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The quadrature of the circle

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Thesis

THE QUADRATURE OF THE CIRCLE

Submitted by

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The Quadrature of the Circle

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  3 Its relation to the number π.

II Historical survey of the problem.

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PREFACE

The purpose of this thesis is to develop methods whereby approximate numerical values of a definite integral may be obtained when the analytic form of a function is unknown or where it is impossible or too difficult to integrate the function. Many functions of this type must be integrated, especially in the field of physics and engineering.

Each chapter is introduced by a paragraph explaining the purpose of the chapter. The formulas are then derived and examples given to illustrate the use of the formulas. In certain cases examples have been used which could be easily integrated. This has been done so that the approximate answer may be compared to the true value.

As it is important to know the reliability of the results obtained a chapter has been devoted to the accuracy of the integration formulas. In this chapter formulas for inherent error in the integration formulas have been derived and examples have been worked out to show the use of these formulas. The examples used are those from the chapter on integration formulas so that the inherent error in the answers to those examples may be shown.

The formulas have been developed in as simple a manner as possible, but as an understanding of their development depends on an understanding of the calculus of finite differences and of interpolation a chapter has been devoted to each as an introduction to the work on numerical integration. In these chapters only what is essential to the understanding of numerical integration has been given.

A small amount of the work is dependent upon infinite series, but as the development of this field would necessarily cover a large amount of material it has not been included in this thesis. References have been given which cover that field of work more completely.

The formulas have been numbered by the number of the article and the number of the formula in the article. For convenience the important formulas have been listed in the appendix.

A very useful table giving values needed in solving examples by Gauss' integration formula has also been included in the appendix.

There is much labor involved in solving many of the examples and references have given to tables which make the work simpler. A calculating machine should also be used, especially when using Gauss' integration formula.

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CHAPTER I

CALCULUS OF FINITE DIFFERENCES.

1. Introduction. The calculus of finite differences is the study of changes which take place in the value of a function when the independent variable suffers a finite change.

For example, if x is increased by a then the function x^3 increases to $(x + a)^3$ and the increment is $(x + a)^3 - x^3 = 3ax^2 + 3a^2x + a^3$.

It consists of two parts:

a. The direct calculus of finite differences where the function is given and the successive orders of differences are to be found.

b. The inverse calculus of finite differences where the successive orders of differences are given and the function from which these differences might have been derived is to be determined.

2. Difference Tables. Let $y = f(x)$. If $x_0 = a$, $x_1 = a + w$, $x_2 = a + 2w$, ..., then $y_0 = f(a)$, $y_1 = f(a + w)$, $y_2 = f(a + 2w)$, ... and so

$y_1 - y_0 = \Delta y_0$ or $f(a + w) - f(a) = \Delta f(a)$

$y_2 - y_1 = \Delta y_1$ or $f(a + 2w) - f(a + w) = \Delta f(a + w)$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$y_n - y_{n-1} = \Delta y_{n-1}$ or $f(a + nw) - f(a + \overline{n-1}w) = \Delta f(a + \overline{n-1}w)$

Then $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, ... $y_n - y_{n-1}$ are called the first differences of the function y and are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$.

The differences of the first differences are called second differences and are denoted by $\Delta^2 y_0$, $\Delta^2 y_1$, $\Delta^2 y_2$, ...

Thus $\Delta y_1 - \Delta y_0 = \Delta^2 y_0$

$$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta y_n - \Delta y_{n-1} = \Delta^2 y_{n-1}$$

In the same way the third, fourth, fifth, etc. differences may be given. These differences may be arranged either in a diagonal or horizontal difference table as follows:

DIAGONAL DIFFERENCE TABLE

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

HORIZONTAL DIFFERENCE TABLE

x	y	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$	$\Delta_4 y$	$\Delta_5 y$	$\Delta_6 y$
x_0	y_0						
x_1	y_1	$\Delta_1 y_1$					
x_2	y_2	$\Delta_1 y_2$	$\Delta_2 y_2$				
x_3	y_3	$\Delta_1 y_3$	$\Delta_2 y_3$	$\Delta_3 y_3$			
x_4	y_4	$\Delta_1 y_4$	$\Delta_2 y_4$	$\Delta_3 y_4$	$\Delta_4 y_4$		
x_5	y_5	$\Delta_1 y_5$	$\Delta_2 y_5$	$\Delta_3 y_5$	$\Delta_4 y_5$	$\Delta_5 y_5$	
x_6	y_6	$\Delta_1 y_6$	$\Delta_2 y_6$	$\Delta_3 y_6$	$\Delta_4 y_6$	$\Delta_5 y_6$	$\Delta_6 y_6$

The relation between the two tables is given by

$$\Delta^m y_k \text{ (diagonal table)} = \Delta_m y_{k+m} \text{ (horizontal table)}$$

e. g. $$\Delta^3 y_1 = \Delta_3 y_4$$

or $$\Delta_m y_n \text{ (horizontal table)} = \Delta^m y_{n-m} \text{ (diagonal table)}$$

e. g. $$\Delta_3 y_4 = \Delta^3 y_1$$

where m denotes the order of differences and k and n the number of the tabulated value going forward for y_k and backward for y_n.

The horizontal table is more compact than the diagonal table.

3. <u>Effect of an Error in Tabulated Values of Function in Difference Table.</u>

Often the tabulated values of a function are found by experiment and, therefore, may not be true values.

