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Limit theorems for sums of independent random variables

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LIMIT THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

by

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I. INTRODUCTION

The present thesis is concerned with a discussion of the more important aspects of the development of the theory of distribution laws for sums of independent random variables.

We shall begin our analysis with a discussion of the binomial probability law and its two limiting forms : the Poisson probability law and the normal probability law. The Central Limit Theorem will be considered along with work regarding the accuracy of the approximating normal distribution in this theorem.

Finally, the concept of an infinitely divisible law will be introduced and related to our earlier considerations as well as to the class of limit laws for suitable sums of independent random variables.

2. NOTATION AND AUXILIARY THEOREMS

We shall for the most part use capital letters to denote random variables. $P(A)$ will stand for the probability of the event A. $E(X)$ and $D^2(X)$ will stand for the mean and variance respectively, of the random variable X : $E(X) = \int_{-\infty}^{\infty} x \, dF(x)$, and $D^2(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 dF(x)$, where $F(x)$ is the distribution function of X ; i.e., $F(x) = P(X \leq x)$. The so-called characteristic function of X will be denoted by $f(t)$, where $f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) = E(e^{itx})$.

The following theorems, stated without proof, belong to the classical realm of probability and will find application in the sequel :

Theorem 2.1 : If $F(x)$ is the distribution function of the random variable X, then the distribution function of $b + \frac{X}{a}$ (where $a > 0$ and b are arbitrary real numbers) is $F[a(x-b)]$ and its characteristic function is $e^{itb} f\left(\frac{t}{a}\right)$.

Theorem 2.2 : Characteristic functions are continuous and satisfy the conditions $f(0) = 1$, and $|f(t)| \leq 1$ for all real t.

Theorem 2.3 : A distribution function is uniquely determined by its characteristic function.

Theorem 2.4 : If $E|X^k| = \beta_k$ exists, then the following holds: $f(t) = \sum_{j=1}^{k-1} \frac{\alpha_j (it)^j}{j!} + \frac{\vartheta \beta_k t^k}{k!}$, where $|\vartheta| \leq 1$ and $\alpha_k = E(X^k)$.

Theorem 2.5 : A necessary and sufficient condition for the convergence of the sequence of distribution functions $\{F_n(t)\}$ to a limiting distribution function, $F(x)$, is that in any finite interval $|t| \leq T$, the characteristic functions $f_n(t)$ converge uniformly to a limiting function. This limiting function is the characteristic function of $F(x)$.

Theorem 2.6 : The characteristic function of the sum of any finite number of independent random variables is equal to the product of the corresponding characteristic functions.

3. THE CENTRAL LIMIT THEOREM

Consider the situation wherein an experiment is to be carried out independently a certain number, n, of times and at each trial there are only two possible outcomes, s and f. Attached to the outcome s there is a probability p, and to f a probability 1 - p = q. We shall assume that these probabilities remain constant throughout the repetitions of the experiment. We shall be interested in the total number, without regard to order, of s's appearing in the n trials. The probability of exactly k s's and n - k f's appearing in the n trials is given by the following :

$b(k;n,p) = \binom{n}{k} p^k q^{n-k}$, where k = 0,1,2,...,n. This is the binomial probability law.[1] We introduce the random variables $X_1, X_2, \ldots, X_n$ defined as follows :

$$X_k = \begin{cases} 1 & \text{if s appears at the kth trial} \\ 0 & \text{if f appears at the kth trial} \end{cases}$$

We may then introduce the variable $S_n = X_1 + X_2 + \ldots + X_n$ which equals the number of s's appearing in n independent trials. Note that we have a sum of n identically distributed random variables; i.e., each X_k is distributed by the

[1] Emanuel Parzen, Modern Probability Theory and its Applications (New York, John Wiley & Sons., Inc., 1960), p.102.

same Bernoulli probability law :[2]

$P(X_k = 1) = p$
$P(X_k = 0) = q$ for $k = 1,2,\ldots,n$.

In very many applications of the binomial probability law the computations would be quite impractical were it not for the fact that we can approximate the binomial law by either of two other probability laws : the Poisson probability law, or the normal probability law.

In situations where, in dealing with Bernoulli trials, we are using a large n and a small p, whereas the product $\lambda = np$ is of moderate magnitude, we may use the Poisson approximation to the binomial distribution :

$b(k;n,p) \approx \frac{\lambda^k e^{-\lambda}}{k!}$, which is a uniform approximation when $n \to \infty$ and $p \to 0$ in such a way that $\lambda = np$ remains bounded.[3]

We are often interested in the probability that the number of successes lies between preassigned limits, α and β. If α and β are integers ($\alpha < \beta$), then we define the event as $\alpha \leq S_n \leq \beta$. Its probability is $P(\alpha \leq S_n \leq \beta) = b(\alpha;n,p) + b(\alpha+1;n,p) + \ldots + b(\beta;n,p)$. If n is sufficiently large, we have recourse to the following theorem due to DeMoivre and Laplace :

[2] Parzen, p.101.

[3] William Feller, An Introduction to Probability Theory and its Applications, 2nd ed. (New York, John Wiley & Sons, Inc., 1958), p.143.

Theorem 3.1[4]: The probability that a random phenomenon obeying the binomial probability law, with parameters n and p, will yield an observed value lying between α and β, inclusive, for any two such integers is given approximately by

$$\sum_{k=\alpha}^{\beta} \binom{n}{k} p^k q^{n-k} \doteq \Phi\left(\frac{\beta - np + 1/2}{\sqrt{npq}}\right) - \Phi\left(\frac{\alpha - np - 1/2}{\sqrt{npq}}\right)$$

where we have $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$ as the normal distribution function with mean value equal to 0 and with standard deviation equal to 1.

If λ = np is large, we cah use either the normal or the Poisson approximation to the binomial distribution law. The implication here is that perhaps it is possible to approximate the Poisson by means of the normal; indeed, for fixed values of α, β, and k, the following difference tends to 0 as $\lambda \to \infty$:[5]

$$\frac{e^{-\lambda}\lambda^k}{k!} - \frac{1}{\sqrt{2\pi}} \int_{\frac{k-\lambda-1/2}{\sqrt{\lambda}}}^{\frac{k-\lambda+1/2}{\sqrt{\lambda}}} e^{-y^2/2} dy \to 0, \quad (\lambda \to \infty).$$

We may consider the above in the following light: We are given the distribution of a sum of independent identically (Bernoulli) distributed random variables, and we have arrived at two limiting distributions for such a sum by means of two essentially different limiting processes. In the case of the normal approximation to the binomial, we

[4] Parzen, p.239.

