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2023-12-15

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D. Glassmeyer, A. Brakoniecki, J.M. Amador. "Comparing Elementary and Secondary Teachers’ Robust Understanding of Proportional Reasoning" International Journal of Science and Mathematics Education. https://doi.org/10.1007/s10763-023-10437-z

https://hdl.handle.net/2144/48714

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Comparing Elementary and Secondary Teachers' Robust Understanding of Proportional Reasoning

Abstract:

Identifying the knowledge resources teachers productively and unproductively draw upon can provide a means by which to create support structures to develop a more robust understanding of the content. To provide more informed grade-level support structures in teacher education programs, this study examined the knowledge resources 20 secondary pre-service teachers (PSTs) and 13 elementary PSTs drew upon when solving a comparison proportional reasoning problem. Data from written work and videos of PSTs' explanations were analyzed using the Robust Understanding of Proportional Reasoning for Teaching framework. Both elementary and secondary PSTs ubiquitously drew upon the same four knowledge resources (comparison of quantities, ratios, proportional situation, and ratio as measure). Elementary PSTs were more apt to counterproductively draw upon the knowledge resource ratios ≠ fractions, while secondary PSTs more often counterproductively drew upon equivalence. Mathematics educators can leverage the knowledge resources afforded by this task and strategically highlight productive and counterproductive resources to tailor instruction that develops PSTs' robust understanding of proportional reasoning.

Keywords: elementary pre-service teachers; knowledge resources, proportional reasoning; secondary pre-service teachers

Introduction

Ratios and proportions are essential to understand mathematics and science content spanning the secondary and post-secondary curriculum, including scale, probability, percent, rate, trigonometry, equivalence, measurement, algebra, planar geometry, density, molarity, speed, acceleration, and force (Burgos & Godino, 2020; Dole et al. 2012). This importance is reflected by educational systems in countries emphasizing reasoning about ratios and proportions in national teaching and learning standards (Australian Curriculum, Assessment & Reporting Authority, 2017; NCTM, 2020). Despite the importance of proportional reasoning as a mathematical domain, research on the topic is underdeveloped and has primarily focused on student development, negating a focus on how teachers reason about proportions (Lamon, 2007; Weiland et al., 2021). An increased focus on knowing teachers' understanding of proportional reasoning is needed since their understanding relates to students' proportional reasoning (Hilton et al., 2016). Knowing how teachers understand and reason about proportions can inform teacher education and professional development efforts, supporting a deeper and more connected understanding of proportional reasoning as well as them knowing and using explicit teaching strategies to enhance their students' proportional reasoning (Copur-Gencturk et al., 2022; Hilton et al., 2016).

Understanding how pre-service teachers (PSTs) reason proportionally is also needed, as PSTs' understanding of mathematics concepts can be drastically different from in-service teachers, given the different levels of understanding developed through years of teaching (Copur Gencturk & Doleck, 2021; Copur-Gencturk et al., 2022). Having an understanding of how PSTs reason proportionally can help mathematics educators develop support structures specific to PST populations to better prepare them to support students' proportional reasoning (Copur-Gencturk et al., 2020, 2022).

While much work exists detailing PSTs' proportional reasoning, specifically their difficulties or shortcomings (Cabero-Fayos et al., 2020; Copur-Gencturk et al., 2022; Johnson, 2017), more research is needed that: (a) describes PSTs' proportional reasoning by identifying the knowledge invoked in a situation and how that knowledge is interconnected to a network of understanding (Arican, 2021; Brown et al., 2019, 2020a; Weiland et al., 2021); (b) identifies opportunities to develop PSTs' proportional reasoning in relation to the mathematical content taken in various teacher education programs (Joshua & Lee, 2022), specifically about how PSTs in elementary education programs differ in their understanding of proportional reasoning about PSTs in secondary education programs (Son, 2013); and (c) details teachers' reasoning about multiplicative relationships when more than two quantities are present (Authors, year; Arican, 2018; Copur-Gencturk et al., 2022). This study was designed to address these three calls for research by (a) using a framework that acknowledges the knowledge resources PSTs invoke (Weiland et al., 2021); (b) classifying and comparing proportional reasoning knowledge resources from PSTs in an elementary education program with PSTs in a secondary education program; and (c) examining PSTs' proportional reasoning on a problem with more than two quantities, called a comparison proportional problem. Specifically, our research question was: What knowledge resources do elementary and secondary PSTs draw upon when solving a comparison proportional problem?

To answer the research question, PST responses to a comparison proportional problem were collected using a self-recorded video explaining their thinking, and documents of their written work on the problem were obtained. Knowledge resources of 20 secondary PSTs and 13 elementary PSTs were analyzed using the Robust Understanding of Proportional Reasoning for Teaching framework (Weiland et al., 2020) to produce a comparison of what kinds of knowledge resources each group of PSTs drew upon. To provide context for this study, we synthesize what is known about proportional reasoning, teachers' proportional reasoning knowledge, and the established framework developed to identify knowledge resources productive for teachers to draw upon when engaging in proportional reasoning.

Conceptual Framing and Related Literature

Proportional reasoning

We use Vergnaud's (1983) definition of proportions as the infinitely many equivalent ratios within an equivalence class. Lamon (2007) builds on this definition to define proportional reasoning as "supplying reasons in support of claims made about the structural relationships among four quantities, (say a, b, c, d) in a context simultaneously involving covariance of quantities and invariance of ratios or products" (pp. 637–638), including "detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships" (p. 647). Proportional reasoning is typically measured using either missing value problems or comparison problems (Cramer et al., 1993). Missing value problems provide three of the four quantities and ask for the unknown quantity in a proportional relationship. Comparison problems provide two ratios and ask for the relation between the ratios, which one ratio is greater than, less than, or equal to the other (Lobato et al., 2010).

