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Model reduction for multi-scale partial differential equations

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MODEL REDUCTION FOR MULTI-SCALE PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this thesis, a new method is developed for model reduction in multi-scale systems of coupled reaction-diffusion equations. The new method uses scaling variables which are naturally suggested by the classical diffusion operator. The main differential operators have a spectral gap and an infinite basis of natural modes associated to the point spectrum. The first principal results consist of establishing the existence of nonlinear slow modes for reaction-diffusion systems and of rigorously studying the algebraic and exponential decay of general solutions toward them. The new method exploits separations of time scales between slow and fast species. It is illustrated on two prototypical examples, the Davis-Skodje model and the Michaelis-Menten-Henri (MMH) model with diffusion of both species. The solutions of the Davis-Skodje and MMH models are decomposed completely into slow parts, which we label as the nonlinear slow mode, and fast parts which exhibit short term algebraic decay and long term exponential decay. The second principal result consists of introducing a new class of multi-scale reaction-diffusion equations that possess closed-form, low-dimensional, invariant manifolds. This new class of PDEs is of interest in its own right, since there are currently very few examples of PDEs known to have manifolds

expressed in closed form. Also, it provides a useful set of benchmark problems for testing and comparing numerical methods for model reduction in nonlinear PDEs. The third principal result consists of a mathematical analysis of the Approximate Slow Invariant Manifold (ASIM) method developed by Powers and Paolucci. The accuracy of the ASIM method is proven for systems of reaction-diffusion equations with slow and fast reaction kinetics.

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List of Abbreviations

ASIM	....	Approximate Slow Invariant Manifold
MMH	....	Michaelis-Menten-Henri
ODE	....	Ordinary Differential Equation
PDE	....	Partial Differential Equation
R-D	....	Reaction-Diffusion

Chapter 1

Introduction

1.1 Overview of Model Reduction for Ordinary Differential Equations

Complex chemical reactions and reactive flows occur in many problems in atmospheric science, biology, chemistry, combustion science, enzyme kinetics, and other areas of science and engineering. Methods for understanding and quantifying complex reactions include a broad class known as model reduction methods, such as the Quasi-Steady State Approximation (A. Goeke, 2015; Palsson, 2011; Stiefenhofer, 1998; Segel and Slemrod, 1989; Turányi et al., 1993), Principal Component Analysis (Vajda et al., 1985), the Intrinsic Low-Dimensional Manifold (ILDM) Method (Maas and Pope, 1992; Maas and Pope, 1994), analytical singular perturbation methods (Heineken et al., 1967; Li et al., 1993; Segel and Slemrod, 1989), the Computational Singular Perturbation (CSP) Method (Kaper et al., 2015; S.H. Lam, ; Lam and Goussis, 1994), the Functional Iteration Method (Roussel and Fraser, 1990), the Straightening out of Fibers Method (Uldall Kristiansen et al., 2014), the Computational Algebra Method (Romanovski et al., 2013; Romanovski and Shafer, 2009), the Method of Invariant Manifolds (Gorban and Karlin, 2003; Gorban et al., 2004), the Zero-Derivative Principle (Benoît et al., 2015; Gear et al., 2005; Hardin et al., 2009; Zagaris et al., 2012), and entropy-based methods (Gorban and Karlin, 1992; Hangos, 2010; Lebiedz, 2010). In the spaces of the species concentrations, these methods and others have been designed to locate and approximate the lower-dimensional slow invariant manifolds to which the fast kinetics rapidly relax and which govern the long-term system dynamics.

The ILDM Method is a commonly-used and computationally-efficient method based on linear projections determined by the slow and fast subspaces of the Jacobian of the reaction kinetics. It is exact for systems with linear slow manifolds, and the error between ILDMs and exact slow invariant manifolds has been shown to be proportional to the curvature of the slow manifold, see (Davis and Skodje, 1999; Kaper and Kaper, 2002). The Functional Iteration Method, the CSP Method, and the Straightening out of Fibers Method are iterative methods in which each step of the iteration generates a successively more accurate approximation to the slow invariant manifolds in systems of chemical kinetics, as shown in (Kaper and Kaper, 2002; Kaper et al., 2015). They incorporate the effects of curvature; and, while computationally more involved than the ILDM Method, they have proven to be useful for highly-accurate model reduction in many applications. Moreover, in practice, some combination of methods is often used, with the ILDM Method being used to derive an initial approximation of the exact slow invariant manifold and then some type of iteration is used to improve the estimate.

1.2 Overview of Model Reduction in Systems of Partial Differential Equations

More recently, a number of methods have been developed to achieve model reduction in systems of reactive flow problems,

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}(\boldsymbol{u}) - \frac{\partial \boldsymbol{h}(\boldsymbol{u}, \boldsymbol{u}_x)}{\partial x}. \tag{1.2.1}$$

Here, x and t denote the independent variables for space and time; $\boldsymbol{u} = \boldsymbol{u}(x,t)$ denotes the species concentrations and $\boldsymbol{u}$ is a map from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}^{m+n}$, where m and n are positive integers; $\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u}(x,t))$ is the reaction term, where the components of $\boldsymbol{F}$ are smooth functions, and $\boldsymbol{h}(\boldsymbol{u}, \boldsymbol{u}_x)$ is the convective and diffusive flux vector, where $\boldsymbol{u}_x$ is the partial derivative on x. For system (1.2.1), the chemical kinetics are coupled to active and passive transport processes, such as convection and diffusion. These PDE reduction methods include extensions of the ILDM Method (Bykov and Maas, 2007; Maas, 1995; Maas, 2001; Maas and Pope, 1994), operator splitting methods (Singh et al., 2001; Yang and Pope, 1998), extensions of the CSP Method to include diffusion (Hadjinicolaou and Goussis, 1998), an inertial manifold method (Mengers and Powers, 2013; Yannacopoulos et al., 1995), a geometric approach to study a hierarchy of lower-dimensional manifolds during the approach to equilibrium (Davis, 2006a; Davis, 2006b), and the Approximate Slow Invariant Manifold (ASIM) Method (Singh et al., 2002; Singh et al., 2001), among others.

In the operator splitting approach, at each time step, one alternately solves the ODE for the chemical kinetics, $\partial \boldsymbol{u}/\partial t = \boldsymbol{F}(\boldsymbol{u})$, finding an approximation of the exact slow invariant manifold (for example with the ILDM Method), and then integrates that data forward using the transport terms, $\partial \boldsymbol{u}/\partial t = -\partial \boldsymbol{h}/\partial x$. This is based on the general idea of Strang for the solution of general PDEs, and can be designed to be second-order accurate using the appropriate fractional time steps in the integration. Note, however, that the number of PDEs to be solved with this approach is the same as in the full system.

The inertial manifold method has also been applied for model reduction in reaction-diffusion systems, see (Yannacopoulos et al., 1995). This approach selects inertial manifolds as the approximations of the exact slow manifolds, and it explicitly demonstrates that the reduced dynamics are typically infinite-dimensional, even in the case when only the slow species diffuse. As shown in (Yannacopoulos et al., 1995) for example for the Michaelis-Menten-Henri mechanism in the presence of diffusion, the reduced equation is an R-D equation with a density-dependent diffusivity. A more-recent development involving inertial manifolds centers on finding one-dimensional slow invariant manifolds for reaction-diffusion equations (Mengers and Powers, 2013). This method employs Galerkin projection to obtain a finite-dimensional approximate inertial manifold, and then it employs the saddle-connection method of (Davis and Skodje, 1999) to find a one-dimensional heteroclinic orbit which connects a saddle point to the equilibrium point in the phase space of this system of ODEs.

To date, these methods for model reduction in nonlinear PDEs have primarily been developed for numerical implementation. The application of the inertial manifold method (Yannacopoulos et al., 1995) provided some theory of the Michaelis-Menten-Henri (MMH) example. The analysis of the ASIM method is presented in Chapter 7 of this thesis, see also (Wu and Kaper, 2017).

1.3 Summary of Dissertation

This dissertation is divided into six chapters. We begin with introducing a set of scaling variables for multi-scale reaction-diffusion PDEs, the corresponding linear operators, and a weighted Sobolev space that the operators act on. The main differential operators have a spectral gap and an infinite basis of natural modes associated to the point spectrum. The spectral stability of the solutions will be determined by the isolated eigenvalues of the operators.

In Chapter 2, we establish the existence of the N-th order nonlinear slow modes expressed as an asymptotic expansion in the powers of ϵ for the Davis-Skodje model with diffusion of both the slow and fast species. We rigorously study the algebraic and exponential decay of general solutions toward them. Numerical solutions are also presented to illustrate the slow modes and the decay toward them.

In Chapter 3, the slow modes of the Davis-Skodje model are expressed into basis functions with regard to the weighted Sobolev space $L^2(m)$. We discuss the projection onto the neutral mode and the algebraic decay in the Davis-Skodje model. The equation of the slow species is linear and decoupled from the fast species.

Chapter 4 is parallel to Chapter 2. The Michaelis-Menten-Henri (MMH) model with diffusion of both the slow and fast species is a coupled PDE system with mixed nonlinear reaction terms. We push the dependency of the fast species on the slow species to higher order and reduce the model to a simpler system where the slow species is governed by a decoupled nonlinear PDE. In Chapter 4, we establish the existence of the N-th order nonlinear slow modes for the reduced MMH model, and we rigorously study the algebraic and exponential decay of general solutions toward them. They give an approximate solution to the original MMH PDE.

Chapter 5 introduces a new class of multi-scale reaction-diffusion systems that possess closed-form, low-dimensional, invariant manifolds. These manifolds are given in closed form by the graphs of elementary functions. The manifolds are exponentially attracting in the normal directions to the manifold for all positive values of the diffusivities, and they govern the long-term system dynamics. For general solutions in the basins of attraction of the manifolds, these invariant manifolds and the exponential attraction provide a geometric decomposition into a low-dimensional slow component along the invariant manifolds and an infinite-dimensional component consisting of an invariant family of fast stable fibers along which the fast exponential decay takes place. In turn, this geometric decomposition enables a quantitative study of how the evolution toward the low-dimensional manifolds depends on the magnitudes of the diffusivities. Furthermore, this new class of multi-scale reaction-diffusion equations provides a useful set of benchmark problems for testing and comparing methods for model reduction in nonlinear PDEs.

Finally, the last chapter (Ch.6) presents a mathematical analysis of the Approximate Slow Invariant Manifold (ASIM) method developed by Powers and Paolucci. Beginning with systems of two species (one slow and one fast), and then treating general systems with multiple slow and fast species, we explicitly determine the accuracy of the Approximate Slow Invariant Manifold method. We find that it is correct up to and including the terms of first order in the small parameter that measures the separation of the kinetics time scales, and that it captures many of the terms at second order, as well. Our analysis includes precise statements of the errors at second order, and we find that these are proportional to the slow components of the reaction-diffusion equation, as well as to the curvature of the critical manifold. We illustrate the results analytically on the MMH model with diffusion of the slow species and the Davis-Skodje model in which both the slow and fast species diffuse.

Chapter 2

Nonlinear Slow Modes of the Davis-Skodje Model

In this chapter, we begin by using a set of scaling variables which were presented in (Wayne, 2008) for multi-scale reaction-diffusion PDEs, the corresponding linear operators, and a weighted Sobolev space that the operators act on. We present the analytical solution of the Davis-Skodje Model as a finite series plus rapidly decaying remainders, and we illustrate the solution numerically.

2.1 Davis-Skodje Model in Scaling Variables

We start with the Davis-Skodje model on the fast time scale, with diffusion of both the slow and fast species,

$$\begin{aligned} \frac{\partial y}{\partial t} &= -\epsilon y + \epsilon d_1 \frac{\partial^2 y}{\partial x^2} \\ \frac{\partial z}{\partial t} &= -z + \frac{y}{1+y} - \epsilon \frac{y}{(1+y)^2} + \epsilon d_2 \frac{\partial^2 z}{\partial x^2}. \end{aligned} \tag{2.1.1}$$

Here, $x \in \mathbb{R}$ and $t \geq 0$. The variables $y = y(x,t)$ and $z = z(x,t)$ denote non-negative species concentrations. The kinetics terms are those in the Davis-Skodje mechanism (Davis and Skodje, 1999). The parameter ϵ satisfies $0 < \epsilon \ll 1$. It measures the separation of the two time scales. The diffusivities are $\mathcal{O}(\epsilon)$ on both slow and fast species, with diffusion coefficients $d_1, d_2 \geq 0$.

We use the scaling variables used in (Wayne, 2008),

$$\xi = \frac{x}{\sqrt{1+t}} \quad \text{and} \quad \tau = \log(1+t). \tag{2.1.2}$$

These variables are inspired by the form of the fundamental solution of the heat equation, where it is natural to study the system in terms of $\frac{x}{\sqrt{t}}$. Here, ξ is defined as $\xi = \frac{x}{\sqrt{1+t}}$ to avoid the singularity at $t = 0$. From the form of the fundamental solution of the heat equation, it is also natural to introduce the dependent variables

$$y(x,t) = e^{-\frac{\tau}{2}}\tilde{y}(\xi,\tau) \quad \text{and} \quad z(x,t) = e^{-\frac{\tau}{2}}\tilde{z}(\xi,\tau). \tag{2.1.3}$$

$\eta = e^{-\frac{\tau}{2}}$ is introduced to make the system autonomous. In terms of these scaling variables, equation (2.1.1) becomes

$$\begin{aligned}
\frac{\partial \tilde{y}}{\partial \tau} &= \mathcal{L}_1\tilde{y} - \epsilon\eta^{-2}\tilde{y} \\
\frac{\partial \tilde{z}}{\partial \tau} &= \mathcal{L}_2\tilde{z} + \eta^{-2}\left(-\tilde{z} + \frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right) \\
\frac{\partial \eta}{\partial \tau} &= -\frac{\eta}{2},
\end{aligned} \tag{2.1.4}$$

where the linear operators are $\mathcal{L}_i \cdot = \epsilon d_i \partial_\xi^2 \cdot + \frac{1}{2}\partial_\xi(\xi\cdot)$, $i = 1, 2$. Compared to the Laplacian, $\frac{\partial^2}{\partial x^2}$, whose spectrum is the negative real line, this form of the equation has the advantage that a gap in the spectrum appears which separates the slowly decaying modes from the more rapidly decaying ones and allows us to do model reduction. $\mathcal{L}_1$ and $\mathcal{L}_2$ are inspired by the linear operator $\mathcal{L}$ (see (Wayne, 2008)) that converts the heat equation $u_t = u_{xx}$ into $\tilde{u}_\tau = \mathcal{L}\tilde{u}$. In these scaling variables, the solutions of the heat equation tend toward a Gaussian profile as t tends toward infinity. Also, in quantum mechanics, the operator $-(\phi^0)^{-1/2}\mathcal{L}(\phi^0)^{1/2}$ is the Hamiltonian of the harmonic oscillator up to numerical constants.

For all $m > \frac{1}{2}$, we define the weighted Sobolev space

$$L^2(m) = \left\{ f \in L^2(\mathbb{R}) | \, \|f\|_m < \infty \right\},$$
$$\|f\|_m = \left(\int_{\mathbb{R}} \left(1 + \left| \frac{\xi}{\sqrt{\epsilon}} \right|^m \right)^2 (f(\xi))^2 d\xi \right)^{1/2}, \tag{2.1.5}$$
$$H^s(m) = \{\partial^\alpha f \in L^2(m) | \text{ for all } \alpha \leq s\}.$$

Let the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ act on their maximal domain in $L^2(m)$. Then, $\mathcal{L}_1$ and $\mathcal{L}_2$ have point spectrum $\sigma = \{\lambda_k = -\frac{k}{2}, k = 0, 1, 2, \ldots\}$, and essential spectrum $\sigma_{\text{ess}} = \left\{ \lambda \in \mathbb{C} | \text{Re}(\lambda) \leq \frac{1}{4} - \frac{\text{m}}{2} \right\}$. The modes corresponding to eigenvalues $\lambda = -\frac{k}{2}, k = 1, 2, \ldots$, decay at a rate of $e^{-\frac{k}{2}\tau}$. The modes lying in σ_{ess} all decay at least with a rate of $e^{\left(\frac{1}{4} - \frac{m}{2}\right)\tau}$. Moreover, if $K \in \mathbb{N}$ satisfies $K + \frac{1}{2} < m$, then $\lambda_k = -\frac{k}{2}, k = 0, 1, 2, \ldots, K$ are isolated eigenvalues of $\mathcal{L}$. In particular, we can always push the essential spectrum far away from the imaginary axis by taking m sufficiently large. Thus, the spectral stability of the solutions will be determined by the isolated eigenvalues of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $L^2(m)$.

Each eigenvalue λ_k has an associated eigenfunction

$$\begin{aligned} \phi^k(\xi) &= \frac{k!}{2^k} \frac{1}{\sqrt{4\pi\epsilon d_1}} H_k \left(\frac{\xi}{\sqrt{\epsilon d_1}} \right) e^{-\frac{\xi^2}{4\epsilon d_1}} \\ \psi^k(\xi) &= \frac{k!}{2^k} \frac{1}{\sqrt{4\pi\epsilon d_2}} H_k \left(\frac{\xi}{\sqrt{\epsilon d_2}} \right) e^{-\frac{\xi^2}{4\epsilon d_2}} \end{aligned} \tag{2.1.6}$$

for the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The corresponding eigenfunctions have a Gaussian decay at infinity. Here, $H_k(\xi)$ denotes the Hermite polynomial of degree k,

$$H_k(\xi) = \frac{2^k}{k!} e^{\frac{\xi^2}{4}} \partial_\xi^k \left(e^{-\frac{\xi^2}{4}} \right), k = 0, 1, 2 \ldots. \tag{2.1.7}$$

The components of these eigenvectors satisfy the following orthogonality properties for any m and n:

$$\int_{\mathbb{R}} \phi^m(\xi) H_n\left(\frac{\xi}{\sqrt{\epsilon d_1}}\right) d\xi = \delta_{mn}, \quad \int_{\mathbb{R}} \psi^m(\xi) H_n\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right) d\xi = \delta_{mn}. \tag{2.1.8}$$

Then, based on these orthogonality properties, it is natural to define the projection operator $P_n : L^2(m) \times L^2(m) \to L^2(m) \times L^2(m)$ by

$$P_n \begin{pmatrix} f \\ g \end{pmatrix} (\xi) = \sum_{k=0}^{n} \begin{pmatrix} \left(\int_{\mathbb{R}} H_k\left(\frac{\xi'}{\sqrt{\epsilon d_1}}\right) f(\xi') d\xi'\right) \phi^k(\xi) \\ \left(\int_{\mathbb{R}} H_k\left(\frac{\xi'}{\sqrt{\epsilon d_2}}\right) g(\xi') d\xi'\right) \psi^k(\xi) \end{pmatrix}. \tag{2.1.9}$$

Also, the complementary projection operator is defined as $Q_n = \mathbf{1} - P_n$:

$$Q_n \begin{pmatrix} f \\ g \end{pmatrix} (\xi) = \begin{pmatrix} f - \sum_{k=0}^{n} \left(\int_{\mathbb{R}} H_k\left(\frac{\xi'}{\sqrt{\epsilon d_1}}\right) f(\xi') d\xi'\right) \phi^k(\xi) \\ g - \sum_{k=0}^{n} \left(\int_{\mathbb{R}} H_k\left(\frac{\xi'}{\sqrt{\epsilon d_2}}\right) g(\xi') d\xi'\right) \psi^k(\xi) \end{pmatrix}. \tag{2.1.10}$$

We will use $P_{n,1}$, $Q_{n,1}$, $P_{n,2}$, and $Q_{n,2}$ in Chapter 3 to denote the first and second components of P_n and Q_n, respectively.

2.2 Analytical Solution with Nonlinear Slow Mode and Remainder

In this section, we define the N-th nonlinear slow mode of the Davis-Skodje model (2.1.1) as an asymptotic expansion in ϵ and express the solution as a finite series with remainders.

First, for any $0 \leq N < \infty$, the N-th nonlinear slow mode is defined to be

$$z_{\text{slow},N}(x, t; \epsilon) = \sum_{j=0}^{N} \epsilon^j \zeta_j(x, t), \tag{2.2.1}$$

with

$$\begin{aligned}\zeta_0 =& \frac{y}{1+y},\\ \zeta_1 =& d_2 \frac{\partial^2}{\partial x^2}\zeta_0(x,t) - \frac{1}{\epsilon}\frac{\partial}{\partial t}\zeta_0(x,t) - \frac{y}{(1+y)^2} = \frac{(d_2-d_1)y_{xx}}{(1+y)^2} - \frac{2d_2 y_x^2}{(1+y)^3},\end{aligned} \tag{2.2.2}$$

and

$$\zeta_j(x,t) = d_2 \frac{\partial^2}{\partial x^2}\zeta_{j-1}(x,t) - \frac{1}{\epsilon}\frac{\partial}{\partial t}\zeta_{j-1}(x,t), \text{ for all } j \geq 2. \tag{2.2.3}$$

We will use the modal decomposition of the two components $\tilde{y}$ and $\tilde{z}$ naturally given by the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ in the scaling variables in (2.1.4) to prove the following theorem for the Davis-Skodje Model (2.1.1).

Theorem 2.2.1. *Fix $m \geq \frac{9}{2}$, $3 \leq M \leq m - \frac{3}{2}$, and $\epsilon_0 > 0$ sufficiently small. Then, there exists $1 \leq N_0(M) < \infty$, such that for all $0 \leq N < N_0(M)$ and $0 < \epsilon < \epsilon_0$, the solution of the Davis-Skodje model with general initial data $y_0(x;\epsilon)$ and $z_0(x;\epsilon)$ in the weighted Sobolev space $H^{2N_0}(m)$ may be written as*

$$\begin{aligned} y(x,t;\epsilon) &= e^{-\epsilon t}\frac{e^{-\frac{x^2}{4\epsilon d_1(1+t)}}}{\sqrt{4\pi\epsilon d_1(1+t)}}\sum_{j=0}^{M}\frac{j!}{2^j}y^j(0)(1+t)^{-\frac{j}{2}}H_j\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right) + \mathcal{R}_{M,1}(x,t;\epsilon)\\ z(x,t;\epsilon) &= z_{slow,N}(x,t;\epsilon) + e^{-t}w_N(x,t;\epsilon),\end{aligned} \tag{2.2.4}$$

where $z_{slow,N}(x,t;\epsilon)$ is the N-th nonlinear slow mode (2.2.1). In addition, for any $0 \leq N < N_0(M)$, $w_N(x,t;\epsilon)$ in (2.2.4)(b) is given by

$$\begin{aligned} w_N(x,t;\epsilon) =& \frac{e^{-\frac{x^2}{4\epsilon d_2(1+t)}}}{\sqrt{4\pi\epsilon d_2(1+t)}}\sum_{j=0}^{M}\frac{j!}{2^j}z^j(0)(1+t)^{-\frac{j}{2}}H_j\left(\frac{x}{\sqrt{\epsilon d_2(1+t)}}\right) + \mathcal{R}_{M,2}(x,t;\epsilon)\\ &- \int_{\mathbb{R}} G(x-x',t)\left(\sum_{j=0}^{N}\epsilon^j\zeta_j(x',0)\right)dx' + \mathcal{R}_N(x,t;\epsilon),\end{aligned} \tag{2.2.5}$$

where $G(x,t)$ is the Green's function $G(x,t) = \frac{1}{\sqrt{4\pi\epsilon d_2 t}}e^{-\frac{x^2}{4\epsilon d_2 t}}$. $\mathcal{R}_{M,1}$ and $\mathcal{R}_{M,2}$ are the *fast decaying remainders,*

$$\begin{aligned} ||\mathcal{R}_{M,1}|| &\leq C_1 e^{-\epsilon t}(1+t)^{-\frac{M}{2}+1}||\mathcal{R}_{M,1}(x,0)|| \\ ||\mathcal{R}_{M,2}|| &\leq C_2 (1+t)^{-\frac{M}{2}+1}||\mathcal{R}_{M,2}(x,0)||, \end{aligned} \tag{2.2.6}$$

and $\mathcal{R}_N$ *is uniformly* $\mathcal{O}(\epsilon^{N+1})$.

By Theorem 2.2.1, the y component of the solution is governed by the first equation in (2.2.4), which we label as (2.2.4)(a). It decays slowly in time as $e^{-\epsilon t}$. The z component of the solution is governed by the second equation in (2.2.4), which we label as (2.2.4)(b). We observe that the z component is decomposed completely into a slow part $z_{\text{slow},N}(x,t;\epsilon)$ given by (2.2.1), which we label as the N-th nonlinear slow mode, and the fast parts given by (2.2.5). The terms in w_N are uniformly bounded in space, and the terms in $e^{-t}w_N$ decay at least as fast as e^{-t}. Hence, the solutions decay exponentially toward the N-th nonlinear slow mode.

It is useful to verify that if one evaluates formula (2.2.4)(b) for $z(x,t;\epsilon)$ at $t=0$, one recovers the initial data $z_0(x;\epsilon)$. This follows by observing that the Green's function acts as a δ-function,

$$\lim_{\sigma\to 0^+} \int_{\mathbb{R}} G(x-x',\sigma)\left(\sum_{j=0}^{N} \epsilon^j \zeta_j(x',0)\right) dx' = \sum_{j=0}^{N} \epsilon^j \zeta_j(x,0).$$

Also, we observe that the initial data $z_0(x;\epsilon)$ lies on the N-th nonlinear slow mode if and only if $w_N(x,0;\epsilon)=0$. Moreover, we note that requiring the initial data to lie in H^{2N_0} insures that it has sufficiently many derivatives, though the diffusion operator may naturally smoothen some rougher data.

Finally, while the central idea underlying the proof is to use the modal decomposition in terms of the Gaussians and Hermite functions naturally induced by the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ in scaling variables, we will use each of the representations (2.1.1) and (2.1.4) at different stages of the proof, since there are advantages to both representations.

Proof of Theorem 2.2.1.

The equation (2.1.1)(a) for y decouples. For $\tilde{y}(\xi,\tau)$ in the scaling variables ξ and τ, we rewrite (2.1.4)(a) as

$$\tilde{y}_\tau + \epsilon e^\tau \tilde{y} = \mathcal{L}_1 \tilde{y}.$$

In the weighted norm space $L^2(m)$, we fix $m \geq \frac{9}{2}$. Then, the linear operator $\mathcal{L}_1$ has $M+1$ isolated eigenvalues $0, -\frac{1}{2}, \ldots, -\frac{M}{2}$ for $M \in \mathbb{N}$, $3 \leq M \leq m - \frac{3}{2}$. Using the modal decomposition, we find that $\tilde{y}(\xi,\tau)$ is given by

$$\tilde{y}(\xi,\tau) = e^{\epsilon(1-e^\tau)} \sum_{j=0}^{M} \tilde{y}^j(0) e^{-\frac{j}{2}\tau} \phi^j(\xi) + \tilde{\mathcal{R}}_{M,1}(x,t;\epsilon), \tag{2.2.7}$$

where the remainder has estimate

$$\left\| \tilde{\mathcal{R}}_{M,1} \right\|_m \leq C_1 e^{-\frac{M+1}{2}\tau} e^{\epsilon(1-e^\tau)} \left\| \tilde{\mathcal{R}}^0_{M,1} \right\|_m .$$

The proof is similar to the analysis of remainders (3.2.20) and (3.5.20) in sections 3.2 and 3.5, and those proofs were developed first.

Transforming back to the original variables, we find

$$y(x,t) = e^{-\epsilon t} \sum_{j=0}^{M} \frac{j!}{2^j} \frac{y^j(0)}{\sqrt{4\pi\epsilon d_1}} (1+t)^{-\frac{j+1}{2}} H_j\left(\frac{x}{\sqrt{\epsilon d_1 (1+t)}} \right) e^{-\frac{x^2}{4\epsilon d_1(1+t)}} + \mathcal{R}_{M,1}(x,t;\epsilon). \tag{2.2.8}$$

This establishes (2.2.4)(a) in the Theorem for the $y(x,t)$ component of the general solution of (2.1.1).

Before presenting the analysis for the $\tilde{z}$ component of (2.1.4), we use the solution (2.2.8) for $y(x,t)$ to establish the following Lemma about the functions $\zeta_j(x,t)$ given in (2.2.2) and (2.2.3):

Lemma 2.2.2. *Fix any $m \geq \frac{9}{2}$ and $3 \leq M \leq m - \frac{3}{2}$, there exists $1 \leq N_0(M) < \infty$ such that for each $j = 0, 1, 2, \ldots, N_0(M)$, the function $\zeta_j(x,t)$ is uniformly $\mathcal{O}(1)$ for all x and t.*

Proof of Lemma 2.2.2 Fix any $m \geq \frac{9}{2}$, and let $3 \leq M \leq m - \frac{3}{2}$. We will prove that the functions $\zeta_j(x,t)$ are uniformly $\mathcal{O}(1)$ up to some finite j and for all x and t.

We work with the decomposition of y (2.2.8) as the M-th partial sum and the remainder $\mathcal{R}_{M,1}$. To begin, we observe that by choosing the initial data in the weighted Sobolev space $H^{2N_0}(m)$, the spatial derivatives of the remainder term $\mathcal{R}_{M,1}(x,t;\epsilon)$ in (2.2.8) satisfy

$$\partial_t(\partial_x^n \mathcal{R}_{M,1}) = -\epsilon \partial_x^n \mathcal{R}_{M,1} + \epsilon d_1 \partial_{xx}(\partial_x^n \mathcal{R}_{M,1}) \quad \text{for any } n \leq 2N_0(M).$$

They are uniformly bounded with estimates

$$||\partial_x^n \mathcal{R}_{M,1}|| \leq Ce^{-\epsilon t}(1+t)^{-\frac{M}{2}+1}||\partial_x^n \mathcal{R}_{M,1}(x,0)||,$$

where C is a constant independent of ϵ, and $\partial_x^n \mathcal{R}_{M,1}(x,0)$ is in the weighted norm space $L^2(m)$. This is in the similar fashion as the estimate of $\mathcal{R}_{M,1}$. Hence, in the rest of this proof, we work mainly on the M-th partial sum to bound terms such as $\frac{y}{1+y}$, $\frac{y_x^2}{(1+y)^3}$, $\frac{y_{xx}}{(1+y)^2}$, etc.

