $Q_\lambda(T')$ is included in a rotated copy U' of U whose lines are not parallel with those of U. Thus $Q_\lambda(T') \subseteq U \cap U'$, which is uniformly discrete. But now $Q_\lambda(T')$ is the perpendicular projection of $Q_\lambda(T)$ onto the x,y-plane (by Lemma 12.20 again), and so $Q_\lambda(T)$ is confined to the union V of a discrete set of lines in $\mathbb{R}^3$ parallel to the z-axis with a uniform positive lower bound on interline spacing. Projecting $Q_\lambda(T)$

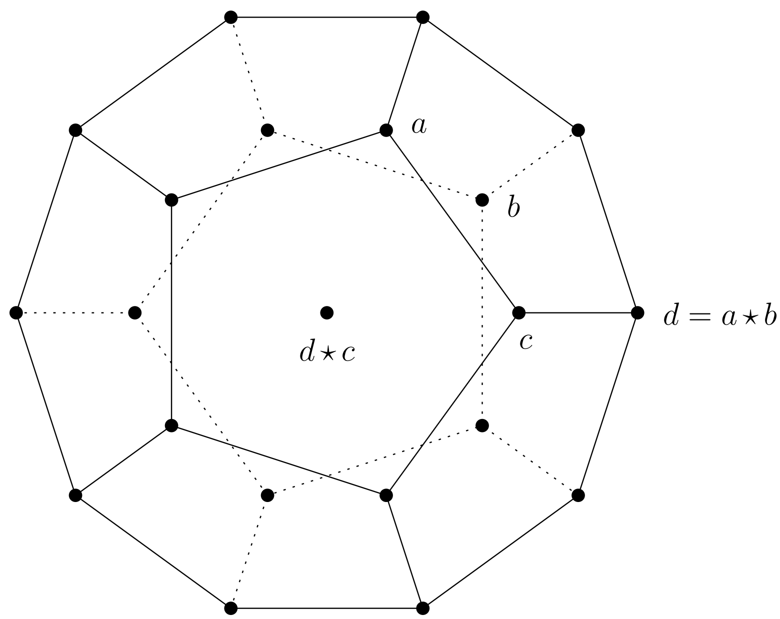


Figure 23: The projection of a dodecahedron D onto the plane of one of its faces. The point in the middle is not part of the projection of D itself, but of a point in $Q_\lambda(D)$ lying directly above the top face. Here, $\star = \star_{1+\varphi}$.

perpendicularly into the plane containing one of T's other faces shows that $Q_\lambda(T)$ is included in a rotated copy V' of V. Thus $Q_\lambda(T) \subseteq V \cap V'$, which is uniformly discrete. $\square$

We turn now to geometry-based proofs of the cases of the dodecahedron and the icosahedron.

Proposition 14.11 (Berman & Moody [4]). *The* $(1+\varphi)$*-convex closure of the regular dodecahedron and regular icosahedron in* $\mathbb{R}^3$ *are both uniformly discrete.*

Proof. We set $\lambda := 1 + \varphi = (3 + \sqrt{5})/2$. Let D be the vertices of a regular dodecahedron in $\mathbb{R}^3$ (its orientation and location are not important). It turns out, quite fortunately, that if we project $Q_\lambda(D)$ perpendicularly into the plane containing one of D's faces, we get a point-set equal (up to similarity) to $Q_\lambda(P_{10})$, shown in Figure 11, which is discrete by Propositions 14.4 and 14.5. (Note that $\lambda = \lambda_5$.)

To see this, consider the perpendicular projection of D into the plane containing one of its faces, shown in Figure 23. Call this projection $D' \subseteq \mathbb{R}^2$. D' consists of two concentric decagons. Points in the outer decagon are λ-extrapolants of pairs of adjacent points of the inner decagon; for example, $d = a \star b$ in the figure. Letting S be the inner decagon (a copy of P_{10}), we see that all the points of D' are therefore in $Q_\lambda(S)$. Moreover, we know that $Q_\lambda(S)$ is uniformly discrete by Propositions 14.4 and 14.5. Thus $Q_\lambda(D')$ is uniformly discrete. Notice that the center point $d \star c$ shown in Figure 23 is also in $Q_\lambda(D')$ (though not in D' itself).

As with previous arguments, we conclude that $Q_\lambda(D)$ is confined to the union U of a discrete set of parallel lines in $\mathbb{R}^3$ with a uniform positive lower bound on interline spacing. By symmetry, we can project onto another, nonparallel face to see that $Q_\lambda(D)$ is also contained in a rotated copy of U, hence $Q_\lambda(D)$ must be uniformly discrete. This finishes the case of the dodecahedron.

Consider the top pentagonal face of D whose projection is shown in Figure 23 centered and bounded by solid lines. Call this face F. The center point $d \star c \in Q_\lambda(D')$, shown in Figure 23, is actually the projection of a point $e \in \mathbb{R}^3$ lying directly above F. The point e is the intersection of lines extending the five edges of D that are incident to the vertices of F but are not edges of F itself (their projections appear as the solid radial lines in the figure). The point e is in $Q_\lambda(D)$, because e is on each of these lines, and its projection $d \star c$ is the λ-extrapolation of points in $Q_\lambda(D')$. By symmetry, there are points lying similarly above the other eleven faces of D. These twelve points, all in $Q_\lambda(D)$ are the vertices of a regular icosahedron I. Since $I \subseteq Q_\lambda(D)$, we have $Q_\lambda(I) \subseteq Q_\lambda(D)$, and thus $Q_\lambda(I)$ is uniformly discrete. $\square$

14.5 Other results

In this section we give some *ad hoc* results about specific point sets. For example, we get a lower bound of 1/2 on the packing distances of some of the uniformly discrete sets we considered in the last subsection. We make no attempt at generalizing our results, although such generalizations are most likely possible.

If $\lambda = 2$, then $R_2(P_6) = Q_2(P_6)$ is a proper subset of the Eisenstein integers (the regular triangular lattice) that has both translational symmetry (in six directions) and D_6 rotational symmetry[14] about the center of the hexagon. For another example, if $S = \{0, 1, i, 1+i\}$ is the set of vertices of a square (similarly positioned), then $R_2(S) = Q_2(S) = \mathbb{Z}[i]$, the set of Gaussian integers. If $L = \{0, 1, i\}$, then $R_2(L)$ is a proper subset of $\mathbb{Z}[i]$, however, as the following lemma shows:

Lemma 14.12. *Let γ be any element of $\mathbb{C} \setminus \mathbb{Q}$. Then*

$$Q_2(\{0,1,\gamma\}) = \{a + b\gamma \mid a, b \in \mathbb{Z} \ \& \ ab \equiv 0 \pmod 2\} . \tag{43}$$

If $\gamma \in \mathbb{R}$, then $R_2(\{0,1,\gamma\}) = \mathbb{R}$; otherwise, $R_2(\{0,1,\gamma\}) = Q_2(\{0,1,\gamma\})$, which is a discrete set.

Proof. It is easy to check, using the fact that $x \star_2 y = 2y - x$ for any x and y, that the right-hand side of (43) contains $\{0, 1, \gamma\}$ and is 2-convex. This proves $\subseteq$.

For the reverse inclusion, we first note that $Q_2(\{0,1\}) = \mathbb{Z}$ and $Q_2(\{0,\gamma\}) = \gamma\mathbb{Z}$, and thus $\mathbb{Z} \cup \gamma\mathbb{Z} \subseteq Q_2(\{0,1,\gamma\})$. Now let $a, b \in \mathbb{Z}$ be arbitrary. If b is even, then $a + b\gamma = (-a) \star_2 (b\gamma/2)$. If a is even, then $a + b\gamma = (-b\gamma) \star_2 (a/2)$. In either case, this shows that $a + b\gamma \in Q_2(\mathbb{Z} \cup \gamma\mathbb{Z}) = Q_2(\{0,1,\gamma\})$, which establishes the reverse containment.

If γ is a real, irrational number, then (43) implies that $Q_2(\{0,1,\gamma\})$ is a dense subset of $\mathbb{R}$, and so $R_2(\{0,1,\gamma\}) = \mathbb{R}$. If $\gamma \notin \mathbb{R}$, then $Q_2(\{0,1,\gamma\})$ is a subset of the discrete lattice generated by 1 and γ. $\square$

Considering $Q_2(P_6)$ again, one can easily check that $P_6 \subseteq Q(\{0,1,\eta\})$ where $\eta := e^{i\tau/3}$ is the principal third root of unity. Thus, by Lemma 14.12, $Q_2(P_6) = Q_2(\{0,1,\eta\}) = \{a + b\eta \mid a, b \in \mathbb{Z} \ \& \ ab \equiv 0 \pmod 2\}$, and this is one of the sets shown in Figure 22.

We now reconsider the set $Q_{1+\varphi}(P_5)$, illustrated in Figure 10. Like $Q_2(P_6)$, this set is uniformly discrete by Propositions 14.4 and 14.5 and has D_5 symmetry about the center of P_5. Unlike $Q_2(P_6)$, however, $Q_{1+\varphi}(P_5)$ has no translational symmetry.

[14] D_n is the dihedral group of order $2n$.

Theorem 14.13. *Except for adjacent points in P_5, all points of $Q_{1+\varphi}(P_5)$ are farther than unit distance apart.*

Proof. Let $\lambda := 1 + \varphi$ and let $\star$ mean $\star_\lambda$ as usual. Let $S := Q_\lambda(P_5)$. We again use the projection idea from the proof of Proposition 14.3, but for technical convenience, we project horizontally onto the imaginary axis via Im rather than vertically onto the real axis via Re. A little elementary geometry shows that $\mathrm{Im}(P_5) = \{0, \zeta, \kappa\}$, where

$$\zeta := \sin(\tau/5) = \frac{\sqrt{2}}{4}\sqrt{5+\sqrt{5}} \approx 0.951 , \qquad \kappa := \varphi\zeta = \frac{\sqrt{5+2\sqrt{5}}}{2} \approx 1.539 .$$

From this we readily get $0 = -\varphi\kappa + \lambda\zeta = \kappa \star \zeta \in Q_\lambda(\{\kappa, \zeta\})$, and therefore

$$\mathrm{Im}(S) = Q_\lambda(\mathrm{Im}(P_5)) = Q_\lambda(\{0, \zeta, \kappa\}) = Q_\lambda(\{\kappa, \zeta\}) = \rho_{\kappa,\zeta}(Q_\lambda) .$$

The right-hand side is discrete by Proposition 6.1, which shows that $\mathrm{Im}(S)$ is discrete.

Fix any $a, b \in S$ such that $0 < |b - a| \le 1$. We now show that $|b - a| = 1$ and $a, b \in P$. Let C be the center of P_5, and let G be the group of rotations about C through angles that are multiples of $\tau/5$. (G is the 5-element cyclic group generated by $\rho_{\xi,0}$, where $\xi := e^{i3\tau/10}$.) By symmetry, each element of G leaves P_5, and therefore S, invariant.

Claim 14.14. *There exist distinct $g_1, g_2 \in G$ such that*

1. $\{\mathrm{Im}(g_1(a)), \mathrm{Im}(g_1(b))\} = \{\mathrm{Im}(g_2(a)), \mathrm{Im}(g_2(b))\} = \{\zeta, \kappa\}$,
2. *the slope of the line through $g_1(a)$ and $g_1(b)$ is $\tan(\tau/10)$, and*
3. *the slope of the line through $g_2(a)$ and $g_2(b)$ is $-\tan(\tau/10)$.*

Furthermore, $|b - a| = 1$.

Proof of the Claim. Let θ be the argument of $b - a$, and let $r = |b - a|$. By assumption, $0 < r \le 1$. The elements of G rotate the line segment connecting a with b to form line segments with length r and with arguments $\theta + k\tau/5$, for $k \in \{0, 1, 2, 3, 4\}$. The vertical displacement of each such line segment (that is, the absolute difference between the imaginary parts of the two endpoints) is thus $r|\sin(\theta + k\tau/5)|$. A simple geometric argument shows that there exist distinct $k_1, k_2 \in \{0, 1, 2, 3, 4\}$ such that $0 < |\sin(\theta + k_1\tau/5)| \le \sin(\tau/10)$ and $0 < |\sin(\theta + k_2\tau/5)| \le \sin(\tau/10)$. Let g_1 and g_2 be the corresponding elements of G, respectively. Now we note that

$$\sin(\tau/10) = \frac{\sqrt{2}}{4}\sqrt{5-\sqrt{5}} = \kappa - \zeta \approx 0.588 .$$

We thus have

$$0 < |\mathrm{Im}(g_1(b)) - \mathrm{Im}(g_1(a))| \le \kappa - \zeta , \qquad 0 < |\mathrm{Im}(g_2(b)) - \mathrm{Im}(g_2(a))| \le \kappa - \zeta . \tag{44}$$

We know that $g_1(a), g_1(b), g_2(a), g_2(b) \in S$, and we established earlier that $\mathrm{Im}(S) = \rho_{\kappa,\zeta}(Q_\lambda)$. Thus $\mathrm{Im}(g_j(a))$ and $\mathrm{Im}(g_j(b))$ are both in $\rho_{\kappa,\zeta}(Q_\lambda)$ for $j \in \{1, 2\}$. Now Proposition 6.1 implies that, except for κ and ζ, any two points of $\rho_{\kappa,\zeta}(Q_\lambda)$ differ by at least $(\kappa - \zeta)\varphi = \zeta$, which is strictly greater than $\kappa - \zeta$. This establishes the first item of the Claim, and it also implies that equality must hold for both inequalities in (44). Therefore, for both $j \in \{1, 2\}$,

$$\sin(\tau/10) = \kappa - \zeta = |\mathrm{Im}(g_j(b)) - \mathrm{Im}(g_j(a))| = r|\sin(\theta + k_j\tau/5)| \leq |\sin(\theta + k_j\tau/5)| \leq \sin(\tau/10) .$$

The only way this can occur is when $r = 1$ and $|\sin(\theta + k_j\tau/5)| = \sin(\tau/10)$. Since $g_1 \neq g_2$, it must then be that the line through $g_1(a)$ and $g_1(b)$ and the line through $g_2(a)$ and $g_2(b)$ have oppositely signed slopes, both with absolute value $\tan(\tau/10)$. By swapping g_1 and g_2 if necessary, we can assume that the line through $g_1(a)$ and $g_1(b)$ has positive slope. This establishes the rest of the Claim. □

It remains to show that $a, b \in P_5$, and for this it suffices that $g_1(a)$ and $g_1(b)$ are both in P_5. By swapping a and b if necessary, we can assume that $\mathrm{Im}(g_1(a)) = \zeta$ and $\mathrm{Im}(g_1(b)) = \kappa$. To get the slope of the line between $g_2(a)$ and $g_2(b)$ to be $-\tan(\tau/10)$, as asserted by the Claim, it must be that g_2 results from applying g_1 followed by a clockwise rotation around C through angle $\tau/5$. Thus we have $\mathrm{Im}(g_2(a)) = \kappa$ and $\mathrm{Im}(g_2(b)) = \zeta$, and this is only possible if $g_1(b) = g_2(a)$ is the apex of P_5 and $g_1(a)$ is the element of P_5 immediately to its left. This finishes the proof of Theorem 14.13. □

Corollary 14.15. *All $\mathbb{R}$-affine transformations of $\mathbb{C}$ that are symmetries of $Q_{1+\varphi}(P_5)$ are length-preserving and leave P_5 invariant.*

We turn to the case of the regular octagon. The $(2+\sqrt{2})$-clonvex closure of P_8 is illustrated in the upper right portion of Figure 22. As with $Q_{1+\varphi}(P_5)$, this set is discrete and has no translational symmetry, although it has D_8 symmetry about the center of P_8. The same techniques are used to prove the following, whose proof we only sketch:

Theorem 14.16. *All points in $Q_{2+\sqrt{2}}(P_8)$ are at least unit distance apart.*

Unlike $Q_{1+\varphi}(P_5)$, $Q_{2+\sqrt{2}}(P_8)$ has infinitely many pairs of points that are unit distance apart; they radiate out from P_8 in the eight directions whose angles are multiples of $\tau/8$, as can be seen in Figure 22.

Proof sketch of Theorem 14.16. Let $S := S_\lambda(P_8)$. $\mathrm{Im}(S)$ is a subset of $\rho_{\kappa,1}(Q_\lambda)$, where this time, $\kappa := \sqrt{2}/2$. Similarly, $\mathrm{Re}(S)$ is a subset of $\rho_{0,1-\kappa}(Q_\lambda)$. Q_λ is uniformly discrete by Corollary 13.5, where $m := -2$ and $n := 1$. This makes both the real and imaginary parts of elements of S drawn from discrete sets. Thus S is discrete.

By Corollary 13.5 with $\lambda' = 1 - \lambda = -1 - \sqrt{2}$, any two distinct elements of $\mathrm{Im}(S)$ or of $\mathrm{Re}(S)$ differ by at least $(1-\kappa)(-\lambda') = (2-\sqrt{2})(1+\sqrt{2})/2 = \sqrt{2}/2$. It follows that if distinct $x, y \in S$ have different real parts and different imaginary parts, then $|x - y| \geq 1$. If x and y have the same real part and different imaginary parts, or vice versa, then by symmetry we can rotate the plane about the center of P_8 through angle $\tau/8$ to obtain images x' and y', both in S, whose real parts and imaginary parts both differ. Then we have $|x - y| = |x' - y'| \geq 1$. □

We conjecture that $Q_{2+\sqrt{2}}(P_8)$ has no $\mathbb{R}$-affine symmetries except those that leave P_8 invariant.

Theorem 14.17. *All points in $Q_{2+\sqrt{3}}(P_{12})$ are at least unit distance apart.*

Proof. Let $\lambda := -1-\sqrt{3}$. Then $Q_{2+\sqrt{3}}(P_{12}) = Q_{1-\lambda}(P_{12}) = Q_\lambda(P_{12})$ by Fact 2.13. As in previous proofs we consider the projection $\mathrm{Re}(Q_\lambda(P_{12})) = Q_\lambda(\mathrm{Re}(P_{12}))$ onto the real axis. We have

$$\mathrm{Re}(P_{12}) = \{-(1+\sqrt{3})/2, -\sqrt{3}/2, 0, 1, (2+\sqrt{3})/2, (3+\sqrt{3})/2\} = \frac{1}{2}S\ ,$$

where $S := \{\lambda, 1+\lambda, 0, 2, 1-\lambda, 2-\lambda\} \subseteq \mathbb{Z}[\lambda]$. Since $\frac{1}{2}S$ is a finite subset of $\mathbb{Q}(\lambda)$, we know that $Q_\lambda(\mathrm{Re}(P_{12})) = Q_\lambda(\mathrm{Re}(P_{12})) = Q_\lambda\left(\frac{1}{2}S\right)$ is uniformly discrete by Theorem 12.14. We first get a lower bound on the interpoint distances of $Q_\lambda\left(\frac{1}{2}S\right)$, then combine this with symmetry to get a lower bound on the interpoint distances of $Q_\lambda(P_{12})$.

The only conjugate of λ is $\mu := \sqrt{3}-1 \approx 0.732$. Now we can apply Fact 12.29 with $d := 2$, $k := 1$, $\mu_0 := \mu$, and

$$S_0 := \{a_0 + a_1\mu : a_0, a_1 \in \mathbb{Z}\ \ \&\ \ a_0 + a_1\lambda \in S\} = \{\mu, 1+\mu, 0, 2, 1-\mu, 2-\mu\}\ ,$$

and this makes $\ell_0 = \min(S_0) = 0$ and $h_0 = \max(S_0) = 2$. Thus by Equation (37), we have

$$Q_\lambda(S) \subseteq \{a_0 + a_1\lambda : a_0, a_1 \in \mathbb{Z}\ \ \&\ \ 0 \le a_0 + a_1\mu \le 2\}\ . \tag{45}$$

Now consider two distinct points of $Q_\lambda(S)$, say $a = a_0+a_1\lambda$ and $b = b_0+b_1\lambda$, where $a_0, a_1, b_0, b_1 \in \mathbb{Z}$. We show here that $|b-a| \ge 1$. Let $\delta := b-a = \delta_0 + \delta_1\lambda$ where $\delta_0 = b_0 - a_0$ and $\delta_1 = b_1 - a_1$ are not both zero. If $\delta_1 = 0$, then δ is a nonzero integer and we are done, so we can assume that $\delta_1 \ne 0$. Also assume without loss of generality that $\delta_0 + \delta_1\mu > 0$ (otherwise swap the roles of a and b). Since $a_0 + a_1\mu$ and $b_0 + b_1\mu$ are both in $[0,2]$, we then have $0 < \delta_0 + \delta_1\mu < 2$, or equivalently, $-\delta_1\mu < \delta_0 < 2 - \delta_1\mu$. Adding $\delta_1\lambda$ to all sides gives us

$$-\delta_1\mu + \delta_1\lambda < \delta_0 + \delta_1\lambda = \delta < 2 - \delta_1\mu + \delta_1\lambda\ ,$$

that is,

$$-2\sqrt{3}\,\delta_1 < \delta < 2 - 2\sqrt{3}\,\delta_1\ ,$$

using the fact that $\lambda - \mu = -2\sqrt{3}$. If $\delta_1 < 0$, then $\delta > 2\sqrt{3} \ge 1$. If $\delta_1 > 0$, then $\delta < 2 - 2\sqrt{3} \le -1$. Thus in any case, $|\delta| = |b-a| \ge 1$.

It follows immediately that the distance between any two distinct points of $Q_\lambda\left(\frac{1}{2}S\right) = \frac{1}{2}Q_\lambda(S)$ is at least $1/2$. Thus $Q_\lambda(P_{12})$ is confined to the union of a discrete set of vertical lines with interline distance at least $1/2$. Now consider any two distinct points $x, y \in Q_\lambda(P_{12})$ and let $z := y - x$. Writing $z = re^{i\theta}$ where $r := |z| > 0$ and $\theta := \arg z$, it suffices to show that $r \ge 1$. We see by the symmetry of P_{12} that $Q_\lambda(P_{12})$ is invariant under rotation about the center of P_{12} through angle $\tau/12 = \pi/6$. Thus for each $k \in \mathbb{Z}$, $Q_\lambda(P_{12})$ contains a pair of points that differ by $e^{ik\pi/6}z = re^{i(\theta+k\pi/6)}$. Choose k to minimize $|\mathrm{Re}(e^{ik\pi/6}z)| = r|\cos(\theta + k\pi/6)|$, and set $\psi := \theta + k\pi/6$ for our chosen k. Let $x', y' \in Q_\lambda(P_{12})$ be such that $y' - x' = e^{ik\pi/6}z = re^{i\psi}$ (whence $\mathrm{Re}(y'-x') = r\cos\psi$). Then either $\cos\psi = 0$ (if x' and y' lie on the same vertical line) or $|\cos\psi| \ge 1/(2r)$ (if x' and y' lie on different vertical lines). By swapping x' and y' if necessary, we can assume that $0 \le \psi < \pi$. Since k—and thus ψ—was chosen to minimize $|\cos\psi|$, we get that $5\pi/12 \le \psi \le 7\pi/12$.

We have two cases:

Case 1: $\cos\psi \ne 0$. Then putting the two inequalities above together, we get

$$\frac{1}{2r} \le |\cos\psi| \le \cos(5\pi/12)\ ,$$

and thus

$$r \geq \frac{1}{2\cos(5\pi/12)} > \frac{1}{2\cos(\pi/3)} = 1 \; .$$

Case 2: $\cos\psi = 0$, that is, $\psi = \pi/2$, and x' and y' lie on the same vertical line. Using the rotational symmetry of $Q_\lambda(P_{12})$ again, we can find points x'' and y'' such that $y'' - x'' = e^{-i\pi/6}(y' - x') = re^{i\pi/3}$. Thus

$$\mathrm{Re}(y'' - x'') = r\cos(\pi/3) = \frac{r}{2} \geq \frac{1}{2} \; ,$$

because x'' and y'' lie on different vertical lines. This also implies $r \geq 1$.

□

15 Relative density of discrete $Q_\lambda(S)$

Proposition 12.32 says that $R_\lambda(S)$ is a subset of the model set of a certain cut-and-project scheme, where λ is sPV and $S \subseteq \mathbb{Z}[\lambda]$ is finite. One of our chief conjectures is that these $R_\lambda(S)$ are all Meyer sets. To establish the conjecture it suffices by Fact 12.11 to show that $R_\lambda(S)$ is relatively dense (in either $\mathbb{R}$ or $\mathbb{C}$ as appropriate). While we have not succeeded in finding a general proof, we have a number of partial results, as well as a sufficient condition for R_λ to be Meyer: This is the case if λ is sPV and R_λ contains a unit of $\mathbb{Z}[\lambda]$ other than 1 (see Corollary 15.29).

We begin with the case of real quadratic sPV numbers (Section 15.1), and, generalizing techniques of Berman and Moody [4] and Masáková et al. [23], characterize all such numbers for which R_λ *equals* a naturally corresponding model set (which is stronger than the Meyer property). As model sets are relatively dense, relative density is immediate. We find that the model set property can only hold for four values of λ (or the corresponding $1-\lambda$): $-\varphi$, $-1-\sqrt{2}$, $-1-\sqrt{3}$, and $\frac{-3-\sqrt{17}}{2}$. We then prove it holds for each of these values; the proof for $\frac{-3-\sqrt{17}}{2}$ is somewhat more difficult and appears separately, as outlined in the next paragraph. We thus obtain a characterization that strictly strengthens a previous result of Masáková et al. [23], which gave a similar characterization for quadratic unitary Pisot numbers (a subset of quadratic sPV numbers).