[5] Ibid., p.248.

hold p constant and let $n \to \infty$. For the Poisson approximation to the binomial, we allow both n and p to vary, but in such a way that $n \to \infty$ and $p \to 0$ in a manner that leaves λ bounded. Consider the following configuration:

p_1 : X_{11}
p_2 : X_{21} , X_{22}
p_3 : X_{31} , X_{32} , X_{33}
. . . .
p_n : X_{n1}, X_{n2}, . . . , X_{nn}
. . . .

In the first case above we are changing n alone as we consider successive rows of mutually independent variables; i.e., $p_i = p$ (i = 1,2,...). In the second of the cases we must consider the fact that the p's are changing as we move to different rows with increasing n. Let it be pointed out at this time that this configuration is a special case of the more general situation with a double sequence X_{n1}, X_{n2}, . . . , X_{nn}, . . . (n = 1,2,...) of random variables, independent for each choice of n, but not necessarily identically distributed.

The approximation of the binomial probability law by means of the normal law is of major importance because it provided the foot-hold needed to attain the heights afforded by the Central Limit Theorem. The DeMoivre-Laplace theorem is a particular case of this theorem. Our X's were Bernoulli variables, so that $S_n = X_1 + X_2 + \ldots + X_n$ was a binomial variable. Let us proceed to the more general case wherein we have a set of n observations, X_1, X_2, X_3, . . . , X_n. They are said to constitute a random sample of the random variable X if X_1, X_2, . . . , X_n are independent random variables identically distributed as X. Let $S_n = X_1 + X_2 + \ldots + X_n$ be the sum of the random variables.

The Central Limit Theorem states that the sum of a large number of independent and identically distributed random variables with finite variances and expectations,or means, normalized to have mean 0 and standard deviation 1, is approximately normally distributed. (By normalized we mean that from the sum we subtract its mean value and divide the resulting difference by the square root of the variance; i.e., the standard deviation, of the sum.) It can be shown that $E(S_n) = n\,E(X)$, and that $\sigma(S_n) = D(S_n) = \sqrt{n}\,D(X) = \sqrt{n}\,\sigma(X)$. Restated, the theorem is that for any real numbers $\alpha < \beta$,

$$P(\alpha \leq S_n \leq \beta) = P\left[\frac{\alpha - E(S_n)}{\sigma(S_n)} \leq \frac{S_n - E(S_n)}{\sigma(S_n)} \leq \frac{\beta - E(S_n)}{\sigma(S_n)}\right] \doteq$$

$$\doteq \Phi\left[\frac{\beta - E(S_n)}{\sigma(S_n)}\right] - \Phi\left[\frac{\alpha - E(S_n)}{\sigma(S_n)}\right].$$ [6]

Much of the energy expended in probability theory up

[6] Parzen, p.372.

to the 1930's was directed to the determination of the conditions under which this above relation would hold. In the case of identically distributed independent variables it was found that the mere existence of the expectations and variances is sufficient to justify the application of the Central Limit Theorem. This is no longer the case when this restriction is relaxed.

We have assumed identically distributed random variables. Let us retain the assumption of independence of the variables, but discard the assumption of identical distributions. We introduce the following notation :

$$S_n = X_1 + X_2 + \cdots + X_n ,$$

$$a_i = E(X_i) \quad , \quad b_i^2 = D^2(X_i) \; ,$$

$$A_n = \sum_{i=1}^{n} a_i \quad , \quad B_n^2 = \sum_{i=1}^{n} b_i^2 \; .$$

The above-mentioned inquiries usually start with the question of what additional conditions ensure the Central Limit Theorem in the form

$$P\left[\frac{S_n - A_n}{B_n} < z\right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt \, , \; n \to \infty \, , \text{ UNIFORMLY IN } z.$$

In this less restricted case, it is not sufficient to require that the means and the variances of the individual summands merely exist. It is necessary that we ensure that in the passage to the limit no individual summand be permitted to dominate the total sum, S_n.

Lindeberg's condition[7] that for every $\varepsilon > 0$, $\lim_{n \to \infty} \sum_{k=1}^{n} P(|X_k - a_k| \geq \varepsilon B_n) = 0$ is more general and perhaps logically simpler than the condition whose sufficiancy we shall prove. In anticipation of this proof, let us introduce some helpful terminology. A random variable X, depending on n, for which two quantities, m_0 and s_0 (not necessarily depending on n), can be found such that the distribution function of $\frac{X - m_0}{s_0}$ tends to $\Phi(x)$ is said to be asymptotically normally distributed. This is the same as stating that for any two numbers, α and β , independent of n, $\lim_{n \to \infty} P(\alpha \leq S_n \leq \beta) = \Phi(\beta) - \Phi(\alpha)$. We may again paraphrase the Central Limit Theorem as follows : Under certain very general conditions the sums $S_n = X_1 + X_2 + \ldots + X_n$ is asymptotically normal, (A_n, B_n), whatever the distributions of the independent X_i. The following condition for this is due to Lyapunov :

Theorem 3.2 [8]: Let $X_1 , X_2 , \ldots , X_n$ be a set of n independent random variables and denote by a_i and b_i the mean and standard deviation of X_i. Suppose that the third absolute moment of X_i about its mean, $\rho_i^3 = E(|X_i - a_i|^3)$ is finite for every i. Write $\rho^3 = \rho_1^3 + \rho_2^3 + \cdots + \rho_n^3$.

7 B.V. Gnedenko and A.N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables (Cambridge, Addison - Wesley, 1954), p.5.

8 Harald Cramér, Mathematical Methods of Statistics, 8th ed. (Princeton University Press, 1958), p.215.

If the condition $\lim_{n\to\infty} \frac{\rho}{B_n} = 0$ is satisfied, then the sum $S_n = \sum_{i=1}^{n} X_i$ is asymptotically normally distributed with respect to the quantities A_n and B_n defined above.

proof : Let $f_i(t)$ denote the characteristic function of the ith deviation, $X_i - a_i$, and let $f(t)$ denote the characteristic function of the normalized sum, $\frac{S_n - A_n}{B_n} =$ $= \frac{1}{B_n} \sum_{i=1}^{n} (X_i - a_i)$. It follows from Theorem 2.1 that $f(t) = \prod_{j=1}^{n} f_j\left(\frac{t}{B_n}\right)$. From Theorem 2.5 we see that it suffices to show that $f(t) \longrightarrow e^{-\frac{1}{2}t^2}$, the characteristic function of the normal probability law, when $n \to \infty$, for fixed t.

