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How does the Introduction of Hidden Orders Affect Limit Order Markets?

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Abstract

Hidden orders are widely used in major exchanges. This paper studies how hidden orders affect markets by introducing these orders to the limit order market model in Foucault et al. (2005). We investigate the equilibrium outcomes in an infinite-time horizon model with multiple price levels and unknown hidden order queues by extending the algorithms in Pakes and McGuire (2001). The model suggests that the introduction of hidden orders (including the hidden limit orders and the midpoint peg orders) increases the profit of patient traders, slightly worsens impatient traders' average executed market prices, mitigates price impact by reducing both the effective spread and the Amihud illiquidity measure, and decreases the mid-quote volatility.

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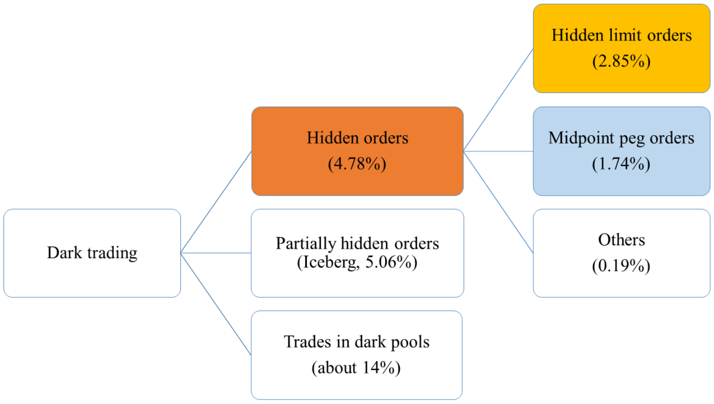
1. Introduction

In dark trading, all or part of the order information may be hidden from the marketplace. Nearly all major exchanges, such as NASDAQ and NYSE, allow dark trading, which has three types: hidden orders that are entirely invisible, iceberg orders that are partially invisible with fixed volumes displayed, and dark pools. Figure 1 depicts the usages of the three types along with the two most popular kinds of hidden orders, namely, (i) midpoint peg orders pegged to the midpoint of the national best bid and ask prices and (ii) hidden limit orders submitted to one of the available limit price levels with volume completely invisible.

Figure 1: **Different types of dark trading. The percentage for each type is the usage of each type.** The shares for each type of hidden orders and partially hidden orders are obtained by using the average percentage of matched volumes of each type over the total matched volume during the second half of the year 2019 in NYSE Arca and NYSE Tape A; the share of dark pools comes from the market structure reports from Rosenblatt Securities Inc. as the averaged percentage of executed volumes in the dark pools over the total executed volume during the second half of the year 2019 in the US equity market.



In this paper, we focus on the hidden limit orders and the midpoint peg orders—the two most important types of hidden orders— in markets with limit orders. More specifically, by introducing hidden orders to the model of markets with limit orders in Foucault et al. (2005), the research question of this paper is how the introduction of midpoint peg orders and hidden limit orders affects traders' profits and market liquidity.

As pointed out by Bloomfield et al. (2015), there is relatively little theoretical literature on hidden orders, although of great importance, unlike dark pools and iceberg orders. An exception is Boulatov and George (2013), which proposes a novel equilibrium model of hidden orders in a dealer market with a single price level in a one-period economy. They conclude that the ability to hide will impel some informed traders to submit hidden orders as liquidity suppliers instead of using market orders; thus, the liquidity demanders, who submit market orders, would benefit from the allowance of hidden orders. However, endogenous decisions on switching between orders with different types of transparency, such as limit orders, hidden limit orders, and midpoint peg orders, are not allowed in their paper. Thus, two main issues remain.

(1) They study the impact of hidden orders on dealer markets. What is the effect of hidden orders on markets with limit order books?

(2) They study a one-period model. What about a multi-period dynamic model?

There are two main difficulties if one wants to address the above research issues. First, hidden orders not only interact with visible orders in the limit order market but also form a non-observable order queue regarding the trade priority. Indeed, according to the "price-visibility-time" priority rule in exchanges (see, e.g., Service and Description (2009)), hidden orders submitted to the same price level will be "queued" to guarantee that the one submitted earlier will have a higher trading priority. Thus, one must study dynamic games with imperfect information for the hidden order queue.

Second, different types of traders potentially want to use hidden orders. Chakrabarty et al. (2020) separates the traders submitting hidden orders into two types by motivation: High-frequency algorithmic traders submit hidden orders to mitigate competition from other venues, such as limit order books, and agency algorithmic traders, who mainly execute large orders on behalf of clients, use hidden orders to protect information. Agency algorithmic traders are less sensitive to the trading time uncertainty in hidden orders but benefit more from concealing information. High-frequency algorithmic traders are more susceptible to trading time uncertainty in hidden orders. They would use hidden orders only if the endogenously determined expected waiting time of hidden orders is significantly shorter than that of limited orders.

To the best of our knowledge, the paper is the first that proposes a dynamic equilibrium model to study the impact of hidden orders on markets with limit order books. To circumvent the first difficulty, we study the interaction with visible orders in non-observable order queues by extending the original model in Foucault et al. (2005), where hidden orders are not considered, to include both hidden limit orders and the midpoint peg orders. We investigate the equilibrium problem of an infinite time horizon game with imperfect information for hidden order queues.

To circumvent the second difficulty of incorporating different types of traders using hidden orders, we significantly extend the algorithm in Pakes and McGuire (2001) to find the Markov perfect equilibrium for an infinite-time horizon game with heterogeneous impacts and multiple discrete price levels. The original algorithm in Pakes and McGuire (2001) for their industry dynamics model is further generalized by Goettler et al. (2005) to the setting of limit order books (without hidden orders) by introducing overly optimistic initial expectations and forced start points to avoid ending up in non-equilibrium states. Due to the multiple discrete price levels and heterogeneous impacts, a direct application of the two algorithms may not yield the correct equilibrium strategies. Our algorithm complements the two algorithms by using forced explorations and additional strategy updates to ensure that correct equilibrium states and strategies can be found, and by introducing a stopping rule so that the algorithm can test and re-evaluate strategies to avoid ending up in non-equilibrium states. We also prove that the stopping rule indeed leads to equilibrium states.

By comparing the equilibrium results of the benchmark limit order market model with and without hidden orders, we investigate the effects of introducing hidden orders on different traders and the whole market. Our model suggests that the allowance of hidden orders will increase the expected profit (consisting of the expected trading profit minus the expected waiting cost) of patient traders, both sensitive ones (such as high-frequency traders) and insensitive ones (such as agency algorithmic traders), due to the new option of using hidden orders. However, the introduction of hidden orders may slightly worsen impatient traders' average executed market prices.[1] The intuition is that introducing hidden orders to a limit order market induces liquidity suppliers to submit orders at less aggressive levels, and impatient traders thus suffer from the introduction. Furthermore, introducing hidden orders mitigates price impact by reducing both the effective spread and the Amihud illiquid measure, and decreases the mid-quote volatility. This is mainly because most hidden orders are initially submitted to the best prices or inside the bid-ask spread to earn the trading priority, although hidden orders may later be outside the bid-ask spread before their execution due to the possible undercuts by limit orders in the future.

In summary, the contribution of this paper is threefold. First, from a modeling viewpoint, we study how hidden orders affect markets by introducing these orders to the model of limit order markets in Foucault et al. (2005). Second, in terms of economic insights, our model indicates that the introduction of hidden orders (including the hidden limit orders and the midpoint peg orders) increases the profit of patient traders, slightly harms that of the impatient traders, decreases the effective spread and the mid-quote volatility, and reduces price impact. Third, from an algorithm viewpoint, we investigate the equilibrium outcomes in an infinite-time horizon model with multiple price levels and unknown hidden order queues by extending the algorithms in Pakes and McGuire (2001) and Goettler et al. (2005).

To the best of our knowledge, there is no empirical study specifically on the liquidity benefit of introducing hidden orders to limit order markets. However, there are studies about the liquidity benefit of introducing other types of dark trading. Using both the midpoint trades in the limit order market and those in the dark pool, Foley and Putniņš

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[1] An exception is when the waiting cost of patient traders is so high that limit orders are not used at all. Then the model is comparable to the case of Bloomfield et al. (2015), in which only market orders and hidden orders are allowed. In this case, the introduction of hidden orders increases the expected profits of impatient traders because some patient traders become liquidity suppliers by using hidden orders instead of competing as liquidity demanders with impatient traders.

(2016) show that the two-sided dark pool trading benefits liquidity, while the one-sided dark pool trading does not significantly affect liquidity. On the other hand, using the data on trades in the midpoint dark pool only (without the limit order market) Degryse et al. (2015) find that the midpoint dark pool has a detrimental effect on liquidity. Using the data from Canadian markets, Comerton-Forde et al. (2018) also observe improvement in the liquidity of lit markets after introducing restrictions on trading in dark pools. Our theoretical model suggests that hidden orders in the limit order market benefit liquidity. Therefore, combining all these studies, it is suggested that different combinations of dark trading (two-sided and one-sided dark pools and hidden orders) and lit trading (with or without limit order markets) could have different effects on liquidity and that none of the conclusions may contradict each other.

There is one experimental study of how hidden orders affect trading strategies. Bloomfield et al. (2015) investigate how introducing iceberg and hidden orders can affect traders' strategies and market outcomes by inviting students to trade in their simulated markets. They conclude that, with increasing opaqueness, traders would switch from displayed orders to non-displayed ones. We complement their experimental results by presenting optimal strategies in the market with hidden orders, showing how traders switch from displayed orders to non-displayed ones. In particular, we find that both sensitive and insensitive patient traders may choose hidden orders, while the insensitive ones could submit hidden orders more frequently.

In a broader sense, our paper is also related to theoretical models for other types of dark trading, such as iceberg orders and dark pools. In terms of dark pools, Hendershott and Mendelson (2000), Degryse et al. (2009), Ye (2011), Zhu (2014), Ye (2016) and Iyer et al. (2018) consider markets with both a dealer market as the lit market and a midpoint dark pool as dark trading, where a dark trade occurs at the midpoint of the best bid and ask prices; Buti et al. (2017) and Brolley (2019) investigate markets with both a limit order market as the lit market and a dark pool as dark trading. In terms of iceberg orders, Buti and Rindi

(2013) study a game among big and small traders in a limit order market to conclude that the introduction of icebergs would increase the total welfare, and Moinas (2006) presents a three-period game between the limit order traders and the market order traders in a limit order market with iceberg orders.

The rest of the paper is organized as follows. After the basic setting of the model is given in Section 2, some analytical solutions for the special case of a homogeneous impact are provided in Section 3. The main results for heterogeneous impacts are given in Section 4. We provide empirical implications in Section 5. Robustness checks are presented in Section 6. Section 7 concludes. All technical proofs are relegated to the online supplement.

2. Basic Model

In this section, we will present the market setting, order placement strategies, and the definition of equilibrium in our model.

2.1. Market Settings

We shall first introduce the market setting adopted from Foucault et al. (2005), and then describe the new market settings to include hidden orders.

Settings Adopted from Foucault et al. (2005). This is an infinite time horizon model in continuous time with discrete price levels. Consider a market for a single security, organized as a limit order book within a price range $[B, A]$, where $A > B > 0$. In other words, we assume that the book depth is unlimited for a price higher than A or lower than B. The tick size of the order book is $\Delta > 0$.

Risk-neutral traders arrive at the market according to a Poisson process with parameter $\lambda > 0$. The valuations of buyers and sellers are V_{buyer} and V_{seller}. We assume that each trader is a natural buyer or seller, i.e., $V_{buyer} > A$ and $V_{seller} < B$. As a result, submitting a market order is more profitable than no-trading. Every trader has a waiting cost proportional to the length of time elapsed between the arrival and the completion of the order. The waiting time of each order is endogenous. Traders are either patient (Type P) or impatient (Type

I), with waiting costs of δ_P and δ_I per unit of time, respectively, where $\delta_I \geq \delta_P \geq 0$. The proportion of patient traders in the population is $\theta_P \in (0, 1)$, while that of impatient traders is $\theta_I = 1 - \theta_P$.

As in Foucault et al. (2005), the following assumptions are imposed on the market structure: (i) Each trader arrives only once with one unit to trade. (ii) New limit orders must undercut the existing limit orders. (iii) Buyers and sellers alternate with certainty, i.e., given the current arrival is a buyer (seller), the subsequent arrival must be a seller (buyer).[2]

Rules of Hidden Orders. The price-visibility-time rule specifies: (i) A hidden order can only be executed after executing all the visible orders at the same price level. (ii) A hidden order submitted earlier has a higher trade priority than the other hidden orders at the same price level. Thus, submitting a hidden order to the price levels where there are existing limit orders may entail considerable waiting time. We first consider hidden orders inside the spread upon arrival, although in our model the hidden order may later be outside the bid-ask spread before its execution due to possible undercuts by limit orders in the future. This assumption is also consistent with the fact that most hidden orders are submitted to the best prices or the prices inside the spread to earn the trading priority; see, e.g., Holcomb and Upson (2015). Later, this assumption will be dropped when we do the robustness check in Section 6.

2.2. Order Placement Strategies with Hidden Orders

Our model includes four types of orders: market, limit, hidden limit, and midpoint peg orders. Visible trading strategies for two lit market placements (i.e., market and limit orders) are adopted from Foucault et al. (2005), while invisible strategies for two types of hidden

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[2] Although cancellation and different order sizes are standard in the market, the first assumption is made to simplify the setting to focus on the equilibrium and market spread evolution in our model, not the individual trading problem. The second assumption is imposed to emphasize a critical difference between hidden orders and limit orders: Compared to hidden orders, a limit order reveals the trade intention and thus may attract the undercutting via the arrivals of new limit orders. The third assumption is to facilitate some analytical derivations, which is not essential when we apply our algorithm later to solve equilibrium strategies numerically.

orders (i.e., hidden limit and midpoint peg orders) are newly introduced in our model. In the following two subsections, we shall introduce the formulation and the expected profits of the visible and invisible strategies.

Visible Strategies for Market and Limit Orders Adopted from Foucault et al. (2005). A limit order, whose price is j ticks away from the best price on the opposite side at its submission time, is denoted as $j-$limit order.[3] For an arbitrary fixed initial market state, the expected profit of a $j-$limit order for a buyer of type $i \in \{P, I\}$, facing a current bid-ask spread $s = a - b$, is

$$\hat{U}_i(s,0) = V_{buyer} - (a - \bar{m}(s))\Delta, \quad \hat{U}_i(s,j) = V_{buyer} - (a-j)\Delta - \delta_i T(s,j),\ j = 1, \cdots, s-1,$$

where $T(s,j)$ is the (endogenous) expected waiting time of a $j-$limit order submitted at a spread of s, δ_i is the unit waiting cost per time period (for either $i = P$ or $i = I$), $\bar{m}(s)$ is the (endogenous) expected execution price of a market buy order submitted at a spread s, relative to the best ask price. Both $T(\cdot)$ and $\bar{m}(\cdot)$ are to be determined endogenously in equilibrium. The profits for sellers are defined likewise.[4] Because the term $V_{buyer} - a\Delta$ does not involve the decision variable j, we can take that term out to define the expected utility function of a j-limit order submitted at a spread of s as

$$U_i(s,0) = \bar{m}(s)\Delta,\ U_i(s,j) = j\Delta - \delta_i T(s,j),\ j = 1, ..., s-1. \tag{1}$$

New Invisible Strategies with Heterogeneous Impacts and Hidden Order Queue. We denote $j-$hidden order as the hidden order whose price is j ticks away from the best

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[3] For example, a buy $1-$limit order represents a buy limit order whose price is one tick size lower than the best ask price at its submission time, and a $0-$limit order is a market order. Therefore, a trader arriving at a market—with the best bid price b (in ticks), the best ask price a (in ticks), and the bid-ask spread $s = a-b$—would have s choices of limit orders, namely $j-$limit order, $j = 0, 1, \ldots, s-1$. Thus, the execution prices of buyers' and sellers' $j-$limit orders are $a - j$ and $b + j$, respectively, for $j = 1, \ldots, s-1$.

[4] There are two differences between our formulation and that in Foucault et al. (2005) due to hidden orders inside the spread in our model: First, the expected waiting time depends on not only the strategy j but also the current spread s. Second, the relative execution price of a market order may not be zero. Thus, the market order could be executed at a price better than the best visible price, and the existing hidden orders could affect the expected waiting time of the $j-$limit order if the hidden orders have a better price than the $j-$limit order.

price on the opposite side at its submission time, $j \in \{1, \dots, s-1\}$. Similar to the visible strategies, for an arbitrary fixed initial market state, the expected gain of a $j-$hidden order, denoted by $U^H_{i,\nu}(s, j)$, $j \in \{1, \dots, s-1\}$, and a midpoint peg order, denoted by $U^H_{i,\nu}(s, \frac{s}{2})$, for a buyer of patient type $i \in \{P, I\}$ and impact type $\nu \in \Lambda$, facing a current bid-ask spread s, are

$$\begin{cases} U^H_{i,\nu}(s, \dfrac{s}{2}) = \bar{m}^H(s)\Delta - \delta_i \beta_\nu T^H(s, \dfrac{s}{2}), \\ U^H_{i,\nu}(s, j) = j\Delta - \delta_i \beta_\nu T^H(s, j), \; j = 1, ..., s-1. \end{cases} \tag{2}$$

There are two types of patient traders as in Chakrabarty et al. (2020). A patient trader with a multiplier $\beta_\nu < 1$, e.g., a agency algorithmic trader, benefits from the information protection by hidden orders and is less sensitive to the uncertainty in the expected waiting time of the submitted hidden order. As a result, the unit waiting-time cost of a hidden order with a multiplier $\beta_\nu < 1$ would be less than that of a limit order, should the endogenously determined $T_H(s, j)$ and $T(s, j)$ be the same, reflecting the reduced cost by concealing information. On the contrary, a patient trader with $\beta_\nu > 1$, e.g., a high-frequency algorithmic trader, is sensitive to the uncertainty in the expected waiting time due to the rapid changes of trading opportunities.

If the expected waiting time T^H and the execution price j or $\bar{m}^H$ are the same as the limit order's time and price, T and j, the insensitive traders would submit hidden orders with a multiplier $\beta_\nu < 1$, while the sensitive ones submit visible orders, according to (1)-(2). However, both the times $T^H(\cdot)$ and $T(\cdot)$ and the price $\bar{m}^H(\cdot)$ are to be determined endogenously in equilibrium.[5] As a result, although the trader with a large β_ν would intuitively prefer hiding, both sensitive and insensitive traders could possibly choose limit or hidden orders in the equilibrium under certain market conditions. If the impacts of the waiting time are homogeneous, then the set Λ only has one element; for heterogeneous impacts, the set

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[5] Two differences exist between the profits of newly introduced hidden orders in (2) and those of limit orders in (1). First, as the midpoint peg order is pegged to the midpoint price, the expected execution price $\bar{m}^H(\cdot)$ of the midpoint peg order moves with the market and could be different from the midpoint price at its submission time. Second, there is the multiplier β_ν to the expected waiting times of hidden orders.

Λ has more than one element.

Due to the price-visibility-time rule, the length of the existing hidden order queue at the same price level would affect the expected waiting time of a newly submitted hidden order. However, unlike limit orders, hidden orders are invisible. Thus, traders cannot distinguish the number of existing hidden orders or the position of a submitted hidden order. Indeed, in our model traders can only identify the average waiting time (in terms of positions) of a $j-$hidden order, $T^H(s,j)$, which is common knowledge.

Summary of the Optimal Trading Strategy Problem. With impact type $\nu \in \Lambda$, and patient type $i \in \{P, I\}$, and a common knowledge on the averaged expectations on waiting times and execution prices, $\{T(\cdot), T^H(\cdot), \bar{m}(\cdot), \bar{m}^H(\cdot)\}$, each trader arriving at a bid-ask spread s chooses to submit one of the following orders: a market order, a $j-$limit/hidden order, or a midpoint peg order, to maximize the expected gain. Specifically, a trader of patient type $i \in \{P, I\}$ and impact type $\nu \in \Lambda$ attempts to solve[6]

$$\max_{j \in \{0,\cdots,s-1\}} \{U_i(s,j),\ U^H_{i,\nu}(s,j),\ U^H_{i,\nu}(s,\frac{s}{2})\}, \tag{3}$$

with the utilities defined in (1)-(2) and the averaged expectations on waiting times and execution prices, $\{T(\cdot), T^H(\cdot), \bar{m}(\cdot), \bar{m}^H(\cdot)\}$, as common knowledge.