Certain aspects of Berman and Moody's technique, which we call the *covering method*, are then progressively strengthened in Section 15.2, where we complete the proof of our characterization of Section 15.1 for the remaining case of $\frac{-3-\sqrt{17}}{2}$. Sufficient conditions for obtaining a model set are further extended to higher degree λ. This allows us to conclude that R_λ is relatively dense for three cubic λ's: two non-real values, and λ_7, the latter being the appropriate parameter for generating the set based on the regular heptagon (see Figure 12). These proofs explicitly show that R_λ *contains* (but does not necessarily equal) a model set.

Another notion, which we call *affine embedding*, provides a flexible tool for proving both relative density *and* uniform discreteness (Section 15.3). Affine embeddings combined with the covering method leads to the strongest sufficient condition we have for R_λ to be a Meyer set (Corollary 15.29, as mentioned above). It also yields a method to determine if certain elements of Σ_P are *not* contained in R_λ, for certain λ (Section 15.3.2).

Finally, in Section 15.4 we use the 1-dimensional (real) results outlined above to prove relative density results in higher dimensions. Thus, for example, the relative density of R_{λ_7} implies the relative density of the set of Figure 12 derived from the heptagon.

15.1 Relative Density in $\mathbb{R}$ of R_λ for Real Quadratic sPV Numbers λ

We assume throughout this subsection that λ is a real quadratic sPV number. These numbers are characterized in Corollary 13.5.

In Proposition 12.32, we showed that R_λ is a subset of a cut-and-project, or model, set. In the quadratic case, according to Eq. (39) in Corollary 13.5 (for $\lambda < 0$ without loss of generality), we can write this relation as follows:

$$R_\lambda \subseteq \{a - b\lambda \mid a, b \in \mathbb{Z} \text{ and } a - b\mu \in [0,1]\}$$

where $\mu = \lambda'$ is the Galois conjugate of λ. (Following Berman & Moody, we use $x \mapsto x'$ as the field automorphism of $\mathbb{Q}(\lambda) = \mathbb{Q}(\mu)$ that swaps λ with μ.) Thus, more compactly,

$$R_\lambda \subseteq \{x \mid x \in \mathbb{Z}[\lambda] \text{ and } x' \in [0,1]\}.$$

Following the notation of Definition 12.26, which is based in turn on the notation of Berman and Moody [4], we denote $P := [0,1]$ and $\Sigma_P := \{x \mid x \in \mathbb{Z}[\lambda] \text{ and } x' \in [0,1]\}$. Then the above containment is stated simply as,

$$R_\lambda \subseteq \Sigma_P\ .$$

Our goal in this section is to determine for which λ the above containment is an equality. This we do in Theorem 15.7. When equality holds, given that Σ_P is a model set, it then follows that R_λ is relatively dense. This is one of only two ways we have been able to prove that R_λ is relatively dense; the other is used for Corollary 15.34 later on.

The development closely follows prior work of Berman and Moody [4] and Masáková et al. [23].

Recall the concept of the *field norm* $N(x) = xx'$, for $x \in \mathbb{Z}[\lambda]$.[15] More concretely, if $x = a + b\lambda$, then $N(x) = (a + b\lambda)(a + b\mu)$. It is known that N is multiplicative, and that x is a unit (has an inverse) in $\mathbb{Z}[\lambda]$ iff $N(x) = \pm 1$.

Definition 15.1. An algebraic integer α *unitary* iff α is a unit of $\mathbb{Z}[\alpha]$. A quadratic integer $\alpha \in \mathbb{R}$ is *unitary Pisot* iff it is a PV number which is a unit in $\mathbb{Z}[\alpha]$.

Most quadratic sPV numbers are not unitary Pisot (as they can be irrationals with non-unit field norm, cf. Corollary 13.5 for $n > 1$). In [23], Masáková et al. characterize the real quadratic unitary Pisot numbers that lead to model sets. By a result of Pinch (in [30], Theorem 15) for real quadratic integer α, R_α is discrete iff α is sPV. Thus real quadratic unitary Pisots can lead to model sets only if they are also sPV. Our characterization below (Theorem 15.7) thus strengthens that of [23], as it deals with a strict superset of those unitary Pisots that may result in model sets.

Let R'_λ denote the set of all conjugates of elements in R_λ. It is clear that $R'_\lambda = R_{\lambda'} = R_\mu$. Similarly, $\Sigma'_P = \{x' \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\}$. Since μ generates the same ring as λ, $x \in \mathbb{Z}[\lambda]$ iff $x' \in \mathbb{Z}[\lambda]$, so it is plain that $\Sigma'_P = \mathbb{Z}[\lambda] \cap P$. Because the algebraic structures are the same, that implies $R_\lambda \subseteq \Sigma_P$ iff $R_\mu \subseteq \mathbb{Z}[\lambda] \cap P$. Further, $R_\lambda = \Sigma_P$ iff $R_\mu = \mathbb{Z}[\lambda] \cap P = \Sigma'_P$.

The following result and proof is identical to an analogous one in [23]. The only observation we make here is that it's not necessary to assume that λ is unitary Pisot.

Lemma 15.2. *If $R_\mu = \Sigma'_P$, then for any $y \in \Sigma'_P$, both of the following hold:*

[15] More generally, the field norm (over $\mathbb{Q}$) of an algebraic number x is the product of the conjugates of x over $\mathbb{Q}$. If x is an algebraic integer, then $N(x)$ is also an integer.

(i) $\mu \mid y$ *or* $\mu \mid (y-1)$.

(ii) $(1-\mu) \mid y$ *or* $(1-\mu) \mid (y-1)$.

(Divisibility is with respect to $\mathbb{Z}[\lambda] = \mathbb{Z}[\mu]$*.)*

Proof. By the hypothesis, $y \in \Sigma'_P$ implies that $y \in R_\mu$. By Lemma 7.8, this implies that $y = \sum_{i=0}^{n} b_i \mu^i (1-\mu)^{n-i}$ for some $n \geq 0$ and $0 \leq b_i \leq \binom{n}{i}$. If $b_0 = 0$, then clearly $\mu \mid y$. If $b_0 = 1$, then $y = 1 +$ terms divisible by μ, so $\mu \mid (y-1)$, which yields item (i). For item (ii), the proof is the same, but based on the $i = n$ term in the sum. □

We recall some well-known facts about rings of real quadratic integers (i.e., rings of algebraic integers in real quadratic fields) that are of the form $\mathbb{Z}[\sqrt{d}]$, where $d > 0$ is square-free. See [11] and [12] for further details and proofs. It is known that all such rings have infinitely many units. Even for modest values of d, the so-called *fundamental units* (those of smallest size and > 1) can already be huge. In fact, all units in $\mathbb{Z}[\sqrt{d}]$ are positive or negative powers $\pm u_d^k$ of the fundamental unit u_d. Now our rings $\mathbb{Z}[\lambda]$ are not quite of this form, since (in the negative case), $\lambda = \frac{-m-\sqrt{D}}{2}$ where $D = m^2 + 4n$, and D might not be square-free. Clearly, $\mathbb{Z}[\sqrt{D}] \subseteq \mathbb{Z}[\lambda]$, and any units in $\mathbb{Z}[\sqrt{D}]$ are also units in $\mathbb{Z}[\lambda]$. Indeed, the rings $\mathbb{Z}[\sqrt{D}]$ also have infinitely many units. This is because any element $a + b\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$ is a unit iff $N(a+b\sqrt{D}) = (a+b\sqrt{D})(a-b\sqrt{D}) = a^2 - Db^2 = 1$. This is Pell's equation, which is known by a theorem of Lagrange to have infinitely many integer solutions when D is not a perfect square (as is true when λ is real quadratic sPV). The solution $a_0, b_0 \in \mathbb{Z}$ with the smallest value $a_0 + b_0\sqrt{D} > 1$ corresponds to the fundamental unit u_D in $\mathbb{Z}[\sqrt{D}]$. Suppose $D = s^2 d$ where d is square-free. Since $\mathbb{Z}[\sqrt{D}] = \mathbb{Z}[s\sqrt{d}] \subseteq \mathbb{Z}[\sqrt{d}]$, and all units in $\mathbb{Z}[\sqrt{d}]$ are powers of a fundamental unit $u_d \in \mathbb{Z}[\sqrt{d}]$, it follows that the fundamental unit u_D of $\mathbb{Z}[\sqrt{D}]$ is a power of $u_d > 1$: for some $k > 0$, $u_D = u_d^k$, and the units of $\mathbb{Z}[\sqrt{D}]$ are powers of u_D. This fact is used in the next proof.

Given these considerations, in the next lemma we need only rely on the existence of arbitrarily large units in $\mathbb{Z}[\lambda]$, which usually coincide with neither λ nor $1-\lambda$ (so that λ need not be unitary). Otherwise the result and the proof are essentially those of a similar lemma in [23].

Lemma 15.3. *Let* λ *be a real quadratic sPV number. Suppose that* $R_\mu = \Sigma'_P$*. Then in the ring* $\mathbb{Z}[\lambda]$*,* $\mu \mid 2$ *and* $(1-\mu) \mid 2$*.*

Proof. For any unit $z \in \mathbb{Z}[\lambda]$, we know that it has an inverse $z^{-1} \in \mathbb{Z}[\lambda]$ so that $zz^{-1} = 1$. We may take z to be a power of a fundamental unit > 1, and $z^{-1} > 0$. Since $z, z^{-1} \in \mathbb{R}$ and $z > 1$, the equation $zz^{-1} = 1$ then implies that $z^{-1} \in (0,1)$. Furthermore, we choose z sufficiently large that $z^{-1} \in (0, 1/2)$. Let $u = z^{-1}$ denote this unit. Thus $u, 2u \in \mathbb{Z}[\lambda] \cap P = \Sigma'_P$, and Lemma 15.2 can be applied to $y = u$ and $y = 2u$.

If μ is a unit, then $\mu \mid 2$ and we are done. Suppose then that μ is not a unit, and that $\mu \nmid 2$. Since μ is not a unit, we also have $\mu \nmid u$. Now if $\mu \mid 2u$, given that u is a unit, we would have $2 = \mu y u^{-1}$, $u^{-1} \in \mathbb{Z}[\lambda]$, for some $y \in \mathbb{Z}[\lambda]$, so that $\mu \mid 2$, a contradiction. Thus $\mu \nmid 2$ and $\mu \nmid u$ also imply that $\mu \nmid 2u$.

Since $\mu \nmid 2u$, by Lemma 15.2 part (i), $\mu \mid (2u-1)$. From the fact that $\mu \nmid u$ we conclude from the same Lemma that $\mu \mid (u-1)$. Then $\mu \mid (2u - 1 - u + 1)$, i.e., $\mu \mid u$, a contradiction. Hence $\mu \mid 2$.

The exact same argument applies to $1 - \mu$, appealing to part (ii) of Lemma 15.2. □

Lemma 15.4. *For any real quadratic sPV number* λ *such that in the ring* $\mathbb{Z}[\lambda]$*,* $\mu \mid 2$ *and* $(1-\mu) \mid 2$ *(equivalently,* $\lambda \mid 2$ *and* $(1-\lambda) \mid 2$*), one of the following must hold:*

(a) $\lambda = -\varphi$ *or* $\lambda = 1 + \varphi$

(b) $\lambda = -1 - \sqrt{2}$ *or* $\lambda = 2 + \sqrt{2}$

(c) $\lambda = -1 - \sqrt{3}$ *or* $\lambda = 2 + \sqrt{3}$

(d) $\lambda = \frac{-3-\sqrt{17}}{2}$ *or* $\lambda = \frac{5+\sqrt{17}}{2}$

Proof. We consider $\lambda < 0$; the postive case is similar. Then by Corollary 13.5, λ and μ have the minimal polynomial $x^2 + mx - n$, for $0 < n \leq m$. Since $\mu \mid 2$, there are $a, b \in \mathbb{Z}$ such that $(a + b\mu)\mu = 2$. Given that $\mu^2 + m\mu - n = 0$, a little calculation gives the constraint $nb = 2$. This is only possible if $n = 1$ or $n = 2$.

Since we also have $(1 - \mu) \mid 2$, there are $c, d \in \mathbb{Z}$ such that $(c + d\mu)(1 - \mu) = 2$. Again, a little calculation yields $d = \frac{2}{m-n+1}$. Since $d \in \mathbb{Z}$, this can only hold if $m - n = 0$ or 1.

If $n = 1$, we have the unitary Pisot case, which is essentially treated in [23]. For the sake of completeness, we give the argument here: From $n = 1$ and the constraint $m - n = 0$ or 1, we conclude that $m = 1$ or $m = 2$. In the first case we obtain $\lambda = -\varphi$ (item (a)), and in the second, $\lambda = -1 - \sqrt{2}$ (item (b)).

Now suppose $n = 2$. Then $m = 2$ or $m = 3$. The former case corresponds to $\lambda = -1 - \sqrt{3}$, item (c). The latter case corresponds to $\lambda = \frac{-3-\sqrt{17}}{2}$, item (d). ☐

From Lemmas 15.3 and 15.4 we immediately get the following:

Lemma 15.5. *For any real quadratic sPV number λ such that $R_\mu = \Sigma'_P$ (equivalently, $R_\lambda = \Sigma_P$), one of the items (a), (b), (c), (d) of Lemma 15.4 must hold.*

In each of the items (a), (b), and (c) in Lemma 15.4, at least one of the two alternative values of λ is unitary Pisot. We next prove that $R_\lambda = \Sigma_P$ for these values of λ. This was first proved in [23] using the notion of β-expansions due to Rényi [31]. Here we include a self-contained proof that does not use β-expansions. The original method of Berman and Moody [4] can only be applied directly to $\lambda = -\varphi$ or $1 + \varphi$. We extend the technique for the other two values.

Theorem 15.6 (Masáková et al. [23])**.** *If λ or $1 - \lambda$ is as in Lemma 15.4 items (a), (b), or (c), then $R_\lambda = \Sigma_P$.*

Proof. We find it convenient to have $\mu > 1/2$. This can always be arranged either by using λ or $1 - \lambda$, and since $R_\lambda = R_{1-\lambda}$, there is no loss of generality. Furthermore, whenever we make this choice for the relevant λ's (Lemma 15.4), then $1 - \lambda$ is a unit. To be concrete, this dictates that we use $\lambda = 2 + \sqrt{2}$ or $\lambda = -1 - \sqrt{3}$. In those cases, λ is *not* a unit. Only when $\lambda = 1 + \varphi$ does it hold that *both* λ and $1 - \lambda$ are units; that is what makes that case easier.

For any set A, define $x \star A = \{x \star a \mid a \in A\}$, and similarly $A \star x = \{a \star x \mid a \in A\}$.

Our primary task is to prove that

$$\Sigma_P = 0 \star \Sigma_P \cup 1 \star \Sigma_P \cup \Sigma_P \star 0 \cup \Sigma_P \star 1. \tag{46}$$

This is a generalization of Berman and Moody's "replication" property. It says that any element in Σ_P can be obtained by extrapolation on the left or the right by 0 or 1 with another element in Σ_P. The main result then follows easily, as explained below.

We compute $0 \star \Sigma_P$, $1 \star \Sigma_P$, $\Sigma_P \star 0$, and $\Sigma_P \star 1$ in turn, starting with $0 \star \Sigma_P$:

$$\begin{aligned} 0 \star \Sigma_P &= \{0 \star x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{\lambda x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{\lambda x \mid \lambda x \in \lambda\mathbb{Z}[\lambda] \wedge (\lambda x)' \in \mu P\} \end{aligned}$$

Note at this point that, because λ is not necessarily a unit (unless $\lambda = -\varphi$ or $1 + \varphi$), the ideal $\lambda\mathbb{Z}[\lambda]$ is not equal to the ring $\mathbb{Z}[\lambda]$. We address this issue later. After changing variables $\lambda x \mapsto x$,

$$0 \star \Sigma_P = \{x \mid x \in \lambda\mathbb{Z}[\lambda] \wedge x' \in \mu P\}. \tag{47}$$

Note the important fact that $\mu P = [0, \mu]$.

Proceeding similarly,

$$\begin{aligned} 1 \star \Sigma_P &= \{1 \star x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{1 - \lambda + \lambda x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{1 - \lambda + \lambda x \mid 1 - \lambda + \lambda x \in 1 - \lambda + \lambda\mathbb{Z}[\lambda] \wedge (1 - \lambda + \lambda x)' \in 1 - \mu + \mu P\}, \end{aligned}$$

and hence,

$$1 \star \Sigma_P = \{x \mid x \in 1 - \lambda + \lambda\mathbb{Z}[\lambda] \wedge x' \in 1 - \mu + \mu P\}. \tag{48}$$

Observe that $1 - \mu + \mu P = [1 - \mu, 1]$. Further note that, because $\mu > 1/2$,

$$\mu P \cup (1 - \mu + \mu P) = [0, \mu] \cup [1 - \mu, 1] = [0, 1] = P.$$

It is crucial that the intervals μP and $1 - \mu + \mu P$ cover P.

Next we extrapolate Σ_P on the *right* with 0:

$$\begin{aligned} \Sigma_P \star 0 &= \{x \star 0 \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{(1 - \lambda)x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{(1 - \lambda)x \mid (1 - \lambda)x \in (1 - \lambda)\mathbb{Z}[\lambda] \wedge ((1 - \lambda)x)' \in (1 - \mu)P\}. \end{aligned}$$

Because $1 - \lambda$ *is* a unit, we have $(1 - \lambda)\mathbb{Z}[\lambda] = \mathbb{Z}[\lambda]$. Therefore,

$$\Sigma_P \star 0 = \{x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in (1 - \mu)P\}. \tag{49}$$

Observe that $(1 - \mu)P = [0, 1 - \mu]$ is a *proper* subset of $[0, 1/2]$ because of $\mu > 1/2$.

Finally we extrapolate Σ_P on the right with 1:

$$\begin{aligned} \Sigma_P \star 1 &= \{x \star 1 \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{\lambda + (1 - \lambda)x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in P\} \\ &= \{\lambda + (1 - \lambda)x \mid \lambda + (1 - \lambda)x \in \lambda + (1 - \lambda)\mathbb{Z}[\lambda] \wedge (\lambda + (1 - \lambda)x)' \in \mu + (1 - \mu)P\}. \end{aligned}$$

Using the fact that $\lambda + (1 - \lambda)\mathbb{Z}[\lambda] = \mathbb{Z}[\lambda]$,

$$\Sigma_P \star 1 = \{x \mid x \in \mathbb{Z}[\lambda] \wedge x' \in \mu + (1 - \mu)P\}, \tag{50}$$

Observe that $\mu + (1-\mu)P = [\mu, 1]$. Further note that, because $\mu > 1/2$,

$$(1-\mu)P \cup (\mu + (1-\mu)P) = [0, 1-\mu] \cup [\mu, 1] = P \setminus (1-\mu, \mu).$$

The fact that $(1-\mu)P$ and $\mu + (1-\mu)P$ fail to cover P in general necessitates extrapolation on the *left* as well as the right.

We now address the nature of the ideal $\lambda\mathbb{Z}[\lambda]$. From Lemma 15.3, we know that $\lambda \mid 2$. As mentioned previously, λ *is not a unit*, unless $\lambda = 1+\varphi$ or $-\varphi$. When $\lambda = 1+\varphi$ or $-\varphi$, λ is a unit, so $\lambda\mathbb{Z}[\lambda] = \mathbb{Z}[\lambda]$, and the problem goes away (more on this below). But in the other cases $N(\lambda) \neq 1$: For $\lambda = 2+\sqrt{2}$, $N(\lambda) = 2$, and for $\lambda = -1-\sqrt{3}$, $N(\lambda) = -2$. Now for any $x = a + b\lambda \in \mathbb{Z}[\lambda]$, we have $x = a + b\lambda \equiv a \pmod{\lambda}$. That is, $x \equiv k \pmod{\lambda}$ where k is an integer[16]. However, because $\lambda \mid 2$, we find that $2 \equiv 0 \pmod{\lambda}$. Hence for any $x \in \mathbb{Z}[\lambda]$, we have either $x \equiv 0 \pmod{\lambda}$ or $x \equiv 1 \pmod{\lambda}$. Hence the ideal $\lambda\mathbb{Z}[\lambda]$ partitions $\mathbb{Z}[\lambda]$ into two classes, one equivalent to 0 mod λ, one equivalent to 1 mod λ, and preserving addition and multiplication when $N(\lambda) = \pm 2$. Then $\mathbb{Z}[\lambda]/\lambda\mathbb{Z}[\lambda] \cong \mathbb{Z}_2$. As this is a field, $\lambda\mathbb{Z}[\lambda]$ is a maximal ideal; it is also possible to prove the latter fact directly. However, in this proof we really only need the partition of $\mathbb{Z}[\lambda]$ into the "0" class and the "1" class.

The fact that $\lambda\mathbb{Z}[\lambda]$ induces this partitioning (for $\lambda \neq -\varphi$ or $1+\varphi$) leads to the necessity of extrapolating on the *right* as well as the left.

We now turn to the task of establishing Eq. (46). First, because each is a subset of $\mathbb{Z}[\lambda]$, and the relevant convex sets (e.g., P, μP, etc.) are contained in P, it is clear that each of $0 \star \Sigma_P$, $1 \star \Sigma_P$, $\Sigma_P \star 0$, $\Sigma_P \star 1$ are contained in Σ_P. For the reverse containment, consider any $x \in \Sigma_P$. There are three cases:

(i) $x' \in [0, 1-\mu]$: By Eq. (49), $\Sigma_P \star 0$ contains all $x \in \mathbb{Z}[\lambda]$ such that $x' \in [0, 1-\mu]$. Thus this implies that $x \in \Sigma_P \star 0$.

(ii) $x' \in [\mu, 1]$: By Eq. (50), $\Sigma_P \star 1$ contains all $x \in \mathbb{Z}[\lambda]$ such that $x' \in [\mu, 1]$. Thus this implies that $x \in \Sigma_P \star 1$.

(iii) $x' \in (1-\mu, \mu)$: By the partition of $\mathbb{Z}[\lambda]$, we have that either $x \equiv 0 \pmod{\lambda}$ or $x \equiv 1 \pmod{\lambda}$. If the former, then $x \in \lambda\mathbb{Z}[\lambda]$, and hence (since $x' \in \mu P = [0, \mu]$), by Eq. (47), we have $x \in 0 \star \Sigma_P$. If the latter, then $x \in 1 - \lambda + \lambda\mathbb{Z}[\lambda]$, and hence (since $x' \in 1 - \mu + \mu P = [1-\mu, 1]$), by Eq. (48), we have $x \in 1 \star \Sigma_P$.

This establishes Eq. (46). If $\lambda = 1+\varphi$, we can take $\mu < 1/2$, and we would only have cases (i) and (ii) above. That is, for $\lambda = 1+\varphi$, we find $\Sigma_P = \Sigma_P \star 0 \cup \Sigma_P \star 1$.

The proof is completed by an induction to show that $\Sigma_P \subseteq R_\lambda$. We first consider $\lambda = 2+\sqrt{2}$ and $-1-\sqrt{3}$. For these values, $|\lambda|, |1-\lambda| > 2$. Let x be any element of Σ_P. By Equation (46), there exists $y \in \Sigma_P$ such that $x \in \{0 \star y, 1 \star y, y \star 0, y \star 1\}$. Suppose for the moment that $|x| > \max(|\lambda|, |1-\lambda|)$. There are four cases:

1. If $x = 0 \star y$, then $x = \lambda y$ and $|\lambda| \cdot |y| = |x|$, hence $|y| = \frac{|x|}{|\lambda|} < |x|$.
2. If $x = 1 \star y$, then $x = 1 - \lambda + \lambda y$, so $\lambda y = x - (1-\lambda)$, and $|\lambda| \cdot |y| \leq |x| + |1-\lambda| < 2|x|$. Thus $|y| < \frac{2}{|\lambda|}|x| < |x|$.

[16] To clarify, the mod λ notation means equivalence up to λ-multiples of elements of $\mathbb{Z}[\lambda]$. Thus a more precise (but also more cumbersome) notation would be $x \equiv k \pmod{\lambda\mathbb{Z}[\lambda]}$.

3. If $x = y \star 0$ then $x = (1-\lambda)y$, and $|y| = \frac{|x|}{|1-\lambda|} < |x|$.

4. If $x = y \star 1$, then $x = (1-\lambda)y + \lambda$. This implies $(1-\lambda)y = x - \lambda$, and hence $|y| \leq \frac{|x|+|\lambda|}{|1-\lambda|} < \frac{2|x|}{|1-\lambda|} < |x|$.

Thus for any $x \in \Sigma_P$ with $|x| > \max(|\lambda|, |1-\lambda|)$, we obtain a $y \in \Sigma_P$ with norm less than $|x|$ which yields x under extrapolation with 0 or 1 on the left or right (and thus $y \in R_\lambda$ implies $x \in R_\lambda$). Since Σ_P is uniformly discrete, we can repeat this process a finite number of times to reduce to a $y \in \Sigma_P$ with $|y| \leq \max(|\lambda|, |1-\lambda|)$. It only remains to verify that the finite number of $y \in \Sigma_P$ satisfying $|y| \leq \max(|\lambda|, |1-\lambda|)$ are in R_λ. But these are exactly the points $0, 1, 1-\lambda$, and λ itself, all of which are in R_λ.