Teachers' proportional reasoning

Distinct research groups have studied elementary and secondary in-service teachers' proportional reasoning, citing opportunities for growth, interventions that have supported understanding, and emphasizing the importance of content knowledge in the domain. Berk and colleagues (2009) found that elementary teachers demonstrated opportunities for growth in flexibility about thinking about proportion but made considerable gains in understanding with appropriate support. Studying secondary grade teachers, Weiland et al. (2021) found teachers are relatively proficient at identifying proportional situations, but were prone to over-identify situations as proportional, even when the situations were not proportional (Weiland et al., 2021). In another secondary grades study, Copur-Gencturk et al. (2022) found PSTs with a mathematics degree showed evidence of a greater ability to reason propositionally than did those with degrees in multiple subject areas, which is common among elementary school teachers—further emphasizing the importance of understanding how teachers of various grades reason about proportion, to support teachers' robust understanding of proportionality. Orrill and Millett (2021) examined 32 in-service secondary school teachers' ability to make sense of proportional reasoning using variable parts and concluded that teachers' proportional reasoning understanding is related to their ability to understand students' responses related to proportion. In yet another study of secondary teachers, (Authors, year) found a need to support awareness and accuracy of teachers' proportional reasoning understanding. Taken collectively, in-service teachers' knowledge related to proportional reasoning influences their abilities to support students' understanding.

Similar to recommendations for in-service teachers, PSTs need in-depth knowledge of the concepts within proportional reasoning that extend far beyond simplistic procedures for solving tasks, such as cross-multiplication. To provide mathematical foundations for their students, PSTs should have a keen understanding of the foundations of proportional reasoning to make sense of the mathematical domain themselves, before instructing students (Cabero-Fayos et al., 2020). At the same time, in-service teachers' and PSTs' problem-solving approaches and mathematical understanding can often differ, further highlighting the importance of supporting PSTs' mathematical understanding (Berk et al., 2009; Copur-Gencturk & Doleck, 2021).

Researchers have documented elementary and secondary PSTs' difficulty in distinguishing between proportional and non-proportional situations (Arican, 2019a, 2019b; Weiland et al., 2019), and their often over-reliance on algorithms such as cross-multiplication, to solve problems at the expense of overlooking the multiplicative structures within proportional relationships (Brown et al., 2020b; Lamon, 2007; Lobato et al. 2011). Knowing more about PSTs' understanding related to proportional reasoning would illuminate an increased understanding of how to improve instructional design related to proportion (Dole, 2008; Livy & Herbert, 2013; Livy & Vale, 2011; Joshua & Lee, 2022).

Most researchers on teachers' proportional reasoning have not directly compared elementary and secondary PSTs' understandings. One exception is Son (2013), who examined how 57 elementary, secondary, or special education PSTs solved a similar rectangles proportional reasoning problem. Son (2013) found elementary and secondary PSTs used different approaches (within ratio approach and between ratio approach). Son argued for the need to collect data from PSTs with different focal grade levels to better explore content knowledge and related understanding, highlighting the benefits of focusing research on approaches of more than one population.

The Robust Understanding of Proportional Reasoning for Teaching Framework

Teachers' proportional reasoning influences students' proportional reasoning, so there is a need to understand teachers' proportional reasoning (Copur-Gencturk et al., 2022). Drawing upon a knowledge-in-pieces perspective (diSessa et al. 2016), Weiland et al. (2020) developed a robust understanding of proportional reasoning for teaching framework, which is an analytical tool to describe what knowledge resources teachers employ when making sense of proportional reasoning. Knowledge resources are defined as elements of complex knowledge systems that may or may not be activated in situations. An individual may draw upon a knowledge resource productively when the resource moves them forward in their understanding or solving of a problem. An individual may counterproductively draw upon a knowledge resource when the resource is applied in a way that cannot or does not move them forward in their understanding or solving of a problem (e.g., applying cross-multiplication to a non-proportional situation). The knowledge resources can be used by researchers to describe grain-sized elements of teachers' proportional reasoning knowledge. This framework defines robust understanding "as a well-developed system of connected knowledge resources that can be drawn upon in different combinations across a wide array of proportional contexts" (Weiland et al., 2020, p. 4). The focus on the connections between knowledge resources aligns with research definitions and recommendations in the previous section (Cabero-Fayos et al., 2020; Lobato et al., 2011).

Weiland et al. (2020) used prior research to develop an observable set of 19 operationalized knowledge resources for a robust understanding of proportional reasoning for teachers, in alphabetical order: (1) Batches, (2) Comparison of Quantities, (3) Constant Ratio, (4) Covariance, (5) Distortion, (6) Equivalence, (7) Fluidity with Symbolic Representations, (8) Horizon Knowledge, (9) Multiplicative Comparison, (10) Partitioning & Tiling, (11) Proportional Situation, (12) Ratios (P:P/P:W), (13) Ratio as Measure, (14) Ratios ≠ Fractions, (15) Relative Thinking, (16) Rules, (17) Scaling Up/Down, (18) Unit Rate, and (19) Variable Parts.

Researchers have used and refined these knowledge resources to determine what knowledge resources teachers draw upon. For example, Brown et al. (2020a) gave secondary grades a paper and pencil task as well as a similar dynamic task and found differences in their knowledge resources drawn, despite the two tasks being mathematically similar. Arican and Özçakir (2021) also found using dynamic sketches supported PSTs' proportional reasoning when solving proportional problems involving irregular geometric figures. Brown et al. (2020b) used this framework to examine the knowledge resources a secondary grades mathematics teacher drew upon in a professional development setting, finding that while she had a variety of knowledge resources, these knowledge resources were disconnected, and this disconnection was easily overlooked despite being a focus in the professional development. They offer three suggestions for professional development: (a) prioritizing and problematizing key ideas and shared definitions, (b) identifying isolated or under-developed knowledge resources, and (c) connecting teachers' isolated knowledge resources to advance teachers' robust understanding of proportional reasoning.