Let $y_0, y_1, \ldots, y_n$ be true values of a function and suppose that y_6 is affected by an error so that its value as found by experiment is $y_6 + \epsilon$. The effect of this error is found in the following table:

y	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$	$\Delta_4 y$	$\Delta_5 y$
y_0					
y_1	$\Delta_1 y_1$				
y_2	$\Delta_1 y_2$	$\Delta_2 y_2$			
y_3	$\Delta_1 y_3$	$\Delta_2 y_3$	$\Delta_3 y_3$		
y_4	$\Delta_1 y_4$	$\Delta_2 y_4$	$\Delta_3 y_4$	$\Delta_4 y_4$	
y_5	$\Delta_1 y_5$	$\Delta_2 y_5$	$\Delta_3 y_5$	$\Delta_4 y_5$	$\Delta_5 y_5$
$y_6 + \epsilon$	$\Delta_1 y_6 + \epsilon$	$\Delta_2 y_6 + \epsilon$	$\Delta_3 y_6 + \epsilon$	$\Delta_4 y_6 + \epsilon$	$\Delta_5 y_6 + \epsilon$
y_7	$\Delta_1 y_7 - \epsilon$	$\Delta_2 y_7 - 2\epsilon$	$\Delta_3 y_7 - 3\epsilon$	$\Delta_4 y_7 - 4\epsilon$	$\Delta_5 y_7 - 5\epsilon$
y_8	$\Delta_1 y_8$	$\Delta_2 y_8 + \epsilon$	$\Delta_3 y_8 + 3\epsilon$	$\Delta_4 y_8 + 6\epsilon$	$\Delta_5 y_8 + 10\epsilon$
y_9	$\Delta_1 y_9$	$\Delta_2 y_9$	$\Delta_3 y_9 - \epsilon$	$\Delta_4 y_9 - 4\epsilon$	$\Delta_5 y_9 - 10\epsilon$
y_{10}	$\Delta_1 y_{10}$	$\Delta_2 y_{10}$	$\Delta_3 y_{10}$	$\Delta_4 y_{10} + \epsilon$	$\Delta_5 y_{10} + 5\epsilon$
y_{11}	$\Delta_1 y_{11}$	$\Delta_2 y_{11}$	$\Delta_3 y_{11}$	$\Delta_4 y_{11}$	$\Delta_5 y_{11} - \epsilon$
y_{12}	$\Delta_1 y_{12}$	$\Delta_2 y_{12}$	$\Delta_3 y_{12}$	$\Delta_4 y_{12}$	$\Delta_5 y_{12}$

Thus it can be seen that :

a. The error increases as the successive differences increase.

b. The coefficients of ϵ's are binomial coefficients with alternating signs.

c. The algebraic sum of the difference of any order is in the same horizontal line as the error in the tabulated value in a horizontal difference table.

Because of the above facts it is possible to find the error, trace it back to its source, and to correct it. To do this the differencing is continued until a column is found where there is a complete alternation of signs. If the column is $\Delta^5 y$, for example, the binomial coefficients of the error will be 1, -5, 10, -10, 5, -1. The values in this column should be true differences plus these binomial coifficients multiplied by some number ϵ, which is the error. To locate the error in the function in a horizontal table trace backward along a horizontal line from the first error. We can check by tracing upward and backward from the last ϵ. The incorrect value of the function is the true value plus the error, so to correct it subtract the positive or negative error from the incorrect function.

Example. - - - - - - Moon's latitude.

Date 1943	Moon's Lat.	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$		
	° ' "	° ' "	' "	' "	' "	' "	ϵ is about 45"	
Feb. 16.0	-3 44 29.4							
16.5	-3 20 08.7	+0 24 20.7						
17.0	-2 53 38.7	+0 26 30.0	+2 09.3					
17.5	-2 25 14.0	+0 28 24.7	+1 54.7	-0 14.6				
18.0	-1 55 10.5	+0 30 03.5	+1 38.8	-0 15.9	-0 01.3			
18.5	-1 23 45.1	+0 31 25.4	+1 21.9	-0 16.9	-0 01.0	+0 00.3		
19.0	-0 52 01.1	+0 31 44.0	+0 18.6	-1 03.3	-0 46.4	-0 45.4	1.	$-\epsilon$
19.5	-0 18 03.0	+0 33 58.1	+2 14.1	+1 55.5	+2 58.8	+3 45.2	-5.	$-\epsilon$
20.0	+0 15 33.6	+0 33 36.6	-0 21.5	-2 35.6	-4 31.1	-7 29.9	10.	$-\epsilon$
20.5	+0 49 11.9	+0 33 38.3	+0 01.7	+0 23.2	+2 58.8	+7 29.9	-10.	$-\epsilon$
21.0	+1 22 29.3	+0 33 17.4	-0 20.9	-0 22.6	-0 45.8	-3 44.6	+5.	$-\epsilon$
21.5	+1 55 02.7	+0 32 33.4	-0 44.0	-0 23.1	-0 00.5	+0 45.3	-1.	$-\epsilon$

In this example the error is found in $\Delta^5 y$ for which order the binomial coefficients are 1, -5, 10, -10, 5, -1. The error is about -45". When the correction is applied to latitude for February 19.0 the corrected latitude will read $-0^\circ\ 51'\ 16''.1$

4. Representation of a Polynomial by Factorials and Differences. The polynomial of n^{th} degree, $x(x-1)(x-2)\ldots(x-n+1)$ may be written as $[x]^n$ and is called a factorial x taken to n factors. If the interval of the argument in the difference table of $[x]^n$ is taken as unity then

$$[x]^n = x(x-1)(x-2)\dots(x-n+1)$$

$$[x+1]^n = (x+1)(x)(x-1)(x-2)\dots(x-n+2)$$

$$[x+2]^n = (x+2)(x+1)(x)(x-1)\dots(x-n+3)$$

$$\Delta[x]^n = [x(x-1)\dots(x-n+2)][(x+1)-(x-n+1)]$$

$$\therefore \Delta[x]^n = n[x]^{n-1}$$

$$\Delta[x+1]^n = [x+2]^n - [x+1]^n = [(x+1)(x)(x-1)\dots(x-n+3)][(x+2)-(x-n+2)]$$

$$= n[x+1]^{n-1}$$

$$\Delta^2[x]^n = \Delta[x+1]^n - \Delta[x]^n = n[x+1]^{n-1} - n[x]^{n-1}$$

$$= [n(x+1)(x)(x-1)(x-2)\dots(x-n+3)] - [n(x)(x-1)\dots(x-n+2)]$$

$$= [n(x)(x-1)(x-2)\dots(x-n+3)][(x+1)-(x-n+2)]$$

$$= n(n-1)[x]^{n-2}$$

This method could be continued until

$$\Delta^n[x]^n = [n]^n[x]^{n-n} = [n]^n[x]^0$$

which is a constant.