2.3. Definition of Equilibrium

We shall use the concept of Markov-perfect equilibrium, which is used in Foucault et al. (2005) and Pakes and McGuire (2001). The Markov specification requires traders to make decisions based only on the current visible order book and not on any previous book or prior trade information. Thus, the equilibrium is stationary as the optimal strategy is a function

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[6] For convenience, the following rules are used to break possible ties: (i) If a patient trader is indifferent between two limit orders at different price levels, in terms of the expected gain defined in (1), the trader submits the limit order creating the larger spread. (ii) If a patient trader is indifferent between a limit and a hidden order, the trader submits the hidden order. (iii) If a patient (impatient) trader is indifferent between a market and a limit/hidden order, the non-market (resp. market) order is submitted. (iv) Traders arriving at a spread $s = 1$ always submit a market order. Note that these assumptions are to simplify expressions, but not crucial to numerical results.

of the order book state rather than time. In other words, all the traders of an identical type and facing an identical limit order book should have an identical optimal strategy. In addition, as a trading game, the equilibrium strategies solve Problem (3) when the expected waiting time is computed endogenously. For each subgame, the traders' strategies would represent a Nash equilibrium.

More precisely, the expected waiting times and execution prices, $\{T(\cdot), T^H(\cdot), \bar{m}(\cdot), \bar{m}^H(\cdot)\}$, are common knowledge. Based on the common knowledge, each trader of type (β_ν, i) solves Problem (3) to get his/her optimal trading strategy. With these optimal strategies and the arrival process of traders, we can calculate the expected waiting times and prices analytically or numerically. The equilibrium is identified if these calculated expected waiting times and prices conform to the common knowledge.

Equilibrium with a Homogeneous Impact. For the case with a homogeneous impact, take the expected waiting time of a $j-$hidden order submitted at the bid-ask spread s as an illustration: with the optimal trading strategies and arrival process of traders, we can analytically derive the probability that the next arrival after submitting a $j-$hidden order would trigger an execution of a $j-$hidden order, and the long-run probability of the submitted order taking the $k-$th position in the hidden order queue. Denote these two probabilities by p and p_k, respectively. For $k > 1$, the expected waiting time of the $j-$hidden order taking the k-th position in the queue, $T^H(s, j, k)$, satisfies

$$\begin{aligned} T^H(s,j,k) &= p \cdot \left(\frac{1}{\lambda} + \frac{1}{\lambda} + T^H(s,j,k-1)\right) + (1-p) \cdot \left(\frac{1}{\lambda} + \frac{1}{\lambda} + T^H(s,j,k)\right) \\ &= \frac{2}{p\lambda} + T^H(s,j,k-1). \\ T^H(s,j,1) &= p \cdot \frac{1}{\lambda} + (1-p) \cdot \left(\frac{1}{\lambda} + \frac{1}{\lambda} + T^H(s,j,1)\right) = \frac{2-p}{p\lambda}. \end{aligned}$$

The term $\frac{2}{p\lambda}$ reflects the extent to which the order at $(k-1)-$th position is better than the one at $k-$th position in terms of the expected waiting time. The expected waiting time is defined by,

$$T^H(s,j) = \sum_{k=1}^{\infty} p_k \cdot T^H(s,j,k),$$

which is the average in terms of all the positions in the queue. Similar calculations can be applied to other expected waiting times and execution prices. The equilibrium is found if these calculated expected waiting times and execution prices are equal to these in common knowledge.

Equilibrium with Heterogeneous Impacts. Each type of traders identifies the optimal trading strategy by solving Problem (3) with the common knowledge $\{T(\cdot), T^H(\cdot), \bar{m}(\cdot), \bar{m}^H(\cdot)\}$, Since it is challenging to get analytical solutions, we simulate the market with optimal strategies and the Poisson arrival process to output the averages of realized outcomes as the expected times and prices. The equilibrium is found if each trader plays the best response, the averages of the realized waiting times, and execution prices match the expected values used in Problem (3).

3. Equilibrium Results for the Case of a Homogeneous Impact

We first consider the case where all the traders share an identical impact in this section. Although this case is not general enough, it leads to analytical solutions for the equilibrium strategies so that we can use them to compare our results with existing results in the literature, and as a benchmark to evaluate the results for heterogeneous impact obtained in the next section from our new algorithm.

In this case, all the traders have an identical parameter β in the profit formulation (2), and the value of β is common knowledge. According to the definitions of the patient and impatient traders and the problem formulation in (3), the market order, which is immediately executed, provides the same profit for both patient and impatient traders. Therefore, if the optimal strategy of an impatient trader arriving at spread s is submitting a non-market order, then that of a patient trader arriving at spread s must also be submitting a non-market order.

First, we shall eliminate some trivial cases. (i) For the spreads where an impatient trader's optimal strategy is to submit a non-market order, there would not exist any market order. This corresponds to the states where the spread is so large that no market order is arriving, but some limit orders may come to narrow the spread until market orders are placed again. Denote s_c as the cutoff spread for such a case. Because there is no market order when the spread is $s_c + 1$, one can understand this cutoff spread s_c as a spread beyond which the book depth is unlimited. (ii) Because we require traders arriving at the spread $s = 1$ to submit a market order, we do not discuss the optimal strategy for patient traders arriving at the spread $s = 1$. Therefore, we shall focus on the trading strategies of patient traders arriving at spreads $s \in [2, s_c]$.

3.1. Solving for Equilibrium

We first present the expected waiting times of hidden orders in the equilibrium in the following lemma to facilitate the derivation of our equilibrium strategies with a homogeneous impact.

Lemma 1. *In the equilibrium with a homogeneous impact, the expected execution price and waiting time of a midpoint peg order, and the expected waiting time of a $j-$hidden order, submitting at a spread s, $j = 1, 2, \cdots, s-1$, are given by*

$$T^H(s, \frac{s}{2}) = \frac{1}{\lambda(1+\theta_P)}, \quad \bar{m}(s) = \frac{s}{2}, \tag{4}$$

$$T^H(s, j) = \begin{cases} \frac{1}{\lambda(\theta_I - \theta_P)}, & \text{if } \theta_I > \theta_P, \\ \infty, & \text{otherwise.} \end{cases} \tag{5}$$

There are two observations from the lemma. (i) The expected waiting time of a hidden limit order goes to infinity if the proportion of impatient traders is no higher than that of patient traders. The impatient trader would submit a market order and thus act as a liquidity demander. In other words, in a market where the liquidity supplier dominates with a homogeneous impact, placing a hidden limit order is not optimal; otherwise, a long hidden-order queue would be formed, whose average waiting time approaches infinity. (ii) The expected waiting times of the midpoint peg order and the $j-$hidden order are independent of the spread upon their submissions. The reason is that all the patient traders share an identical impact and thus choose to use an identical strategy for a given spread. In particular, if a patient trader arriving at a spread s find it optimal to submit a hidden order, the submission of this trader would not change the visible limit order book. With a homogeneous impact, the next arriving patient trader would still submit the hidden order. The expected waiting time of a hidden order thus depends on the expected arrival time of market orders from impatient traders. Based on this lemma, we present the equilibrium spreads and strategies for the equilibrium with a homogeneous impact.

Theorem 1. *In the equilibrium with a homogeneous impact, the equilibrium spreads and strategies are given as follows, with the analytical expressions of the spreads n_j and the indexes k_i are provided in the online supplement.*

(1) *If $\theta_I > \theta_P$, the set of equilibrium spreads $\{n_1, n_2, \cdots, n_q\}$ is $\{n_i : \ n_i \leq s_c\} \cup \{s_c\}$, where the spreads n_i (in ticks), $i = 1, 2, \cdots, k_t$, are defined as:*

$$n_1, \cdots, n_{k_1-1}, n_{k_1}, n_{k_1+1}, \cdots, n_{k_2-1}, n_{k_2}, n_{k_2+1}, \cdots, n_{k_t}.$$

(i) The patient trader arriving at the spread $s \leq n_1$ would submit a market order.

(ii) The patient trader arriving at the spread n_{k_i} would submit a midpoint peg order, for $i = 1, \ldots, t-1$.

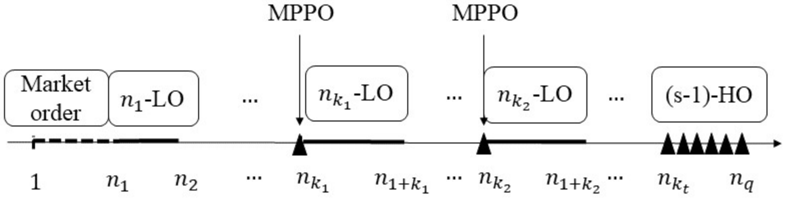
(iii) The patient trader arriving at the spread $s \in (n_j, n_{j+1})$ would submit an n_j-limit order, for all $j \in \{1, \ldots, q-1\}$; the patient trader arriving at the spread n_{j+1}, $j \in \{1, \ldots, q-1\}/\{k_1 - 1, \ldots, k_t - 1\}$ would submit an n_j-limit order.

(iv) The patient trader arriving at the spread $s \in [n_{k_t}, s_c]$ submits an $(s-1)-$hidden order.

(2) *If $\theta_I \leq \theta_P$, the equilibrium spreads and the equilibrium strategy are the same as those* *stated in Part 1, except* $t = \infty$ *and* $n_{k_t} > s_c$*. In other words, the traders would never submit a hidden limit order in such a case.*

Figure 2 illustrates the general pattern of the equilibrium strategies. The trader chooses to submit market orders when the current spread is relatively small, as the price benefits of non-market orders are relatively insignificant compared with the waiting costs. As the spread becomes larger, the price benefits start to motivate traders to choose limit orders. However, when the spread gets even larger, the trader may find that the waiting time of a limit order is too long and that the execution price of a market order is also unsatisfactory, leading the trader to submit a midpoint peg order, whose waiting time and execution price lie between those of a market order and those of a limit order. When the bid-ask spread is so large that submitting a midpoint peg order is not optimal, the trader would finally turn to a hidden limit order. This result is consistent with the empirical finding by Bessembinder et al. (2009) for iceberg orders that the traders would choose dark trading more when the bid-ask spread is wide, although the case here is for hidden orders, not iceberg orders.

Figure 2: **Equilibrium strategy with a homogeneous impact.** A patient trader's submission of optimal orders based on the bid-ask spread, according to Theorem 1. The axis is the bid-ask spread in ticks. The dotted line corresponds to the interval of spreads where arriving patient traders would submit market orders. In contrast, the solid line corresponds to the interval of spreads where arriving patient traders would submit limit orders. The black triangles correspond to the spreads where arriving patient traders would submit hidden orders. The MPPO stands for a midpoint peg order, while a $j-$LO/HO stands for a $j-$limit/hidden order.



Note that the midpoint peg and limit orders alternate when the spread is relatively large. More precisely, traders arriving at the cutoff spreads n_{k_i}, $i = 1, 2, \dots, t-1$, choose to submit a midpoint peg order (instead of a limit orders), thus shortening the expected waiting time; the optimal strategy at a spread $s \in (n_{k_i}, n_{k_i+1})$ is to submit an $n_{k_i}-$limit order. In other words, the limit orders creating spreads no smaller than n_{k_i} become attractive with the shortened waiting time, and motivate the traders to choose limit orders until the spread is so large that the waiting time of a limit order is too long to compete with a midpoint peg order again. As a result, this phenomenon reveals that the introduction of midpoint peg order enhances the market's liquidity by providing traders a shortcut to execute their orders with each other.

There are key differences between our equilibrium results with a homogeneous impact and other existing equilibrium results on dark trading. (i) Due to the consideration of the infinite time horizon, traders in our model consider limit/hidden orders at multiple discrete price levels, which is also different from the game by Brolley (2019) where traders only submit orders to the best price level in the limit order market. (ii) The equilibrium market in our case can have different bid-ask spreads in different periods, which is not considered by the one- or two-period games (see, e.g., Hendershott and Mendelson (2000), Iyer et al. (2018), Ye (2011), Ye (2016) and Zhu (2014)). (iii) We consider the hidden order queue with expected waiting times, quite different from the existing stationary equilibrium strategy by Degryse et al. (2009) for dark pools which do not have the time priority rule.

3.2. An Example with a Homogeneous Impact

If we set the common impact β to be infinity (i.e., traders are too sensitive about the uncertainty in the execution time of hidden orders to submit any hidden order), hidden orders disappear and the equilibrium spreads and strategies in Theorem 1 are the same as those in Propositions 4 and 5 of Foucault et al. (2005). More precisely, Theorem 1 yields that

$$\begin{cases} s_0 = \dfrac{\delta_P}{\lambda\Delta},\ n_1 = CF(s_0), \\ n_2 = n_1 + CF(2s_0\rho),\ n_{i+2} = n_2 + \displaystyle\sum_{\nu=1}^{i} CF(2s_0\rho^{\nu+1}), \end{cases} \tag{6}$$

and that the patient traders arriving at a spread $s \in (n_i, n_{i+1}]$, $i = 1, 2, \cdots, q-1$ would submit an n_i-limit order. See our online supplement for the details.

Next we introduce hidden orders to the limit order market ($\beta < \infty$) with the parameters in Example 3 by Foucault et al. (2005), as shown in Table 1. Note that when the common impact $\beta = 1$, all traders have the same waiting cost from hiding orders as from visible limit orders.

Table 1: The parameter settings in Example 3 by Foucault et al. (2005). PT stands for the patient trader.

tick size	arrival rate	proportion of PT	waiting cost per unit time for PT
$\Delta = \$1/16$	$\lambda = 1$	$\theta_P = 0.45$	$\delta_P = 0.05$

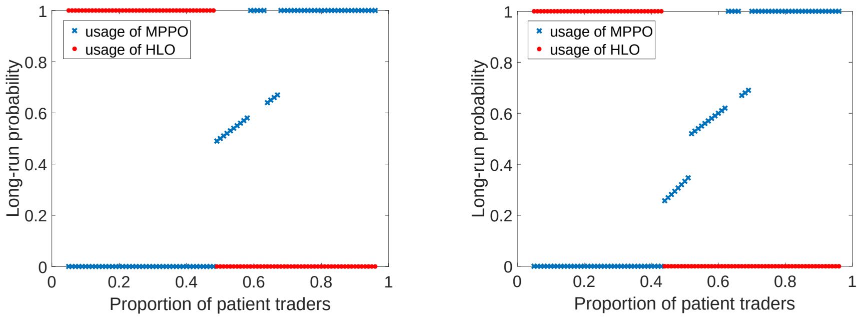
For an illustrative purpose, we set the maximum spread as $s_c = 4$. According to the equilibrium spreads given in Theorem 1, we can calculate the equilibrium spreads as $n_i = i$, $i = 1, \cdots, 4$, and $k_i = i + 1$ for $i = 1, 2, 3$. The equilibrium strategy is thus to submit a midpoint peg order at all the spread points $s = 2, 3, 4$. The detailed derivation can be found in the online supplement.

With specific θ_P (the proportion of patient traders) and β (the homogeneous impact parameter), one can derive the equilibrium spreads and strategies according to Theorem 1, and thus the long-run probabilities of the market staying at these spreads and using a particular type of order. In particular, we consider an empty initial market with the maximum bid-ask spread being $s_c = 10$ and keep all the other parameters the same as in Table 1.

Figure 3 depicts two long-run steady-state probabilities of midpoint peg orders and hidden limit orders for a patient trader. Note the summation of these two probabilities may be less than one because limit orders could be optimal in some states.

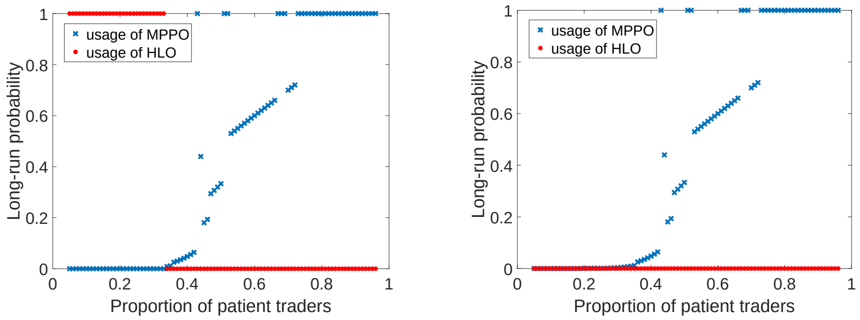
With a homogeneous impact, a large β means that all patient traders are sensitive to the uncertainty in the waiting time of submitted hidden orders. According to the profits defined in (1)-(2), the trader with a large β would prefer limit orders over hidden ones. When the impact parameter β is sufficiently large, all the traders would submit limit orders, but not any hidden orders, as shown in Figure 3(f). In fact, one can apply Theorem 1 to prove that the hidden limit order is not optimal in the equilibrium with $\beta \geq 1$. The intuition is that the hidden limit order is submitted to a price less aggressive than the midpoint price and

Figure 3: **Long-run probabilities of different types of hidden orders with a homogeneous impact, calculated under equilibrium spreads and optimal strategies presented in Theorem 1.** The x-axis represents the proportion of patient traders θ_P. The crosses stand for the long-run probability of using a midpoint peg order (MPPO), while the dots stand for that of a hidden limit order (HLO). A smaller β indicates less sensitivity to the execution uncertainty of submitting hidden orders. A larger θ_P implies fewer market order traders, thus reducing the usage of hidden limit orders and promoting the use of midpoint peg orders.



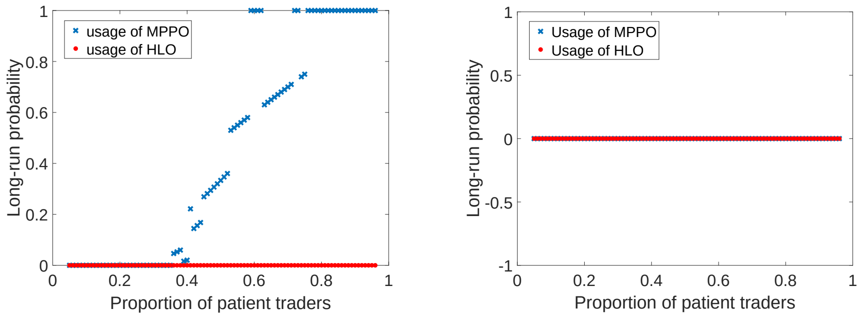
(a) long-run usage of hidden orders with a homogeneous impact $\beta = 0.1$.

(b) long-run usage of hidden orders with a homogeneous impact $\beta = 0.5$.



(c) long-run usage of hidden orders with a homogeneous impact $\beta = 0.99$.

(d) long-run usage of hidden orders with a homogeneous impact $\beta = 1$.



(e) long-run usage of hidden orders with a homogeneous impact $\beta = 2$.

(f) long-run usage of hidden orders with a homogeneous impact $\beta = 12$.

could form a relatively long hidden order queue. By contrast, midpoint peg orders are likely to be quickly executed with rather aggressive prices even with $\beta \geq 1$.

When traders are less sensitive to the uncertainty in the waiting time of hidden orders with $\beta < 1$, the hidden limit order would be optimal in the equilibrium when the proportion of patient traders, θ_P, is low because impatient traders would submit market orders and the hidden limit order would be more likely to be executed when there are more impatient traders than patient traders. Furthermore, according to Theorem 1, once the hidden limit order is optimal for the patient trader arriving at a bid-ask spread s, it would be optimal for all the patient traders arriving at the spreads from s to s_c. As a result, once the hidden limit order is optimal, with an empty initial market, the patient trader would submit a hidden limit order. In contrast, the impatient trader would submit a market order. The visible market state would stay unchanged. The long-run probability of the market state, where patient traders submit a hidden limit order, is one.