Arican (2021) used Weiland et al.'s (2020) framework when interviewing six PSTs to determine the productive and counterproductive knowledge resources they drew upon when responding to two proportion tasks and two linear graphs. Arican found these teachers productively drew upon many knowledge resources as well as cross-multiplication that hindered PSTs' ability to solve nonproportional problems. This conclusion supports prior work (Arican, 2018, 2019a) that PSTs may over-rely on rote computations and memorization of rules when solving proportions.

To build on these prior studies, Authors (year) used the Robust Understanding of Proportional Reasoning for Teaching framework to identify how in-service secondary in-service teachers solved a comparison proportions problem. The authors recommend mathematics teacher educators "consider and build upon knowledge resources teachers productively draw upon…focus on developing knowledge resources afforded by the problem but not drawn upon by all teachers…support awareness and accuracy of the mathematical terminology…[and] anticipate and address counterproductive statements to support teachers' proportional reasoning" (Authors, year, page number), all aspects that are similarly important for the PST population.

Methods

Participants

This study examined both elementary and secondary PSTs' proportional reasoning knowledge resources, with participants coming from two different settings. The first group of participants was 13 elementary PSTs in a methods course at a large western university in the United States. The PSTs were all preparing to teach multiple subject areas and were in their last year of coursework as part of a bachelor's degree and licensure program. Course objectives were to learn to make sense of students' mathematical thinking and to design instruction responsive to students' needs, all with a focus on various mathematics content domains, including proportional reasoning. These PSTs did not have prior teaching experience and had all taken at least three prior mathematics content courses that included algebra and two courses with mathematics content specific for elementary teachers, although no courses they had taken focused exclusively on proportional reasoning.

The second group of participants was 20 secondary PSTs at a large eastern university in the United States. The objectives of this course were to develop secondary teachers' mathematical knowledge for teaching Algebra at the secondary level. The course began with a focus on key mathematics concepts for students entering secondary school classrooms and extended through topics often included in Algebra II courses. Some of these topics included variables, algebraic relationships, functions, and families of functions. Most PSTs in the course were at the beginning of a 1-year teacher preparation masters program and had not yet taken any courses focused on mathematics content or mathematical knowledge for teaching. Most of these PSTs did not have formal teaching experience and all had taken a variety of advanced undergraduate mathematics classes.

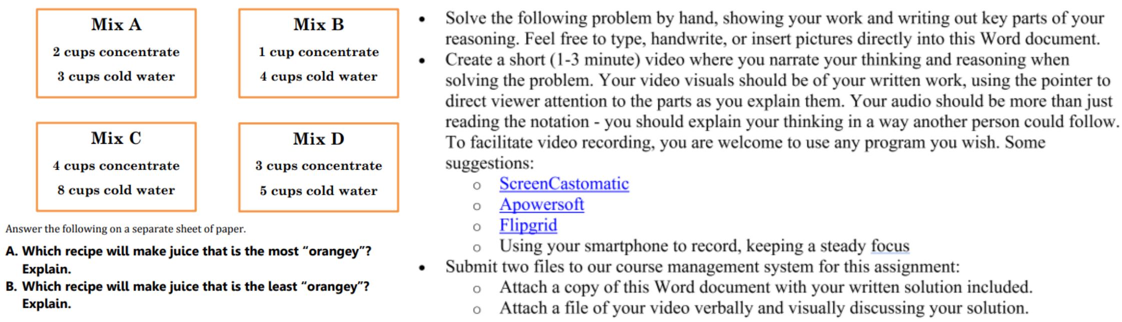
Data collection and analysis

The same data were collected from both groups of participants near the beginning of the semester before any instruction on proportional reasoning had begun. PSTs were given the following comparison proportional reasoning problem, adapted from the Connected Mathematics Project materials (Lappan et al. 1998/2002/2005), and associated prompt

(Figure 1). Data were comprised of their written work depicted in the participants' submitted Word document and their verbal and visual discussion captured in their self-recorded video. Participants were first asked to solve the problem and record their process in writing. Then they were asked to create and record a video where they narrated their thinking and reasoning as they were solving the problem. These videos typically lasted between one to three minutes and showed the PST work visually as they described their process and rationale for their decisions. PSTs were explicitly asked to not just read aloud every written step but to explain their thinking in ways that another person could follow. PSTs were given one week to complete the task as part of their weekly homework. The videos were transcribed for data analysis purposes but viewed during analysis for a more comprehensive understanding of the description.

Figure 1

The prompt and proportional reasoning task was given to all participants.


Mix A
2 cups concentrate
3 cups cold water

Mix B
1 cup concentrate
4 cups cold water

Mix C
4 cups concentrate
8 cups cold water

Mix D
3 cups concentrate
5 cups cold water

Answer the following on a separate sheet of paper.

A. Which recipe will make juice that is the most "orangey"? Explain.
B. Which recipe will make juice that is the least "orangey"? Explain.

- Solve the following problem by hand, showing your work and writing out key parts of your reasoning. Feel free to type, handwrite, or insert pictures directly into this Word document.
- Create a short (1-3 minute) video where you narrate your thinking and reasoning when solving the problem. Your video visuals should be of your written work, using the pointer to direct viewer attention to the parts as you explain them. Your audio should be more than just reading the notation - you should explain your thinking in a way another person could follow. To facilitate video recording, you are welcome to use any program you wish. Some suggestions:
  - ScreenCastomatic
  - Apowersoft
  - Flipgrid
  - Using your smartphone to record, keeping a steady focus
- Submit two files to our course management system for this assignment:
  - Attach a copy of this Word document with your written solution included.
  - Attach a file of your video verbally and visually discussing your solution.