From Theorem 2.4, with ϑ used as the symbol for some quantity — real or complex — with modulus $\leqslant 1$, we get $f(t) = E(e^{it(X_i - a_i)}) = 1 - \frac{1}{2} b_i^2 t^2 + \frac{1}{6} \vartheta \rho_i^3 t^3$, where we are careful to distinguish i as subscript and i as complex entity.

Further, $\log f_i\left(\frac{t}{B_n}\right) = \log\left(1 - \frac{b_i^2 t^2}{2B_n^2} + \frac{1}{6} \frac{\rho_i^3 t^3}{B_n^3}\right) =$

$$= \log(1 + z), \text{ where } z = -\frac{b_i^2 t^2}{2B_n^2} + \frac{1}{6} \frac{\vartheta \rho_i^3 t^3}{B_n^3} .$$

From the condition of Lyapunov, we have for all sufficiently large values of n, $\frac{\rho_i}{B_n} \leqslant \frac{\rho}{B_n} < 1$, and thus, from the fact that $\beta_i^{1/i} \leqslant \beta_i^{1/i+1}$, (i = 1,2,...), where we have $\beta_i =$ $E(|Y|^i)$, we get $b_i \leqslant \rho_i$ for every i and the following holds :

$$z = \frac{\vartheta \rho_i^2 t^2}{2 B_n^2} + \frac{\vartheta \rho_i^3 t^3}{6 B_n^3} = \frac{\vartheta \rho_i^2}{B_n^2}\left(\frac{t^2}{2} + \frac{|t|^3}{6}\right).$$ Our

condition leads now to the fact that for every fixed t, we have $z \to 0$ as $n \to \infty$. Thus, for sufficiently large n, $|z| < \frac{1}{2}$. For $|z| < \frac{1}{2}$ we have

$$\log(1 + z) = z - \frac{z^2}{2}\left(1 - \frac{2}{3} z + \frac{2}{4} z^2 - \ldots\right) =$$

$$= z + \tfrac{1}{2} \vartheta z^2 \left(1 + \frac{1}{2} + \frac{1}{2}2 + \ldots\right) = z + \vartheta z^2 .$$

Hence, $$\log f_i\left(\frac{t}{B_n}\right) = -\frac{b_i^2 t^2}{2B_n^2} + \frac{\vartheta \rho_i^3 t^3}{6B_n^3} + \frac{\vartheta \rho_i^4}{B_n^4}\left(\frac{t^2}{2} + \frac{|t|^3}{6}\right)^2 = -\frac{b_i^2 t^2}{2B_n^2} + \frac{\vartheta \rho_i^3}{B_n^3}\left[\frac{|t|^3}{6} + \left(\frac{t^2}{2} + \frac{|t|^3}{6}\right)^2\right].$$

Sum over $i = 1,2,\ldots,n$. From the expression for $f(t)$ we obtain $$\log f(t) = -\frac{t^2}{2} + \frac{\vartheta \rho^3}{B_n^3}\left[\frac{|t|^3}{6} + \left(\frac{t^2}{2} + \frac{|t|^3}{6}\right)^2\right].$$

As $n \to \infty$, it follows from the condition that $\log f(t) \to$ $\to -\frac{t^2}{2}$ for every fixed t. Our theorem is proved.

Let us now consider the problem of determining some numerical bounds for the degrees of approximation that are involved in the above-mentioned cases of the Central Limit Theorem ascribed to Lindeberg and to Lyapunov, as well as the additional case of Feller to be mentioned below.

Define the variable $\lambda(X_k)$ as follows :

$$\lambda(X_K) = \begin{cases} \rho_k^3 / b_k^2 & \text{if } b_k^2 \neq 0 \\ 0 & \text{if } b_k^2 = 0 \end{cases}$$, where we

assume at least one $b_k^2 \neq 0$; this will be termed the moment ratio of X .

Set $\Lambda = \max.\left[\lambda(X_1), \lambda(X_2), \ldots, \lambda(X_m)\right]$ and $\varepsilon = \frac{\Lambda}{B_n}$, where in the special case of identically distributed random variables, $\varepsilon = \frac{c}{\sqrt{n}}$, where c is some function of the third order absolute moments. A normally distributed variable with mean value A_n and standard deviation B_n has the distribution function $\Phi\left(\frac{x - A_n}{B_n}\right)$. Let our sum have the distribution function F(x).

Let $M = \sup_{-\infty < x < \infty} \left| F(x) - \Phi\left(\frac{x - A_n}{B_n}\right)\right|$. Theorem 3.2 states essentially that $\lim_{\varepsilon \to 0} M = 0$.

Theorem 3.3 [9]: Let $X_1, X_2, \ldots, X_m$ be independent random variables for which $\rho_i^3 = E(|X_i - a_i|^3)$ are finite. This is sufficient for the inequality $M \leqslant 1.88\,\varepsilon$ to hold.

From this it follows that the ratios M/ε arising from admissible sets of variates form a bounded group of real numbers. A more general condition for the Central

[9] A.C.Berry,"The Accuracy of the Gaussian Approximation to the Sum of Independent Variates", Transactions of the American Math. Soc. (New York, 1941),49, p.124.

Limit Theorem is due to Feller. It is as follows : We have given any real numbers $s > 0$ and $a_1, a_2, \ldots, a_m$. For given $\varepsilon > 0$ we introduce the quantities

$$\varepsilon_0 = \sum_{k=1}^{m} P\left(| X_k - a_k | > \varepsilon s \right)$$

$$\varepsilon_1 = \frac{1}{s} \sum_{k=1}^{m} \left| \int_{a_k - \varepsilon s}^{a_k + \varepsilon s} (x - a_k)\, dF_k(x) \right|$$

$$\varepsilon_2 = \left| 1 - \frac{1}{s^2} \sum_{k=1}^{m} \int_{a_k - \varepsilon s}^{a_k + \varepsilon s} (x - a_k)^2\, dF_k(x) \right| .$$

If these three entities are small, then our M is small.