Note that $\phi^j\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right)$ is $\mathcal{O}(\epsilon^{-(j+1)/2})$, $\phi_x^j\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right)$ is $\mathcal{O}(\epsilon^{-(j+3)/2})$, and $\phi_{xx}^j\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right)$ is $\mathcal{O}\left(\epsilon^{-(j+5)/2}\right)$. In this manner, given M, we have $\phi^M\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right)$

is $\mathcal{O}(\epsilon^{-(M+1)/2})$ and $\frac{\partial^n \phi^M}{\partial x^n}\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right)$ is $\mathcal{O}\left(\epsilon^{-\frac{M+1}{2}-n}\right)$ for any $n \in \mathbb{N}$.

One may verify that $\zeta_0 = \frac{y}{1+y}$ is uniformly $\mathcal{O}(1)$ for any M. This follows from the representation $y(x,t;\epsilon) = e^{-\epsilon t}\sum_{j=0}^{M} y^j(0)(1+t)^{-\frac{j}{2}}\phi^j\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right) + \mathcal{R}_{M,1}$ and the fact that $\phi^M\left(\frac{x}{\sqrt{\epsilon d_1(1+t)}}\right)$ is $\mathcal{O}(\epsilon^{-(M+1)/2})$ and $\mathcal{R}_{M,1}$ is uniformly bounded. ζ_1 has terms $\frac{y_x^2}{(1+y)^3}$ and $\frac{y_{xx}}{(1+y)^2}$, which are $\mathcal{O}\left(\epsilon^{\frac{M+1}{2}-2}\right)$. Hence, they are uniformly bounded when $\frac{M+1}{2} \geq 2$. For $j \geq 2$, one can also calculate $\left[d_2\frac{\partial^2}{\partial x^2} - \frac{1}{\epsilon}\frac{\partial}{\partial t}\right]^{j-1}\zeta_1$ to verify that ζ_j is $\mathcal{O}(\epsilon^{\frac{M+1}{2}-2j})$. Hence, by letting $N_0(M)$ be the largest integer such that for each $j = 0,1,2,\ldots,N_0(M)$, $2j \leq \frac{M+1}{2}$, we have ζ_j is uniformly bounded. This completes the proof of Lemma 2.2.2. □

Remark. If one works with $N = 0$, i.e. one has $z_{\text{slow},0}$, then it suffices to take $m = \frac{9}{2}$, so that $M = 3$ and hence $N_0(3) = 1$, where we recall that one needs $2N_0(M) \leq \frac{M+1}{2}$. If one wants to work instead with $N = 1$, i.e. one has $z_{\text{slow},1}$, then it suffices to take $m = \frac{17}{2}$, so that $M = 7$ and hence $N_0(7) = 2$. We note that in chapter 3, the requirements on m will be simpler.

With this Lemma in hand, we now turn to analyze the $\tilde{z}$ component of (2.1.4), which may be rewritten as

$$\tilde{z}_\tau + e^\tau \tilde{z} = \mathcal{L}_2\tilde{z} + e^\tau\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right).$$

Let

$$\tilde{v}(\xi,\tau) = \tilde{z}(\xi,\tau)e^{e^\tau - 1}. \tag{2.2.9}$$

Then, $\tilde{v}$ satisfies the inhomogeneous heat equation

$$\tilde{v}_\tau = \mathcal{L}_2 \tilde{v} + e^{e^\tau - 1 + \tau} \left(\frac{\tilde{y}}{1 + \eta \tilde{y}} - \epsilon \frac{\tilde{y}}{(1 + \eta \tilde{y})^2} \right). \tag{2.2.10}$$

Moreover, the initial condition is $\tilde{v}(\xi, 0) = \tilde{z}(\xi, 0)$.

We use Duhamel's Principle to solve (2.2.10) by separating $\tilde{v}$ into two parts,

$$\tilde{v} = \tilde{\nu} + \tilde{\omega}, \tag{2.2.11}$$

where $\tilde{\nu}$ and $\tilde{\omega}$ satisfy

$$\tilde{\nu}_\tau = \mathcal{L}_2 \tilde{\nu} + e^{e^\tau - 1 + \tau} \left(\frac{\tilde{y}}{1 + \eta \tilde{y}} - \epsilon \frac{\tilde{y}}{(1 + \eta \tilde{y})^2} \right), \qquad \tilde{\nu}(\xi, 0) = 0, \tag{2.2.12}$$

$$\tilde{\omega}_\tau = \mathcal{L}_2 \tilde{\omega}, \qquad \tilde{\omega}(\xi, 0) = \tilde{v}(\xi, 0). \tag{2.2.13}$$

We will show that $\tilde{\nu}$ consists of the slow varying part of $\tilde{v}$ and a fast decaying part of $\tilde{v}$, and $\tilde{\omega}$ gives the remaining fast decaying part of $\tilde{v}$.

First, we observe that the solution of the equation for $\tilde{\omega}$ may be written as

$$\tilde{\omega}(\xi, \tau) = \sum_{j=0}^{M} \tilde{\omega}^j(0) e^{-\frac{j}{2}\tau} \psi^j(\xi) + \tilde{\mathcal{R}}_{M,2}(x, t; \epsilon), \tag{2.2.14}$$

where

$$\left\| \tilde{\mathcal{R}}_{M,2} \right\|_m \leq C_1 e^{-\frac{M+1}{2}\tau} \left\| \tilde{\mathcal{R}}^0_{M,2} \right\|_m .$$

Again, the proof is similar to the analysis of remainders (3.2.20) and (3.5.20) in sections 3.2 and 3.5. Translating back to (x, t) variables and letting $\omega = \eta \tilde{\omega}$, we have

$$\omega(x, t) = \sum_{j=0}^{M} z^j(0)(1+t)^{-\frac{j+1}{2}} \psi^j \left(\frac{x}{\sqrt{1+t}} \right) + \mathcal{R}_{M,2}(x, t; \epsilon), \tag{2.2.15}$$

with $\omega(x,0) = z(x,0)$. This yields the first two terms in (2.2.5).

Next, for $\tilde{\nu}$, we let $\nu = \eta\tilde{\nu}$. In the original (x,t) variables, (2.2.12) is transformed into

$$\nu_t = \epsilon d_2 \nu_{xx} + e^t \left(\frac{y}{1+y} - \epsilon \frac{y}{(1+y)^2} \right), \qquad \nu(x,0) = 0. \tag{2.2.16}$$

This equation can be solved exactly as

$$\nu(x,t) = \int_0^t \int_{\mathbb{R}} G(x-x', t-t') e^{t'} \left(\frac{y}{1+y} - \epsilon \frac{y}{(1+y)^2} \right) dx' dt', \tag{2.2.17}$$

where $y = y(x',t')$ and G denotes the Green's function

$$G(x,t) = \frac{1}{\sqrt{4\pi\epsilon d_2 t}} e^{-\frac{x^2}{4\epsilon d_2 t}},$$

which is the solution of the heat equation $G_t = \epsilon d_2 G_{xx}$.

The expression $\frac{y(x,t)}{1+y(x,t)}$ occurs throughout the analysis. Hence, (recall (2.2.2)) we use the following quantity

$$\zeta_0(x,t) = \frac{y(x,t)}{1+y(x,t)}.$$

We evaluate the solution $\nu(x,t)$ in (2.2.17) explicitly as an expansion in ϵ using a sequence of integration by parts steps in t'. In each of these steps, "dv" is $e^{t'} dt'$.

Changing the order of integration (which is permitted due to the exponential decay in space), we begin with the first step of integration by parts on the first term in the integrand in (2.2.17),

$$
\begin{aligned}
&\int\limits_0^t G(x-x',t-t')e^{t'}\zeta_0(x',t')dt' \\
&= \lim_{\sigma\to 0^+} G(x-x',t-t')e^{t'}\zeta_0(x',t')\Big|_0^{t-\sigma} - \int\limits_0^t e^{t'}\left(G_{t'}\zeta_0(x',t') + G\frac{\partial}{\partial t'}\zeta_0(x',t')\right)dt' \\
&= \lim_{\sigma\to 0^+} G(x-x',\sigma)e^{t-\sigma}\zeta_0(x',t-\sigma) - G(x-x',t)\zeta_0(x',0) \\
&\quad + \int\limits_0^t e^{t'}\left(\epsilon d_2 G_{x'x'}\zeta_0(x',t') - G\frac{\partial}{\partial t'}\zeta_0(x',t')\right)dt'.
\end{aligned}
\tag{2.2.18}
$$

Here, in the last term, we used $G_{t'}(x-x',t-t') = -\epsilon d_2 G_{x'x'}(x-x',t-t')$. Next, we observe that $\lim_{\sigma\to 0^+} G(x-x',\sigma)$ is a δ-function, i.e.

$$
\lim_{\sigma\to 0^+}\int\limits_{\mathbb{R}} G(x-x',\sigma)e^{t-\sigma}\zeta_0(x',t-\sigma)dx' = e^t\zeta_0(x,t).
$$

Substituting this and (2.2.18) into the solution (2.2.17), we have

$$
\begin{aligned}
\nu(x,t) &= e^t\zeta_0(x,t) - \int\limits_{\mathbb{R}} G(x-x',t)\zeta_0(x',0)dx' \\
&+ \int\limits_{\mathbb{R}}\int\limits_0^t e^{t'}\left(\epsilon d_2 G_{x'x'}\zeta_0(x',t') - G\left(\frac{\partial}{\partial t'}\zeta_0(x',t') + \frac{\epsilon y}{(1+y)^2}\right)\right)dt'dx'.
\end{aligned}
\tag{2.2.19}
$$

The first two terms in (2.2.19) are already in the desired form. It remains to analyze the integrand in the double integral in (2.2.19) and express it in a simpler form. In particular, we use two steps of integration by parts in x' to rewrite the first term in the integrand, as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} G_{x'x'}(x-x',t-t')\zeta_0(x',t')dx' \\
&= -G_{x'}(x-x',t-t')\zeta_0(x',t')\big|_{-\infty}^{+\infty} - G(x-x',t-t')\frac{\partial}{\partial x'}\zeta_0(x',t')\bigg|_{-\infty}^{+\infty} \\
&\quad + \int_{\mathbb{R}} G(x-x',t-t')\frac{\partial^2}{\partial x'^2}\zeta_0(x',t')dx' \\
&= \int_{\mathbb{R}} G(x-x',t-t')\frac{\partial^2}{\partial x'^2}\zeta_0(x',t')dx'.
\end{aligned} \tag{2.2.20}$$

Here, we used that both $\zeta_0(x,t)$ and $\frac{\partial}{\partial x'}\zeta_0(x,t)$ vanish at infinity by (2.2.2) and (2.2.8).

To simplify the calculation, we write (recall (2.2.2))

$$\zeta_1(x,t) = d_2\frac{\partial^2}{\partial x^2}\zeta_0(x,t) - \frac{1}{\epsilon}\frac{\partial}{\partial t}\zeta_0(x,t) - \frac{y(x,t)}{(1+y(x,t))^2}.$$

Hence, the double integral in (2.2.19) becomes

$$\epsilon\int_{\mathbb{R}}\int_0^t e^{t'}G(x-x',t-t')\zeta_1(x',t')dt'dx'. \tag{2.2.21}$$

Combining (2.2.19) and (2.2.21), we obtain

$$\nu(x,t) = e^t\zeta_0(x,t) - \int_{\mathbb{R}} G(x-x',t)\zeta_0(x',0)dx' + \epsilon\int_{\mathbb{R}}\int_0^t e^{t'}G(x-x',t-t')\zeta_1(x',t')dt'dx'. \tag{2.2.22}$$

This completes the first step of the integration by parts in t'. Moreover, we observe that the last term in (2.2.22) is uniformly bounded, since the double integral is uniformly bounded of $\mathcal{O}(\epsilon)$ due to standard properties of the Gaussian function and the boundedness of ζ_1 established in the Lemma.

Now, we perform the second step in the sequence of integration by parts steps in t'. In particular, in order to evaluate the double integral in (2.2.22), we use integration by parts on t' in the inner integral in (2.2.22), following exactly the procedure shown in the derivation of (2.2.19) and again using the delta function property of the Green's function in the limit as $\sigma \to 0$. This yields

$$\nu(x,t) = e^t\left(\zeta_0(x,t) + \epsilon\zeta_1(x,t)\right) - \int\limits_{\mathbb{R}} G(x-x',t)\left(\zeta_0(x',0) + \epsilon\zeta_1(x',0)\right)dx' + \int\limits_{\mathbb{R}}\int\limits_0^t e^{t'}\left(\epsilon^2 d_2 G_{x'x'}(x-x',t-t')\zeta_1(x',t') - \epsilon G(x-x',t-t')\frac{\partial}{\partial t'}\zeta_1(x',t')\right)dt'dx', \tag{2.2.23}$$

which is the analog of (2.2.19) from the first step of integration by parts. The first two terms are already in the desired form. Next, we express the term involving $G_{x'x'}$ in the double integral in terms of G using two steps of integration by parts, exactly as in the analysis shown above in (2.2.20) for step 1, and we recall from (2.2.3) that

$$\zeta_2(x,t) = d_2\frac{\partial^2}{\partial x^2}\zeta_1(x,t) - \frac{1}{\epsilon}\frac{\partial}{\partial t}\zeta_1(x,t).$$

This yields

$$\begin{aligned}\nu(x,t) =& e^t\left(\zeta_0(x,t) + \epsilon\zeta_1(x,t)\right) - \int\limits_{\mathbb{R}} G(x-x',t)\left(\zeta_0(x',0) + \epsilon\zeta_1(x',0)\right)dx' \\ &+ \epsilon^2\int\limits_{\mathbb{R}}\int\limits_0^t e^{t'}G(x-x',t-t')\zeta_2(x',t')dt'dx'.\end{aligned} \tag{2.2.24}$$

This completes the second step of the sequence of integration by parts in t', and here we note that the last term in (2.2.24) is uniformly bounded of $\mathcal{O}(\epsilon^2)$.

Next, we perform the induction step, that is we assume that that the n-th step of integration by parts in t' has been completed, and we perform the $(n+1)$-st step.

The double integral at the end of the n-th step may be evaluated as follows:

$$
\begin{aligned}
&\epsilon^n \int\limits_{\mathbb{R}} \int\limits_0^t e^{t'} G(x-x', t-t')\zeta_n(x', t')dt'dx' \\
=&\epsilon^n e^t \zeta_n(x,t) - \epsilon^n \int\limits_{\mathbb{R}} G(x-x', t)\zeta_n(x', 0)dx' \\
&+ \int\limits_{\mathbb{R}} \int\limits_0^t e^{t'} \left(\epsilon^{n+1} d_2 G_{x'x'}\zeta_n(x', t') - \epsilon^n G \frac{\partial}{\partial t'}\zeta_n(x', t')\right) dt'dx' \\
=&\epsilon^n e^t \zeta_n(x,t) - \epsilon^n \int\limits_{\mathbb{R}} G(x-x', t)\zeta_n(x', 0)dx' \\
&+ \epsilon^{n+1} \int\limits_{\mathbb{R}} \int\limits_0^t e^{t'} G(x-x', t-t')\zeta_{n+1}(x', t')dt'dx',
\end{aligned}
\tag{2.2.25}
$$

where we recall the recursive definition (2.2.3) of ζ_{n+1}. Hence, at the end of the $(n+1)$-st step of integration by parts in t', we have

$$
\begin{aligned}
\nu(x,t) =& e^t \sum_{j=0}^{n} \epsilon^j \zeta_j(x,t) - \int\limits_{\mathbb{R}} G(x-x', t)\left(\sum_{j=0}^{n} \epsilon^j \zeta_j(x', 0)\right) dx' \\
&+ \epsilon^{n+1} \int\limits_{\mathbb{R}} \int\limits_0^t e^{t'} G(x-x', t-t')\zeta_{n+1}(x', t')dt'dx'.
\end{aligned}
\tag{2.2.26}
$$

The expression (2.2.26) for $\nu(x,t)$ represents the solution of the full system up to and including $\mathcal{O}(\epsilon^n)$. At each power of ϵ, it contains the terms that give the asymptotic expansion of the N-th nonlinear slow mode and the spatial integral terms that involve the initial condition $y(x', 0)$ and its derivatives. The double integral at the end of (2.2.26) is uniformly bounded of $\mathcal{O}(\epsilon^{n+1})$ due to standard properties of the Gaussian function and to the Lemma when taking m sufficiently large.

Therefore, by induction on n, we have established that by taking m sufficiently large so that $N_0(M)$ is sufficiently large, for any $0 \leq N < N_0(M)$,

$$\nu(x,t) = e^t \sum_{j=0}^{N} \epsilon^j \zeta_j(x,t) - \int_{\mathbb{R}} G(x-x',t) \left(\sum_{j=0}^{N} \epsilon^j \zeta_j(x',0) \right) dx' + \mathcal{R}_N(x,t;\epsilon), \quad (2.2.27)$$

with $\nu(x,0) = 0$.

Finally, translating from ν and ω back to z by recalling $z = e^{-t}(\nu+\omega)$ from (2.2.9) and (2.2.11) and using ω from (2.2.15), we obtain

$$z(x,t) = z_{\text{slow},N}(x,t;\epsilon) + e^{-t} w_N(x,t;\epsilon), \quad (2.2.28)$$

where we recall that $z_{\text{slow},N}$ and w_N are given, respectively, by

$$z_{\text{slow},N}(x,t;\epsilon) = \sum_{j=0}^{N} \epsilon^j \zeta_j(x,t) \quad (2.2.29)$$

and

$$\begin{aligned} w_N(x,t;\epsilon) = &\sum_{j=0}^{M} z^j(0)(1+t)^{-\frac{j+1}{2}} \psi^j \left(\frac{x}{\sqrt{1+t}} \right) + \mathcal{R}_{M,2}(x,t;\epsilon) \\ &- \int_{\mathbb{R}} G(x-x',t) \left(\sum_{j=0}^{N} \epsilon^j \zeta_j(x',0) \right) dx' + \mathcal{R}_N(x,t;\epsilon). \end{aligned} \quad (2.2.30)$$

Hence, we have established (2.2.4)(b), (2.2.1), and (2.2.5). This completes the proof of Theorem 2.2.1.

□

Remark. For $m \geq 17/2$, ζ_2 is uniformly bounded in x and t. The formula for $\zeta_2(x,t)$

is given by

$$\zeta_2 = d_2 \frac{\partial^2}{\partial x^2}\left(\frac{(d_2-d_1)y_{xx}}{(1+y)^2} - \frac{2d_2 y_x^2}{(1+y)^3}\right) + \left(\frac{2(d_2-d_1)y_{xx}}{(1+y)^3} - \frac{6d_2 y_x^2}{(1+y)^4}\right)(-y + d_1 y_{xx})$$
$$- \frac{(d_2-d_1)}{(1+y)^2}(-y_{xx} + d_1 y_{xxxx}) + \frac{4d_2 y_x}{(1+y)^3}(-y_x + d_1 y_{xxx}). \tag{2.2.31}$$

2.3 Numerical Results

In this section, we illustrate the results of Theorem 2.2.1. In particular, we obtain numerical representations of the solutions $(y(x,t), z(x,t))$ of system (2.1.1), and we use these to illustrate the slow modes and the decay toward them, recalling (2.2.4). We work on the interval $[0, L]$, with Neumann boundary conditions for the z component. Also, we take general, non-negative initial conditions $y(x,0) = y_0(x)$ and $z(x,0) = z_0(x)$.

The representations of the solutions of (2.1.1) consist of the exact solution for $y(x,t)$ as in (2.2.4)(a), and a numerically-derived representation for $z(x,t)$ obtained from the z equation in (2.1.1) in which y is represented by the exact solution. Then, we compare the dynamics of the fast component $z(x,t)$ with the 1st nonlinear slow mode $z_{\text{slow},1}$ in (2.2.1), to illustrate the fast decay $e^{-t}w$ in (2.2.4)(b).

For illustrative purposes, we first chose the initial data for y to be the eigenfunction corresponding to the 0 eigenvalue from (2.1.6)(a) at $\xi = \frac{x}{\sqrt{1+t}}$ and $t = 0$,

$$y_0(x) = \frac{1}{\sqrt{4\pi\epsilon d_1}} e^{-\frac{x^2}{4\epsilon d_1}}. \tag{2.3.1}$$

In this case, $y^j(0)$ in (2.2.4) can be computed by

$$y^j(0) = \int_{\mathbb{R}} y_0(\xi) H_j\left(\frac{\xi}{\sqrt{\epsilon d_1}}\right) d\xi = \delta_{0j}. \tag{2.3.2}$$

The exact solution for the slow component $y(x,t)$ is

$$y(x,t) = e^{-\epsilon t} \frac{1}{\sqrt{4\pi\epsilon d_1(1+t)}} e^{-\frac{x^2}{4\epsilon d_1(1+t)}}. \tag{2.3.3}$$

The nonlinear slow mode up to and including $\mathcal{O}(\epsilon)$ can be computed using (2.3.3) where $N = 1$,

$$z_{\text{slow},1}(x,t;\epsilon) = \frac{y}{1+y} + \epsilon\left(\frac{(d_2 - d_1)y_{xx}}{(1+y)^2} - \frac{2d_2 y_x^2}{(1+y)^3}\right). \tag{2.3.4}$$

The numerical representation of z is obtained by solving the equation (2.1.1)(b),

$$\frac{\partial z}{\partial t} = -z + \frac{y}{1+y} - \epsilon\frac{y}{(1+y)^2} + \epsilon d_2 \frac{\partial^2 z}{\partial x^2}, \tag{2.3.5}$$

using the method of lines and the finite element method in Mathematica, where y is represented by its analytical solution (2.3.3). Here, for illustrative purposes, we chose the initial condition

$$z(x,0) = \frac{1+\cos(x)}{2}, \tag{2.3.6}$$

which is away from the nonlinear slow mode $z_{\text{slow},1}(x,0;\epsilon)$, so that we can observe the decay of z to $z_{\text{slow},1}$.

We illustrate the theory when ϵ is small and the diffusion coefficients d_1 and d_2 have $\mathcal{O}(1)$ values. We set $L = 2$, $d_1 = 1, d_2 = 2$ and $\epsilon = 0.05$. Figure 2·1 depicts the y solution of (2.1.1). Figure 2·2 depicts the z solution of (2.1.1) and the nonlinear slow mode $z_{\text{slow},1}$.

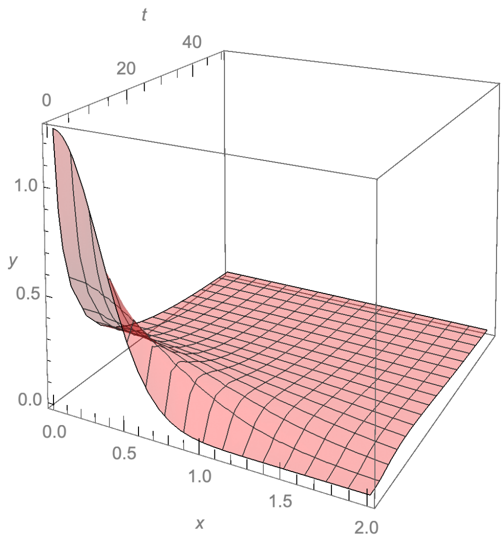
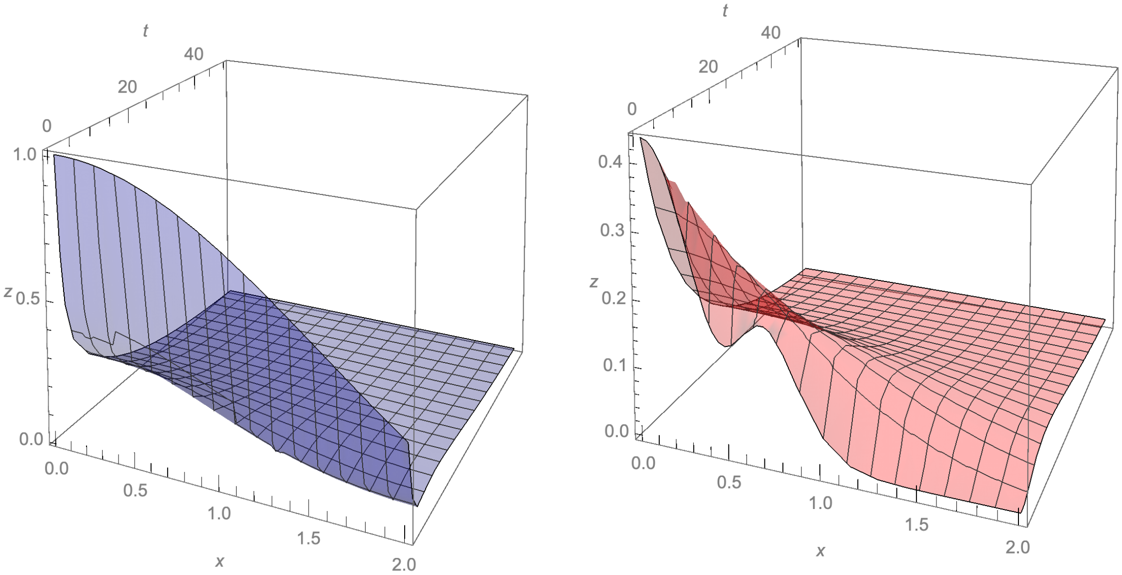


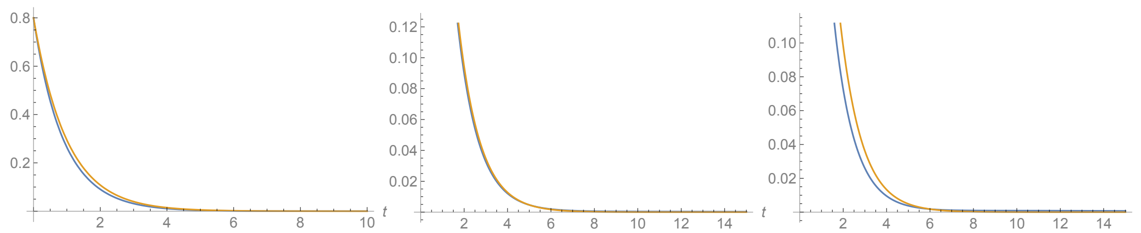
Figure 2·1: y solution of (2.1.1) with initial condition (2.3.1).



(a) (b)

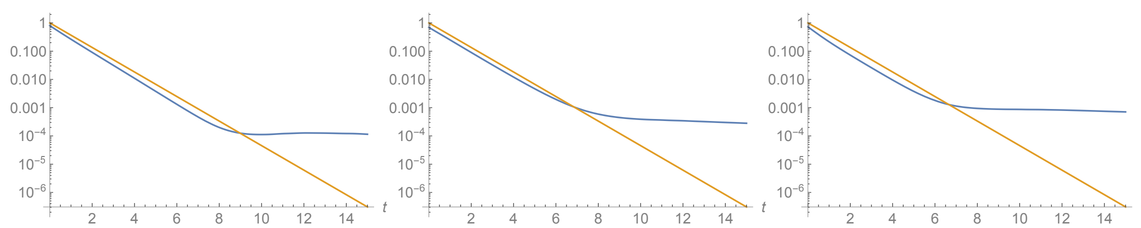
Figure 2·2: z component of (2.1.1): (a) z solution with initial condition (2.3.6); and (b) Nonlinear slow mode $z_{\text{slow},1}$.

Figure 2·3 shows comparison of the difference $|z - z_{\text{slow},1}|$ and the theoretical estimate, Ce^{-t} (for some $C > 0$) at three points: $x = 0.45, x = 0.7, x = 1$. Figure 2·4 is the LogPlot of $|z - z_{\text{slow},1}|$ and e^{-t} at the three points. From both Figure 2·3 and 2·4, we see that $|z - z_{\text{slow},1}|$ decays faster than e^{-t} until it reaches an $\mathcal{O}(\epsilon^2)$ error, since we truncated $z_{\text{slow},N}$ up to and including $\mathcal{O}(\epsilon)$ by taking $N = 1$.



(a) (b) (c)

Figure 2·3: comparison of $|z - z_{\text{slow},1}|$ (blue) and $|z(x,0) - z_{\text{slow},1}(x,0;\epsilon)|e^{-t}$ (orange) as functions of t at: (a) $x = 0.45$; (b) $x = 0.7$; and (c) $x = 1$.



(a) (b) (c)

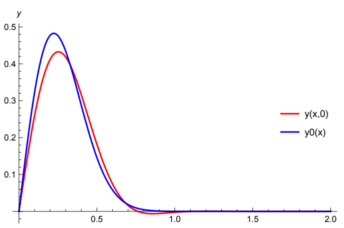
Figure 2·4: LogPlot of $|z - z_{\text{slow},1}|$ (blue) and e^{-t} (orange) as functions of t at: (a) $x = 0.45$; (b) $x = 0.7$; and (c) $x = 1$.

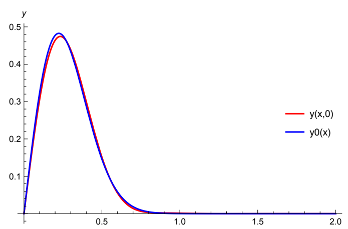
The numerical simulations show that over the entire domain $x \in [0, L]$, the full solution z of system (2.1.1) approaches the nonlinear slow modes z_{slow} given by (2.2.1), in a short time. Also, the difference is bounded by e^{-t}, which is the decaying prefactor of $w(x,t;\epsilon)$ in (2.2.4)(b) . This concludes the illustration of the results of Theorem 2.2.1. The z solution of system (2.1.1) is attracted to the nonlinear slow modes exponentially in t.