Both the written and video data were analyzed using the Robust Understanding of Proportional Reasoning for Teaching framework (Weiland et al., 2019). We also drew upon the nuances described by Authors (year) to differentiate between the 19 knowledge resources described in the framework to identify the productive and counterproductive ways teachers can draw upon the knowledge resources. To begin data analysis, we watched the video data, transcribed verbal utterances, then analyzed each statement in conjunction with the mathematical work being referenced in the video and submitted it in written form. We coded each statement by having Authors 1 and 2 independently apply the Proportional Reasoning Knowledge Resources Relation Framework and associated definitions for each code. After individually coding data for a particular participant, we compared codes, discussed discrepancies, and came to an agreement, including Author 3's perspective when a discrepancy remained. The researcher triangulation in coding supported the credibility and confirmability of the study by incorporating multiple possible interpretations of the PSTs written and verbal statements (Lincoln & Guba, 1985).

Results

Knowledge resources PSTs drew upon productively

The most frequent five productive knowledge resources that each group of PSTs drew upon are given in Table 1. PSTs productively drew upon 13 of the 19 proportional reasoning knowledge resources. All 33 PSTs drew upon two knowledge resources: (2) Comparison of Quantities and (13) Ratio as Measure. Almost all PSTs (32 of 33) drew upon (12) Ratios (P:P/P:W) and almost all PSTs (31 of 33) drew upon (11) Proportional Situation. The only other knowledge resource the majority of PSTs drew upon was (18) Unit Rate (28 of 33).

Table 1

The knowledge resources PSTs productively drew upon.

Code	# (%) of Elementary PSTs (n=13)	# (%) of Secondary PSTs (n=20)	Total # (%) of PSTs (n=33)
2. Comparison of Quantities	13 (100%)	20 (100%)	33 (100%)
13. Ratio as Measure	13 (100%)	20 (100%)	33 (100%)
12. Ratios (P:P/P:W)	12 (92%)	20 (100%)	32 (97%)
11. Proportional Situation	11 (85%)	20 (100%)	31 (94%)
18. Unit Rate	9 (69%)	17 (85%)	26 (79%)

Comparison of Quantities

Every PST (33 of 33) showed evidence of drawing upon the knowledge resource (2) Comparison of Quantities, by naming the specific quantities they used in a ratio (typically cups of water, cups of concentrate, or cups of total juice mixture). For example, PST Breanne began their video by identifying the quantities in the ratio they wrote down, saying “I have mix A which is 2 cups concentrate to 3 cups cold water, so that's 2 to 3”, and writing down “Mix A: 2/3” (Figure 2).

Figure 2

Breanne’s transcript and written response provide evidence of how this elementary PST drew upon popular knowledge resources (e.g., 2, 11, 12, 13).

I took all of these mixes and I created ratios with them…I have mix A which is 2 cups concentrate to 3 cups cold water, so that's 2 to 3. Mix B here is 1 to 4. So, I have 1 to 4 there, 4 to 8, and then 3 to 5 right here.

Mix A: 2/3
Mix B: ¼
Mix C: 4/8
Mix D: 3/5

We're comparing the most concentration and then the least amount of concentration. So, knowing that I have these different fractions with different denominators I knew that we had to get those all the same. That's one of the areas that I kind of struggle with sometimes trying to find the least common denominator so usually I just find one that works.

Multiply everything out and then reduce later if needed. So, the common denominator of these numbers here is 120. Typically, I just kind of multiply them together to figure out which one works. So, then I have the conversion is here. 2/3 is equal to 80 over 120. 1/4 is equal to 30 over 120. 4/8 is equal to 60 over 120. 3/5 is equal to 48 over 120. All I did was take 120 divided by this denominator, and then find that answer, multiply it there by the numerator, in order to get the similar fraction.

Common Denominator
120

Mix A:2/3---→ 80/120
Mix B: ¼--→ 30/120
Mix C: 4/8-→ 60/120
Mix D: 3/5-→ 48/120

Once you have that it's really easy to see which item has the most concentration, which mix has the most concentration. The one with the higher numerator is going to be more concentrated because there is more of that in the mix. Mix A here is the most orangey. [That] is how the question asked, which means it has a most concentrated mix and you can easily see that mix B here is the least orangey or the least concentrated mix.

Patty’s response (Figure 3) also names the specific quantities used to solve the problem, beginning their video response by saying “What is shown here is the cups concentrate per the cups amount of water and that's a proportion” and adding “but to compare them, we need to compare a part to a whole. So we need to compare the parts concentrate to the total amount of mix.” While this response uses the word “proportion” to describe a ratio (something we detail later in the counterproductive knowledge resource section), Patty identifies the quantities in the part-to-part ratio and part-to-whole fraction they use, therefore showing evidence of drawing upon knowledge resource (2) Comparison of Quantities.

Figure 3

A portion of Patty's response provides evidence of how this secondary PST drew upon drew upon popular knowledge resources (e.g., 2, 11, 12, 13).

To determine which is the most orangey we need to determine which mixture has the most amount of concentrate per total amount of mixture. What is shown here is the cups concentrate per the cups amount of water and that's a proportion.

A	B	C	D
2 cups concentrate / 3 cups water or 2 cups concentrate / 5 cups total mix	1 cup concentrate / 4 cups water or 1 cup concentrate / 5 cups total mix	4 cups concentrate / 8 cups water or 4 cup concentrate / 12 cups total mix	3 cups concentrate / 5 cups water or 3 cups concentrate / 8 cups total mix

- First, because they have the same denominator, compare Mix A and Mix B.
- 2/5 > 1/5 so Mix A is more orangey than Mix B

- Next, compare Mix C and Mix D.
- To compare 4/12 and 3/8, we must find equivalent fractions with common denominators.
- 4/12 = 8/24 and 3/8 = 9/24
- Because 9/24 > 8/24, Mix D is more orangey than Mix C

But what we need to determine is the fraction of parts concentrate to total mixture…this 1st row shows the proportions of the amount of concentrate per cups of water.

A. Which recipe will make juice that is the most orangey?

Because Mix A is <u>more</u> orangey than Mix B and Mix D is <u>more</u> orangey than Mix C, We can compare compare Mix A to Mix D to figure out which is the <u>most</u> orangey of the four mixes.