Correspondingly, we have the following theorem :

Theorem 3.4 [10]: If $\varepsilon_0 \leq \varepsilon$, $\varepsilon_1 \leq \varepsilon$, $\varepsilon_2 \leq \varepsilon$ then $M \leq 5.8\varepsilon$.

Note : It can be proved, more generally, that if $\varepsilon_1^2 + \varepsilon_2 < 1$

we have $M \leq \dfrac{C(\varepsilon + \varepsilon_1)}{(1 - \varepsilon_1^2 - \varepsilon_2)^{1/2}} + \varepsilon_0 + \dfrac{\varepsilon_1}{\sqrt{2\pi}} + \dfrac{1}{\sqrt{2\pi}} \log \dfrac{1}{(1 - \varepsilon_1^2 - \varepsilon_2)^{1/2}}$

where $C = \sup \dfrac{M}{\varepsilon} \leq 1.88$ for admissible sets of variates.

More particularly, if we restrict ourselves to the case where our s and a_i's are our earlier B_n and individual variable means, respectively, we can reformulate our ε_i's as follows :

$$\varepsilon_0 = \sum_{k=1}^{m} P\left(| X_k - a_k | > \varepsilon B_m \right)$$

$$\varepsilon_1 = \frac{1}{B_m} \sum_{k=1}^{m} \left| \left(\int_{-\infty}^{a_k - \varepsilon B_m} + \int_{a_k + \varepsilon B_m}^{\infty} \right) (x - a_k)\, dF_k(x) \right|$$

$$\varepsilon_2 = \frac{1}{B_m^2} \sum_{k=1}^{m} \left(\int_{-\infty}^{a_k - \varepsilon B_m} + \int_{a_k + \varepsilon B_m}^{\infty} \right) (x - a_k)^2\, dF_k(x)$$

since the following results are valid :

[10] Berry, p.134.

i.) $$\sum_{k=1}^{n} \left| \int_{a_k-\varepsilon B_n}^{a_k+\varepsilon B_n} (x-a_k)\, dF_k(x) \right| = \sum_{k=1}^{n} \left| \int_{-\infty}^{\infty} (x-a_k)\, dF_k(x) - \right.$$

$$\left. - \left(\int_{-\infty}^{a_k-\varepsilon B_n} + \int_{a_k+\varepsilon B_n}^{\infty} \right) (x-a_k)\, dF_k(x) \right|$$

where the first entity within the absolute value signs on the right is equal to 0 by our restriction.

ii.) $$\left| 1 - \frac{1}{B_n^2} \sum_{k=1}^{n} \int_{a_k-\varepsilon B_n}^{a_k+\varepsilon B_n} (x-a_k)^2\, dF_k(x) \right| = \left| 1 - \frac{1}{B_n^2} \left[\right.\right.$$

$$\left.\left. \sum_{k=1}^{n} \int_{-\infty}^{\infty} (x-a_k)^2\, dF_k(x) - \sum_{k=1}^{n} \left(\int_{-\infty}^{a_k-\varepsilon B_n} + \int_{a_k+\varepsilon B_n}^{\infty} \right) (x-a_k)^2\, dF_k(x) \right] \right| =$$

$$= \frac{1}{B_n^2} \sum_{k=1}^{n} \left(\int_{-\infty}^{a_k-\varepsilon B_n} + \int_{a_k+\varepsilon B_n}^{\infty} \right) (x-a_k)^2\, dF_k(x) .$$

Lindeberg's condition for the Central Limit Theorem, stated earlier, may be restated for the case of independent random variables with 0 means and finite variances as follows : for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^{n} \left(\int_{-\infty}^{-\varepsilon B_n} + \int_{\varepsilon B_n}^{\infty} \right) x^2\, dF_k(x) = 0 \text{ [11]}.$$

Bringing in our restrictions above and relaxing the present condition that the a_i's are all equal to 0, we see that Lindeberg's condition is $\lim_{n\to\infty} \varepsilon_2 = 0$. Our estimate of the degree of approximation attained in this case is as follows :

[11] Gnedenko and Kolmogorov, p.103.

Theorem 3.5 [12]: If $\varepsilon_2 \leq \varepsilon^3$, then $M \leqslant 3.6\,\varepsilon$.

Finally, consider again the Lyapunov condition. Reformulated and generalized, it is that the following holds : $\lim_{n\to\infty} \frac{1}{B_n^m} \sum_{k=1}^{n} E(|X_k - a_k|^m) = 0$, where m is greater than 2 and not necessarily integral. We are assuming in this final case that each X_k does have a finite absolute moment of the necessary order. From our point of view, the condition would be used to justify the statement that M is small when $\eta =$

$= \frac{1}{B_n^m} \sum_{k=1}^{n} E(|X_k - a_k|^m)$ is small. Letting $\varepsilon = \eta^{1/m+1}$, we can prove that $\varepsilon_2 \leqslant \varepsilon^3$ and hence, using Theorem 3.5 above, we get the following estimate corresponding to the Lyapunov condition :

Theorem 3.6 [13]: $M \leqslant 3.6\,\eta^{1/m+1}$

In conclusion, we can say that we have found explicit numerical upper bounds for the least upper bound, M, of the absolute difference between F(x), the distribution function of our sum, S_n, and $\Phi\left(\frac{x - A_n}{B_n}\right)$, the distribution function of a normal variable with mean value equal to A_n and standard deviation equal to

[12] Berry, p.136

[13] Berry, p.136.

B_n , under various conditions on the independent variables X_i and their moments, corresponding to the conditions under which hold the Feller, Lindeberg, and Lyapunov versions of the Central Limit Theorem of probability theory.

4 INFINITELY DIVISIBLE DISTRIBUTIONS

To attain our most general conclusions regarding limit laws for sums of independent random variables it becomes necessary at this point to introduce the idea of an infinitely divisible random variable.

Let us call a random variable, η , infinitely divisible if for every natural number n, η can be represented as the sum $\eta = X_{n1} + X_{n2} + \ldots + X_{nn}$ of n independent identically distributed random variables taken from our generalized configuration of section 3. The distribution of an infinitely divisible random variable will be called an infinitely divisible distribution. We see from the definition and Theorem 2.6 that a distribution is infinitely divisible if and only if the corresponding characteristic function is the n^{th} power of some characteristic function, $f_n(t)$, depending on n; i.e., $f(t) = [f_n(t)]^n$.