As shown in Figure 3, the long-run probability of the usage of the midpoint peg order tends to increase with the proportion of patient traders. The intuition is as follows. A high proportion of patient traders leads to a market with few incoming market order traders, making both the hidden limit order and the non-aggressive limit order difficult to be crossed by a market order. The midpoint peg order provides a shortcut for patient traders to execute their orders with each other at the midpoint price, which is still better than the price of a market order. [7]

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[7] It is not obvious how introducing hidden orders to a limit order market affects the impatient (market order) traders. First, introducing hidden orders provides an impatient trader opportunities to trade at the midpoint price, which seems to benefit the impatient trader. Second, the introduction of hidden orders prevents some patient traders from submitting aggressive limit orders and thus enlarges the visible spread. We will discuss this topic in Section 5 in our welfare analysis for the more general case of heterogeneous impacts.

4. Equilibrium Results for the Case of Heterogeneous Impacts

In this section, we consider the market with heterogeneous impacts. In particular, the impact parameter in (2) of hidden orders could take one of the values in the set $\{\beta_\nu : \nu \in \Lambda\}$. As discussed in the last section, we focus on the trading strategies of patient traders arriving at spreads $s \in [2, s_c]$, where impatient traders act as the market order trader. Thus, each patient trader with an impact type β_ν, arriving at a bid-ask spread $s \in [2, s_c]$, would compare (1)-(2) to solve the value function as

$$v(s, \nu) = \max_{j \in \{0, \cdots, s-1\}} \{U_P(s, j),\ U^H_{P,\nu}(s, j),\ U^H_{P,\nu}(s, \frac{s}{2})\}. \tag{7}$$

4.1. Solving for Equilibrium

Since it is challenging to get analytical solutions for the case of heterogeneous impacts, we attempt to find a numerical algorithm to solve for the equilibrium with heterogeneous impacts.

To avoid the curse of dimensionality, Pakes and McGuire (2001) introduces a stochastic algorithm for computing Markov perfect equilibrium by approximating the unknown continuation values by a simple average of returns from past outcomes of the algorithm[8]. In a finite state ergodic Markov Process, every sample path enters into recurrent states in finite time. Thus, their algorithm focuses on the policies in these recurrent states.

However, in models for limit order books, multiple discrete price levels and order types result in numerous discrete feasible strategies for each state. If one of the feasible strategies, which should be optimal, is unluckily believed to have a long-expected waiting time by traders, no trader would choose that strategy. Afterward, the expected waiting time of

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[8] Because the existing hidden orders would affect the expected waiting times of incoming orders, the expected waiting times depend not only on the visible market state but also on the invisible market state. Meanwhile, the invisible market state might depend on the historical order flow in the heterogeneous case. This fact makes our problem non-Markovian in the first place. Note that the initial and current market states determine the expected invisible market state. We thus add the initial market state as a state variable to make our problem Markovian. Without additional explanation, we will consider the initial market state as an empty market with the bid-ask spread being s_c, beyond which there is unlimited depth.

the order would never be updated. The corresponding state, to which the strategy would lead, becomes a “non-recurrent” state. The strategy thus remains too time-consuming to be submitted through the whole algorithm. To avoid such a case, Goettler et al. (2005) improve the original algorithm by specifying the initial expected values to be overly optimistic and forcing updates of the “non-recurrent” states every 10 million periods via re-starting the simulation with these states.

The Need to Modify Existing Algorithms. In our numerical experiments, however, the two additional adjustments in Goettler et al. (2005) still fail to ensure the correctness of estimated expected waiting times on the “non-recurrent” states, leading to incorrect equilibrium trading strategies. The intuition is that some states, even forced as the starting states with optimistic expected waiting time, maybe really unlucky and take a relatively long time for the first several executions. Based on the updated large waiting time, the trader arriving afterward would no longer choose to submit that order. The state thus becomes “non-recurrent” again and would be ignored in the equilibrium test, resulting in the wrong equilibrium. For an illustrative purpose, we apply the original algorithms in Pakes and McGuire (2001) and Goettler et al. (2005) to the equilibrium problem in the following counterexample, to demonstrate the need to introduce a new algorithm.

A Counterexample. Consider Example 3 in Foucault et al. (2005), where only limit and market orders are feasible to all the traders. All the parameters are shown in Table 1. For simplicity, we take the maximal spread as $s_c = 4$, i.e., traders only need to choose their optimal strategies for spreads $s \leq 4$. According to the analytical solution, the patient trader would submit a $1-$limit order if the spread upon his arrival is $s = 2$ or 3, and a $3-$limit order if the spread is $s = 4$. We first directly apply the algorithm by Pakes and McGuire (2001) with the optimistic initial settings to this problem, letting the initial expected waiting times of all the limit orders be the expected duration of one trader’s arrival. However, the trader could deviate from the optimal strategy to choose a $1-$limit order at the spread $s = 4$ if the feedback of submitting a $3-$limit order is unluckily bad. Afterward, if we start simulation every 10 million periods, as suggested in Goettler et al. (2005), at the less frequently visited state $s = 3$ a bad feedback could still easily make traders turn to a $1-$limit order. Furthermore, even if we conduct an equilibrium test, in which the expected waiting times on the recurrent states are recorded and checked, s the algorithm would still report no error. As a result, the algorithm is likely to conclude wrongly that the equilibrium strategy as $\Pi(4) = 1-$limit order. We apply the algorithm ten times independently to this simplified problem and find only two of these ten experiments converge to the correct equilibrium strategies. In summary, the counterexample shows that there is a need to improve the existing algorithms due to the incorrect policies on the "non-recurrent" states.

The New Algorithm. To guarantee the correct update of "non-recurrent" states, in addition to forced updates of the "non-recurrent" states in every 10 million periods as in Pakes and McGuire (2001), we use additional updates of feasible strategies and a new stopping rule to evaluate correct equilibrium. More precisely, the new algorithm does the following:

(1) During the k-th iteration, traders identify the strategy set Π_k by optimizing their expected profits (7) based on the k-th waiting time system; we then simulate the market with N traders' arrivals and all the patient traders following the strategy set Π_k to update the waiting time system. The iteration ends if the difference between the updated waiting time system and the original one does not exceed the tolerance level.

(2) After the iterations, we conduct an equilibrium test using a stopping rule. At the $t-$th period, keeping all the other strategies $\Pi_{k+t-1,-}(s, \nu)$ in the strategy set unchanged, the trader of impact type $\nu \in \Lambda$ arriving at the spread s re-estimates the expected waiting times for all feasible strategies to decide whether to deviate. If the strategy set satisfies the stopping rules in Definition 1 within a tolerant level, the algorithm ends and reports the strategy set as the equilibrium strategy.

Definition 1 (A Stopping Rule). *(i) After the update of the expected waiting time system by simulating the market with the strategy set* $\Pi(\cdot)$*, the expected waiting times and execution* *prices of the strategies in* $\Pi(\cdot)$ *stay unchanged. (ii) For each trader type* ν *and bid-ask spread* s*, keeping all the other strategy* $\Pi_{-}(s,\nu)$ *in the strategy set unchanged, the trader re-estimates the expected waiting time and execution price for all his feasible strategies to find that* $\Pi(s,\nu)$ *is still optimal to the problem* (7) *with the changed expected waiting time system.*

We present the details of our algorithm in the online supplement. The strategy satisfying our stopping rules is a perfect equilibrium strategy, as shown in the following theorem.

Theorem 2. *The strategy set* $\Pi^*(\cdot)$ *is an equilibrium trading strategy if it satisfies the stopping rules in Definition 1.*

The Counterexample (continued). We apply our algorithm to Example 3 in Foucault et al. (2005) with maximum spread $s_c = 4$ and a homogeneous impact with either $\beta = \infty$ or $\beta = 1$, respectively. The parameters are shown in Table 1. The algorithm stopped with the equilibrium strategy documented in Table 2. All the equilibrium strategies given by the algorithm successfully match the correct ones given by Theorem 1. Note that no matter which spread $s \in [1, s_c]$ the market starts with, the equilibrium spreads would be $n_1 = 1$, $n_2 = 3$ and $n_3 = s_c = 4$. Thus, the spread $s = 2$ is a "non-recurrent" state, since the 2−limit order is never optimal in the equilibrium. In the numerical experiment, even if we deviate from the true equilibrium strategy by empowering traders to re-estimate the expected profit of the 2−limit order, our numerical algorithm still gets correctly that it would choose not to submit the 2−limit order.

Table 2: Equilibrium strategy outputted by our algorithm for Example 3 in Foucault et al. (2005) with a homogeneous impact $\beta = 1$ or ∞. The strategy $i-$LO stands for the $i-$limit order, $i = 1, 3$. The strategy MPPO stands for a midpoint peg order. The results are the same as the ones derived by Theorem 1.

bid-ask spread	2	3	4
equilibrium strategy for $\beta = \infty$	1-LO	1-LO	3-LO
equilibrium strategy for $\beta = 1$	MPPO	MPPO	MPPO

4.2. Equilibrium Strategies with Heterogeneous Impacts

We apply our algorithm to different sets of parameters to illustrate various equilibrium outcomes in the heterogeneous case. In particular, we consider three settings as follows.

Baseline case. We adopt the set of parameters in Example 2 by Foucault et al. (2005) where patient traders dominate with $\theta_P = 0.55$. The other parameters are shown in Table 1. Each trader has, with equal probability, either an impact parameter $\beta_1 < 1$ or $\beta_2 \geq 1$. In particular, we run our algorithm with β_1 varying from 0.1 to 0.95 and β_2 ranging from 1.0 to 2.0. For illustrative purposes, we only consider the maximum spread as $s_c = 3$; the results still hold for larger maximum spreads in our numerical experiments.

Demand-dominance case. We keep all the other settings the same as the baseline case, except for $\theta_P = 0.45$, which corresponds to the setting in Example 3 by Foucault et al. (2005). In such a case, there are more impatient traders than patient traders. Since the impatient trader would submit market orders and thus act as a liquidity demander, we call this case demand dominance.

Extreme case. Keeping all the other settings the same as the base case, except either the impact parameters are changed to $(\beta_1, \beta_2) = (0.1, 4)$, or the proportion of patient traders is extreme with $\theta_P < 0.45$ or $\theta_P > 0.55$. In such extreme cases, the traders with different impacts may behave more diversely due to significant sensitivity to the execution risk or the extreme market condition.

Table 3 presents the equilibrium strategies in some representative examples of the three cases for the following four different markets; other numerical examples give similar results.

- Market I: Traders choose to submit either a market or a limit order.
- Market II: Traders choose from limit, market, and midpoint peg orders.
- Market III: Traders choose from limit, market, and hidden limit orders.
- Market IV: Traders choose from limit, market, midpoint peg, and hidden limit orders.

Market I corresponds to the market without introducing any hidden orders, which is also precisely the market modeled by Foucault et al. (2005). Markets II and III are the markets where either midpoint peg orders or hidden limit orders are introduced. In contrast, Market IV allows the usage of both midpoint peg orders and hidden limit orders.

Table 3: **Equilibrium strategies of patient traders with heterogeneous impact in Markets I-IV.** We denote the midpoint peg order by MPPO and the j-limit/hidden order by the $j-$LO/HO, $j = 1, 2$. Market I corresponds to a limit order market without hidden orders. In contrast, Market II-IV corresponds to introducing midpoint peg orders, hidden limit orders, or both kinds of hidden orders to Market I, respectively.

		Market I		Market II		Market III		Market IV	
Settings	spread impact	$s = 2$	$s = 3$	$s = 2$	$s = 3$	$s = 2$	$s = 3$	$s = 2$	$s = 3$
$\theta_P = 0.55$	$\beta_1 = 0.9$	1-LO	1-LO	MPPO	MPPO	1-HO	2-HO	MPPO	MPPO
	$\beta_2 = 1.1$	1-LO	1-LO	MPPO	MPPO	1-HO	1-HO	MPPO	MPPO
$\theta_P = 0.55$	$\beta_1 = 0.9$	1-LO	1-LO	MPPO	MPPO	1-HO	1-HO	1-HO	MPPO
	$\beta_2 = 2.0$	1-LO	1-LO	1-LO	1-LO	1-LO	1-LO	1-LO	1-LO
$\theta_P = 0.45$	$\beta_1 = 0.9$	1-LO	1-LO	MPPO	MPPO	1-HO	2-HO	MPPO	MPPO
	$\beta_2 = 1.1$	1-LO	1-LO	MPPO	MPPO	1-HO	1-HO	MPPO	MPPO
$\theta_P = 0.55$	$\beta_1 = 0.1$	1-LO	1-LO	MPPO	MPPO	1-HO	2-HO	1-HO	2-HO
	$\beta_2 = 4.0$	1-LO	1-LO	1-LO	1-LO	1-LO	1-LO	1-LO	1-LO
$\theta_P = 0.20$	$\beta_1 = 0.9$	1-LO	2-LO	MPPO	2-LO	1-LO	2-HO	1-LO	2-HO
	$\beta_2 = 1.1$	1-LO	2-LO	1-LO	2-LO	1-LO	2-LO	1-LO	2-LO

Not surprisingly, in Table 3 the equilibrium strategies (obtained numerically via our algorithm) in Market I under the base case and the demand dominance case are the same as the ones (obtained analytically) in Examples 2 and 3 in Foucault et al. (2005). Note that the values of impact $\{\beta_\nu : \nu = 1, 2\}$ do not affect the equilibrium in Market I because no hidden order is allowed.

Table 3 shows that a patient trader may change the optimal strategy from submitting a limit order to a midpoint peg or hidden limit order. This is consistent with the laboratory finding in Bloomfield et al. (2015) that with the availability of different degrees of opaqueness in trading, traders would switch from displayed orders to non-displayed ones. With the introduction of hidden orders, Table 3 indicates that the patient trader would submit a less aggressive order. For example, the optimal strategy of the patient trader arriving at the spread $s = 3$ may change from the $1-$limit order to the midpoint peg order or the $2-$hidden order. In addition, Table 3 suggests the insensitive trader with $\beta_1 < 1$ would be more likely to submit a hidden order than the sensitive ones. This result is consistent with the empirical finding in Chakrabarty et al. (2020) that agency algorithmic traders are more likely to submit hidden orders than high-frequency algorithmic traders.

Table 4: The differences between the optimal strategy with a homogeneous impact (*Homo.*) and that with heterogeneous impacts (*Heter.*). HLO stands for the hidden limit order. The main difference is that HLO could never be optimal in equilibrium with a homogeneous impact if the majority is patient.

	Homo.	Heter.
HLO is preferred by the optimistic trader	✓	✓
HLO could be optimal in the equilibrium if the majority is patient	×	✓

With heterogeneous impacts, in the base case where the majority are patient traders, hidden limit orders are possibly used as the equilibrium strategies by the patient trader. This is different from the case with a homogeneous impact, as summarized in Table 4. The intuition is as follows. In the case of the homogeneous impact, all the traders share an identical attitude towards the execution risks of hiding and thus choose the same strategy. Therefore, if the majority is patient, all patient traders would expect a long queue of hidden limit orders should all patient traders submit hidden orders. Thus, the same endogenous strategy of all patient traders is not to submit hidden limit orders. On the contrary, in the heterogeneous case, different types of patient traders choose different strategies. As a result, the insensitive traders would turn to invisible orders, while the sensitive traders submit visible ones.

5. Implications of Introducing Hidden Orders

In this section, we examine how the introduction of hidden orders affect the market participators' profits and the market liquidity. In particular, we simulate Markets I-IV with one million trader arrivals, where all patient traders follow equilibrium strategies with various parameter settings. We compare several descriptive statistics of these four markets to study the effects of the introduction of hidden orders in terms of price impact, effective spread, expected profits, execution price, and mid-quote volatility.

5.1. Aggressiveness and the Expected Profits of Patient Traders

In our model, the patient traders would choose their strategy to maximize the expected profits, which includes both the execution price and the expected waiting time. In equilibrium, we define the expected profit of patient traders with impact β_ν, $\nu \in \Lambda$ as

$$W_\nu = \frac{\sum_{i=1}^{N} \mathbf{1}_{i=\nu} \sum_{j=1}^{q} \mathbf{1}_{s=n_j} v(n_j, \nu)}{\sum_{i=1}^{N} \mathbf{1}_{i=\nu}}, \tag{8}$$

where N is the number of traders, $n_1, ..., n_q$ are equilibrium spreads, $\mathbf{1}_{i=\nu}$ is an indicator function that equals to either 1 if the arriving trader is a patient trader with impact β_ν or zero otherwise. And the indicator function $\mathbf{1}_{s=n_j}$ equals to either 1 if the bid-ask spread is n_j or zero otherwise. The value $v(n_j, \nu)$ is the optimal profit, defined in (7), which considers both the execution price and the expected waiting time.

We measure the order aggressiveness of patient traders with impact β_ν, $\nu \in \Lambda$ by using the relative price tick of the submitted buy (resp. sell) orders to the best ask (resp. bid) prices, i.e.,

$$RP_\nu = \frac{\sum_{i=1}^{N} \mathbf{1}_{i=\nu} \sum_{j=1}^{q} \mathbf{1}_{s=n_j} RP(n_j, \nu)}{\sum_{i=1}^{N} \mathbf{1}_{i=\nu}}, \tag{9}$$

where the value $RP(n_j, \nu)$ is the number of ticks of the order, which is submitted by the patient trader with impact β_ν arriving at the spread n_j, away from the best price on the opposite side.

By comparing the profits columns of Market I with those of Markets II-IV in Table 5, it appears that the introduction of hidden orders would largely increase the expected profit of patient traders by a factor of two to seven times. In addition, the increase is not limited to the insensitive patient trader with the impact parameter $\beta_1 < 1$; the sensitive patient trader with the impact parameter $\beta_2 > 1$, who may not submit any hidden order in the equilibrium[9], also

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[9] See the result form parameter setting $\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0), \rho_1 = 0.5, s_c = 3$ for instance.

receives significantly higher expected profit in the markets allowing hidden orders (although less significantly than the insensitive ones). The intuition is that the introduction of hidden orders benefits the expected profit of insensitive patient traders directly by providing a new option to submit an order and benefits the expected of sensitive patient traders indirectly by providing a liquid market with shorter expected waiting times, as some orders can even be executed upon their submissions against the hidden orders. We summarize this profit implication as follows.

Empirical Implication 1. *The introduction of hidden orders, including midpoint peg orders and hidden limit orders, to a limit order market increases the expected profit of patient traders.*

Table 5: Summary statistics (mean and standard deviation) of the expected profits of patient traders in Markets I-IV. Starting with an empty initial market, we report the expected profits of patient traders, defined in (8), W_1, W_2, with two different impacts β_1 and β_2. ρ_1 is the proportion of patient traders with impact β_1 to all the patient traders. For an illustrative purpose, we multiply the expected profits by 100. By comparing the first column with the last three columns, we can observe that introducing hidden orders, no matter the midpoint peg orders or limit hidden orders, would increase the expected profit of patient traders.