→ To Compare 2/5 to 3/8, we must find equivalent fractions with common denominators.

Mix A
2/5 = 16/40

Mix D
3/8 = 15/40

Because 16/40 > 15/40,
Mix A is more orangey than Mix D

So, if Mix A > Mix B, Mix D>C and Mix A > Mix D, then Mix A must be more orangey than Mix C as well, so Mix A is the most orangey of all the mixes.

B. Which recipe will make juice that is the least orangey?

Because Mix B is <u>less</u> orangey than Mix A and Mix C is <u>less</u> orangey than Mix D, We can compare compare Mix B to Mix C to figure out which is the least orangey of the four mixes.

→ To Compare 1/5 to 4/12, we must find equivalent fractions with common denominators.

Mix B
1/5 = 3/15

Mix C
4/12 = 1/3 = 5/15

Because 3/15 < 5/15,
Mix B is less orangey than Mix C

So, if Mix B < Mix A , Mix C < Mix D and Mix B < Mix C, then Mix B must be less orangey than Mix D as well, so Mix B must be the least orangey of all the mixes.

Ratio as Measure

All 33 PSTs showed evidence of drawing upon the knowledge resource (13) Ratio as Measure, by identifying an abstractable quantity created from the combination of the two quantities, namely how the quantities of water and concentrate were being combined to form the new quantity of taste (also called “mixture”, “juice”, “flavor”, or “orangey”). The final sentence in Patty’s work (Figure 3) references this created quantity.

Ratios (P:P/P:W)

Almost every PST (32 of 33) showed evidence of drawing upon the knowledge resource (12) Ratios (P:P/P:W), by describing a part-to-part or part-to-whole comparison in words or visual representations. Breanne’s first sentence, “I took all of these mixes and I created ratios with them” (Figure 2), Patty’s statement of “we need to compare a part to a whole” and the accompanying mathematical statement “4/12”, and Patty’s visual cups depicting concentrate to water (Figure 3) show these PSTs drawing upon knowledge resource (12) Ratios (P:P/P:W). The one teacher (Toni) who did not show evidence of drawing upon knowledge resource (12) Ratios (P:P/P:W) is detailed in the counterproductive section.

Proportional Situation

Nearly all PSTs (31 of 33) drew upon knowledge resource (11) Proportional Situation by recognizing this problem involves proportional reasoning and working with and justifying the structural relationships among quantities involving ratios or products (Lamon, 2007). For example, both Breanne and Patty used proportional reasoning when they created part-to-whole ratios, attained a common denominator, and used these ratios to justify comparisons to determine the mix that was most and least orangey (Figures 3, 4). The two teachers (Toni and Alfonso) who did not show evidence of drawing upon knowledge resource (11) Proportional Situation are detailed in the Absolute Differences counterproductive section.

Unit Rate

The knowledge resource (18) Unit Rate was evidenced by 26 (79%) PSTs who created sharing-like relationships such as amount-per-one or amount-per-x. Unit rates could be characterized as either part-to-part comparisons or part-to-whole, with both options being drawn upon commonly by PSTs (Table 2). This table also provides a breakdown as to how within each subcode, we examined how frequently PSTs used a particular unit rate. Most commonly, nine PSTs (27%) drew upon knowledge resource (18) Unit Rate, by creating a percentage of units of water per 1 unit of mixture. Five PSTs (15%) created a percentage of units of water per 1 unit of concentrate. The second most common unit rate was using 120 for creating part-to-part comparisons (4 PSTs, 12%) and for creating part-to-whole comparisons (3 PSTs, 9%). The only other unit rates observed were used by a single PST.

Table 2

Details on how teachers drew upon knowledge resource 18. Unit Rate.

Subcode of 18. Unit Rate	Total # (%) of PSTs (n=33)	# (%) of Elementary PSTs (n=13)	# (%) of Secondary PSTs (n=20)	Unit rate used
Part-to-Part	12 (36%)	2 (15%)	3 (15%)	Percentage
		2 (15%)	2 (10%)	120
		1 (7%)		12, 60
			1 (5%)	15 and 4
			1 (5%)	1 cup concentrate
Part-to-Whole	14 (42%)	4 (31%)	5 (25%)	Percentage
			3 (15%)	120
			1 (5%)	15 and 40
			1 (5%)	1 cup mixture

Knowledge resources PSTs drew upon counterproductively

PSTs provided evidence of drawing upon 10 different knowledge resources counterproductively (Tables 3 and 4).

Table 3

The knowledge resources PSTs counterproductively drew upon (codes not listed were now drawn upon counterproductively.

Code	# (%) of Elementary PSTs (n=13)	# (%) of Secondary PSTs (n=20)	Total # (%) of PSTs (n=33)
14. Ratios ≠ Fractions	6 (46%)	2 (10%)	8 (24%)
6. Equivalence	1 (8%)	4 (20%)	5 (15%)
2. Comparison of Quantities	2 (15%)	3 (15%)	5 (15%)
12. Ratios (P:P/P:W)	3 (23%)	0	3 (9%)
7. Fluidity with Symbolic Representation	2 (15%)	0	2 (6%)
Absolute Differences	2 (15%)	0	2 (6%)
18. Unit Rate	1 (8%)	0	1 (3%)
1. Batches	0	1 (5%)	1 (3%)
17. Scaling Up/Down	0	1 (5%)	1 (3%)

Table 4

The knowledge resources PSTs drew upon that had both productive (P) and counterproductive (CP) codes.