We note the following two properties :

Lemma 4.1 [14]: The characteristic function of an infinitely divisible law never vanishes.

[14] B.V.Gnedenko, Limit Theorems for Sums of Independent Random Variables, trans.45, American Math. Soc.(New York,1951), p.14.

Lemma 4.2 [15]: The limiting distribution for a sequence of infinitely divisible distributions is itself infinitely divisible.

Consider the second lemma: If $F^{(k)}(x) \to F(x)$, by Theorem 2.5 $f^{(k)}(t) \longrightarrow f(t)$. Since $f^{(k)}(t)$ is the characteristic function for an infinitely divisible distribution, $f^{(k)}(t) = \left[f_n^{(k)}(t) \right]^n$ for every n, or $f_n^{(k)}(t) = \sqrt[n]{f^{(k)}(t)}$, where the principal branch of the root is intended here. This is a characteristic function and never vanishes for any t, by Lemma 4.1. From these relations we may conclude that $f_n^{(k)}(t) = f_n(t)$; now applying Theorem 2.5 in the con - verse direction, we see that for some unknown distribution functions this $f_n(t)$ must be a characteristic function. Finally, since $f(t) = \left[f_n(t) \right]^n$, by definition, f(t) is the characteristic function of an infinitely divisible law.

The two limiting distributions considered in section 3 are related to the ideas under discussion by means of the following two theorems :

Theorem 4.1 [16]: A normally distributed random variable is infinitely divisible.

[15] Gnedenko, p.14.

[16] Gnedenko and Kolmogorov, p.71.

proof : Suppose $E(X) = a$ and $D^2(X) = \sigma^2$. The characteristic function of X is given by $f(t) = E(e^{itx}) =$ $= e^{iat - \frac{1}{2}\sigma^2 t^2}$. Since for every $n > 0$, we have $f_n(t) =$ $= e^{i\frac{a}{n}t - \frac{1}{2}\frac{\sigma^2}{n}t^2}$ is the characteristic function of a normal law, our result follows as desired.

Theorem 4.2 [17]: A Poisson distributed random variable is infinitely divisible.

proof : We assume a point of reference a. The possible values of X then have the form $a + kh$ $(k = 0,1,\ldots)$, and $P(X = a + kh) = \frac{\lambda^k e^{-\lambda}}{k!}$ $(\lambda > 0)$. The characteristic function of X is $f(t) = e^{iat + \lambda(e^{ith} - 1)}$. From this we see that for every $n > 0$, $\sqrt[n]{f(t)} =$ $= e^{i\frac{a}{n} + \frac{\lambda}{n}(e^{ith} - 1)}$ is the characteristic function of a Poisson distributed random variable also. This implies our desired result.

For our purposes we shall need information which is more explicit concerning these infinitely divisible laws, namely formulas for the characteristic functions, or, more conveniently as the sequel will show, formulas for the logarithms of these characteristic functions. Kolmogorov found that a distribution $F(x)$ with finite variances is infinitely divisible if and only if the

[17] Gnedenko and Kolmogorov, p.71.

logarithm of the characteristic function takes the following form :

(1) [18] $$\log f(t) = i\gamma t + \int_{-\infty}^{\infty} \left\{ e^{itu} - 1 - itu \right\} \frac{1}{u^2}\, dG(u),$$

where γ is a constant and $G(u)$ is a non - decreasing function of bounded variation. If we assume — as we may, without loss of generality — that $G(-\infty) = 0$, the representation is unique.

Lévy and Khintchine were later able to generalize upon this result as follows :

Theorem 4.3 [19] : A necessary and sufficient condition that the function f(t) be the characteristic function of an infinitely divisible distribution is that its logarithm be representable in the form

(2) $$\log f(t) = i\gamma t + \int_{-\infty}^{\infty} \left\{ e^{itu} - 1 - \frac{itu}{1+u^2} \right\} \frac{1+u^2}{u^2}\, dG(u),$$

where γ is a real constant, $G(u)$ is a non - decreasing function of bounded variation, and the integrand at $u = 0$ is given by $\lim_{u \to 0} \left[\left\{ e^{itu} - 1 - \frac{itu}{1+u^2} \right\} \frac{1+u^2}{u^2} \right] = -\frac{t^2}{2}$.

Further, the representation is unique.

If we define the functions $M(u)$ and $N(u)$ and the constant σ^2 by setting up the following relations :

[18] Gnedenko, p.12.

[19] Gnedenko and Kolmogorov, p.76.

$$M(u) = \int_{-\infty}^{u} \frac{1+z^2}{z^2}\, dG(z) \qquad u < 0$$

(3) $$N(u) = -\int_{u}^{\infty} \frac{1+z^2}{z^2}\, dG(z) \qquad u > 0$$

$\sigma^2 = G(+0) - G(-0)$, formula (2) for

log f(t) can be written in the following way :

(4) $$\log f(t) = i\gamma t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{-0} \left\{ e^{itu} - 1 - \frac{itu}{1+u^2} \right\} dM(u) + $$
$$+ \int_{+0}^{\infty} \left\{ e^{itu} - 1 - \frac{itu}{1+u^2} \right\} dN(u) \, .$$

From (3) and (4) we can conclude that M(u) and N(u) have the following properties :

(a) They are non - decreasing in the intervals $(-\infty, -0)$ and $(+0, \infty)$, respectively.

(b) They are continuous at those and only those points at which G(u) is continuous.

(c) They satisfy the relations $M(-\infty) = N(\infty) = 0$

(d) They satisfy $\int_{-\varepsilon}^{0} u^2 dM(u) + \int_{0}^{\varepsilon} u^2 dN(u) < \infty$.

Finally, we arrive at the following form for log f(t) :

(5) $$\log f(t) = i\gamma(\tau) t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{-\tau} (e^{itu} - 1)\, dM(u) + $$
$$+ \int_{\tau}^{\infty} (e^{itu} - 1)\, dN(u) + \int_{-\tau}^{-0} (e^{itu} - 1 - itu)\, dM(u) + \int_{+0}^{\tau} (e^{itu} - 1 - itu)\, dN(u)$$

where τ is an arbitrary constant chosen so that τ and $-\tau$ are continuity points of N(u) and M(u), respectively.