Parameter Setting	Market I W_1, W_2	Market II W_1, W_2	Market III W_1, W_2	Market IV W_1, W_2
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	1.2460, 1.2460 (0.0000)(0.0000)	6.4787,5.8351 (0.0000)(0.0000)	5.5612, 4.9156 (0.0000)(0.0000)	6.4732, 5.8284 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.1,\ s_c = 3;$	1.2460, 1.2460 (0.0000)(0.0000)	6.4737, 5.8290 (0.0000)(0.0000)	9.0489,4.7238 (0.0000)(0.0000)	6.4715, 5.8262 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.9,\ s_c = 3;$	1.2460, 1.2460 (0.0000)(0.0000)	6.4746, 5.3440 (0.0000)(0.0000)	6.6474, 8.3090 (0.0000)(0.0000)	6.4745, 5.8299 (0.0000)(0.0000)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	1.2431, 1.2431 (0.0000)(0.0000)	7.0668, 6.5538 (0.0000)(0.0000)	6.9900, 6.4779 (0.0000)(0.0000)	7.0650, 6.5517 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 4;$	1.3900, 1.3901 (0.0004)(0.0003)	9.5925, 8.9464 (0.0000)(0.0000)	9.6013, 8.9572 (0.0000)(0.0000)	9.5896, 8.9428 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 5;$	1.3686, 1.3685 (0.0003)(0.0003)	12.7315, 12.0884 (0.0088)(0.0055)	11.8171,11.1725 (0.0000)(0.0000)	12.7152, 12.0686 (0.0079)(0.0045)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	1.2490, 1.2490 (0.0000)(0.0000)	6.2786, 5.5905 (0.0000)(0.0000)	5.4563, 3.7529 (0.0000)(0.0000)	6.2690, 5.5788 (0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.1,\ s_c = 3;$	1.2490, 1.2490 (0.0000)(0.0000)	6.2740,5.5849 (0.0000)(0.0000)	8.9384,4.2081 (0.0000)(0.0000)	6.2631, 5.5715 (0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.9,\ s_c = 3;$	1.2490, 1.2490 (0.0000)(0.0000)	6.2728, 5.5834 (0.0000)(0.0000)	4.8897, 8.1509 (0.0000)(0.0000)	6.2771, 5.5887 (0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 5;$	3.6258 3.6270 (0.0053)(0.0050)	10.5106, 10.1362 (0.0028)(0.0034)	10.4947, 10.1204 (0.0027)(0.0037)	10.5133, 10.1381 (0.0027)(0.0035)

$\theta_P = 0.20, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.5$, $s_c = 3$;	4.2534, 4.2531 (0.0051)(0.0044)	4.3966, 4.2935 (0.0040)(0.0040)	4.6749, 4.5849 (0.0042)(0.0039)	4.6870, 4.5923 (0.0041)(0.0042)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$, $\rho_1 = 0.5$, $s_c = 3$;	1.2460, 1.2460 (0.0000)(0.0000)	5.8516, 2.9843 (0.0000)(0.0000)	4.0742, 3.6981 (0.0000)(0.0000)	5.8458, 2.9988 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$, $\rho_1 = 0.1$, $s_c = 3$;	1.2460, 1.2460 (0.0000)(0.0000)	6.4590, 2.8950 (0.0000)(0.0000)	9.0567,1.8162 (0.0000)(0.0000)	6.4721, 2.9240 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$, $\rho_1 = 0.8$, $s_c = 3$;	1.2460, 1.2460 (0.0000)(0.0000)	6.2541, 3.7374 (0.0000)(0.0000)	5.0364, 4.7410 (0.0000)(0.0000)	6.2651, 3.7306 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$, $\rho_1 = 0.5$, $s_c = 4$;	1.3900, 1.3900 (0.0004)(0.0003)	9.5958, 6.0462 (0.0000)(0.0000)	9.5979, 6.0509 (0.0000)(0.0000)	9.5987, 6.0528 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0)$, $\rho_1 = 0.5$, $s_c = 4$;	1.3901, 1.3900 (0.0004)(0.0003)	10.888043 6.0522 (0.0000)(0.0000))	10.887218 6.0489 (0.0000)(0.0000)	10.887492 6.0500 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0)$, $\rho_1 = 0.5$, $s_c = 5$;	1.3685, 1.3685 (0.0003)(0.0003)	10.478063 7.3697 (0.0089)(0.0043)	13.992099 8.1071 (0.0000)(0.0000)	9.866400 7.5890 (0.0075)(0.0045)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.5, 2.0)$, $\rho_1 = 0.5$, $s_c = 5$;	17.979251 17.9776 (0.0109)(0.0109)	18.001111 17.9951 (0.0110)(0.0117)	21.486047 18.3681 (0.0085)(0.0072)	21.490998 18.3830 (0.0082)(0.0085)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 3$;	1.2544, 1.2544 (0.0000)(0.0000)	8.9830, 3.0199 (0.0000)(0.0000)	11.3862, 3.1514 (0.0000)(0.0000)	11.3813, 3.1592 (0.0000)(0.0000)
$\theta_P = 0.50, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.1$, $s_c = 4$;	2.4811, 2.4812 (0.0024)(0.0029)	10.9756, 7.9118 (0.0041)(0.0056)	17.7435, 2.9187 (0.0000)(0.0000)	17.7476, 2.9218 (0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 4$;	3.624017 3.6231 (0.0043)(0.0052)	11.0802, 8.4486 (0.0045)(0.0061)	14.5117, 3.4781 (0.0108)(0.0013)	14.5079, 3.4599 (0.0092)(0.0012)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 3$;	5.8721, 5.8725 (0.0064)(0.0068)	8.6772, 6.0920 (0.0036)(0.0061)	11.5387, 6.1199 (0.0070)(0.0050)	11.5407, 6.0965 (0.0065)(0.0048)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 3$;	1.2637, 1.2637 (0.0000)(0.0000)	9.0356, 3.8671 (0.0000)(0.0000)	7.9223, 4.8713 (0.0000)(0.0000)	9.0347, 3.8595 (0.0000)(0.0000)

From the results shown in Table 6, it is observed that both types of patient traders submit orders at less aggressive prices after introducing hidden orders. We summarize this implication as follows.

Empirical Implication 2. *The introduction of hidden orders, including midpoint peg orders and hidden limit orders, to a limit order market decreases the order aggressiveness of patient traders.*

To the best of our knowledge, there is no empirical work on potential traders' welfare benefits from introducing hidden orders to a limit order market. In terms of other types of dark trading, Boulatov and George (2013) shows that for dealer markets if a dealer market hides the orders providing liquidity the uninformed liquidity demander would have a lower trading cost, and for iceberg orders the option to hide in iceberg orders is valuable to patient traders. We complement Boulatov and George (2013) by showing that introducing hidden orders would increase the expected profit of patient traders by facilitating them to submit orders at less aggressive price levels.

Table 6: Summary statistics (mean and standard deviation) of the order aggressiveness of patient traders in Markets I-IV. Starting with an empty initial market, we report the aggressiveness of patient traders, defined in (9), RP_1, RP_2, with two different impact β_1 and β_2. ρ_1 is the proportion of patient traders with impact β_1 to all the patient traders. By comparing the first column with the last three columns, we can observe that introducing hidden orders, no matter the midpoint peg orders or limit hidden orders, would encourage the patient traders to submit less aggressive orders.

Parameter Setting	Market I RP_1, RP_2	Market II RP_1, RP_2	Market III RP_1, RP_2	Market IV RP_1, RP_2
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.0000, 1.0000	1.5000, 1.5000	1.3546, 1.3546	1.5000, 1.5000
$\rho_1 = 0.5$, $s_c = 3$; (base case)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.0000, 1.0000	1.5000, 1.5000	2.0000, 1.3180	1.5000, 1.5000
$\rho_1 = 0.1$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.0000, 1.0000	1.5000, 1.5000	1.3178, 2.0000	1.5000, 1.5000
$\rho_1 = 0.9$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.0000, 1.0000	1.5000, 1.5000	1.4871, 1.4871	1.5000, 1.5000
$\rho_1 = 0.5$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.9000, 1.9004	2.0000, 2.0000	2.0000, 2.0000	2.0000, 2.0000
$\rho_1 = 0.5$, $s_c = 4$;	(0.0024)(0.0023)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.9001, 1.8999	2.5000, 2.5000	2.3548, 2.3548	2.5000, 2.5000
$\rho_1 = 0.5$, $s_c = 5$;	(0.0025)(0.0022)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.0000, 1.0000	1.5000, 1.5000	2.0000, 1.1376	1.5000, 1.5000
$\rho_1 = 0.5$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.0000, 1.0000	1.5000, 1.5000	2.0000, 1.2785	1.5000, 1.5000
$\rho_1 = 0.1$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.0000, 1.0000	1.5000, 1.5000	1.2784, 2.0000	1.5000, 1.5000
$\rho_1 = 0.9$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1)$,	2.0999, 2.1004	2.9148, 2.9146	2.9149, 2.9149	2.9146, 2.9144
$\rho_1 = 0.5$, $s_c = 5$;	(0.0024)(0.0023)	(0.0023)(0.0022)	(0.0022)(0.0024)	(0.0022)(0.0022)
$\theta_P = 0.20, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.8001, 1.8000	1.7844, 1.7844	1.8889, 1.8890	1.8888, 1.8889
$\rho_1 = 0.5$, $s_c = 3$;	(0.0014)(0.0012)	(0.0014)(0.0012)	(0.0011)(0.0010)	(0.0011)(0.0011)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$,	1.0000, 1.0000	1.5000, 1.1072	1.2157, 1.2165	1.5000, 1.1071
$\rho_1 = 0.5$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$,	1.0000, 1.0000	1.5000, 1.5000	2.0000, 1.3179	1.5000, 1.5000
$\rho_1 = 0.1$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$,	1.0000, 1.0000	1.5000, 1.1533	1.3057, 1.3092	1.5000, 1.1527
$\rho_1 = 0.8$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$,	1.9001, 1.8999	2.0000, 2.0000	2.0000, 2.0000	2.0000, 2.0000
$\rho_1 = 0.5$, $s_c = 4$;	(0.0024)(0.0019)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0)$,	1.9001, 1.8997	2.0000, 2.0000	2.0000, 2.0000	2.0000, 2.0000

$\rho_1 = 0.5,\ s_c = 4;$	(0.0103)(0.0120)	(0.0080)(0.0087)	(0.0083)(0.0073)	(0.0078)(0.0074)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$	1.9000, 1.9001	2.1789, 2.1273	3.0000, 2.1453	2.0815, 2.1627
$\rho_1 = 0.5,\ s_c = 5;$	(0.0022)(0.0023)	(0.0014)(0.0014)	(0.0000)(0.0000)	(0.0015)(0.0016)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.5, 2.0),$	3.8758, 3.8756	3.8702, 3.8762	3.9377, 3.9376	3.9376, 3.9376
$\rho_1 = 0.5,\ s_c = 5;$	(0.0018)(0.0018)	(0.0020)(0.0019)	(0.0013)(0.0012)	(0.0013)(0.0014)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.1, 4.0),$	1.0000, 1.0000	1.5000, 1.1088	2.0000, 1.0000	2.0000, 1.0000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.50, (\beta_1, \beta_2) = (0.1, 4.0),$	1.9999, 1.9999	1.8547, 2.2058	3.0000, 1.0000	3.0000, 1.0000
$\rho_1 = 0.1,\ s_c = 4;$	(0.0019)(0.0023)	(0.0007)(0.0018)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0),$	2.1003, 2.0999	1.8700, 2.2855	2.6345, 2.2687	2.6344, 2.2686
$\rho_1 = 0.5,\ s_c = 4;$	(0.0020)(0.0024)	(0.0007)(0.0018)	(0.0018)(0.0021)	(0.0016)(0.0020)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.1, 4.0),$	1.8998, 1.8999	1.4723, 1.9449	1.9448, 1.9450	1.9449, 1.9447
$\rho_1 = 0.5,\ s_c = 3;$	(0.0012)(0.0013)	(0.0006)(0.0011)	(0.0011)(0.0010)	(0.0011)(0.0010)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.1, 4.0),$	1.0000, 1.0000	1.5000, 1.1609	1.3219, 1.3225	1.5000, 1.1610
$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)

5.2. Executed Market Prices for Impatient Traders

Because the optimal strategy of the impatient trader arriving at n_j, $j = 1, \ldots, q$, is always to submit a market order, we directly take the average executed prices of impatient buyers and sellers as the indicator of the impatient trader's benefit.

According to Table 7, impatient traders, who submit market orders and act as the liquidity demanders in our model, could receive a worse executed market price after introducing hidden orders. The intuition is that with the introduction of hidden orders, the patient traders would choose less aggressive price levels to submit orders, as shown in Table 6 and Empirical Implication 2. As a result, the introduction of hidden orders could increase the possibility of the market order being executed against less aggressive orders, thus worsening impatient traders' average executed market prices.

However, given a typical tick size being \$1/16, the magnitude of the average impatient trade price change in Table 7 is less than 0.5 tick, which is relatively small. This leads to the following implication.

Empirical Implication 3. *The introduction of hidden orders, including midpoint peg orders and hidden limit orders, to a limit order market can slightly worsen impatient traders' average executed market prices.*

Table 7: Summary statistics (mean and standard deviation) of the execution prices of impatient traders Markets I-IV. Starting with an empty initial market and limit price levels from \$10 to \$$(10{+}s_c * \Delta)$, where the tick size $\Delta = 1/16$, we report the average buy and sell prices of impatient buyers and sellers (*Mbuy* and *Msell* in \$). ρ_1 is the proportion of patient traders with impact β_1. By comparing the first column with the last three columns, we can observe that introducing hidden orders, no matter the midpoint peg orders or limit hidden orders, could slightly increase the buy price and decrease the sell price of impatient traders.

Parameter Setting	Market I *Mbuy, Msell*	Market II *Mbuy, Msell*	Market III *Mbuy, Msell*	Market IV *Mbuy, Msell*
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	10.1187, 10.0687 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)	10.1431, 10.0443 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.1,\ s_c = 3;$	10.1187, 10.0688 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)	10.1453, 10.0422 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.9,\ s_c = 3;$	10.1187, 10.0688 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)	10.1453, 10.0422 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	10.0687, 10.1187 (0.0001)(0.0000)	10.1419, 10.0457 (0.0003)(0.0003)	10.1267, 10.0609 (0.0004)(0.0004)	10.1418, 10.0457 (0.0003)(0.0003)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 4;$	10.1792, 10.0708 (0.0002)(0.0000)	10.2056, 10.0444 (0.0001)(0.0001)	10.2056, 10.0444 (0.0001)(0.0001)	10.2057, 10.0444 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 5;$	10.2212, 10.0913 (0.0001)(0.0001)	10.2570, 10.0555 (0.0001)(0.0002)	10.2459, 10.0665 (0.0002)(0.0002)	10.2571, 10.0555 (0.0002)(0.0002)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	10.1313, 10.0562 (0.0001)(0.0001)	10.1584, 10.0291 (0.0001)(0.0001)	10.1546, 10.0329 (0.0001)(0.0001)	10.1584, 10.0291 (0.0001)(0.0001)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.1,\ s_c = 3;$	10.1313, 10.0563 (0.0001)(0.0001)	10.1584, 10.0291 (0.0001)(0.0001)	10.1506, 10.0369 (0.0001)(0.0001)	10.1584, 10.0291 (0.0001)(0.0001)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.9,\ s_c = 3;$	10.1312, 10.0563 (0.0001)(0.0001)	10.1584, 10.0291 (0.0001)(0.0001)	10.1506, 10.0369 (0.0001)(0.0001)	10.1584, 10.0291 (0.0001)(0.0001)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 5;$	10.2377, 10.0747 (0.0001)(0.0001)	10.2680, 10.0444 (0.0001)(0.0001)	10.2681, 10.0444 (0.0001)(0.0001)	10.2681 10.0444 (0.0001)(0.0001)
$\theta_P = 0.20, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	10.1726, 10.0149 (0.0000)(0.0000)	10.1729, 10.0146 (0.0000)(0.0000)	10.1723, 10.0152 (0.0000)(0.0000)	10.1723, 10.0152 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.5,\ s_c = 3;$	10.1187, 10.0688 (0.0001)(0.0001)	10.1333, 10.0542 (0.0001)(0.0001)	10.1288, 10.0587 (0.0001)(0.0001)	10.1333, 10.0542 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.1,\ s_c = 3;$	10.1188, 10.0688 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)	10.1453, 10.0422 (0.0001)(0.0001)	10.1542, 10.0333 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.8,\ s_c = 3;$	10.1187, 10.0688 (0.0001)(0.0001)	10.1449, 10.0426 (0.0001)(0.0001)	10.1367, 10.0508 (0.0001)(0.0001)	10.1449, 10.0426 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.5,\ s_c = 4;$	10.1792, 10.0708 (0.0001)(0.0001)	10.2057, 10.0444 (0.0001)(0.0001)	10.2056, 10.0443 (0.0001)(0.0001)	10.2056, 10.0444 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$ $\rho_1 = 0.5,\ s_c = 4;$	10.1792, 10.0708 (0.0001)(0.0001)	10.2056, 10.0444 (0.0001)(0.0001)	10.2056, 10.0444 (0.0001)(0.0001)	10.2057, 10.0443 (0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$ $\rho_1 = 0.5,\ s_c = 5;$	10.2211, 10.0914 (0.0001)(0.0001)	10.2450, 10.0675 (0.0001)(0.0001)	10.2447, 10.0678 (0.0002)(0.0001)	10.2428, 10.0697 (0.0001)(0.0001)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.5, 2.0),$ $\rho_1 = 0.5,\ s_c = 5;$	10.3055, 10.0070 (0.0000)(0.0000)	10.3055, 10.0070 (0.0000)(0.0000)	10.3055, 10.0070 (0.0000)(0.0000)	10.3055, 10.0070 (0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.1, 4.0),$ $\rho_1 = 0.5,\ s_c = 3;$	10.1187, 10.0688 (0.0001)(0.0001)	10.1333, 10.0542 (0.0001)(0.0001)	10.1359, 10.0516 (0.0001)(0.0001)	10.1359, 10.0516 (0.0001)(0.0001)
$\theta_P = 0.50, (\beta_1, \beta_2) = (0.1, 4.0),$	10.1875, 10.0625	10.2077, 10.0423	10.1875, 10.0625	10.1875, 10.0625

$\rho_1 = 0.1$, $s_c = 4$;	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0)$,	10.1958, 10.0542	10.2114, 10.0386	10.2013, 10.0487	10.2013, 10.0487
$\rho_1 = 0.5$, $s_c = 4$;	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.1, 4.0)$,	10.1806, 10.0069	10.1795, 10.0080	10.1806, 10.0069	10.1806, 10.0069
$\rho_1 = 0.5$, $s_c = 3$;	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.1, 4.0)$,	10.0688, 10.1188	10.0752, 10.1123	10.0739, 10.1136	10.0752, 10.1122
$\rho_1 = 0.5$, $s_c = 3$;	(0.0000)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)

On the surface, this implication seems to contradict the theoretical finding in Boulatov and George (2013) for dealer markets, which shows that the allowance of hiding in a dealer market (with market orders only and without limit orders) would benefit liquidity demanders. To make a fair comparison, we set up a special case in Section A in our online supplement where the waiting cost of patient traders is so high that they only submit market orders in Market I. With the introduction of hidden orders to such a market, we find that the impatient trader would benefit, similar to Boulatov and George (2013). Intuitively, if patient traders submit market orders in Market I as liquidity demanders, then the introduction of hidden orders could induce patient traders to submit hidden orders, thus becoming liquidity suppliers. As a result, there are fewer liquidity demanders and more liquidity suppliers. The impatient traders, who always act as liquidity demanders, thus benefit from introducing hidden orders.

However, we want to emphasize that one of the main features of limit order markets, compared with dealer markets, is the submission of limit orders. The patient traders would submit limit orders and thus act as liquidity suppliers in Market I, as shown in Table 3. Introducing hidden orders to a limit order market induces the suppliers to submit orders at less aggressive prices. The impatient traders thus suffer from the introduction, as shown in Table 7.

5.3. Price Impact

We use the effective spread and the Amihud illiquidity measure, both used in Amihud (2002), to measure price impact. The effective spread is defined as

$$S_E = 2\alpha \cdot \frac{1}{N} \sum_{\tau=1}^{N} \frac{\phi_\tau - M_\tau}{M_\tau}, \tag{10}$$

where α indicates the direction of the trade ($+1$ for buyer-initiated trades and -1 for seller-initiated trades), ϕ_τ and M_τ are the transaction and midpoint prices of the $\tau-$th trade, respectively. Note that the effective spread is not equal to the spread; for example the effective spread can be negative. The Amihud illiquidity measure (in the log scale), a measure of price impact scaled by traded dollar volume, is defined as

$$Q = \log\left(1 + \frac{10^5}{H} \sum_{h=1}^{H} \frac{|r_h|}{\$Vol_h}\right), \tag{11}$$

where r_h and $\$Vol_h$ are the mid quote return and traded dollar volume, respectively, during the $h-$th time period.[10] A smaller S_E or a smaller Q implies a more liquid market.