For $\lambda = 1 + \varphi$, while $|\lambda| = 1 + \varphi > 2$, we have $1 < |1-\lambda| = \varphi < 2$. Thus the above argument doesn't work. This case is still simpler, however, since we only need cases (i) and (ii) above. This is Berman and Moody's argument, which we review here. We consider $x \in \Sigma_P$ with $|x| > |1-\lambda|^3 = \varphi^3$. If $x = y \star 0$, then $x = (1-\lambda)y$, so $|y| = \frac{|x|}{|1-\lambda|} < |x|$. If $x = y \star 1$, then $x = (1-\lambda)y + \lambda$ and $|y| \leq \frac{|x|+|\lambda|}{|1-\lambda|} = \frac{|x|+1+\varphi}{\varphi} < \frac{1}{\varphi}\left(1 + \frac{1+\varphi}{\varphi^3}\right)|x|$. Now observe that,

$$\frac{1}{\varphi}\left(1 + \frac{1+\varphi}{\varphi^3}\right) = \frac{\varphi^3 + \varphi + 1}{\varphi^4} = \frac{\varphi^3 + \varphi^2}{\varphi^2(\varphi+1)} = \frac{\varphi^2(\varphi+1)}{\varphi^2(\varphi+1)} = 1,$$

so $|y| < |x|$. As before, we reduce to a $y \in \Sigma_P$ with norm $\leq \varphi^3 = 1 + 2\varphi$. The relevant points in Σ_P are $0, 1, \lambda, 1-\lambda$ and $-1-\lambda = \lambda \star 0$, all of which are in R_λ. $\square$

The foregoing proof requires either λ or $1-\lambda$ to be a unit. This does not hold for $\lambda = \frac{-3-\sqrt{17}}{2}$, and hence this requires a different technique (although still not relying on β-expansions), developed in Section 15.2. It turns out that $R_\lambda = \Sigma_P$ for $\lambda = \frac{-3-\sqrt{17}}{2}$ as well (see Proposition 15.18 in Section 15.2.4 below). Thus from Lemma 15.5, Theorem 15.6, and Proposition 15.18, the following is immediate:

Theorem 15.7. *If λ is a real quadratic sPV number, then $R_\lambda = \Sigma_P$ if and only if λ is one of the cases (a), (b), (c), or (d) of Lemma 15.4.*

This can be further strengthened. In [30, Theorem 15], Pinch shows that if λ is real and not an integer, and R_λ is discrete, then λ has a conjugate in $(0,1)$. Hence if λ is real and quadratic and R_λ is discrete, we can conclude that λ is sPV. Conversely, we know that if λ is sPV, then R_λ is discrete. Thus for real, quadratic λ, R_λ is discrete iff λ is sPV. We thus have the following corollary of Theorem 15.7 (recall that if $\lambda \in \mathbb{R}$ and R_λ is discrete, then λ is an algebraic integer by Theorem 8.1):

Corollary 15.8. *If λ is real and quadratic, and R_λ is discrete, then $R_\lambda = \Sigma_P$ if and only if λ is one of the cases (a), (b), (c), or (d) of Lemma 15.4.*

As per the discussion just after Definition 15.1, Theorem 15.7 also implies the following immediately:

Corollary 15.9 (Masáková et al. [23])**.** *If λ is unitary Pisot, then $R_\lambda = \Sigma_P$ iff λ is one of the cases (a), (b), and (c) of Lemma 15.4.*

15.2 The Covering Method and its Applications

The underlying strategy of the proof of Theorem 15.6 is to find a finite cover of the interval $P = [0, 1]$ (in that case, simply the intervals $[0, 1-\mu]$, $[1-\mu, \mu]$, $[\mu, 1]$), such that for any $x \in \Sigma_P$, depending on which element of the cover x' is contained in, we can write x as $b \star_\lambda y$ or $y \star_\lambda b$, for $b \in \{0, 1\}$. Since, in all of these four cases, $|y| < |x|$, an inductive proof then suffices to show that $\Sigma_P \subseteq R_\lambda$.

In this section we generalize that proof, which yields further relative density results, including for case (d) of Theorem 15.7, $\lambda = \frac{-3-\sqrt{17}}{2}$. That case is covered in Proposition 15.18 below.

In particular, the technique allows us to use a fixed set of "seed" points other than $\{0, 1\}$. Indeed, it turns out that when Σ_P has a unit, Theorem 15.10 below shows that it is always possible to adjoin finitely many points Y in Σ_P to R_λ such that it equals Σ_P, i.e., for some $Y \subseteq \Sigma_P$, $\Sigma_P = R_\lambda(Y)$. Thus, if we can explicitly find such a set Y that is a subset of R_λ, then R_λ itself is cut and project: $\Sigma_P = R_\lambda$. However, the result yields more than relative density results. As we will see later on, even when $Y \not\subseteq R_\lambda$, the results of this section enable us to prove that certain elements of Σ_P are *not* in R_λ, again in certain cases.

The proof of Theorem 15.10 extends that of Theorem 15.6 in two ways. First, the size of the cover of P is larger, but still finite (depending on the size of the fundamental unit in Σ_P). Secondly, the proof that $\Sigma_P \subseteq R_\lambda$ proceeds by inducting on $|z|$ as before, but now a larger finite set Y of seed points plays the role that $\{0, 1\}$ played in the proof of Theorem 15.6. In one respect, the induction is simpler, since we find z can be written as $x \star_\lambda y$ where $x \in Y$ and hence (as shown below) $|z| > |y|$. That is, z is the extrapolant of one of a finite number of seed points on the *left only* with a smaller point on the *right*.

15.2.1 Quadratics and Non-Real Cubics

We begin by considering quadratic sPV numbers and non-real cubic (degree 3) sPV numbers. In the latter case—as with the real quadratic case—λ has a unique conjugate $\mu \in (0, 1)$. It also has λ^* as its other conjugate. In this section, we again let $x \mapsto x'$ be the unique field isomorphism[17] $\mathbb{Q}(\lambda) \to \mathbb{Q}(\mu)$ that maps λ to μ, and we define $P := [0, 1]$ and $\Sigma_P = \Sigma_{[0,1]} := \{x \in \mathbb{Z}[\lambda] \mid x' \in [0, 1]\}$ as before. So if $\lambda \notin \mathbb{R}$, we have

$$\Sigma_P = \{a + b\lambda + c\lambda^2 \mid a, b, c \in \mathbb{Z} \ \ \& \ \ 0 \le a + b\mu + c\mu^2 \le 1\}\,. \tag{51}$$

Since Σ_P is a cut-and-project (model) set, it is relatively dense in $\mathbb{C} \cong \mathbb{R}^2$.

The next theorem shows that, for any sPV λ with a unique conjugate in $(0, 1)$, if there exists an $\alpha \in \Sigma_P$ that is a unit of $\mathbb{Z}[\lambda]$ other than 1, then there exists a finite set $Y \subseteq \Sigma_P$ such that $R_\alpha(Y) = \Sigma_P$. If, in addition, $\alpha \in R_\lambda$, then $R_\lambda(Y) = \Sigma_P$. The theorem also gives sufficient conditions on Y such that $R_\alpha(Y) = \Sigma_P$. These conditions are sometimes not completely necessary, however, as one can often get by with a smaller "seed" set Y. This will be true, for example, in the case where $\lambda = -(3 + \sqrt{13})/2$ (Proposition 15.32, below). This theorem is our main tool for proving relative density of R_λ for such λ.

Theorem 15.10. *Let λ be a strong PV number with a unique conjugate in $(0, 1)$ (which implies λ is either real quadratic or non-real cubic). Let $P := [0, 1]$ and $\Sigma_P := \{x \in \mathbb{Z}[\lambda] \mid x' \in [0, 1]\}$, where the map $x \mapsto x'$ is as defined above. Suppose Σ_P contains a unit α of $\mathbb{Z}[\lambda]$ other than 1. Let*

[17] If $\lambda \notin \mathbb{R}$, then $\mathbb{Q}(\lambda) \neq \mathbb{Q}(\mu)$, and thus this map is not an automorphism as in the real quadratic case.

$\beta := \alpha' \in P$. Then $|\alpha| > 1,\ \ 0 < \beta < 1$, and there exists a finite set $X \subseteq \Sigma_P$ such that

$$[0,1] \subseteq \bigcup_{x \in X} \left[(1-\beta)x', (1-\beta)x' + \beta\right] . \tag{52}$$

Moreover, for any such X, letting

$$M := \left(\frac{|\alpha - 1|}{|\alpha| - 1}\right) \max_{x \in X} |x| , \tag{53}$$

the set

$$Y := \Sigma_P \cap [-M, M] \tag{54}$$

is finite, and $R_\alpha(Y) = \Sigma_P$. If, in addition, $\alpha \in R_\lambda$, then $R_\lambda(Y) = \Sigma_P$.

Proof. Observe that $\beta \in (0,1)$, since $\alpha \in \Sigma_P$ and $\alpha \notin \{0,1\}$. This in turn implies $\alpha \notin \mathbb{Z}$, and thus, since λ has prime degree and $\alpha \in \mathbb{Z}[\lambda]$, it must be that α has the same degree as λ, and by standard facts of algebra, the conjugates of α are α, β, and α^* (see, e.g., Corollary A.4). If λ is real quadratic, then $\alpha \in \mathbb{R}$ and so $\alpha = \alpha^*$ making α real quadratic. If λ is non-real qubic, then α is also qubic, whence $\alpha \neq \alpha^*$ and so $\alpha \notin \mathbb{R}$. Since α is a unit, we have $N(\alpha) = \pm 1$. If $\alpha \in \mathbb{R}$, then $N(\alpha) = \alpha\beta$, and $N(\alpha) = \alpha\alpha^*\beta$ otherwise. Thus $\beta = 1/|\alpha|$ if $\alpha \in \mathbb{R}$ and $\beta = 1/|\alpha|^2$ otherwise. In either case, we have $|\alpha| > 1$.

Also observe that Σ_P is α-convex: for any $u, v \in \Sigma_P,\ \ u \star_\alpha v \in \mathbb{Z}[\lambda]$ and $(u \star_\alpha v)' = u' \star_\beta v' \in P$ (because $u', v', \beta \in P$). Then since $Y \subseteq \Sigma_P$, we have $R_\alpha(Y) \subseteq \Sigma_P$. It therefore suffices for the first part to show that $\Sigma_P \subseteq R_\alpha(Y)$.

There exists a finite X satisfying Equation (52), because Σ'_P is dense in $[0,1]$.[18] (Notice, however, that X must contain both 0 and 1.) From the fact that $|\beta| < 1$ and either $|\alpha||\beta| = 1$ or $|\alpha|^2|\beta| = 1$, we must have $|\alpha| > 1$. Then M is well-defined, and in addition, $X \subseteq Y$ because $|\alpha - 1|/(|\alpha| - 1) \geq 1$.

We show for any $z \in \Sigma_P$ that z is in $R_\alpha(Y)$. We do this by induction on $|z|$, which is allowed, because Σ_P is uniformly discrete. If $|z| \leq M$, then $z \in Y$ and we are done, so suppose $|z| > M$. Choose $x \in X$ such that $z' \in [(1-\beta)x', (1-\beta)x' + \beta]$, and let $y := x \star_{1/\alpha} z$. One can quickly check that $x \star_\alpha y = z$. Moreover, $y \in \mathbb{Z}[\lambda]$, since α is a unit in $\mathbb{Z}[\lambda]$. We next show that $|y| < |z|$. The condition $|y| < |z|$ is equivalent to

$$\left|\left(1 - \frac{1}{\alpha}\right) x + \frac{1}{\alpha} z\right| < |z| ,$$

This holds provided $|(1-1/\alpha)x| + |z/\alpha| < |z|$, or equivalently, $|\alpha - 1||x| + |z| < |\alpha||z|$. Since $x \in X$, we have $|x| \leq M(|\alpha| - 1)/|\alpha - 1|$, and so the inequality is satisfied.

It remains to show that $0 \leq y' \leq 1$, thus putting y in Σ_P. This suffices: applying the inductive hypothesis to y to get $y \in R_\alpha(Y)$, we get $z = x \star_\alpha y \in R_\alpha(Y)$ as desired. We have

$$y' = (x \star_{1/\alpha} z)' = x' \star_{1/\beta} z' = \frac{1}{\beta}\left((\beta - 1)x' + z'\right) .$$

Since $z' \in [(1-\beta)x', (1-\beta)x' + \beta]$, we then have

$$\frac{1}{\beta}\left((\beta - 1)x' + (1-\beta)x'\right) \leq y' \leq \frac{1}{\beta}\left((\beta - 1)x' + (1-\beta)x' + \beta\right) ;$$

[18] See the discussion following this proof for bounds on how big X needs to be in the case where λ is real quadratic.

that is, $0 \leq y' \leq 1$. Thus by induction, we get $\Sigma_P \subseteq R_\alpha(Y)$, and hence, $\Sigma_P = R_\alpha(Y)$.

If $\alpha \in R_\lambda$, then by Lemma 2.25 and the fact that Σ_P is λ-convex,

$$\Sigma_P = R_\alpha(Y) \subseteq R_\lambda(Y) \subseteq \Sigma_P \ ,$$

and thus all sets above are equal. $\square$

Remark. We have proved something stronger. Define $Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots$ inductively as follows: $Y^{(0)} := Y$ and $Y^{(n+1)} := X \star_\alpha Y^{(n)}$ for $n \geq 0$. Then the proof shows that $\Sigma_P = \bigcup_{n=0}^{\infty} Y^{(n)}$. This was shown for the case where $\lambda = \alpha = -\varphi$ by Berman & Moody [4]. $\square$

If λ is real quadratic, we can get sufficient bounds on the size of X based on results in the theory of uniformly distributed distributions (see, for example, Allouche & Shallit [2]). The following concepts and most of the following facts are from [2, Chapter 2]. Let $[a_0, a_1, a_2, \ldots]$ be the continued fraction expansion of $\mu := \lambda'$ (noting that $a_0 = 0$).[19] For $i \geq 0$, define $A_i := \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix}$, and for $k \geq -1$ define

$$\begin{bmatrix} p_k \\ q_k \end{bmatrix} := A_0 A_1 \cdots A_k \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{55}$$

The q_k satisfy the recurrence $q_{k+1} = a_{k+1} q_k + q_{k-1}$ and form a strictly increasing sequence that is bounded below by the Fibonacci sequence; also, $\lim_{k\to\infty} p_k/q_k = \mu$. Letting

$$\eta_k := (-1)^k (q_k \mu - p_k) \ , \tag{56}$$

one can show that $\eta_k > 0$ for all $k \geq -1$ and that $\eta_{k+1} < \eta_k$ for all $k \geq 1$. In fact, $1/q_{k+2} < \eta_k < 1/q_{k+1}$ for all $k \geq 1$.

Then we have

$$\Sigma'_P = \{1\} \cup \{\lceil b\mu \rceil - b\mu \mid b \in \mathbb{Z}\} = \{1\} \cup \{\{a\mu\} \mid a \in \mathbb{Z}\} \ ,$$

where $\{a\mu\} := a\mu - \lfloor a\mu \rfloor$ is the fractional part of $a\mu$. (To see this, set $a := -b$.) For integer $n \geq 0$, let

$$X_n := \{1\} \cup \{\lceil b\mu \rceil - b\lambda \mid b \in \mathbb{Z} \ \& \ -n \leq b \leq 0\} \subseteq \Sigma_P \ .$$

Then $X'_n = \{1\} \cup \{\{a\mu\} \mid a \in \mathbb{Z} \ \& \ 0 \leq a \leq n\}$. Much is known about sets of this form for $n \geq 0$. By the three-distance theorem (see [2, Section 2.6]), there are at most three possible distances between adjacent points of X'_n—the smallest distance is η_k, where $k \geq -1$ is largest such that $q_k \leq n$ [2, Theorem 2.6.2]; the largest distance d_n satisfies the bound

$$d_n \leq \eta_k + \eta_{k-1} \ , \tag{57}$$

where $k \geq 0$ is largest such that $q_k \leq n + 1$ [2, Theorem 2.6.3].

We now consider choosing X to be X_n for n sufficiently large so that Equation (52) is satisfied. The intervals on the right cover $[0, 1]$ just when there are no gaps between adjacent intervals. That is, for any $x, y \in X_n$ such that $x' < y'$ are adjacent in X'_n, is suffices that $(1-\beta)x' + \beta \geq (1-\beta)y'$,

[19] Generally speaking, $a_0 = \lfloor \mu \rfloor$ and $\mu = a_0 + 1/[a_1, a_2, \ldots]$, etc.

or equivalently, $y' - x' \le \beta/(1-\beta)$. Thus it suffices to choose n so that $d_n \le \beta/(1-\beta)$, and by Equation (57) this will be true if

$$\eta_k + \eta_{k-1} \le \frac{\beta}{1-\beta}\ , \tag{58}$$

where $k \ge 0$ is largest such that $q_k \le n+1$. Thus we first choose the least k satisfying (58), then we set $n := q_k - 1$.

Letting X be X_n, although sufficient, is not an optimal choice, because it does not minimize $\max_{x\in X}|x|$ and thus the value of M in Equation (53). To minimize M, notice that any set of the form $W_{m,n} := \{1\} \cup \{\lceil b\mu\rceil - b\lambda \mid b \in \mathbb{Z}\ \&\ m \le b \le m+n\}$ for $m \in \mathbb{Z} \cap [-n, 0]$ works just as well as X_n in covering $[0,1]$. This is because $W_{m,n}$ is the image of X_n under the cyclic shift permutation $c : [0,1] \to [0,1]$ defined for $x \in [0,1]$ by

$$c(x) = \begin{cases} 1 & \text{if } x = 1, \\ (x + \{m\mu\}) \bmod 1 & \text{if } x < 1. \end{cases}$$

which preserves the set of distances between adjacent points in X_n versus those in $W_{m,n}$. This means that, given our choice of n above, we can set $X := W_{m,n}$ where m minimizes $\max_{x\in X}|x| = \max_{x\in W_{m,n}}|x|$. If $\lambda > 0$, then one can verify that $m = \lceil -n/2\rceil$, and likewise if $\lambda < 0$, then $m = \lfloor -n/2\rfloor$. In the former case,

$$\max_{x\in X}|x| = \begin{cases} \lambda n/2 - \lfloor \mu n/2\rfloor & \text{if } n \text{ is even and } n > 0, \\ \lambda(n+1)/2 - \lceil \mu(n+1)/2\rceil & \text{if } n \text{ is odd.} \end{cases} \tag{59}$$

In the latter case,

$$\max_{x\in X}|x| = \begin{cases} \lceil \mu n/2\rceil - \lambda n/2 & \text{if } n \text{ is even and } n > 0, \\ \lfloor \mu(n+1)/2\rfloor - \lambda(n+1)/2 & \text{if } n \text{ is odd.} \end{cases} \tag{60}$$

(Verifying the above is made easier by noting that $\lambda \notin [-1, 2]$.) In either case, if $n = 0$, then $W_{m,n} = \{0, 1\}$, so $\max_{x\in X}|x| = 1$.

We summarize the foregoing in the following proposition.

Proposition 15.11. *In Theorem 15.10 where λ is real quadratic, it suffices to let*

$$X := \{1\} \cup \{\lceil b\mu\rceil - b\lambda \mid b \in \mathbb{Z}\ \&\ m \le b \le m+n\}\ ,$$

where $n = q_k - 1$ for the least k satisfying (58) for η_k defined by (56) and q_k defined by (55) based on the continued fraction expansion of λ', and $m = \lceil -n/2\rceil$ if $\lambda > 0$ and $m = \lfloor -n/2\rfloor$ otherwise. For this X, the value $\max_{x\in X}|x|$ is given by Equation (59) or (60), provided $n > 0$.

15.2.2 Higher Degrees

Theorem 15.10 is specific to quadratic sPV's or non-real sPV's of degree 3. In this section we turn to real sPV's of arbitrary degree, real or complex.

For any subset R of a topological space, we let R^{int} denote the interior of R.

Fix a strong PV number λ with k many conjugates $0 < \mu_0 < \mu_1 < \cdots < \mu_{k-1} < 1$ in $(0,1)$. Here we assume that λ is nontrivial, so that $k \ge 1$. (We handle the case of trivial sPV λ in Proposition 15.46, below). For any $z \in \mathbb{R}^k$ and $0 \le i < k$, we let z_i be the $(i+1)^{st}$ component of z.

Particularly, for any $x \in \mathbb{Z}[\lambda]$ and $0 \le i < k$, we let x'_i denote the conjugate of x that is the image of x under the unique ring homomorphism $\mathbb{Z}[\lambda] \to \mathbb{Z}[\mu_i]$ mapping λ to μ_i (cf. Definition 12.26), and so we let $x' := (x'_0, x'_1, \ldots, x'_{k-1})$, viewed as a column vector. We also let $\Lambda_x := \operatorname{diag}(x'_0, x'_1, \ldots, x'_{k-1})$, the diagonal matrix with the x'_i's along the diagonal. Set $P := [0,1]^{\times k}$ and for any $S \subseteq \mathbb{R}^k$ let Σ_S denote $\Sigma_S^{(\lambda)}$, recalling that this is $\{x \in \mathbb{Z}[\lambda] \mid x' \in S\}$.

The following generalizes Theorem 15.10, but with any sub-box S of P that can be arbitrarily close to, but never equals, P.

Theorem 15.12. *Suppose Σ_P contains an $\alpha \neq 1$ that is a unit of $\mathbb{Z}[\lambda]$. Fix any $0 < \varepsilon < 1/2$, and define $S := [\varepsilon, 1-\varepsilon]^{\times k}$. There exists a finite set $Y \subseteq \Sigma_P$ such that $\Sigma_S \subseteq Q_\alpha(Y)$. If, in addition, $\alpha \in Q_\lambda$, then $\Sigma_S \subseteq Q_\lambda(Y)$.*

Proof. This proof follows the general approach of that of Theorem 15.10. Since $\alpha \in \Sigma_P$, we have $0 < \alpha'_i < 1$ for all $0 \le i < k$. Since α is a unit, $1/\alpha \in \mathbb{Z}[\lambda]$, and $(1/\alpha)' = (1/\alpha'_0, 1/\alpha'_1, \ldots, 1/\alpha'_k)$. Set $\Lambda := \Lambda_\alpha = \operatorname{diag}(\alpha'_0, \ldots, \alpha'_{k-1})$. For any $x \in \mathbb{R}^k$, define

$$J_x := x \star_\Lambda S = \{z \in \mathbb{R}^k \mid (\exists y \in S)\ z = x \star_\Lambda y\} = \{z \in \mathbb{R}^k \mid x \star_{\Lambda^{-1}} z \in S\} .$$

For the last equality, one can readily check that for all $x, y, z \in \mathbb{R}^k$, if $x \star_\Lambda y = z$ then $y = x \star_{\Lambda^{-1}} z$ and conversely.

Claim 15.13. *For all $x \in \mathbb{R}^k$, $J_x = \Big[x_0 \star_{\alpha'_0} \varepsilon,\ x_0 \star_{\alpha'_0} (1-\varepsilon)\Big] \times \cdots \times \Big[x_{k-1} \star_{\alpha'_{k-1}} \varepsilon,\ x_{k-1} \star_{\alpha'_{k-1}} (1-\varepsilon)\Big]$.*

Proof. For $z \in \mathbb{R}^k$, set $y := x \star_{\Lambda^{-1}} z$, so that $z = x \star_\Lambda y$. Then we have $z \in J_x$ if and only if $x \star_\Lambda y \in J_x$, if and only if $y \in S$ (since $\star_\Lambda$ is one-to-one in its second argument), if and only if $\varepsilon \le y_i \le 1-\varepsilon$ for all $0 \le i < k$, if and only if $x_i \star_{\alpha'_i} \varepsilon \le x_i \star_{\alpha'_i} y_i \le x_i \star_{\alpha'_i} (1-\varepsilon)$ for all i (since $\star_{\alpha'_i}$ is monotone increasing in its second argument), if and only if $x_i \star_{\alpha'_i} \varepsilon \le z_i \le x_i \star_{\alpha'_i} (1-\varepsilon)$ for all i, if and only if z belongs to the right-hand side. ☐

Note then that $J_x^{\mathrm{int}} = \Big(x_0 \star_{\alpha'_0} \varepsilon,\ x_0 \star_{\alpha'_0} (1-\varepsilon)\Big) \times \cdots \times \Big(x_{k-1} \star_{\alpha'_{k-1}} \varepsilon,\ x_{k-1} \star_{\alpha'_{k-1}} (1-\varepsilon)\Big)$.