The relationship that exists between γ of (2) and

$\gamma(\tau)$ of (5) in the representation of log f(t) is obtained by comparison of (4) and (5). Since (4) and (5) are equal, it follows that

$$i\gamma t + \int_{-\infty}^{-\tau}(e^{iut}-1)\,dM(u) - it\int_{-\infty}^{-\tau}\frac{u}{1+u^2}\,dM(u) + \int_{\tau}^{\infty}(e^{iut}-1)\,dN(u) - it\int_{\tau}^{\infty}\frac{u}{1+u^2}\,dN(u) + \int_{-\tau}^{-0}(e^{iut}-1-iut)\,dM(u) +$$

$$+ it\int_{-\tau}^{-0}\frac{u^3}{1+u^2}\,dM(u) + \int_{+0}^{\tau}(e^{iut}-1-iut)\,dN(u) + it\int_{+0}^{\tau}\frac{u^3}{1+u^2}\,dN(u) =$$

$$= i\gamma(\tau)t + \int_{-\infty}^{-\tau}(e^{iut}-1)\,dM(u) + \int_{\tau}^{\infty}(e^{iut}-1)\,dN(u) +$$

$$+ \int_{-\tau}^{-0}(e^{iut}-1-iut)\,dM(u) + \int_{+0}^{\tau}(e^{iut}-1-iut)\,dN(u).$$

It follows then that

$$\gamma(\tau) = \gamma - \int_{-\infty}^{-\tau}\frac{u}{1+u^2}\,dM(u) - \int_{\tau}^{\infty}\frac{u}{1+u^2}\,dN(u) + \int_{-\tau}^{-0}\frac{u^3}{1+u^2}\,dM(u) + \int_{+0}^{\tau}\frac{u^3}{1+u^2}\,dN(u).$$

Using our relations, (3), we conclude that

$$(6)\quad \gamma(\tau) = \gamma - \int_{|u|\geq\tau}\frac{1}{u}\,dG(u) + \int_{|u|<\tau}u\,dG(u),$$

We shall now consider conditions for the convergence of sequences of infinitely divisible laws to a limit law.

Theorem 4.4 [20] : In order that a sequence, $\{F_n(x)\}$, of infinitely divisible laws converge to a limit law, F(x), it is necessary and sufficient that as $n \longrightarrow \infty$,

i.) $G_n(x) \longrightarrow G(x)$ in every point of continuity of G(x)

[20] Gnedenko, p.14.

ii.) $G_n(+\infty) \longrightarrow G(+\infty)$

iii.) $\gamma_n \rightarrow \gamma$,

where the functions $G_n(x)$ and $G(x)$, and the constants γ_n and γ are determined by the formula (2) of Lévy and Khintchine for the distributions F(x) and F(x), respectively.

We shall find the following form of Theorem 4.4 useful in the sequel :

Theorem 4.5 [21] : In order that a sequence, $\{F_n(x)\}$, of infinitely divisible laws converge to a limit law, F(x), it is necessary and sufficient that as $n \rightarrow \infty$,

i.) $M_n(u) \rightarrow M(u)$, $N_n(u) \rightarrow N(u)$ at the points of continuity of M(u) and N(u).

ii.) $\gamma_n(\tau) \longrightarrow \gamma(\tau)$

iii.) $\lim\limits_{\varepsilon \to 0} \overline{\lim\limits_{n \to \infty}} \left[\int_{-\varepsilon}^{-0} u^2 dM_n(u) + \sigma_n^2 + \int_{+0}^{\varepsilon} u^2 dN_n(u) \right] =$

$= \lim\limits_{\varepsilon \to 0} \underline{\lim\limits_{n \to \infty}} \left[\int_{-\varepsilon}^{-0} u^2 dM_n(u) + \sigma_n^2 + \int_{+0}^{\varepsilon} u^2 dN_n(u) \right] = \sigma^2$

where the functions M(u), $M_n(u)$, N(u), $N_n(u)$ and the constants $\sigma_n, \sigma, \gamma_n(\tau), \gamma(\tau)$ are determined by (5) for the distributions $F_n(x)$ and F(x).

[21] Gnedenko, p.18.

5 THE LIMIT THEOREMS FOR SUMS OF INDEPENDENT VARIABLES

We shall be concerned in this section with the discussion of the more important theorems concerning the limit distributions for sums of independent random variables, and the relationships that exist between the latter and the class of infinitely divisible limit laws.

After the foundations laid by Kolmogorov, Khintchine, and Levy, two main areas of research lay open to investigators on the subject :

a.) We have given a double sequence X_{n1}, X_{n2}, . . . X_{nk_n} (n = 1 , 2 , . . .) of random variables which are independent in sequence for every choice of n. Our problem is to find those distributions which can occur as limit distributions for the sums $\mu_n = X_{n1} + X_{n2} + \dots + X_{nk_n} - A_n$, where $\{A_n\}$ is a suitably chosen sequence of constants. We shall consider this problem in the light of the requirement that the role of the individual summands occurring above becomes vanishingly small as $n \rightarrow \infty$.

This requirement may be paraphrased in a definition : our restriction is that the summands be infinitesimal, i.e., we require that for every $\epsilon > 0$,

$P(|X_{nk}| \geqslant \varepsilon) \rightarrow 0$, $(n \rightarrow \infty)$, uniformly in k for $1 \leqslant k \leqslant k_n$.

b.) We can step back from our first question above and consider conditions for the existence of such limiting distributions, and we can step forward to ask under what conditions we would have convergence of the distributions of our sums to particular limit laws. This latter question will not be considered here; it could form the basis of a new inquiry.

We have already introduced the concept of an infinitely divisible distribution. We shall now investigate the relationship between the class of such distributions and the class of limit distributions for sums of independent infinitesimal random variables. Indeed, after Khintchine, we shallsee that the two classes coincide.

5_1 DISTRIBUTIONS WITH FINITE VARIANCES

Two restrictions are imposed at the outset of this phase of the inquiry. First, we shall consider those sums whose individual summands are mutually independent and satisfy the conditions

a.) For any $\varepsilon > 0$, $P(|X_{nk} - E(X_{nk})| \geqslant \varepsilon) \rightarrow 0$, as $n \to \infty$, uniformly in k for $1 \leq k \leq k_n$, and

b.) The X_{nk} have finite variances, and

$D^2\left(\sum_{k=1}^{k_n} X_{nk}\right) = \sum_{k=1}^{k_n} D^2(X_{nk}) \leq C$, independent of n.