Code	# (%) of Elementary PSTs (n=13)	# (%) of Secondary PSTs (n=20)	Total # (%) of PSTs (n=33)
2. Comparison of Quantities	13 P, 2 CP	20 P, 3 CP	33 P, 5 CP
12. Ratios (P:P/P:W)	12 P, 3 CP	20 P, 0 CP	32 P, 3 CP
18. Unit Rate	9 P, 1 CP	17 P, 0 CP	26 P, 1 CP
17. Scaling Up/Down	6 P, 0 CP	9 P, 1 CP	15 P, 1 CP
6. Equivalence	7 P, 1 CP	6 P, 4 CP	13 P, 5 CP
7. Fluidity with Symbolic Representation	3 P, 2 CP	8 P, 0 CP	11 P, 2 CP
14. Ratios ≠ Fractions	2 P, 6 CP	5 P, 2 CP	7 P, 8 CP
1. Batches	0 P, 0 CP	1 P, 1 CP	1 P, 1 CP

Ratios ≠ Fractions

The most frequently occurring counterproductive knowledge resource was (14) Ratios ≠ Fractions, of which eight PSTs (24%) provided evidence of drawing upon, usually by referencing a part-to-part relationship using the word “fraction”, which is misleading given Weiland et al.’s (2020) framework indicating fractions are only part-to-whole relationships while ratios are either part-to-part or part-to-whole relationships. Six of the eight PSTs who did this were elementary. For example, Mateo described his work:

> I ended up doing pictures and then also solving the fractions into decimals form to show 2 different ways of solving it. To start off with, I did a drawing of what I believe would be 2 cups for the concentration and then I made this top part right here water. I believe this would be 2 and then all that together would be water to make up for the 2/3 and then I changed the fraction into decimal which is 0.67.

Mateo refers to part-to-part relationships (concentration to water, represented as 2/3) as fractions. Figure 2 illustrates how Breanne also counterproductively draws upon the knowledge resource (14) Ratios ≠ Fractions, by referencing part-to-part relationships using the word “fractions”.

Equivalence

The second most frequently occurring counterproductive knowledge resource was (6) Equivalence, which five PSTs (15%) provided evidence of drawing upon, usually by using the word “proportion” to reference a single ratio. Four of the five PSTs who did this were secondary, such as Patty (Figure 3) who stated, “What is shown here is the cups concentrate per the cups amount of water and that's a proportion… what we need to determine is the fraction of parts concentrate to total mixture…this 1st row shows the proportions of the amount of concentrate per cups of water.” Breanne also drew upon knowledge resource (6) Equivalence counterproductively by using the term “similar fraction” rather than “equivalent fraction”, saying “All I did was take 120 divided by this denominator, and then find that answer, multiply it there by the numerator, in order to get the similar fraction.” Breanne’s work and transcript (Figure 2) is interesting in three ways: (a) it provides an example of the only elementary PST who counterproductively drew upon knowledge resource (6) Equivalence; (b) it provides an example of a different way this knowledge resource could counterproductively be drawn upon; and (c) it provides an example of how a PST can draw upon a single knowledge resource both productively and counterproductively.

Comparison of Quantities

The third most frequently occurring counterproductive knowledge resource was (2) Comparison of Quantities, which five PSTs (12%) provided evidence of drawing upon by mislabeling what quantities they were comparing. For example, Tracy made part-to-part comparisons and correctly identified the quantities in the ratio as concentrate to water, but then stated "mix a was the most orangey, it had…the most concentrate in it. And the next b is the least orangey and had the least concentrate to water ratio." Even though the second sentence accurately described the interpretation of the ratio, the first sentence in this video transcript communicated a description of single quantities being compared rather than the entire ratio and was thus coded as both productively and counterproductively drawing upon knowledge resource (2) Comparison of Quantities.

Ratios (P:P/P:W)

The fourth most frequently occurring counterproductive knowledge resource was (12) Ratios (P:P/P:W), which three PSTs (9%) counterproductively drew upon. All three PSTs were elementary (Mari, Eleon, Mateo) and showed evidence of counterproductively drawing upon this knowledge resource by describing or drawing a part:part as a part:whole or a part:whole comparison as a part:part. For example, Mari used a ratio of 2 cups of concentrate to the 3 cups of water for Mix A and then calculated what percentage of the mix was orange concentrate, explaining:

> I typed in 2 divided by 3 and I got .6 repeating and I know I would normally multiply this by 100 to get 66.6 repeating percent… Through my calculator, I was

> able to tell for sure that mix A had the most orange concentrate in the entire mix. It had the largest percentage of orange concentrate out of the whole mix.

This description indicates Mari interpreted their part-to-part relationship (2:3, or 66.6 repeating percent) as a part-to-whole relationship. Eleon and Mateo evidenced this knowledge resource by drawing diagrams with undescribed ratios.

Comparing knowledge resources between elementary and secondary PSTs

Secondary PSTs productively drew upon 13 knowledge resources that largely overlapped with the 9 knowledge resources the elementary PSTs drew upon (Table 1). This close alignment provides evidence that elementary and secondary PSTs productively drew upon mostly the same knowledge resources, the most prevalent of which were (2) Comparison of Quantities, (13) Ratio as Measure, (12) Ratios (P:P/P:W), and (11) Proportional Situation. Using this cutoff indicated three knowledge resources with the largest variation between how the two groups of PSTs productively drew upon them: more secondary PSTs (17 of 20, 85%) drew upon (18) Unit Rate than elementary PSTs (9 of 13, 69%); more elementary PSTs (7 of 13, 54%) drew upon (6) Equivalence than secondary PSTs (6 of 20, 30%); and more secondary PSTs (8 of 20, 40%) drew upon (7) Fluidity with Symbolic Representation than elementary PSTs (3 of 13, 23%).

Secondary PSTs counterproductively drew upon five different knowledge resources, which varied compared to the seven knowledge resources elementary PSTs counterproductively drew upon (Table 3). One similarity was how both groups counterproductively drew upon (2) Comparison of Quantities (15% of elementary and secondary PSTs). Three knowledge resources were counterproductively drawn upon by one group more than the other: more elementary PSTs counterproductively drew upon (14) Ratios ≠ Fractions (6 of 13, 46%) than secondary PSTs (2 of 20, 10%); more elementary PSTs counterproductively upon (12) Ratios (3 of 13, 23%) compared to no secondary PSTs; and two elementary teachers (15%) counterproductively drew upon (7) Fluidity with Symbolic Representation. Additionally, two elementary PSTs counterproductively drew upon Absolute Differences compared to no secondary PSTs. We also noticed a slight difference in how more secondary PSTs counterproductively upon (6) Equivalence (4 or 20, 20%) as compared to elementary PSTs (1 of 13, 8%).