For $z \in \mathbb{R}^k$, define

$$J_z^{-1} := \{x \in \mathbb{R}^k \mid z \in J_x^{\mathrm{int}}\} .$$

For any $x \in \mathbb{R}^k$, we have $x \in J_z^{-1}$ if and only if $z \in J_x^{\mathrm{int}}$, if and only if $x_i \star_{\alpha'_i} \varepsilon < z_i < x_i \star_{\alpha'_i} (1-\varepsilon)$ for all i, by the claim. That is, $x \in J_z^{-1}$ if and only if, for all $0 \le i < k$,

$$(1-\alpha'_i)x_i + \alpha'_i\varepsilon < z_i < (1-\alpha'_i)x_i + \alpha'_i(1-\varepsilon) .$$

Solving for x_i, this pair of inequalities is seen to be equivalent to

$$\frac{z_i - \alpha'_i(1-\varepsilon)}{1-\alpha'_i} < x_i < \frac{z_i - \alpha'_i\varepsilon}{1-\alpha'_i}$$

or alternatively,

$$(1-\varepsilon) \star_{1/(1-\alpha'_i)} z_i < x_i < \varepsilon \star_{1/(1-\alpha'_i)} z_i ,$$

Thus we obtain

$$J_z^{-1} = \Big((1-\varepsilon) \star_{1/(1-\alpha'_0)} z_0,\ \varepsilon \star_{1/(1-\alpha'_0)} z_0\Big) \times \cdots \times \Big((1-\varepsilon) \star_{1/(1-\alpha'_{k-1})} z_{k-1},\ \varepsilon \star_{1/(1-\alpha'_{k-1})} z_{k-1}\Big) .$$

Remark. Letting $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^k$, we see that J_z^{-1} is an open box with opposite corners $(1-\varepsilon)\mathbf{1} \star_{(I-\Lambda)^{-1}} z$ and $\varepsilon\mathbf{1} \star_{(I-\Lambda)^{-1}} z$. $\square$

Claim 15.14. *$J_z^{-1} \cap P^{\text{int}} \neq \emptyset$ for all $z \in S$.*

Proof. It suffices to show for every $0 \leq i < k$ that if $\varepsilon \leq z_i \leq 1 - \varepsilon$, then

$$(1-\varepsilon) \star_{1/(1-\alpha_i')} z_i < 1 \quad \text{and} \quad \varepsilon \star_{1/(1-\alpha_i')} z_i > 0 \, .$$

Fixing i and letting $\beta := \alpha_i'$, we get the following chain of equivalences for the first inequality above:

$$\begin{aligned}\left(1 - \frac{1}{1-\beta}\right)(1-\varepsilon) + \frac{1}{1-\beta} z_i &< 1 \\ z_i - \beta(1-\varepsilon) &< 1 - \beta \\ z_i &< 1 - \beta\varepsilon \, ,\end{aligned}$$

and the last inequality is certainly true if $z_i \leq 1 - \varepsilon$ (because $\beta < 1$). Similarly, for the second inequality:

$$\begin{aligned}\left(1 - \frac{1}{1-\beta}\right)\varepsilon + \frac{1}{1-\beta} z_i &> 0 \\ z_i - \beta\varepsilon &> 0 \\ z_i &> \beta\varepsilon \, ,\end{aligned}$$

which is again true if $z_i \geq \varepsilon$. $\square$

We know that Σ_P' is dense in P (and in fact, $Q_\alpha' = Q_\Lambda(\{0, \mathbf{1}\})$ is dense in P by Lemma 12.22). For every $z \in S$, choose an element x_z of $J_z^{-1} \cap \Sigma_P'$ in some standard way. Such an element exists by density and because $J_z^{-1} \cap P^{\text{int}}$ is open and nonempty by Claim 15.14. Then $z \in {J_{x_z}}^{\text{int}}$ for all $z \in S$, meaning that the family $\{{J_{x_z}}^{\text{int}} \mid z \in S\}$ is an open cover of S. Since S is compact, we may choose a finite subcover, indexed by a finite set $X \subseteq \Sigma_P$ so that $\{{J_{x'}}^{\text{int}} \mid x \in X\}$ covers S.

Remark. It is only necessary that S be covered by the $J_{x'}$ themselves, not their interiors. The interiors were only used to apply the compactness argument. Thus one may be able to find a smaller set X such that the $J_{x'}$ cover S. $\square$

The rest of the proof is very close to that of Theorem 15.10, with only some subtle differences. An argument similar to the one used in the proof of that theorem shows that $|\alpha| > 1$. Let

$$Y := X \cup \{y \in \Sigma_S : |y| \leq M\} \, ,$$

where

$$M := \left(\frac{|\alpha - 1|}{|\alpha| - 1}\right) \max_{x \in X} |x| \, .$$

Y is certainly finite, because Σ_P is uniformly discrete. We now apply Lemma 15.15, given below, to obtain the first part of the theorem.

If $\alpha \in Q_\lambda$, then $Q_\alpha(Y) \subseteq Q_\lambda(Y)$ by Lemma 2.25, and the rest of the theorem follows. $\square$

Remark. There is a possible trade-off between the choice of X versus Y. It is not necessary that $X \subseteq Y$, only that $X \subseteq Q_\alpha(Y)$. So, for example, if we choose X so that $X \subseteq Q_\alpha$ (which is always possible by Lemma 12.22), then we can just let $Y := \{y \in \Sigma_S : |y| \leq M\}$ (albeit with a possibly bigger value of M). $\square$

Remark. Theorems 15.6 and 15.10 use essentially the same technique to induct on $|z|$. We abstract that process in the following Lemma, which we applied in Theorem 15.12, and will also be used in subsequent case studies. $\square$

Lemma 15.15. *Let λ, P, S, α, and k be as in Theorem 15.12. Suppose there exists a finite set $X \subseteq \mathbb{Z}[\lambda]$ such that, setting $\Lambda := \operatorname{diag}(\alpha'_0, \ldots, \alpha'_{k-1})$ and $J_{x'} := x' \star_\Lambda S$ for each $x \in X$, it holds that $S \subseteq \bigcup_{x \in X} J_{x'}$. Let $Y = X \cup \{y \in \Sigma_S : |y| \leq M\}$, where $M = \frac{|\alpha-1|}{|\alpha|-1} \max_{x \in X} |x|$. Then $\Sigma_S \subseteq Q_\alpha(Y)$.*

Proof. We show for any $z \in \Sigma_S$ that z is in $Q_\alpha(Y)$. We do this by induction on $|z|$ as in Theorem 15.10. The only difference here is that, due to the arbitrary degree of λ, there are multiple conjugates. If $|z| \leq M$, then $z \in Y \subseteq Q_\alpha(Y)$ and we are done, so suppose $z \in \Sigma_S$ with $|z| > M$. By definition of Σ_S, $z' \in S$. Since $\{J_{x'} | x \in X\}$ covers S, for some $x \in X$ we have $z' \in J_{x'}$. Let $y := x \star_{1/\alpha} z$, so that $x \star_\alpha y = z$. As before, since α is a unit in $\mathbb{Z}[\lambda]$, we have $y \in \mathbb{Z}[\lambda]$. We also have that $|y| < |z|$ by an argument identical to the one for Theorem 15.10, repeated here in more detail:

$$\begin{aligned}
|y| &= \left| \left(1 - \frac{1}{\alpha}\right) x + \frac{1}{\alpha} z \right| \\
&\leq \left| 1 - \frac{1}{\alpha} \right| \cdot |x| + \frac{|z|}{|\alpha|} \\
&= \frac{1}{|\alpha|} \left(|\alpha - 1| \cdot |x| + |z| \right) \\
&\leq \frac{1}{|\alpha|} \left(|\alpha - 1| \cdot \frac{|\alpha| - 1}{|\alpha - 1|} M + |z| \right) \\
&= \frac{1}{|\alpha|} ((|\alpha| - 1)M + |z|) \\
&< \frac{1}{|\alpha|} ((|\alpha| - 1)|z| + |z|) \\
&= |z|.
\end{aligned}$$

It remains to show that $y' \in S$, thus putting y in Σ_S. This suffices: applying the inductive hypothesis to y to get $y \in Q_\alpha(Y)$, and given that $x \in Y \subseteq Q_\alpha(Y)$, we get $z = x \star_\alpha y \in Q_\alpha(Y)$ as desired. We have

$$y' = (x \star_{1/\alpha} z)' = x' \star_{\Lambda^{-1}} z' .$$

Since $z' \in J_{x'}$, we then have $y' \in x' \star_{\Lambda^{-1}} J_{x'} = x' \star_{\Lambda^{-1}} (x' \star_\Lambda S) = S$ by the definition of $J_{x'}$ and a simple calculation. We conclude by induction that $\Sigma_S \subseteq Q_\alpha(Y)$. $\square$

In the remainder of this subsection, we apply Theorem 15.10 to three values of λ. We first apply it to the two non-real values of λ depicted in Figure 9, showing that these two sets are both model sets and hence relatively dense Meyer sets. Secondly, we apply the theorem to the one real quadratic case left unresolved by Theorem 15.7, showing that $Q_\lambda = \Sigma^{(\lambda)}_{[0,1]}$ for $\lambda := (-3-\sqrt{17})/2$. This last case also makes use of Proposition 15.11. Finally, we apply Theorem 15.12 to one *real* λ of degree 3. This is more complicated than the non-real complex λ's, since there are two conjugates in $(0,1)$.

15.2.3 Case Study: Two Non-Real Values of λ

We first consider the two non-real values of λ given in Figure 9.

Proposition 15.16. *Let λ be the root of the polynomial x^3+x^2-1 closest to the point $-0.877+0.745i$ (see Figure 9). Then $R_\lambda = \Sigma_P$.*

Proof. We have that λ is cubic sPV with conjugate $\mu \approx 0.754878$. (The exact form of μ is not important.) As in other cases, λ is itself a unit of $\mathbb{Z}[\lambda]$, so we set $\alpha := \lambda$ whence $\beta = \mu$. Since $\mu > 1/2$, we may take $X := \{0,1\}$. Plugging into Equation (53), we get $M \approx 13.379361$. Letting Y be as in Equation (54), we get $R_\lambda(Y) = \Sigma_P$. Thus it suffices to show that $Y \subseteq R_\lambda$.

We can rewrite Equation (51) as

$$\Sigma_P = \{1\} \cup \{\lceil b\mu + c\mu^2 \rceil - b\lambda - c\lambda^2 \mid b,c \in \mathbb{Z}\}\ .$$

We check for all $b,c \in \mathbb{Z}$ such that $\left|\lceil b\mu + c\mu^2\rceil - b\lambda - c\lambda^2\right| \le M$ that the resulting point is in R_λ. Assuming this inequality, we have

$$\begin{aligned} M &\ge \left|\lceil b\mu + c\mu^2\rceil - b\lambda - c\lambda^2\right| \ge \left|b\mu + c\mu^2 - b\lambda - c\lambda^2\right| - 1 = \left|b(\mu-\lambda) + c(\mu^2-\lambda^2)\right| - 1 \\ &= |\mu-\lambda|\,|b + c(\mu+\lambda)| - 1 = |\mu-\lambda|\sqrt{(b+xc)^2+(yc)^2} - 1\ , \end{aligned}$$

where $x := \mathrm{Re}(\mu+\lambda) = \mu + \mathrm{Re}(\lambda)$ and $y := \mathrm{Im}(\mu+\lambda) = \mathrm{Im}(\lambda) > 0$. Set $r := |\mu+\lambda|$. Letting b and c be arbitrary real numbers for a moment, a little calculus shows that for fixed b the quantity $(b+xc)^2+(yc)^2$ is minimized by setting $c := -xb/r^2$. Then we have

$$\sqrt{(b+xc)^2+(yc)^2} \ge \sqrt{\left(b - \frac{x^2b}{r^2}\right)^2 + \frac{y^2x^2b^2}{r^4}} = \frac{|b|}{r^2}\sqrt{y^4+y^2x^2} = \frac{|b|y}{r}\ .$$

Thus $|\mu-\lambda|\sqrt{(b+xc)^2+(yc)^2} - 1 \ge |\mu-\lambda||b|y/r - 1$, which means it suffices to consider only those $b \in \mathbb{Z}$ such that $|\mu-\lambda||b|y/r - 1 \le M$, or equivalently,

$$|b| \le \frac{r(M+1)}{y|\mu-\lambda|} \approx 11.41\ .$$

Similarly, fixing c, the quantity $(b+xc)^2+(yc)^2$ is minimized by setting $b := -xc$, which gives $\sqrt{(b+xc)^2+(yc)^2} \ge |c|y$, so it suffices to consider only those $c \in \mathbb{Z}$ such that $|\mu-\lambda||c|y - 1 \le M$, or equivalently,

$$|c| \le \frac{M+1}{y|\mu-\lambda|} \approx 10.76\ .$$

A check by computer verifies that all values of the form $\lceil b\mu + c\mu^2\rceil - b\lambda - c\lambda^2$ are in R_λ for integers $-11 \le b \le 11$ and $-10 \le c \le 10$. $\square$

Proposition 15.17. *Let λ be the root of the polynomial x^3+x-1 closest to the point $-0.341+1.162i$ (see Figure 9). Then $R_\lambda = \Sigma_P$.*

Proof. This proof proceeds just as the previous proof but with different values. We have that λ is cubic sPV with conjugate $\mu \approx 0.682328$. Again, we can set $\alpha := \lambda$. Since $\mu > 1/2$, we may again take $X := \{0, 1\}$. This time we get $M \approx 8.424341$, yielding the upper bound of approximately 7.06 for $|b|$ and 5.24 for $|c|$.

As before, check by computer verifies that all values of the form $\lceil b\mu + c\mu^2 \rceil - b\lambda - c\lambda^2$ are in R_λ for integers $-7 \le b \le 7$ and $-5 \le c \le 5$. ☐

15.2.4 Case Study: $-(3+\sqrt{17})/2$

Here we prove that $R_\lambda = \Sigma^{(\lambda)}_{[0,1]}$ for $\lambda := (-3-\sqrt{17})/2$. We make use of both Theorem 15.10 and Proposition 15.11.

Proposition 15.18. *Let $\lambda := -(3+\sqrt{17})/2$. Then $Q_\lambda = \Sigma_P$, where $P = [0, 1]$.*

Proof. The minimal polynomial of λ and its conjugate $\mu := (\sqrt{17}-3)/2$ is x^2+3x-2. In this case, λ is not a unit of $\mathbb{Z}[\lambda]$, but Q_λ contains the value $\alpha := 9 - 16\lambda \approx 65.9848$, which *is* a unit of $\mathbb{Z}[\lambda]$ (the fundamental unit, actually). A derivation showing that $\alpha \in Q_\lambda$ was found by computer and is given in Appendix B. The conjugate of α is then $\beta = 1/\alpha = 9 - 16\mu \approx 0.015155$.

We will apply Proposition 15.11 in this case. We start with the continued fraction expansion of μ, which is $[a_0, a_1, a_2, \ldots] = [0, \overline{1, 1, 3}]$, where the bar indicates that the pattern $1, 1, 3$ repeats forever. The table below gives the values needed to calculate n and m in that proposition. To get p_k and q_k for $k \ge 0$, we can use the recurrences $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$. The values of p_k and q_k for $k < 0$ are given in order to obtain the correct initial values when $k = 0$ and $k = 1$. Recall that $\eta_k = (-1)^k (q_k\mu - p_k)$.

k	-2	-1	0	1	2	3	4	5	6	7
a_k			0	1	1	3	1	1	3	1
p_k	0	1	0	1	1	4	5	9	32	41
q_k	1	0	1	1	2	7	9	16	57	73
η_k	μ	1	μ	$1-\mu$	$2\mu-1$	$4-7\mu$	$9\mu-5$	$9-16\mu$	$57\mu-32$	$41-73\mu$

From the table, we get that $\eta_7 + \eta_6 = 9 - 16\mu = \beta < \beta/(1-\beta)$, so $k = 7$ satisfies Equation (58). Furthermore, this is the least such k-value: $0.023665 \approx \eta_6+\eta_5 = 41\mu-23 > \beta/(1-\beta) = (2\mu-1)/8 \approx 0.015388$. Thus we set $n := q_7 - 1 = 72$, whence $m = -36$. Then, by Equation (60), we have

$$\max_{x\in X} |x| = \lceil 36\mu \rceil - 36\lambda = 20 - 36\lambda \approx 148.215901$$

Thus we can set $X := \{1\} \cup \{\lceil b\mu \rceil - b\lambda \mid b \in \mathbb{Z} \ \& \ -36 \le b \le 36\}$, and this satisfies Equation (52).[20] Since $\alpha > 1$, we have, from Equation (53), $M = \max_{x\in X} |x| = 20 - 36\lambda$ and thus $Y = X$ in (54).

To summarize, we have $\Sigma_P = Q_\lambda(X)$ by Theorem 15.10. Finally, a computer run shows that in fact, $X \subseteq Q_\lambda$, and thus $\Sigma_P = Q_\lambda$. ☐

[20] A computer check reveals that X is minimal, i.e., no proper subset of X satisfies (52).

15.2.5 Case Study: λ_7

Recall Proposition 14.5, in which we characterize those λ_n (Eq. (42)) that are sPV (thus giving rise to uniformly discrete Q_{λ_n}), for odd n. We know that Q_{λ_3} and Q_{λ_5} are also relatively dense. In this section we use Theorem 15.12 to prove that Q_{λ_7} is also relatively dense.

For the remainder of this section let λ denote $\lambda_7 \approx 5.0489$, whose minimal polynomial is $x^3 - 6x^2 + 5x - 1$. Denote the conjugates of λ by $\mu, \nu \in [0,1]$, where $\mu \approx 0.30798$ and $\nu \approx 0.64310$. For $x \in \mathbb{Z}[\lambda]$, where x is of the form $a_0 + a_1\lambda + a_2\lambda^2$ for $a_0, a_1, a_2 \in \mathbb{Z}$, we denote its conjugates by $x' = a_0 + a_1\mu + a_2\mu^2$ and $x'' = a_0 + a_1\nu + a_2\nu^2$. By Fact 12.29, we have $Q_\lambda \subseteq \Sigma_P$, where $P = [0,1]^{\times 2}$. Let $S \subseteq P$ denote the $(2\nu - 1) \times (2\nu - 1)$ square with corners at $(1-\nu, 1-\nu)$ and (ν, ν). It is convenient to introduce the parallelogram T with corners at $(0,0), (\mu, \nu), (1-\mu, 1-\nu)$, and $(1,1)$. It is easy to see that $S \subseteq T \subseteq P$ and hence $\Sigma_S \subseteq \Sigma_T \subseteq \Sigma_P$. We have,

$$\Sigma_T = \{x \in \mathbb{Z}[\lambda] \mid (x', x'') \in T\}.$$

We can characterize the points of Σ_T as follows. Let $\beta = \mu + \nu - \mu\nu$ and define

$$\begin{aligned} n(m) &:= \lceil m/\lambda \rceil = \lceil \mu\nu m \rceil \\ \ell(m) &:= \lfloor \beta m \rfloor + n(m) \\ p(m) &:= n(m) - \ell(m)\lambda + m\lambda^2. \end{aligned}$$

Proposition 15.19. $\Sigma_T = \{1, \lambda, 1-\lambda\} \cup \{p(m) \mid m \in \mathbb{Z}\}$.

Proof. It is easiest to see this by transforming P via the matrix W defined in Fact 12.29. In this instance,

$$W = \begin{bmatrix} 1 & \mu \\ 1 & \nu \end{bmatrix}, \qquad W^{-1} = \frac{1}{\nu - \mu} \begin{bmatrix} \nu & -\mu \\ -1 & 1 \end{bmatrix}.$$

We interpret the column vector $[x', x'']^T$ as the point (x', x''). Let $\Omega = W^{-1}T$. It is easy to see that W^{-1} maps $(0,0) \mapsto (0,0)$, $(\mu, \nu) \mapsto (0,1)$, $(1,1) \mapsto (1,0)$, and $(1-\mu, 1-\nu) \mapsto (1,-1)$. Thus Ω is a parallelogram with corners at $(0,0), (0,1), (1,0)$, and $(1,-1)$. More generally, W^{-1} maps $(x', x'') = (a_0 + a_1\mu + a_2\mu^2, a_0 + a_1\nu + a_2\nu^2)$ to $(a_0 - \mu\nu a_2, a_1 + (\mu+\nu)a_2)$. Thus $(x', x'') \in T$ iff the following conditions hold:

$$\begin{array}{ccccc} 0 & \leq & a_0 - \mu\nu a_2 & \leq & 1 \\ -a_0 + \mu\nu a_2 & \leq & a_1 + (\mu+\nu)a_2 & \leq & -a_0 + \mu\nu a_2 + 1 \end{array} \tag{61}$$

Note that for $a_2 \neq 0$, all inequalities above are proper because μ and ν are irrational. The first inequality shows that $a_0 = \lceil \mu\nu a_2 \rceil$, and the second $a_1 = -\lceil \mu\nu a_2 \rceil + \lceil \mu\nu - (\mu+\nu)a_2 \rceil = -\lceil \mu\nu a_2 \rceil - \lfloor \beta a_2 \rfloor$. Letting m denote a_2, this yields $a_0 = n(m)$ and $a_1 = -\ell(m)$. Thus the points in Σ_T are of the form $p(m)$, $m \in \mathbb{Z}$. The exceptions are the nonzero corners of T, namely $(1,0)$, $(0,1)$ and $(1,-1)$, which correspond to the values 1, λ and $1-\lambda$. □

Remark. The apparent "outlier" nature of the points $1, \lambda$, and $1-\lambda$ is an artifact of the definition of $p(m)$, which has a λ^2 term for any $m \neq 0$. In that case, the inequalities in Eq. (61) are proper. However, for $m = a_2 = 0$, the equalities can be met, which accounts for the points $1, \lambda$, and $1-\lambda$.

The one value of $p(m)$ that works out for one of the initial points is $p(0)$, all of whose terms are 0. $\square$

We find it is possible to choose a finite number of points in Σ_T to construct the set Y of Theorem 15.12. We remark that all the points we use are in $T \setminus S$.

We now prove the main result of this section. Since λ is a unit, by Theorem 15.12, there exists a finite $Y \subseteq \Sigma_P$ such that $\Sigma_S \subseteq Q_\lambda(Y)$. Our task here is to explicitly construct such a Y such that $Y \subseteq Q_\lambda$. That, in turn, requires finding an explicit finite cover of the set S, indexed by a finite set X. Once X is found, the problem reduces to the tedious but routine verification that $Y \subseteq Q_\lambda$, and finally appealing to Lemma 15.15.

Lemma 15.20. $\Sigma_S \subseteq Q_\lambda$.

Proof. We adopt the notations and conventions of the proof of Theorem 15.12. We construct the set Y and show that $Y \subseteq Q_\lambda$. Since $Q_\lambda(Y) = Q_\lambda$, the result follows easily by Theorem 15.12.

Let $X = \{\lambda, 1-\lambda\} \cup \{p(m) | m = \pm 12, \pm 13, \pm 16, \pm 17\}$. We claim that $C = \{J_x | x \in X\}$ forms a cover of S. Following the notation of Theorem 15.12, we set $\varepsilon = 1 - \nu$. Hence the sets in C are of the form,

$$J_x = \left[x' \star_\mu (1-\nu),\ x' \star_\mu \nu\right] \times \left[x'' \star_\nu (1-\nu),\ x'' \star_\nu \nu\right]. \tag{62}$$

These are rectangles of width $(2\nu - 1)\mu$ and height $(2\nu - 1)\nu$ extrapolated from (x', x'') with $1-\nu$ and ν. In Figure 24, we show the set S in red and the sets J_x in blue (displaying the index x, which is either λ, $1-\lambda$ or m for the appropriate points $p(m)$). We slightly abuse notation and denote $J_{p(m)}$ by J_m. Using decimal approximations, we list the sets J_x in Table 1, each one specified by the coordinates (x'_0, x''_0) of its lower left-hand corner and (x'_1, x''_1) of its upper right-hand corner.

x	(x'_0, x''_0)	(x'_1, x''_1)
λ	(0.32304408,0.45904241)	(0.41119002,0.64310413)
$p(16)$	(0.39367557,0.46947633)	(0.48182156,0.65353804)
$p(-13)$	(0.48175166,0.46483280)	(0.56989766,0.64889452)
$p(12)$	(0.49574111,0.49871152)	(0.58388711,0.68277324)
$p(-17)$	(0.58381721,0.49406799)	(0.67196321,0.67812971)
$p(17)$	(0.32803679,0.32187029)	(0.41618279,0.50593201)
$p(-12)$	(0.41611289,0.31722676)	(0.50425889,0.50128848)
$p(13)$	(0.43010234,0.35110548)	(0.51824834,0.53516720)
$p(-16)$	(0.51817843,0.34646195)	(0.60632443,0.53052367)
$1-\lambda$	(0.58880998,0.35689587)	(0.67695598,0.54095759)

Table 1: Coordinates of the J_x of Figure 24.