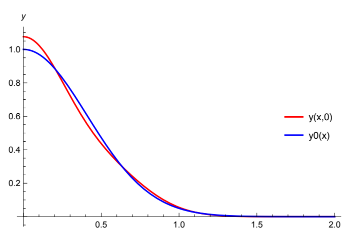
We may also choose several different Gaussian related initial data $y_0(x)$, and illustrate the modal expansion of y by comparing these $y_0(x)$ with

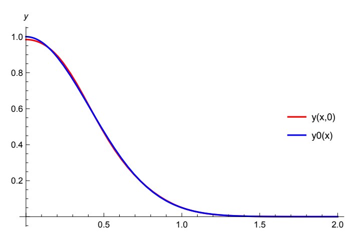
$$y(x,0) = \frac{1}{\sqrt{4\pi\epsilon d_1}} e^{-\frac{x^2}{4\epsilon d_1}} \sum_{j=0}^{J} y^j(0) H_j\left(\frac{x}{\sqrt{\epsilon d_1}}\right)$$

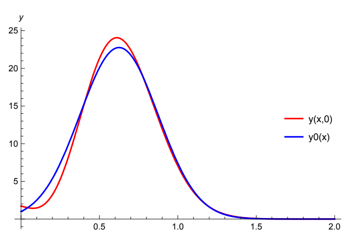
for sufficiently large J, as shown in Table 2.1:

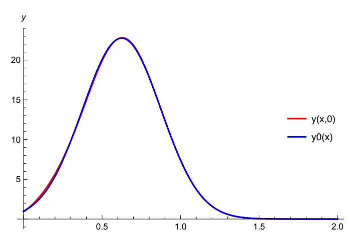
<table>
<tr><th>$y_0(x)$</th><th colspan="2">Comparison of $y_0(x)$ with $y(x,0)$ on $[0,2]$</th></tr>
<tr><td>$\sum_{j=0}^{4} \frac{(-1)^j}{j+1}\phi^j(x)$</td><td colspan="2">We verified that the modal expansion gives the exact solution for $J \geq 4$.</td></tr>
<tr><td>$-\phi^0(x)\phi^1(x)$</td><td>

(a) $J = 5$</td><td>

(b) $J = 10$</td></tr>
<tr><td>$e^{-3\xi^2}$</td><td>

(c)$J = 5$</td><td>

(d)$J = 10$</td></tr>
<tr><td>$e^{-8\xi^2+10\xi}$</td><td>

(e)$J = 5$</td><td>

(f)$J = 10$</td></tr>
</table>

Table 2.1: Choice of initial data $y_0(x)$.

Chapter 3

Projection onto the Neutral Mode and Algebraic Decay in the Davis-Skodje Model

In the previous chapter, we studied the Davis-Skodje PDE and established the existence of the N-th order nonlinear slow mode $z_{\text{slow},N}$, which are based on the asymptotic expansion in powers of ϵ up to and including $\mathcal{O}(N)$. Hence, in this chapter, we also study the same Davis-Skodje PDE. However, here, we project directly onto the modes naturally given by the operators $\mathcal{L}_1$ and $\mathcal{L}_2$. In particular, in chapter 3.1 - 3.3, we project onto the 0th mode and carry out a complete decomposition of solutions in terms of the 0th mode and the algebraic and exponential decay toward it. Then, in chapter 3.4 - 3.6, we analyze the projection onto the sum of the 0th and 1st modes. We note that the requirements on m in this chapter are much weaker than in chapter 2. Finally, in chapter 3.7, we present the results of numerical simulations to illustrate the general theory.

3.1 Dynamics of the Zero Mode and Decay toward it

We first recall the Davis-Skodje PDE with general diffusion coefficients $d_1, d_2 > 0$ given by

$$\begin{aligned} y_t &= -\epsilon y + \epsilon d_1 y_{xx} \\ z_t &= -z + \frac{y}{1+y} - \epsilon \frac{y}{(1+y)^2} + \epsilon d_2 z_{xx}. \end{aligned} \tag{3.1.1}$$

Here, $x \in \mathbb{R}$, $t \in [0, \infty)$, $y = y(x,t)$, $z = z(x,t)$, and $0 < \epsilon \ll 1$. We study solutions of (3.1.1) with initial conditions $y(x,0)$ and $z(x,0)$ in the weighted norm space $L^2(m)$. Here, the norm is defined by $\|f\| = \left(\int_{\mathbb{R}} (1 + |x/\sqrt{\epsilon(1+t)}|^m)^2 f^2(x/\sqrt{1+t})dx\right)^{1/2}$ for any $f \in L^2(\mathbb{R}), m \geq 1$.

Recall from section 2.1 that in the scaling variables,

$$
\begin{aligned}
&\xi = \frac{x}{\sqrt{1+t}}, \qquad \tau = \log{(1+t)}, \\
&y(x,t) = e^{-\frac{\tau}{2}}\tilde{y}(\xi,\tau), \quad z(x,t) = e^{-\frac{\tau}{2}}\tilde{z}(\xi,\tau), \quad \eta = e^{-\frac{\tau}{2}},
\end{aligned} \tag{3.1.2}
$$

(3.1.1) is written as

$$
\begin{aligned}
\tilde{y}_\tau &= (\mathcal{L}_1 - \epsilon\eta^{-2})\tilde{y} \\
\tilde{z}_\tau &= (\mathcal{L}_2 - \eta^{-2})\tilde{z} + \eta^{-2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right) \\
\eta_\tau &= -\frac{\eta}{2}.
\end{aligned} \tag{3.1.3}
$$

The linear operators are $\mathcal{L}_i \cdot = \epsilon d_i \partial_\xi^2 \cdot + \frac{1}{2}\partial_\xi(\xi \cdot),\ i = 1,2$. The initial data $\tilde{y}(\xi,0)$ and $\tilde{z}^1(\xi,0)$ are in the weighted Sobolev space

$$
L^2(m) = \left\{ f \in L^2(\mathbb{R}) |\, \|f\|_m = \left(\int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 (f(\xi))^2 d\xi \right)^{1/2} < \infty \right\}. \tag{3.1.4}
$$

The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ have point spectrum $\sigma = \{\lambda_k = -\frac{k}{2}, k = 0,1,2,\ldots\}$, and the essential spectrum is given by $\sigma_{\text{ess}} = \left\{\lambda \in \mathbb{C} | \mathrm{Re}(\lambda) \leq \frac{1}{4} - \frac{\mathrm{m}}{2}\right\}$. Each eigenvalue λ_k has an associated eigenfunction $\phi^k(\xi)$ and $\psi^k(\xi)$ from (2.1.6) for the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

We also recall the projection operators P_n and Q_n from (2.1.9) and (2.1.10) respectively, based on the orthogonality properties (2.1.8). We will use $P_{n,1}$, $Q_{n,1}$, $P_{n,2}$, and $Q_{n,2}$ to denote the first and second components of P_n and Q_n, respectively.

In this section, we fix $m \geq 1$ in (2.1.5). This choice is made so that the eigenvalue at zero is isolated. The central idea in the analysis is to separate the solution of (3.1.3) into three components (see (3.1.5) and (3.1.9)), one corresponding to the projection onto the zero mode, another corresponding to the fast-slow structure of the equations, and the final component corresponding to remainder terms, which we will show decay exponentially. In particular, we express the solutions as follows:

$$\tilde{y}(\xi,\tau) = \alpha^0(\tau)\phi^0(\xi) + \tilde{r}_1(\xi,\tau), \quad \tilde{z}(\xi,\tau) = \beta^0(\tau)\psi^0(\xi) + \tilde{r}_2(\xi,\tau). \tag{3.1.5}$$

Theorem 3.1.1. *Fix $m \geq 1$. Let the solution $\tilde{y}(\xi,\tau)$ and $\tilde{z}(\xi,\tau)$ be decomposed as in* (3.1.5)*. Then, $\alpha^0(\tau)$ is given exactly by*

$$\alpha^0(\tau) = \alpha^0(0)e^{\epsilon(1-e^{\tau})}, \tag{3.1.6}$$

and $\beta^0(\tau)\psi^0(\xi)$ is given by

$$\beta^0(\tau)\psi^0(\xi) = P_{0,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \mathcal{O}\left(e^{-\frac{3}{2}\tau}e^{2\epsilon(1-e^{\tau})}\right). \tag{3.1.7}$$

The complementary component $\tilde{r}_1(\xi,\tau)$ is the remainder term which decays exponentially fast in τ, i.e., there exists a constant $C > 0$ independent of ϵ such that

$$\|\tilde{r}_1\|_m \leq Ce^{-\frac{\tau}{4}}e^{\epsilon(1-e^{\tau})}\left\|\tilde{r}_1^0\right\|_m. \tag{3.1.8}$$

Finally, the other complementary component $\tilde{r}_2$ may be written as

$$\tilde{r}_2(\xi,\tau) = Q_{0,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \tilde{w}_2(\xi,\tau), \tag{3.1.9}$$

where $\tilde{w}_2(\xi,\tau)$ is the remainder term which decays exponentially fast in τ, i.e., there exists a constant $K > 0$ independent of ϵ such that

$$\|\tilde{w}_2\|_m \leq K\left(e^{-\frac{\tau}{4}}e^{1-e^{\tau}}\left\|\tilde{w}_2^0\right\|_m + e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}\left\|\mathcal{L}_2\phi^0\right\|_m\right). \tag{3.1.10}$$

Here,

$$\left\|\mathcal{L}_2\phi^0\right\|_m = \left\|(\mathcal{L}_2 - \mathcal{L}_1)\phi^0\right\|_m = \epsilon(d_2 - d_1)\left\|\partial_\xi^2\phi^0\right\|_m. \quad (3.1.11)$$

Theorem 3.1.1 provides a complete understanding of the dynamics of the solutions (3.1.5) of the Davis-Skodje PDE (3.1.1). First, for $\tilde{y}$, we observe that (3.1.6) completely determines the evolution of $\alpha^0(\tau)$. In particular, on the initial time interval $\tau \in [0, \log(1/\epsilon)]$, $\alpha^0(\tau)$ is $\mathcal{O}(1)$. Then, for $\tau > c\log(1/\epsilon)$ with any $c > 1$, $\alpha^0(\tau) \le e^{\epsilon - \epsilon^{1-c}}$, i.e., it is exponentially small. Also, (3.1.8) shows that the complementary term $\tilde{r}_1$ is the remainder, which decays $\mathcal{O}(e^{-\frac{\tau}{4}})$ faster than $\alpha^0(\tau)$.

Second, for $\tilde{z}$, we observe that (3.1.7), (3.1.9), and (3.1.10) provide a complete picture. In particular,

$$\tilde{z}(\xi,\tau) = P_{0,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \mathcal{O}\left(e^{-\frac{3}{2}\tau}e^{2\epsilon(1-e^{\tau})}\right) + Q_{0,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \tilde{w}_2(\xi,\tau). \quad (3.1.12)$$

Hence, to leading order,

$$\tilde{z}_0 = P_{0,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + Q_{0,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) = \frac{\tilde{y}}{1+\eta\tilde{y}}. \quad (3.1.13)$$

It is labeled as the leading order slow mode. Moreover, we observe that in the analysis of the equation for β^0 (see (3.1.23)), one may determine the higher order terms successively. For example, after (3.1.13), the next term in the expansion of $\tilde{z}$ is $-\frac{\eta^3}{2}\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2}$ and higher order terms are $\mathcal{O}\left(e^{-\frac{5}{2}\tau}e^{2\epsilon(1-e^{\tau})}\right)$.

Finally, (3.1.10) establishes that the remainder $\tilde{w}_2$ decays faster than $\tilde{y}$ and $\tilde{z}^0$. Also, it contains a term decaying like $e^{-\frac{\tau}{4}}e^{1-e^{\tau}}$, which is faster than the remainder $\tilde{r}_1$ of the slow component, as well as a term slowly decaying like $e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}$ and proportional to the difference of diffusion coefficients $d_2 - d_1$.

Short term decay rates of the two terms are illustrated in figure 3·1 for fixed $\epsilon = 0.1$ and $\epsilon = 0.01$. Long term decay rates of the two terms are illustrated in figure 3·2 for fixed $\epsilon = 0.1$ and $\epsilon = 0.01$.

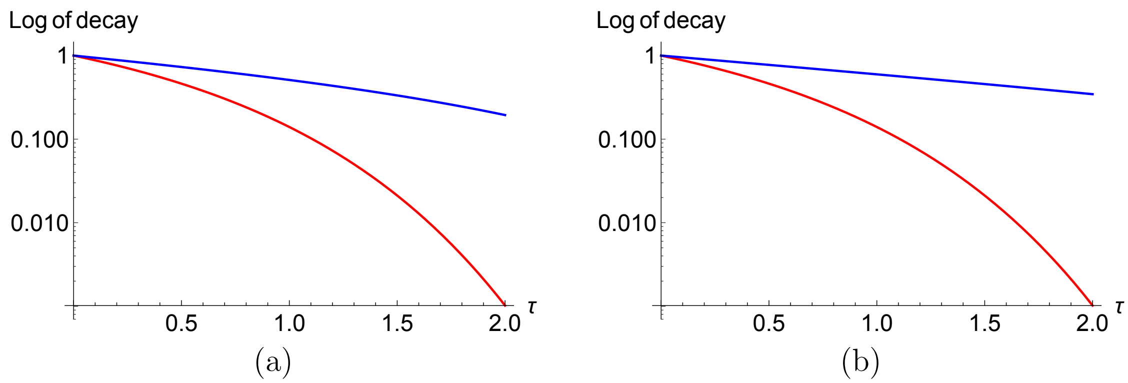


Figure 3·1: LogPlots of $e^{-\frac{\tau}{4}}e^{1-e^{\tau}}$ (red) and $e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}$ (blue) against τ on short term for fixed ϵ: (a) $\epsilon = 0.1$, (b) $\epsilon = 0.01$.

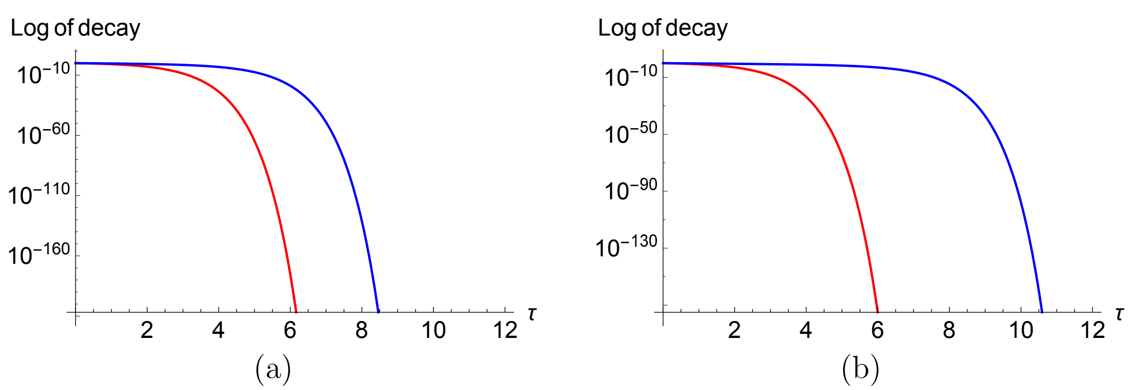


Figure 3·2: LogPlots of $e^{-\frac{\tau}{4}}e^{1-e^{\tau}}$ (red) and $e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}$ (blue) against τ on long term for fixed ϵ: (a) $\epsilon = 0.1$, (b) $\epsilon = 0.01$.

In this manner, Theorem 3.1.1 establishes that the dynamics of the solutions are approximated to within an exponentially decaying error relative to $\alpha^0(\tau)\phi^0(\xi)$ and the slow mode $\tilde{z}_0 = \frac{\tilde{y}}{1+\eta\tilde{y}}$.

We recall from (3.1.2) that $\xi = \frac{x}{\sqrt{1+t}}$ and $\tau = \log(1+t)$, and we let $r_1 = \eta\tilde{r}_1$ and $w_2 = \eta\tilde{w}_2$. Then, we translate the solutions (3.1.5), (3.1.6), (3.1.7), (3.1.9) and the estimates (3.1.8), (3.1.10) back into the original space and time variables x and t.

Corollary 3.1.2. *The zero mode approximation of y and the leading order slow mode are given by*

$$\begin{aligned} y(x,t) &= \frac{\alpha^0(0)}{\sqrt{4\pi\epsilon d_1}}\frac{e^{-\epsilon t}}{\sqrt{1+t}}e^{-\frac{x^2}{4\epsilon d_1(1+t)}}, \\ z(x,t) &= \frac{y}{1+y}. \end{aligned} \tag{3.1.14}$$

In terms of the original variables, the two remainder terms decay algebraically on the fast time scale (t), and exponentially on the slow time scale (ϵt),

$$\begin{aligned} &\|r_1\| \le C(1+t)^{-\frac{3}{4}}e^{-\epsilon t}\|r_1(x,0)\|, \\ &\|w_2\| \le K\left((1+t)^{-\frac{3}{4}}e^{-t}\|w_2(x,0)\| + (1+t)^{-1}e^{-\epsilon t}\left\|\mathcal{L}_2\phi^0\right\|\right), \end{aligned} \tag{3.1.15}$$

where $\|\cdot\|$ denotes the norm $\|f\| = \left(\int_{\mathbb{R}}(1+|x/\sqrt{\epsilon(1+t)}|^m)^2 f^2(x/\sqrt{1+t})dx\right)^{1/2}$ for any $f \in L^2(\mathbb{R}), m \ge 1$.

Corollary 3.1.2 shows that both $y(x,t)$ and $z(x,t)$ undergo algebraic and exponential decay in time. The algebraic decay is at the rate $(1+t)^{-\frac{3}{4}}$ for $y(x,t)$ and $(1+t)^{-1}$ for $z(x,t)$. It governs the short time decay of z towards the leading order slow mode z_0. For longer terms, exponential decay $e^{-\epsilon t}$ dominates the behavior of remainders. The long-time behavior of Davis-Skodje model solutions can be determined by the zero mode consisting of the solution of an ordinary differential equation and the leading order slow mode.

Proof of Theorem 3.1.1. We begin the proof by deriving and analyzing the equations for $\alpha^0(\tau)$ and $\beta^0(\tau)$.

We substitute (3.1.5) into (3.1.3) and apply the projection operator P_0 to obtain the following equations for α^0 and β^0:

$$\begin{aligned}
\alpha^0_\tau &= -\epsilon\eta^{-2}\alpha^0,\\
\beta^0_\tau &= -\eta^{-2}\beta^0 + \eta^{-2}\int_{\mathbb{R}}\left[\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right]d\xi,\\
\eta_\tau &= -\frac{\eta}{2}.
\end{aligned} \tag{3.1.16}$$

Here, we used that $\mathcal{L}_1\phi^0 = 0, \mathcal{L}_2\psi^0 = 0, P_{0,2}(g) = \left(\int_{\mathbb{R}} g d\xi\right)\psi^0$, and $P_0\begin{pmatrix}\tilde{r}_1\\ \tilde{r}_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. The ODE (3.1.16)(a) for α^0 decouples and may be solved exactly,

$$\alpha^0(\tau) = \alpha^0(0)e^{\epsilon(1-e^\tau)}. \tag{3.1.17}$$

This establishes (3.1.6).

The ODE (3.1.16)(b) for $\beta^0(\tau)$ may be solved using an integrating factor,

$$\begin{aligned}
\beta^0(\tau) &= e^{-\eta^{-2}}\left[\int_0^\tau e^{\eta^{-2}}\eta^{-2}\int_{\mathbb{R}}\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right)d\xi d\hat{\tau} + C\right]\\
&= e^{-\eta^{-2}}\left[\int_{\mathbb{R}}\int_0^\tau e^{\eta^{-2}}\eta^{-2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right)d\hat{\tau} d\xi + C\right].
\end{aligned} \tag{3.1.18}$$

For the inner integral on $\hat{\tau}$, we use integration by parts on the first term, recalling that $\frac{\partial e^{\eta^{-2}}}{\partial\hat{\tau}} = \eta^{-2}e^{\eta^{-2}}$ and noting that the boundary term at $\hat{\tau} = 0$ can be included in the constant C,

$$\begin{aligned}
\beta^0(\tau) =& e^{-\eta^{-2}}\left[\int_{\mathbb{R}}\left(e^{\eta^{-2}}\frac{\tilde{y}}{1+\eta\tilde{y}} - \int_0^\tau e^{\eta^{-2}}\frac{\tilde{y}_\tau + \frac{\eta}{2}\tilde{y}^2}{(1+\eta\tilde{y})^2}d\hat{\tau}\right.\right.\\
&\left.\left. - \int_0^\tau e^{\eta^{-2}}\eta^{-2}\epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}d\hat{\tau}\right)d\xi + C\right].
\end{aligned}$$

Next, we use (3.1.3)(a) to substitute for $\tilde{y}_\tau$ and to simplify the integrand:

$$\beta^0(\tau) = e^{-\eta^{-2}} \left[\int_{\mathbb{R}} \left(e^{\eta^{-2}} \frac{\tilde{y}}{1+\eta\tilde{y}} - \int_0^\tau e^{\eta^{-2}} \frac{\mathcal{L}_1\tilde{y} + \frac{\eta}{2}\tilde{y}^2}{(1+\eta\tilde{y})^2} d\hat{\tau} \right) d\xi + C \right]. \tag{3.1.19}$$

Next, we recall that $\tilde{y} = \alpha^0\phi^0 + \tilde{r}_1, \mathcal{L}_1\phi^0 = 0$, and $\int_{\mathbb{R}} \tilde{r}_1 d\xi = 0$. Hence, we have

$$\int_{\mathbb{R}} \int_0^\tau e^{\eta^{-2}} \mathcal{L}_1\tilde{y} d\hat{\tau} d\xi = \int_0^\tau e^{\eta^{-2}} \alpha^0 d\hat{\tau} \int_{\mathbb{R}} \mathcal{L}_1\phi^0 d\xi + \mathcal{L}_1 \int_0^\tau e^{\eta^{-2}} \int_{\mathbb{R}} \tilde{r}_1 d\xi d\hat{\tau} = 0. \tag{3.1.20}$$

Then, $\beta^0(\tau)$ becomes

$$\beta^0(\tau) = \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{1}{2} e^{-\eta^{-2}} \left[\int_{\mathbb{R}} \int_0^\tau e^{\eta^{-2}} \frac{\eta\tilde{y}^2}{(1+\eta\tilde{y})^2} d\hat{\tau} d\xi + C \right]. \tag{3.1.21}$$

For the second integral, we use integration by parts again on the inner integral on $\hat{\tau}$,

$$\begin{aligned} \beta^0(\tau) = & \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{1}{2} e^{-\eta^{-2}} \int_{\mathbb{R}} e^{\eta^{-2}} \frac{\eta^3\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi \\ & + \frac{1}{2} e^{-\eta^{-2}} \left[\int_{\mathbb{R}} \int_0^\tau e^{\eta^{-2}} \left(\frac{\eta^3\tilde{y}^2}{(1+\eta\tilde{y})^2} \right)_{\hat{\tau}} d\hat{\tau} d\xi + C \right]. \end{aligned} \tag{3.1.22}$$

We observe that the last integral in (3.1.22) decays $\mathcal{O}(e^{-\tau})$ faster than the last integral in (3.1.21), and the constant C can be absorbed into the higher order term. Thus, to the leading order, $\beta^0(\tau)$ may be written as

$$\begin{aligned} \beta^0(\tau) &= \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{\eta^3}{2} \int_{\mathbb{R}} \frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi \left(1 + \mathcal{O}(e^{-\tau})\right) \\ &= \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{\eta^3}{2} \int_{\mathbb{R}} \frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi + \mathcal{O}\left(e^{-\frac{5}{2}\tau} e^{2\epsilon(1-e^\tau)}\right). \end{aligned} \tag{3.1.23}$$

This establishes (3.1.7).

Application of Q_0 to system (3.1.3) yields the following system for $\tilde{r}_1$ and $\tilde{r}_2$,

$$\begin{aligned}\frac{\partial \tilde{r}_1}{\partial \tau} &= \mathcal{L}_1 \tilde{r}_1 - \epsilon \eta^{-2} \tilde{r}_1, \\ \frac{\partial \tilde{r}_2}{\partial \tau} &= \mathcal{L}_2 \tilde{r}_2 - \eta^{-2} \tilde{r}_2 + \eta^{-2} Q_{0,2} \left(\frac{\tilde{y}}{1+\eta \tilde{y}} - \epsilon \frac{\tilde{y}}{(1+\eta\tilde{y})^2} \right),\end{aligned} \tag{3.1.24}$$

where we recall that $Q_{0,2}$ denotes the second component of Q_0.

The analysis of the complementary components $\tilde{r}_1$ and $\tilde{r}_2$, which will establish (3.1.8) - (3.1.10), is carried out in sections 3.2 and 3.3, respectively.

3.2 Analysis of the Remainder Term $\tilde{r}_1$ (slow component)

In this section, we follow the ideas presented in (Wayne, 2008) to analyze the remainder term $\tilde{r}_1$ and establish the decay estimate (3.1.8). It turns out to be easier to analyze the equation for $\tilde{r}_1$ in the original variables x and t. From (3.1.24)(a), we find that $r_1 = \eta \tilde{r}_1$ satisfies

$$\frac{\partial r_1}{\partial t} = \epsilon d_1 \frac{\partial^2 r_1}{\partial x^2} - \epsilon r_1. \tag{3.2.1}$$

Let $r_1 = e^{-\epsilon t} \rho_1$. Then, ρ_1 satisfies the heat equation

$$\frac{\partial \rho_1}{\partial t} = \epsilon d_1 \frac{\partial^2 \rho_1}{\partial x^2}, \tag{3.2.2}$$

with solution

$$\rho_1(x,t) = \frac{1}{\sqrt{4\pi \epsilon d_1 t}} \int_{\mathbb{R}} e^{-\frac{(x-x')^2}{4\epsilon d_1 t}} \rho_1^0(x') dx', \tag{3.2.3}$$

where $\rho_1^0(x) = \rho_1(x,0)$.

Next, by noting that $\rho_1(x,0) = r_1(x,0) = \tilde{r}_1(e^{\frac{\tau}{2}}\xi, 0) = \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi)$, we find that

$\tilde{r}_1(\xi,\tau)$ is given by

$$\begin{aligned}\tilde{r}_1(\xi,\tau) &= e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \rho_1(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) \\ &= \frac{e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})}}{\sqrt{4\pi\epsilon d_1(1-e^{-\tau})}} \int_{\mathbb{R}} e^{-\frac{(\xi-\xi')^2}{4\epsilon d_1(1-e^{-\tau})}} \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'.\end{aligned} \tag{3.2.4}$$

We write $\Phi(\xi) \equiv \frac{1}{\sqrt{4\pi\epsilon d_1(1-e^{-\tau})}} \exp\left[-\frac{\xi^2}{4\epsilon d_1(1-e^{-\tau})}\right]$, so that $\tilde{r}_1$ may be written succinctly as

$$\tilde{r}_1(\xi,\tau) = e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \int_{\mathbb{R}} \Phi(\xi-\xi')\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'.$$

By definition of the projection operator,

$$\int_{\mathbb{R}} \tilde{r}_1(\xi,\tau)d\xi = 0 \text{ for all } \tau, \quad Q_0^1\tilde{r}_1 = \tilde{r}_1. \tag{3.2.5}$$

Then, by subtracting zero and by splitting $\mathbb{R}$ into segments $|\xi'| \geq \sqrt{\epsilon}$ and $|\xi'| \leq \sqrt{\epsilon}$, we find that $\tilde{r}_1$ may be written as

$$\begin{aligned}\tilde{r}_1(\xi,\tau) =& e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \int_{\mathbb{R}} (\Phi(\xi-\xi') - \Phi(\xi))\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi' \\ =& e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \left[\int_{|\xi'|\geq\sqrt{\epsilon}} \Phi(\xi-\xi')\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi' - \Phi(\xi) \int_{|\xi'|\geq\sqrt{\epsilon}} \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi' \right. \\ &\left. + \int_{|\xi'|\leq\sqrt{\epsilon}} (\Phi(\xi-\xi') - \Phi(\xi))\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi \right].\end{aligned} \tag{3.2.6}$$

In compact form, we write (3.2.6) as

$$\tilde{r}_1(\xi,\tau) = e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \left[R_1 + R_2 + R_3\right], \tag{3.2.7}$$

where R_1, R_2, and R_3 are given by

$$\begin{aligned} R_1 =& \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \Phi(\xi-\xi')\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi', \\ R_2 =& -\Phi(\xi) \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi', \\ R_3 =& \int\limits_{|\xi'|\leq\sqrt{\epsilon}} \hat{\Phi}(\xi,\xi')\xi'\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'. \end{aligned} \tag{3.2.8}$$

Note that in R_3 we also used the following Taylor expansion for $|\xi'| \leq \sqrt{\epsilon}$,

$$\Phi(\xi-\xi') = \Phi(\xi) - \hat{\Phi}(\xi,\xi')\xi', \text{ where } \hat{\Phi}(\xi,\xi') = \int\limits_0^1 \Phi'(\xi-s\xi')ds. \tag{3.2.9}$$

In the rest of this section, we estimate the $L^2(m)$ norms of each of the terms R_1, R_2, and R_3 individually.