Discussion

Proportional reasoning as a domain positioned within the trajectory of mathematics education curriculum can serve as a bridge, linking topics from the elementary grades to the topics of the secondary grades. This study was born out of a desire to better understand how elementary and secondary teachers made sense and reasoning about ratios and proportions, and to try and identify if there were any noticeable or significant differences in how teachers reasoned proportionally (which could have implications for working with students).

When comparing what knowledge resources elementary and secondary PSTs draw upon when solving this comparison proportional problem, we found PSTs largely did not differ between the knowledge resources they drew upon, which adds to existing literature that focused on individual participant groups (e.g., Berk et al., 2009; Copur-Gencturk et al., 2022; Weiland et al., 2021).

Of the 19 knowledge resources, both groups productively drew upon the following four knowledge resources most frequently: (2) Comparison of Quantities, (13) Ratio as Measure, (12) Ratios (P:P/P:W), and (11) Proportional Situation. The high frequency of these four knowledge resources can be explained in light of the comparison of the proportional reasoning problem given to the PSTs, which asked them to compare the quantity of orangeyness between multiple juices whose components were listed in a near-ratio format. Knowledge resource (2) Comparison of Quantities is needed to identify and compare the two ingredients in the recipe. Knowledge resource (13) Ratio as Measure is needed to identify the quantity of orangeyness. Knowledge resource (12) Ratios (P:P/P:W) is needed when using ratios to determine which recipe was the most and least orangey. Knowledge resource (11) Proportional Situation is needed when considering a ratio, comparing a ratio, creating a proportion, etc.

In relation to prior work, this study indicates PSTs may draw upon similar knowledge resources as in-service teachers when solving the same comparison proportional reasoning problem. When Authors (year) examined mostly in-service middle and secondary in-service teachers' knowledge resources on the same orange juice task, the same proportional reasoning knowledge resources were drawn upon by the in-service teachers in comparison to what was found with PSTs in this study. The similarity between knowledge resources drawn upon by pre- and in-service teachers indicates minor differences between these two populations of teachers, which differs from prior work describing drastic differences between pre- versus in-service teachers' proportional reasoning (Copur-Gencturk & Doleck, 2021), though more research is needed to determine whether these patterns exist in other pre- and in-service teacher populations.

Additionally, we found similarities between PSTs and the in-service teachers Authors (year) also in terms of which knowledge resources were not drawn upon when solving the orange juice task. We agree with Arican's (2021) explanation that the types of knowledge resources drawn upon "might be the result of using different types of problems (missing value vs comparison) with differing contexts (gears vs mixture)" (p. 22). This conclusion further supports the need for teacher education efforts to expose

PSTs to a variety of proportional reasoning problems to draw upon a variety of knowledge resources (Cabero-Fayos et al., 2020).

When comparing the knowledge resources counterproductively drawn upon, PSTs in this study drew upon the same knowledge resources that in-service teachers counterproductively drew upon within the same task. Both groups counterproductively drew upon the knowledge resource (14) Ratios ≠ Fractions the most, followed by lesser instances of counterproductively drawn upon (6) Equivalence, (2) Comparison of Quantities, (7) Fluidity with Symbolic Representation, Absolute Differences, and (18) Unit Rate. PSTs misinterpreting a part-to-part ratio as a part-to-whole ratio is not uncommon, and this suggests greater importance should be given in teacher education efforts for PSTs to explain part-to-part and part-to-whole relationships created in problem solving (Johnson, 2017). Teacher educators can also use these counterproductive statements as learning experiences by drawing teachers' attention to them (Authors, year).

Differing from the prior study, this study found every PST productively drew upon (2) Comparison of Quantities, while Authors (year) found 37 of the 51 in-service teachers productively drew upon that knowledge resource. Additionally, more PSTs (15 of 33) drew upon (17) Scaling Up/Down, in comparison to how Authors (year) found that no in-service teachers productively drew upon that particular knowledge resource. One possible explanation for these differences is that this study collected video evidence of PSTs explaining their work, whereas the prior study only had written documents to analyze. This explanation, supported by other work indicating the use of video data in conjunction with written work, can provide a more authentic capture of teacher thinking (Authors, year; Santagata et al., 2021; Walkoe et al., 2023).

While no existing research seems to directly compare elementary and secondary PSTs' proportional reasoning knowledge resources, we can situate findings from this current research with existing extant literature. We found elementary and secondary PSTs differed on three knowledge resources they productively drew upon. First, more secondary PSTs (17 of 20, 85%) drew upon (18) Unit Rate, than elementary PSTs (9 of 13, 69%). This difference may be explained by prior research suggesting elementary and middle PSTs do not interpret rate concepts created by division operations (Alenazi, 2016; Jansen & Hohensee, 2016), while secondary PSTs' understanding of unit rate could be influenced by their experiences with unit rates occurring in more advanced mathematical coursework (Arican, 2021).

Second, more elementary PSTs (7 of 13, 54%) drew upon (6) Equivalence than secondary PSTs (6 of 20, 30%). This difference may have been because the elementary PSTs were more drawn toward explaining how the proportion was created and why ratios were equal, whereas secondary PSTs may have tended towards more symbolic solutions methods without explanation (Son, 2013).