Table 8 suggests that introducing hidden orders would mitigate price impact by reducing both the effective spread and the Amihud illiquidity measure. Note that we focus on the hidden order inside the spread. As a result, the usage of hidden orders implies hidden liquidity inside the spread, whose price is closer to the midpoint price than the visible orders at the best price levels. The average trade price is thus closer to the midpoint price.[11] The trades are possibly executed against hidden orders inside the spread and thus would have less effects on the visible order book in the market with hidden orders than the benchmark Market I without hidden orders. Furthermore, the effects of the introduction of midpoint

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[10] We take 11000 trader arrivals as one time period during a one-million-arrival simulation to approximate one hour in a day, which is the setting used in Foley and Putniņš (2016).

[11] The effective spread could even be negative in an extreme market with few liquidity demander arrivals. Under such a market condition, the liquidity providers may submit hidden orders to the price level even more aggressively than the mid-price to earn the trade priority. Take the result in the parameter setting with $\theta_P = 0.95, (\beta_1, \beta_2) = (0.9, 1.1)$ as an example.

peg orders are not significantly different from those of hidden limit orders. The introduction of both kinds of hidden orders also does not differ significantly from the introduction of either one of them in terms of the effects on the trading profits and market liquidity. We summarize this as follows.

Empirical Implication 4. *The introduction of hidden orders, including midpoint peg orders and hidden limit orders, to a limit order market can mitigate price impact by reducing both the effective spread and the Amihud illiquidity measure.*

To the best of our knowledge, there are no empirical studies specifically on the liquidity benefits of introducing hidden orders in the limit order market. In other types of dark trading, Foley and Putniņš (2016) empirically show that the two-sided dark trading benefits liquidity, while the one-sided dark trading does not significantly affect liquidity; Zhu (2014) and Degryse et al. (2015) theoretically and empirically, respectively, show that the midpoint dark pool has a detrimental effect on liquidity. It is worth mentioning that the one-sided dark trading data in Foley and Putniņš (2016) includes both the midpoint trades in the limit order market and those in the dark pool, while the data in Degryse et al. (2015) consists of trades in the midpoint dark pool only.

Table 8: Summary statistics (mean and standard deviation) of Markets I-IV. Starting with an empty initial market, we report the effective spread S_E defined in (10) and the illiquidity metric Q in (11). ρ_1 is the proportion of patient traders with impact β_1. For an illustrative purpose, we multiply the effective spread by 100. By comparing the first column with the last three columns, we can observe that introducing hidden orders, no matter the midpoint peg orders or limit hidden orders, would decrease the effective spread and reduce price impact.

Parameter Setting	Market I $S_E,\ Q$	Market II $S_E,\ Q$	Market III $S_E,\ Q$	Market IV $S_E,\ Q$
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.5$, $s_c = 3$;	1.1765, 0.0592 (0.0008)(0.0039)	0.8358, 0.0000 (0.0010)(0.0000)	0.4953, 0.0000 (0.0015)(0.0000)	0.8359, 0.0000 (0.0012)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.1$, $s_c = 3$;	1.1764, 0.0594 (0.0008)(0.0039)	0.8358, 0.0000 (0.0012)(0.0000)	0.5636, 0.0000 (0.0015)(0.0000)	0.8361, 0.0000 (0.0011)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.9$, $s_c = 3$;	1.1764, 0.0600 (0.0009)(0.0041)	0.8359, 0.0000 (0.0012)(0.0000)	0.5637, 0.0000 (0.0016)(0.0000)	0.8360, 0.0000 (0.0011)(0.0000)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.5$, $s_c = 3$;	0.6811, 0.1012 (0.0004)(0.0015)	0.0930, 0.0000 (0.0006)(0.0000)	-0.4953, 0.0000 (0.0007)(0.0000)	0.0928, 0.0000 (0.0005)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$,	1.5219, 0.0612	1.1109, 0.0000	1.1108, 0.0000	1.1113, 0.0000

$\rho_1 = 0.5,\ s_c = 4;$	(0.0014)(0.0035)	(0.0016)(0.0000)	(0.0016)(0.0000)	(0.0014)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 5;$	1.6829, 0.0780 (0.0020)(0.0037)	1.3845, 0.0000 (0.0019)(0.0000)	1.0459, 0.0000 (0.0022)(0.0000)	1.3845, 0.0000 (0.0018)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	1.3004, 0.0494 (0.0009)(0.0041)	1.0217, 0.0000 (0.0011)(0.0000)	0.9878, 0.0000 (0.0012)(0.0000)	1.0217, 0.0000 (0.0011)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.1,\ s_c = 3;$	1.3004, 0.0494 (0.0007)(0.0035)	1.0217, 0.0000 (0.0011)(0.0000)	0.7983, 0.0000 (0.0015)(0.0000)	1.0216, 0.0000 (0.0011)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.9,\ s_c = 3;$	1.3003, 0.0493 (0.0007)(0.0040)	1.0216, 0.0000 (0.0010)(0.0000)	0.7984, 0.0000 (0.0016)(0.0000)	1.0216, 0.000000 (0.0011)(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 5;$	2.0098, 0.0644 (0.0018)(0.0044)	2.0464, 0.0365 (0.0022)(0.0030)	2.0467, 0.0368 (0.0020)(0.0028)	2.0463, 0.0364 (0.0024)(0.0030)
$\theta_P = 0.20, (\beta_1, \beta_2) = (0.9, 1.1),$ $\rho_1 = 0.5,\ s_c = 3;$	1.6807, 0.0134 (0.0005)(0.0017)	1.6660, 0.0142 (0.0005)(0.0018)	1.6232, 0.0077 (0.0006)(0.0015)	1.6234, 0.0076 (0.0007)(0.0016)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.5,\ s_c = 3;$	1.1764, 0.0599 (0.0007)(0.0042)	1.0062, 0.0306 (0.0011)(0.0050)	0.8360, 0.0308 (0.0012)(0.0047)	1.0061, 0.0305 (0.0010)(0.0042)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.1,\ s_c = 3;$	1.1765, 0.0606 (0.0008)(0.0038)	0.8360, 0.0000 (0.0011)(0.0000)	0.5635, 0.0000 (0.0016)(0.0000)	0.8359, 0.0000 (0.0012)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.8,\ s_c = 3;$	1.1764, 0.0595 (0.0009)(0.0035)	0.9041, 0.0124 (0.0012)(0.0035)	0.6316, 0.0116 (0.0017)(0.0032)	0.9040, 0.0116 (0.0011)(0.0029)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$ $\rho_1 = 0.5,\ s_c = 4;$	1.5217, 0.0613 (0.0014)(0.0035)	1.1112, 0.0000 (0.0017)(0.0000)	1.1111, 0.0000 (0.0014)(0.0000)	1.1111, 0.0000 (0.0015)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$ $\rho_1 = 0.5,\ s_c = 4;$	1.5215, 0.0615 (0.0016)(0.0031)	1.1110, 0.0000 (0.0016)(0.0000)	1.1111, 0.0000 (0.0018)(0.0000)	1.1110, 0.0000 (0.0016)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$ $\rho_1 = 0.5,\ s_c = 5;$	1.6826, 0.0779 (0.0021)(0.0033)	1.6706, 0.0511 (0.0018)(0.0053)	1.2744, 0.0000 (0.0019)(0.0000)	1.5994, 0.0515 (0.0020)(0.0057)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.5, 2.0),$ $\rho_1 = 0.5,\ s_c = 5;$	3.0001, 0.0062 (0.0003)(0.0014)	2.9994, 0.0060 (0.0003)(0.0014)	2.9701, 0.0035 (0.0004)(0.0012)	2.9702, 0.0036 (0.0003)(0.0011)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.1, 4.0),$ $\rho_1 = 0.5,\ s_c = 3;$	1.1764, 0.0593 (0.0008)(0.0038)	1.0062, 0.0305 (0.0009)(0.0043)	1.0963, 0.0262 (0.0009)(0.0041)	1.0962, 0.0262 (0.0009)(0.0043)
$\theta_P = 0.50, (\beta_1, \beta_2) = (0.1, 4.0),$ $\rho_1 = 0.1,\ s_c = 4;$	1.6461, 0.0545 (0.0013)(0.0032)	1.5503, 0.0234 (0.0015)(0.0028)	1.6461, 0.0366 (0.0009)(0.0068)	1.6460, 0.0365 (0.0009)(0.0062)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0),$ $\rho_1 = 0.5,\ s_c = 4;$	1.7679, 0.0470 (0.0013)(0.0034)	1.6491, 0.0205 (0.0015)(0.0024)	1.6392, 0.0319 (0.0015)(0.0033)	1.6391, 0.0320 (0.0015)(0.0031)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.1, 4.0),$ $\rho_1 = 0.5,\ s_c = 3;$	1.7827, 0.0062 (0.0003)(0.0014)	1.7242, 0.0034 (0.0005)(0.0011)	1.7514, 0.0036 (0.0004)(0.0011)	1.7514, 0.0034 (0.0003)(0.0011)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.1, 4.0),$ $\rho_1 = 0.5,\ s_c = 3;$	0.6811, 0.1011 (0.0004)(0.0017)	0.3870, 0.0512 (0.0006)(0.0045)	0.0928, 0.0517 (0.0010)(0.0053)	0.3871, 0.0522 (0.0007)(0.0005)

Our theoretical result complements these studies by suggesting that hidden orders in the limit order market benefit the liquidity by mitigating price impact. The reason the existing empirical studies do not agree on the effects of dark trading is perhaps that the different combinations of sub-types of dark trading and various types of lit markets could have different effects. Our theoretical result is for the hidden orders in limit order markets, while the other papers cited above study different combinations.

5.4. Mid-quote Volatility

To examine the effect of the introduction of hidden orders on volatility, we define mid-quote volatility as the standard deviation of the log returns of the mid-quotes. In particular, the mid-quote volatility is defined as

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(r_i - \bar{r})^2}{N-1}}, \tag{12}$$

where r_i is the log return of the mid-quotes at the $i-$th arrival and $\bar{r}$ is the mean of the log return.

From the mid-quote volatility reported in Table 9, introducing hidden orders would reduce the mid-quote volatility. Note that we focus on the hidden order inside the spread. The trades are possibly executed against hidden orders inside the spread and thus would have less effects on the visible bid and ask quotes in the market allowing hidden orders than the benchmark Market I without hidden orders. We summarize this as follows.

Empirical Implication 5. *The introduction of hidden orders, including midpoint peg orders and hidden limit orders, to a limit order market would decrease the mid-quote volatility.*

Table 9: Summary statistics (mean and standard deviation) of the mid-quote volatility in Markets I-IV. Starting with an empty initial market, we report the mid-quote volatility defined in (12). ρ_1 is the proportion of patient traders with impact β_1. By comparing the first column with the last three columns, we can observe that introducing hidden orders, no matter the midpoint peg orders or limit hidden orders, would decrease the mid-quote volatility.

Parameter Setting	Market I	Market II	Market III	Market IV
	σ	σ	σ	σ
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$	0.0052	0.0000	0.0000	0.0000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)	(0.0000)	(0.0000)	(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$	0.0046	0.0000	0.0000	0.0000
$\rho_1 = 0.5,\ s_c = 4;$	(0.0000)	(0.0000)	(0.0000)	(0.0000)

$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.5$, $s_c = 5$;	0.0057 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.5$, $s_c = 5$;	0.0053 (0.0000)	0.0020 (0.0000)	0.0020 (0.0000)	0.0020 (0.0000)
$\theta_P = 0.20, (\beta_1, \beta_2) = (0.9, 1.1)$, $\rho_1 = 0.5$, $s_c = 3$;	0.0019 (0.0000)	0.0018 (0.0000)	0.0014 (0.0000)	0.0014 (0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$, $\rho_1 = 0.5$, $s_c = 3$;	0.0052 (0.0000)	0.0037 (0.0000)	0.0037 (0.0000)	0.0037 (0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0)$, $\rho_1 = 0.8$, $s_c = 3$;	0.0052 (0.0000)	0.0023 (0.0000)	0.0023 (0.0000)	0.0023 (0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0)$, $\rho_1 = 0.5$, $s_c = 4$;	0.0046 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)	0.0000 (0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0)$, $\rho_1 = 0.5$, $s_c = 5$;	0.0057 (0.0000)	0.0037 (0.0000)	0.0000 (0.0000)	0.0037 (0.0000)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.5, 2.0)$, $\rho_1 = 0.5$, $s_c = 5$;	0.0014 (0.0000)	0.0014 (0.0000)	0.0010 (0.0000)	0.0010 (0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 3$;	0.0052 (0.0000)	0.0037 (0.0000)	0.0033 (0.0000)	0.0033 (0.0000)
$\theta_P = 0.50, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.1$, $s_c = 4$;	0.0044 (0.0000)	0.0018 (0.0000)	0.0049 (0.0000)	0.0049 (0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 4$;	0.0041 (0.0000)	0.0018 (0.0000)	0.0026 (0.0000)	0.0026 (0.0000)
$\theta_P = 0.10, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 3$;	0.0014 (0.0000)	0.0010 (0.0000)	0.0010 (0.0000)	0.0010 (0.0000)
$\theta_P = 0.95, (\beta_1, \beta_2) = (0.1, 4.0)$, $\rho_1 = 0.5$, $s_c = 3$;	0.0061 (0.0000)	0.0043 (0.0000)	0.0043 (0.0000)	0.0043 (0.0000)

6. Robustness Checks

Besides the assumptions adopted from Foucault et al. (2005), we previously assumed that hidden orders are submitted to prices inside the spread upon arrival. According to empirical studies, most hidden orders are submitted to the prices inside the spread or at the best prices. We thus relax our assumption to allow the hidden orders to be submitted to prices inside the spread and at the best prices.

In this extended case, the traders arriving at an identical spread could choose different strategies.[12] As a result, we have to change the state variable from the bid-ask spread to the

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[12] For example, if there are unlimited limit orders outside the price levels [\$10, \$10.1875] with the tick

whole limit order book, which consists of all the submitted limit order volume at available price levels. In particular, for a market with the maximum spread being s, the state is a vector of $s-1$ dimension with the $i-$th element being the number of buy/sell limit orders at the $i-$th available price level. In addition, a buyer and a seller arriving at an identical limit order book could choose different strategies.[13] Therefore, the extended state variables and increased number of representative players post some challenges to our numerical study[14].

To be concise, we present just one representative example in our numerical study, where the hidden orders at the best prices are possibly optimal in the equilibrium, with the extreme impact $\beta_1 = 0.1$ and $\beta_2 = 4.0$, and the proportion of patient traders being $\theta_P = 0.45$. We present market statistics under different parameter settings in Table 10. It appears that the main conclusions regarding the trader profit and market liquidity still hold.

Table 10: Summary statistics (mean and standard deviation) of Markets I-IV. Starting with an empty initial market and limit price levels being \$10, \$10.0625, \$10.125 and \$10.1875, we report the expected profit of patient traders defined in (8), the average trade prices of impatient buyers and sellers (*Mbuy* and *Msell* in \$), the effective spread S_E defined in (10) and the illiquidity metric Q in (11). For an illustrative purpose, we multiply both the expected profit and the effective spread by 100. ρ_1 is the proportion of patient traders with impact β_1. By comparing the first column with the last three columns, we can observe that the introduction of hidden orders, no matter the midpoint peg orders or limit hidden orders, would increase the expected profit of patient traders, decrease the effective spread, and reduce market impact.

Parameter Setting	Market I	Market II	Market III	Market IV
Expected profits of patient traders (W_1, W_2)				
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$	0.8066, 0.8063	6.4726, 5.8276	5.5663, 4.9213	6.4733, 5.8284
$\rho_1 = 0.5,\ s_c = 3;$	(0.0009)(0.0010)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$	0.8064, 0.8065	5.1675, 2.8231	3.8256, 3.5050	5.1692, 2.8273
$\rho_1 = 0.5,\ s_c = 3;$	(0.0010)(0.0009)	(0.0030)(0.0055)	(0.0011)(0.0006)	(0.0030)(0.0005)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$	0.8065, 0.8065	6.4582, 2.8297	6.2659, 3.0280	6.4576, 2.8308

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size being \$1/16, a buyer arriving at a market spread $s = 2$ with the best bid price being \$10 would never consider submitting a hidden order at the best bid, due to the unlimited limit orders at a price \$10. In contrast, a buyer arriving at a spread $s = 2$ with the best bid price being \$10.125 may submit a hidden order at the best bid price.

[13] For example, a buyer coming at a market $(0, -1)$ would never consider submitting a hidden order at the best bid price \$10, while a seller could consider a hidden order at the best ask price \$10.125.

[14] With the increase of the representative players, we find that the pure-strategy Nash equilibrium may not exist under certain parameter settings after some time-consuming numerical work. However, as long as our algorithm generates the equilibrium strategy, the empirical implications of our model still hold.

$\rho_1 = 0.5,\ s_c = 3;$	(0.0010)(0.0009)	(0.0043)(0.0060)	(0.0035)(0.0004)	(0.0043)(0.0006)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0),$	0.8619, 0.8622	7.8418, 2.5860	10.2443, 2.5807	10.2426, 2.5815
$\rho_1 = 0.5,\ s_c = 3;$	(0.0010)(0.0012)	(0.0054)(0.0007)	(0.0070)(0.0006)	(0.0070)(0.0006)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$	0.8621, 0.8621	6.2687, 5.5784	5.4638, 3.7466	6.2709, 5.5811
$\rho_1 = 0.5,\ s_c = 3;$	(0.0010)(0.0010)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
Order aggressiveness of patient traders (RP_1, RP_2)(to be updated)				
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$	0.6451, 0.6452	1.5000, 1.5000	1.3550, 1.3550	1.5000, 1.5000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0007)(0.0007)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$	0.6451, 0.6452	1.2911, 0.9683	1.0765, 1.0768	1.2913, 0.9687
$\rho_1 = 0.5,\ s_c = 3;$	(0.0006)(0.0007)	(0.0008)(0.0004)	(0.0006)(0.0004)	(0.0008)(0.0004)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$	0.6453, 0.6450	1.2914, 0.9688	1.7638, 0.9096	1.2912, 0.9688
$\rho_1 = 0.5,\ s_c = 3;$	(0.0007)(0.0007)	(0.008)(0.0004)	(0.0013)(0.0004)	(0.0008)(0.0004)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0),$	0.6896, 0.6896	1.3100, 0.9652	1.7647, 0.9023	1.7646, 0.9023
$\rho_1 = 0.5,\ s_c = 3;$	(0.0009)(0.0008)	(0.0011)(0.0006)	(0.0015)(0.0005)	(0.0015)(0.0005)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$	0.6896, 0.6897	1.5000, 1.5000	2.0000, 1.1379	1.5000, 1.5000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0008)(0.0008)	(0.0000)(0.0000)	(0.0000)(0.0000)	(0.0000)(0.0000)
Impatient traders' execution price $(Mbuy,\ \ Msell)$				
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$	10.1432, 10.0444	10.1542, 10.0333	10.1432, 10.0444	10.1542, 10.0333
$\rho_1 = 0.5,\ s_c = 3;$	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$	10.1431, 10.0444	10.1487, 10.0388	10.1431, 10.0444	10.1487, 10.0388
$\rho_1 = 0.5,\ s_c = 3;$	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$	10.1431, 10.0444	10.1487, 10.0388	10.1490, 10.0385	10.1487, 10.0388
$\rho_1 = 0.5,\ s_c = 3;$	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0),$	10.1487, 10.0388	10.1536, 10.0340	10.1546, 10.0329	10.1546, 10.0329
$\rho_1 = 0.5,\ s_c = 3;$	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$	10.1487, 10.0388	10.1584, 10.0291	10.1546, 10.0329	10.1584, 10.0291
$\rho_1 = 0.5,\ s_c = 3;$	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)	(0.0001)(0.0001)
Liquidity Measurement $(S_E,\ q)$				
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$	1.1765, 0.0594	0.8359, 0.0000	0.4953, 0.0000	0.8359, 0.0000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0009)(0.0037)	(0.0013)(0.0000)	(0.0017)(0.0000)	(0.0013)(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$	1.1764, 0.0600	1.0061, 0.0303	0.8358, 0.0303	1.0061, 0.0303
$\rho_1 = 0.5,\ s_c = 3;$	(0.0008)(0.0044)	(0.0010)(0.0044)	(0.0013)(0.0044)	(0.0010)(0.0044)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$	1.176479 0.060082	1.0062, 0.0309	1.0963, 0.0266	1.0062, 0.0309
$\rho_1 = 0.5,\ s_c = 3;$	(0.0008)(0.0035)	(0.0011)(0.0041)	(0.0010)(0.0038)	(0.0011)(0.0041)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0),$	1.300332 0.048998	1.1610, 0.0244	1.2428, 0.0227	1.2428, 0.0227
$\rho_1 = 0.5,\ s_c = 3;$	(0.0007)(0.0040)	(0.0009)(0.0036)	(0.0008)(0.0035)	(0.0008)(0.0035)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$	1.300281 0.049137	1.0216, 0.0000	0.9878, 0.0000	1.0216, 0.0000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0007)(0.0042)	(0.0010)(0.0000)	(0.0011)(0.0000)	(0.0010)(0.0000)
Mid-quote volatility σ				
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 1.1),$	0.0052	0.0000	0.0000	0.0000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)	(0.0000)	(0.0000)	(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.9, 2.0),$	0.0052	0.0037	0.0037	0.0037
$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)	(0.0000)	(0.0000)	(0.0000)
$\theta_P = 0.55, (\beta_1, \beta_2) = (0.5, 2.0),$	0.0052	0.0037	0.0033	0.0037
$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)	(0.0000)	(0.0000)	(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0),$	0.0049	0.0035	0.0033	0.0033

$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)	(0.0000)	(0.0000)	(0.0000)
$\theta_P = 0.45, (\beta_1, \beta_2) = (0.9, 1.1),$	0.0049	0.0000	0.0000	0.0000
$\rho_1 = 0.5,\ s_c = 3;$	(0.0000)	(0.0000)	(0.0000)	(0.0000)

In addition, the equilibrium strategies are presented in Table 11. One can observe that with the majority being impatient traders submitting market orders, the optimistic traders would submit hidden orders at the best prices.