Estimation of the $L^2(m)$ norm of R_1:

Since $1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m \leq C_1\left(1+\left|\frac{\xi-\xi'}{\sqrt{\epsilon}}\right|^m\right)\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m$ for any $|\xi'| \geq \sqrt{\epsilon}$, we have

$$\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) R_1 \leq C_1 \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \left(1+\left|\frac{\xi-\xi'}{\sqrt{\epsilon}}\right|^m\right)\Phi(\xi-\xi')\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'.$$

Applying the convolution version of Young's inequality $\|g*h\|_{L^2} \leq \|g\|_{L^1}\|h\|_{L^2}$, we obtain

$$\begin{aligned} \|R_1\|_m =& \left\|\left(1+\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m\right) R_1\right\|_{L^2} \\ \leq& C_1\left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}} \left(1+\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m\right)\Phi(\xi')d\xi'\right)\left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}} \left|\frac{\xi'}{\sqrt{\epsilon}}\right|^{2m}\left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2 d\xi'\right)^{1/2}. \end{aligned} \tag{3.2.10}$$

Then, we estimate the first term on the right hand side of (3.2.10) by changing variables to $\zeta' = \frac{\xi'}{\sqrt{\epsilon}}$ and observing that

$$\int\limits_{|\xi'|\geq\sqrt{\epsilon}} \left(1+\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m\right)\Phi(\xi')d\xi' \leq C_2,$$

where C_2 is independent of ϵ, since $\int_{\mathbb{R}} e^{-z^2}dz = \sqrt{\pi}$ and $\int_{|z|\geq 1}(1+|z|^m)e^{-z^2}dz$ is bounded. Then, we estimate the second term by changing variables to $\zeta' = e^{\frac{\tau}{2}}\xi'$ and noting that $\left|\frac{\zeta'}{\sqrt{\epsilon}}\right|^{2m} \leq \left(1+\left|\frac{\zeta'}{\sqrt{\epsilon}}\right|^m\right)^2$. We obtain

$$\int\limits_{|\xi'|\geq\sqrt{\epsilon}} \left|\frac{\xi'}{\sqrt{\epsilon}}\right|^{2m} \left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2 d\xi' = \int\limits_{|\zeta'|\geq e^{\tau/2}\sqrt{\epsilon}} e^{-(m+\frac{1}{2})\tau}\left|\frac{\zeta'}{\sqrt{\epsilon}}\right|^{2m}\tilde{r}_1^0(\zeta')^2 d\zeta' \leq e^{-(m+\frac{1}{2})\tau}\left\|\tilde{r}_1^0\right\|_m^2. \tag{3.2.11}$$

Therefore, we have established that

$$\|R_1\|_m \leq C_3 e^{-(m+\frac{1}{2})\frac{\tau}{2}}\left\|\tilde{r}_1^0\right\|_m. \tag{3.2.12}$$

Estimation of the $L^2(m)$ norm of R_2:

Using the Cauchy-Schwartz inequality, we find

$$\begin{aligned}
&\left|\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)R_2\right| \\
=&\left|\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)\Phi(\xi)\int\limits_{|\xi'|\geq\sqrt{\epsilon}}\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^{-m}\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'\right| \\
\leq&\epsilon^{\frac{m}{2}}\left|\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)\Phi(\xi)\right|\left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}}|\xi'|^{-2m}d\xi'\right)^{1/2}\left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}}\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^{2m}\left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2 d\xi'\right)^{1/2}.
\end{aligned}$$

Noticing that

$$\int\limits_{|\xi'|\geq\sqrt{\epsilon}}|\xi'|^{-2m}d\xi' = \frac{2}{2m-1}\epsilon^{-m+\frac{1}{2}},$$

and using (3.2.11), we obtain

$$\left|\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)R_2\right| \le C_4\epsilon^{\frac{1}{4}}e^{-(m+\frac{1}{2})\frac{\tau}{2}}\left\|\tilde{r}_1^0\right\|_m\left|\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)\Phi(\xi)\right|. \tag{3.2.13}$$

Next, by changing variables to $\zeta' = \frac{\xi'}{\sqrt{\epsilon}}$, we observe that

$$\int_{\mathbb{R}}\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2\Phi^2(\xi)d\xi \le \frac{C_5}{\sqrt{\epsilon}},$$

where C_5 is independent of ϵ, since $\int_{\mathbb{R}} e^{-z^2}dz = \sqrt{\pi}$ and $\int_{\mathbb{R}}(1+|z|^m)^2e^{-z^2}dz$ is bounded.

Thus,

$$\|R_2\|_m \le C_6 e^{-(m+\frac{1}{2})\frac{\tau}{2}}\left\|\tilde{r}_1^0\right\|_m. \tag{3.2.14}$$

Estimation of the $L^2(m)$ norm of R_3:

By the Cauchy-Schwartz inequality,

$$\begin{aligned}\|R_3\|_m^2 &= \int_{\mathbb{R}}\left(\int_{|\xi'|\le\sqrt{\epsilon}}\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)\hat{\Phi}(\xi,\xi')\xi'\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'\right)^2 d\xi \\ &\le \int_{\mathbb{R}}\epsilon\left(\int_{|\xi'|\le\sqrt{\epsilon}}\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2\hat{\Phi}^2(\xi,\xi')d\xi'\right)\left(\int_{|\xi'|\le\sqrt{\epsilon}}\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^2\left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2d\xi'\right)d\xi.\end{aligned} \tag{3.2.15}$$

Now, we make the substitution $\zeta' = e^{\frac{\tau}{2}}\xi'$ in the second integral so that it is bounded by

$$\int_{|\xi'|\le\sqrt{\epsilon}}\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^2\left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2d\xi' = e^{-\frac{3}{2}\tau}\int_{|\zeta'|\le e^{\frac{\tau}{2}}\sqrt{\epsilon}}\left|\frac{\zeta'}{\sqrt{\epsilon}}\right|^2\left(\tilde{r}_1^0(\zeta')\right)^2d\zeta' \le e^{-\frac{3}{2}\tau}\left\|\tilde{r}_1^0\right\|_m^2. \tag{3.2.16}$$

Also, we change the order of integration in the first integral on the right hand side of

(3.2.15). Hence, the $L^2(m)$ norm of R_3 satisfies

$$\|R_3\|_m \leq e^{-\frac{3}{4}\tau} \left\|\tilde{r}_1^0\right\|_m \sqrt{\epsilon} \left(\int\limits_{|\xi'|\leq\sqrt{\epsilon}} \int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \hat{\Phi}^2(\xi,\xi') d\xi d\xi' \right)^{1/2} . \tag{3.2.17}$$

We recall that

$$\Phi(\xi) = \frac{1}{\sqrt{4\pi\epsilon d_1(1-e^{-\tau})}} e^{-\frac{\xi^2}{4\epsilon d_1(1-e^{-\tau})}}.$$

Hence,

$$\hat{\Phi}(\xi,\xi') = -\int_0^1 \frac{\xi - s\xi'}{4\sqrt{\pi}(\epsilon d_1(1-e^{-\tau}))^{3/2}} e^{-\frac{(\xi-s\xi')^2}{4\epsilon d_1(1-e^{-\tau})}} ds.$$

Since $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$ and $\int_{\mathbb{R}} (1+|z|^m)^2 e^{-z^2} dz$ is bounded, by changing variables to $\zeta' = \frac{\xi'}{\sqrt{\epsilon}}$, it can be proved that

$$\int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \hat{\Phi}^2(\xi,\xi') d\xi \leq C_7 \epsilon^{-\frac{3}{2}},$$

where C_7 is independent of ϵ.

Thus,

$$\int\limits_{|\xi'|\leq\sqrt{\epsilon}} \int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \hat{\Phi}^2(\xi,\xi') d\xi d\xi' \leq \frac{2C_7}{\epsilon}, \tag{3.2.18}$$

which bounds the last term on the right hand side of (3.2.17).

Hence, by (3.2.15), (3.2.16), and (3.2.18), we have established that

$$\|R_3\|_m \leq C_8 e^{-\frac{3}{4}\tau} \left\|\tilde{r}_1^0\right\|_m, \text{ for } m \geq 1. \tag{3.2.19}$$

Finally, by combining the estimates of R_1, R_2, and R_3 in (3.2.12), (3.2.14), and

(3.2.19), we have that for any $m \geq 1$,

$$\|\tilde{r}_1\|_m \leq e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \left[\|R_1\|_m + \|R_2\|_m + \|R_3\|_m\right] \leq C e^{-\frac{\tau}{4}} e^{\epsilon(1-e^{\tau})} \left\|\tilde{r}_1^0\right\|_m, \qquad (3.2.20)$$

where C is independent of ϵ. This establishes (3.1.8).

3.3 Analysis of the Remainder Term $\tilde{w}_2$ (fast component)

In this section, we analyze and estimate the remainder term $\tilde{w}_2$ in the $L^2(m)$ norm. We start by recalling (3.1.9),

$$\tilde{r}_2 = Q_{0,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \tilde{w}_2. \qquad (3.3.1)$$

Plugging (3.3.1) into equation (3.1.24)(b), we obtain the equation for $\tilde{w}_2$,

$$\frac{\partial \tilde{w}_2}{\partial \tau} = (\mathcal{L}_2 - \eta^{-2})\tilde{w}_2 + Q_{0,2}\left(-\frac{\partial}{\partial \tau}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \mathcal{L}_2\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) - \epsilon\eta^{-2}\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right), \qquad (3.3.2)$$

where we note that $\mathcal{L}_2$ and $Q_{0,2}$ commute. Also, we see from this calculation that the first term in (3.3.1) was chosen to remove the first projected term in (3.1.24)(b).

By simplifying and using (3.1.3)(a), we find

$$\frac{\partial}{\partial \tau}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) = \frac{\tilde{y}_\tau}{(1+\eta\tilde{y})^2} - \frac{\tilde{y}^2\eta_\tau}{(1+\eta\tilde{y})^2} = \frac{(\mathcal{L}_1 - \epsilon\eta^{-2})\tilde{y}}{(1+\eta\tilde{y})^2} + \frac{\eta}{2}\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2}. \qquad (3.3.3)$$

Thus, the equation for $\tilde{w}_2$ may be simplified as

$$\frac{\partial \tilde{w}_2}{\partial \tau} = (\mathcal{L}_2 - \eta^{-2})\tilde{w}_2 + Q_{0,2}\left(-\frac{\mathcal{L}_1\tilde{y}}{(1+\eta\tilde{y})^2} - \frac{\eta}{2}\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} + \mathcal{L}_2\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right)\right). \qquad (3.3.4)$$

Here, from the definition (2.1.10) of $Q_{0,2}$ and from $\mathcal{L}_2\psi^0 = 0$, we have

$$\begin{aligned} Q_{0,2}\tilde{y} &= (1 - P_{0,2})\tilde{y} = \alpha^0(\phi^0 - \psi^0) + \tilde{r}_1, \\ Q_{0,2}\mathcal{L}_2\tilde{y} &= \alpha^0\mathcal{L}_2\phi^0 + \mathcal{L}_2\tilde{r}_1. \end{aligned} \tag{3.3.5}$$

Thus, we see that $\tilde{w}_2$ satisfies

$$\frac{\partial \tilde{w}_2}{\partial \tau} = (\mathcal{L}_2 - \eta^{-2})\tilde{w}_2 + \alpha^0\mathcal{L}_2\phi^0 + \mathcal{O}(e^{-\frac{\tau}{4}}e^{\epsilon(1-e^\tau)}), \tag{3.3.6}$$

where we used the fact that $\mathcal{L}_1\tilde{y}, -\eta^2\tilde{y}^2$, and $\mathcal{L}_2\tilde{r}_1$ are $\mathcal{O}(e^{-\frac{\tau}{4}}e^{\epsilon(1-e^\tau)})$. This is the equation that we will analyze to obtain the decay estimate on $\tilde{w}_2$.

Rewriting (3.3.6) in terms of the x and t variables, we find that $w_2 = \eta\tilde{w}_2$ satisfies

$$\frac{\partial w_2}{\partial t} = \epsilon d_2 \frac{\partial^2 w_2}{\partial x^2} - w_2 + (1+t)^{-3/2}\alpha^0\mathcal{L}_2\phi^0 + h.o.t. \tag{3.3.7}$$

Let $w_2 = e^{-t}\rho_2$, then ρ_2 satisfies the inhomogeneous heat equation

$$\frac{\partial \rho_2}{\partial t} = \epsilon d_2 \frac{\partial^2 \rho_2}{\partial x^2} + e^t(1+t)^{-3/2}\alpha^0\mathcal{L}_2\phi^0 + h.o.t. \tag{3.3.8}$$

We use a standard method to write $\rho_2 = u + v$, where u satisfies a homogeneous heat equation with the full initial condition,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \epsilon d_2 \frac{\partial^2 u}{\partial x^2} \\ u(x, 0) &= \rho_2^0(x), \end{aligned} \tag{3.3.9}$$

and v satisfies an inhomogeneous equation with zero initial condition,

$$\begin{aligned} \frac{\partial v}{\partial t} &= \epsilon d_2 \frac{\partial^2 v}{\partial x^2} + e^t(1+t)^{-3/2}\alpha^0\mathcal{L}_2\phi^0 + h.o.t \\ v(x, 0) &= 0. \end{aligned} \tag{3.3.10}$$

The solution $u(x,t)$ of equation (3.3.9) is

$$u(x,t) = \frac{1}{\sqrt{4\pi\epsilon d_2 t}} \int_{\mathbb{R}} e^{-\frac{(x-x')^2}{4\epsilon d_2 t}} \rho_2^0(x') dx'. \tag{3.3.11}$$

By noting that $\rho_2^0(x) = \tilde{w}_2^0(\xi e^{\frac{\tau}{2}})$, we find that in the scaling variables ξ and τ,

$$u(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) = \frac{1}{\sqrt{4\pi\epsilon d_2(1-e^{-\tau})}} \int_{\mathbb{R}} e^{-\frac{(\xi-\xi')^2}{4\epsilon d_2(1-e^{-\tau})}} \tilde{w}_2^0(e^{\frac{\tau}{2}}\xi') d\xi'. \tag{3.3.12}$$

We apply the same analysis as we did in the previous section for $\tilde{r}_1$. First, we write the right hand side of (3.3.12) as a summation of three terms, in the same manner as (3.2.7). Then, following the same steps as used in section 3.2, one can show that the $L^2(m)$ norms of each of the three terms decay exponentially. Hence, by combining the three estimates as was done above in (3.2.20), we obtain

$$\left\| u(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) \right\|_m \leq K_0 e^{-\frac{3}{4}\tau} \left\| \tilde{w}_2^0 \right\|_m, \tag{3.3.13}$$

where K_0 is some constant independent of ϵ.

Next, the solution $v(x,t)$ of equation (4.3.15) is

$$v(x,t) = \int_0^t \frac{1}{\sqrt{4\pi\epsilon d_1(t-t')}} \int_{\mathbb{R}} e^{-\frac{(x-x')^2}{4\epsilon d_1(t-t')}} e^{t'}(1+t')^{-3/2} \alpha^0 \mathcal{L}_2 \phi^0 dx' dt' + h.o.t., \tag{3.3.14}$$

which in the ξ and τ variables (noting $dx' = e^{\tau/2} d\xi'$ and $dt' = e^{\tau'} d\tau'$) becomes,

$$v(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) = \int_0^{\tau} \frac{e^{-\frac{\tau'}{2}} e^{e^{\tau'}-1}}{\sqrt{4\pi\epsilon d_1(1-e^{\tau'-\tau})}} \int_{\mathbb{R}} e^{-\frac{(\xi-\xi')^2}{4\epsilon d_1(1-e^{\tau'-\tau})}} \alpha^0 \mathcal{L}_2 \phi^0 d\xi' d\tau' + h.o.t.. \tag{3.3.15}$$

We write

$$T(\tau) = e^{-\frac{\tau}{2}} e^{e^{\tau}-1} \alpha^0(\tau),$$

$$\Phi(\xi) \equiv \frac{1}{\sqrt{4\pi\epsilon d_1(1-e^{\tau'-\tau})}} \exp\left[-\frac{\xi^2}{4\epsilon d_1(1-e^{\tau'-\tau})}\right],$$

and change the order of integration, so that v may be written succinctly as

$$v(e^{\frac{\tau}{2}}\xi, e^\tau - 1) = \int\limits_{\mathbb{R}} \int\limits_0^\tau T(\tau')\Phi(\xi-\xi')\mathcal{L}_2\phi^0(\xi')d\tau' d\xi' + h.o.t..$$

We observe that

$$\begin{aligned} \int\limits_{\mathbb{R}} \mathcal{L}_2\phi^0(\xi')d\xi' &= \int\limits_{\mathbb{R}} (\mathcal{L}_2 - \mathcal{L}_1)\phi^0(\xi')d\xi' = (\epsilon d_2 - \epsilon d_1)\int\limits_{\mathbb{R}} \partial^2_{\xi'}\phi^0(\xi')d\xi' \\ &= \epsilon(d_2-d_1)\partial_{\xi'}\phi^0(\xi')|^{+\infty}_{-\infty} = 0. \end{aligned} \tag{3.3.16}$$

Then, by subtracting zero and by splitting $\mathbb{R}$ into segments $|\xi'| \geq \sqrt{\epsilon}$ and $|\xi'| \leq \sqrt{\epsilon}$, we find that $v(e^{\frac{\tau}{2}}\xi, e^\tau - 1)$ may be written as

$$\begin{aligned} v(e^{\frac{\tau}{2}}\xi, e^\tau - 1) &= \int\limits_{\mathbb{R}} \int\limits_0^\tau T(\tau')(\Phi(\xi-\xi') - \Phi(\xi))\mathcal{L}_2\phi^0(\xi')d\tau' d\xi' + h.o.t. \\ &= \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \int\limits_0^\tau T(\tau')\Phi(\xi-\xi')d\tau'\mathcal{L}_2\phi^0(\xi')d\xi' \\ &\quad - \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \int\limits_0^\tau T(\tau')\Phi(\xi)d\tau'\mathcal{L}_2\phi^0(\xi')d\xi' \\ &\quad + \int\limits_{|\xi'|\leq\sqrt{\epsilon}} \int\limits_0^\tau T(\tau')(\Phi(\xi-\xi') - \Phi(\xi))\mathcal{L}_2\phi^0(\xi')d\tau' d\xi' + h.o.t.. \end{aligned} \tag{3.3.17}$$

In compact form, we write (3.3.17) as

$$v(e^{\frac{\tau}{2}}\xi, e^\tau - 1) = v_1(\xi,\tau) + v_2(\xi,\tau) + v_3(\xi,\tau) + h.o.t., \tag{3.3.18}$$

where v_1, v_2, and v_3 are given by

$$
\begin{aligned}
v_1 &= \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \int\limits_0^{\tau} T(\tau')\Phi(\xi-\xi')d\tau'\mathcal{L}_2\phi^0(\xi')d\xi', \\
v_2 &= -\int\limits_{|\xi'|\geq\sqrt{\epsilon}} \int\limits_0^{\tau} T(\tau')\Phi(\xi)d\tau'\mathcal{L}_2\phi^0(\xi')d\xi', \\
v_3 &= \int\limits_{|\xi'|\leq\sqrt{\epsilon}} \int\limits_0^{\tau} T(\tau')\Phi'(\xi-\xi')d\tau'\xi'\mathcal{L}_2\phi^0(\xi')d\xi'.
\end{aligned}
\tag{3.3.19}
$$

Note that in the third integral, we also used the following Taylor expansion

$$
\Phi(\xi-\xi') \approx \Phi(\xi) - \Phi'(\xi-\xi')\xi', \text{ for } |\xi'| \leq \sqrt{\epsilon}. \tag{3.3.20}
$$

In the rest of this section, we estimate the $L^2(m)$ norms of each of the terms v_1, v_2, and v_3 individually.

Estimation of the $L^2(m)$ norm of v_1:

Since $1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m \leq K_1\left(1+\left|\frac{\xi-\xi'}{\sqrt{\epsilon}}\right|^m\right)\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m$ for any $|\xi'| \geq \sqrt{\epsilon}$, we have

$$
\left(1+\left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)v_1 \leq K_1 \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \int\limits_0^{\tau} T(\tau')\left(1+\left|\frac{\xi-\xi'}{\sqrt{\epsilon}}\right|^m\right)\Phi(\xi-\xi')d\tau'\left|\frac{\xi'}{\sqrt{\epsilon}}\right|^m \mathcal{L}_2\phi^0(\xi')d\xi'.
$$

Applying the convolution version of Young's inequality $\|g*h\|_{L^2} \leq \|g\|_{L^1}\|h\|_{L^2}$, we obtain

$$\begin{aligned} \|v_1\|_m \leq & K_1 \left(\int\limits_{|\xi'| \geq \sqrt{\epsilon}} \int\limits_0^{\tau} T(\tau') \left(1 + \left| \frac{\xi'}{\sqrt{\epsilon}} \right|^m \right) \Phi(\xi') d\tau' d\xi' \right) \\ & \cdot \left(\int\limits_{|\xi'| \geq \sqrt{\epsilon}} \left| \frac{\xi'}{\sqrt{\epsilon}} \right|^{2m} \left(\mathcal{L}_2 \phi^0(\xi') \right)^2 d\xi' \right)^{1/2} . \end{aligned} \tag{3.3.21}$$

Then, we estimate the first term on the right hand side of (3.3.21) by observing that

$$\begin{aligned} & \int\limits_{|\xi'| \geq \sqrt{\epsilon}} \int\limits_0^{\tau} T(\tau') \left(1 + \left| \frac{\xi'}{\sqrt{\epsilon}} \right|^m \right) \Phi(\xi') d\tau' d\xi' = \int\limits_0^{\tau} T(\tau') d\tau' \int\limits_{|\xi'| \geq \sqrt{\epsilon}} \left(1 + \left| \frac{\xi'}{\sqrt{\epsilon}} \right|^m \right) \Phi(\xi') d\xi' \\ \leq & K_2 \int\limits_0^{\tau} T(\tau') d\tau' \leq K_3 \left(\int\limits_0^{\tau} T^2(\tau') d\tau' \right)^{1/2} , \end{aligned}$$

where K_2 and K_3 are independent of ϵ, since $\int_{\mathbb{R}} \frac{1}{\sqrt{\epsilon}} e^{-\frac{z^2}{\epsilon}} dz = \sqrt{\pi}$. Then, we estimate the second term by observing that

$$\int\limits_{|\xi'| \geq \sqrt{\epsilon}} \left| \frac{\xi'}{\sqrt{\epsilon}} \right|^{2m} \left(\mathcal{L}_2 \phi^0(\xi') \right)^2 d\xi' \leq \left\| \mathcal{L}_2 \phi^0 \right\|_m^2 . \tag{3.3.22}$$

Therefore, we have established that

$$\|v_1\|_m \leq K_3 \left(\int\limits_0^{\tau} T^2(\tau') d\tau' \right)^{1/2} \left\| \mathcal{L}_2 \phi^0 \right\|_m . \tag{3.3.23}$$

Estimation of the $L^2(m)$ norm of v_2:

Using the Cauchy-Schwartz inequality, we find that

$$\int\limits_{|\xi'| \geq \sqrt{\epsilon}} \mathcal{L}_2 \phi^0(\xi') d\xi' \leq \left(\int\limits_{|\xi'| \geq \sqrt{\epsilon}} |\xi'|^{-2} d\xi' \right)^{1/2} \left(\int\limits_{|\xi'| \geq \sqrt{\epsilon}} |\xi'|^2 \left(\mathcal{L}_2 \phi^0(\xi') \right)^2 d\xi' \right)^{1/2} ,$$

where

$$\left(\int_{|\xi'| \geq \sqrt{\epsilon}} |\xi'|^{-2} d\xi' \right)^{1/2} \leq K_4 \epsilon^{-1/4},$$

and

$$\left(\int_{|\xi'| \geq \sqrt{\epsilon}} |\xi'|^2 \left(\mathcal{L}_2 \phi^0(\xi') \right)^2 d\xi' \right)^{1/2} \leq \sqrt{\epsilon} \left\| \mathcal{L}_2 \phi^0 \right\|_m .$$

Thus, we obtain

$$|v_2| \leq K_4 \epsilon^{1/4} \left\| \mathcal{L}_2 \phi^0 \right\|_m \int_0^\tau T(\tau') \Phi(\xi) d\tau'. \tag{3.3.24}$$

Then,

$$\begin{aligned} \|v_2\|_m &\leq K_4 \epsilon^{1/4} \left\| \mathcal{L}_2 \phi^0 \right\|_m \left(\int_{\mathbb{R}} \left(\int_0^\tau T(\tau') \Phi(\xi) \left(1 + \left| \frac{\xi}{\sqrt{\epsilon}} \right|^m \right) d\tau' \right)^2 d\xi \right)^{1/2} \\ &\leq K_4 \epsilon^{1/4} \left\| \mathcal{L}_2 \phi^0 \right\|_m \left(\int_{\mathbb{R}} \int_0^\tau T^2(\tau') \Phi^2(\xi) \left(1 + \left| \frac{\xi}{\sqrt{\epsilon}} \right|^m \right)^2 d\tau' d\xi \right)^{1/2} \\ &= K_4 \epsilon^{1/4} \left\| \mathcal{L}_2 \phi^0 \right\|_m \left(\int_0^\tau T^2(\tau') d\tau' \right)^{1/2} \left(\int_{\mathbb{R}} \Phi^2(\xi) \left(1 + \left| \frac{\xi}{\sqrt{\epsilon}} \right|^m \right)^2 d\xi \right)^{1/2} . \end{aligned} \tag{3.3.25}$$

Here, by changing variable to $\zeta = \frac{\xi}{\sqrt{\epsilon}}$ we observe that

$$\int_{\mathbb{R}} \Phi^2(\xi) \left(1 + \left| \frac{\xi}{\sqrt{\epsilon}} \right|^m \right)^2 d\xi \leq \frac{K_5}{\sqrt{\epsilon}},$$

where K_5 is independent of ϵ, since $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$ and $\int_{\mathbb{R}} (1+|z|^m) e^{-z^2} dz$ is bounded. Thus, we have established that

$$\|v_2\|_m \leq K_6 \left(\int_0^\tau T^2(\tau') d\tau' \right)^{1/2} \left\| \mathcal{L}_2 \phi^0 \right\|_m . \tag{3.3.26}$$

Estimation of the $L^2(m)$ norm of v_3:

By the Cauchy-Schwartz inequality,

$$\begin{aligned}\|v_3\|_m^2 &= \int_{\mathbb{R}} \left(\int_{|\xi'|\le\sqrt{\epsilon}} \int_0^{\tau} T(\tau') \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi'(\xi-\xi') d\tau' \xi' \mathcal{L}_2\phi^0(\xi') d\xi' \right)^2 d\xi \\ &\le \int_{\mathbb{R}} \left(\int_{|\xi'|\le\sqrt{\epsilon}} \left(\int_0^{\tau} T(\tau') \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi'(\xi-\xi') d\tau' \right)^2 d\xi' \right) \\ &\quad \cdot \left(\int_{|\xi'|\le\sqrt{\epsilon}} |\xi'|^2 \left(\mathcal{L}_2\phi^0(\xi')\right)^2 d\xi' \right) d\xi.\end{aligned} \tag{3.3.27}$$

Now, we observe that in the first integral,

$$\left(\int_0^{\tau} T(\tau') \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi'(\xi-\xi') d\tau' \right)^2 \le \int_0^{\tau} T^2(\tau') \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \Phi'^2(\xi-\xi') d\tau', \tag{3.3.28}$$

and in the second integral,

$$\int_{|\xi'|\le\sqrt{\epsilon}} |\xi'|^2 \left(\mathcal{L}_2\phi^0(\xi')\right)^2 d\xi' \le \epsilon \left\|\mathcal{L}_2\phi^0\right\|_m^2. \tag{3.3.29}$$

Then, changing the order of integration, the $L^2(m)$ norm of v_3 satisfies

$$\|v_3\|_m^2 \le \epsilon \left\|\mathcal{L}_2\phi^0\right\|_m^2 \int_0^{\tau} T^2(\tau') \int_{|\xi'|\le\sqrt{\epsilon}} \int_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \Phi'^2(\xi-\xi') d\xi d\xi' d\tau'. \tag{3.3.30}$$

We recall that

$$\Phi(\xi) = \frac{1}{\sqrt{4\pi\epsilon d_1(1-e^{\tau'-\tau})}} e^{-\frac{\xi^2}{4\epsilon d_1(1-e^{\tau'-\tau})}}.$$

Hence

$$\Phi'(\xi-\xi') = -\frac{\xi-\xi'}{4\sqrt{\pi}(\epsilon d_1(1-e^{-\tau}))^{3/2}} e^{-\frac{(\xi-\xi')^2}{4\epsilon d_1(1-e^{-\tau})}}.$$

Since $\int_{\mathbb{R}} \frac{1}{\sqrt{\epsilon}} e^{-\frac{z^2}{\epsilon}} dz = \sqrt{\pi}$, it can be proved that

$$\int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \Phi'^2(\xi - \xi')d\xi \leq K_7 \epsilon^{-\frac{3}{2}},$$

where K_7 is independent of ϵ.

Thus,

$$\int\limits_{|\xi'| \leq \sqrt{\epsilon}} \int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \Phi'^2(\xi - \xi')d\xi d\xi' \leq \frac{2K_7}{\epsilon}, \tag{3.3.31}$$

which bounds the inner integral (which is a double integral on ξ and ξ') on the right hand side of (3.3.30).

Hence, we have established that

$$\|v_3\|_m \leq K_8 \left(\int\limits_0^\tau T^2(\tau')d\tau'\right)^{1/2} \left\|\mathcal{L}_2\phi^0\right\|_m. \tag{3.3.32}$$

Therefore, combining (3.3.23), (3.3.26), and (3.3.32), we see that the estimate of v in the $L^2(m)$ norm becomes

$$\left\|v(e^{\frac{\tau}{2}}\xi, e^\tau - 1)\right\|_m \leq K \left(\int\limits_0^\tau T^2(\tau')d\tau'\right)^{1/2} \left\|\mathcal{L}_2\phi^0\right\|_m, \tag{3.3.33}$$

where K is independent of ϵ.

Next, we estimate $\int_0^\tau T^2(\tau')d\tau'$ in (3.3.33),

$$\int\limits_0^\tau T^2(\tau')d\tau' = \int\limits_0^\tau e^{-\tau'} e^{2(e^{\tau'}-1)} \left(\alpha^0(\tau')\right)^2 d\tau' = \left(\alpha^0(0)\right)^2 \int\limits_0^\tau e^{-\tau'} e^{2(1-\epsilon)(e^{\tau'}-1)} d\tau', \tag{3.3.34}$$

where we recall $\alpha^0(\tau)$ from (3.1.6).

Letting $s' = e^{\tau'}$ and integrating by parts, we obtain

$$\begin{aligned}\int_0^{\tau} T^2(\tau')d\tau' &= \left(\alpha^0(0)\right)^2 \int_1^{e^{\tau}} s'^{-2} e^{2(1-\epsilon)(s'-1)} ds' \\ &= \frac{\left(\alpha^0(0)\right)^2}{2(1-\epsilon)} \left(e^{-2\tau} e^{2(1-\epsilon)(e^{\tau}-1)} - 1\right) + \frac{\left(\alpha^0(0)\right)^2}{1-\epsilon} \int_1^{e^{\tau}} s'^{-3} e^{2(1-\epsilon)(s'-1)} ds' \\ &= \frac{\left(\alpha^0(0)\right)^2}{2(1-\epsilon)} e^{-2\tau} e^{2(1-\epsilon)(e^{\tau}-1)} (1 + \mathcal{O}(e^{-\tau})),\end{aligned} \tag{3.3.35}$$

where in the last step, we observed that the second term from the integration by parts decays $\mathcal{O}(e^{-\tau})$ faster than the leading order term.

Hence, combining (3.3.33) and (3.3.35), we have established that

$$\left\| v(e^{\frac{\tau}{2}}\xi, e^{\tau} - 1) \right\|_m \leq K e^{-\tau} e^{(1-\epsilon)(e^{\tau}-1)} \left\| \mathcal{L}_2 \phi^0 \right\|_m , \tag{3.3.36}$$

Now, since $w_2 = \eta \tilde{w}_2$, $w_2 = e^{-t}\rho_2$, and $\rho_2 = u+v$, we find that in the scaling variables,

$$\tilde{w_2}(\xi, \tau) = e^{\frac{\tau}{2}} e^{1-e^{\tau}} \left(u(e^{\frac{\tau}{2}}\xi, e^{\tau} - 1) + v(e^{\frac{\tau}{2}}\xi, e^{\tau} - 1) \right). \tag{3.3.37}$$

Therefore, combining (3.3.13) and (3.3.36), we obtain the following decay estimate:

$$\|\tilde{w}_2\|_m = \left\| e^{\frac{\tau}{2}} e^{1-e^{\tau}} (u+v) \right\|_m \leq K \left(e^{-\frac{\tau}{4}} e^{1-e^{\tau}} \left\| \tilde{w}_2^0 \right\|_m + e^{-\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \left\| \mathcal{L}_2 \phi^0 \right\|_m \right). \tag{3.3.38}$$

This establishes (3.1.10) and completes the proof of Theorem 3.1.1.