Thus the first differences of the polynomials of n^{th} degree, $[x]^n$, $[x+1]^n$, ... are polynomials of $(n-1)$ degree and the n^{th} differences are constant, providing the independent variables are in arithmetic progression. (Case for $n \neq 1$.)

Theorem 1: Any polynomial may be represented by factorials and the differences found by the formula $\Delta[x]^n = n[x]^{n-1}$. The n^{th} differences will be constant.

Proof: If $F_p(x)$ represents a polynomial in x of degree p, then $F_p(x) = R+(x-n+p)F_{p-1}(x)$ where R is the remainder and $F_{p-1}(x)$ is the quotient when $F_p(x)$ is divided by $(x-n+p)$. $F_{p-1}(x)$ is a polynomial of degree $(p-1)$.

By repeating the above process a polynomial of n^{th} degree may be transformed into one expressed in terms of factorials.

Thus $\quad F_n(x) = \alpha + [x]F_{n-1}(x)$

$$= \alpha + \beta[x] + [x]^2F_{n-2}(x)$$

$$= \alpha + \beta[x] + \gamma[x]^2 + [x]^3F_{n-3}x$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= \alpha + \beta[x] + \gamma[x]^2 + \delta[x]^3 + \dots [x]^n F_0(x)$$

$\alpha, \beta, \gamma, \delta, \dots$ are constants as is also $F_0(x)$. Let $F_0(x)$ be k then

$$(4{:}1)\, F_n(x) = \alpha + \beta[x] + \gamma[x]^2 + \delta[x]^3 + \epsilon[x]^4 + \dots + k[x]^n$$

$$\Delta F_n(x) = \beta + 2\gamma[x] + 3\delta[x]^2 + 4\epsilon[x]^3 + \dots + nk[x]^{n-1}$$

$$\Delta^2 F_n(x) = 2\gamma + 6\delta[x] + 12\epsilon[x]^2 + \dots + n(n-1)k[x]^{n-2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Delta^n F_n(x) = [n]^n k[x]^0 \text{ which is a constant term.}$$

Thus any polynomial may be represented in terms of factorials, the successive differences found by $\Delta[x]^n = n[x]^{n-1}$ and the n^{th} differences will be constant.

Example. Find the differences, in terms of factorials, of

$$y = 15x^4 + 32x^3 - 57x^2 + 8x + 6$$

1	15	+ 32	- 57	+ 8	6
	0	+ 15	+ 47	-10	
2	15	+ 47	- 10	- 2	
	0	+ 30	+154		
3	15	+ 77	+144		
	0	+ 45			
	15	+122			

Therefore, in factorial notation $y = 15[x]^4 + 122[x]^3 + 144[x]^2 - 2[x] + 6$

$$\Delta y = 60[x]^3 + 366[x]^2 + 288[x]$$

$$\Delta^2 y = 180[x]^2 + 732[x] + 288$$

$$\Delta^3 y = 360[x] + 732$$

$$\Delta^4 y = 360$$

If a is a tabulated value of an argument of a polynomial of degree n and w is the interval between the successive values, then $f(a + xw)$ corresponds to the argument $(a + xw)$ and if $f(a + xw)$ stands for $F_n(x)$ in 4:1, then by applying $\Delta[x]^n = n[x]^{n-1}$ to both sides it is found that

(4:2) $\Delta f(a+xw) = \beta + 2\gamma[x] + 3\delta[x]^2 + 4\epsilon[x]^3 + \ldots + nk[x]^{n-1}$

(4:3) $\Delta^2 f(a+xw) = 2\gamma + 2.3\delta[x] + 3.4\epsilon[x]^2 + \ldots + n(n-1)k[x]^{n-2}$

(4:4) $\Delta^3 f(a+xw) = 2.3\delta + 2.3.4\epsilon[x] + \ldots + n(n-1)(n-2)k[x]^{n-3}$

To find the value of $\alpha, \beta, \gamma, \ldots$ let $x = 0$ in equations 4:1, 4:2 4:3, 4:4. Then

$$f(a) = \alpha$$

$$\Delta f(a) = \beta$$

$$\Delta^2 f(a) = 2\gamma \text{ or } \frac{\Delta^2 f(a)}{2} = \gamma$$

$$\Delta^3 f(a) = 2\cdot 3\delta \text{ or } \frac{\Delta^3 f(a)}{2\cdot 3} = \delta$$

Substituting in 4:1, the following formula is given:

$$(4:5)\; f(a+xw) = f(a) + \Delta f(a)[x] + \frac{\Delta^2 f(a)}{2!}[x]^2 + \frac{\Delta^3 f(a)}{3!}[x]^3 + \ldots + \frac{\Delta^n f(a)}{n!}[x]^n$$

Thus the polynomial $f(a+xw)$ may be expressed in terms of factorials and differences.

Theorem 2: If the n^{th} differences of a tabulated function are constant, when the values of the independent variable are taken in arithmetic progression, the function is a polynomial of degree n.*

5. Central Difference Notation. The following system was introduced by W. F. Sheppard. A symbol δ is used to represent $\Delta E^{-\frac{1}{2}}$ where $E^{\frac{1}{2}}$ is used to denote a change from u_n to $u_{n+\frac{1}{2}}$ so that $E^{\frac{1}{2}}u_n = u_{n+\frac{1}{2}}$

As $\delta = \Delta E^{-\frac{1}{2}}$ or $\Delta = \delta E^{\frac{1}{2}}$ we may write

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* The proof is found in Rice's Theory and Practice of Interpolation. p. 24.