Third, more secondary PSTs (8 of 20, 40%) drew upon (7) Fluidity with Symbolic Representation than elementary PSTs (3 of 13, 23%). An explanation for this difference is again likely because the secondary PSTs had taken more prior mathematics courses (Copur-Gencturkm et al., 2022; Orrill & Millett, 2021). Considering the amount of algebraic and symbolic representations used in undergraduate mathematics, the secondary PSTs may have been more likely to use multiple symbolic representations due to their experience connecting those representations. In any case, interpreting and explaining symbolic representations of percentages, fractions, and units can be challenging for teachers, and consequently, teacher education efforts must support teachers to understand the fractions, decimals, and percentages they and their future students will use (Copur-Gencturk & Ölmez, 2022; Lee et al., 2011; Parker & Leinhardt, 1995).

In comparison to productive knowledge resources, PSTs were much less likely to counterproductively draw upon knowledge resources, with less than half of the PSTs in either group counterproductively drawing upon any single knowledge resource.

The groups differed on how they counterproductively drew upon two knowledge resources. More elementary PSTs counterproductively drew upon (14) Ratios ≠ Fractions (6 of 13, 46%) than secondary PSTs (2 of 20, 10%), which may suggest elementary PSTs could benefit by better attending to mathematics vocabulary, specifically in what they call ratios versus fractions (Lamon, 2020; Vanluydt et al., 2021). More elementary PSTs counterproductively upon (12) Ratios (3 of 13, 23%) compared to no secondary PSTs, suggesting elementary PSTs may benefit from better distinguishing between part-to-part and part-to-whole comparisons (Johnson, 2017).

Additionally, considering that proportional reasoning is difficult to teach and learn (e.g., Lamon, 2007) recognizing that support for PSTs could be beneficial is not altogether surprising. What is noteworthy is that this research has examined both elementary and secondary PSTs' understanding of proportional reasoning with an unprecedented level of granularity, articulating the exact kinds of knowledge resources from which elementary *and* secondary PSTs draw, and from that identify counterproductive knowledge to point to specific areas where support may help.

Limitations

Similar to prior work (Authors, year), one limitation of this study is relying on a single comparison proportional reasoning problem to categorize the knowledge resources drawn upon by PSTs. While using the same single problem can help make comparisons across studies examining in-service and PST populations, the knowledge resources described here may be specific to the orange juice task (Arican, 2021) and not intended to describe the knowledge resources PSTs would use when solving other comparison proportional reasoning problems. Another limitation is the number of participants within each group of PSTs. The knowledge resources drawn upon are specific to the group of students in this study and do not necessarily inform about how elementary or secondary PSTs reason proportionally in general. Larger groups of elementary and secondary PSTs would have provided a more reliable depiction of how often the group of PSTs drew upon specific knowledge resources. We recognize that the findings are specific to these particular PSTs and likely not representative of all PSTs.

Implications

The practical implications of this study build off Brown et al.'s (2020b) three recommendations for developing teachers' robust understanding in professional development. Their first recommendation is to prioritize and problematize key ideas and shared definitions. This study suggests mathematics teacher educators working with both elementary and secondary PSTs should problematize what quantities within ratios represent and thus support knowledge resource (2) Comparison of Quantities. Mathematics teacher educators working with secondary PSTs should consider prioritizing the shared definition of proportion and ratios with PSTs to support knowledge resource (6) Equivalence. Mathematics teacher educators working with elementary PSTs should establish a shared definition of ratio and fraction to support knowledge resource (14) Ratios ≠ Fractions. Additionally, MTEs should problematize what any percentages used in elementary PSTs problem solutions mean to support knowledge resource (7) Fluidity with Symbolic Representation by understanding the fractions, decimals, and percentages they and their future students use (Copur-Gencturk & Ölmez, 2022; Joshua & Lee, 2022; Lee et al., 2011; Parker & Leinhardt, 1995).

Brown et al.'s (2020b) second recommendation is for mathematics teacher educators to identify isolated or under-developed knowledge resources. This study found both elementary and secondary PSTs might benefit from developing knowledge resource (18) Unit Rate. Mathematics teacher educators should consider challenging elementary PSTs to explain how the proportion was created and why ratios were equal to better develop knowledge resource (6) Equivalence. Given the higher number of elementary PSTs counterproductively drawing upon knowledge resource (12) Ratios, mathematics teacher educators should have elementary PSTs explicitly distinguish between part-to-whole and part-to-whole comparisons (Johnson, 2017).

The third recommendation by Brown et al. (2020b) is connecting teachers' isolated knowledge resources to advance teachers' robust understanding of proportional reasoning. This study offers anecdotal evidence for how this might occur. Mathematics teacher educators can highlight rarely occurring knowledge resources, such as (1) Batches or (8) Horizon Knowledge, to discuss. Showing examples of absolute difference thinking, including diagrams, could help elementary PSTs consider how to address and avoid having their students draw upon this counterproductive knowledge resource when solving proportional problems (Authors, year; Johnson, 2017; Son, 2013). Attending to PSTs' counterproductive knowledge resources is important because these statements may become an issue with another proportional reasoning problem or teaching context, and addressing these counterproductive statements detailed by these PSTs promotes coherent mathematical thinking by attending to precision in communication (Australian Curriculum, Assessment & Reporting Authority, 2017; Bishop et al., 2022; CCSSM, 2010; NCTM, 2000).

Conclusion

Mathematics educators have been tasked to better support PSTs' proportional reasoning (Johnson, 2017), which involves a better understanding of how PSTs reason proportionally (Arican, 2021). Knowing what proportional reasoning knowledge resources elementary and secondary PSTs draw upon can help make more informed grade-level support structures in teacher education programs. This study offers new insight into how elementary and secondary PSTs reason about a comparison proportional reasoning problem and offers a Proportional Reasoning Knowledge Resources Relation Framework for others to build upon. Beneficial future research could examine the knowledge resources drawn upon by other PSTs solving this problem or the knowledge resources drawn upon when solving other proportional reasoning problems, and might further refine the framework for use with mathematics education researchers, mathematics teacher educators, and mathematics teachers. These efforts could further support the proportional reasoning of students of all levels.

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