□

3.4 Dynamics of the Zero and First Mode and Decay toward them

In this section, we fix $m \geq 2$ in (2.1.5). This choice is made so that the first two eigenvalues at 0 and $-\frac{1}{2}$ are isolated, and the essential spectrum lies in the left plane $\sigma_{\text{ess}} = \{\lambda \in \mathbb{C} | \text{Re}(\lambda) \leq -\frac{3}{4}\}$. The central idea in the analysis is to separate the solution of (3.1.3) into components, one corresponding to the projection onto the 0 and $-\frac{1}{2}$ modes, another corresponding to the fast-slow structure of the equations, and the final component corresponding to remainder terms, which we will show decay exponentially. In particular, we express the solutions as follows:

$$\begin{aligned} \tilde{y}(\xi,\tau) &= \alpha^0(\tau)\phi^0(\xi) + \alpha^1(\tau)\phi^1(\xi) + \tilde{r}_1(\xi,\tau), \\ \tilde{z}(\xi,\tau) &= \beta^0(\tau)\psi^0(\xi) + \beta^1(\tau)\psi^1(\xi) + \tilde{r}_2(\xi,\tau). \end{aligned} \tag{3.4.1}$$

Theorem 3.4.1. *Fix $m \geq 2$. Let the solution $\tilde{y}(\xi,\tau)$ and $\tilde{z}(\xi,\tau)$ be decomposed as in (3.4.1). Then, $\alpha^0(\tau)$ and $\alpha^1(\tau)$ are given exactly by*

$$\begin{aligned} \alpha^0(\tau) &= \alpha^0(0)e^{\epsilon(1-e^{\tau})}. \\ \alpha^1(\tau) &= \alpha^1(0)e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}. \end{aligned} \tag{3.4.2}$$

and $\beta^0(\tau)\psi^0(\xi) + \beta^1(\tau)\psi^1(\xi)$ is given by

$$\beta^0(\tau)\psi^0(\xi) + \beta^1(\tau)\psi^1(\xi) = P_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \mathcal{O}\left(e^{-\frac{3}{2}\tau}e^{2\epsilon(1-e^{\tau})}\right). \tag{3.4.3}$$

The complementary component $\tilde{r}_1(\xi,\tau)$ is the remainder term which decays exponentially fast in τ, i.e., there exists a constant $C > 0$ independent of ϵ such that

$$\|\tilde{r}_1\|_m \leq Ce^{-\frac{3}{4}\tau}e^{\epsilon(1-e^{\tau})}\left\|\tilde{r}_1^0\right\|_m. \tag{3.4.4}$$

Finally, the other complementary component $\tilde{r}_2$ may be written as

$$\tilde{r}_2(\xi,\tau) = Q_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \tilde{w}_2(\xi,\tau), \tag{3.4.5}$$

where $\tilde{w}_2(\xi,\tau)$ is the remainder term which decays exponentially fast in τ, i.e., there exists a constant $K>0$ independent of ϵ such that

$$\|\tilde{w}_2\|_m \le C\left(e^{-\frac{3}{4}\tau}e^{1-e^{\tau}}\left\|\tilde{w}_2^0\right\|_m + e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}\left\|\mathcal{L}_2\phi^0\right\|_m\right). \tag{3.4.6}$$

Theorem 3.4.1 provides a complete understanding of the dynamics of the solutions (3.4.1) of the Davis-Skodje PDE (3.1.1). First, for $\tilde{y}$, we observe that (3.4.2) completely determines the evolution of $\alpha^0(\tau)$ and $\alpha^1(\tau)$. Also, (3.4.4) shows that the complementary term $\tilde{r}_1$ is a remainder, which decays $\mathcal{O}(e^{-\frac{3}{4}\tau})$ faster than $\alpha^0(\tau)$ and $\alpha^1(\tau)$.

Second, for $\tilde{z}$, we observe that (3.4.3), (3.4.5), and (3.4.6) provide a complete picture. In particular,

$$\tilde{z}(\xi,\tau) = P_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \mathcal{O}\left(e^{-\frac{3}{2}\tau}e^{2\epsilon(1-e^{\tau})}\right) + Q_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \tilde{w}_2(\xi,\tau). \tag{3.4.7}$$

Hence, to leading order,

$$\tilde{z}_0 = P_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + Q_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) = \frac{\tilde{y}}{1+\eta\tilde{y}}, \tag{3.4.8}$$

which is the leading order slow mode. Moreover, we observe that in the analysis of the equation for β^0 (see (3.4.18) and (3.4.21)), one may determine the higher order terms successively. For example, after (3.4.8), the next term in the expansion of $\tilde{z}$ is $-\frac{\eta^3}{2}\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2}$ and higher order terms are $\mathcal{O}\left(e^{-\frac{5}{2}\tau}e^{2\epsilon(1-e^{\tau})}\right)$.

Finally, (3.4.6) establishes that the remainder $\tilde{w}_2$ decays faster than $\tilde{y}$ and $\tilde{z}_0$. Also, it contains a term decaying like $e^{-\frac{3}{4}\tau}e^{1-e^{\tau}}$, which is faster than the remainder $\tilde{r}_1$ of the slow component, as well as a term slowly decaying like $e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}$ and proportional to the difference of diffusion coefficients d_2-d_1.

In this manner, Theorem 3.4.1 establishes that the dynamics of the solutions are approximated to within an exponentially decaying relative error $\alpha^0(\tau)\phi^0(\xi) + \alpha^1(\tau)\phi^1(\xi)$ and the leading order slow mode $\tilde{z}_0 = \frac{\tilde{y}}{1+\eta\tilde{y}}$.

We recall from (3.1.2) that $\xi = \frac{x}{\sqrt{1+t}}$ and $\tau = \log(1+t)$, and we let $r_1 = \eta\tilde{r}_1$ and $w_2 = \eta\tilde{w}_2$. Then, we translate the solutions (3.4.1), (3.4.2), (3.4.3), (3.4.5) and the estimates (3.4.4), (3.4.6) back into the original space and time variables x and t.

Corollary 3.4.2. *In terms of the original variables x and t, the zero and first mode approximation of y and the leading order slow mode are given by*

$$\begin{aligned} y(x,t) &= \frac{1}{\sqrt{4\pi\epsilon d_1}}\left(\alpha^0(0) + \alpha^1(0)\frac{x}{\sqrt{1+t}}\right)\frac{e^{-\epsilon t}}{\sqrt{1+t}}e^{-\frac{x^2}{4\epsilon d_1(1+t)}}, \\ z(x,t) &= \frac{y}{1+y}, \end{aligned} \tag{3.4.9}$$

Also, in terms of the original variables, the remainder terms $r_1(x,t)$ and $w_2(x,t)$ decay algebraically,

$$\begin{aligned} \|r_1\| &\leq C(1+t)^{-\frac{5}{4}}e^{-\epsilon t}\|r_1(x,0)\|, \\ \|w_2\| &\leq K\left((1+t)^{-\frac{5}{4}}e^{-t}\|w_2(x,0)\| + (1+t)^{-1}e^{-\epsilon t}\left\|\mathcal{L}_2\phi^0\right\|_m\right), \end{aligned} \tag{3.4.10}$$

where $\|\cdot\|$ denotes the norm $\|f\| = \left(\int_{\mathbb{R}}(1+|x/\sqrt{\epsilon(1+t)}|^m)^2 f^2(x/\sqrt{1+t})dx\right)^{1/2}$ for any $f \in L^2(\mathbb{R}), m \geq 2$.

Proof of Theorem 3.4.1. We begin the proof by deriving and analyzing the equations for $\alpha^0(\tau)$, $\alpha^1(\tau)$, $\beta^0(\tau)$ and $\beta^1(\tau)$.

We substitute (3.4.1) into (3.1.3) and apply the projection operators P_0 and $P_1 - P_0$

respectively to obtain the following equations for α^0, β^0, α^1 and β^1:

$$\begin{aligned}
\alpha^0_\tau &= -\epsilon\eta^{-2}\alpha^0, \\
\alpha^1_\tau &= -\left(\frac{1}{2} + \epsilon\eta^{-2}\right)\alpha^1, \\
\beta^0_\tau &= -\eta^{-2}\beta^0 + \eta^{-2}\int\limits_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2} d\xi, \\
\beta^1_\tau &= -\left(\frac{1}{2} + \eta^{-2}\right)\beta^1 + \eta^{-2}\int\limits_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right) d\xi, \\
\eta_\tau &= -\frac{\eta}{2}.
\end{aligned} \tag{3.4.11}$$

Here, we used that $\mathcal{L}_1\phi^0 = 0, \mathcal{L}_1\phi^1 = -\frac{1}{2}\phi^1, \mathcal{L}_2\psi^0 = 0, \mathcal{L}_2\psi^1 = -\frac{1}{2}\psi^1, P_{0,2}(g) = \left(\int_{\mathbb{R}} g d\xi\right)\psi^0$,
$(P_{1,2} - P_{0,2})(g) = \left(\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right) g d\xi\right)\psi^1, P_0\begin{pmatrix}\tilde{r}_1\\ \tilde{r}_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, and $P_1\begin{pmatrix}\tilde{r}_1\\ \tilde{r}_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

The ODEs (3.4.11)(a) for α^0 and (3.4.11)(b) for α^1 decouple and may be solved exactly,

$$\begin{aligned}
\alpha^0(\tau) &= \alpha^0(0)e^{\epsilon(1-e^\tau)}. \\
\alpha^1(\tau) &= \alpha^1(0)e^{-\frac{\tau}{2}}e^{\epsilon(1-e^\tau)}.
\end{aligned} \tag{3.4.12}$$

The ODE (3.4.11)(c) for $\beta^0(\tau)$ may be solved using an integrating factor,

$$\begin{aligned}
\beta^0(\tau) &= e^{-\eta^{-2}}\left[\int\limits_0^\tau e^{\eta^{-2}}\eta^{-2}\int\limits_{\mathbb{R}}\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right) d\xi d\hat{\tau} + C\right] \\
&= e^{-\eta^{-2}}\left[\int\limits_{\mathbb{R}}\int\limits_0^\tau e^{\eta^{-2}}\eta^{-2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right) d\hat{\tau} d\xi + C\right].
\end{aligned} \tag{3.4.13}$$

For the inner integral on $\hat{\tau}$, we use integration by parts on the first term, recalling that $\frac{\partial e^{\eta^{-2}}}{\partial\hat{\tau}} = \eta^{-2}e^{\eta^{-2}}$ and noting that the boundary term at $\hat{\tau} = 0$ can be included in the constant C,

$$\beta^0(\tau) = e^{-\eta^{-2}} \left[\int_{\mathbb{R}} \left(e^{\eta^{-2}} \frac{\tilde{y}}{1+\eta\tilde{y}} - \int_0^\tau e^{\eta^{-2}} \frac{\tilde{y}_\tau + \frac{\eta}{2}\tilde{y}^2}{(1+\eta\tilde{y})^2} d\hat{\tau} \right.\right.$$
$$\left.\left. - \int_0^\tau e^{\eta^{-2}} \eta^{-2} \epsilon \frac{\tilde{y}}{(1+\eta\tilde{y})^2} d\hat{\tau} \right) d\xi + C \right].$$

Next, we use (3.1.3)(a) to substitute for $\tilde{y}_\tau$ and to simplify the integrand:

$$\beta^0(\tau) = e^{-\eta^{-2}} \left[\int_{\mathbb{R}} \left(e^{\eta^{-2}} \frac{\tilde{y}}{1+\eta\tilde{y}} - \int_0^\tau e^{\eta^{-2}} \frac{\mathcal{L}_1\tilde{y} + \frac{\eta}{2}\tilde{y}^2}{(1+\eta\tilde{y})^2} d\hat{\tau} \right) d\xi + C \right]. \tag{3.4.14}$$

We recall that $\tilde{y} = \alpha^0\phi^0 + \tilde{r}_1, \mathcal{L}_1\phi^0 = 0$, and $\int_{\mathbb{R}} \tilde{r}_1 d\xi = 0$. Hence, we have

$$\int_{\mathbb{R}} \int_0^\tau e^{\eta^{-2}} \mathcal{L}_1\tilde{y} d\hat{\tau} d\xi = \int_0^\tau e^{\eta^{-2}} \alpha^0 d\hat{\tau} \int_{\mathbb{R}} \mathcal{L}_1\phi^0 d\xi + \mathcal{L}_1 \int_0^\tau e^{\eta^{-2}} \int_{\mathbb{R}} \tilde{r}_1 d\xi d\hat{\tau} = 0. \tag{3.4.15}$$

Then, $\beta^0(\tau)$ becomes

$$\beta^0(\tau) = \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{1}{2} e^{-\eta^{-2}} \left[\int_{\mathbb{R}} \int_0^\tau e^{\eta^{-2}} \frac{\eta\tilde{y}^2}{(1+\eta\tilde{y})^2} d\hat{\tau} d\xi + C \right]. \tag{3.4.16}$$

For the second integral, we use integration by parts again on the inner integral on $\hat{\tau}$,

$$\begin{aligned} \beta^0(\tau) = & \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{1}{2} e^{-\eta^{-2}} \int_{\mathbb{R}} e^{\eta^{-2}} \frac{\eta^3\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi \\ & + \frac{1}{2} e^{-\eta^{-2}} \left[\int_{\mathbb{R}} \int_0^\tau e^{\eta^{-2}} \left(\frac{\eta^3\tilde{y}^2}{(1+\eta\tilde{y})^2} \right)_\tau d\hat{\tau} d\xi + C \right]. \end{aligned} \tag{3.4.17}$$

We observe that the last integral in (3.4.17) decays $\mathcal{O}(e^{-\tau})$ faster than the last integral in (3.4.16), and the constant C can be absorbed into the higher order term. Thus, to the leading order, $\beta^0(\tau)$ may be written as

$$\begin{aligned}\beta^0(\tau) &= \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{\eta^3}{2}\int_{\mathbb{R}} \frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi \left(1+\mathcal{O}(e^{-\tau})\right) \\ &= \int_{\mathbb{R}} \frac{\tilde{y}}{1+\eta\tilde{y}} d\xi - \frac{\eta^3}{2}\int_{\mathbb{R}} \frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi + \mathcal{O}\left(e^{-\frac{5}{2}\tau} e^{2\epsilon(1-e^{\tau})}\right).\end{aligned} \tag{3.4.18}$$

The ODE (3.4.11)(d) for $\beta^1(\tau)$ may be solved as

$$\begin{aligned}\beta^1(\tau) =& e^{-\eta^{-2}}\eta \left[\int_0^{\tau} e^{\eta^{-2}}\eta^{-3}\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right) d\xi d\hat{\tau} + C\right] \\ =& e^{-\eta^{-2}}\eta \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left[\int_0^{\tau} e^{\eta^{-2}}\left(\eta^{-3}+\frac{1}{2}\eta^{-1}\right)\frac{\tilde{y}}{1+\eta\tilde{y}} d\hat{\tau}\right. \\ & \left. - \int_0^{\tau}\frac{1}{2}e^{\eta^{-2}}\eta^{-1}\frac{\tilde{y}}{1+\eta\tilde{y}} d\hat{\tau} - \int_0^{\tau} e^{\eta^{-2}}\eta^{-3}\frac{\epsilon\tilde{y}}{(1+\eta\tilde{y})^2} d\hat{\tau}\right] d\xi + Ce^{-\eta^{-2}}\eta.\end{aligned} \tag{3.4.19}$$

For the inner integral on $\hat{\tau}$, we use integration by parts on the first term, recalling that $\frac{\partial e^{\eta^{-2}}\eta^{-1}}{\partial\hat{\tau}} = e^{\eta^{-2}}\left(\eta^{-3}+\frac{1}{2}\eta^{-1}\right)$ and noting that the boundary term at $\hat{\tau}=0$ can be included in the constant C,

$$\begin{aligned}\beta^1(\tau) =& e^{-\eta^{-2}}\eta \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left[e^{\eta^{-2}}\eta^{-1}\frac{\tilde{y}}{1+\eta\tilde{y}} - \int_0^{\tau} e^{\eta^{-2}}\eta^{-1}\frac{\tilde{y}_{\hat{\tau}}+\frac{\eta}{2}\tilde{y}^2}{(1+\eta\tilde{y})^2} d\tau\right. \\ & \left. -\frac{1}{2}\int e^{\eta^{-2}}\eta^{-1}\frac{\tilde{y}}{1+\eta\tilde{y}} - \int_0^{\tau} e^{\eta^{-2}}\eta^{-3}\frac{\epsilon\tilde{y}}{(1+\eta\tilde{y})^2} d\hat{\tau}\right] d\xi + Ce^{-\eta^{-2}}\eta.\end{aligned} \tag{3.4.20}$$

Next, we use (3.1.3)(a) to substitute for $\tilde{y}_\tau$ and to simplify the integrand:

$$\begin{aligned}\beta^1(\tau) =& \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) d\xi \\ & - e^{-\eta^{-2}}\eta\left[\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\int_0^{\tau} e^{\eta^{-2}}\eta^{-1}\frac{\left(\mathcal{L}_1+\frac{1}{2}\right)\tilde{y}+\frac{\eta}{2}\tilde{y}^2}{(1+\eta\tilde{y})^2} d\hat{\tau} d\xi + C\right].\end{aligned} \tag{3.4.21}$$

We recall that $\tilde{y} = \alpha^0\phi^0 + \alpha^1\phi^1 + \tilde{r}_1, \mathcal{L}_1\phi^0 = 0, \mathcal{L}_1\phi^1 = -\frac{1}{2}, \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\phi^0 d\xi = 0$, and $\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\tilde{r}_1 d\xi = 0$. Hence, we have

$$\begin{aligned} &\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\left(\mathcal{L}_1 + \frac{1}{2}\right)\tilde{y}\right) d\xi \\ &= \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(-\frac{1}{2}\alpha^1\phi^1 + \mathcal{L}_1\tilde{r}_1 + \frac{1}{2}(\alpha^0\phi^0 + \alpha^1\phi^1 + \tilde{r}_1)\right) d\xi = 0. \end{aligned} \tag{3.4.22}$$

Then, $\beta^1(\tau)$ becomes

$$\begin{aligned} \beta^1(\tau) = &\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) d\xi \\ &- \frac{1}{2}e^{-\eta^{-2}}\eta\left[\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\int_0^{\tau} e^{\eta^{-2}}\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} d\hat{\tau}\right) d\xi + C\right]. \end{aligned} \tag{3.4.23}$$

For the second integral, we use integration by parts again on the inner integral on $\hat{\tau}$,

$$\begin{aligned} \beta^1(\tau) = &\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) d\xi - \frac{1}{2}e^{-\eta^{-2}}\eta\left[\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(e^{\eta^{-2}}\eta^2\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2}\right.\right. \\ &\left.\left. - \int_0^{\tau} e^{\eta^{-2}}\eta^2\left(\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2}\right)_{\hat{\tau}} d\hat{\tau}\right) d\xi + C\right]. \end{aligned} \tag{3.4.24}$$

We observe that the last integral in (3.4.24) decays $\mathcal{O}(e^{-\tau})$ faster than the last integral in (3.4.23), and the constant C can be absorbed into the higher order term. Thus, to the leading order, $\beta^1(\tau)$ may be written as

$$\begin{aligned}\beta^1(\tau) &= \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) d\xi \\ &\quad - \frac{1}{2}e^{-\eta^{-2}}\eta \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right) e^{\eta^{-2}}\eta^2 \frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi \left(1+\mathcal{O}(e^{-\tau})\right). \\ &= \int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right)\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) d\xi \\ &\quad - \frac{\eta^3}{2}\int_{\mathbb{R}} H_1\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right) \frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} d\xi + \mathcal{O}\left(e^{-\frac{5}{2}\tau} e^{2\epsilon(1-e^{\tau})}\right).\end{aligned} \tag{3.4.25}$$

These established (3.4.2) and (3.4.3).

Application of Q_1 to system (3.1.3) yields the following system for $\tilde{r}_1$ and $\tilde{r}_2$,

$$\begin{aligned}\frac{\partial \tilde{r}_1}{\partial \tau} &= \mathcal{L}_1 \tilde{r}_1 - \epsilon\eta^{-2}\tilde{r}_1, \\ \frac{\partial \tilde{r}_2}{\partial \tau} &= \mathcal{L}_2 \tilde{r}_2 - \eta^{-2}\tilde{r}_2 + \eta^{-2} Q_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}} - \epsilon\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right),\end{aligned} \tag{3.4.26}$$

where we recall that $Q_{1,2}$ denotes the second component of Q_1.

The analysis of the complementary components $\tilde{r}_1$ and $\tilde{r}_2$, which will establish (3.4.4) - (3.4.6), is carried out in sections 3.5 and 3.6, respectively.

3.5 Analysis of the Remainder Term $\tilde{r}_1$ (slow component)

Most of the analysis here follows that of 3.2. One of the main differences is that in the third term in (3.5.7), we keep also the linear term in the Taylor expansion. Note also that $m \geq 2$ here. In this section, we analyze the remainder term $\tilde{r}_1$ and establish the decay estimate (3.4.4). As in section 3.2, it turns out to be easier to analyze the equation for $\tilde{r}_1$ in the original variables x and t. For (3.4.26)(a), we find that $r_1 = \eta\tilde{r}_1$

satisfies

$$\frac{\partial r_1}{\partial t} = \epsilon d_1 \frac{\partial^2 r_1}{\partial x^2} - \epsilon r_1. \tag{3.5.1}$$

Let $r_1 = e^{-\epsilon t}\rho_1$. Then ρ_1 satisfies heat equation

$$\frac{\partial \rho_1}{\partial t} = \epsilon d_1 \frac{\partial^2 \rho_1}{\partial x^2} \tag{3.5.2}$$

with solution

$$\rho_1(x,t) = \frac{1}{\sqrt{4\pi\epsilon d_1 t}} \int_{\mathbb{R}} e^{-\frac{(x-x')^2}{4\epsilon d_1 t}} \rho_1^0(x')dx', \tag{3.5.3}$$

where $\rho_1^0(x) = \rho_1(x,0)$.

Next, by noting that $\rho_1(x,0) = r_1(x,0) = \tilde{r}_1(e^{\frac{\tau}{2}}\xi, 0) = \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi)$, we find that $\tilde{r}_1(\xi,\tau)$ is

$$\begin{aligned} \tilde{r}_1(\xi,\tau) &= e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \rho_1(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) \\ &= \frac{e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})}}{\sqrt{4\pi\epsilon d_1(1-e^{-\tau})}} \int_{\mathbb{R}} e^{-\frac{(\xi-\xi')^2}{4\epsilon d_1(1-e^{-\tau})}} \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'. \end{aligned} \tag{3.5.4}$$

We write $\Phi(\xi) \equiv \frac{1}{\sqrt{4\pi\epsilon d_1(1-e^{-\tau})}} \exp\left[-\frac{\xi^2}{4\epsilon d_1(1-e^{-\tau})}\right]$, so that $\tilde{r}_1$ becomes

$$\tilde{r}_1(\xi,\tau) = e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \int_{\mathbb{R}} \Phi(\xi-\xi')\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi'.$$

By definition of the projection operator,

$$\begin{aligned} P_0^1\tilde{r}_1 = 0, &\quad \int_{\mathbb{R}} \tilde{r}_1(\xi,\tau)d\xi = 0, \text{ for all } \tau, \\ P_1^1\tilde{r}_1 = 0, &\quad \int_{\mathbb{R}} \xi\tilde{r}_1(\xi,\tau)d\xi = 0, \text{ for all } \tau. \end{aligned} \tag{3.5.5}$$

Then, by subtracting zero and by splitting $\mathbb{R}$ into segments $|\xi'| \geq \sqrt{\epsilon}$ and $|\xi'| \leq \sqrt{\epsilon}$, we find that $\tilde{r}_1$ may be written as

$$\begin{aligned}\tilde{r}_1(\xi,\tau) =& e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \int_{\mathbb{R}} [\Phi(\xi-\xi') - (\Phi(\xi) - \Phi'(\xi)\xi')] \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi' \\ =& e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \Bigg[\int_{|\xi'|\geq\sqrt{\epsilon}} \Phi(\xi-\xi') \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi' - \int_{|\xi'|\geq\sqrt{\epsilon}} (\Phi(\xi) - \Phi'(\xi)\xi') \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi' \\ &+ \int_{|\xi'|\leq\sqrt{\epsilon}} [\Phi(\xi-\xi') - (\Phi(\xi) - \Phi'(\xi)\xi')] \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi' \Bigg].\end{aligned} \tag{3.5.6}$$

In compact form, we write (3.5.6) as

$$\tilde{r}_1(\xi,\tau) = e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \left[R_1 + R_2 + R_3\right], \tag{3.5.7}$$

where R_1, R_2, and R_3 are given by

$$\begin{aligned} R_1 =& \int_{|\xi'|\geq\sqrt{\epsilon}} \Phi(\xi-\xi') \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi', \\ R_2 =& -\int_{|\xi'|\geq\sqrt{\epsilon}} (\Phi(\xi) - \Phi'(\xi)\xi') \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi', \\ R_3 =& \frac{1}{2} \int_{|\xi'|\leq\sqrt{\epsilon}} \hat{\Phi}(\xi,\xi') (\xi')^2 \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi'. \end{aligned} \tag{3.5.8}$$

Note that we also used the following Taylor expansion for $|\xi'| \leq \sqrt{\epsilon}$

$$\Phi(\xi-\xi') = \Phi(\xi) - \Phi'(\xi)\xi' + \frac{1}{2}\hat{\Phi}(\xi,\xi')(\xi')^2, \text{ where } \hat{\Phi}(\xi,\xi') = \int_0^1 (1-s)\Phi''(\xi - s\xi') ds \tag{3.5.9}$$

in the third integral. Here, Φ' and Φ'' denote the first and second order derivative of Φ on ξ.

In the rest of this section, we estimate the $L^2(m)$ norms of each of the terms

R_1, R_2, and R_3 individually.

Estimation of the $L^2(m)$ norm of R_1:

Following the estimation of the $L^2(m)$ norm of R_1 for the zero mode in section 3.2, we find that R_1 for the zero and first mode here has the same estimate as (3.2.12)

$$\|R_1\|_m \leq C_1 e^{-(m+\frac{1}{2})\frac{\tau}{2}} \left\|\tilde{r}_1^0\right\|_m . \tag{3.5.10}$$

Estimation of the $L^2(m)$ norm of R_2:

Using the Cauchy-Schwartz inequality, we find

$$\begin{aligned}
&\left| \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi(\xi) \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi' \right| \\
\leq& \epsilon^{\frac{m}{2}} \left| \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi(\xi) \right| \left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}} |\xi'|^{-2m} d\xi' \right)^{1/2} \left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}} \left|\frac{\xi}{\sqrt{\epsilon}}\right|^{2m} \left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2 d\xi' \right)^{1/2} \\
\leq& C_2 \epsilon^{\frac{1}{4}} e^{-(m+\frac{1}{2})\frac{\tau}{2}} \left\|\tilde{r}_1^0\right\|_m \left| \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi(\xi) \right|,
\end{aligned} \tag{3.5.11}$$

and

$$\left| \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi'(\xi) \int\limits_{|\xi'|\geq\sqrt{\epsilon}} \xi' \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')d\xi' \right|$$

$$\leq \epsilon^{\frac{m}{2}} \left| \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi'(\xi) \right| \left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}} |\xi'|^{2-2m} d\xi' \right)^{1/2} \left(\int\limits_{|\xi'|\geq\sqrt{\epsilon}} \left|\frac{\xi}{\sqrt{\epsilon}}\right|^{2m} \left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2 d\xi' \right)^{1/2}$$

$$\leq C_3 \epsilon^{\frac{3}{4}} e^{-(m+\frac{1}{2})\frac{\tau}{2}} \left\|\tilde{r}_1^0\right\|_m \left| \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \Phi'(\xi) \right|. \tag{3.5.12}$$

Also, using $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$ and $\int_{|z|\geq 1} (1 + |z|^m)^2 e^{-z^2} dz$ being bounded, we observe that

$$\int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \Phi^2(\xi) d\xi \leq C_4 \epsilon^{-\frac{1}{2}}, \text{ and } \int\limits_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \Phi'^2(\xi) d\xi \leq C_5 \epsilon^{-\frac{3}{2}}. \tag{3.5.13}$$

Thus,

$$\|R_2\|_m \leq C_6 e^{-(m+\frac{1}{2})\frac{\tau}{2}} \left\|\tilde{r}_1^0\right\|_m. \tag{3.5.14}$$

Estimation of the $L^2(m)$ norm of R_3:

By the Cauchy-Schwartz inequality,

$$\|R_3\|_m^2 = \frac{1}{4} \int\limits_{\mathbb{R}} \left(\int\limits_{|\xi'|\leq\sqrt{\epsilon}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right) \hat{\Phi}(\xi,\xi') \xi'^2 \tilde{r}_1^0(e^{\frac{\tau}{2}}\xi') d\xi' \right)^2 d\xi$$

$$\leq \frac{1}{4}\epsilon^2 \int\limits_{\mathbb{R}} \left(\int\limits_{|\xi'|\leq\sqrt{\epsilon}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \hat{\Phi}^2(\xi,\xi') d\xi' \right) \left(\int\limits_{|\xi'|\leq\sqrt{\epsilon}} \left|\frac{\xi'}{\sqrt{\epsilon}}\right|^4 \left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2 d\xi' \right) d\xi. \tag{3.5.15}$$

Now, we make the substitution $\zeta' = e^{\frac{\tau}{2}}\xi'$ in the second integral so that it is bounded by

$$\int_{|\xi'|\leq\sqrt{\epsilon}} \left|\frac{\xi'}{\sqrt{\epsilon}}\right|^4 \left(\tilde{r}_1^0(e^{\frac{\tau}{2}}\xi')\right)^2 d\xi' = e^{-\frac{5}{2}\tau} \int_{|\zeta'|\leq e^{\frac{\tau}{2}}\sqrt{\epsilon}} \left|\frac{\zeta'}{\sqrt{\epsilon}}\right|^4 \left(\tilde{r}_1^0(\zeta')\right)^2 d\zeta' \leq e^{-\frac{5}{2}\tau} \left\|\tilde{r}_1^0\right\|_m^2, \tag{3.5.16}$$

where $m \geq 2$. Also, we change the order of integration in the first integral on the right hand side of (3.5.16). Hence, the $L^2(m)$ norm of R_3 satisfies

$$\|R_3\|_m \leq \frac{1}{2}e^{-\frac{5}{4}\tau} \left\|\tilde{r}_1^0\right\|_m \epsilon \left(\int_{|\xi'|\leq\sqrt{\epsilon}} \int_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \hat{\Phi}^2(\xi,\xi')d\xi d\xi' \right)^{1/2}. \tag{3.5.17}$$

We recall that

$$\Phi(\xi) = \frac{1}{\sqrt{4\pi\epsilon d_1(1-e^{-\tau})}} e^{-\frac{\xi^2}{4\epsilon d_1(1-e^{-\tau})}}.$$

Hence,

$$\hat{\Phi}(\xi,\xi') = \int_0^1 (1-s)\left[\frac{1}{4\sqrt{\pi}(\epsilon d_1(1-e^{-\tau}))^{3/2}} - \frac{(\xi - s\xi')^2}{8\sqrt{\pi}(\epsilon d_1(1-e^{-\tau}))^{5/2}}\right] e^{-\frac{(\xi-s\xi')^2}{4\epsilon d_1(1-e^{-\tau})}} ds.$$

Since $\int_{\mathbb{R}} e^{-z^2}dz = \sqrt{\pi}$ and $\int_{|z|\geq 1}(1+|z|^m)^2 e^{-z^2}dz$ is bounded, it can be proved that

$$\int_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \hat{\Phi}^2(\xi,\xi')d\xi \leq C_7\epsilon^{-\frac{5}{2}},$$

where C_7 is independent of ϵ.