$\Delta u_0 = \delta E^{\frac{1}{2}} u_0 = \delta u_{\frac{1}{2}} = u_1 - u_0$, $\Delta^2 u_0 = \delta^2 u_1 = \Delta u_1 - \Delta u_0$, $\Delta^3 u_0 = \delta^3 u_{3/2}$ ...,
$\Delta^n u_0 = \delta^n u_{n/2}$

Thus the difference table for $f(x) = y$ becomes

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$a-2w$	u_{-2}				
		$\delta u_{-3/2}$			
$a-w$	u_{-1}		$\delta^2 u_{-1}$		
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
a	u_0		$\delta^2 u_0$		$\delta^4 u_0$
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
$a+w$	u_1		$\delta^2 u_1$		$\delta^4 u_1$
		$\delta u_{3/2}$		$\delta^3 u_{3/2}$	
$a+2w$	u_2		$\delta^2 u_2$		$\delta^4 u_2$

The arithmetic mean of two adjacent entries in the same column of differences is represented by μ. Thus

$\mu\delta u_0 = \frac{1}{2}(\delta u_{-\frac{1}{2}} + \delta u_{\frac{1}{2}})$, $\mu\delta^3 u_0 = \frac{1}{2}(\delta^3 u_{-\frac{1}{2}} + \delta^3 u_{\frac{1}{2}})$, ...

The table is then written

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$a-2w$	u_{-2}				
	$\mu u_{-3/2}$	$\delta u_{-3/2}$			
$a-w$	u_{-1}	$\mu\delta u_{-1}$	$\delta^2 u_{-1}$		
	$\mu u_{-\frac{1}{2}}$	$\delta u_{-\frac{1}{2}}$	$\mu\delta^2 u_{-\frac{1}{2}}$	$\delta^3 u_{-\frac{1}{2}}$	
a	u_0	$\mu\delta u_0$	$\delta^2 u_0$	$\mu\delta^3 u_0$	$\delta^4 u_0$
	$\mu u_{\frac{1}{2}}$	$\delta u_{\frac{1}{2}}$	$\mu\delta^2 u_{\frac{1}{2}}$	$\delta^3 u_{\frac{1}{2}}$	$\mu\delta^4 u_{\frac{1}{2}}$
$a+w$	u_1	$\mu\delta u_1$	$\delta^2 u_1$	$\mu\delta^3 u_1$	$\delta^4 u_1$
	$\mu u_{3/2}$	$\delta u_{3/2}$	$\mu\delta^2 u_{3/2}$	$\delta^3 u_{3/2}$	$\mu\delta^4 u_{3/2}$
$a+2w$	u_2	$\mu\delta u_2$	$\delta^2 u_2$	$\mu\delta^3 u_2$	$\delta^4 u_2$

This notation is used to simplify the development of the interpolation formulas.

CHAPTER II

INTERPOLATION

6. Introduction. In simple form interpolation is the finding of an intermediate value of a function from a set of given values of that function.

In general, however, the problem is more complex because the analytical form of the function may be unknown or too complicated to handle and so must be replaced by a function easier to handle. The process of replacing a given function by a simpler one is known as interpolation.

7. Newton's Formulas. It has already been stated that any function may be written as a polynomial of degree n providing the n^{th} differences are constant. Now suppose the function $y = f(u)$ is given and assume that the n^{th} differences are constant. If u has the values a, $(a+w)$, $(a+2w)$, ... then $y = f(a)$, $f(a+w)$, $f(a+2w)$, ... The problem of interpolation is to find a value for the function $y = f(a+xw)$ where x is any part of the given interval. As, for example, if the problem is to find a value of the function for a value of u half way between $(a+w)$ and $(a+2w)$ the function would be $y = f(a+\frac{3}{2}w)$. Because the n^{th} differences are constant $f(a+xw)$ may be written as polynomial of degree n in the following form: 4:5 :

$$f(u) = f(a+xw) = f(a) + [x]\Delta f(a) + \frac{[x]^2}{2!}\Delta^2 f(a) + \dots \quad \text{or}$$

$$(7{:}1)\quad y = f(u) = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 + \dots$$

In central difference notation it may be written as

$$(7{:}2)\quad y = f(a+xw) = y_0 + [x]\delta y_{1/2} + \frac{[x]^2}{2!}\delta^2 y_1 + \frac{[x]^3}{3!}\delta^3 y_{3/2} + \frac{[x]^4}{4!}\delta^4 y_2 + \frac{[x]^5}{5!}\delta^5 y_{5/2} + \dots$$

This is known as Newton's interpolation formula

8. Central Difference Formulas.

a. Stirling's Interpolation Formula. By transforming Newton's forward interpolation formula a formula may be derived that uses differences which lie on a horizontal line. A formula of this type is called a central difference formula and is used for finding values of a function for a value of the argument near the middle of the tabulated sets of values. It is more rapidly convergent, and more accurate than Newton's formulas except for values near the beginning or end of the tabulated values.

$$\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}} = 2\mu\delta y_0$$
$$\delta y_{\frac{1}{2}} - \delta y_{-\frac{1}{2}} = \delta^2 y_0$$

By addition

$$2\delta y_{\frac{1}{2}} = 2\mu\delta y_0 + \delta^2 y_0 \quad \text{or} \quad \delta y_{\frac{1}{2}} = \mu\delta y_0 + \tfrac{1}{2}\delta^2 y_0$$
$$\delta^2 y_1 - \delta^2 y_0 = \delta^3 y_{\frac{1}{2}} = \mu\delta^3 y_0 + \tfrac{1}{2}\delta^4 y_0$$
$$\therefore \delta^2 y_1 = \delta^2 y_0 + \mu\delta^3 y_0 + \tfrac{1}{2}\delta^4 y_0$$
$$\delta^3 y_{3/2} = \delta^3 y_{\frac{1}{2}} + \delta^4 y_1\ ,\ \delta^3 y_{\frac{1}{2}} = \mu\delta^3 y_0 + \tfrac{1}{2}\delta^4 y_0\ ,\ \delta^4 y_1 = \delta^4 y_0 + \mu\delta^5 y_0 + \tfrac{1}{2}\delta^6 y_0$$
$$\therefore \delta^3 y_{3/2} = \mu\delta^3 y_0 + 3/2\,\delta^4 y_0 + \mu\delta^5 y_0 + \tfrac{1}{2}\delta^6 y_0$$