Second, we shall be interested in the case of convergence of the distributions of the sums, $\mathcal{U}_n$, defined above, where the variances of these distributions also converge to the variance of the limit law.

Variables which are such that for some sequence, $\{a_{nk}\}$, of constants, $P(|X_{nk} - a_{nk}| \geq \varepsilon) \to 0$, as $n \to \infty$, uniformly in k for $1 \leq k \leq k_n$, are said to be asymptotically constant. Note that in a.) above, we have $a_{nk} = E(X_{nk})$.

Theorem $5_1.1$ [22] : In order that for suitably chosen (this is perhaps ambiguous since for any given A_n the distribution functions of sums will converge if and only if th "accompanying" distribution functions involving the same A_n converge. We shall show later how to choose the A_n) constants A_n the distributions of the sums, $\mathcal{U}_n = X_{n1} + \ldots + X_{nk_n} - A_n$ of independent random variables satisfying a.) and b.) above, converge to a limiting distribution, it is necessary and sufficient that the infinitely divisible distributions for which the logarithms, $\psi_n(t)$, of the characteristic functions are determined by the formula

[22] Gnedenko and Kolmogorov, p. 98.

$$(1)\ \psi_{\mu}(t) = -iA_{\mu}t + \sum_{k=1}^{k_{\mu}} \left\{ it\,E(X_{\mu k}) + \int_{-\infty}^{\infty} (e^{itx} - 1)\, dF_{\mu k}(x + E(X_{\mu k})) \right\}$$

converge to a limiting distribution. The limiting distributions of both sequences coincide.

Corollary $5_1.1$: The limiting distributions for sums $\mathcal{M}_n = X_{\mu 1} + X_{\mu 2} + \ldots + X_{\mu k_\mu} - A_n$ of independent random variables satisfying conditions a.) and b.) are infinitely divisible.

Using the relation $G_n(u) = \sum_{k=1}^{k_\mu} \int_{-\infty}^{\mu} x^2 dF_{\mu k}(x + E(X_{\mu k}))$,

where for all n, $G_n(u)$ is non - decreasing, satisfies the condition $G_n(-\infty) = 0$, and by b.) is uniformly bounded, we see that the function $\psi_\mu(t)$ can be written as follows :

$$(2)\ \psi_{\mu}(t) = -iA_{\mu}t + it\sum_{k=1}^{k_{\mu}} E(X_{\mu k}) + \int_{-\infty}^{\infty} (e^{itx} - 1 - itx)\frac{1}{x^2}\, dG_{\mu}(x) .$$

We recall that we began with the problem of considering conditions not only for the convergence of certain distributions to a limiting distribution, but also for the convergence of their variances to the variance of the limit law. From the work just completed we remark that the variances of our sums are the same as the variances of the distributions determined by (1). Indeed, the variance of the distribution $F_n(x)$ equals

$$G_n(+\infty) = \sum_{k=1}^{k_\mu} \int_{-\infty}^{\infty} x^2 dF_{\mu k}(x + E(X_{\mu k})) = \sum_{k=1}^{k_\mu} D^2(X_{\mu k}) = D^2(\mathcal{M}_\mu) .$$

Theorem $5_1.2$ [23] : In order that for suitably chosen constants A_n the distributions of the sums, $\mathcal{U}_n$, defined above, of independent random variables satisfyimg a.) and b.) converge to a limiting distribution and that their variances converge to the variance of the limit law, it is necessary and sufficient that there exist a function $G(u)$ such that, as $n \to \infty$,

i.) $G_n(u) \to G(u)$ in the points of continuity of $G(u)$

ii.) $G_n(+\infty) \to G(+\infty)$

The constants A_n may be determined by the formula $A_n = \sum_{k=1}^{k_n} E(X_{nk}) - \gamma + o(1)$, where γ is an arbitrary constant. The logarithm of the characteristic function of the limit law is given by Kolmogorov's representation, using the function $G(u)$ and the constant γ.

As a result of the theorem we may conclude that in order that the distributions of the sums, $\mathcal{U}_n$, of independent random variables satisfying a.) and b.) converge to a limit distribution and that their variances converge to the variance of the limit law, it is necsesary and sufficient that we add to conditions i.) and ii.) above, a third condition, viz.,

iii.) $\sum_{k=1}^{k_n} \int_{-\infty}^{\infty} x\, dF_{nk}(x) \to \gamma$, $(n \to \infty)$.

[23] Gnedenko, p.31.

5_2 THE GENERAL FORM OF THE LIMIT THEOREMS

Turning now to the general problem of determining the limit laws for sums of independent random variables, we first reconsider the concept of an asymptotically constant random variable. We shall confine ourselves in what follows to the consideration of variables of this type. We bring in this restriction for the following reason : if we do not assume that the variables in any given row of our general configuration of a double sequence are identically distributed, the problem of determining all possible limit laws is meaningless. Our limit laws could be absolutely arbitrary since in the sum, $\mathcal{A}_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$, a single summand in any row could dominate the limiting procedure. We are requiring the condition of " asymptotic negligibility " of the variation of each individual summand in comparison with the B_n for the sum, $\mathcal{A}_n$.

Recalling that an asymptotically constant random variable, X_{nk} , is one for which it is possible to find a sequence of constants, $\{b_{nk}\}$, such that for any $\epsilon > 0$, $\sup_{1 \leq k \leq k_n} P(|X_{nk} - b_{nk}| > \epsilon) \rightarrow 0$, $(n \rightarrow \infty)$, we find that it is conveniently possible to let the

$b_{nk} = m_{nk}$ [24], where m_{nk} is the median of the variable X_{nk} ; i.e., m_{nk} is a number such that the following holds : $P(X_{nk} \geqslant m_{nk}) \geqslant \frac{1}{2}$, $P(X_{nk} \leqslant m_{nk}) \geqslant \frac{1}{2}$.