Thus,

$$\int_{|\xi'|\leq\sqrt{\epsilon}} \int_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 \hat{\Phi}^2(\xi,\xi')d\xi d\xi' \leq \frac{2C_7}{\epsilon^2}, \tag{3.5.18}$$

which bounds the last term on the right hand side of (3.5.17).

Hence, by (3.5.15), (3.5.16), and (3.5.18), we have established that

$$\|R_3\|_m \le C_8 e^{-\frac{5}{4}\tau} \left\|\tilde{r}_1^0\right\|_m, \text{ for } m \ge 2. \tag{3.5.19}$$

Finally, by combining the estimates of $R_1, R_2,$ and R_3 in (3.5.10), (3.5.14), and (3.5.19), we have that for any $m \ge 2$,

$$\|\tilde{r}_1\|_m \le e^{\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \left[\|R_1\|_m + \|R_2\|_m + \|R_3\|_m\right] \le C e^{-\frac{3}{4}\tau} e^{\epsilon(1-e^{\tau})} \left\|\tilde{r}_1^0\right\|_m, \tag{3.5.20}$$

where C is independent of ϵ. This establishes (3.4.4).

3.6 Analysis of the Remainder Term $\tilde{w}_2$ (fast component)

In this section, we analyze and estimate the remainder term $\tilde{w}_2$ in the $L^2(m)$ norm. We start by recalling (3.4.5),

$$\tilde{r}_2 = Q_{1,2}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \tilde{w}_2. \tag{3.6.1}$$

Plugging (3.6.1) into equation (3.1.24)(b), we obtain the equation for $\tilde{w}_2$,

$$\frac{\partial \tilde{w}_2}{\partial \tau} = (\mathcal{L}_2 - \eta^{-2})\tilde{w}_2 + Q_{1,2}\left(-\frac{\partial}{\partial \tau}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) + \mathcal{L}_2\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) - \epsilon\eta^{-2}\frac{\tilde{y}}{(1+\eta\tilde{y})^2}\right), \tag{3.6.2}$$

where we note that $\mathcal{L}_2$ and $Q_{1,2}$ commute.

By simplifying and using (3.1.3)(a), we find

$$\frac{\partial}{\partial \tau}\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right) = \frac{\tilde{y}_\tau}{(1+\eta\tilde{y})^2} - \frac{\tilde{y}^2\eta_\tau}{(1+\eta\tilde{y})^2} = \frac{(\mathcal{L}_1 - \epsilon\eta^{-2})\tilde{y}}{(1+\eta\tilde{y})^2} + \frac{\eta}{2}\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2}. \tag{3.6.3}$$

Thus, the equation for $\tilde{w}_2$ may be simplified as

$$\frac{\partial \tilde{w}_2}{\partial \tau} = (\mathcal{L}_2 - \eta^{-2})\tilde{w}_2 + Q_{1,2}\left(-\frac{\mathcal{L}_1\tilde{y}}{(1+\eta\tilde{y})^2} - \frac{\eta}{2}\frac{\tilde{y}^2}{(1+\eta\tilde{y})^2} + \mathcal{L}_2\left(\frac{\tilde{y}}{1+\eta\tilde{y}}\right)\right). \quad (3.6.4)$$

Here, we analyze the inhomogeneous terms in (3.6.4) in the following manner:

$$\mathcal{L}_1\tilde{y} = \mathcal{L}_1(\alpha^0\phi^0 + \alpha^1\phi^1 + \tilde{r}_1) = -\frac{1}{2}\alpha^1\phi^1 + \mathcal{L}_1\tilde{r}_1 \quad (3.6.5)$$

because of the spectrum of $\mathcal{L}_1$. Recalling the decay rate (3.4.12) of $\alpha^1(\tau)$ and the decay rate (3.5.20) of $\tilde{r}_1(\xi,\tau)$, the first inhomogeneous term is of $\mathcal{O}(\eta)$ and the first two terms will be kept in the remainder. Then, in order to evaluate the third inhomogeneous term in (3.6.4), using the definition of $Q_{1,2}$ from (2.1.10) and $\mathcal{L}_2\psi^0 = 0, \mathcal{L}_2\psi^1 = -\frac{1}{2}\psi^1$, we have

$$Q_{1,2}\tilde{y} = \alpha^0(\phi^0 - \psi^0) + \alpha^1(\phi^1 - \psi^1) + \tilde{r}_1, \quad (3.6.6)$$

$$Q_{1,2}\mathcal{L}_2\tilde{y} = \alpha^0\mathcal{L}_2\phi^0 + \alpha^1(\mathcal{L}_2\phi^1 + \frac{1}{2}\psi^1) + \mathcal{L}_2\tilde{r}_1. \quad (3.6.7)$$

Thus, only keeping the leading order term in the inhomogeneous part, recalling again the decay rate of $\alpha^1(\tau)$ (3.4.12) and $\tilde{r}_1(\xi,\tau)$ (3.5.20), we see that equation $\tilde{w}_2$ satisfies

$$\frac{\partial \tilde{w}_2}{\partial \tau} = (\mathcal{L}_2 - \eta^{-2})\tilde{w}_2 + \alpha^0\mathcal{L}_2\phi^0 + \mathcal{O}(e^{-\frac{\tau}{2}}e^{\epsilon(1-e^{\tau})}). \quad (3.6.8)$$

We will show that the solution for $\tilde{w}_2$ decays exponentially.

Note that the $\tilde{w}_2$ equation (3.6.8) is same as (3.6.4) when we only separate the zero mode except that $m \geq 2$ here. Now, following section 3.3, we rewrite (3.6.8) in terms of x and t variables to show that $w_2 = \eta\tilde{w}_2$ satisfies

$$\frac{\partial w_2}{\partial t} = \epsilon d_2\frac{\partial^2 w_2}{\partial x^2} - w_2 + (1+t)^{-3/2}\alpha^0\mathcal{L}_2\phi^0 + h.o.t. \quad (3.6.9)$$

Let $w_2 = e^{-t}\rho_2$, then ρ_2 satisfies inhomogeneous heat equation

$$\frac{\partial \rho_2}{\partial t} = \epsilon d_2 \frac{\partial^2 \rho_2}{\partial x^2} + e^t (1+t)^{-3/2} \alpha^0 \mathcal{L}_2 \phi^0 + h.o.t. \tag{3.6.10}$$

We use a standard trick to split the solution for ρ_2 into $u+v$, where u solves a homogeneous heat equation with Dirichlet initial condition, and v solves a inhomogeneous equation with zero initial condition as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \epsilon d_2 \frac{\partial^2 u}{\partial x^2} \\ u(x,0) &= \rho_2^0(x) = \tilde{r}_2^0(\xi e^{\frac{\tau}{2}}), \end{aligned} \tag{3.6.11}$$

and

$$\begin{aligned} \frac{\partial v}{\partial t} &= \epsilon d_2 \frac{\partial^2 v}{\partial x^2} + e^t (1+t)^{-3/2} \alpha^0 \mathcal{L}_2 \phi^0 + h.o.t \\ v(x,0) &= 0. \end{aligned} \tag{3.6.12}$$

Then, in the scaling variables,

$$\tilde{w_2}(\xi,\tau) = e^{\frac{\tau}{2}} e^{1-e^{\tau}} \left(u(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) + v(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) \right). \tag{3.6.13}$$

Estimate of $u(\xi,\tau)$ depends on the choice of m. Similar to the analysis for $\tilde{r}_1$ in section 3.5, as was done above in (3.5.20), we have

$$\left\| e^{\frac{\tau}{2}} e^{1-e^{\tau}} u(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) \right\|_m \le K_1 e^{-\frac{3}{4}\tau} e^{1-e^{\tau}} \left\| \tilde{w}_2^0 \right\|_m. \tag{3.6.14}$$

Estimate of $v(\xi,\tau)$ follows from (3.3.36) in Section 3.3,

$$\left\| v(e^{\frac{\tau}{2}}\xi, e^{\tau}-1) \right\|_m \le K_2 e^{-\tau} e^{(1-\epsilon)(e^{\tau}-1)} \left\| \mathcal{L}_2 \phi^0 \right\|_m. \tag{3.6.15}$$

Finally, we obtain that $\tilde{w}_2$ decays like

$$\|\tilde{w}_2\|_m = \left\| e^{\frac{\tau}{2}} e^{1-e^{\tau}} (u+v) \right\|_m \le K \left(e^{-\frac{3}{4}\tau} e^{1-e^{\tau}} \left\| \tilde{w}_2^0 \right\|_m + e^{-\frac{\tau}{2}} e^{\epsilon(1-e^{\tau})} \left\| \mathcal{L}_2 \phi^0 \right\|_m \right). \tag{3.6.16}$$

This establishes (3.4.6) and completes the proof of Theorem 3.4.1.

□

3.7 Numerical Results

In this section, we illustrate the results of Theorems 3.1.1 and 3.4.1, and Corollaries 3.1.2 and 3.4.2. In particular, we obtain numerical representations of the solutions $(y(x,t), z(x,t))$ of system (3.1.1), and we use these to illustrate the slow modes and the decay toward them, recalling (3.1.14), (3.1.15), (3.4.9), and (3.4.10). We work on the interval $[0, L]$, with Neumann boundary conditions for both components. Also, we take general, non-negative initial conditions $y(x,0) = y_0(x)$ and $z(x,0) = z_0(x)$.

The representations of the solutions of (3.1.1) consist of the exact solution for $y(x,t)$, found using a Fourier cosine series for any non-negative $y_0(x)$, and then a numerically-derived representation for $z(x,t)$ obtained from the z equation in (3.1.1) in which y is represented by the exact solution. This is an advantage of the Davis-Skodje model, but we observe that this special aspect is not needed at all for the theory.

For illustrative purposes, we chose

$$y_0(x) = \frac{6}{L^3}\left(-\frac{x^3}{3} + L\frac{x^2}{2}\right). \tag{3.7.1}$$

In this case, the exact solution for the slow component $y(x,t)$ is

$$y(x,t) = \frac{1}{2}e^{-\epsilon t} + \sum_{k=1}^{\infty} \frac{24(\cos(k\pi) - 1))}{(k\pi)^4} e^{-\epsilon\left(1+d_1(\frac{k\pi}{L})^2\right)t} \cos\left(\frac{k\pi}{L}x\right). \tag{3.7.2}$$

The numerical representation of z is obtained by solving the equation (3.1.1)(b),

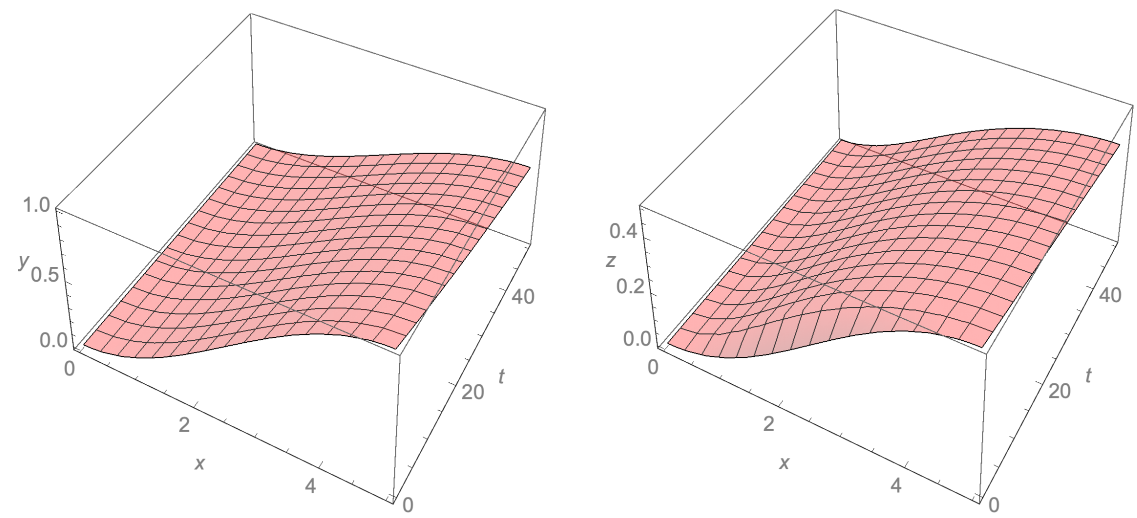
$$\frac{\partial z}{\partial t} = -z + \frac{y}{1+y} - \epsilon \frac{y}{(1+y)^2} + \epsilon d_2 \frac{\partial^2 z}{\partial x^2}, \tag{3.7.3}$$

using the method of lines and the finite element method in Mathematica, where y is represented by its analytical solution (3.7.2). Here, for illustrative purposes, we chose the following initial condition:

$$z(x,0) = \frac{1}{4}\left(1 - \cos\left(\frac{\pi}{L}x\right)\right). \tag{3.7.4}$$

In these illustrations, we first establish a base case in which ϵ is small and the values of d_1 and d_2 are also small. See Figures 3·3-3·5. Second, we illustrate the theory also when d_1 and d_2 have $\mathcal{O}(1)$ values but the difference $d_2 - d_1$ is small. This is the key term in (3.1.15) and (3.4.10), where the remainder terms are bounded by $\|\mathcal{L}_2\phi^0\|$. See Figures 3·6-3·8. Next, we illustrate the theory in the case when d_1 and d_2 are $\mathcal{O}(1)$ and the difference $d_2 - d_1$ is also $\mathcal{O}(1)$. See Figures 3·9 and 3·10. Finally, we illustrate the results for a larger value of ϵ. See Figures 3·11 and 3·12.

The base case is illustrated in Figure 3·3 for small d_1 and d_2.



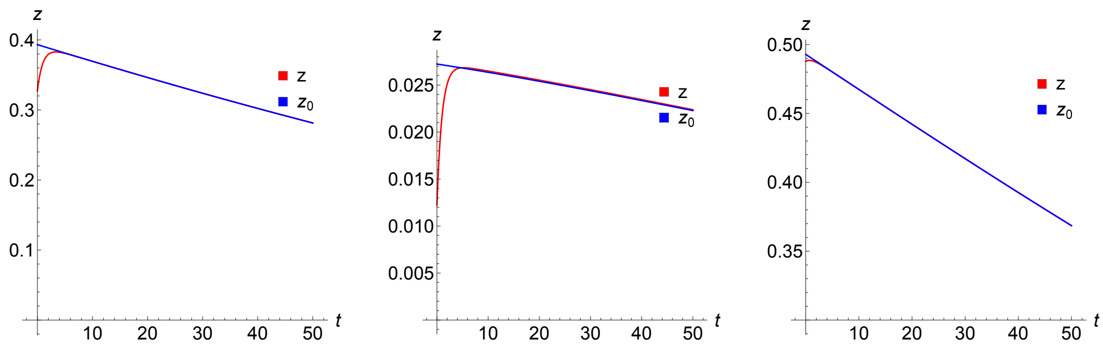
(a) (b)

Figure 3·3: Plots of the solutions of (3.7.2) and (3.7.3) with initial data given by (3.7.1) and (3.7.4), respectively.
(a) y(x,t), (b) z(x,t) for $x \in [0, L]$ with $L = 5$, $t \in [0, T]$ with $T = 50$, and $d_1 = 0.1, d_2 = 0.2$, and $\epsilon = 0.01$.

With these numerical representations of the solutions in hand, we now compare the dynamics of the fast component $z(x, t)$ with the leading order slow mode z_0,

$$z_0(x,t) = \frac{y(x,t)}{1 + y(x,t)}, \tag{3.7.5}$$

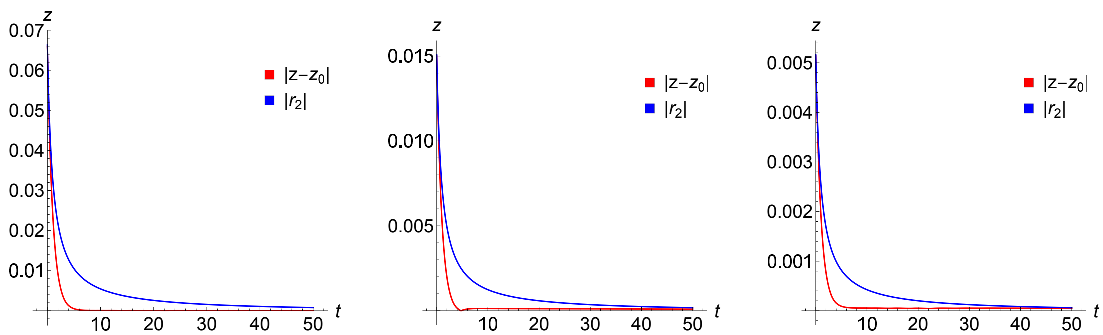
to illustrate the fast algebraic decay and slow exponential decay established in Corollaries 3.1.2 and 3.4.2. We report on the results for $L = 5$, and observe that similar results were obtained for $L = 1$ and $L = 10$. Also, we set the diffusion coefficients $d_1 = 0.1, d_2 = 0.2$ and $\epsilon = 0.01$. Figure 3·4 depicts the dynamics of the leading order slow mode z_0 and the decay of z toward z_0 at three representative value of x: (a) in the interior of the interval at $x = 3$, (b) near the left boundary at $x = 0.5$, and (c) near the right boundary at $x = 4.5$.



(a) (b) (c)

Figure 3·4: Comparison of the full solution z and the leading order slow mode z_0 both as functions of t, for $L = 5, T = 50, d_1 = 0.1, d_2 = 0.2$, and $\epsilon = 0.01$ at (a) $x = 3$, (b) $x = 0.5$, (c) $x = 4.5$.

Figure 3·5 shows comparison of the difference $|z - z_0|$ and the theoretical estimate, $C(1+t)^{-1}e^{-\epsilon t}$ (for some $C > 0$), of the error $|w_2|$ (3.1.15).



(a) (b) (c)

Figure 3·5: Estimates of the difference $|z-z_0|$ between the full solution z and the leading order slow mode z_0, and of $|r_2|$, for $L = 5, T = 50, d_1 = 0.1, d_2 = 0.2$, and $\epsilon = 0.01$, at (a) $x = 3$, (b) $x = 0.5$, (c) $x = 4.5$.

We have similar numerical results for $\mathcal{O}(1)$ values of d_1 and d_2 but with $d_2 - d_2 = 0.1$. For example, when $d_1 = 2$, $d_2 = 2.1$, while other parameters remain the same, i.e. $x \in [0, L]$ with $L = 5$, $t \in [0, T]$ with $T = 50$, and $\epsilon = 0.01$, the full solutions and the comparison of the full and the leading order slow mode are shown in Figures 3·6-3·8:

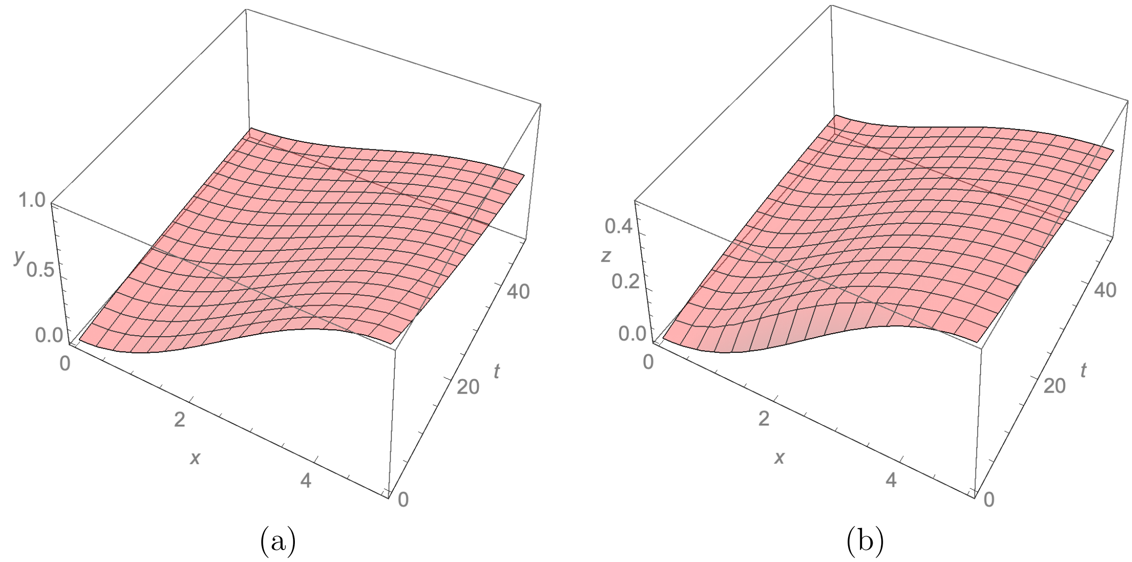


Figure 3·6: Plots of the solutions of (3.7.2) and (3.7.3) with initial data given by (3.7.1) and (3.7.4), respectively.
(a) y(x,t), (b) z(x,t) for $x \in [0, L]$ with $L = 5$, $t \in [0, T]$ with $T = 50$, and $d_1 = 2, d_2 = 2.1$, and $\epsilon = 0.01$.

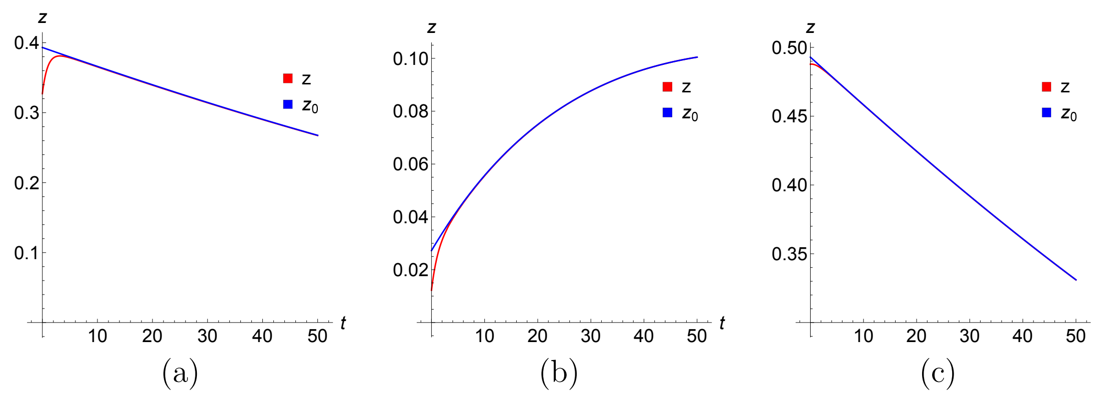
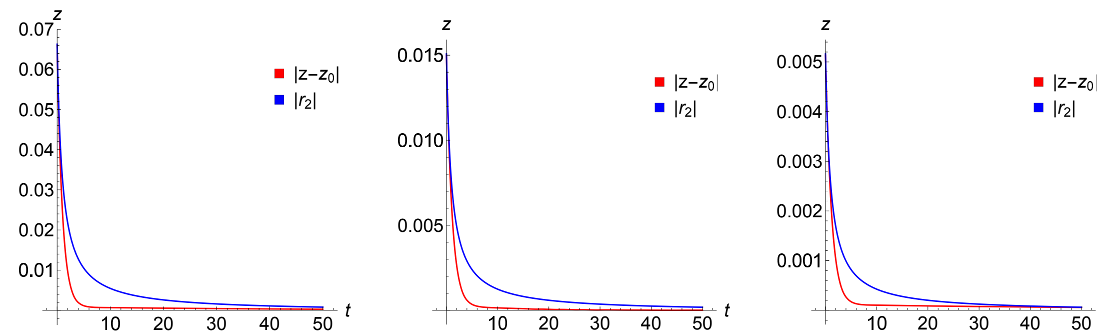


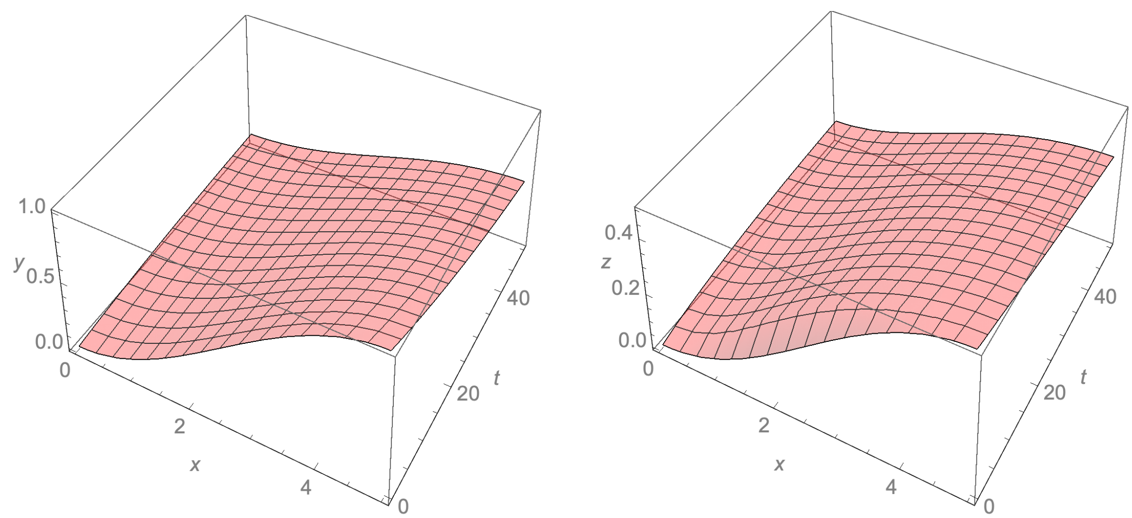
Figure 3·7: Comparison of the full solution z and the approximate slow solution z_0, for $L = 5, T = 50, d_1 = 2, d_2 = 2.1$, and $\epsilon = 0.01$ at (a) $x = 3$, (b) $x = 0.5$, (c) $x = 4.5$.



(a) (b) (c)

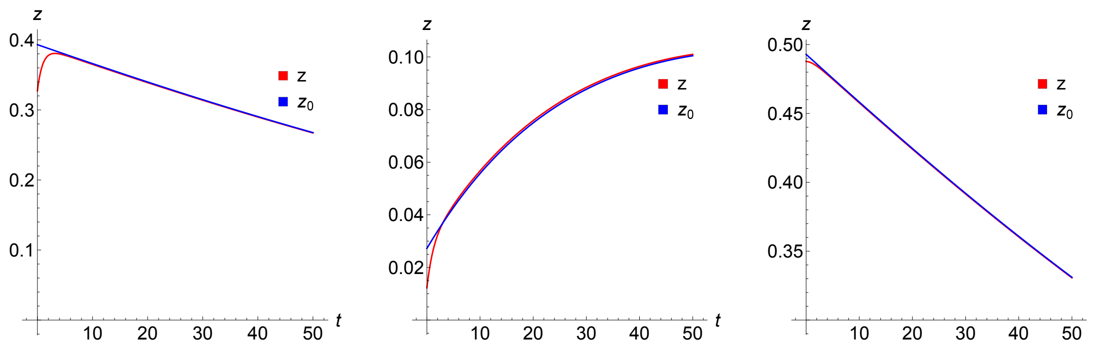
Figure 3·8: Plots of $|z - z_0|$ and $|w_2|$ (3.1.15) for $L = 5, T = 50, d_1 = 2, d_2 = 2.1$, and $\epsilon = 0.01$ at (a) $x = 3$, (b) $x = 0.5$, (c) $x = 4.5$.