Observe that for each x such that $x''_0 \leq 1 - \nu \approx 0.35689587$, the value of x''_1 exceeds $1/2$ (these are the points $p(m)$ for $m = 17, -12, 13, -16$ and $1-\lambda$; for that last point, $x''_0 = 1-\nu$ exactly). Similarly, for each x such that $x''_1 \geq \nu \approx 0.64310413$, the value of x''_0 is less than $1/2$ (these are the reflections $p(m)$ for $m = -17, 12, -13, 16$ and λ; for that last point, $x''_1 = \nu$ exactly). We also note that those pairs of J_x which appear to touch along the vertical boundaries actually intersect. For example, for $p(17)$, $x'_1 = 0.41618279$, which exceeds 0.41611289, the value of x'_0 for $p(-12)$. This

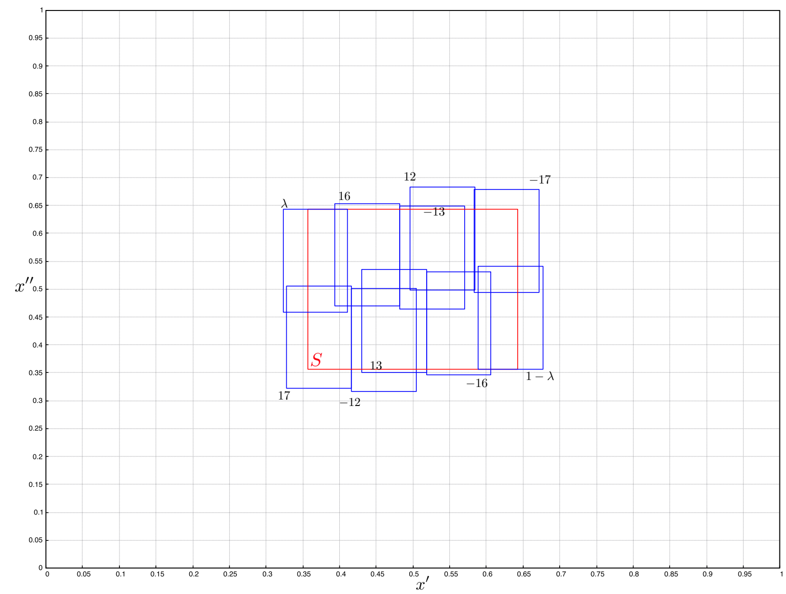


Figure 24: A cover of S. S is in red, the sets J_λ, $J_{1-\lambda}$, and J_m for $|m| = 12, 13, 16, 17$ in blue (labelled with the value of m rather than $p(m)$ for the latter sets). The entire graph represents P.

can be similarly verified for the pairs $(p(16), p(-13))$, $(p(12), p(-17))$, $(p(13), p(-16))$, so that the sets $J_{17} \cup J_{-12}$, $J_{16} \cup J_{-13}$, $J_{12} \cup J_{-17}$, and $J_{13} \cup J_{-16}$ are each connected. Thus we claim that if we divide S into the two subsets $L = \{(x', x'') \in S | 1-\nu \le x' \le \nu \text{ and } 1-\nu \le x'' \le 1/2\}$ and $U = \{(x', x'') \in S | 1-\nu \le x' \le \nu \text{ and } 1/2 \le x'' \le \nu\}$, we find that,

$$L \subseteq (J_{17} \cup J_{-12}) \cup (J_{13} \cup J_{-16}) \cup J_{1-\lambda}, \tag{63}$$

$$U \subseteq J_\lambda \cup (J_{16} \cup J_{-13}) \cup (J_{12} \cup J_{-17}), \tag{64}$$

which shows that $S \subseteq \bigcup C = \bigcup_{x \in X} J_x$.

To verify Eq. (63), given that $x_0'' \le 1-\nu$ and $x_1'' > 1/2$ for each set on the RHS, we only need to verify that each point (x', x'') with $1-\nu \le x' \le \nu$ and $1-\nu \le x'' \le 1/2$ is contained in one of $J_{17} \cup J_{-12}$, $J_{13} \cup J_{-16}$, or $J_{1-\lambda}$. Using Table 1, we enumerate the cases in the following table:

$1-\nu \le x'' \le 1/2$ and x' in:	(x', x'') in:
$[1-\nu, 0.50425889]$	$J_{17} \cup J_{-12}$
$[0.43010234, 0.60632443]$	$J_{13} \cup J_{-16}$
$[0.58880998, \nu]$	$J_{1-\lambda}$

The union of the intervals in the left column is $[1-\nu, \nu]$, so Eq. (63) follows.
Eq. (64) is established similarly, for $x'' \ge 1/2$:
We conclude that C covers S.

We now determine the set $Y := X \cup \{y \in \Sigma_S : |y| \le M\}$. Let $Y' := X \cup \{y \in \Sigma_T : |y| \le M\}$. In this case $M = \frac{|\lambda-1|}{|\lambda|-1} \max_{x \in X} |x| = \max_{x \in X} |x|$, since $\lambda > 1$. Observing that the λ^2 term dominates, it is not hard to see that $|p(m)|$ is monotonically increasing in $|m|$. Hence $Y' = \{1, \lambda, 1-\lambda\} \cup \{p(m) :$

$1/2 \leq x'' \leq \nu$ and x' in:	(x', x'') in:
$[1-\nu, 0.41119002]$	J_λ
$[0.39367557, 0.56989766]$	$J_{16} \cup J_{-13}$
$[0.49574111, \nu]$	$J_{12} \cup J_{-17}$

$|m| \leq 17\}$. A derivation of each of these points from the starting points $p_0 = 0$ and $p_1 = 1$ is given in Appendix B, Table 2. This proves that $Y' \subseteq Q_\lambda$, and hence $Q_\lambda(Y') = Q_\lambda$. It follows that $Q_\lambda(Y) = Q_\lambda$, since $\{0,1\} \subseteq Y \subseteq Y'$. Now Y has been constructed to satisfy the properties of the Y of Lemma 15.15 and, furthermore, λ is a unit so we can use it in place of α in that lemma. Hence $\Sigma_S \subseteq Q_\lambda$. □

By Lemma 15.20, $\Sigma_S \subseteq Q_\lambda$, hence since Σ_S is relatively dense, so is Q_λ. Moreover, by Fact 12.29, $Q_\lambda \subseteq \Sigma_P$, so we have,

Theorem 15.21. *Q_{λ_7} is a Meyer set.*

This, and many similar results, can be proved via Corollary 15.29 below, although that does not show explicitly that Q_λ contains a model set.

15.3 Affine Embedding

15.3.1 Definitions and General Results

Recall that for $x, y \in \mathbb{C}$, the map $\rho_{x,y} : \mathbb{C} \to \mathbb{C}$ maps $z \in \mathbb{C}$ to $x \star_z y$ (Definition 2.6). This definition extends naturally to any module over a commutative ring (Definition 12.15).

Definition 15.22. Let R be a commutative ring and A, B subsets of R. If $\rho_{a,b}(A) \subseteq B$ for some $a, b \in R$ such that $x(a-b) \neq 0$ for all $x \in R \setminus \{0\}$, then we say A *affinely embeds* in B, and write $A \hookrightarrow B$. If $A \hookrightarrow B$ and $B \hookrightarrow A$, we say A and B are *affine-equivalent*, and write $A \leftrightharpoons B$.

The condition on a and b in Definition 15.22 is equivalent to $\rho_{a,b}$ being one-to-one, hence the use of the term "embedding." If R has no zero-divisors, then $\rho_{a,b}$ is a one-to-one map just when $a \neq b$. We will only apply Definition 15.22 when R has no zero-divisors.

The $\hookrightarrow$ relation is clearly reflexive and transitive (cf. Fact 2.7(4) for $R = \mathbb{C}$), making $\leftrightharpoons$ an equivalence relation.

Fact 15.23. *Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let A and B be subsets of F such that $A \hookrightarrow B$.*

1. *If B is uniformly discrete, then A is uniformly discrete.*

2. *If A is relatively dense in F, then B is relatively dense in F.*

3. *For any $\lambda \in F$, $Q_\lambda(A) \hookrightarrow Q_\lambda(B)$ and $R_\lambda(A) \hookrightarrow R_\lambda(B)$. (This follows from Lemma 2.12.)*

Lemma 15.24. *Let T be any finite subset of $\mathbb{Q}[x]$. Then $T \hookrightarrow Q_{[x]}$. Furthermore, there exist distinct $a, b \in \mathbb{Z}[x]$ such that $\rho_{a,b}(T) \subseteq Q_{[x]}$ and the only irreducible monic polynomials in $\mathbb{Q}[x]$ dividing $b - a$ are x and $x - 1$.*

Proof. Suppose $T = \{f_1, \ldots, f_n\}$, where $f_i \in \mathbb{Q}[x]$ for each $1 \leq i \leq n$. Choose $d, \ell, k \in \mathbb{Z}^+$ sufficiently large that for all $1 \leq i \leq n$, $df_i \in \mathbb{Z}[x]$ and $(df_i(\lambda) + \ell) \cdot \lambda^k(1-\lambda)^k \in (0,1)$ for any $\lambda \in (0,1)$. Then, writing $h(x) := x^k(1-x)^k$ and $g_i := (df_i + \ell) \cdot h$, we find that $g_i \in \mathbb{Z}[x]$ and $g_i(\lambda) \in (0,1)$ for any $\lambda \in (0,1)$. By the characterization given in Theorem 11.1, this implies $g_i \in Q_{[x]}$. Now define the polynomials,

$$\begin{aligned} a &:= \ell h \ , \\ b &:= a + dh = (\ell + d)h \ . \end{aligned}$$

Then $a, b \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$ and $a \neq b$. In fact, $b - a = dh$, whose only irreducible monic factors in $\mathbb{Q}[x]$ are x and $x - 1$. Moreover, for $1 \leq i \leq n$,

$$\rho_{a,b}(f_i) = (1 - f_i)a + f_i b = a + f_i(b - a) = \ell h + df_i h = (df_i + \ell)h = g_i \ .$$

Thus $\rho_{a,b}(T) \subseteq Q_{[x]}$, and hence $T \hookrightarrow Q_{[x]}$ via a and b satisfying the additional hypothesis. $\square$

Remark. Lemma 15.24 holds just as well for infinite T, provided d, ℓ, and k exist with the requisite properties, i.e., $\bigcup_{f \in T}\{|f(\lambda)| : \lambda \in (0,1)\}$ is bounded and all coefficients in elements of T share a common denominator. $\square$

Lemma 15.25. *Let $\lambda \in \mathbb{C} \setminus \{0,1\}$ and S be any finite subset of $\mathbb{Q}[\lambda]$. Then $S \hookrightarrow Q_\lambda$.*

Proof. Let $f_1, \ldots, f_n \in \mathbb{Q}[x]$ be such that $S = \{f_1(\lambda), \ldots, f_n(\lambda)\}$. Set $T := \{f_1, \ldots, f_n\}$. Then let $a, b \in \mathbb{Z}[x]$ be as in Lemma 15.24, and set

$$c := a(\lambda) \ , \qquad\qquad d := b(\lambda) \ .$$

We claim first that $c \neq d$: If λ is transcendental over $\mathbb{Q}$, then $c \neq d$ follows immediately from the fact that $a \neq b$. If λ is algebraic over $\mathbb{Q}$, then let $p \in \mathbb{Q}[x]$ be the minimal (monic) polynomial of λ. Since $\lambda \notin \{0,1\}$, p is neither x nor $x - 1$, and thus $p \nmid b - a$. It follows that $a(\lambda) \neq b(\lambda)$, i.e., $c \neq d$. This proves the claim.

Finally, we have $\rho_{a,b}(T) \subseteq Q_{[x]}$ by Lemma 15.24. Evaluating both sides at λ, we then get $\rho_{c,d}(S) \subseteq Q_\lambda$ by Lemma 7.3. Thus $S \hookrightarrow Q_\lambda$. $\square$

Theorem 15.26. *For any nontrivial sPV λ such that Q_λ contains a unit of $\mathbb{Z}[\lambda]$ other than 1, and for any S defined as in Theorem 15.12, $\Sigma_S \hookrightarrow Q_\lambda$.*

Proof. By Theorem 15.12, there exists a finite set $Y \subseteq \Sigma_P \subseteq \mathbb{Z}[\lambda]$ such that $\Sigma_S \subseteq Q_\lambda(Y)$. By Lemma 15.25, $Y \hookrightarrow Q_\lambda$. By Fact 15.23(3), $Q_\lambda(Y) \hookrightarrow Q_\lambda(Q_\lambda) = Q_\lambda$. Hence $\Sigma_S \hookrightarrow Q_\lambda$. $\square$

Using Fact 15.23(2),

Corollary 15.27. *For any nontrivial sPV λ such that Q_λ contains a unit of $\mathbb{Z}[\lambda]$ other than 1, Q_λ is relatively dense (in either $\mathbb{R}$ or $\mathbb{C}$, as $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C} \setminus \mathbb{R}$, respectively).*

Since $\{0,1\} \subseteq \Sigma_P$, it follows trivially that $\{0,1\} \hookrightarrow \Sigma_P$. Therefore $Q_\lambda(\{0,1\}) = Q_\lambda \hookrightarrow Q_\lambda(\Sigma_P)$. However, Σ_P is closed under λ-extrapolation, i.e., $Q_\lambda(\Sigma_P) = \Sigma_P$. Hence $Q_\lambda \hookrightarrow \Sigma_P$. Since Σ_P is uniformly discrete, by Fact 15.23(1) we can use this as an alternative proof that Q_λ is uniformly discrete.

Corollary 15.28. *For any sPV λ such that Q_λ contains a unit of $\mathbb{Z}[\lambda]$ other than 1, Q_λ is a Delone set.*

Because $Q_\lambda \subseteq \Sigma_P$, and Σ_P is a Meyer set, we have,

Corollary 15.29. *For any nontrivial sPV λ such that Q_λ contains a unit of $\mathbb{Z}[\lambda]$ other than 1, Q_λ is a Meyer set.*

Remark. Alternatively, we note that if $A \hookrightarrow B \hookrightarrow C$ and A, C are Meyer sets, then B is also Meyer. In this case we have $\Sigma_S \hookrightarrow Q_\lambda \hookrightarrow \Sigma_P$, and hence Q_λ is a Meyer set. □

The next lemma (actually its corollary) has the same conclusion as Corollary 15.29 but with different hypotheses; it does not assume that λ is sPV.

Lemma 15.30. *For any $\lambda \in \mathbb{C}$, if R_λ is discrete, then $R_\lambda - R_\lambda$ is uniformly discrete.*

Proof. The lemma is trivial if $\lambda \in \{0, 1\}$, so assume otherwise. If R_λ is discrete, then R_λ is uniformly discrete by Corollary 3.5 (and furthermore, $R_\lambda = Q_\lambda$). We now have

$$R_\lambda - R_\lambda = Q_\lambda - Q_\lambda = Q_\lambda(\{0,1\} - \{0,1\}) = Q_\lambda(\{-1,0,1\}) \ ,$$

the last equality due to Lemma 2.11. R_λ contains the three distinct values 0, $\lambda(1-\lambda)$, and $2\lambda(1-\lambda)$, which form a 3-arithmetic progression (see Corollary 11.4), and thus it is clear that $\{-1,0,1\} \hookrightarrow \{0, \lambda(1-\lambda), 2\lambda(1-\lambda)\}$. We therefore get $R_\lambda - R_\lambda \hookrightarrow R_\lambda(\{0, \lambda(1-\lambda), 2\lambda(1-\lambda)\}) = R_\lambda$ by Fact 15.23. It then follows from the same Fact that $R_\lambda - R_\lambda$ is uniformly discrete. □

Corollary 15.31. *For F being either $\mathbb{R}$ or $\mathbb{C}$, and for any $\lambda \in F$, if R_λ is discrete and relatively dense in F, then R_λ is a Meyer set.*

Proof. By Lemma 15.30 and Definition 12.8. □

15.3.2 Case Study: $\lambda := -(3+\sqrt{13})/2$

In this subsection we use Theorems 15.7 and 15.10 to identify points that are in Σ_P but not in Q_λ, for $\lambda := -(3+\sqrt{13})/2$ and $P := [0,1]$. Again making use of Theorem 15.10, as well as affine embedding, we also show that Q_λ is relatively dense in $\mathbb{R}$ and hence a Meyer set.

Proposition 15.32. *Letting $\lambda := -(3+\sqrt{13})/2$, we have*

$$Q_\lambda(\{0,1,2\lambda\}) = Q_\lambda(\{0,1,1-2\lambda\}) = \Sigma_P \ ,$$

where $\Sigma_P = \Sigma_P^{(\lambda)}$ is the cut-and-project (model) set $\{x \in \mathbb{Z}[\lambda] : x' \in [0,1]\}$ of Definition 12.26.

Proof. We let $\mu := \lambda' = (\sqrt{13}-3)/2 \approx 0.302776$ be the conjugate of λ, and we note that $\mu^2 = 1 - 3\mu$. The fact that λ is already a unit of $\mathbb{Z}[\lambda]$ means we can set $\alpha := \lambda$ (and thus $\beta = \mu$) in Theorem 15.10 (and obviously, $\alpha \in Q_\lambda$ in this case). We have $(2\lambda)' = 2\mu \in [0,1]$, so $2\lambda \in \Sigma_P$. In order to apply Theorem 15.10, we let $X := \{0, 1, \lambda, 2\lambda\}$. One can readily check using decimal approximations that

$$[0,1] = [0,\mu] \cup [1-\mu, 1] \cup [4\mu - 1, 5\mu - 1] \cup [8\mu - 2, 9\mu - 2] = \bigcup_{x \in X} \left[(1-\mu)x', (1-\mu)x' + \mu\right] \ ,$$

and thus X satisfies Eq. (52). Letting M be as in Eq. (53), we have $M = \frac{\lambda-1}{\lambda+1}\max_{x\in X}|x| = \frac{2+\sqrt{13}}{3}|2\lambda| = \frac{2+\sqrt{13}}{3}(-2\lambda) = (19+5\sqrt{13})/3 \approx 12.342585$. Then by Eqs. (39) and (54),

$$Y = \Sigma_P \cap [-M, M] = \{3\lambda,\ 2\lambda,\ \lambda,\ 0,\ 1,\ 1-\lambda,\ 1-2\lambda,\ 1-3\lambda\}\ .$$

From Theorem 15.10, we know that $Q_\lambda(Y) = \Sigma_P$, and thus we are done if we can show that $Y \subseteq Q_\lambda(\{0,1,2\lambda\})$. Clearly, $\{2\lambda,\lambda,0,1,1-\lambda\} \subseteq Q_\lambda(\{0,1,2\lambda\})$. For the other three elements of Y, we have, noting that $\lambda^2 = 1-3\lambda$,

$$\begin{aligned} 3\lambda &= 1-(1-3\lambda) = 1-\lambda^2 = (1-\lambda)+\lambda(1-\lambda) = 1\star_\lambda(1-\lambda)\ ,\\ 1-2\lambda &= \lambda+(1-3\lambda) = \lambda+\lambda^2 = (1-\lambda)\lambda+2\lambda^2 = \lambda\star_\lambda(2\lambda)\ ,\\ 1-3\lambda &= \lambda^2 = 0\star_\lambda\lambda\ . \end{aligned}$$

Thus $\Sigma_P = Q_\lambda(Y) \subseteq Q_\lambda(Q_\lambda(\{0,1,2\lambda\})) = Q_\lambda(\{0,1,2\lambda\}) \subseteq \Sigma_P$, since $\{0,1,2\lambda\} \subseteq \Sigma_P$ and Σ_P is λ-convex. Thus $Q_\lambda(\{0,1,2\lambda\}) = \Sigma_P$. To see that $Q_\lambda(\{0,1,1-2\lambda\}) = \Sigma_P$, observe that $1-2\lambda \in \Sigma_P$ and that $2\lambda \in Q_\lambda(\{0,1,1-2\lambda\})$; indeed,

$$2\lambda = 1-\lambda-(1-3\lambda) = 1-\lambda-\lambda^2 = (1-\lambda)(1-\lambda)+\lambda(1-2\lambda) = (1-\lambda)\star_\lambda(1-2\lambda)\ .$$

Thus we have $Q_\lambda(\{0,1,1-2\lambda\}) = Q_\lambda(\{0,1,2\lambda\}) = \Sigma_P$. ☐

We know from Theorem 15.7 that $Q_{-(3+\sqrt{13})/2} \neq \Sigma_P$. Proposition 15.32 gives us two specific points that are in Σ_P but not in Q_λ:

Corollary 15.33. *Let $\lambda := -(3+\sqrt{13})/2$. Then $\{2\lambda, 1-2\lambda\} \subseteq \Sigma_P \setminus Q_\lambda$.*

Proof. One readily checks that 2λ and $1-2\lambda$ are both in Σ_P. But if either one (say, 2λ) is in Q_λ, then we would have $Q_\lambda = Q_\lambda(\{0,1,2\lambda\}) = \Sigma_P$, and we know by Theorem 15.7 that $Q_\lambda \neq \Sigma_P$. ☐

Remark. Another way to see that $2\lambda \notin Q_\lambda$ without using Proposition 15.32 is to notice that $2\lambda \in (1-\lambda)\mathbb{Z}[\lambda]+2$. This implies $2\lambda \notin (1-\lambda)\mathbb{Z}[\lambda]+\{0,1\}$ because $1-\lambda \nmid 2$. Then apply Corollary 10.5 with $D = \mathbb{Z}[\lambda]$. ☐

Using the idea of affine embedding, it follows easily from Proposition 15.32 that $Q_{-(3+\sqrt{13})/2}$ is relatively dense in $\mathbb{R}$.

Corollary 15.34 (to Proposition 15.32)**.** *Q_λ is relatively dense in $\mathbb{R}$, where $\lambda := -(3+\sqrt{13})/2$.*

Proof. Observe that $\{0,1,2\lambda\} \hookrightarrow Q_\lambda$. In fact,

$$\rho_{0,1-\lambda}(\{0,1,2\lambda\}) = \{0, 1-\lambda, 2\lambda(1-\lambda)\} = \{0, 1-\lambda, \lambda\star_\lambda(1-\lambda)\} \subseteq Q_\lambda\ .$$

Thus by Proposition 15.32 and Fact 15.23 (with $F = \mathbb{R}$), we have $\Sigma_P = Q_\lambda(\{0,1,2\lambda\}) \hookrightarrow Q_\lambda(Q_\lambda) = Q_\lambda$, and since Σ_P is relatively dense in $\mathbb{R}$, it follows that Q_λ is relatively dense in $\mathbb{R}$. ☐

Remark. The last proof shows that $\Sigma_P \leftrightarrows Q_\lambda$ for $\lambda := -(3+\sqrt{13})/2$, because clearly, $Q_\lambda \subseteq \Sigma_P$ and thus $Q_\lambda \hookrightarrow \Sigma_P$. ☐

15.3.3 A Concise Proof of Theorem 12.14

In this subsection, we use affine embedding to give an alternate, shorter proof of Theorem 12.14. The proof we give here is less informative than the original one, as it completely obscures any connection with cut-and-project sets.

Definition 15.35. For $n \geq 1$, we will say that a point set $S \subseteq \mathbb{R}^n$ is *sub-Meyer* if $S - S$ is uniformly discrete.

Fact 12.11 implies that if $S \subseteq \mathbb{R}^n$ is sub-Meyer and relatively dense in $\mathbb{R}^n$, then S is a Meyer set. We can identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way.

Recall that $\mathcal{D} = \mathbb{C} \setminus \mathcal{C} = \{\lambda \in \mathbb{C} \mid R_\lambda \text{ is not convex}\} = \{\lambda \in \mathbb{C} \mid R_\lambda \text{ is discrete}\}$ (Definition 2.21 and Corollary 3.5).

The next theorem is a strengthened version of Lemma 15.30, and some of the proof is cribbed from there.

Theorem 15.36. *$Q_\lambda(S)$ is sub-Meyer for any $\lambda \in \mathcal{D}$ and finite $S \subseteq \mathbb{Q}(\lambda)$.*

Proof. Fix $\lambda \in \mathcal{D}$ and finite $S \subseteq \mathbb{Q}(\lambda)$. We have that Q_λ is uniformly discrete by Corollary 3.5. Letting $T := S - S$, we see that T is a finite subset of $\mathbb{Q}(\lambda)$. We have that λ is algebraic by Theorem 7.11 (which is subsumed by Theorem 8.2), and thus $\mathbb{Q}(\lambda) = \mathbb{Q}[\lambda]$. It follows by Lemma 15.25 that $T \hookrightarrow Q_\lambda$. Applying $Q_\lambda(\cdot)$ to both sides and using Fact 15.23, we get $Q_\lambda(T) \hookrightarrow Q_\lambda(Q_\lambda) = Q_\lambda$, and thus $Q_\lambda(T)$ is uniformly discrete (again by Fact 15.23). Finally, using Lemma 2.11, we have $Q_\lambda(S) - Q_\lambda(S) = Q_\lambda(S - S) = Q_\lambda(T)$, which proves the theorem. □

Theorem 15.37. *$Q_\lambda(S)$ is sub-Meyer for any sPV λ and finite $S \subseteq \mathbb{Q}(\lambda)$.*

Proof. Let λ be any sPV number. By Theorem 15.36 it suffices to show that $\lambda \in \mathcal{D}$. The case where $\lambda \in \mathbb{R}$ was proven by Pinch and is Proposition 12.4. The case where $\lambda \notin \mathbb{R}$ is by Proposition 12.5. □

15.4 Higher Dimensions

For certain λ, we can use relative density results in $\mathbb{R}$ to prove relative density in higher dimensions. We have already shown (Proposition 6.1, and again in Theorem 15.7) that $Q_{1+\varphi}$ ($= R_{1+\varphi}$) is relatively dense, where φ is the golden ratio, and similarly for $Q_{2+\sqrt{2}}$ and $Q_{2+\sqrt{3}}$ in Theorem 15.7.

The next two lemmas allow us in some cases to prove relative density in higher dimensions based on relative density in $\mathbb{R}$.