$\delta^5 y_{5/2}$ ... can be found in a similar way so that Newton's formula becomes

$$(8:1)\quad f(a+xw) = y_0 + x\left[\mu\delta y_0 + \tfrac{1}{2}\delta^2 y_0\right] + \frac{[x]^2}{2!}\left[\delta^2 y_0 + \mu\delta^3 y_0 + \tfrac{1}{2}\delta^4 y_0\right]$$
$$+ \frac{[x]^3}{3!}\left[\mu\delta^3 y_0 + 3/2\,\delta^4 y_0 + \mu\delta^5 y_0 + \tfrac{1}{2}\delta^6 y_0\right] + \frac{[x]^4}{4!}\left[\delta^4 y_0 + 2\mu\delta^5 y_0 + 2\delta^6 y_0 + \mu\delta^7 y_0 + \tfrac{1}{2}\delta^8 y_0\right] + \frac{[x]^5}{5!}\left[\mu\delta y_0 \cdots\right.$$
$$= y_0 + x\mu\delta y_0 + \frac{x^2}{2!}\delta^2 y_0 + \frac{x(x^2-1)}{3!}\mu\delta^3 y_0 + \frac{x^2(x^2-1)}{4!}\delta^4 y_0 + \frac{x(x^2-1)(x^2-4)}{5!}\mu\delta^5 y_0 + \cdots$$

This is known as <u>Stirling's interpolation formula.</u>

As
$$\mu\delta y_0 = \frac{\Delta y_{-1} + \Delta y_0}{2}$$
$$\delta^2 y_0 = \Delta^2 y_{-1}$$
$$\mu\delta^3 y_0 = \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}$$
$$\vdots \qquad \vdots$$

the formula may be written without central difference notation as

$$(8:2)\quad y = f(a+xw) = y_0 + \frac{x(\Delta y_{-1} + \Delta y_0)}{2} + \frac{x^2}{2!}\Delta^2 y_{-1}$$

$$+ \frac{x(x^2-1)}{3!}\cdot\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{x^2(x^2-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{x(x^2-1)(x^2-4)}{5!}\cdot\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} + \cdots +$$

$$\frac{x(x^2-1)(x^2-4)x^2-9)\cdots(x^2-\overline{n-1}^2)}{(2n-1)!}\cdot\frac{\Delta^{2n-1}y_{-n} + \Delta^{2n-1}y_{-(n-1)}}{2}$$

$$+ \frac{x^2(x^2-1)(x^2-4)(x^2-9)\cdots[x^2-(n-1)^2]}{2n!}\Delta^{2n}y_{-n}$$

This formula assumes that the equal intervals are unity. If not, then u is used in place of x where $u = \frac{x-x_0}{w}$ and w denotes the equal increments between x_0 and x_1 , x_1 and x_2 , ...

In this formula the arithmetic means of the odd differences immediately above and below the horizontal line through y_0 and the even differences on this line are used.

b. Bessel's Interpolation Formula. Bessel's formula uses differences which occur on a horizontal line midway between the lines on which y_0 and y_1 stand.

Since

$$y_1 + y_0 = 2\mu y_{\frac{1}{2}}$$
$$y_1 - y_0 = \delta y_{\frac{1}{2}}$$
$$y_0 = \mu y_{\frac{1}{2}} - \tfrac{1}{2}\delta y_{\frac{1}{2}}$$

$$\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}} = 2\mu\delta y_0$$
$$\delta y_{\frac{1}{2}} - \delta y_{-\frac{1}{2}} = \delta^2 y_0$$
$$\delta y_{\frac{1}{2}} = \mu\delta y_0 + \tfrac{1}{2}\delta^2 y_0$$

and $\delta^2 y_0 = \mu\delta^2 y_{\frac{1}{2}} - \frac{1}{2}\delta^3 y_{\frac{1}{2}}$

so that $\mu\delta y_0 = \delta y_{\frac{1}{2}} - \frac{1}{2}\mu\delta^2 y_{\frac{1}{2}} + \frac{1}{4}\delta^3 y_{\frac{1}{2}}$

$$\mu\delta^3 y_0 = \delta^3 y_{\frac{1}{2}} - \tfrac{1}{2}\delta^4 y_0$$
$$\delta^4 y_0 = \mu\delta^4 y_{\frac{1}{2}} - \tfrac{1}{2}\delta^5 y_{\frac{1}{2}}$$

Then, by substituting these values in Stirling's formula (8:1), the following formula is obtained:

$$f(x) = (\mu y_{\frac{1}{2}} - \tfrac{1}{2}\delta y_{\frac{1}{2}}) + x(\delta y_{\frac{1}{2}} - \tfrac{1}{2}\mu\delta^2 y_{\frac{1}{2}} + \tfrac{1}{4}\delta^3 y_{\frac{1}{2}}) + \frac{x^2}{2!}(\mu\delta^2 y_{\frac{1}{2}} - \tfrac{1}{2}\delta^3 y_{\frac{1}{2}})$$
$$+ \frac{x(x^2-1)}{3!}(\delta^3 y_{\frac{1}{2}} - \tfrac{1}{2}\mu\delta^4 y_{\frac{1}{2}} + \tfrac{1}{4}\delta^5 y_{\frac{1}{2}})$$
$$+ \frac{x^2(x^2-1)}{4!}(\mu\delta^4 y_{\frac{1}{2}} - \tfrac{1}{2}\delta^5 y_{\frac{1}{2}}) + \frac{x(x^2-1)(x^2-4)}{5!}(\delta^5 y_{\frac{1}{2}} - \tfrac{1}{2}\mu\delta^6 y_{\frac{1}{2}} + \tfrac{1}{4}\delta^7 y_{\frac{1}{2}}) +$$

(8:3) or $$f(x) = \mu y_{\frac{1}{2}} + (x-\tfrac{1}{2})\delta y_{\frac{1}{2}} + \frac{x(x-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{x(x-1)(x-\frac{1}{2})}{3!}\delta^3 y_{\frac{1}{2}}$$
$$+ \frac{x(x-1)(x+1)(x-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \frac{x(x-1)(x+1)(x-2)(x-\frac{1}{2})}{5!}\delta^5 y_{\frac{1}{2}} + \cdots$$