Next, we set $d_1 = 2$ and $d_2 = 3$, while other parameters remain the same, i.e. $x \in [0, L]$ with $L = 5$, $t \in [0, T]$ with $T = 50$, and $\epsilon = 0.01$. The full solution still approaches the slow mode z_0 but with a relatively larger difference, due to the larger value of $d_2 - d_1$. The full solutions and the comparison of solutions are shown in the following figures:



(a) (b)

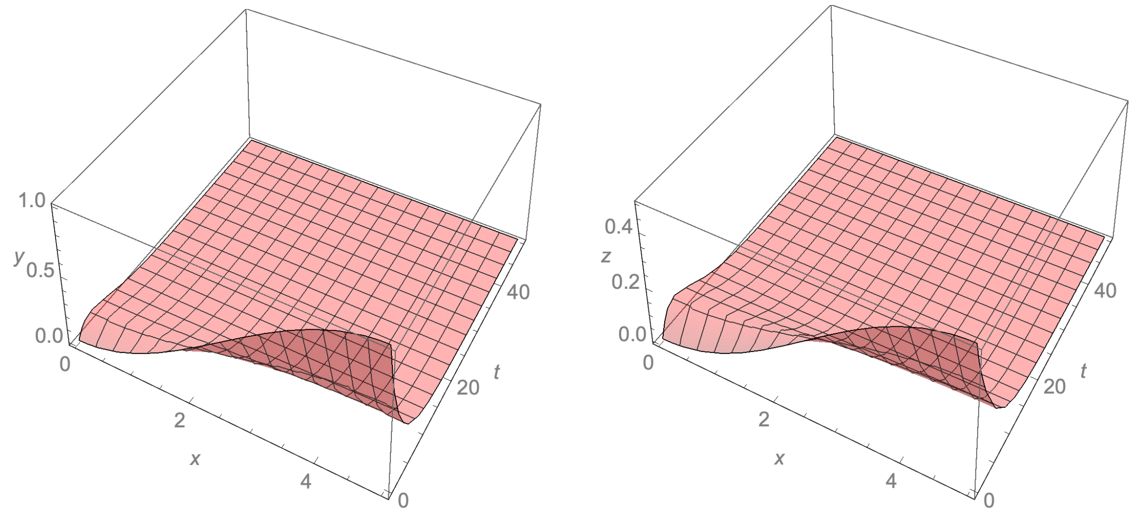
Figure 3·9: Plots of the solutions of (3.7.2) and (3.7.3) with initial data given by (3.7.1) and (3.7.4), respectively.
(a) y(x,t), (b) z(x,t) for $x \in [0, L]$ with $L = 5$, $t \in [0, T]$ with $T = 50$, and $d_1 = 2, d_2 = 3$, and $\epsilon = 0.01$.



(a) (b) (c)

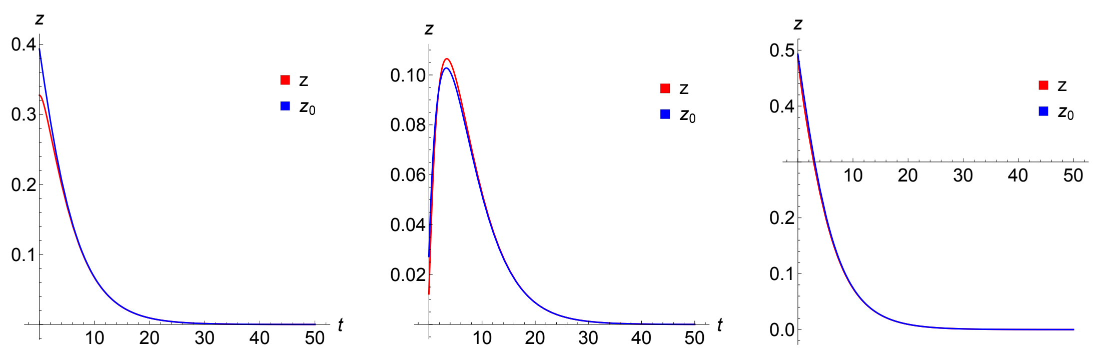
Figure 3·10: Comparison of the full solution z and the leading order slow mode z_0, for $L = 5, T = 50, d_1 = 2, d_2 = 3$, and $\epsilon = 0.01$ at (a) $x = 3$, (b) $x = 0.5$, (c) $x = 4.5$.

When there is not large separation of time scales, i.e. for ϵ relatively large, we observe that the full solution z still approaches the slow mode z_0, but at a slower rate. Figures 3·11 and 3·12 show the full solution and the comparison of the full and approximate solutions for $x \in [0, L]$ with $L = 5$, $t \in [0, T]$ with $T = 50$, $d_1 = 2, d_2 = 3$, and $\epsilon = 0.2$.



(a) (b)

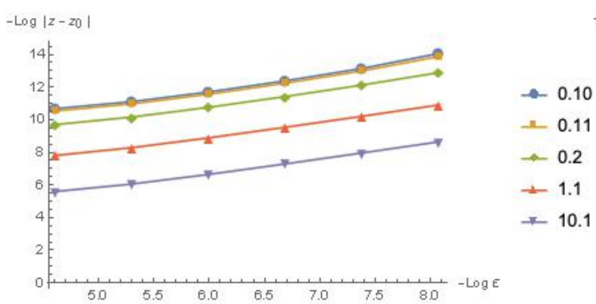
Figure 3·11: Plots of the solutions of (3.7.2) and (3.7.3) with initial data given by (3.7.1) and (3.7.4), respectively.
(a) y(x,t), (b) z(x,t) for $x \in [0, L]$ with $L = 5$, $t \in [0, T]$ with $T = 50$, and $d_1 = 2, d_2 = 3$, and $\epsilon = 0.2$.



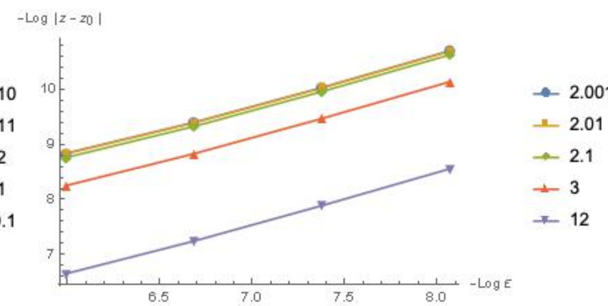
(a) (b) (c)

Figure 3·12: Comparison of the full solution z and the leading order slow mode z_0
for $L = 5, T = 50, d_1 = 2, d_2 = 3$, and $\epsilon = 0.2$ at (a) $x = 3$, (b) $x = 0.5$, (c) $x = 4.5$.

Next, we illustrate the dependence on $d_2 - d_1$ of the results in Theorems 3.1.1 and 3.4.1, and Corollaries 3.1.2 and 3.4.2. We fix $x = 3$ and $t = 50$, and consider the sequence of ϵ values $0.0003125, 0.000625, 0.00125, 0.0025, 0.005, 0.01$. We plot $-\log|z - z_0|$ against $-\log\epsilon$ for different pairs of d_1 and d_2. Figure 3·11 shows the dependence of the difference between the full and approximate solution $|z - z_0|$ on $d_2 - d_1$ with (a) $d_1 = 0.1, d_2 = 0.101, 0.11, 0.2, 1.1, 10.1$, and (b) $d_1 = 2, d_2 = 2.001, 2.01, 2.1, 3, 12$. The figure shows that the difference $|z - z_0|$ increases as $d_2 - d_1$ increases. This dependence agrees with our estimate on w_2, which is proportional to $\|\mathcal{L}_2\phi^0\| = \epsilon(d_2 - d_1)\left\|\partial_\xi^2\phi^0\right\|$.



(a)



(b)

Figure 3·13: Plots of$-\log|z - z_0|$ against $-\log\epsilon$ at $x = 3, t = 50$ for $L = 5, T = 50$, for (a) $d_1 = 0.1, d_2 = 0.101, 0.11, 0.2, 1.1, 10.1$, (b)$d_1 = 2, d_2 = 2.001, 2.01, 2.1, 3, 12$.

The numerical simulations show that over the entire domain $x \in [0, L]$, the full solution z of system (3.1.1) approaches the leading order slow mode z_0 given by (3.7.5), which is derived from the zero mode, in a short time. Also, the difference is bounded by the estimates given in Corollaries 3.1.2 and 3.4.2, $|z - z_0| \leq K(1 + t)^{-1}e^{-\epsilon t}$, for all times, for some $K > 0$. This concludes the illustration of the results of Theorems 3.1.1 and 3.4.1, and Corollaries 3.1.2 and 3.4.2. The full solution of system (3.1.1) is attracted to the slow modes algebraically fast in t and exponentially in ϵt.

Chapter 4

Nonlinear Slow Modes of the MMH Model

In this chapter, we study the MMH Model on the fast time scale, with diffusion of both the slow and fast species,

$$\begin{aligned} y_t &= -\epsilon y + \epsilon(y + a - b)z + \epsilon d_1 y_{xx} \\ z_t &= y - (y + a)z + \epsilon d_2 z_{xx}. \end{aligned} \tag{4.0.1}$$

We begin (chapter 4.1) by transforming (4.0.1) using the scaling variables and by recalling the corresponding linear operators, and the weighted Sobolev space from chapter 2.1. Then, in chapter 4.2, in the equation for y, we push the z dependence in the y component to higher order in ϵ to derive a reduced MMH model in which the y component is decoupled and the z component is linear with inhomogeneous terms depending on y and its derivatives. Chapter 4.3 is to derive the analytical solution of the reduced MMH Model (4.3.3) as a finite series plus rapidly decaying remainders.

4.1 The MMH model in Scaling Variables

In the scaling variables, $\xi = \frac{x}{\sqrt{1+t}}, \tau = \log(1+t),\ y(x,t) = e^{-\frac{\tau}{2}}\tilde{y}(\xi,\tau),\ z(x,t) = e^{-\frac{\tau}{2}}\tilde{z}(\xi,\tau)$, and $\eta = e^{-\frac{\tau}{2}}$, (4.0.1) is written as

$$\begin{aligned}
\tilde{y}_\tau &= (\mathcal{L}_1 - \epsilon\eta^{-2})\tilde{y} + \epsilon\eta^{-2}(a-b)\tilde{z} + \epsilon\eta^{-1}\tilde{y}\tilde{z} \\
\tilde{z}_\tau &= (\mathcal{L}_2 - a\eta^{-2})\tilde{z} + \eta^{-2}\tilde{y} - \eta^{-1}\tilde{y}\tilde{z} \\
\eta_\tau &= -\frac{\eta}{2},
\end{aligned} \tag{4.1.1}$$

where the linear operators are $\mathcal{L}_i\cdot = \epsilon d_i\partial_\xi^2\cdot + \frac{1}{2}\partial_\xi(\xi\cdot),\ i = 1,2.$

We recall from the Davis-Skodje model (section 2.1) that the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ act on their maximal domain in the weighted Sobolev space

$$L^2(m) = \left\{ f \in L^2(\mathbb{R}) |\ \|f\|_m = \left(\int_{\mathbb{R}} \left(1 + \left|\frac{\xi}{\sqrt{\epsilon}}\right|^m\right)^2 (f(\xi))^2 d\xi \right)^{1/2} < \infty \right\}.$$

The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ have point spectrum $\sigma = \{\lambda_k = -\frac{k}{2}, k = 0,1,2,\ldots\}$, and the essential spectrum is given by $\sigma_{\text{ess}} = \left\{\lambda \in \mathbb{C} | \text{Re}(\lambda) \le \frac{1}{4} - \frac{\text{m}}{2}\right\}$. Each eigenvalue λ_k has an associated eigenvector

$$\begin{aligned}
\phi^k(\xi) &= \frac{k!}{2^k}\frac{1}{\sqrt{4\pi\epsilon d_1}} H_k\left(\frac{\xi}{\sqrt{\epsilon d_1}}\right) e^{-\frac{\xi^2}{4\epsilon d_1}} \\
\psi^k(\xi) &= \frac{k!}{2^k}\frac{1}{\sqrt{4\pi\epsilon d_2}} H_k\left(\frac{\xi}{\sqrt{\epsilon d_2}}\right) e^{-\frac{\xi^2}{4\epsilon d_2}}
\end{aligned} \tag{4.1.2}$$

for the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Here, $H_k(\xi)$ denotes the Hermite polynomial of degree k,

$$H_k(\xi) = \frac{2^k}{k!} e^{\frac{\xi^2}{4}} \partial_\xi^k \left(e^{-\frac{\xi^2}{4}}\right), k = 0,1,2,\ldots. \tag{4.1.3}$$

We also recall the projection operator $P_n : L^2(m) \times L^2(m) \to L^2(m) \times L^2(m)$ defined by

$$P_n \begin{pmatrix} f \\ g \end{pmatrix} (\xi) = \sum_{k=0}^{n} \begin{pmatrix} \left(\int_{\mathbb{R}} H_k \left(\frac{\xi'}{\sqrt{\epsilon d_1}} \right) f(\xi') d\xi' \right) \phi^k(\xi) \\ \left(\int_{\mathbb{R}} H_k \left(\frac{\xi'}{\sqrt{\epsilon d_2}} \right) g(\xi') d\xi' \right) \psi^k(\xi) \end{pmatrix}, \tag{4.1.4}$$

and the complementary projection operator defined as $Q_n = \mathbf{1} - P_n$:

$$Q_n \begin{pmatrix} f \\ g \end{pmatrix} (\xi) = \begin{pmatrix} f - \sum_{k=0}^{n} \left(\int_{\mathbb{R}} H_k \left(\frac{\xi'}{\sqrt{\epsilon d_1}} \right) f(\xi') d\xi' \right) \phi^k(\xi) \\ g - \sum_{k=0}^{n} \left(\int_{\mathbb{R}} H_k \left(\frac{\xi'}{\sqrt{\epsilon d_2}} \right) g(\xi') d\xi' \right) \psi^k(\xi) \end{pmatrix}. \tag{4.1.5}$$

We will use $P_{n,1}$, $Q_{n,1}$, $P_{n,2}$, and $Q_{n,2}$ to denote the first and second components of P_n and Q_n, respectively.

4.2 The Reduced MMH Model

The MMH model (4.1.1) is a system of coupled nonlinear PDEs. We start by approximating (4.1.1) by a reduced PDE system for $\tilde{y}$ and $\tilde{z}$ so that the equation for $\tilde{y}$ decouples. This can be achieved by pushing the $\tilde{z}$ dependence in the $\tilde{y}$ component of the system to higher order in ϵ.

We let

$$\tilde{z} = \frac{\tilde{y}}{a + \eta\tilde{y}} + \epsilon\tilde{z}^1. \tag{4.2.1}$$

Then, equation (4.1.1) in terms of $\tilde{y}$ and $\tilde{z}^1$ is

$$\begin{aligned} \tilde{y}_\tau =& \mathcal{L}_1\tilde{y} - \epsilon\eta^{-2} b \frac{\tilde{y}}{a + \eta\tilde{y}} + \epsilon^2\eta^{-2}(\eta\tilde{y} + a - b)\tilde{z}^1 \\ \tilde{z}^1_\tau =& \left(\mathcal{L}_2 - a\eta^{-2} - \eta^{-1}\tilde{y} - \epsilon\eta^{-2} \frac{a(\eta\tilde{y} + a - b)}{(a + \eta\tilde{y})^2} \right) \tilde{z}^1 \\ &+ \eta^{-2} \left(\frac{ab\tilde{y}}{(a + \eta\tilde{y})^3} + \eta^2 \frac{a(d_2 - d_1)\tilde{y}_{\xi\xi}}{(a + \eta\tilde{y})^2} - \eta \frac{2ad_2\tilde{y}_\xi^2}{(a + \eta\tilde{y})^3} \right). \end{aligned} \tag{4.2.2}$$

We have chosen $\tilde{z}^1$ in (4.2.1) so that in (4.2.2)(a) the kinetics is dependent on $\tilde{z}$ only at $\mathcal{O}(\epsilon^2)$. Also, the term $-\epsilon\eta^{-2}\frac{b\tilde{y}}{a+\eta\tilde{y}}$ can be separated into a linear part $-\epsilon\eta^{-2}\frac{b}{a}\tilde{y}$ and a nonlinear part $\epsilon\eta^{-1}\frac{b}{a}\frac{\tilde{y}^2}{a+\eta\tilde{y}}$. Hence, we consider solving $\tilde{y}$ from a decoupled PDE

$$\tilde{y}_\tau = \left(\mathcal{L}_1 - \epsilon\eta^{-2}\frac{b}{a}\right)\tilde{y} + \epsilon\eta^{-1}\frac{b}{a}\frac{\tilde{y}^2}{a+\eta\tilde{y}}, \tag{4.2.3}$$

where $\left(\mathcal{L}_1 - \epsilon\eta^{-2}\frac{b}{a}\right)$ is the effective linear operator.

In (4.2.2)(b), the operator on $\tilde{z}^1$ can be separated into a linear operator $\mathcal{L}_2 - \eta^{-2}\left(a + \epsilon\frac{a-b}{a}\right)$ which is independent of $\tilde{y}$, and a part $-\eta^{-2}f(\eta\tilde{y})$ that depends on $\tilde{y}$ (hence nonlinear in $\tilde{y}$ and $\tilde{z}$), where

$$f(\eta\tilde{y}) = \eta\tilde{y} + \epsilon\frac{a(\eta\tilde{y}+a-b)}{(a+\eta\tilde{y})^2} - \epsilon\frac{a-b}{a} = \eta\tilde{y}\left(1 + \epsilon\frac{1}{a+\eta\tilde{y}}\left(1 - \frac{b}{a} - \frac{b}{a+\eta\tilde{y}}\right)\right).$$

We notice that by assuming $\tilde{y}(\xi,\tau)$ evolves slower than $e^{\frac{\tau}{2}}$, i.e. $\eta\tilde{y}$ is at most $\mathcal{O}(1)$, we have $f(\eta\tilde{y})$ is at most $\mathcal{O}(1)$. Also, by choosing $\epsilon \ll 1$, we have $f(\eta\tilde{y}) > 0$ because $a + \eta\tilde{y} \geq a$. Hence, $-\eta^{-2}f(\eta\tilde{y})$ in the operator on $\tilde{z}^1$ is negative, and is at most $\mathcal{O}(\eta^{-2})$. It pushes the spectrum of the linear operator $\mathcal{L}_2 - \eta^{-2}\left(a + \epsilon\frac{a-b}{a}\right)$ to the left.

In the equation for $\tilde{z}^1(\xi,\tau)$ (4.2.2)(b), we neglect the $-\eta^{-2}f(\eta\tilde{y})$ term in the coefficient of $\tilde{z}^1$ and use the effective linear operator $\mathcal{L}_2 - \eta^{-2}\left(a + \epsilon\frac{a-b}{a}\right)$ to further reduce the MMH PDE model. Hence, we consider the following equation as an approximation of (4.2.2)(b):

$$\tilde{z}^1_\tau = \left(\mathcal{L}_2 - c\eta^{-2}\right)\tilde{z}^1 + \eta^{-2}g(\eta, \tilde{y}, \tilde{y}_\xi, \tilde{y}_{\xi\xi}), \tag{4.2.4}$$

where

$$\begin{aligned} c &= a + \epsilon\left(1 - \frac{b}{a}\right) \\ g(\eta, \tilde{y}, \tilde{y}_\xi, \tilde{y}_{\xi\xi}) &= \frac{ab\tilde{y}}{(a+\eta\tilde{y})^3} + \eta^2\frac{a(d_2-d_1)\tilde{y}_{\xi\xi}}{(a+\eta\tilde{y})^2} - \eta\frac{2ad_2\tilde{y}_\xi^2}{(a+\eta\tilde{y})^3}. \end{aligned} \tag{4.2.5}$$

System (4.2.3) and (4.2.4) forms a reduced version of (4.2.2). We shall refer to it as the reduced MMH PDE. The $\tilde{y}$ component of the reduced PDE is decoupled. The equation for $\tilde{z}^1$ is linear with inhomogeneous terms depending on $\tilde{y}, \tilde{y}_\xi$ and $\tilde{y}_{\xi\xi}$. We will separate the $\tilde{z}^1$ component into a nonlinear slow mode and decay towards it, as in the Davis-Skodje model.

Remark. To get a higher order of accuracy in ϵ, we can also push the $\tilde{z}$ dependence of $\tilde{y}$ to $\mathcal{O}(\epsilon^3)$ by letting

$$\tilde{z}^1 = \frac{g(\eta, \tilde{y}, \tilde{y}_\xi, \tilde{y}_{\xi\xi})}{a + \eta\tilde{y}} + \epsilon\tilde{z}^2. \tag{4.2.6}$$

Note that the superscript here denotes an index, not a power. Then, substituting (4.2.6) into (4.2.2), we have $\tilde{y}$ and $\tilde{z}^2$ satisfy

$$\begin{aligned}
\tilde{y}_\tau &= \left(\mathcal{L}_1 - \epsilon\eta^{-2}\frac{b}{a}\right)\tilde{y} + \epsilon\eta^{-1}\frac{b}{a}\frac{\tilde{y}^2}{a+\eta\tilde{y}} + \epsilon^2\eta^{-2}\frac{\eta\tilde{y}+a-b}{a+\eta\tilde{y}}g + \epsilon^3\eta^{-2}(\eta\tilde{y}+a-b)\tilde{z}^2 \\
\tilde{z}^2_\tau &= \left(\mathcal{L}_2 - a\eta^{-2} - \eta^{-1}\tilde{y} - \epsilon\eta^{-2}\frac{a(\eta\tilde{y}+a-b)}{(a+\eta\tilde{y})^2}\right)\tilde{z}^2 \\
&\quad + \frac{1}{\epsilon}\left(\mathcal{L}_2 - \frac{\partial}{\partial\tau} - \epsilon\eta^{-2}\frac{a(\eta\tilde{y}+a-b)}{(a+\eta\tilde{y})^2}\right)\left(\frac{g}{a+\eta\tilde{y}}\right).
\end{aligned} \tag{4.2.7}$$

Notice that the operator on $\tilde{z}^2$ in (4.2.7)(b) is the same as in (4.2.2)(b), and the inhomogeneous term in (4.2.7)(b) can be proven to be $\mathcal{O}(\eta^{-2})$. Hence, we can reduce (4.2.7) to

$$\begin{aligned}
\tilde{y}_\tau &= \left(\mathcal{L}_1 - \epsilon\eta^{-2}\frac{b}{a}\right)\tilde{y} + \epsilon\eta^{-1}\frac{b}{a}\frac{\tilde{y}^2}{a+\eta\tilde{y}} + \epsilon^2\eta^{-2}\frac{\eta\tilde{y}+a-b}{a+\eta\tilde{y}}g \\
\tilde{z}^2_\tau &= \left(\mathcal{L}_2 - c\eta^{-2}\right)\tilde{z}^2 + \eta^{-2}g^2(\eta, \tilde{y}, \tilde{y}_\xi, \tilde{y}_{\xi\xi}),
\end{aligned} \tag{4.2.8}$$

where we recall c and g from (4.2.5), and g^2 is given by

$$g^2(\eta, \tilde{y}, \tilde{y}_\xi, \ldots \tilde{y}_{\xi\xi}) = \frac{1}{\epsilon}\eta^2\left(\mathcal{L}_2 - \frac{\partial}{\partial\tau} - \epsilon\eta^{-2}\frac{a(\eta\tilde{y}+a-b)}{(a+\eta\tilde{y})^2}\right)\left(\frac{g(\eta, \tilde{y}, \tilde{y}_\xi, \tilde{y}_{\xi\xi})}{a+\eta\tilde{y}}\right).$$

Formally, we can continue pushing the $\tilde{z}$ dependence of $\tilde{y}$ to higher and higher orders in ϵ and derive reduced systems in this manner. However, we have not made an attempt to rigorously control the inhomogeneous terms.

4.3 Analytical Solution with Nonlinear Slow Mode and Remainder

In this section, we analyze the reduced PDE system formed by (4.2.3) and (4.2.4), which we recall are given by

$$\begin{aligned} \tilde{y}_\tau &= \left(\mathcal{L}_1 - \epsilon\eta^{-2}\frac{b}{a}\right)\tilde{y} + \epsilon\eta^{-1}\frac{b}{a}\frac{\tilde{y}^2}{a+\eta\tilde{y}} \\ \tilde{z}^1_\tau &= \left(\mathcal{L}_2 - c\eta^{-2}\right)\tilde{z}^1 + \eta^{-2}g(\eta,\tilde{y},\tilde{y}_\xi,\tilde{y}_{\xi\xi}), \end{aligned} \tag{4.3.1}$$

where

$$\begin{aligned} c &= a + \epsilon\left(1 - \frac{b}{a}\right) \\ g(\eta,\tilde{y},\tilde{y}_\xi,\tilde{y}_{\xi\xi}) &= \frac{ab\tilde{y}}{(a+\eta\tilde{y})^3} + \eta^2\frac{a(d_2-d_1)\tilde{y}_{\xi\xi}}{(a+\eta\tilde{y})^2} - \eta\frac{2ad_2\tilde{y}_\xi^2}{(a+\eta\tilde{y})^3}. \end{aligned} \tag{4.3.2}$$

In the original x and t variables, the reduced MMH model is

$$\begin{aligned} y_t &= -\epsilon b\frac{y}{a+y} + \epsilon d_1 y_{xx} \\ z^1_t &= -cz^1 + \epsilon d_2 z_{xx} + \frac{aby}{(y+a)^3} + \frac{a(d_2-d_1)y_{xx}}{(y+a)^2} - \frac{2ad_2y_x^2}{(y+a)^3}. \end{aligned} \tag{4.3.3}$$

Fix J such that $4 \le J < \infty$. We make the following assumption about y and the first J derivatives:

Assumption. We assume that there exists an upper bound κ such that $0 \le y(x,t) \le \kappa$ for all x and t. Moreover, we assume that the x derivatives of y up to order J are uniformly bounded.

By analyzing the derivatives of (4.3.3)(a), one obtains an equation for y_x,

$$(y_x)_t = \left(\epsilon d_1 \partial_x^2 - \epsilon \frac{ab}{(a+y)^2}\right) y_x. \tag{4.3.4}$$

The solution y_x is bounded by the solution of a simpler linear PDE

$$(y_x)_t = \left(\epsilon d_1 \partial_x^2 - \epsilon \frac{ab}{(a+\kappa)^2}\right) y_x.$$

We can also analyze the higher order derivatives of (4.3.3)(a) to obtain the equations for $y_{xx}, \ldots$

Based on the boundedness assumption of $y(x,t)$ and a finite number of its derivatives, we define the N-th nonlinear slow mode of the reduced MMH model (4.3.3) and express the solution into a finite series with remainders.

For any $0 \leq N < \infty$, the N-th nonlinear slow mode is defined to be

$$z^1_{\text{slow},N}(x,t;\epsilon) = \sum_{j=0}^{N} \epsilon^j \zeta_j(x,t), \tag{4.3.5}$$

with

$$\begin{aligned} \zeta_0 &= \frac{1}{c}\left[\frac{aby}{(y+a)^3} + \frac{a(d_2-d_1)y_{xx}}{(y+a)^2} - \frac{2ad_2 y_x^2}{(y+a)^3}\right], \\ \zeta_j(x,t) &= \frac{1}{c}\left[d_2 \frac{\partial^2}{\partial x^2}\zeta_{j-1}(x,t) - \frac{1}{\epsilon}\frac{\partial}{\partial t}\zeta_{j-1}(x,t)\right], \text{ for all } j \geq 1. \end{aligned} \tag{4.3.6}$$

We need the following preliminary result about the terms $\zeta_j(x,t)$ in the nonlinear slow mode (4.3.5).

Lemma 4.3.1. *For the J fixed above, there exists $1 \le N_0 < \infty$ such that for each $j = 0, 1, 2, \ldots, N_0$, the function $\zeta_j(x,t)$ is uniformly $\mathcal{O}(1)$ for all x and t.*

The proof of Lemma 4.3.1 follows immediately from the **assumption** that the first J-th derivatives of $y(x,t)$ are uniformly bounded. The partial derivatives $\frac{\partial}{\partial t}$ are evaluated using the y equation (4.3.3)(a). We express the terms as spatial derivatives that are all uniformly bounded by assumption. N_0 is chosen such that $2N_0 + 2 \le J$.

Now, in the reduced MMH PDE model (4.3.3), the z^1 component satisfies an inhomogeneous linear PDE. Following the N-th nonlinear slow mode analysis of the Davis-Skodje model, we will use the modal decomposition of $\tilde{z}^1$ naturally given by the operator $\mathcal{L}_2$ in the scaling variables in (4.3.1) to prove the following theorem for the reduced MMH Model (4.3.3) in the original variables.

Theorem 4.3.2. *Fix $m \ge \frac{3}{2}$, $0 \le M \le m - \frac{3}{2}$, and $\epsilon_0 > 0$ sufficiently small. Then, for all $0 < \epsilon < \epsilon_0$, under the major **assumption** on y and its derivatives, there exists $1 \le N_0 < \infty$, such that for all $0 \le N < N_0$, the z^1 component solution of the reduced MMH model (4.3.3) with general initial data $z_0^1(x;\epsilon)$ in the weighted norm space may be written as*

$$z^1(x,t;\epsilon) = z^1_{slow,N}(x,t;\epsilon) + e^{-ct} w^1_N(x,t;\epsilon), \tag{4.3.7}$$

where $z^1_{slow,N}(x,t;\epsilon)$ is the N-th nonlinear slow mode (4.3.5). In addition, for any $0 \le N < N_0$, $w^1_N(x,t;\epsilon)$ in (4.3.7) is given by

$$\begin{aligned} w^1_N(x,t;\epsilon) =& \frac{e^{-\frac{x^2}{4\epsilon d_2(1+t)}}}{\sqrt{4\pi\epsilon d_2(1+t)}} \sum_{j=0}^{M} \frac{j!}{2^j} z^{1,j}(0)(1+t)^{-\frac{j}{2}} H_j\left(\frac{x}{\sqrt{\epsilon d_2(1+t)}}\right) + \mathcal{R}_{M,2}(x,t;\epsilon) \\ & - \int_{\mathbb{R}} G(x-x',t)\left(\sum_{j=0}^{N} \epsilon^j \zeta_j(x',0)\right) dx' + \mathcal{R}_N(x,t;\epsilon), \end{aligned} \tag{4.3.8}$$

where $G(x,t)$ is the Green's function $G(x,t) = \frac{1}{\sqrt{4\pi\epsilon d_2 t}} e^{-\frac{x^2}{4\epsilon d_2 t}}$. $\mathcal{R}_{M,2}(x,t;\epsilon)$ is the fast *decaying remainder,*

$$||\mathcal{R}_{M,2}|| \leq C_2(1+t)^{-\frac{M}{2}+1}||\mathcal{R}_{M,2}(x,0)||, \tag{4.3.9}$$

and $\mathcal{R}_N(x,t;\epsilon)$ *is* $\mathcal{O}(\epsilon^{N+1})$.

By Theorem 4.3.2, the z^1 component of the solution is governed by (4.3.7) and is decomposed completely into a slow part $z^1_{\text{slow},N}(x,t;\epsilon)$ given by (4.3.5), and the fast parts given by (4.3.8). The terms in w^1_N are uniformly bounded in space, and the terms in $e^{-ct}w^1_N$ decay at least as fast as e^{-ct}.