Lemma 15.38. *For any $\lambda \in \mathbb{R}$ and any $S \subseteq \mathbb{R}^n$ (for $n \geq 1$), if Q_λ is relatively dense in $\mathbb{R}$ and $Q_\lambda(S)$ contains the vertices of a parallelepiped with nonzero n-dimensional volume, then $Q_\lambda(S)$ is relatively dense in $\mathbb{R}^n$.*

Proof. We first prove the special case where $S = C_n := \{0,1\}^n \subseteq \mathbb{R}^n$, the corners of the unit hypercube in $\mathbb{R}^n$. In this case, we show that $Q_\lambda(C_n) \supseteq Q_\lambda \times \cdots \times Q_\lambda$ (n-fold Cartesian product) by induction on n.[21] Assuming Q_λ is relatively dense in $\mathbb{R}$, the right-hand side is clearly relatively dense in $\mathbb{R}^n$, which proves the lemma in this special case. Afterwards, we show how the general case follows easily.

[21] In fact, equality holds, but we will not need this.

In the special case, the statement above is trivial for $n = 1$, where $C_1 = \{0, 1\}$ and hence $Q_\lambda(C_1) = Q_\lambda$. Now assume $n \geq 1$ and that the statement above holds for n, i.e., $Q_\lambda(C_n) \supseteq (Q_\lambda)^{\times n}$. In $\mathbb{R}^{n+1}$, it is easily checked that $Q_\lambda(C_n \times \{b\}) = Q_\lambda(C_n) \times \{b\}$ for $b \in \{0, 1\}$. Whence

$$Q_\lambda(C_{n+1}) = Q_\lambda(C_n \times \{0, 1\}) \supseteq Q_\lambda(C_n) \times \{0, 1\} .$$

That is, for every $v \in Q_\lambda(C_n)$, we have $v \times \{0, 1\} \subseteq Q_\lambda(C_{n+1})$, and from this we get

$$Q_\lambda(\{v\} \times \{0, 1\}) = \{v\} \times Q_\lambda \subseteq Q_\lambda(C_{n+1}) .$$

This holds for all $v \in Q_\lambda(C_n)$, and so by the inductive hypothesis, it holds for all $v \in (Q_\lambda)^{\times n}$. Thus,

$$(Q_\lambda)^{\times(n+1)} = (Q_\lambda)^{\times n} \times Q_\lambda \subseteq Q_\lambda(C_{n+1}) ,$$

as we wished to show.

Now in the general case, we can assume without loss of generality that $S = \{x_b \mid b \in \{0, 1\}^n\} \subseteq \mathbb{R}^n$, where the x_b form the corners of a nondegenerate parallelepiped. We can index the points so that there exists an invertible $\mathbb{R}$-linear map $\ell : \mathbb{R}^n \to \mathbb{R}^n$ that maps each $b \in \{0, 1\}^n$ to $x_b - x_0$. Letting $S' := \ell^{-1}(S - x_0)$, we see that $S' = \{0, 1\}^n$, and thus $Q_\lambda(S')$ is relatively dense in $\mathbb{R}^n$ by the special case proved above. By Lemma 12.17[22] we have

$$x_0 + \ell(Q_\lambda(S')) = x_0 + Q_\lambda(\ell(S')) = x_0 + Q_\lambda(S - x_0) = Q_\lambda(S) .$$

The left-hand side is seen to be relatively dense in $\mathbb{R}^n$ because this property is preserved under invertible $\mathbb{R}$-linear maps and under translation. □

Lemma 15.39. *For any $\lambda \in \mathbb{R}$ and $S \subseteq \mathbb{C}$, if Q_λ is relatively dense in $\mathbb{R}$ and S includes three non-colinear points, then $Q_\lambda(S)$ is relatively dense in $\mathbb{C}$.*

Proof. For any $z \in \mathbb{C}$ such that $\mathrm{Im}(z) > 0$, define $\Delta_z := \{0, 1, z\}$. To say that S contains three non-colinear points is equivalent to saying that $\Delta_z \hookrightarrow S$ for some such z. Thus it suffices by Fact 15.23 to show that $Q_\lambda(\Delta_z)$ is relatively dense in $\mathbb{C}$ for all such z.

Fix a $z := x + yi$ for some arbitrary $x, y \in \mathbb{R}$ with $y > 0$. Since Q_λ is relatively dense in $\mathbb{R}$, we must have $\lambda \notin [0, 1]$. Let $\sigma := \lambda(1 - \lambda)$. Then $\sigma < 0$, and both σ and 2σ are in Q_λ:

$$\sigma = \lambda \star_\lambda 0 , \qquad\qquad 2\sigma = \lambda \star_\lambda (1 - \lambda) .$$

Consider the four points

$$a := 0 \star_\sigma 1 \qquad\qquad b := 1 \star_\sigma 0$$
$$c := 0 \star_{2\sigma} z \qquad\qquad d := 1 \star_{2\sigma} z$$

All four points are in $Q_\lambda(\Delta_z)$ by Lemma 2.25(1): a and b are in $Q_\sigma \subseteq Q_\lambda \subseteq Q_\lambda(\Delta_z)$, and c and d are in $Q_{2\sigma}(\Delta_z) \subseteq Q_\lambda(\Delta_z)$. A little calculation shows that

$$a = \sigma \qquad\qquad b = 1 - \sigma$$
$$c = 2\sigma z = 2\sigma x + 2\sigma y i \qquad\qquad d = (1 - 2\sigma) + 2\sigma z = 1 - 2\sigma + 2\sigma x + 2\sigma y i$$

Thus a, b, c, and d form the vertices of a parallelogram in $\mathbb{C}$, and this parallelogram is nondegenerate, since $\sigma \neq 1 - \sigma$ and $2\sigma y \neq 0$. It follows that $Q_\lambda(\Delta_z)$ is relatively dense in $\mathbb{C}$ by Lemma 15.38. □

[22] where $R = \mathbb{R}$, $M = N = \mathbb{R}^n$, and $f = \ell$

Recall from Section 14.1 that for integer $n \geq 3$, the point set $P_n \subseteq \mathbb{C}$ is the set of vertices of the regular n-gon, located so that $\{0,1\} \subseteq P_n$ and $\mathrm{Im}(z) \geq 0$ for all $z \in P_n$ (and so the unit interval $[0,1]$ forms the bottom base of the n-gon).

Corollary 15.40. *For any $\lambda \in \mathbb{R}$, if Q_λ is relatively dense in $\mathbb{R}$, then $Q_\lambda(P_n)$ is relatively dense in $\mathbb{C}$ for all $n \geq 3$.*

The following corollary covers all the sets shown in Figures 10–22, showing that they are all relatively dense in $\mathbb{C}$.

Corollary 15.41. *The following sets are all relatively dense in $\mathbb{C}$:*

1. *$Q_{\lambda_5}(P_5)$ and $Q_{\lambda_5}(P_{10})$ (recall $\lambda_5 = 1 + \varphi = (3 + \sqrt{5})/2$)*
2. *$Q_{\lambda_7}(P_7)$ and $Q_{\lambda_7}(P_{14})$ (cf. Equation (42))*
3. *$Q_{2+\sqrt{2}}(P_8)$*
4. *$Q_{2+\sqrt{3}}(P_{12})$*
5. *$Q_{\lambda_9}(P_9)$ and $Q_{\lambda_9}(P_{18})$ (cf. Equation (42))*
6. *$Q_{\lambda_{15}}(P_{15})$ and $Q_{\lambda_{15}}(P_{30})$ (cf. Equation (42))*

Proof. Consulting the table in the proof of Proposition 14.5, we see that λ_n has field norm ± 1 for $n \in \{5, 7, 9, 15\}$ and is thus a unit of $\mathbb{Z}[\lambda_n]$. Applying Corollary 15.29, we get that Q_{λ_n} is relatively dense in $\mathbb{R}$ for these n.[23] Theorem 15.6 also implies $Q_{2+\sqrt{2}}$ and $Q_{2+\sqrt{3}}$ are relatively dense in $\mathbb{R}$. We apply Corollary 15.40 to all these sets. $\square$

Corollary 15.42. *All the sets mentioned in Corollary 15.41 are Meyer sets.*

Proof. For S be any of these sets, we have that $S - S$ is uniformly discrete by Propositions 14.4, 14.5, and 14.9. Now we apply Fact 12.11(2) to S. $\square$

Proposition 15.43. *The $(1+\varphi)$-convex closure of a regular dodecahedron in $\mathbb{R}^3$ is relatively dense in $\mathbb{R}^3$.*

Proof. Let $\lambda := 1 + \varphi$ and let D be a regular dodecahedron in $\mathbb{R}^3$. Choose some pair of opposite pentagonal faces F and F' of D. The set of corners of each of F and F' is congruent to P_5, and so $Q_\lambda(D)$ includes a congruent copy of $Q_\lambda(P_5)$ in each of the (parallel) planes containing F and F'. By the previous proposition, each copy contains the corners of a nondegenerate parallelogram (rhombus, actually), and these rhombi can be chosen so that the one in F' is a translation of the one in F. See Figure 25. $\square$

[23] Isolated cases of these results are proved elsewhere in the paper. We have that $Q_{1+\varphi}$ is relatively dense in $\mathbb{R}$, either by Proposition 6.1 or by Theorem 15.6. By Theorem 15.21, Q_{λ_7} is relatively dense in $\mathbb{R}$.

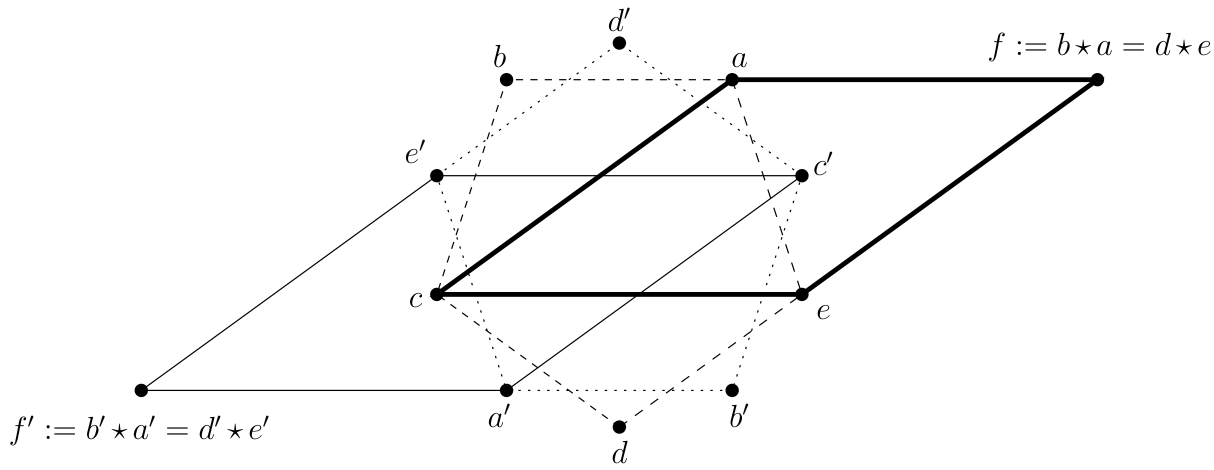


Figure 25: Two opposite pentagonal faces $F = \{a, b, c, d, e\}$ (dashed lines) and $F' = \{a', b', c', d', e'\}$ (dotted lines) of a regular dodecahedron D are shown together with two rhombi extending out from each in opposite directions. The rhombus on the right (thick solid lines) lies in the plane of F and is formed from points a, c, e, and f, the latter of which is seen to be in $Q_\lambda(F)$. Similarly, the rhombus on the left (thin solid lines) lies in the (parallel) plane of F' and is formed from the points a', c', e', and $f' \in Q_\lambda(F')$. (These points are antipodal with respect to D to the points a, c, e, and f, respectively.) Thus the points $\{a, c, e, f, a', c', e', f'\}$ are all in $Q_\lambda(D)$ and form the corners of a nondegenerate parallelepiped in $\mathbb{R}^3$. Here, $\lambda = 1 + \varphi$, and $\star$ means $\star_\lambda$.

15.5 Trivial sPV λ

In this subsection, we show relative density of Q_λ in some cases for trivial sPV λ.

Proposition 15.44. *Let $\lambda \in \mathbb{C}$ be a trivial strong PV number. If Q_λ contains an integer other than 0 or 1, then Q_λ is relatively dense in either $\mathbb{R}$ or $\mathbb{C}$ as $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C} \setminus \mathbb{R}$, respectively.*

Proof. Suppose α is in $Q_\lambda \cap (\mathbb{Z} \setminus \{0, 1\})$. Then Q_α is relatively dense in $\mathbb{R}$ because it is periodic (Theorem 10.12). Further, $Q_\alpha \subseteq Q_\lambda$ (Corollary 2.26). By Fact 12.3, λ is either a $\mathbb{Z}$-integer or a nonreal quadratic integer. In the former case, Q_λ is clearly relatively dense in $\mathbb{R}$. In the latter case, Q_λ contains three non-colinear points, and thus $Q_\alpha(Q_\lambda)$ is relatively dense in $\mathbb{C}$ by Lemma 15.39. Since $Q_\alpha(Q_\lambda) \subseteq Q_\lambda(Q_\lambda) = Q_\lambda$ (Lemma 2.25), the result follows. ☐

Proposition 15.45. *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is trivial sPV and Q_λ contains a point $z \neq 0$ such that $\operatorname{Re}(z) \in \{0, 1/2\}$, then Q_λ is relatively dense in $\mathbb{C}$.*

Proof. By Fact 12.3 (q.v. Lemma 10.2), z is a non-real quadratic integer. If $\operatorname{Re}(z) = 0$, then $z = i\sqrt{n}$ for some positive $n \in \mathbb{Z}$. But then, $0 \star_z z = z^2 = -n \in Q_z \subseteq Q_\lambda$ (Corollary 2.26), and hence Q_λ is relatively dense in $\mathbb{C}$ by Proposition 15.44. Similarly, if $\operatorname{Re}(z) = 1/2$, then $z = (1 + ib\sqrt{4n-1})/2$ for some integers b and n with b odd and $n > 0$. It follows that $z \star_z 0 = (1-z)z = (1 + b^2(4n-1))/4 \in Q_z \cap \mathbb{Z} \subseteq Q_\lambda \cap \mathbb{Z}$. If $b \neq \pm 1$ or $n > 1$, then $z \star_z 0 \notin \{0, 1\}$, which implies Q_λ is relatively dense in $\mathbb{C}$ by Proposition 15.44. If $n = 1 = \pm b$, then $z = e^{\pm i\tau/6}$, whence $-1 = z^3 = 0 \star_z (0 \star_z z) \in Q_z \subseteq Q_\lambda$, and again by Proposition 15.44, Q_λ is relatively dense in $\mathbb{C}$. ☐

The next proposition mirrors Corollary 15.27 but for trivial sPV λ. (In Theorem 15.12, if λ is trivial, then $k = 0$, and thus $\Sigma_P^{(\lambda)} = \mathbb{Z}[\lambda]$ by Definition 12.26.)

Proposition 15.46. *Let $\lambda \in \mathbb{C}$ be a trivial sPV number. If $\mathbb{Z}[\lambda]$ contains an $\alpha \neq 1$ that is a unit of $\mathbb{Z}[\lambda]$, then Q_λ is relatively dense in either $\mathbb{R}$ or $\mathbb{C}$ as $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C} \setminus \mathbb{R}$, respectively.*

Proof. It follows from Fact 12.3 that $\mathbb{Z}[\lambda]$ contains no points $z \in \mathbb{C}$ such that $|z| < 1$. Thus if α is a unit of $\mathbb{Z}[\lambda]$, it must be that $|\alpha| = 1$, and since α is either a $\mathbb{Z}$-integer or non-real quadratic integer, we have $\mathrm{Re}(\alpha) \in \{-1, -1/2, 0, 1/2\}$. Proposition 15.45 applies if $\mathrm{Re}(\alpha) \in \{0, 1\}$. Proposition 15.44 applies if $\mathrm{Re}(\alpha) = -1$ (so $\alpha = -1$). If $\mathrm{Re}(\alpha) = -1/2$, then $\alpha = (-1 \pm i\sqrt{3})/2$, whence $\alpha \star_\alpha 0 = (1 - \alpha)\alpha = 4i\sqrt{3} \in Q_\alpha \subseteq Q_\lambda$ (Corollary 2.26), so Proposition 15.45 applies. $\square$

Part III: Bent Paths

16 The λ-convex closure of a bent path

We continue to use $\star$ without subscript to denote $\star_\lambda$.

This part of the paper is dedicated to proving Theorem 3.9. We sequester the proof in this way because it uses some concepts and techniques that are not used anywhere else in the paper, particularly, winding number and some basic homology and homotopy theory. We only need a few facts about these:

- For every loop ℓ in $\mathbb{C}$ (see the start of Section 3 for definitions) and every point x not on ℓ, ℓ has a well defined *winding number* about x, which is an integer indicating the number of times ℓ "wraps around" x—positive for counterclockwise, negative for clockwise.
- If two loops are homologous in $\mathbb{C} \setminus \{x\}$ (that is, their difference can be expressed as the sum of boundaries of continuous images of disks in $\mathbb{C} \setminus \{x\}$), then they have the same winding number about x.
- Any two loops that are homotopic in $\mathbb{C} \setminus \{x\}$ (that is, one can continuously deform one loop into the other, staying within $\mathbb{C} \setminus \{x\}$ and keeping the endpoint fixed) are also homologous in $\mathbb{C} \setminus \{x\}$, and thus have the same winding number about x.
- The winding number of a finite sum of loops—about some x not on any of the loops—is the sum of the winding numbers of the individual loops about x.
- If $x, y \in \mathbb{C}$ and ℓ is a loop in $\mathbb{C} \setminus \{x, y\}$ such that there is a path from x to y that does not intersect ℓ, then ℓ has the same winding number about y as it has about x.

Definition 16.1. A path in $\mathbb{C}$ is *bent* if it does not lie within any single straight line.

Theorem 3.9 can then be restated as follows:

Theorem 16.2. *$Q_\lambda(c) = \mathbb{C}$ for any $\lambda \in \mathbb{C} \setminus [0, 1]$ and any bent path c.*

Definition 16.3. Let $c : [0, 1] \to \mathbb{C}$ be a path. A *subpath* of c is any path $d : [0, 1] \to \mathbb{C}$ that starts at some point $c(a)$, follows c, and ends at some point $c(b)$, where $0 \le a \le b \le 1$. That is, there exist $0 \le a \le b \le 1$ such that $d(x) = c(\rho_{a,b}(x))$ for all $x \in [0, 1]$.

The *loop closure* of c, denoted $lcl(c)$, is the loop obtained by first following c (double speed) from $c(0)$ to $c(1)$, then along a straight line from $c(1)$ back to $c(0)$. It can be parameterized thus:

$$lcl(c)(x) = \begin{cases} c(2x) & \text{if } 0 \le x \le 1/2, \\ \rho_{c(1),c(0)}(2x - 1) & \text{if } 1/2 \le x \le 1. \end{cases}$$

The proof of Theorem 16.2 uses the following lemma:

Lemma 16.4. *If $c : [0, 1] \to \mathbb{C}$ is a bent path that does not include any nonempty open subset of $\mathbb{C}$, then c includes a subpath $d : [0, 1] \to \mathbb{C}$ with the following properties:*

1. *d lies entirely in a closed half-plane whose boundary passes through its endpoints $d(0)$ and $d(1)$, and*

2. *there exists a point $x \in \mathbb{C}$ such that lcl(d) has nonzero winding number about x (which is not on lcl(d)).*

Proof. Let A, B, and C be three noncolinear points along c. We can assume without loss of generality that $A = c(0)$, $B = c(1/2)$, and $C = c(1)$ (otherwise, take an appropriately reparameterized subpath of c). By our assumption about c not filling any space, we can choose a point x in the interior of the triangle $\triangle ABC$ that does not lie on c. We first show that there is a subpath e of c that satisfies the second condition of the lemma with respect to x. Let t be the (oriented) loop formed by tracing the perimeter of $\triangle ABC$, starting at A, going to B, then to C, then back to A. Clearly, t has winding number ± 1 about x. (It is not necessary, but we can assume that t goes counterclockwise, so its winding number about x is $+1$.) Now, t is evidently homologous to the sum of the following three loops (in fact, the difference is empty):

- the loop closure ℓ_1 of c,
- the loop closure ℓ_2 of the path obtained by following c backwards from B to A, and
- the loop closure ℓ_3 of the path obtained by following c backwards from C to B.

Since the winding number around x is invariant under homology of loops in $\mathbb{C} \setminus \{x\}$, and the winding number of a sum is the sum of the winding numbers, it follows that the winding numbers (around x) of ℓ_1, ℓ_2, and ℓ_3 sum to 1. Thus at least one of these three loops has nonzero winding number around x. If it is ℓ_1, then we take $e := c$; otherwise, if it is ℓ_2, then we let e be c restricted to $[0, 1/2]$ (reparameterized); and otherwise (if it is ℓ_3) we take e to be c restricted to $[1/2, 1]$ (reparameterized). Then e and x satisfy the second condition of the lemma with $d := e$ (but not necessarily the first). See Figure 26. (Note that the orientation of e does not matter here, because the winding number will be nonzero regardless of orientation.)

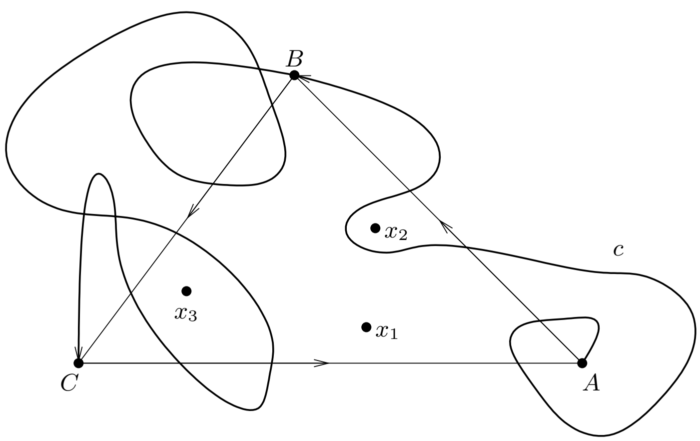


Figure 26: The curve c and the triangle $\triangle ABC$. For each $i \in \{1, 2, 3\}$, the loop ℓ_i has nonzero winding number around the point x_i.

We now find a subpath d of e that satisfies both conditions. Let x be as above (so that lcl(e) has nonzero winding number about x), let L be the line through $P := e(0)$ and $Q := e(1)$, and let L' be the line through x parallel to L. The situation might look like Figure 27.

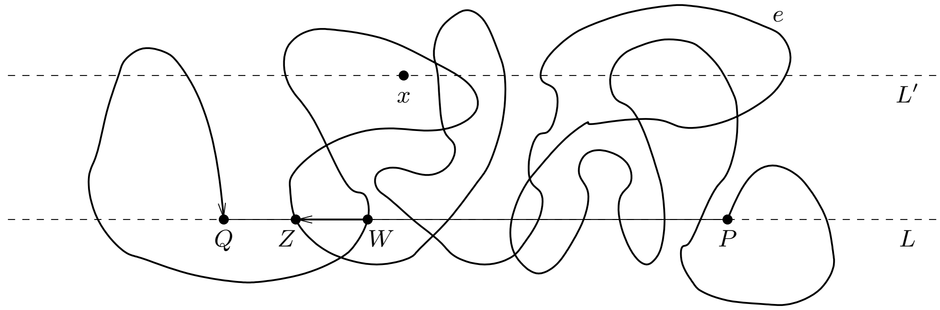


Figure 27: The curve e and the point x. (The small loop that follows the curve e from Z to W then straight back to Z has nonzero winding number around x.)

Let H be the open halfplane of $\mathbb{C}$ with boundary L and containing x. Then by continuity, $e^{-1}(H)$ is an open subset of $(0,1)$, and hence is the disjoint union of at most countably many open intervals $I_0, I_1, I_2, \ldots \subseteq (0,1)$. For $i = 0,1,2,\ldots$, let e_i be the subpath of e restricted to I_i. Now we claim that there can be only finitely many i such that e_i intersects L'. Indeed, suppose there were infinitely many such i, say $i_0, i_1, i_2, \ldots$. For all $j \in \{0,1,2\ldots\}$ let a_j be the left boundary of I_{i_j}, and pick some $z_j \in I_{i_j}$ such that $e(z_j) \in L'$. By the continuity of e, we must have $e(a_j) \in L$. The sequence $z_0, z_1, z_2, \ldots$ has some accumulation point $z \in [0,1]$. Take some monotone subsequence $z_{j_0}, z_{j_1}, z_{j_2}, \ldots$ converging to z, where $j_0 < j_1 < j_2 < \cdots$. If this sequence is increasing, then we have $z_{j_k} < a_{j_{k+1}} < z_{j_{k+1}}$ for all k, and if it is decreasing, then we have $z_{j_{k+1}} < a_{j_k} < z_{j_k}$ for all k. In either case, the sequence $a_{j_0}, a_{j_1}, a_{j_2}, \ldots$ also converges to z, but then since $a_j \in e^{-1}(L)$ and $z_j \in e^{-1}(L')$ for all j, and both $e^{-1}(L)$ and $e^{-1}(L')$ are closed, we have $z \in e^{-1}(L) \cap e^{-1}(L') = \emptyset$. Contradiction. This establishes the claim.