Table 11: Equilibrium strategies of patient traders with heterogeneous impacts in Markets I-IV. We denote the midpoint peg order by MPPO and the j-limit/hidden order by the $j-$LO/HO, $j = 1, 2$. Market I corresponds to a limit order market without hidden orders. In contrast, Market II-IV corresponds to introducing midpoint peg orders, hidden limit orders, or both kinds of hidden orders to Market I, respectively.

		Extreme impact ($\theta_P = 0.45, (\beta_1, \beta_2) = (0.1, 4.0), \rho_1 = 0.5, s_c = 3$)	
Markets	impact LOB	$\beta_1 = 0.1$ Buyer v.s. Seller	$\beta_2 = 4.0$ Buyer v.s. Seller
Market I	$(0, -1)$	1-LO v.s. 1-LO	1-LO v.s. 1-LO
	$(1, 0)$	1-LO v.s. 1-LO	1-LO v.s. 1-LO
	$(0, 0)$	1-LO v.s. 1-LO	1-LO v.s. 1-LO
Market II	$(0, -1)$	MPPO v.s. MPPO	MPPO v.s. MPPO
	$(1, 0)$	MPPO v.s. MPPO	MPPO v.s. MPPO
	$(0, 0)$	MPPO v.s. MPPO	MPPO v.s. MPPO
Market III	$(0, -1)$	1-HO v.s. 2-HO	1-LO v.s. 1-HO
	$(1, 0)$	2-HO v.s. 1-HO	1-HO v.s. 1-LO
	$(0, 0)$	2-HO v.s. 2-HO	1-LO v.s. 1-LO
Market IV	$(0, -1)$	1-HO v.s. 2-HO	1-LO v.s. 1-LO
	$(1, 0)$	2-HO v.s. 1-HO	1-LO v.s. 1-LO
	$(0, 0)$	2-HO v.s. 2-HO	1-LO v.s. 1-LO

7. Conclusion

To investigate how the introduction of hidden orders affects the trader profits and market liquidity, we introduce hidden orders to the limit order market modeled by Foucault et al. (2005). Traders differ in their impatience and impacts of the execution uncertainty of submitted hidden orders. Upon arrival, each trader chooses among a market order, a limit order, a midpoint peg order, and a hidden limit order to submit. The choice is driven by a trade-off between the execution price and the cost of delayed execution. We derive equilibrium strategies analytically for the infinite horizon game with a homogeneous impact and propose a new numerical algorithm to find equilibrium strategies with heterogeneous impacts by modifying the algorithm in Pakes and McGuire (2001) with additional forced explorations and a new stopping rule. We find that the introduction of hidden orders (including the hidden limit orders and the midpoint peg orders) increases the profit of patient traders, slightly worsens impatient traders' average executed market prices, mitigates price impact by reducing both the effective spread and the Amihud illiquidity measure, and decreases the mid-quote volatility.

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ONLINE SUPPLEMENT

How does the Introduction of Hidden Orders Affect Limit Order Markets?

This online supplement contains a special case of our model, our algorithm for the equilibrium with heterogeneous belief, the proofs of Lemma 1 and Theorems 1-2 and the derivations for the examples in Section 3.2 in the paper. To facilitate the notations in this supplement, we denote $\rho = \theta_P/\theta_I$. We also denote MPPO and $j-$HO/LO as the midpoint peg order and the $j-$hidden/limit order, respectively.

A. Effects of Introducing Hidden orders to a Market where Traders Submitting Market Orders Only

Patient traders could submit limit orders to act as liquidity suppliers in our limit order market. To make a fair comparison with the model by Boulatov and George (2013), in this section, we consider a special case where the traders are not patient and thus only submit market orders in Market I. Specifically, we adopt similar parameter settings as the cases in our main paper, except for the high waiting cost of patient traders. In particular, the tick size is $\Delta = \$1/16$, the arrival rate is $\lambda = 1$, the proportion of patient traders is $\theta_P = 0.55$ or 0.45, and the waiting costs per unit time for patient and impatient traders are $\delta_P = 0.135$ and $\delta_I = 0.2$. The patient traders share an identical impact with $\beta = 0.5, 1.0$ or 2.0.

As shown in Table A.1, both the patient and impatient traders submit market orders in Market I under such a case. With the allowance of hidden orders, the patient traders with small impacts ($\beta = 0.5$ or 1.0) might turn from a market order to a hidden order. This finding is consistent with that by Boulatov and George (2013). If the patient traders are sensitive to the execution uncertainty with parameter $\beta \geq 2$, they will keep submitting market orders even after the introduction since the traders are not that patient and believe their submitted hidden orders would have a long waiting time.

In such a case, we investigate the effects of introducing hidden orders to Market I and

Table A.1: Equilibrium strategies of patient traders with a homogeneous impact in Markets I-IV under the case where traders submit market orders only in Market I: we denote the midpoint peg order by MPPO, the market order by MO, and the 1−hidden order by the 1−HO. Market I corresponds to a limit order market without hidden orders. In contrast, Market II-IV corresponds to introducing midpoint peg orders, hidden limit orders, or both kinds of hidden orders to Market I. Unlike those in the base case (shown in Table 3), patient traders in this case have a high waiting cost and thus submit market orders in Market I.

		$\theta_P = 0.55$			$\theta_P = 0.45$		
markets	impact spread	0.5	1.0	2.0	0.5	1.0	2.0
Market I	$s = 2$	MO	MO	MO	MO	MO	MO
	$s = 3$	MO	MO	MO	MO	MO	MO
Market II	$s = 2$	MPPO	MO	MO	MPPO	MO	MO
	$s = 3$	MPPO	MO	MO	MPPO	MPPO	MO
Market III	$s = 2$	1-HO	MO	MO	1-HO	MO	MO
	$s = 3$	1-HO	MO	MO	1-HO	MO	MO
Market IV	$s = 2$	MPPO	MO	MO	MPPO	MO	MO
	$s = 3$	MPPO	MO	MO	MPPO	MPPO	MO

document the result for the parameters $\theta_P = 0.55$ and $\beta = 0.5$ in Table A.2. The effects are similar in the other cases where patient traders turn from market orders to hidden orders. We find that the introduction of hidden orders still increases the profit of patient traders and benefits the market liquidity, which is the same as the other cases in this paper. In addition, the metric Q is always zero in this case since there is no limit order submission and thus no midpoint price change in the markets. The introduction would improve the average trade prices of impatient traders, which is the same as the finding in Boulatov and George (2013) but different from the result in the main body of this paper, where patient traders submit limit orders in Market I.

Intuitively, introducing hidden orders provides a new option and induces the patient trader to submit less aggressive orders. In such a case, patient traders submit market orders in Market I and could turn to hidden orders after the introduction. As a result, there are fewer liquidity demanders and more liquidity suppliers. The impatient traders, who always act as liquidity demanders, thus benefit from the introduction.

Table A.2: Market characteristics of Markets I-IV in the case where traders only submit market orders in Market I: starting with an empty initial market and limit price levels being \$10, \$10.0625, \$10.125, and \$10.1875, we document the expected profits of patient traders defined in (8), the average trade prices of impatient buyers and sellers (market buy and market sell), the effective spread S_E defined in (10) and the illiquidity metric Q in (11) for markets I-IV, which differ by the policies on the allowance of hidden orders. For an illustrative purpose, we multiply the profit and the effective spread by 100. We can observe that introducing hidden orders increases the patient trader's expected profit and benefits the liquidity. However, unlike the case of limit order markets, this special case reveals that the market buy (sell) price of impatient traders is decreased (increased) after the introduction to a market where traders submit market orders only.

	expected profit	market buy	market sell	S_E	Q
Market I	0	\$10.1875	\$10.0000	1.8576	0
	(0)	(0)	(0)	(0.0000)	(0)
Market II	5.0224	\$10.1542	\$10.0333	0.8359	0
	(0.0000)	(0.0001)	(0.0001)	(0.0011)	(0)
Market III	4.1132	\$10.1431	\$10.0443	0.4954	0
	(0.0000)	(0.0001)	(0.0001)	(0.0014)	(0)
Market IV	5.0210	\$10.1543	\$10.0332	0.8362	0
	(0.0000)	(0.0001)	(0.0001)	(0.0012)	(0)

B. Details of the Numerical Algorithm

We present our algorithm in Algorithm 1. In particular, we first set the expected waiting times of all feasible strategies, $\{T(\cdot), T^H(\cdot)\}$, initially to be $1/\lambda$, which is the expected duration of one trader's arrival. These expected waiting times are optimistic since they imply that all the orders can be executed upon the next trader's arrival. In addition, the initial expected execution prices of a midpoint peg order and a market order submitted at a spread s are set to be $\bar{m}^H(s) = s/2$ and $\bar{m}(s) = 0$, respectively. We document the expected execution prices and waiting times in our expected waiting time system $\{T_1(\cdot), T_1^H(\cdot), \bar{m}_1(\cdot), \bar{m}_1^H(\cdot)\}$. Initialize the iteration time as $k = 1$. For the $k-$th iteration, we iterate over the following two steps.

Step 1: Based on the expected waiting time system $\{T_k(\cdot), T_k^H(\cdot), \bar{m}_k(\cdot), \bar{m}_k^H(\cdot)\}$, traders optimize their expected profit (7) to identify the optimal strategy system Π_k.

Step 2: Starting with the initial market where the bid-ask spread is the maximum spread s_c and there is unlimited depth at the best bid and ask price levels, we simulate the market with $N = 1,000,000$ traders' arrivals. Traders could be patient/impatient buyers/sellers with probabilities defined as the parameter settings. The patient traders arrive and submit strategies according to the optimal trading strategy Π_k. We document the averaged waiting times and execution prices as the expected waiting time system $\{T_{k+1}, T^H_{k+1}, \bar{m}_{k+1}, \bar{m}^H_{k+1}\}$. Stop the iteration and go to step of equilibrium test if k is larger than the maximum iteration number ite_{num}[15] or if the difference between the updated waiting time system and the original one is less than the tolerance level,

$$\left\|(T_{k+1}, T^H_{k+1}, \bar{m}_{k+1}, \bar{m}^H_{k+1}) - (T_k, T^H_k, \bar{m}_k, \bar{m}^H_k)\right\|_1 \leq \ 0.005 * len,$$

where $\|\cdot\|_1$ is the L1 norm and len is the number of updated rows.

We next show the details on how to update the expected waiting time system during the simulation. Document the executed units of all feasible strategies during the simulation by n. Initialize $T_{k,1} = T_k$, $T^H_{k,1} = T^H_k$, $\bar{m}_{k,1} = \bar{m}_k$, $\bar{m}^H_{k,1} = \bar{m}^H_k$ and $n = 1$.

For each execution of a $j-$limit/hidden order submitted at a spread s, $j \in \{1, 2, \cdots, s-1, s/2\}$[16], we update the expected waiting time for the executed limit/hidden order as

$$\begin{aligned}
n(s,j) &= n(s,j) + 1, \\
T_{k,n(s,j)}(s,j) &= \frac{n(s,j)-1}{n(s,j)} T_{k,n(s,j)-1}(s,j) + \frac{D}{n(s,j)}, \\
T^H_{k,n(s,j)}(s,j) &= \frac{n(s,j)-1}{n(s,j)} T^H_{k,n(s,j)-1}(s,j) + \frac{D}{n(s,j)},
\end{aligned}$$

where $n(s,j)$ is the number of executed units of the j-limit/hidden order submitted at the bid-ask spread s. D is the duration between the submission time and the execution time of this executed order. Note that D could possibly be zero in our problem, since the submitted limit/hidden order may be executed immediately against an existing hidden order in the market. For each transaction, we must trace the executed order back to its submission time

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[15] The maximum iteration number is set to be $ite_{num} = 3$ in our algorithm for the equilibrium with the market maximum bid-ask spread being $s_c = 3$.

[16] The price level $s/2$ is only available for the midpoint peg order.

to figure out the expected waiting time and expected execution price to be updated. For each execution of a midpoint peg order, besides the expected waiting time, we also update its execution price as

$$\bar{m}^H_{k,n(s,s/2)}(s) = \frac{n(s,s/2)-1}{n(s,s/2)}\bar{m}^H_{k,n(s,s/2)-1}(s) + \frac{M}{n(s,s/2)},$$

where M is the relative executed price level of the midpoint peg order, compared with the best price on the opposite side at its submission time. In addition, for the market order, i.e., $0-$limit order, in our model, it could be executed against hidden orders inside the spread and thus has a non-zero execution price. We also update its execution price as

$$\bar{m}_{k,n(s,0)}(s) = \frac{n(s,0)-1}{n(s,0)}\bar{m}_{k,n(s,0)-1}(s) + \frac{M}{n(s,0)},$$

where M is the (relative) executed price of the market order.

For each submission, we update the limit order book L by adding the order to an appropriate queue position at that price level, according to the price-visibility-time priority.

There might be some states not visited by traders during the iteration with the strategy system Π_k. However, if the traders wander to these states, we also want to know their optimal strategies to form a complete optimal strategy system like the one shown in Theorem 1. We thus also force the update of these states in our numerical experiment. However, this update can be omitted if the complete optimal strategy system is not needed, without any effects on the perfection of our outputted strategy set. Denote the largest non-visited spread as s_1. To update the waiting time of the s_1-limit order, we force the trader arriving at the maximum spread s_c to submit an s_1-limit order and start the simulation again from the initial market with bid-ask spread being s_c. We then update the expected waiting time for the s_1-limit order and mark it as a visited spread. Similarly, after the simulation, we force the update of the largest non-visited spread until all the spreads are visited. Up till now, we document the estimated expected waiting time system as $\{T_{k+1}(\cdot), T^H_{k+1}(\cdot), \bar{m}_{k+1}(\cdot), \bar{m}^H_{k+1}(\cdot)\}$.

Step of equilibrium test: After the iteration ends, we test the equilibrium by empowering traders to re-optimize their profit. In particular, keeping all the other optimal strategies