This is known as <u>Bessel's formula</u>. It may also be written as

(8:4) $$y = \frac{y_0 + y_1}{2} + (x-\tfrac{1}{2})\Delta y_0 + \frac{x(x-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{x(x-1)(x-\frac{1}{2})}{3!}\Delta^3 y_{-1}$$
$$+ \frac{x(x-1)(x+1)(x-2)}{4!}\cdot\frac{\Delta^4 y_{-1} + \Delta^4 y_{-1}}{2} + \frac{x(x-1)(x+1)(x-2)(x-\frac{1}{2})}{5!}\Delta^5 y_{-2}$$
$$+\cdots+ \frac{x(x-1)(x+1)(x-2)(x+2)\cdots(x-n)(x+n-1)}{(2n)!}\cdot\frac{\Delta^{2n}y_{-n} + \Delta^{2n}y_{-n+1}}{}$$
$$+ \frac{(x-\frac{1}{2})(x)(x-1)(x+1)(x-2)(x+2)\cdots(x-n)(x+n-1)}{(2n+1)!}\Delta^{2n+1}y_{-n}$$

A more convenient formula is given by putting $x-\frac{1}{2} = v$ or $x = v+\frac{1}{2}$. Then the formula becomes

(8:5) $$y = \frac{y_0 + y_1}{2} + v\Delta y_0 + \frac{(v^2 - \frac{1}{4})}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{v(v^2 - \frac{1}{4})}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(v^2-\frac{1}{4})(v^2-\frac{9}{4})}{4!}\;\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{v(v^2-\frac{1}{4})(v^2-\frac{9}{4})}{5!}\Delta^5 y_{-2} + \cdots$$

c. <u>Remainder Term in Bessel's Formula</u>. Let the arbitrary function

(8:6) $$F_{(z)} = f_{(z)} - \phi_{(z)} - \left[F_{(x)} - \phi_{(x)}\right]\frac{(z-x_0)(z-x_1)(z-x_{-1})\cdots(z-x_n)(z-x_{-n})(z-x_{n+1})}{(x-x_0)(x-x_1)(x-x_{-1})\cdots(x-x_n)(x-x_{-n})(x-x_{n+1})}$$

be taken where $f(x)$ is the given function, $\phi(x)$ a polynomial interpolation formula, and z a real variable. Assume that the function $f(x)$ is continuous and has continuous derivatives and that $f(z)$ and $F(z)$ will also be continuous and have continuous derivatives. $F(z)$ will vanish at $2n+3$ points $z = x, x_0, x_1, x_{-1}, \ldots x_n, x_{-n}, x_{n+1}$. Thus $F'(z)$ will vanish $2n+2$ times, $F''(z)$ will vanish $2n+1$ times, and the $(2n+2)$th derivative of $F(z)$ will vanish at least once at some point for $z = \xi$. $\phi(z)$ corresponding to y in (8:4) is a polynomial of $(2n+1)$ degree so that its $(2n+2)$th derivative will be zero. $(z-x_0)(z-x_1)(z-x_{-1})\ldots(z-x_n)(z-x_{-n})(z-x_{n+1})$ is a polynomial of $(2n+2)$ degree and so its $(2n+2)$th derivative is the same as the $(2n+2)$th derivative of z^{2n+2} which is $(2n+2)!$

Thus if (8:6) is differentiated $2n+2$ times with respect to z and $F^{2n+2}(z) = 0$ is taken for some value of $z = \xi$ then

$$0 = f^{2n+2}(\xi) - 0 - \left[f(x) - \phi(x)\right]\frac{(2n+2)!}{(x-x_0)(x-x_1)(x-x_{-1})\ldots(x-x_n)(x-x_{-n})(x-x_{n+1})}$$

Thus if R_n equals the error then

$$R_n = f(x) - \phi(x) = \frac{f^{2n+2}(\xi)}{(2n+2)!}(x-x_0)(x-x_1)(x-x_{-1})\ldots(x-x_n)(x-x_{-n})(x-x_{n+1})$$

If $x-x_0 = hu$, $x-x_1 = h(u-1)$, $x-x_{-1} = h(u+1), \ldots$

then $R_n = \dfrac{h^{2n+2}f^{2n+2}(\xi)}{(2n+2)!}u(u-1)(u+1)(u-2)\ldots(u-n)(u+n)(u-n-1)$

and if $u = v+\frac{1}{2}$ then

$$(8:7) \quad R_n = \frac{h^{2n+2}f^{(2n+2)}(\xi)}{(2n+2)!}\left(v^2-\tfrac{1}{4}\right)\left(v^2-9/4\right)\ldots\left[v^2-\frac{(2n+1)^2}{4}\right]$$

These are the remainder terms in formula (8:5)

9. Lagrange's Interpolation Formula. This is a formula which may be used when the intervals are or are not equal.

The problem is to find a polynomial of n^{th} degree at most which has the same values as an arbitrary function $f(x)$ for $n+1$ values of x. If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are the $n+1$ pairs of corresponding values then the function may be written as a polynomial in the form

$$(9{:}1)\quad f(x) = A_0(x-x_1)(x-x_2)\dots(x-x_n)$$
$$+ A_1(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n)$$
$$+ A_2(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$+ A_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$$(9{:}2)\quad \text{or}\quad f(x) = \sum_{k=0}^{n} A_k \frac{F(x)}{x-x_k} \quad \text{where}$$

$$F(x) = (x-x_0)(x-x_1)\dots(x-x_n)$$

The constants may be determined by putting $x = x_k$ and all the terms on the right hand side will vanish except the term containing A_k. Formula 9:1 will then read

$$f(x_k) = A_k(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)$$

or

$$A_k = \frac{f(x_k)}{(x_k-x_0)(x_k-x_1)\ \dots\ (x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$$

which may be written as $A_k = \dfrac{f(x_k)}{F'(x_k)}$

Thus

$$(9{:}3)\quad f(x) = \sum_{k=0}^{n} \frac{F(x)}{(x-x_k)\,F'(x_k)}\, f(x_k)$$

This is known as Lagrange's formula.