Now by the above claim, we have $e_{i_1}, \ldots, e_{i_n}$ intersect L' for some natural number n and some indices $i_1, \ldots, i_n$, and no other e_i intersect L'. Set $\ell := \mathit{lcl}(e)$. For $1 \leq j \leq n$, let $r_j < s_j$ be the boundary points of I_{i_j}, and let ℓ_j be the loop closure of the subpath e_{i_j} of e that goes from $e(r_j)$ to $e(s_j)$ (note that $e(r_j)$ and $e(s_j)$ both lie on L). We now can express ℓ as the *finite* sum $\ell_1 + \ell_2 + \cdots + \ell_n + p$, where $p := \ell - (\ell_1 + \ell_2 + \cdots + \ell_n)$. This decomposition (for the curve shown in Figure 27) is illustrated in Figure 28.

The winding number of ℓ about x is nonzero, because e satisfies the second condition of the lemma with respect to x. This winding number is the sum of the winding numbers (about x) of $\ell_1, \ldots, \ell_n$, and p. There is no contribution from p to the winding number, because p stays entirely in the open halfplane bounded by L' and containing L (thus it cannot wrap around x). It follows that there must be some ℓ_j that has a nonzero winding number around x (e.g., the small loop through Z and W in Figure 27). Since e_{i_j} lies entirely in H (except for its endpoints), we can take $d := e_{i_j}$, which then satisfies both conditions of the lemma. $\square$

We first prove the special case of Theorem 16.2 where λ is real. This is Proposition 16.5, below. Afterwards, we will explain how to modify the proof for nonreal λ.

Proposition 16.5. *$Q_\lambda(c) = \mathbb{C}$ for any $\lambda \in \mathbb{R} \setminus [0,1]$ and any bent path c.*

Proof. We can assume that $\lambda > 1$ by Fact 2.13. If c includes a nonempty open subset of $\mathbb{C}$, then we are done by Proposition 3.8, and so from now on we assume that this is not the case. Then

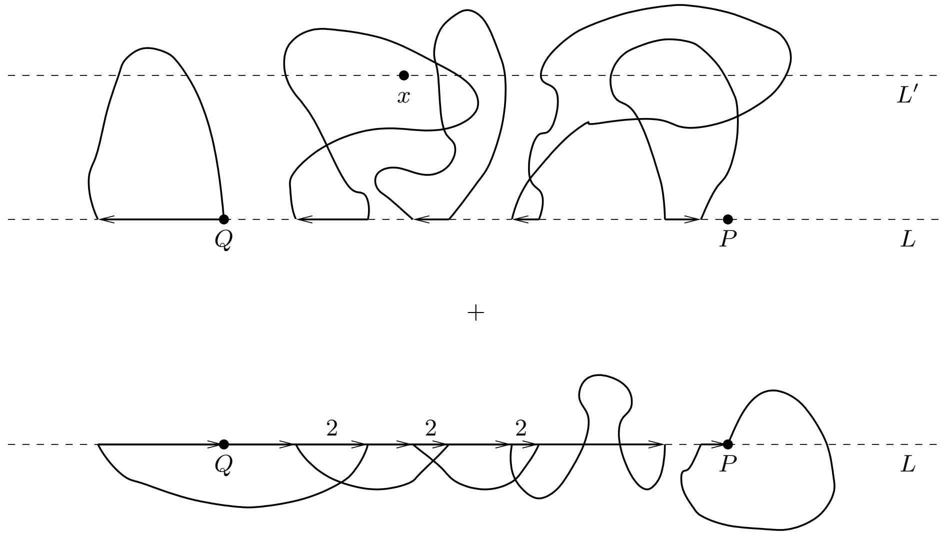


Figure 28: The decomposition of ℓ in the previous figure as a sum $\ell_1 + \cdots + \ell_5$ of five loops that intersect L' (top) and those that don't (p, bottom). The "2" above a segment indicates that it counts double.

Lemma 16.4 implies that we can take a subpath d of c satisfying the two properties of the lemma with respect to some point x, and then it is enough to show that $Q_\lambda(d) = \mathbb{C}$. And for *this* it is enough to show that $Q_\lambda(d)$ contains a nonempty open subset of $\mathbb{C}$, thanks again to Proposition 3.8.

The first property of Lemma 16.4 says that d lies entirely to one side of some line L through $d(0)$ and $d(1)$. (If $d(0) = d(1)$, then L may not be unique.) Letting $\ell := lcl(d)$, the second property of the lemma says that ℓ has nonzero winding number about x.

Since d is compact and hence closed, there is some ball B of radius $\varepsilon > 0$ about x that is disjoint from $d \cup L$. Furthermore, ℓ has the same nonzero winding number about every point $y \in B$ as it has about x. Now notice that the set $B' := \{y \star d(0) \mid y \in B\}$ is an open neighborhood of the point $x' := x \star d(0)$ (in fact, a ball centered at x' with radius $\varepsilon(\lambda - 1)$). Note that B' lies entirely on the side of L opposite B and d. Figure 29 shows a typical situation when $\lambda = 2$.

We finish the proof by showing that $B' \subseteq Q_\lambda(d)$, whence $Q_\lambda(d) = \mathbb{C}$ by Proposition 3.8. We do this in two steps: (i) we define a loop ℓ' entirely included in $Q_\lambda(d)$ that has nonzero winding number about every point $y' \in B'$; and (ii) we exhibit a homotopy h (that stays entirely within $Q_\lambda(d)$) from ℓ' to the constant loop $d(0)$. Assuming for the moment that we can do this, suppose there exists some $y' \in B' \setminus Q_\lambda(d)$. Then since h avoids y', it must keep the winding number about y' invariant throughout the deformation of the loop, but this is impossible, because the winding number of ℓ' about y' is nonzero whereas the winding number of the constant loop $d(0)$ about y' is zero. Thus no such y' can exist, and so $B' \subseteq Q_\lambda(d)$ as desired.

The loop ℓ' is made up of three segments: the first two are similar to d, and the third is d itself in reverse. We define $\ell' : [0,1] \to \mathbb{C}$ formally as follows: for all $s \in [0,1]$,

$$\ell'(s) := \begin{cases} d(3s) \star d(0) & \text{if } 0 \le s \le 1/3, \\ d(1) \star d(3s-1) & \text{if } 1/3 \le s \le 2/3, \\ d(3-3s) & \text{if } 2/3 \le s \le 1. \end{cases}$$

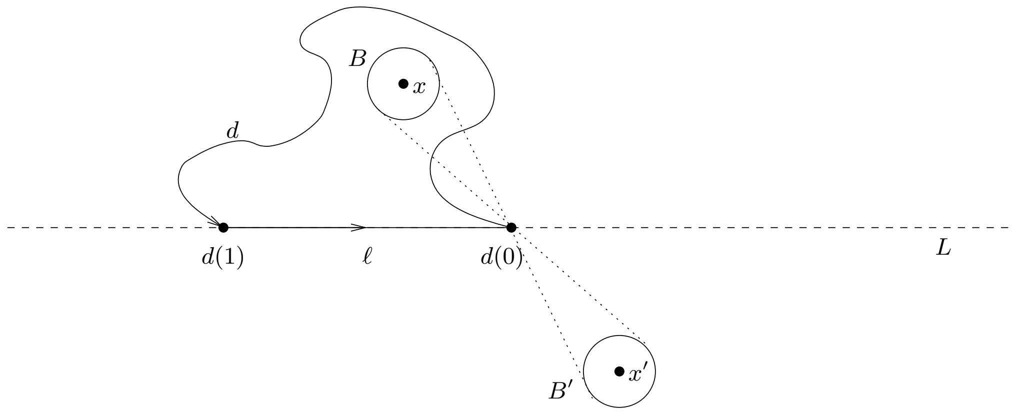


Figure 29: A typical path d satisfying the two properties of Lemma 16.4 when $\lambda = 2$. The loop ℓ has nonzero winding number about every point in the ball B centered at x. The ball B' centered at x' is also shown, and it lies entirely on the opposite side of L from B and d.

One readily checks that $\ell' \subseteq Q_\lambda(d)$, since it contains only λ-extrapolations of points on d. For convenience, we let $a := \ell'(1/3) = d(1) \star d(0)$. Note that a is colinear with $d(0)$ and $d(1)$, since λ is real. Figure 30 shows the ℓ' constructed from the path d of Figure 29.

The loop ℓ' is homologous to the sum of two separate loops: ℓ_1 is the loop closure of the first third of ℓ' (ℓ_1 follows ℓ' from $d(0)$ to a (dashed curve in Figure 30) then a straight line segment along L from a back to $d(0)$); ℓ_2 is the loop closure of the last two thirds of ℓ' (ℓ_2 first follows ℓ' from a around through $d(1)$ to $d(0)$, then goes straight from $d(0)$ to a along L). Notice that B' and ℓ_1 are the images of B and $\ell = lcl(d)$, respectively, under the map $z \mapsto z \star d(0)$. They are both obtained by first rotating by π about the point $d(0)$ followed by dilating about $d(0)$ by a factor of $\lambda - 1$. Thus by similarity, ℓ_1 has the same nonzero winding number about every point in B' as ℓ does about every point in B. Also notice that ℓ_2 lies entirely to the other side of L from B', because d does, and the middle third of ℓ' is just a dilation of d about $d(1)$ by a factor of λ. It follows that ℓ_2 has zero winding number about every point in B', and thus we conclude that ℓ' has the same nonzero winding number around B' as ℓ_1 does.

Finally, we exhibit the promised homotopy $h : [0,1] \times [0,1] \to \mathbb{C}$ from ℓ' to the constant loop $d(0)$ and staying inside $Q_\lambda(d)$: for all $s, t \in [0,1]$, define

$$h(s,t) := \begin{cases} d(3s(1-t)) \star d(0) & \text{if } 0 \le s \le 1/3, \\ d(1-t) \star d((3s-1)(1-t)) & \text{if } 1/3 \le s \le 2/3, \\ d((3-3s)(1-t)) & \text{if } 2/3 \le s \le 1. \end{cases}$$

One checks that h is the desired homotopy, that is, h is continuous, and for all $s, t \in [0,1]$, we have: $h(s,0) = \ell'(s)$; $h(0,t) = h(1,t) = d(0)$; $h(s,1) = d(0)$; and $h(s,t) \in Q_\lambda(d)$. (Geometrically, for fixed t, the loop $h(\cdot, t)$ is defined analogously to the curve ℓ', but using only the first $(1-t)$-fraction of d.) $\square$

We need one more lemma before we prove Theorem 16.2. The preceding proof does not quite work as is when $\lambda \notin \mathbb{R}$, because the point $x' = x \star d(0)$ shown in Figure 29 may not lie below the line L, which means it may be tangled up with d in such a way that the winding numbers of the

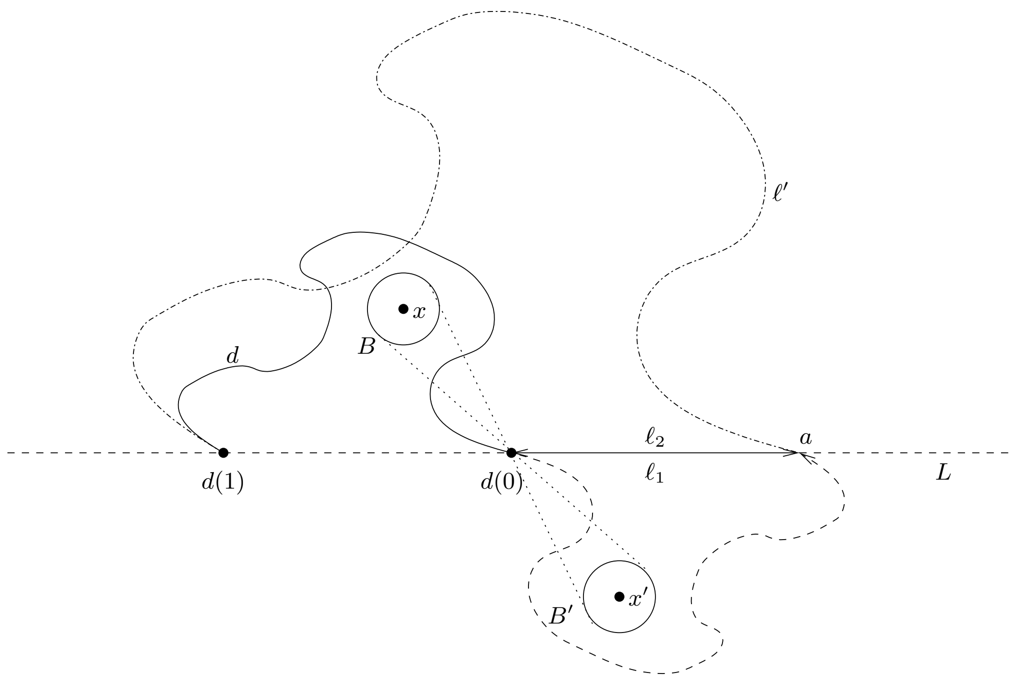


Figure 30: The loop ℓ' constructed from the path d of Figure 29. The loop starts at $d(0)$, follows the dashed curve to the point a, then the dots and dashes to $d(1)$, then the curve d (solid) backwards to $d(0)$. The sets B and B' are also shown. The line segment between $d(0)$ and a is used to split ℓ' into the sum of two loops: ℓ_1 lying below L and ℓ_2 lying above L.

two loops ℓ_1 and ℓ_2 (see Figure 30) may cancel, leaving a zero winding number of ℓ' about x' when we need it to be nonzero. To fix this, we do not use λ but instead use another point $\mu \in Q_\lambda$ that is close enough to being real that the point $x \star_\mu d(0)$ does lie below L. Then the whole proof of Proposition 16.5 goes through with μ replacing λ.

Remark. It is interesting and a bit frustrating to note that Theorem 3.4 almost suffices to prove Theorem 16.2 in the case where $\lambda \notin \mathbb{R}$, because $Q_\lambda(c)$ clearly has an accumulation point and $Q_\lambda(c) \subseteq F_\lambda = \mathbb{C}$. Unfortunately, Theorem 3.4 only applies to λ-clonvex sets, and so it only asserts that $R_\lambda(c) = \mathbb{C}$. This is enough to get $Q_\lambda(c)$ dense in $\mathbb{C}$ but does not quite show equality. $\square$

For any $z \neq 0$, we define $\arg z$ to be the unique $\theta \in [0, \tau)$ such that $z = |z|e^{i\theta}$.

Lemma 16.6. *For any $\lambda \in \mathbb{C}\backslash[0,1]$ and any $\varepsilon > 0$, there exists $\mu \in Q_\lambda\backslash\{1\}$ such that $\arg(\mu-1) < \varepsilon$.*

Proof. Assume, without loss of generality, that $\varepsilon < \pi/2$. It suffices to find a point $\mu \in Q_\lambda$ such that $\mathrm{Re}(\mu) > 2$ and $\arg \mu < \tan^{-1}((\tan \varepsilon)/2)$. That is, μ is somewhere in the shaded region in Figure 31.

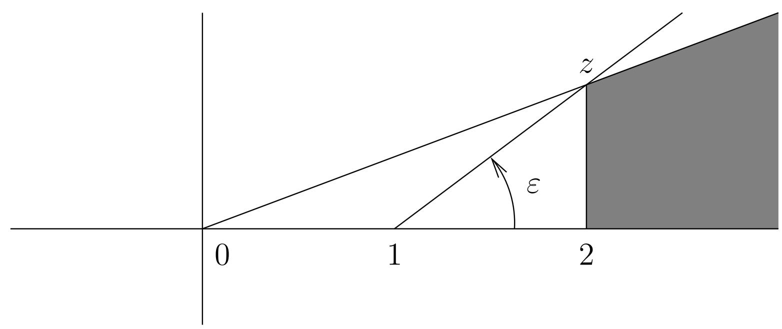


Figure 31: The shaded region is bounded by the real axis, the vertical line connecting 2 and $z = 2 + i \tan \varepsilon$, and the line from the origin through z. If μ is in the closure of this region and $\mu \neq z$, then $\arg(\mu - 1) < \varepsilon$.

We know that Q_λ is unbounded by Corollary 4.5. Fix some $\nu \in Q_\lambda$ with $|\nu| > 1$, and note that all positive powers of ν are in Q_λ. If $(\arg \nu)/\tau$ is rational, then there exists $n_0 \in \mathbb{Z}^+$ such that $\arg(\nu^{kn_0}) = 0$ for all $k \in \mathbb{Z}^+$; then pick k large enough so that $\mu := \nu^{kn_0}$ has real part > 2. If $(\arg \nu)/\tau$ is irrational, then a standard pigeonhole argument shows that the set $\{(n \arg \nu) \bmod \tau \mid n \in \mathbb{Z}^+\}$ is dense in $[0, \tau)$, and so contains infinitely many points in $[0, \varepsilon)$. Thus we can find an $n \in \mathbb{Z}^+$ such that $\arg(\nu^n) = ((n \arg \nu) \bmod \tau) < \varepsilon$ and $|\nu^n| = |\nu|^n$ is large enough to put ν^n in the interior of the shaded region. Set $\mu := \nu^n$. $\square$

Now we prove Theorem 16.2 by modifying the proof of Proposition 16.5 for nonreal λ.

Proof of Theorem 16.2. Let c be a bent path, and let λ be a point in $\mathbb{C} \setminus \mathbb{R}$. As in the proof of Proposition 16.5, we can assume c includes no nonempty open subset of $\mathbb{C}$ and replace c by a subpath d satisfying Lemma 16.4. As before, let x be given by that Lemma, let L be a straight line through $d(0)$ and $d(1)$ not containing x, and let $\ell := lcl(d)$. By extending d a little bit along L

if necessary, we can assume that $d(0) \neq d(1)$. By reversing d if necessary, we can also assume that

$$\mathrm{Im}\left(\frac{x - d(1)}{d(0) - d(1)}\right) > 0\ ,$$

that is, the three points $d(1), d(0), x$ are oriented counterclockwise as they are in Figure 29.

Now let ε be the angle $\angle x, d(0), d(1)$ formed by rays from $d(0)$ through x and $d(1)$, respectively. That is,

$$\varepsilon = \arg\left(\frac{d(1) - d(0)}{x - d(0)}\right)\ .$$

By our choice of orientation, we know that $0 < \varepsilon < \pi$. By Lemma 16.6, there exists a $\mu \in Q_\lambda$ such that $\arg(\mu - 1) < \varepsilon$. By part (1.) of Lemma 2.25, we have $Q_\mu(d) \subseteq Q_\lambda(d)$, and so it suffices to show that $Q_\mu(d)$ contains a nonempty open subset of $\mathbb{C}$. This will be just as we did in the proof of Proposition 16.5 but with the "almost real" μ replacing the real λ. If μ is in fact real, then $\mu > 1$, and we are done by Proposition 16.5, and so we assume $\mu \notin \mathbb{R}$, whence $\mathrm{Im}(\mu) > 0$. It follows that for any $z, w \in \mathbb{C}$, the three points $z, w, z \star_\mu w$ are oriented counterclockwise.

Set $x' := x \star_\mu d(0)$. Then—and this is the crucial point—x' lies opposite the line L from x, as shown in Figure 32.

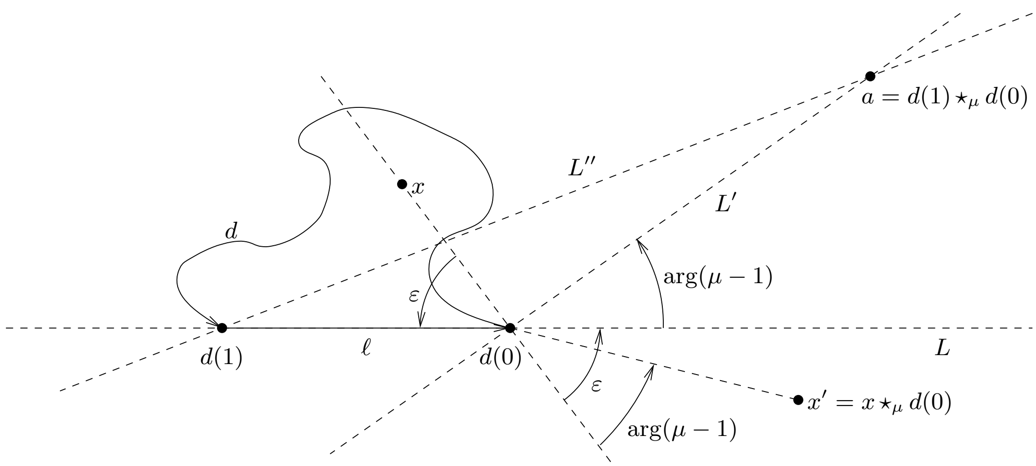


Figure 32: The point $x' = x \star_\mu d(0)$ (lower right) lies below the line L, whereas x lies above L. The point $a = d(1) \star_\mu d(0)$ is also shown, as well as various lines and angles, including the line L' through $d(0)$ and a and the line L'' through $d(1)$ and a. The point x' must also lie on the opposite side of L' as $d(1)$, and hence on the same side of L'' as $d(0)$.

Figure 32 is analogous to Figures 29 and 30. As before, let $a := d(1) \star_\mu d(0)$. Let L' be the line through $d(0)$ and a, and let L'' be the line through $d(1)$ and a (see Figure 32). Notice that x' must be on the opposite side of L' from $d(1)$, and this together with the position of x' with respect to L implies that x' must be on the same side of L'' as $d(0)$. As before, we can find an open ball B surrounding x such that: (i) ℓ has the same nonzero winding number about every $y \in B$ as it has about x; and (ii) the open ball $B' = \{y \star_\mu d(0) \mid y \in B\}$ surrounding x' intersects none of the three lines L, L', or L''.

Now we define the loop $\ell' \subseteq Q_\mu(d)$ similar to the proof of Proposition 16.5.

$$\ell'(s) := \begin{cases} d(3s) \star_\mu d(0) & \text{if } 0 \le s \le 1/3, \\ d(1) \star_\mu d(3s-1) & \text{if } 1/3 \le s \le 2/3, \\ d(3-3s) & \text{if } 2/3 \le s \le 1. \end{cases}$$

The first third of ℓ' runs from $d(0)$ to a and lies opposite L' from $d(1)$. As before, its loop closure ℓ_1 is a rotated, dilated copy of ℓ and so has the same nonzero winding number about every point in B' as ℓ does about x. The middle third of ℓ' runs from a to $d(1)$ and stays on the other side of L'' from $d(0)$, and hence also from B'. Finally, our choice of μ ensures that the last third of ℓ', which coincides with d, stays on the side of L opposite B'. Thus the loop closure ℓ_2 of the final two thirds of ℓ' cannot contribute to the winding number of ℓ' about any point in B'. It follows that ℓ' has the same nonzero winding number about every point in B' as ℓ_1 has.

We define the homotopy h just as before, but with $\star_\mu$ instead of $\star_\lambda$:

$$h(s,t) := \begin{cases} d(3s(1-t)) \star_\mu d(0) & \text{if } 0 \le s \le 1/3, \\ d(1-t) \star_\mu d((3s-1)(1-t)) & \text{if } 1/3 \le s \le 2/3, \\ d((3-3s)(1-t)) & \text{if } 2/3 \le s \le 1. \end{cases}$$

This homotopy stays within $Q_\mu(d)$ and contracts ℓ' to the constant point $d(0)$, whose winding number about any point in B' is zero. Thus the curve must pass through each point in B' sometime during the deformation, and this puts $B' \subseteq Q_\mu(d)$ as before. Hence $\mathbb{C} = Q_\mu(d) = Q_\lambda(c)$. $\square$

Part IV: Concluding Remarks

17 Conjectures, open problems, and future research

We have many more questions than we can investigate in any reasonable length of time. We only give a sampling in this section. Some may be easy, but we have just not looked at them in depth.

Recall the set $\mathcal{C}$ (Definition 2.21) and its complement $\mathcal{D} := \mathbb{C} \setminus \mathcal{C}$. We know that $\mathcal{D}$ is closed, discrete, and contains only algebraic integers (Theorem 8.2). We also know that $\mathcal{D}$ contains all strong PV numbers (Theorem 12.14), but we know of no other elements of $\mathcal{D}$ than these.

Conjecture 17.1. *Q_λ is discrete if and only if λ is a strong PV number.*

To make progress towards this conjecture, we can use various constructions we have developed to carve out more territory for $\mathcal{C}$ in the complex plane. This approach was started in our paper. Pinch [30, Theorem 15] showed that every real, non-integer element of $\mathcal{D}$ must have at least one conjugate in $(0, 1)$, but beyond these facts, we know little about $\mathcal{C}$ and $\mathcal{D}$.

There are a number of open questions about Q_λ when λ belongs to a discrete subring D of $\mathbb{C}$. For example, we conjecture that equality holds in Lemma 10.4 for every such D. We know that it holds for $D := \mathbb{Z}$ (Theorem 10.12).

Conjecture 17.2. *For any discrete subring D of $\mathbb{C}$ and for any $\lambda \in D$,*

$$Q_\lambda = \lambda(1-\lambda)D + \{0, 1, \lambda, 1-\lambda\} . \tag{65}$$

The next open question (and its generalizations) is one of the most interesting.

Open Question 17.3. *For which sPV λ does set equality hold in Eq. (38)?*

We have settled this question for real quadratic sPV λ (Theorem 15.7), strengthening previous results of Masáková et al. [23] obtained via different means. When equality holds, Q_λ is a model set and relative density of Q_λ follows immediately. We thus know Q_λ is not a model set of the form Σ_P for most real quadratic sPV λ. Indeed we have now proved that $4+\sqrt{13} \notin Q_{(5+\sqrt{13})/2}$ (Corollary 15.33), a result hinted at by computer. Nonetheless, we conjecture that relative density holds in general; see Open Question 17.5 below. The resulting sets would thus be Meyer sets but not model sets of the form Σ_P.