Algorithm 1 Our algorithm to solve for the equilibrium.

```
1. Initialize $k = 1, T_1 = T_1^H = 1/\lambda, \bar{m}_1^H = s/2, \bar{m}_1 = 0$.
2. Iterate $k$ until some stopping criteria are met. In the $k$-th iteration, update $\{T_k, T_k^H, \bar{m}_k, \bar{m}_k^H\}$ to $\{T_{k+1}, T_{k+1}^H, \bar{m}_{k+1}, \bar{m}_{k+1}^H\}$ and document $\Pi_k$ as follows:
  (a) For each possible market state, each type of traders optimizes his expected utility,
      $$\max_{j\in\{1,\cdots,s-1\}}\{j\Delta - \delta_P T_k(s,j),\ j\Delta - \delta_P\beta_\nu T_k^H(s,j),\ \bar{m}_k\Delta,\ \bar{m}_k^H(s)\Delta - \delta_P\beta_\nu T_k^H(s,\frac{s}{2})\},$$
      to identify his trading strategy. Document the strategy system as $\Pi_k$.
  (b) Simulate the market with $N$ traders submitting orders according to $\Pi_k$. Initialize $T_{k,1} = T_k$, $T_{k,1}^H = T_k^H$, $\bar{m}_{k,1} = \bar{m}_k$, $\bar{m}_{k,1}^H = \bar{m}_k^H$ and $n = 1$. For each execution, update
      $$T_{k,n}(s,j) = \frac{n-1}{n}T_{k,n-1}(s,j) + \frac{D}{n},\ T_{k,n}^H(s,j) = \frac{n-1}{n}T_{k,n-1}^H(s,j) + \frac{D}{n},$$
      $$\bar{m}_{k,n}(s) = \frac{n-1}{n}\bar{m}_{k,n-1}(s) + \frac{M}{n},\ \bar{m}_{k,n}^H(s) = \frac{n-1}{n}\bar{m}_{k,n-1}^H(s) + \frac{M}{n},$$
      where $n$ is the number of executed units of the executed order, $D$ is the waiting time of the executed order for this execution, and $M$ is the relative executed price. After the simulation, document the waiting time system as $\{T_{k+1}, T_{k+1}^H, \bar{m}_{k+1}, \bar{m}_{k+1}^H\}$.
3. After the iteration ends, do the following equilibrium test until some stopping criteria are met. Initialize $t = 1$.
  (a) for each market state and trader type,
        keeping all the other strategies in $\Pi_{k+t-1}$ unchanged, the trader arriving at the state estimates the expected profits of all the other feasible strategies with $N$ trader arrivals to determine whether to deviate.
        If deviate, break the for loop and update the strategy system to $\Pi_{k+t-1}$. Otherwise, continue the for loop.
      End for loop.
  (b) follow Step 2(b) to update the corresponding waiting time system as $\{T_{k+t}, T_{k+t}^H, \bar{m}_{k+t}, \bar{m}_{k+t}^H\}$.
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unchanged for each possible market state, each type of patient traders arriving at this state would re-estimate[17] the expected waiting times and execution prices for all the feasible strategies and optimize his profit to choose his strategy. If all the traders stick to the

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[17] We assume that traders of one type arriving at the same market state would submit the same strategy. Therefore, for the re-estimation part, not just one trader, but one type of traders arriving at the same market state changes the strategy.

original strategy system, and the strategy system stays unchanged after they re-optimize their utilities based on the corresponding waiting time system, the strategy set is outputted as the equilibrium strategy. Otherwise, update the strategy system and the corresponding expected waiting time system, and iterate again for the equilibrium test in the new system.

C. Proof of Lemma 1

We first consider the expected waiting time of the midpoint peg orders. Because all the patient traders share the same impact, if the midpoint peg order is chosen as the equilibrium strategy by patient traders arriving at a spread s, all the patient traders arriving at the spread would submit a midpoint peg order. Recall that buyers and sellers alternate with certainty. As a result, the next incoming trader would submit either a market or a midpoint peg order from the opponent side. The expected execution price of the midpoint peg order is $\bar{m}(s) = s/2$.

The market at the spread s could take one of the following two states: 1) there is one midpoint peg order inside the spread, and 2) there is no existing midpoint peg order. The midpoint peg order submitted at state 1) would be executed immediately against the existing midpoint peg order, while the order submitted at state 2) would be executed upon the next arrival. Because midpoint peg orders are invisible, traders cannot distinguish these two states. Denote the long-run probability of state 2) by x. The expected waiting time of a midpoint peg order submitted at spread s is

$$T^H(s, \frac{s}{2}) = x \cdot \frac{1}{\lambda} + (1 - x) \cdot 0 = \frac{x}{\lambda}. \tag{C.1}$$

We next calculate the long-run probability x. In equilibrium, state 1) would become state 2) with a probability of 1, while state 2) may transform to state 1) with a probability of θ_P and stay as state 2) with a probability of $(1 - \theta_P)$. It's easy to calculate the long-run probability of each state for this Markov process with

$$\begin{pmatrix} 1-\theta_P & 1 \\ \theta_P & 0 \end{pmatrix} \begin{pmatrix} x \\ 1-x \end{pmatrix} = \begin{pmatrix} x \\ 1-x \end{pmatrix}. \tag{C.2}$$

By solving the linear equation system (C.2), we have $x = 1/(1+\theta_P)$ and put it back into equation (C.1) to conclude

$$T^H(s, \frac{s}{2}) = \frac{1}{\lambda(1+\theta_P)}.$$

We next derive the expected waiting time of the $j-$hidden order submitted at a bid-ask spread s. If all the patient traders arriving at the spread s choose to submit the $j-$hidden order, each submitted hidden limit order would take a position in the hidden order queue. The waiting time of a $j-$hidden order depends on its position in the queue, which is non-observable. We thus first calculate the expected waiting time of a $j-$hidden order taking the $k-$th position in the queue, $T^H(s,j,k)$, and then derive the long-run probability y_k for each position to figure out the expected waiting time of the $j-$hidden order. We start with the first order in the queue to explain the basic idea. Upon the next arrival, the first order in the queue would either be executed if the next arrival is an impatient trader or otherwise stay as the first in the queue. Because buyers and sellers alternate, if the order is not executed, we need to consider one more arrival to guarantee that the next arrival comes from the opposite side. As a result, the expected waiting time of the first order in the queue of $j-$hidden order is

$$T^H(s,j,1) = \theta_I \frac{1}{\lambda} + \theta_P \left(\frac{1}{\lambda} + \frac{1}{\lambda} + T^H(s,j,1) \right) \Rightarrow \; T^H(s,j,1) = \frac{1+\theta_P}{\theta_I \lambda}. \tag{C.3}$$

Similarly, the expected waiting time of the $j-$hidden order taking the $k-$th ($k > 1$) position in the queue is

$$T^H(s,j,k) = \frac{2}{\lambda} + \theta_I T^H(s,j,k-1) + \theta_P T^H(s,j,k), \tag{C.4}$$

since the next incoming trader would take away the first order in the queue if he is impatient, and would not change the queue otherwise. In either one of these two cases, the $j-$hidden order taking the $k-$th ($k > 1$) position can never be executed upon the next arrival, but needs to wait two arrivals for the next trader from the opposite side. With the first term $T^H(s,j,1)$ derived in (C.3) and the recursive formula shown in (C.4), we can deduce that

$$T^H(s,j,k) = T^H(s,j,1) + \frac{2(k-1)}{\theta_I \lambda} = \frac{1+\theta_P+2(k-1)}{\theta_I \lambda}. \tag{C.5}$$

For the long-run probability, We consider the long-run probability from a buyer's side. The result is the same for the probability from a seller's side. Let's consider three processes: X and Y processes, which are the numbers of the existing buy and sell $j-$hidden orders inside the spread, and Z process, which is either 1 if the next arrival is a buyer or -1 otherwise. We set the initial state as $(X_1, Y_1, Z_1) = (0, 0, 1)$. The initial state corresponds to the state where there is no existing hidden order inside the spread; an incoming buyer would arrive at the spread s. Furthermore, according to the facts that the buyers and sellers alternate with certainty and that the proportion of patient traders is θ_P, we know that $Z_t = (-1)^{t-1}$ and that

$$\begin{aligned}
(X_{2k+1}, X_{2k+2}) &= \begin{cases} (X_{2k}, X_{2k+1} + 1) & \text{with prob. } \theta_P, \\ (X_{2k} - 1, X_{2k+1}) & \text{with prob. } \theta_I, \end{cases} \\
(Y_{2k+1}, Y_{2k+2}) &= \begin{cases} (Y_{2k} + 1, Y_{2k+1}) & \text{with prob. } \theta_P, \\ (Y_{2k}, Y_{2k+1} - 1) & \text{with prob. } \theta_I. \end{cases}
\end{aligned}$$

We can transform the dynamics to find the (X, Y, Z) process is a Markov process with the state space $\{(k, k, 1),\ (k+1, k, -1) \mid k \in \mathbb{Z}_+\}$ and the evolution as

$$(X_{t+1}, Y_{t+1}, Z_{t+1}) = \begin{cases} (X_t + \frac{1}{2}Z_t + \frac{1}{2}, Y_t - \frac{1}{2}Z_t + \frac{1}{2}, -Z_t), & \text{with prob. } \theta_P, \\ (X_t + \frac{1}{2}Z_t - \frac{1}{2}, Y_t - \frac{1}{2}Z_t - \frac{1}{2}, -Z_t), & \text{with prob. } \theta_I. \end{cases}$$

Denote x_{2k-1} and x_{2k} as the long-run probabilities of the state $(k, k, 1)$ and the state $(k + 1, k, -1)$, $k \geq 1$. The transition of this Markov process implies that the long-run probabilities satisfy that

$$-\theta_P x_1 + \theta_I x_2 = 0,\ \theta_P x_{t-1} - x_t + \theta_I x_{t+1} = 0,\ t > 1. \tag{C.6}$$

The linear equation system (C.6) can be solved to find

$$x_t = (1 - \rho)\rho^{t-1}. \tag{C.7}$$

Note that the arrived buyer knows that he must stand at one of the $(2k-1)-$th states $(k, k, 1)$, $k \geq 1$. According to (C.7), for the $j-$hidden order submitting at the spread s, the long-run probability of receiving a waiting time $T^H(s, j, k)$ is

$$y_k = \frac{x_{2k-1}}{\sum_{k=1}^{\infty} x_{2k-1}} = \frac{(\theta_I - \theta_P)}{\theta_I^2} \cdot \rho^{2(k-1)}.$$

Taking together this long-run probability with the waiting time derived in (C.5) and the fact that $\rho = \theta_P/\theta_I$, we can deduce the expected waiting time for the $j-$hidden order as

$$T^H(s, j) = \sum_{k=1}^{\infty} T^H(s, j, k) \cdot y_k = \begin{cases} \frac{1}{(\theta_I - \theta_P)\lambda} & \text{if } \theta_I > \theta_P, \\ \infty & \text{otherwise.} \end{cases}$$

As for the expected waiting time of a limit order, as we discussed above, if the equilibrium strategy for patient traders arriving at a spread is to submit a hidden order, the market spread would stay at the spread or be enlarged after the execution of all the existing hidden orders inside the spread. As a result, for an arbitrary spread s where the equilibrium strategy for patient traders is to submit a limit order, there are no hidden orders inside the spread in the equilibrium. The expected waiting time of a limit order, which is optimal in the equilibrium, thus does not depend on the spread at its submission time. □

D. Parameter Definitions and the Proof of Theorem 1

A. Parameter Definitions in Theorem 1

The parameters in Theorem 1 are defined by the following recursive formula. To make it clear, we explain first the recursive formula of n_{k_1} and then the general recursive formula of n_i. To simplify notations, we denote s_0 and T_0 as

$$\begin{aligned} s_0 &= \delta_P/(\lambda\Delta), && \text{(D.1)} \\ T_0 &= \begin{cases} \frac{1}{\theta_I - \theta_P} & if\ \theta_I > \theta_P, \\ \infty & otherwise, \end{cases} && \text{(D.2)} \end{aligned}$$

and the function $CF(\cdot) : \mathbb{R} \to \mathbb{Z}$ as $CF(x) = \min\{n \in \mathbb{Z} : \ n \geq x\}$.

Case 1. If the inequalities

$$\begin{cases} \beta T_0 \leq 1, \\ 1 + \dfrac{6 s_0 \beta T_0 \theta_P}{1 + \theta_P} \leq CF(s_0 \beta T_0) < CF\left(\dfrac{2 s_0 \beta}{1 + \theta_P}\right), \end{cases} \tag{D.3}$$

are satisfied, there would be no usage of limit order or midpoint peg order in the equilibrium, i.e., the patient trader arriving at a spread $s \leq n_1$ would submit a market order, while the one arriving at a spread $s \geq n_2$ would submit an $(s-1)-$hidden order. The cutting points are $n_1 = CF(s_0\beta T_0)$ and $n_2 = n_1 + 1$, with $t = 1$ and $k_t = 2$.

Case 2. If the inequalities

$$\beta T_0 > 1,\ 1 + 2s_0\left(1 - \frac{\beta}{1+\theta_P}\right) < CF(s_0) < CF\left(\frac{2s_0\beta}{1+\theta_P}\right), \tag{D.4}$$

are satisfied, the cutting point is $n_1 = CF(s_0)$. The patient trader arriving at a spread $s \leq n_1$ would submit a market order, while the one arriving at a spread $s \in (n_1, n_2)$ would submit an n_1-limit order. The cutting point n_2 is defined as follows.

Case 2.1. if inequalities (D.4) and the inequalities

$$\begin{cases} n_1 + CF\left(\dfrac{2s_0\beta}{1+\theta_P} - 2s_0\right) \leq CF(2s_0\rho), \\ 2n_1 + CF\left(\dfrac{2s_0\beta}{1+\theta_P} - 2s_0\right) < 2 + \dfrac{6s_0\beta T_0\theta_P}{1+\theta_P}, \end{cases} \tag{D.5}$$

are satisfied, we have $n_2 = 2n_1 + CF\left(\frac{2s_0\beta}{1+\theta_P} - 2s_0\right)$, and $k_1 = 2$. The patient trader arriving at a spread $s = n_2$ would submit a midpoint peg order, while the one arriving at a spread $s \in (n_2, n_3)$ would submit an n_2-limit order.

Case 2.2. if inequalities (D.4) and the inequalities

$$\begin{cases} n_1 + CF(s_0\beta T_0 - s_0) \geq 1 + \dfrac{6s_0\beta T_0\theta_P}{1+\theta_P}, \\ CF\left(s_0\beta T_0 - s_0\right) < CF(2s_0\rho), \\ CF(s_0\beta T_0 - s_0) + 1 \leq n_1 + CF\left(\dfrac{2s_0\beta}{1+\theta_P} - 2s_0\right), \end{cases} \tag{D.6}$$

are satisfied, we have $n_2 = n_1 + CF(\beta s_0 T_0 - s_0) + 1$, $t = 1$ and $k_1 = 2$. The patient trader arriving at a spread $s \geq n_2$ would submit an $(s-1)$-hidden order.

Case 2.3. if the inequalities (D.4) are satisfied, while neither inequalities (D.5) nor (D.6) holds, we have $n_2 = n_1 + CF(2s_0\rho)$. The patient trader arriving at a spread $s = n_2$ would submit an n_1-limit order, while the one arriving at a spread $s \in (n_2, n_3)$ would submit an n_2-limit order.

Case 3. If neither (D.3) nor (D.4) holds, the cutting points are

$$n_1 = \max\left(1, CF\left(\frac{2s_0\beta}{1+\theta_P}\right) - 1\right),\ n_2 = n_1 + 1,$$

with $k_1 = 2$. The patient trader arriving at a spread $s \leq n_1$ would submit a market order, while the one arriving at a spread $s = n_2$ would submit a midpoint peg order. The patient trader arriving at a spread $s \in (n_2, n_3)$ would submit an n_2-limit order.

Note that if $t = 1$ and $k_1 \leq 2$, we have finished the deduction of all the equilibrium spreads and strategies with Case 1 and Case 2.2. Otherwise, we will present the recursive formula for the remaining spreads as follows. In particular, if $k_1 \leq 2$, we have written down the formula for k_1 and n_{k_1}, and will present the recursive formula for $n_{k_1+1}, \cdots, n_{k_t}$ as follows.

For the cases where $k_1 > 2$, to unify the notations, we denote $k_0 = 2$ and present the recursive formula for $n_{k_0+1}, \cdots, n_{k_t}$ in the same way. Define the cutting points from $k_1 + 1$ or $k_0 + 1$ to n_{k_t}, starting with $j = 1$, as

$$n_{k_i+j} = n_{k_i} + \sum_{\mu=1}^{j} CF(2s_0\rho^{\mu+1}),\ i = 0, 1, \cdots, t-1,\ j = 1, \cdots, \bar{j}, \tag{D.7}$$

where $\bar{j}$ is the minimal integer j satisfying the following inequalities,

$$\begin{cases} n_{k_i+j-1} \leq CF(2s_0\rho^{j+1}) + 2s_0(1 + \sum_{\mu=1}^{j} 2\rho^{\mu} - \dfrac{\beta}{1+\theta_P}), \\ n_{k_i+j-1} < 1 + \dfrac{6s_0\beta T_0\theta_P}{1+\theta_P}, \\ n_{k_i+j-1} < 1 + \dfrac{3s_0\beta T_0\theta_P}{1+\theta_P} - \dfrac{1}{2}CF\left(2s_0(\dfrac{\beta}{1+\theta_P} - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right). \end{cases} \tag{D.8}$$

The number k_{i+1} is then defined by $k_i + \bar{j}$, $i = 0, \cdots, t-2$, while the cutting point n_{k_i}, $i = 1, \cdots, t-1$, is defined by

$$n_{k_i} = \max\left(n_{k_i-1} + 1, 2n_{k_i-1} + CF\left(2s_0(\frac{\beta}{1+\theta_P} - 1 - \sum_{\mu=1}^{k_i-k_{i-1}} 2\rho^{\mu})\right)\right),$$

and the index t is the minimal index i such that there exists an integer $j \in \mathbb{Z}_+$ satisfying the inequalities (D.9), the number k_t is then set to be $k_t = k_{t-1} + j$:

$$\begin{cases} CF\left(s_0(\beta T_0 - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right) < CF(2s_0\rho^{j+1}), \\ 1 + \dfrac{6s_0\beta T_0\theta_P}{1+\theta_P} \leq n_{k_{i-1}+j-1} + CF\left(s_0(\beta T_0 - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right), \\ n_{k_i+j-1} \geq 1 + \dfrac{6s_0\beta T_0\theta_P}{1+\theta_P}, \\ CF\left(s_0(\beta T_0 - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right) < n_{k_{i-1}+j-1} + CF\left(2s_0(\dfrac{\beta}{1+\theta_P} - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right). \end{cases} \tag{D.9}$$

The cutting point n_{k_t} is defined by

$$n_{k_t} = 1 + n_{k_t-1} + \max\left(0, CF\left(s_0(\beta T_0 - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right)\right).$$

B. Proof of Theorem 1

To facilitate the proof, we propose the following three lemmas, which will be used more than once in the proof of Theorem 1. The proofs of these lemmas are deferred to the end of this proof.

Lemma 2. *In the equilibrium with a homogeneous impact and $\theta_I > \theta_P$, for an arbitrary bid-ask spread s, if the equilibrium strategy of a patient trader arriving at the spread s is to submit a market order, i.e., $\Pi(s) =$MO, then the market order submitted at the spread s provides a zero expected profit.*

Lemma 3. *For an arbitrary spread s, if the hidden limit order is not optimal for any spread $j \leq s$, there must exist an positive integer k, such that the expected waiting time of an $s-LO$ is*

$$T(s+1, s) = \frac{1}{\lambda}\left(1 + \sum_{i=1}^{k} 2\rho^{i}\right). \tag{D.10}$$

Lemma 4. *In the equilibrium with a homogeneous impact and* $\theta_I > \theta_P$*, for any bid-ask spread* s*, if the equilibrium strategy of a patient trader arriving at the spread* s *is* $\Pi(s) = (s-1)-HO$*, we have* $\Pi(s+1) = s-HO$*.*

With Lemmas 2-4, we prove the first part of theorem 1 where $\theta_I > \theta_P$ as follows. The other part where $\theta_I \leq \theta_P$ can be proved in the same way.

Because all the traders have an identical impact β, according to Lemma 1, we know that for a patient trader arriving at a current spread s, a $j-$hidden order ($j \leq s/2$) is always dominated by a midpoint peg order, whose expected waiting time is the same but execution price is more favorable. In contrast, a $j-$hidden order ($j > s/2$) is always dominated by an $(s-1)-$hidden order. Therefore, for each spread s, we only need to compare the expected profit among MO, $j-$LO ($j = 1, \cdots, (s-1)$), $(s-1)-$HO and MPPO.

According to Lemma 2, the market order would provide a zero expected profit if it is optimal in the equilibrium. According to the definition of the expected profit in (3), the non-market order might have a negative expected profit if the bid-ask spread is small and a non-negative one if the spread is large. In other words, there exists a cutting point n_1 such that the patient trader would submit a market order at spread $s \leq n_1$ and a non-market order at spread $s > n_1$.

According to Lemma 1, in the equilibrium with a homogeneous impact, the expected waiting time of a limit order does not depend on the spread at its submission time. If the optimal non-market order submitted at $s = n_1 + 1$ is a limit order, it must be an n_1-limit order. Otherwise, this $j-$limit order ($j < n_1$) is also available for a trader arriving at spread $s = n_1$, which is to say, the optimal strategy at $s = n_1$ should also be $j-$limit order, which is a contradiction. As a result, all the traders arriving at $s = n_1$ would submit a market order. The expected waiting time of an n_1-limit order is $1/\lambda$. In addition, based on Lemma 1 and the definition of expected profit (3), we have the expected profits of $(s-1)-$LO, $(s-1)-$HO and MPPO, as follows.