CHAPTER III

NUMERICAL INTEGRATION

10. Introduction. Numerical integration is the process of finding the value of the definite integrand $\int_a^b ydx$ in the following cases:

a. When the function is analytically unknown and only a set of numerical values of the integrand are given. This would be true where graphical records have been made from observations and tests or diagrams have been drawn by an engine, as for example the pressure diagram of a steam engine. It is usually required to find the area under the curve between two extreme ordinates and the base line.

b. When the function is analytically known but cannot be integrated.

The solution may be found by representing the integrand y by one of the interpolation formulas and then integrating between the required limits a and b.

11. Integration of Newton's Formula. An integration formula may be developed from Newton's Interpolation formula. If n+1 values of the function $y = f(x)$ are given which correspond to the values of the argument x_0, x_1, x_2, ..., x_n then the function may be expressed as a polynomial of degree n in the form of Newton's interpolation formula (7:1) as follows:

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n y_0$$

where $u = \frac{x-x_0}{w}$ and $w = x_1 - x_0$. The original intervals are equal but not necessarily unity.

The area bounded by the x-axis and the parabola of n^{th} degree

$$y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots + \frac{u(u-1)\cdots(u-n+1)}{n!}\Delta^n y_0$$

where there are n equal intervals of width w has for the limits of integration, x_0 and $x_0 + nw$. Thus the corresponding limits for u are 0 and n.

Now $x = x_0 + wu$, so that $dx = wdu$

Therefore $\int_{x_0}^{x_0+nw} ydx = \int_0^n ywdu = w\int_0^n ydu$

$$= w\int_0^n (y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 \cdots)du$$

$$= w\left[uy_0+\frac{u^2}{2}\Delta y_0+(\frac{u^3}{3}-\frac{u^2}{2})\frac{\Delta^2 y_0}{2!} + (\frac{u^4}{4} - u^3+u^2)\frac{\Delta^3 y_0}{3!} + (\frac{u^5}{5} - \frac{3u^4}{2} + \frac{11u^3}{3} - 3u^2)\frac{\Delta^4 y_0}{4!} + \cdots \right]_0^n$$

(11:1)
$$= w\left[ny_0 + \frac{n^2}{2}\Delta y_0 + (\frac{n^3}{3} - \frac{n^2}{2})\frac{\Delta^2 y_0}{2!} + (\frac{n^4}{4} - n^3 + n^2)\frac{\Delta^3 y_0}{3!} + (\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2)\frac{\Delta^4 y_0}{4!} + \cdots\right]$$

From this general formula (11:1) are derived the following rules.

a. The Trapezoidal Rule. If $n = 1$ is put in the general formula (11:1) and only first differences used, then the following is found for the first interval:*

$$\int_{x_0}^{x_0+w} ydx = w\left[y_0+\frac{\Delta y_0}{2}\right] = w\left[y_0+\frac{y_1-y_0}{2}\right] = w\left[\frac{y_0+y_1}{2}\right] \text{ or } \frac{w}{2}\left[y_0+y_1\right]$$

For the second interval

$$\int_{x_0+w}^{x_0+2w} ydx = w\left[y_1 + \frac{\Delta y_1}{2}\right] = \frac{w}{2}\left[y_1+y_2\right]$$

This procedure may be continued until for the n^{th} interval

$$\int_{x_0+\overline{n-1}w}^{x_0+nw} ydx = \frac{w}{2}\left[y_{n-1}+y_n\right]$$

The area between the limits x_0 to x_0+nw is the sum of these intervals or

(11:2) $$\int_{x_0}^{x_0+nw} ydx = \frac{w}{2}(y_0+2y_1+2y_2+2y_3+\cdots+2y_{n-1}+y_n)$$

*As only two ordinates, y_0 and y_1, are used there can be no differences higher than the first.

This is called the trapezoidal rule and is useful only when y_0, y_1 ... are near together and if a great degree of accuracy is not necessary.

Geometrically, the area under the curve is determined by dividing this area into equal strips by ordinates. Each strip is assumed to be a trapezoid and the total approximate area is given as the sum of the areas of the trapezoids.

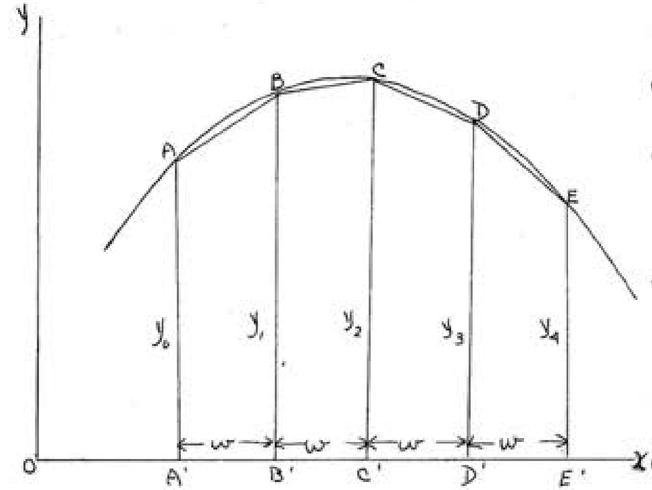

Figure 1. Trapezoidal Rule.