For the consideration of the general case wherein no assumptions are made regarding the variances of these distributions, we shall formulate the following theorem as a necessary step towards our main result:

Theorem $5_2.1$ [25] : In order that for suitably chosen constants A_n , the distributions of the sums, $\mathcal{N}_n$, defined above, of independent and infinitesimal random variables converge to a limiting distribution it is necessary and sufficient that the so - called " accompanying " distributions (infinitely divisible) for which the logs of of the characteristic functions are given by

$$\psi_n(t) = -iA_n t + \sum_{k=1}^{k_n} \left\{ i t \alpha_{nk} + \int_{-\infty}^{\infty} (e^{itx} - 1)\, dF_{nk}(x + \alpha_{nk}) \right.$$

where $\alpha_{nk} = \int_{|x|<\tau} x\, dF_{nk}(x)$ and τ is a positive constant, converge to a limiting distribution. The limiting distributions for both sequences coincide.

Remark : At the beginning of this thesis we considered the special case of a binomial variable $\mathcal{N}_n = X_1 + X_2 + . .$

[24] Gnedenko and Kolmogorov, p.95.

[25] Ibid., p.112.

$\ldots + X_n$, and we analyzed the results of two different limiting operations on the sum. In the Poisson case, where we let $n \to \infty$ and $p_n \to 0$, we see that since each individual summand is equal to 1 with probability p_n , and equal to 0 with probability $1 - p_n$, with our limit operation we are working with infinitesimal random variables, or variables whose individual roles become vanishingly small as $n \to \infty$. Also, in the case of the normal approximation we found that the sum, when normalized, yeilds individual summands $\frac{X_i}{\sqrt{n}\, B_n}$, where $B_n = (D^2(X_1) + \ldots + D^2(X_n))^{1/2}$, which again tend to 0 with probability equal to 1, Although here only n is varying — p is held constant. Again we see that the variables are infinitesimal. Hence, The remarks following will apply to these two special cases in particular.

Theorem 5_2.1 is of great utility, for it allows us to replace the investigation of our original sums, $\mathcal{U}_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$, of independent infinitesimal random variables, X_{nk} , with arbitrarily chosen limit laws, with the investigation of sums of infinitely divisible random variables whose limit laws are those determined by the functions, $\psi_n(t)$, of our theorem.

Our fundamental result is the following theorem by Khintchime, which follows from Theorem $5_2.1$ and the definitions given :

Theorem $5_2.2$ [26] : A necessary and sufficient condition that $F(x)$ be the limit law for the sums, $\mathcal{S}_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$ of infinitesimal random variables which are independent for every choice of n is that $F(x)$ be infinitely divisible.

Khintchine's theorem states, in effect, that the class of limit laws for sums $\mathcal{S}_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$ of independent infinitesimal random variables coincides with the class of infinitely divisible laws. The same statement can be made replacing " infinitesimal" by the expression " asymptotically constant " , since if the X_{nk} are asymptotically constant, then the variables $X_{nk} - m_{nk}$ are infinitesimal.

[26] Gnedenko and Kolmogorov, p.116.

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A COMPREHENSIVE ABSTRACT

As the introduction to this thesis has described it the significant content of the thesis is a consideration of the more important aspects of the theory of limiting distributions for the distributions associated with sequences of sums of independent random variables.

We begin our analysis with the discussion of the relatively common probability law, the binomial probability law. This is defined and related to two further probability laws : the normal law and the Poisson law. It is shown that in the binomial situation when the number, n, of trials approaches ∞, and the probability, p, of success at each trial approaches 0 in such a way that the variable $\lambda = np$ remains bounded, the Poisson approximation to the binomial is a uniform approximation. The DeMoivre - Laplace Limit theorem enables us to see the relation of the normal law to the binomial law. It states that the binomial distribution converges to the normal distribution in the situation wherein we are holding p constant and allowing $n \to \infty$. It is also noted that under favorable conditions the Poisson distribution is itself approximated by means of the

Normal distribution.

We remark for further reference that considerations in these two special cases are but the limiting operations applied to two particular essentially different manifestations of a general configuration in the form of a double sequence of random variables, independent for each choice of n, but not necessarily identically distributed.

The DeMoivre-Laplace Limit theorem is but a special case of the important Central Limit Theorem. We generalize and consider the limiting distribution for a sum of independent identically distributed random variables (normalized), not necessarily binomial. We find that the normal distribution is again the desired limit law. Many questions arise as to how one might change the hypotheses on the variables with what resultant changes in the conditions under which the Central Limit Theorem would hold. For identically distributed independent random variables, it was found that we can apply that theorem under the conditions of the very existence of the means and variances of the summands. As soon as we relax the requirement of identical distribution, more is required than that contained in a statement of existence. Lindeberg's condition is mentioned, and Lyapunov's proof is given.

The paper by Berry is alluded to at this point to enable us to give explicit numerical bounds for the least upper bound, M, of the absolute difference between F(x), the distribution function for our sum, and the distribution function for a normal variable with mean value A_n and with standard deviation B_n . We get such bounds on M under various conditions on the independent variables X_i and their moments, Corresponding to the conditions under which hold the Lindeberg, Lyapunov, and Feller versions of the Central Limit Theorem. We mention at this point the linking concept of an infinitely divisible distribution to provide ourselves with a solid basis from which we can work towards our most general results. We thread our way back to earlier discussions with the result that the normal and Poisson variables are members of the class of infinitely divisible random variables.

The characteristic function of an infinitely divisible distribution is of importance, but for our purposes we find the representations of the log of this characteristic function. The representations of Kolmogorov, Levy and Khintchine are introduced, and a theorem stating necessary and sufficient conditions that a given function be the characteristic function of an infinitely divisible law are given.

From these pursuits it is a natural step to the question of conditions under which these newly defined and

and recognizable infinitely divisible laws converge in sequence to a limit law.

We approach the more general problem of limit theorems for sums with two main avenues of approach. The restricted case wherein the following requirements must be met is considered first : We consider only those sums whose individual summands are mutually independent and asymptotically constant; Secondly, we require that the X_{nk}'s have finite variances.

We then relax these conditions, but consider only infinitesimal variables in the most general case to be considered. We make note of the fact that both of the cases dealt with earlier, the normal and the Poisson approximations to the binomial, yeild infinitesimal random variables. This means that our theorems and subsequent results regarding infinitesimal variables apply to these two particular cases.

Our main result is a theorem by Khintchine giving a necessary and sufficient condition that F(x) be the limit law for sums $u_n = X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n$ of independent infinitesimal random variables. The condition is that F(x) be infinitely divisible. We have found that the class of limit laws for such sums coincides with the class of infinitely divisible laws.