$$U(s,s-1) = (s-1)\Delta - \frac{\delta_P}{\lambda},\ U^H\left(s,\frac{s}{2}\right) = \frac{s}{2}\Delta - \frac{\delta_P\beta}{\lambda(1+\theta_P)}, \tag{D.11}$$

$$U^H(s,s-1) = (s-1)\Delta - \frac{\delta_P\beta T_0}{\lambda} = (s-1)\Delta - \frac{\delta_P\beta}{\lambda(\theta_I-\theta_P)}, \tag{D.12}$$

As the parameters may take different values, such that the first non-market order may take one of the three order types, we can thus classify the results into three corresponding cases.

Case 1. According to the inequalities (D.3), we have $\beta/(\theta_I - \theta_P) \leq 1$. Thus, for $s = n_1+1$, it is easy to show that

$$U^H(s,s-1) = (s-1)\Delta - \frac{\delta_P\beta}{\lambda(\theta_I-\theta_P)} \geq (s-1)\Delta - \frac{\delta_P}{\lambda} = U(s,s-1).$$

That is to say, the $(s-1)-$LO is dominated by the $(s-1)-$HO in such a case. In addition, under the fact that

$$n_1 = CF\left(\frac{s_0\beta}{\theta_I-\theta_P}\right),$$

we can claim that the profit of an $(s-1)-$HO is no less than that of a market order for patient traders arriving at $s = n_1+1$, but not for those arriving at $s = n_1$, since equation (D.12) implies that

$$\begin{cases} U^H(n_1+1,n_1) = n_1\Delta - \dfrac{\delta_P\beta}{\lambda(\theta_I-\theta_P)} \geq 0 \\ U^H(n_1,n_1-1) = (n_1-1)\Delta - \dfrac{\delta_P\beta}{\lambda(\theta_I-\theta_P)} < 0. \end{cases}$$

In addition, it is easy to show that both the profit of an $(s-1)-$HO and that of an MPPO are less than zero at $s \leq n_1$ with the inequalities (D.3). As a result, the optimal strategy of a patient trader arriving at a spread $s \leq n_1$ is to submit a market order. We next compare the profit of $(s-1)-$HO with that of MPPO, as shown in (D.11) and (D.12), to find

$$\begin{aligned} U^H(s,s-1) - U^H\left(s,\frac{s}{2}\right) &= \left(\frac{s}{2}-1\right)\Delta - \frac{3\Delta s_0\beta T_0\theta_P}{1+\theta_P} \geq 0 \\ &\Leftrightarrow\ s \geq 2 + \frac{6s_0\beta T_0\theta_P}{1+\theta_P}. \end{aligned} \tag{D.13}$$

Thus, with the inequalities (D.3), the profit of an $(s-1)-$HO is larger than that of an MPPO for patient traders arriving at $s = n_1 + 1$. Meanwhile, according to Lemma 4, the patient trader would stick to $(s-1)-$HO for all the spreads $s > n_1$. As a result, the patient trader arriving at a spread $s \geq n_1 + 1 = n_2$ would submit an $(s-1)-$HO in such a case, i.e., $t = 1$ and $k_1 = 2$.

Case 2. Similar to the proof in case 1, we can conclude that the $(s-1)-$LO dominates the $(s-1)-$HO in such a case with $\beta/(\theta_I - \theta_P) > 1$. In addition, with a simple comparison among the profit of an $(s-1)-$LO, that of an MPPO and zero, as shown in (D.11) and (D.12), one can conclude that if and only if the inequalities (D.4) hold, the profits of both an $(s-1)-$LO and an MPPO are smaller than zero for $s \leq n_1 := CF(s_0)$, while the profit of an $(s-1)-$LO is larger than both that of an MPPO and zero at $s = n_1 + 1$.

In other words, the patient trader arriving at a spread $s \leq n_1$ would submit a market order, while the one arriving at a spread $s = n_1 + 1$ would submit an n_1-limit order. Afterward, for any spread $s > n_1$, this n_1-limit order is available and better than a market order for the patient trader. Therefore, the trader would stick to it until some non-market order, which is even better than the n_1-limit order, becomes feasible. If there is no hidden order available for traders, one can easily follow the proof in Foucault et al. (2005) to find that the next new optimal strategy would be an $(s-1)-$LO at a spread s such that

$$\begin{aligned} U(s, s-1) &= (s-1)\Delta - \frac{\delta_P}{\lambda}(1+2\rho) > U(s, n_1) = n_1\Delta - \frac{\delta_P}{\lambda} \\ \Leftrightarrow\ s &\geq 1 + n_1 + CF(2s_0\rho). \end{aligned}$$

In other words, the next limit order, which is better than the n_1-LO, would become feasible once the spread is no less than $1 + n_1 + CF(2s_0\rho)$. We thus need to consider whether the profit of an MPPO or an $(s-1)-$HO could be no lower than that of the best feasible limit order, the n_1-LO, for spreads $s \leq n_1 + CF(2s_0\rho)$ to figure out the next cutting point n_2 and the optimal strategy at n_2. According to the expected profit formulations in (D.11) and

(D.12), we have

$$U^H\left(s,\frac{s}{2}\right) \geq U(s,n_1) \quad \Leftrightarrow \quad s \geq 2n_1 + CF\left(\frac{2s_0\beta}{1+\theta_P} - 2s_0\right), \tag{D.14}$$

$$U^H(s,s-1) \geq U(s,n_1) \quad \Leftrightarrow \quad s \geq n_1 + CF(s_0\beta T_0 - s_0) + 1. \tag{D.15}$$

Case 2.1. We first consider in this case that traders turn to a midpoint peg order at a spread $s \leq n_1 + CF(2s_0\rho)$. In particular, in such a case, there exists a spread $n_2 \in (n_1, n_1 + CF(2s_0\rho)]$ such that the profit of an MPPO is no lower than that of the n_1-LO at the spread, while the $(s-1)$-HO provides a lower profit than the n_1-LO at all the spreads $s \in (n_1, n_2)$ and a lower profit than the MPPO at the spread n_2.

According to the comparison between the profits of an MPPO and an $(s-1)$-HO in (D.13), if the $(s-1)$-HO provides a lower profit than the MPPO at the spread n_2, it must also provides a lower profit than the MPPO for all the spreads $s \in (n_1, n_2)$. We thus only need to derive the the necessary and sufficient condition for the existence of the spread $n_2 \in (n_1, n_1 + CF(2s_0\rho)]$ such that the profit of an MPPO is no lower than that of the n_1-LO at the spread and that the profit of $(s-1)$-HO is lower than that of an MPPO at the spread $s = n_2$. Therefore, with the comparisons in (D.13) and (D.14), we can derive the necessary and sufficient condition as the following inequalities,

$$\begin{cases} n_1 + CF\left(\frac{2s_0\beta}{1+\theta_P} - 2s_0\right) < 1 + CF(2s_0\rho), \\ 2n_1 + CF\left(\frac{2s_0\beta}{1+\theta_P} - 2s_0\right) < 2 + \frac{6s_0\beta T_0\theta_P}{1+\theta_P}. \end{cases}$$

Case 2.2. A similar proof as case 2.1 can be applied to case 2.2 with the comparisons in (D.13) and (D.15).

Case 2.3 is corresponding to the case where neither an MPPO nor an $(s-1)-$HO provides a higher profit than the best limit order for any spread $s \in (n_1, n_1 + CF(2s_0\rho)]$. In Case 3, the profit of an MPPO is non-negative and higher than those of all the feasible limit orders and the $(s-1)-$HO at the spread $n_1 + 1$, while all the feasible limit orders and $(s-1)$-HO provides a negative profit at all the spreads $s \leq n_1$.

Afterward, we will discuss in a similar way to identify the formula for new cutting points: for bid-ask spreads $s \in (n_{k_i+j-1}, n_{k_i+j-1} + CF(2s_0\rho^{j+1})]$, we figure out whether there exists a spread such that for patient traders arriving at the spread, an MPPO or an $(s-1)-$HO is better than the best feasible limit order, $n_{k_i+j-1}-$LO, in terms of the expected profit. With Lemma 1 and a similar mathematical induction on the expected waiting time of the $n_{k_i+j-1}-$LO as Lemma 3, we can write down the expected profits as

$$\begin{aligned} U(s, n_{k_i+j-1}) &= n_{k_i+j-1}\Delta - \frac{\delta_P}{\lambda}(1 + \sum_{\mu=1}^{j} 2\rho^{\mu}), \\ U^H(s, s-1) &= (s-1)\Delta - \frac{\delta_P}{\lambda}\beta T_0, \\ U^H(s, \frac{s}{2}) &= \frac{s}{2}\Delta - \frac{\delta_P \beta}{\lambda(1+\theta_P)}. \end{aligned}$$

Therefore, with the comparisons among these profits,

$$U^H\left(s, \frac{s}{2}\right) \geq U(s, n_{k_i+j-1}) \Leftrightarrow s \geq 2n_{n_{k_i+j-1}} + CF\left(2s_0(\frac{\beta}{1+\theta_P} - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right),$$

$$U^H(s, s-1) \geq U(s, n_{k_i+j-1}) \Leftrightarrow s \geq n_{k_i+j-1} + CF\left(s_0(\beta T_0 - 1 - \sum_{\mu=1}^{j} 2\rho^{\mu})\right) + 1,$$

we will identify the cutting point n_{k_i+j}. If there is an $(s-1)-$HO identified as optimal in the equilibrium, Lemma 4 reveals that the trader would stick to an $(s-1)-$HO afterwards. We can thus derive k_t and n_{k_t} to conclude the equilibrium strategy in such a case. Otherwise, if there is an MPPO identified as optimal in the equilibrium, we would have $k_{i+1} = k_i + j$ and a new cutting point as $n_{k_{i+1}}$. □

Proof of Lemma 2. If the optimal strategy for a patient trader arriving at a spread s is to submit a market order, then there must be no existing hidden order inside the bid-ask spread. The reason is that all the patient traders arriving at s would follow the same strategy system with a homogeneous impact and that the hidden orders would never change the bid-ask spread. As a result, the buy/sell market order would be executed immediately at the best ask/bid price. According to the definition of expected profit (3), the market order provides a zero expected profit. □

Proof of Lemma 3. We prove this lemma by mathematical induction. Note that the expected waiting time of a $1-$LO is $T(2,1) = 1/\lambda$, satisfying the equation (D.10) in this induction. Because the hidden limit order is not optimal for any spread $j \leq s$. The optimal strategy for patient traders arriving at the spread s is either a midpoint peg order or a limit order, $\bar{s}-$LO, $\bar{s} < s$.

If the optimal strategy is to submit a midpoint peg order, the next incoming order must be either a market order by an impatient trader or a midpoint peg order by a patient trader. Therefore, the expected waiting time of the $s-$LO is

$$T(s+1,s) = \theta_I \frac{1}{\lambda} + \theta_P \left(\frac{1}{\lambda} + \frac{1}{\lambda} + T(s+1,s) \right) \;\Rightarrow\; T(s+1,s) = \frac{1}{\lambda}\left(1 + 2\rho\right),$$

which follows the formulation in (D.10). If the optimal strategy for the patient trader arriving at the spread s is an $\bar{s}-$LO, its expected waiting time takes the form of (D.10). It is easy to show that the expected waiting time of a limit order, as an optimal strategy in the equilibrium with a homogeneous impact, does not depend on the spread at its submission time. Thus, we have

$$T(s,\bar{s}) = T(\bar{s}+1,\bar{s}) = \frac{1}{\lambda}\left(1 + \sum_{i=1}^{\bar{k}} 2\rho^i\right). \tag{D.16}$$

Because buyers and sellers alternate, the $s-$LO can only be executed after the execution of the $\bar{s}-$LO, the expected waiting time of $s-$LO is

$$T(s+1,s) = \theta_I \frac{1}{\lambda} + \theta_P \left(\frac{1}{\lambda} + T(s,\bar{s}) + T(s+1,s) \right). \tag{D.17}$$

Therefore, with equation (D.16) and a simple rearrangement of equation (D.17), we can derive that the expected waiting time of $s-$LO as

$$\begin{aligned} T(s+1,s) &= \frac{1}{\lambda}\cdot\frac{1}{\theta_I} + \rho\cdot T(s,\bar{s}) = \frac{1}{\lambda}\cdot(1+\rho) + \rho\cdot\frac{1}{\lambda}\left(1 + \sum_{i=1}^{\bar{k}} 2\rho^i\right) \\ &= \frac{1}{\lambda}\left(1 + \sum_{i=1}^{1+\bar{k}} 2\rho^i\right), \end{aligned}$$

which follows the formulation in (D.10). $\square$

Proof of Lemma 4. If the optimal strategy for a patient trader arriving at the spread s is $\Pi(s) = (s-1)-$HO, then $(s-1)-$HO dominates all the $j-$LO with $j \leq s-1$. In addition, the usage of a hidden limit order in the equilibrium, according to Lemma 1, implies that $\theta_I > \theta_P$. For an arbitrary spread s, the expected waiting time of an $(s-1)-$HO is

$$T^H(s, s-1) = \frac{1}{\lambda(\theta_I - \theta_P)}.$$

Furthermore, to calculate the expected waiting time of $s-$LO, let's consider two processes: X process, which is the number of the existing hidden orders on the same side as the $s-$limit order, and Y process, which is either 1 if the next arrival is from the same side or -1 if the next arrival is from the opposite side. We set the initial state as $(X_0, Y_0) = (0, -1)$. Furthermore, according to the facts that the buyers and sellers alternate with certainty and that the proportion of patient traders is θ_P, we know that

$$Y_n = (-1)^n, \ (X_{2n+1}, X_{2n+2}) = \begin{cases} (X_{2n} - 1, X_{2n+1}) & \text{with prob. } \theta_I, \\ (X_{2n}, X_{2n+1} + 1) & \text{with prob. } \theta_P. \end{cases}$$

We can transform the dynamics as the following equation to find the (X, Y) process is a markov chain with the state space $\{(n, 1), \ (n, -1) \mid n \in \mathbb{Z}_+\}$ and the evolution as

$$(X_{n+1}, Y_{n+1}) = \begin{cases} (X_n + \frac{1}{2}Y_n + \frac{1}{2}, -Y_n), & \text{with prob. } \theta_P, \\ (X_n + \frac{1}{2}Y_n - \frac{1}{2}, -Y_n), & \text{with prob. } \theta_I. \end{cases}$$

The expected first passage time of $(X_t, Y_t) = (0, -1)$ given the initial state being $(0, 1)$ is thus $\omega_1 = 1/(\lambda \cdot (\theta_I - \theta_P))$. Therefore, the expected waiting time of the $s-$limit order is

$$T(s+1, s) = \frac{1}{\lambda}\theta_I + \theta_P(\frac{1}{\lambda} + \omega_1 + T(s+1, s)) = \frac{1}{\lambda(\theta_I - \theta_P)}.$$

Meanwhile, we claim that the usage of a hidden limit order in the equilibrium strategy implies that $\beta < 1$. Otherwise, according to Lemma 3, there exists an integer k such that the expected waiting time of $(s-1)-$LO is

$$T(s, s-1) = \frac{1}{\lambda}\left(1 + \sum_{i=1}^{k} 2\rho^i\right) < \frac{1}{\lambda}\left(1 + \sum_{i=1}^{\infty} 2\rho^i\right) = \frac{1}{\lambda(\theta_I - \theta_P)} = T^H(s, s-1).$$

As a result, the $(s-1)-$LO provides a higher profit than the $(s-1)-$HO,

$$U(s,s-1)=(s-1)\Delta-\delta_P\beta T(s,s-1)>U^H(s,s-1)=(s-1)\Delta-\delta_P T^H(s,s-1).$$

This contradicts the fact that $\Pi(s)=(s-1)-$HO. Therefore,

$$U^H(s+1,s)=s\Delta-\frac{\delta_P\beta}{\lambda(\theta_I-\theta_P)}>U(s+1,s)=s\Delta-\frac{\delta_P}{\lambda(\theta_I-\theta_P)}.$$

Therefore, the $s-$HO dominates the $s-$LO in terms of expected profit. We next prove that the $s-$HO also dominates the MPPO. Because the $(s-1)-$HO dominates the MPPO for the patient traders arriving at the spread s, we have

$$\begin{aligned} U^H(s,s-1)-U^H\left(s,\frac{s}{2}\right) &= \left(\frac{s}{2}-1\right)\Delta-\frac{3\Delta s_0\beta\theta_P}{(\theta_I-\theta_P)(1+\theta_P)}\geq 0 \\ &\Leftrightarrow\ s\geq 2+\frac{6s_0\beta\theta_P}{(\theta_I-\theta_P)(1+\theta_P)}. \end{aligned}$$

As a result,

$$s+1>2+\frac{6s_0\beta\theta_P}{(\theta_I-\theta_P)(1+\theta_P)}\ \Rightarrow\ U^H(s+1,s)\geq U^H\left(s+1,\frac{s+1}{2}\right).$$

To conclude, the equilibrium strategy for patient traders arriving at a spread $s+1$ is to submit an $s-$hidden order. □

E. Proof of Theorem 2

Note that if an order cannot be executed upon the next trader's arrival, it must wait until it becomes an order at the best price level on its own side and that the next arrival is from the opposite side to be executed. To facilitate the writing of this proof, we formulate the Bellman equation for the expected optimal profit of a patient trader of type ν arriving at a spread s, $v(s,\nu)$, given the strategy set Π^*, as:

$$v(s,\nu|\Pi^*)=R(s,\nu,\Pi^*)+\sum_{s'}v(s',\nu|\Pi^*)\cdot P(s'|s,\Pi^*). \tag{E.1}$$

Here $R(\cdot)$ is the expected profit of the strategy submitted by the trader and executed upon the next arrival. And $P(s'|s,\Pi^*)$ is the probability of the state being s' when the order becomes an order at the best price level on its own side and the next arrival is from the opposite side to be executed. With the strategy set Π^*, we will have a recurrent state set $\bar{S} \subseteq S$, inside which all the states are visited in the simulated market with the strategy set Π^*. For any $s \in \bar{S}$, the probability of the state staying in $\bar{S}$ is one, i.e.,

$$P(s' \in \bar{S}|s,\Pi^*) = 1.$$

Thus, the first stopping rule implies that the expected profit of a ν type trader arriving at a recurrent spread s, estimated by the algorithm, satisfies the Bellman equation (E.1).

However, the expected waiting times and execution prices of the strategies, which would cause the state evolves from a recurrent state to a non-recurrent state, are not updated during the algorithm. The second stopping rule is thus to guarantee the perfection of our strategy set. More specifically, for an arbitrary (s,ν), the second stopping rule implies that the strategy $\Pi^*(s,\nu)$ solves

$$\max_{\Pi}(R(s,\nu,\Pi^*_-(s,\nu),\Pi(s,\nu)) + \sum_{s'} \tilde{v}(s',\nu) \cdot P(s'|s,\Pi^*_-(s,\nu),\Pi(s,\nu)),$$

where

$$\tilde{v}(s',\nu) = R(s',\nu,\Pi^*_-(s,\nu),\Pi(s,\nu)) + \sum_{s''} \tilde{v}(s'',\nu) \cdot P(s''|s',\Pi^*_-(s,\nu),\Pi(s,\nu)),$$

is estimated by the changed strategy set $(\Pi^*_-(s,\nu),\Pi(s,\nu))$. In other words, the trader would not deviate to the non-recurrent states, even allowed to do so. □

F. Application of Theorem 1 to the Examples in Section 3.2

A. The example without hidden orders by setting $\beta = \infty$

With $\beta = \infty$ and the formulations of the equilibrium spreads presented in Section A, one can easily find that the inequalities (D.4) are satisfied, while neither inequalities (D.5) nor

(D.6) holds. As a result, we can apply the definition in Case 2.3 to have

$$n_1 = CF(s_0),\ n_2 = n_1 + CF(2s_0\rho).$$

Furthermore, it is impossible that the inequalities (D.8) hold, since the fact that $\beta = \infty$ implies that the term,

$$2s_0(1 + \sum_{\mu=1}^{j} 2\rho^{\mu} - \frac{\beta}{1+\theta_P}),$$

approaches $-\infty$ in such a case. As a result, there does not exist the equilibrium spread where the midpoint peg order or the hidden limit order is optimal for the arriving trader. According to (D.7), the equilibrium spreads are

$$n_{2+j} = n_2 + \sum_{\mu=1}^{j} CF(2s_0\rho^{\mu+1}).$$

To conclude, by setting $\beta = \infty$, the equilibrium spreads implied by Theorem 1 are the same as the ones shown in Foucault et al. (2005).

B. Introducing hidden orders to Example 2 by Foucault et al. (2005)

According to the formulations of the equilibrium spreads presented in Section A, we calculate the equilibrium spreads as follows. First, with s_0 and T_0 defined in (D.1)-(D.2), we have

$$s_0 = \frac{\delta_P}{\lambda\Delta} = 0.8,\ T_0 = 10,\ \beta = 1,$$

and thus

$$\beta T_0 = 10 > 1,\ CF(s_0) = 1 < 1 + 2s_0\left(1 - \frac{\beta}{1+\theta_P}\right) = 1.4966.$$

Therefore, neither the inequalities (D.3) nor the inequalities (D.4) hold. We apply the definitions in Case 3 to derive the first two equilibrium spreads in this example as

$$\begin{aligned} n_1 &= \max\left(1, CF\left(\frac{2s_0\beta}{1+\theta_P}\right) - 1\right) = \max\left(1, CF\left(\frac{2\times 0.8}{1+0.45}\right) - 1\right) = 1, \\ n_2 &= n_1 + 1 = 2,\ k_1 = 2. \end{aligned}$$

The inequalities (D.8) hold, because

$$\begin{aligned} n_2 &\leq CF(2s_0\rho^2) + 2s_0\left(1 + 2\rho - \frac{\beta}{1+\theta_P}\right) = 5.1147,\ n_2 < 1 + \frac{6s_0\beta T_0\theta_P}{1+\theta_P} = 15.8966, \\ n_2 &< 1 + \frac{3s_0\beta T_0\theta_P}{1+\theta_P} - \frac{1}{2}CF\left(2s_0(\frac{\beta}{1+\theta_P} - 1 - 2\rho)\right) = 7.9483 - \frac{1}{2}CF(-1.9467) = 8.4483. \end{aligned}$$

Thus, we have $k_2 = 3$ and the third equilibrium spread as

$$n_3 = \max\left(1 + n_2, 2n_2 + CF\left(2s_0(\frac{\beta}{1+\theta_P} - 1 - 2\rho)\right)\right) = 3,$$

In a similar way, because $n_3 < 5.1147, 15.8966$ and 8.4483, we have $k_3 = 4$ and $n_4 = 4$. To conclude, we have $n_i = i$, $i = 1, \cdots, 4$, and $k_i = i + 1$ for $i = 1, 2, 3$.