Let AE be the curve of the equation $y = f(x)$ AA', BB', ... are perpendicular to the x-axis. The area AEE'A' under the curve AE is represented by the sum of the trapezoids ABB'A', BCC'B', CDD'C', and DEE'D'. The area of trapezoid ABB'A' is given by $w\left(\frac{y_0+y_1}{2}\right)$; of trapezoid BCC'B' by $w\left(\frac{y_1+y_2}{2}\right)$, etc. so that the total area is $\frac{w}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$ as represented by formula (11:2).

b. Simpson's One-Third Rule. If $n = 2$ is taken and only first and second differences used then

$$\int_{x_0}^{x_0+2w} y\,dx = w\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3}-2\right)\frac{\Delta^2 y_0}{2!}\right]$$

$$= w\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{w}{3}(y_0 + 4y_1 + y_2)$$

$\int_{x_0+2w}^{x_0+4w} y\,dx, \ldots, \int_{x_0+\overline{n-2}w}^{x_0+nw} y\,dx$ can be found in the same way and then the sum of these intervals taken so that

$$\int_{x_0}^{x_0+nw} ydx = \frac{w}{3}(y_0+4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

$$(11:3) \qquad = \frac{w}{3}\left[y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n\right]$$

This is known as Simpson's one-third rule. In this rule there must be an even number of sub-intervals of width w. *

If the area were represented geometrically by this rule the graph of the given function would be replaced by $\frac{n}{2}$ arcs of second degree parabolas. This is more accurate than the trapezoidal rule.

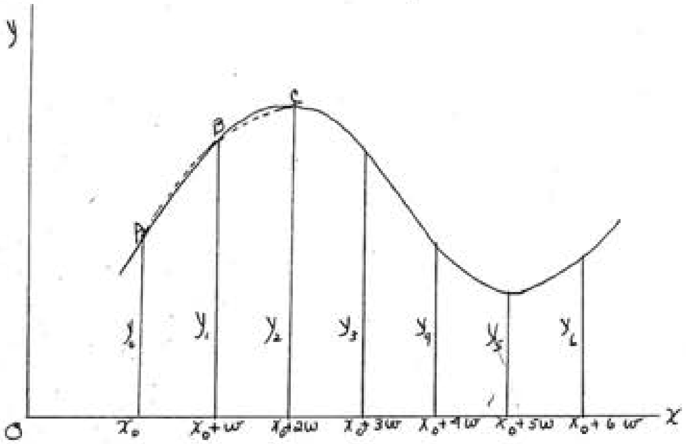

Figure 2. Simpson's One-Third Rule.

The dotted line ABC represents the arc of a second degree parabola. The area under this curve is given by $\frac{1}{3}(y_0+4y_1+y_2)w$. It approximates the area under the heavy line curve ABC. The whole area is found by taking the sum of these $\frac{n}{2}$ areas under the arcs of the second degree parabolas.

c. Weddle's Rule. In this rule if n = 6 is taken and only the first six differences used then

$$\int_{x_0}^{x_0+6w} ydx = w\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right].$$

As $\frac{41}{140}$ is only $\frac{1}{140}$ less than $\frac{3}{10}$ and if w is small so that $\Delta^6 y$ is small then the error caused by changing $\frac{41}{140}\Delta^6 y_0$ to $\frac{3}{10}\Delta^6 y_0$ is unimportant.

*Simpson's three-eighths has been omitted because it is less simple and less accurate than Simpson's one-third rule.

Then $\int_{x_0}^{x_0+6w} ydx = \frac{3}{10} w \Big[20y_0 + 60(y_1-y_0) + 90(y_2-2y_1+y_0) + 80(y_3-3y_2+3y_1-y_0)$

$+ 41(y_4-4y_3+6y_2-4y_1+y_0) + 11(y_5-5y_4+10y_3-10y_2+5y_1-y_0)$

$+ (y_6-6y_5+15y_4-20y_3+15y_2-6y_1+y_0)$

$= \frac{3w}{10}\Big[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\Big]$

If all expressions such as $\int_{x_0}^{x_0+6w} ydx, \int_{x_0+6w}^{x_0+12w} ydx, \ldots, \int_{x_0+\overline{n-6}w}^{x_0+nw} ydx$, where x is a multiple of 6, are added then

$$(11{:}4)\quad \int_{x_0}^{x_0+nw} ydx = \frac{3w}{10}\Big[y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+2y_{12} \cdots + 2y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n\Big]$$

This is known as Weddle's rule.

Examples.

1. The velocity of a train in miles per hour which starts from rest is given by the following table:

t in min.	2	4	6	8	10	12	14	16	18	20	22	24
m.p.h.	7	15	21	27	31	32	30	26	20	15	8	at rest

Estimate the total distance run in 24 minutes.

a. Solution by trapezoidal rule.

$w = \frac{1}{30}$

$D = \int_0^{24} ydt = \frac{1}{60}(14+30+42+54+62+64+60+52+40+30+16)$

$= \frac{1}{60}(464) \doteq$ 7.73 mi.

b. Solution by Simpson's one-third rule.

$w = \frac{1}{30}$

$D = \int_0^{24} ydt = \frac{1}{90}\Big[4(7+21+31+30+20+8)+2(15+27+32+26+15)\Big]$

$= \frac{1}{90}(698) =$ 7.76 mi.

Summary

This problem is particularly interesting in that we have the completed history of a problem which persisted through nearly 4,000 years. The development of this problem may be likened to advances made in other branches of human achievement. There are a few ideas contributed by a small number of great men. Each advance is generally followed by a period in which there are comparatively small improvements and detailed developments of the new ideas by men of lesser rank.

The history of the problem falls into three distinct periods which are marked by differences in methods, immediate aims, and equipment of mathematical knowledge. The first period is the geometric one in which the ideal of an exact construction was always kept in mind. Approximate determinations of the value of π were obtained by calculation of the sides or areas of regular polygons inscribed or circumscribed to the circle. By the end of the period this method was so highly developed that no further progress could be hoped for along the lines of the Greek mathematicians.

In the second period the older geometric forms of investigation gave way to the application of powerful analytic expressions for π which were in the form of convergent series and products. Many calculators of π arose who applied the formula to obtain numerical approximations to the value of π. An important contribution to the solution of the problem was made at the end of the period when the relation between π and e was discovered.

The third and last period extended from the middle of the eighteenth century to the end of the nineteenth century, and was devoted to investigations as to the real nature of π. During this time the transcendence of π was proved, and therefore the impossibility of "squaring the circle", in accordance with Euclidean restrictions, was definitely established.

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