Research Plan 17.4. *Generalize the proof that $4+\sqrt{13} \notin Q_{(5+\sqrt{13})/2}$ to prove nonmembership in Q_λ of as many specific points as possible for as many λ as possible. Use computation to suggest such points.*

All this leads to a general open question:

Open Question 17.5. *Is Q_λ relatively dense in $\mathbb{R}$ for all $\lambda \in \mathbb{R} \setminus [0, 1]$? Is Q_λ relatively dense in $\mathbb{C}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$? (If not, then for which λ?)*

If true, then the discrete Q_λ are all Meyer sets by Corollary 15.31. We conjecture that this is indeed true, but so far we can only prove relative density in the restricted cases addressed in Section 15, particularly Corollary 15.27. Proving relative density more generally will apparently require new techniques. Failing a general proof of relative density, we could at least apply the technique of Corollary 15.34 to show relative density of Q_λ for other individual unitary quadratic sPV λ.

The technique we used to prove Theorem 15.10 only works when applied to quadratic sPV λ such that $\Sigma_P^{(\lambda)}$ contains a non-trivial unit. We believe it holds for any quadratic sPV number λ, either via an alternative proof, or because every $\Sigma_P^{(\lambda)}$ contains a non-trivial unit,

Conjecture 17.6. *If λ is quadratic sPV, the corresponding cut-and-project set Σ_P is finitely generated, that is, there exists a finite $Y \subseteq \Sigma_P$ such that $Q_\lambda(Y) = \Sigma_P$.*

Of far greater consequence is the case of arbitrary degree, which is partially answered in Theorem 15.12, in that case under the stronger hypothesis that Q_λ contains a non-trivial unit. Extending the above, we conjecture the following, further generalizing Theorem 15.12 so that it holds for containment of the entire set Σ_P in $Q_\lambda(Y)$.

Conjecture 17.7. *For any sPV λ, there exists a finite set $Y \subseteq \Sigma_P$ such that $\Sigma_P \subseteq Q_\lambda(Y)$.*

17.1 L-convex, auto-convex, and λ-semiconvex sets

Definition 17.8. Let L and S be any subsets of $\mathbb{C}$.

- S is *L-convex* iff S is λ-convex for all $\lambda \in L$.
- Let $Q_L(S)$ be the least L-convex superset of S.
- S is *auto-convex* iff S is S-convex.

All the Q_λ sets are auto-convex by the following result:[24]

Proposition 17.9. *For any $\lambda, \mu \in \mathbb{C}$, if $\mu \in Q_\lambda$, then Q_λ is μ-convex, and consequently, $Q_\mu \subseteq Q_\lambda$.*

The same goes for all R_λ. The converse does not hold, however: every subring of $\mathbb{C}$ is clearly autoconvex, but there is no $\lambda \in \mathbb{C}$ such that $Q_\lambda = \mathbb{Z}[2i]$, for example. Instead, we hazard a weaker conjecture.

Conjecture 17.10. *For any $\lambda \in \mathbb{C}$, if $S \subseteq R_\lambda$ is auto-convex, then $S = R_\mu$ for some $\mu \in R_\lambda$.*

This conjecture implies that the set $\{R_\lambda \mid \lambda \in \mathbb{C}\}$ is closed under arbitrary intersections, because the intersection of any family of closed, auto-convex sets is clearly auto-convex (and closed).

Definition 17.11. Fixing $\lambda \in \mathbb{C}$, we will say that a set $S \subseteq \mathbb{C}$ is *λ-semiconvex* iff, for every $a, b \in S$, at least one of the points $a \star_\lambda b$ and $b \star_\lambda a$ is in S.

Clearly, if S is λ-semiconvex, then S is also $(1-\lambda)$-semiconvex. Note that the intersection of two λ-semiconvex sets need not be λ-semiconvex.

Research Plan 17.12. *For which $\lambda \in \mathbb{C}$ do there exist nontrivial, bounded λ-semiconvex sets?*

It is not too hard to see that if $|\lambda - 1/2| \leq 1/2$, then any disk (closed or open) is λ-semiconvex. There are at least two values of λ outside this range where bounded λ-semiconvex sets exist: if $\lambda = (1 \pm i\sqrt{3})/2$, then the vertices of any equilateral triangle form a λ-semiconvex set. What other such λ are there?

[24] Pinch proved a more general result for real λ: If $S \subseteq \mathbb{R}$ is auto-convex, then $Q_\lambda(S)$ is auto-convex [30]. The proof generalizes trivially to the complex numbers.

17.2 Computational Questions

Questions such as Research Plan 17.4 lead us to the following line of inquiry: If Q_λ is discrete, then λ is an algebraic integer, and thus each element of Q_λ can be expressed (encoded) as an integer polynomial in λ, of degree bounded by the degree of λ. Q_λ is countably infinite. Furthermore, it is computably enumerable: If $x \in Q_\lambda$, by enumerating all extrapolations beginning with $\{0,1\}$, we will eventually obtain x. It is conceivable by similar reasoning that Q_λ is in NP; however it is not at all obvious that a succinct proof that $x \in Q_\lambda$ (to say nothing of $x \notin Q_\lambda$) can always be found, and its computational complexity is wide open[25]. For those λ that affirmatively answer Open Question 17.3, we have an effective procedure to determine if any x is in Q_λ. For other λ, however, especially if λ is not sPV but Q_λ is nevertheless discrete, we do not know if Q_λ is decidable (although we do know that $Q_{[x]}$ is decidable).

Open Question 17.13. *If Q_λ is discrete, is it decidable? Uniformly in λ? If it is decidable, what is its computational complexity? When is it true that $Q_\lambda \in \mathsf{NP}$, and how does this depend on λ?*

Note that if Conjecture 17.1 is false, it is possible that the answer depends on whether or not λ is a strong PV number. We conjecture that Q_λ is always decidable, but have no intuition regarding its containment in NP, to say nothing of P.

17.3 Miscellaneous Open Problems

It would be interesting to pin down $\mathcal{C} \cap \mathbb{R}$ and $\mathcal{C} \cap \{z \in \mathbb{C} \mid \mathrm{Re}(z) = 1/2\}$. These two cases may be easier than the general case, as they present symmetries not shared by all λ.

Research Plan 17.14. *Determine which $\lambda > 3$ yield $R_\lambda = \mathbb{R}$. Determine which λ with $\mathrm{Re}(\lambda) = 1/2$ yield $R_\lambda = \mathbb{C}$.*

Research Plan 17.15. *Get a reasonably good graphical picture of $\mathcal{D}$.*

Recall that by $Q_{[x]}$ we denote the set of polynomials in $\mathbb{Z}[x]$ generated by the constant polynomials 0 and 1, and by repeated applications of $\star_x$; that is, $Q_{[x]}$ is the smallest set of polynomials containing $0,1$ and closed under the binary operation $(p,q) \mapsto (1-x)p + xq$.

$Q_{[x]}$ has some interesting properties. Recall that any element of Q_λ can be written as $p(\lambda)$ where $p \in Q_{[x]}$, and conversely. Pinch showed [30, Corollary 4.1] that an integer polynomial p is in $Q_{[x]}$ if and only if there exist $n \geq 0$ and integers $b_0, \ldots, b_n$ such that $p(x) = \sum_{i=0}^{n} b_i x^i (1-x)^{n-i}$ and $0 \leq b_i \leq \binom{n}{i}$ for all $0 \leq i \leq n$ (see Lemma 7.8). We have an alternate characterization of $Q_{[x]}$: an integer polynomial p is in $Q_{[x]}$ if and only if either $p \in \{0,1\}$ or $0 < p(\mu) < 1$ for all $0 < \mu < 1$ (see Theorem 11.1). This latter characterization can be used to computably enumerate the integer polynomials *not* in $Q_{[x]}$, leading to a decision procedure for $Q_{[x]}$. What, then, is the complexity of deciding $Q_{[x]}$?

There are other interesting (though noncomputational) questions regarding $Q_{[x]}$. We can list all 14 polynomials in $Q_{[x]}$ of degree ≤ 2, and get a finite upper bound on the number of polynomials in $Q_{[x]}$ of any given degree bound. However, we don't even know how many polynomials there are in $Q_{[x]}$ of degree 3.

Open Question 17.16. *How many elements of $Q_{[x]}$ are there of degree 3?*

[25] See, e.g., [32] for information on the relevant computability and complexity notions discussed here.

Our techniques give an upper bound of 717, and an extensive computer search finds only 90. Perhaps Fact 11.11 can reduce the upper bound.

Other open problems include the following:

Call the triangle with vertices $(0, 1, \lambda)$ the *fundamental triangle*. George McNulty offers the following conjecture [25]:

Conjecture 17.17 (McNulty). *If Q_λ contains a point in the interior of the fundamental triangle, then R_λ is convex.*

Definition 17.18. We will call a discrete set Q_λ *maximal* if it is not a proper subset of any other discrete Q_μ.

Open Question 17.19. *Do maximal Q_λ exist? Is there an easy way to characterize the λ such that Q_λ is maximal? Is there an interesting notion of minimal Q_λ?*

Open Question 17.20. *Is there an easy way to characterize the minimum polynomials of strong PV numbers? Short of that, find such polynomials of higher and higher degree. (We currently can characterize all such polynomials of degree ≤ 4.)*

Open Question 17.21. *We say that a λ-convex set A is* cohesive *if $A = Q_\lambda(A \setminus T)$ for any finite set T. We can show that Σ_P (the right-hand side of Eq. (38)) is cohesive for all unitary quadratic sPV λ—if you remove 0 and 1. This implies that there are infinitely many points "missing" from $Q_{(5+\sqrt{13})/2}$, that is, $\Sigma^{(\lambda)}_{[0,1]} \setminus Q_\lambda$ is infinite for $\lambda = (5 + \sqrt{13})/2$. For which λ is $Q_\lambda \setminus \{0, 1\}$ cohesive? All sPV numbers, perhaps?*

Open Question 17.22. *Let A be λ-convex as in the previous question. An* essential point *of A is some $x \in A$ such that $A \setminus \{x\}$ is λ-convex[26]. For example, 0 and 1 are both essential points of Q_λ for all sPV λ. The question is: if $Q_\lambda \setminus \{0, 1\}$ is not cohesive, must it have an essential point? More generally, what is the smallest size of a set you can remove from a noncohesive set that leaves a λ-convex set?*

[26] And hence $x \notin Q_\lambda(A \setminus \{x\})$

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A Appendix: Supporting Facts

In this appendix we cover a few propositions that support our results but are not central to them.

The next proposition backs up an assertion made in the remark on page 65 following the proof of the main theorem of Section 12—Theorem 12.14.

Proposition A.1. *Let D be any discrete subring of $\mathbb{C}$. Suppose that $p \in D[x]$ is a monic polynomial of degree $d > 0$ such that all but one of its roots lie in the open unit interval $(0, 1)$. Then its remaining root $\lambda \notin (0, 1)$ is a strong PV number.*

Proof. Let $\mu_0, \ldots, \mu_{d-2} \in (0, 1)$ be the roots of p other than λ. We have two cases:

Case 1: $\lambda \in \mathbb{R}$**.** In this case, all the roots of p are real, which implies all the coefficients of p are real. These coefficients are thus integers by Lemma 10.1, and so p is in $\mathbb{Z}[x]$, making λ an algebraic integer. The conjugates of λ are among $\mu_0, \ldots, \mu_{d-2}$ (not necessarily all; we are not assuming p is irreducible). Thus λ is a strong PV number in this case.

Case 2: $\lambda \notin \mathbb{R}$**.** Let p^* be the polynomial obtained by complex-conjugating all the coefficients of p. We have that p^* is also in $D[x]$ (this follows from Lemma 10.2, for example), and the roots of p^* are $\mu_0, \ldots, \mu_{d-2}, \lambda^*$. Letting $q := pp^*$, we see that q is monic, that $q \in D[x]$, and that

$$q(x) = (x - \lambda)(x - \lambda^*)(x - \mu_0)^2 \cdots (x - \mu_{d-2})^2 .$$

The product of the first two factors is in $\mathbb{R}[x]$, as are each of the other factors, and so $q \in \mathbb{R}[x]$. But then $q \in \mathbb{Z}[x]$ by Lemma 10.1, whence λ is an algebraic integer. The conjugates of λ are evidently among $\mu_0, \ldots, \mu_{d-2}$ and λ^*, and so this again makes λ a strong PV number.

□

The next theorem is a standard result in algebra, and its corollary justifies an assertion made in the proof of Proposition 13.2. Background concepts are taken from Chapter 4 of Jacobson [16].

Theorem A.2. *Let F be a field and let polynomial $f \in F[x]$ of degree $d > 0$ be separable*[27] *and irreducible over F. Let E be a splitting field of f over F, and let $R := \{\mu_0, \ldots, \mu_{d-1}\}$ be the set of roots of f in E. Let $p \in F[x]$ be a polynomial over F, and let $z := p(\mu_0) \in E$. Then z is algebraic over F, and its conjugates are $p(\mu_0), \ldots, p(\mu_{d-1}) \in E$ (not necessarily distinct). Furthermore, the mapping p restricted to R is m-to-one for some positive integer m.*

Proof. Note that $|R| = d$ (i.e., the μ_i are pairwise distinct), because f is separable and irreducible.

E is finite dimensional (as a vector space) over F, and so E is an algebraic extension of F, making z algebraic over F. (Indeed, E is a Galois extension of F.) Let $G := \mathrm{Gal}(E/F)$ be the Galois group of E/F.[28] Then G acts on R, i.e., each element of G permutes the elements of R. In fact, G is isomorphic to the group of permutations on R induced by G. We also know that G acts transitively on R, because f is irreducible [16, Theorem 4.14, p. 259]. In particular, $R = \{\eta(\mu_0) : \eta \in G\}$. Let $S := \{p(\mu) : \mu \in R\} = \{p(\mu_i) : 0 \le i < d\}$. Then clearly, G also acts on S. Furthermore, this action is also transitive, which can be seen as follows: We have

$$S = \{p(\eta(\mu_0)) : \eta \in G\} = \{\eta(z) : \eta \in G\} ,$$

[27] A polynomial $f \in F[x]$ is *separable* if each of its irreducible factors has distinct roots in any splitting field of f over F; equivalently, f is coprime with its formal derivative. If $\mathrm{char}\, F = 0$, then every nonzero $f \in F[x]$ is separable.

[28] That is, the group of field automorphisms of E that leave F pointwise fixed.

and so for any $\mu, \nu \in S$, there exist $\eta, \theta \in G$ such that $\mu = \eta(z)$ and $\nu = \theta(z)$. Then $\nu = \theta(z) = (\theta\eta^{-1})(\mu)$.

Let $g \in F[x]$ be the minimum (monic) polynomial of z. For every $\eta \in G$ we have

$$0 = \eta(0) = \eta(g(z)) = g(\eta(z)) = g(\eta(p(\mu_0))) = g(p(\eta(\mu_0))) = g(p(\mu_i))$$

for the unique $0 \le i < d$ such that $\mu_i = \eta(\mu_0)$. Thus $p(\mu_i)$ is a conjugate of z, and because G acts transitively on S, all elements of S are conjugates of z. This shows one direction of the theorem; it remains to show that z has no other conjugates but these.

Let $c > 0$ be the degree of g (and of z). We have $F \subseteq F(z) \subseteq F(\mu_0) \subseteq E$, and $[F(z) : F] = c$.[29] Set $H := \mathrm{Gal}(E/F(z))$. Then H is a subgroup of G (the automorphisms of G that fix $F(z)$ pointwise), and by the fundamental Galois pairing [16, p. 239], we have

$$\frac{|G|}{|H|} = \frac{[E : F]}{[E : F(z)]} = \frac{[E : F]}{[E : F]/[F(z) : F]} = [F(z) : F] = c\ .$$

Now consider how G and H act on S. The group H contains exactly those elements of G that fix z: since z generates $F(z)$, any automorphism of E/F that fixes z also fixes every element of $F(z)$. Thus $H = \mathrm{Stab}\ z$, the stabilizer of z in G. A standard result of group theory (see [16, Theorem 1.10, p. 75] and the text that follows the proof) is that $|S| = |G|/|\mathrm{Stab}\ z|$, because G acts transitively on S. Thus we have

$$|S| = \frac{|G|}{|\mathrm{Stab}\ z|} = \frac{|G|}{|H|} = c\ ,$$

and this implies that all the conjugates of z (c many of them) lie in S.

To prove the last statement, we observe that for all $z \in S$ and $\eta \in G$ we have

$$\eta(\{\mu \in R : p(\mu) = z\}) = \{\mu \in R : p(\mu) = \eta(z)\}\ ,$$

whence the statement follows by the transitivity of G acting on S, and for each $z \in S$ the set $\{\mu \in R : p(\mu) = z\}$ has size $m := |R|/|S| = d/c$. □

Define the *field norm* $N(z)$ of $z \in E$ to be the product of its conjugates. Up to change of sign, $N(z)$ is the constant term in the minimal (monic) polynomial of z.

Corollary A.3. *Let F be a field with characteristic 0, let μ be algebraic over F, and let $p \in F[x]$ be a polynomial. Then $\prod_\nu p(\nu) = N(p(\mu))^m$ for some positive integer m, where ν runs through the conjugates of μ.*

Proof. Let $f \in F \in F[x]$ be the minimal polynomial of μ, and let E be a splitting field of f over F. Let d be the degree of f (i.e., of μ), let c be the degree of $p(\mu)$, and let $m := d/c$. Then by Theorem A.2, each conjugate of $p(\mu)$ has multiplicity m in the multiset $\{p(\nu) : \nu \text{ conjugate to } \mu\}$ of the conjugates of $p(\mu)$. □

[29] For any field extension K of F, $[K : F]$ is the *index* of K over F, i.e., the dimension of K viewed as a vector space over F.

Remark. The theorem and corollary above both go through for any F-rational function $p \in F(x)$, not necessarily a polynomial. $\square$

The corollary below follows by setting $F := \mathbb{Q}$ and recalling that all polynomials are separable in characteristic 0.

Corollary A.4. *If λ is algebraic (over $\mathbb{Q}$) with conjugates $\mu_0, \ldots, \mu_{d-1}$ and $p \in \mathbb{Q}[x]$ is a polynomial, then $p(\lambda)$ is algebraic with conjugates $p(\mu_0), \ldots, p(\mu_{d-1})$.*

Corollary A.5. *If λ is an algebraic integer and $z \in \mathbb{Z}[\lambda]$, then z is an algebraic integer, and for every conjugate c of z, there exists a conjugate μ of λ such that $c = h_\mu(z)$, where $h_\mu : \mathbb{Z}[\lambda] \to \mathbb{Z}[\mu]$ is the unique ring isomorphism mapping λ to μ.*

Proof. This follows from Corollary A.4 and the fact (see Jacobson [16, Theorem 4.23] for example) that the algebraic integers form a subring of $\mathbb{C}$. $\square$

B Appendix: Computer-Aided Derivations

The following table gives a derivation in Q_λ of $\alpha = 9 - 16\lambda$, the fundamental unit of $\mathbb{Z}[\lambda]$, where $\lambda = -(3+\sqrt{17})/2$ (minimal polynomial x^2+3x-2), used in the proof of Proposition 15.18. This may not be the shortest derivation possible; it was found by a program that favors combining points with small absolute value. Here, $\star$ means $\star_\lambda$.

point	value	equals
p_1	$= 1 - \lambda$	$= 1 \star 0$
p_2	$= 3 - 5\lambda$	$= p_1 \star 0$
p_3	$= 13 - 23\lambda$	$= p_2 \star 0$
p_4	$= \lambda$	$= 0 \star 1$
p_5	$= 2 - 3\lambda$	$= 0 \star p_4$
p_6	$= -3 + 6\lambda$	$= p_1 \star p_5$
p_7	$= 12 - 21\lambda$	$= 0 \star p_6$
p_8	$= 17 - 30\lambda$	$= p_3 \star p_7$
p_9	$= -1 + 2\lambda$	$= p_3 \star p_8$
p_{10}	$= -5 + 9\lambda$	$= p_9 \star 0$
p_{11}	$= -23 + 41\lambda$	$= p_{10} \star 0$
p_{12}	$= -105 + 187\lambda$	$= p_{11} \star 0$
p_{13}	$= 4 - 7\lambda$	$= 0 \star p_9$
p_{14}	$= -14 + 25\lambda$	$= 0 \star p_{13}$
p_{15}	$= 27 - 48\lambda$	$= p_{10} \star p_{14}$
p_{16}	$= -96 + 171\lambda$	$= 0 \star p_{15}$
p_{17}	$= -137 + 244\lambda$	$= p_{12} \star p_{16}$
α	$= 9 - 16\lambda$	$= p_{12} \star p_{17}$

The following table gives the derivation of Y (from $\{0,1\}$, so that $Y \subseteq R_\lambda$) of the set Y used in Lemma 15.20.

point	element of Y	value	equals
p_0	$= p(0)$	$= 0$	- -
p_1	$= 1$	$= 1$	- -
p_2	$= \lambda$	$= \lambda$	$= p_0 \star p_1$
p_3	$= 1 - \lambda$	$= 1 - \lambda$	$= p_1 \star p_0$
p_4	$= p(1)$	$= 1 - \lambda + \lambda^2$	$= p_1 \star p_2$
p_5	$= p(-1)$	$= \lambda - \lambda^2$	$= p_0 \star p_3$
p_6	$= p(2)$	$= 1 - 2\lambda + 2\lambda^2$	$= p_3 \star p_2$
p_7	$= p(-2)$	$= 2\lambda - 2\lambda^2$	$= p_2 \star p_3$
p_8	$= p(3)$	$= 1 - 3\lambda + 3\lambda^2$	$= p_5 \star p_3$
p_9	$= p(-3)$	$= 3\lambda - 3\lambda^2$	$= p_4 \star p_2$
p_{10}	$= p(4)$	$= 1 - 4\lambda + 4\lambda^2$	$= p_5 \star p_0$
p_{11}	$= p(-4)$	$= 4\lambda - 4\lambda^2$	$= p_4 \star p_1$
p_{12}	$= p(5)$	$= 1 - 4\lambda + 5\lambda^2$	$= p_0 \star p_4$
p_{13}	$= p(-5)$	$= 4\lambda - 5\lambda^2$	$= p_1 \star p_5$
p_{14}	$= p(6)$	$= 2 - 6\lambda + 6\lambda^2$	$= p_3 \star p_4$
p_{15}	$= p(-6)$	$= -1 + 6\lambda - 6\lambda^2$	$= p_2 \star p_5$
p_{16}	$= p(7)$	$= 2 - 7\lambda + 7\lambda^2$	$= p_7 \star p_3$
p_{17}	$= p(-7)$	$= -1 + 7\lambda - 7\lambda^2$	$= p_6 \star p_2$
p_{18}	$= p(8)$	$= 2 - 8\lambda + 8\lambda^2$	$= p_7 \star p_0$
p_{19}	$= p(-8)$	$= -1 + 8\lambda - 8\lambda^2$	$= p_6 \star p_1$
p_{20}	$= p(9)$	$= 2 - 8\lambda + 9\lambda^2$	$= p_7 \star p_2$
p_{21}	$= p(-9)$	$= -1 + 8\lambda - 9\lambda^2$	$= p_6 \star p_3$
p_{22}	$= p(10)$	$= 2 - 9\lambda + 10\lambda^2$	$= p_0 \star p_6$
p_{23}	$= p(-10)$	$= -1 + 9\lambda - 10\lambda^2$	$= p_1 \star p_5$
p_{24}	$= p(11)$	$= 3 - 11\lambda + 11\lambda^2$	$= p_3 \star p_6$
p_{25}	$= p(-11)$	$= -2 + 11\lambda - 11\lambda^2$	$= p_2 \star p_7$
p_{26}	$= p(12)$	$= 3 - 12\lambda + 12\lambda^2$	$= p_9 \star p_0$
p_{27}	$= p(-12)$	$= -2 + 12\lambda - 12\lambda^2$	$= p_8 \star p_1$
p_{28}	$= p(13)$	$= 3 - 12\lambda + 13\lambda^2$	$= p_9 \star p_2$
p_{29}	$= p(-13)$	$= -2 + 12\lambda - 13\lambda^2$	$= p_8 \star p_3$
p_{30}	$= p(14)$	$= 3 - 13\lambda + 14\lambda^2$	$= p_5 \star p_6$
p_{31}	$= p(-14)$	$= -2 + 13\lambda - 14\lambda^2$	$= p_4 \star p_7$
p_{32}	$= p(15)$	$= 3 - 14\lambda + 15\lambda^2$	$= p_0 \star p_8$
p_{33}	$= p(-15)$	$= -2 + 14\lambda - 15\lambda^2$	$= p_1 \star p_9$
p_{34}	$= p(16)$	$= 4 - 16\lambda + 16\lambda^2$	$= p_3 \star p_8$
p_{35}	$= p(-16)$	$= -3 + 16\lambda - 16\lambda^2$	$= p_2 \star p_9$
p_{36}	$= p(17)$	$= 4 - 16\lambda + 17\lambda^2$	$= p_{11} \star p_2$
p_{37}	$= p(-17)$	$= -3 + 16\lambda - 17\lambda^2$	$= p_{10} \star p_3$

Table 2: Derivation of Y

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