Therefore, Theorem 4.3.2 establishes a solution of the reduced MMH PDE model (4.3.3), as follows. First, we recall from (4.2.1) that

$$z = \frac{y}{y+a} + \epsilon z^1.$$

The approximate solution of the original MMH PDE model (4.0.1) is given by

$$z = z_{\text{slow},N}(x,t;\epsilon) + e^{-ct}w_N(x,t;\epsilon), \tag{4.3.10}$$

where

$$\begin{aligned} z_{\text{slow},N}(x,t;\epsilon) &= \frac{y}{y+a} + \epsilon z^1_{\text{slow},N}(x,t;\epsilon) = \frac{y}{y+a} + \sum_{j=0}^{N} \epsilon^{j+1}\zeta_j(x,t) \\ w_N(x,t;\epsilon) &= \epsilon w^1_N(x,t;\epsilon). \end{aligned} \tag{4.3.11}$$

It is decomposed completely into a slow part $z_{\text{slow},N}(x,t;\epsilon)$, which we label as the N-th nonlinear slow mode, and the fast parts that decay at least as fast as e^{-ct}. Hence, we have shown that the approximate solutions decay exponentially toward the N-th nonlinear slow mode. It still remains to bound the difference between the exact solutions of the full MMH PDE and these approximate solutions; this is beyond the scope of this thesis.

Remark. We briefly compare (4.3.10) and (4.3.11) with the result (2.2.4)(b) and (2.2.5) for the Davis-Skodje PDE. Here, by using (4.2.1), we are looking at solutions whose z components are close to the critical manifold $z = \frac{y}{y+a}$, whereas for the Davis-Skodje PDE we consider general solutions, including those that are $\mathcal{O}(1)$ away from the critical manifold $z = \frac{y}{1+y}$.

Proof of Theorem 4.3.2.

We start by analyzing the $\tilde{z}^1$ component of (4.3.1) in scaling variables. We rewrite (4.3.1)(b) as

$$\tilde{z}^1_\tau + ce^\tau \tilde{z}^1 = \mathcal{L}_2 \tilde{z}^1 + e^\tau g.$$

Let

$$\tilde{v}(\xi, \tau) = \tilde{z}^1(\xi, \tau) e^{c(e^\tau - 1)}. \tag{4.3.12}$$

Then, $\tilde{v}$ satisfies the inhomogeneous heat equation

$$\tilde{v}_\tau = \mathcal{L}_2 \tilde{v} + e^{c(e^\tau - 1) + \tau} g. \tag{4.3.13}$$

Moreover, the initial condition is $\tilde{v}(\xi, 0) = \tilde{z}^1(\xi, 0)$.

We use Duhamel's Principle to solve (4.3.13) by separating $\tilde{v}$ into two parts,

$$\tilde{v} = \tilde{\nu} + \tilde{\omega}, \tag{4.3.14}$$

where $\tilde{\nu}$ and $\tilde{\omega}$ satisfy

$$\tilde{\nu}_\tau = \mathcal{L}_2 \tilde{\nu} + e^{c(e^\tau - 1) + \tau} g, \qquad \tilde{\nu}(\xi, 0) = 0, \tag{4.3.15}$$

$$\tilde{\omega}_\tau = \mathcal{L}_2 \tilde{\omega}, \qquad \tilde{\omega}(\xi, 0) = \tilde{v}(\xi, 0). \tag{4.3.16}$$

First, we observe that the solution of the equation for $\tilde{\omega}$ is

$$\tilde{\omega}(\xi,\tau)=\sum_{j=0}^{M}\tilde{\omega}^j(0)e^{-\frac{j}{2}\tau}\psi^j(\xi)+\tilde{\mathcal{R}}_{M,2}(x,t;\epsilon), \tag{4.3.17}$$

where

$$\left\|\tilde{\mathcal{R}}_{M,2}\right\|_m \le C_1 e^{-\frac{M+1}{2}\tau}\left\|\tilde{\mathcal{R}}^0_{M,2}\right\|_m .$$

Translating back to (x,t) variables and letting $\omega=\eta\tilde{\omega}$, we have

$$\omega(x,t)=\sum_{j=0}^{M}z^j(0)(1+t)^{-\frac{j+1}{2}}\psi^j\left(\frac{x}{\sqrt{1+t}}\right)+\mathcal{R}_{M,2}(x,t;\epsilon), \tag{4.3.18}$$

with $\omega(x,0)=z_0^1(x;\epsilon)$. This yields the first two terms in (4.3.8).

Next, for $\tilde{\nu}$, we let $\nu=\eta\tilde{\nu}$. In the original (x,t) variables, (4.3.15) is transformed back to

$$\nu_t=\epsilon d_2\nu_{xx}+e^{ct}\left[\frac{aby}{(y+a)^3}+\frac{a(d_2-d_1)y_{xx}}{(y+a)^2}-\frac{2ad_2y_x^2}{(y+a)^3}\right],\qquad \nu(x,0)=0. \tag{4.3.19}$$

This equation can be solved exactly as

$$\nu(x,t)=\int_0^t\int_{\mathbb{R}}G(x-x',t-t')e^{ct'}\left[\frac{aby}{(y+a)^3}+\frac{a(d_2-d_1)y_{xx}}{(y+a)^2}-\frac{2ad_2y_x^2}{(y+a)^3}\right]dx'dt', \tag{4.3.20}$$

where y, y_x, and y_{xx} are evaluated at (x',t'). G denotes the Green's function

$$G(x,t)=\frac{1}{\sqrt{4\pi\epsilon d_2 t}}e^{-\frac{x^2}{4\epsilon d_2 t}},$$

which is the solution of the heat equation $G_t=\epsilon d_2 G_{xx}$.

We evaluate the solution $\nu(x,t)$ in (4.3.20) explicitly as an expansion of ϵ using a sequence of integration by parts steps in t'. In each of these steps, "dv" is $ce^{ct'}dt'$.

First, we let (recall (4.3.6)(a))

$$\zeta_0 = \frac{1}{c}\left[\frac{aby}{(y+a)^3} + \frac{a(d_2-d_1)y_{xx}}{(y+a)^2} - \frac{2ad_2y_x^2}{(y+a)^3}\right] \tag{4.3.21}$$

and substitute (4.3.21) into (4.3.20),

$$\nu(x,t) = \int_0^t \int_{\mathbb{R}} G(x-x', t-t')ce^{ct'}\zeta_0(x',t')dx'dt'. \tag{4.3.22}$$

Changing the order of integration (which is permitted due to the exponential decay in space), we carry out the first step of integration by parts in (4.3.22),

$$\begin{aligned} &\int_0^t G(x-x', t-t')ce^{ct'}\zeta_0(x',t')dt' \\ &= \lim_{\sigma\to 0^+} G(x-x',\sigma)e^{c(t-\sigma)}\zeta_0(x', t-\sigma) - G(x-x', t)\zeta_0(x', 0) \\ &+ \int_0^t e^{ct'}\left(\epsilon d_2 G_{x'x'}\zeta_0(x',t') - G\frac{\partial}{\partial t'}\zeta_0(x',t')\right)dt'. \end{aligned} \tag{4.3.23}$$

Here, in the last term, we used $G_{t'}(x-x', t-t') = -\epsilon d_2 G_{x'x'}(x-x', t-t')$. Next, we observe that $\lim_{\sigma\to 0^+} G(x-x',\sigma)$ is a δ-function, i.e.,

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}} G(x-x',\sigma)e^{c(t-\sigma)}\zeta_0(x',t-\sigma)dx' = e^{ct}\zeta_0(x,t).$$

Substituting this and (4.3.23) into the solution (4.3.22), we have

$$\begin{aligned} \nu(x,t) &= e^{ct}\zeta_0(x,t) - \int_{\mathbb{R}} G(x-x',t)\zeta_0(x',0)dx' \\ &+ \int_{\mathbb{R}}\int_0^t e^{ct'}\left(\epsilon d_2 G_{x'x'}\zeta_0(x',t') - G\frac{\partial}{\partial t'}\zeta_0(x',t')\right)dt'dx'. \end{aligned} \tag{4.3.24}$$

The first two terms in (4.3.24) are already in the desired form. It remains to analyze the integrand in the double integral in (4.3.24) and to express it in a simpler form. In particular, we use two steps of integration by parts in x' to rewrite the first term in the integrand, as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} G_{x'x'}(x-x',t-t')\zeta_0(x',t')dx' \\
&= -G_{x'}(x-x',t-t')\zeta_0(x',t')\Big|_{-\infty}^{+\infty} - G(x-x',t-t')\frac{\partial}{\partial x'}\zeta_0(x',t')\Big|_{-\infty}^{+\infty} \\
&\quad + \int_{\mathbb{R}} G(x-x',t-t')\frac{\partial^2}{\partial x'^2}\zeta_0(x',t')dx' \\
&= \int_{\mathbb{R}} G(x-x',t-t')\frac{\partial^2}{\partial x'^2}\zeta_0(x',t')dx'.
\end{aligned} \tag{4.3.25}$$

Here, we used that both $\zeta_0(x,t)$ and $\frac{\partial}{\partial x'}\zeta_0(x,t)$ vanish at infinity.

To simplify the calculation, we write (recall (4.3.6)(b) of ζ_1)

$$\zeta_1(x,t) = \frac{1}{c}\left[d_2\frac{\partial^2}{\partial x^2}\zeta_0(x,t) - \frac{1}{\epsilon}\frac{\partial}{\partial t}\zeta_0(x,t)\right].$$

Hence, the double integral in (4.3.24) becomes

$$\epsilon\int_{\mathbb{R}}\int_0^t ce^{ct'}G(x-x',t-t')\zeta_1(x',t')dt'dx'. \tag{4.3.26}$$

Combining (4.3.24) and (4.3.26), we obtain

$$\nu(x,t) = e^{ct}\zeta_0(x,t) - \int_{\mathbb{R}} G(x-x',t)\zeta_0(x',0)dx' + \epsilon\int_{\mathbb{R}}\int_0^t ce^{ct'}G(x-x',t-t')\zeta_1(x',t')dt'dx'. \tag{4.3.27}$$

This completes the first step of the integration by parts in t'. Moreover, we observe that the last term in (4.3.27) is uniformly bounded, since the double integral is uniformly bounded of $\mathcal{O}(\epsilon)$ due to standard properties of the Gaussian function and the boundedness of ζ_1 established in Lemma 4.3.1.

Next, we perform the induction step, that is we assume that the n-th step of integration by parts in t' has been completed, and we perform the $(n+1)$-st step. The double integral at the end of the n-th step may be evaluated as follows:

$$\begin{aligned}
&\epsilon^n \int_{\mathbb{R}} \int_0^t c e^{ct'} G(x-x', t-t') \zeta_n(x', t') dt' dx' \\
=&\epsilon^n e^{ct} \zeta_n(x,t) - \epsilon^n \int_{\mathbb{R}} G(x-x', t) \zeta_n(x', 0) dx' \\
&+ \int_{\mathbb{R}} \int_0^t e^{ct'} \left(\epsilon^{n+1} d_2 G_{x'x'} \zeta_n(x', t') - \epsilon^n G \frac{\partial}{\partial t'} \zeta_n(x', t') \right) dt' dx' \qquad (4.3.28) \\
=&\epsilon^n e^t \zeta_n(x,t) - \epsilon^n \int_{\mathbb{R}} G(x-x', t) \zeta_n(x', 0) dx' \\
&+ \epsilon^{n+1} \int_{\mathbb{R}} \int_0^t c e^{ct'} G(x-x', t-t') \zeta_{n+1}(x', t') dt' dx',
\end{aligned}$$

where we recall the recursive definition (4.3.6)(b) of ζ_{n+1}. Hence, at the end of the $(n+1)$-st step of integration by parts in t', we have

$$\begin{aligned}
\nu(x,t) =& e^{ct} \sum_{j=0}^{n} \epsilon^j \zeta_j(x,t) - \int_{\mathbb{R}} G(x-x', t) \left(\sum_{j=0}^{n} \epsilon^j \zeta_j(x', 0) \right) dx' \\
&+ \epsilon^{n+1} \int_{\mathbb{R}} \int_0^t e^{ct'} G(x-x', t-t') \zeta_{n+1}(x', t') dt' dx'.
\end{aligned} \qquad (4.3.29)$$

The expression (4.3.29) for $\nu(x,t)$ represents the solution of the full system up to and including $\mathcal{O}(\epsilon^n)$. At each power of ϵ, it contains the terms that give the asymptotic expansion of the N-th nonlinear slow mode and the spatial integral terms that involve the initial condition $y(x', 0)$ and its derivatives. The double integral at the end of (4.3.29) is uniformly bounded of $\mathcal{O}(\epsilon^{n+1})$ due to standard properties of the Gaussian function and to Lemma 4.3.1.

Therefore, by induction on n, we have established that for any $0 \leq N < N_0$,

$$\nu(x,t) = e^{ct}\sum_{j=0}^{N} \epsilon^j \zeta_j(x,t) - \int_{\mathbb{R}} G(x-x',t)\left(\sum_{j=0}^{N} \epsilon^j \zeta_j(x',0)\right) dx' + \mathcal{R}_N(x,t;\epsilon), \quad (4.3.30)$$

with $\nu(x,0) = 0$.

Finally, translating from ν and ω back to z by recalling $z^1 = e^{-ct}(\nu + \omega)$ from (4.3.12) and (4.3.14) and using ω from (4.3.18), we obtain

$$z^1(x,t) = z^1_{\text{slow},N}(x,t;\epsilon) + e^{-ct} w^1_N(x,t;\epsilon), \quad (4.3.31)$$

where we recall that $z^1_{\text{slow},N}$ and w^1_N are given, respectively, by

$$z^1_{\text{slow},N}(x,t;\epsilon) = \sum_{j=0}^{N} \epsilon^j \zeta_j(x,t) \quad (4.3.32)$$

and

$$\begin{aligned} w^1_N(x,t;\epsilon) = &\sum_{j=0}^{M} z^{1,j}(0)(1+t)^{-\frac{j+1}{2}} \psi^j\left(\frac{x}{\sqrt{1+t}}\right) + \mathcal{R}_{M,2}(x,t;\epsilon) \\ &- \int_{\mathbb{R}} G(x-x',t)\left(\sum_{j=0}^{N} \epsilon^j \zeta_j(x',0)\right) dx' + \mathcal{R}_N(x,t;\epsilon). \end{aligned} \quad (4.3.33)$$

Hence, we have established (4.3.7), (4.3.5), and (4.3.8). This completes the proof of Theorem 4.3.2.

□

Chapter 5

A New Class of Multi-Scale Reaction-Diffusion Systems with Closed-form, Low-dimensional, Invariant Manifolds

5.1 A first example: The Davis-Skodje kinetics model with diffusion and cross-diffusion

In this section, we study the following R-D system on the real line:

$$\begin{aligned} y_t &= -\frac{y}{\gamma} + d_1 y_{xx} \\ z_t &= -z + \frac{y}{1+y} - \frac{y}{\gamma(1+y)^2} + d_2 z_{xx} + \frac{(d_1 - d_2) y_{xx}}{(1+y)^2} + \frac{2 d_2 y_x^2}{(1+y)^3}. \end{aligned} \tag{5.1.1}$$

Here, $x \in \mathbb{R}$ and $t \geq 0$. The variables $y = y(x,t)$ and $z = z(x,t)$ denote non-negative species concentrations. The kinetics terms are precisely those in the Davis-Skodje mechanism, see (3.1) in (Davis and Skodje, 1999) (equivalent under time rescaling to the ODE given by (5.1.1) in the case of $d_1, d_2 = 0$). The concentration y satisfies a linear equation, which decouples from the second equation. The concentration z satisfies an equally elementary equation, linear in z with nonlinear y-dependent terms, which are effectively inhomogeneous terms.

The parameter γ satisfies $\gamma > 0$. It measures the separation of the two time scales. For example, with $\gamma > 1$, y is slow, and z is fast, as is the case for the analysis in

(Davis and Skodje, 1999). The parameters $d_1, d_2 \geq 0$ with $d_1 > d_2$ are the diffusivities of the two species, and the additional spatial derivative terms represent cross-diffusion and gradient-squared, as discussed below.

The main results of this section are: (i) the multi-scale R-D system (5.1.1) possesses a closed-form, one-dimensional, invariant manifold; (ii) on the invariant manifold, the system dynamics are governed by a reduced scalar PDE; (iii) all trajectories decay exponentially fast toward the invariant manifold; and, (iv) there exists a geometric decomposition, see (5.1.10), of general solutions of (5.1.1) into a one-dimensional component along the invariant manifold and an infinite-dimensional component in the normal direction, defined as the orthogonal complement of the tangent bundle to the manifold. In effect, the geometric decomposition (5.1.10) shows that the fast stable fibers of the PDE (5.1.1) have been straightened out by the design of the PDE, as may be seen in (5.1.11) and (5.1.12), below.

Remark. As stated above, the equation for y (5.1.1)(a) is linear, and may be solved exactly. In addition, once the solution of y is known, then the second equation in (5.1.1)(b) is a linear equation in z with inhomogeneous terms given by the second, third, fifth, and sixth terms in the equation. Hence it may also be solved exactly. However, neither the decoupling property nor the existence of exact solutions are needed to obtain the model reduction results. Moreover, as shown in Section 6.2.4, the existence of the invariant manifold and the geometric decomposition of general solutions more directly give useful information than the exact solution.

5.1.1 A closed-form, one-dimensional, invariant manifold of (5.1.1)

For this first example PDE (5.1.1), one may verify by direct substitution that the manifold

$$\mathcal{M} = \left\{ (y, z) | \ \ z = \frac{y}{1+y} \right\} \tag{5.1.2}$$

is a closed-form, one-dimensional, invariant manifold of the full PDE system.

Inclusion of the cross-diffusion term $\frac{(d_1-d_2)y_{xx}}{(1+y)^2}$ and the 'gradient-squared' term $\frac{2d_2y_x^2}{(1+y)^3}$ in the z component of the R-D system (5.1.1) precisely achieves this goal of having a closed-form, one-dimensional, invariant manifold. Consider a simpler PDE model with Davis-Skodje kinetics but with only classical diffusion terms,

$$\begin{aligned} y_t &= -\varepsilon y + d_1 y_{xx} \\ z_t &= -z + \frac{y}{1+y} - \varepsilon \frac{y}{(1+y)^2} + d_2 z_{xx}. \end{aligned} \tag{5.1.3}$$

Here, $\varepsilon = \frac{1}{\gamma}$. This more classical PDE, which is referred to as the Davis-Skodje PDE in (Wu and Kaper, 2017), has a nonlinear slow mode (Wu and Kaper, 2017). When $0 < \varepsilon \ll 1$, the asymptotic expansion of the nonlinear slow mode (and of the inertial manifold) is

$$z_{\mathrm{DS}}(x,t) = \frac{y}{1+y} + \varepsilon \left[\frac{(d_2 - d_1) y_{xx}}{(1+y)^2} - \frac{2d_2 y_x^2}{(1+y)^3} \right] + \mathcal{O}(\varepsilon^2) \tag{5.1.4}$$

See Section 5 in (Wu and Kaper, 2017), where also the $\mathcal{O}(\varepsilon^2)$ terms in the asymptotic expansion are given explicitly for $0 < \varepsilon \ll 1$, and where it is shown that these depend on higher-order spatial derivatives of y.

Now, we see from z_{DS} that the first term in the nonlinear slow mode (5.1.4) is the same function that defines $\mathcal{M}$. Moreover, the next terms in the nonlinear slow mode are the same (up to a minus sign, of course) as the cross-diffusion term and 'gradient-squared' term in (5.1.1). Hence, inclusion of these two spatial derivative terms in the PDE (5.1.1) is what eliminates not only these $\mathcal{O}(\varepsilon)$ terms from the formula for $\mathcal{M}$, but also all higher order terms in ε present in the nonlinear slow mode of (5.1.3). Therefore, for the R-D system (5.1.1), the formula for the manifold $\mathcal{M}$ truncates precisely with the closed form (5.1.2).

Most importantly, and surprisingly, the invariance of $\mathcal{M}$ for any $d_1, d_2 \geq 0$ in

(5.1.1) holds for all values of $\gamma > 0$ (as verified by direct substitution), not just large values of γ (i.e., not just when $\varepsilon = 1/\gamma$ is small), even though the insight of adding the two spatial derivative terms to (5.1.1) comes from the large γ analysis.

Also, we see that there is now the outline of a method to generate R-D systems which have closed-form, one-dimensional, invariant manifolds. One starts with a classical R-D system which has multi-scale kinetics, a kinetics slow manifold (known to leading order), and classical diffusion for each species. Then, one finds the terms proportional to the first power of the small parameter (which measures the separation of kinetics time scales) in the asymptotic expansion for the nonlinear slow mode, and adds these terms (with the appropriate signs) to the PDE, as we will pursue in Section 5.2.

We caution that, in introducing the two terms with spatial derivatives of y, no claim is made of relevance to any particular chemical system. This is in analogy to the Davis-Skodje kinetics model in (Davis and Skodje, 1999), where the rational function $\frac{-y}{\gamma(1+y)^2}$ was added for reasons of mathematics and model reduction, without particular chemical motivation, so that $z = \frac{y}{1+y}$ is a closed-form, one-dimensional, invariant manifold of the ODE for all values of $\gamma > 0$. Nevertheless, physically, one might interpret the term $\frac{(d_1-d_2)y_{xx}}{(1+y)^2}$ as a cross-diffusion term, see for example (Jüngel, 2015) for classes of R-D systems with cross-diffusion, and the term $\frac{2d_2y_x^2}{(1+y)^3}$ as a 'gradient-squared' term.

5.1.2 The reduced scalar PDE governing the dynamics of (5.1.1) on $\mathcal{M}$

On the one-dimensional invariant manifold $\mathcal{M}$ defined by (5.1.2), the dynamics of the R-D system (5.1.1) is governed by the decoupled equation for $y(x,t)$, from (5.1.1), with $z(x,t)$ slaved to $y(x,t)$ by (5.1.2). Moreover, there is an exact solution $y(x,t)$ of the decoupled linear equation,

$$y(x,t) = e^{-\epsilon t}\frac{1}{\sqrt{4\pi d_1 t}}\int_{\mathbb{R}} e^{-\frac{(x-x')^2}{4d_1 t}} y(x',0)dx'. \tag{5.1.5}$$

This provides a completely explicit formula for the solutions on $\mathcal{M}$.

5.1.3 $\mathcal{M}$ is normally attracting for (5.1.1)

In this section, we show that the one-dimensional invariant manifold $\mathcal{M} = \{(y,z)|z = \frac{y}{1+y}\}$ of (5.1.1) is exponentially attracting for all $\gamma > 0$ and for all $d_1, d_2 \geq 0$.

We define a new dependent variable $w(x,t)$ via

$$z(x,t) = \frac{y(x,t)}{1+y(x,t)} + w(x,t). \tag{5.1.6}$$

Substitution of (5.1.6) into (5.1.1) shows that w satisfies

$$w_t = -w + d_2 w_{xx}. \tag{5.1.7}$$

The solution of (5.1.7) is

$$w(x,t) = e^{-t}\frac{1}{\sqrt{4\pi d_2 t}}\int_{\mathbb{R}} e^{-\frac{(x-x')^2}{4d_2 t}} w(x',0)dx'. \tag{5.1.8}$$

Now, from the analysis of $\tilde{r}_1$ in scaling variables, we have an estimate for w:

$$\|w(x,t)\| \leq C(1+t)^{-3/4}e^{-t}\|w(x,0)\|, \tag{5.1.9}$$

where $\|\cdot\|$ denotes the norm $\|f\| = \left(\int_{\mathbb{R}}(1+|x/\sqrt{1+t}|^m)^2 f^2(x/\sqrt{1+t})dx\right)^{1/2}$ for any $f \in L^2(\mathbb{R}), m \geq 1$. Hence, solutions are attracted to the one-dimensional invariant manifold $\mathcal{M}$ exponentially fast.

5.1.4 Complete geometric decomposition of general solutions of (5.1.1) in the basin of attraction of $\mathcal{M}$

In this section, we analyze the geometric decomposition of general solutions of (5.1.1). Let $(y(x,0), z(x,0))$ be arbitrary nonnegative initial data in the space $L^2(\mathbb{R})$, bounded in the norm $\|\cdot\|$ introduced above. Let $(y(x,t), z(x,t))$ denote the solution with this data, and write $z(x,0) = \frac{y(x,0)}{1+y(x,0)} + w(x,0)$ for the appropriate function $w(x,0)$. We note that $w(x,0) = 0$ if the initial data lies on $\mathcal{M}$.

Taken together, the results from the previous three sections show that the concentration z in the general solution (y,z) may be decomposed into two components. The first is a one-dimensional component on $\mathcal{M}$, which governs the slow evolution of z. The second is an infinite-dimensional component in the normal bundle to this manifold, governing the exponential decay of the solution toward $\mathcal{M}$ on the fast time scale. In particular, by (5.1.2), (5.1.5), and (5.1.8), we have

$$\begin{aligned} (y(x,t), z(x,t)) &= \left(e^{-\epsilon t} \frac{1}{\sqrt{4\pi d_1 t}} \int\limits_{\mathbb{R}} e^{-\frac{(x-x')^2}{4d_1 t}} y(x',0)dx', \frac{y(x,t)}{1+y(x,t)} + w(x,t) \right), \\ w(x,t) &= e^{-t} \frac{1}{\sqrt{4\pi d_2 t}} \int\limits_{\mathbb{R}} e^{-\frac{(x-x')^2}{4d_2 t}} w(x',0)dx'. \end{aligned} \tag{5.1.10}$$

This provides the geometric decomposition. For the arbitrary initial data $(y(x,0), z(x,0))$, consider the points $b_0(x) = (y(x,0), \frac{y(x,0)}{1+y(x,0)})$ on $\mathcal{M}$ for each $x \in \mathbb{R}$. Also, recall $w(x,0) = z(x,0) - \frac{y(x,0)}{1+y(x,0)}$. Now, for any $b_0(x)$, the set

$$\mathcal{F}_{b_0(x)} = \left\{ (y,z) = \left(y(x,0), \frac{y(x,0)}{1+y(x,0)} + w(x,0) \right) \right\} \tag{5.1.11}$$

for all values of $w(x,0)$ is an infinite-dimensional fiber over $b_0(x)$. Moreover, it may be referred to as a fast, stable fiber over $b_0(x)$, due to the exponential decay of w in the infinite-dimensional function space.

After time t, the image of the point $b_0(x)$ under the flow of (5.1.1) is $b_t(x) = (y(x,t), \frac{y(x,t)}{1+y(x,t)})$, where y is given by (5.1.5). This image is also on $\mathcal{M}$, due to invariance. In addition, we have that

$$(y(x,t), z(x,t)) = b_t(x) + (0, w(x,t)), \tag{5.1.12}$$

which holds for arbitrary $w(x,0)$. Hence, the image of the fiber $\mathcal{F}_{b_0(x)}$ after time t under the flow of (5.1.1) is the fiber $\mathcal{F}_{b_t(x)}$. Finally, along the fast stable fibers, the decay of $w(x,t)$ is given by (5.1.8) and estimated by (5.1.9).

Therefore, taking the union over all $x \in \mathbb{R}$, there is an invariant family $\mathcal{F}_{\mathcal{B}} = \cup_{x\in\mathbb{R}}\mathcal{F}_{b_0(x)}$ of infinite-dimensional fast stable fibers, which foliate the basin of attraction of $\mathcal{M}$ in the infinite-dimensional function space of (5.1.1). This completes the description of the geometric decomposition of general solutions.

We observe that this geometric decomposition extends to the nonlinear PDE (5.1.1) what is known about the Davis-Skodje kinetics ODE (Davis and Skodje, 1999; Kaper and Kaper, 2002), namely that a general solution may be decomposed into a component (base point) which evolves on a slow manifold and a component which decays exponentially toward the manifold in the directions normal to it, along an invariant family of fast stable fibers.

Remark. One may, equivalently, write $w(x,t) = e^{-t}e^{tL_2}w(x,0)$, where $L_2(\cdot) = d_2\frac{\partial^2(\cdot)}{\partial x^2}$, so that the z components of general solutions in the geometric decomposition may also be expressed as

$$\begin{aligned} z(x,t) &= \frac{y(x,t)}{1+y(x,t)} + e^{-t}e^{tL_2}\left(z(x,0) - \frac{y(x,0)}{1+y(x,0f)}\right) \\ &= \frac{y(x,t)}{1+y(x,t)} + e^{-t}\left[w_0(x) + d_2 t w_0(x)_{xx} + \frac{d_2^2t^2}{2!}w_0(x)_{xxxx} + \cdots\right]. \end{aligned}$$

Both formulas for the geometric decomposition provide useful quantitative insights into the dynamics along the fast stable fibers.

Remark. The full R-D system (5.1.1) has an exact solution. The solution of the decoupled linear equation for y is given by (5.1.5), and that for z is readily obtained using Duhamel's Principle:

$$\begin{aligned} y(x,t) =& e^{-\epsilon t}\frac{1}{\sqrt{4\pi d_1 t}}\int\limits_{\mathbb{R}} e^{-\frac{(x-x')^2}{4d_1 t}} y(x',0)dx', \\ z(x,t) =& e^{-t}\frac{1}{\sqrt{4\pi d_2 t}}\int\limits_{\mathbb{R}} e^{-\frac{(x-x')^2}{4d_2 t}} z(x',0)dx' \\ &+ e^{-t}\int\limits_0^t \frac{1}{\sqrt{4\pi d_2(t-t')}}\int\limits_{\mathbb{R}} e^{-\frac{(x-x')^2}{4d_2(t-t')}} e^{t'} I(x',t')dx'dt'. \end{aligned}$$

Here,

$$I(x,t) = \frac{y(x,t)}{1+y(x,t)} - \frac{\varepsilon y(x,t)}{(1+y(x,t))^2} + \frac{(d_1-d_2)y_{xx}(x,t)}{(1+y(x,t))^2} + \frac{2d_2(y_x(x,t))^2}{(1+y(x,t))^3} \quad (5.1.13)$$

denotes the "inhomogeneous" terms in the z-component of (5.1.1). Of course, one may extract (try it!) from this exact solution for z the same complete quantitative information found above in the geometric decomposition (5.1.10), but it is less transparent from the exact solution.

5.1.5 Generalization of the first example R-D system to $\mathbb{R}^N$

In this section, we consider the same PDE (5.1.1) but now on $\mathbb{R}^N$, $N = 2, 3, \ldots$,

$$\begin{aligned} y_t &= \frac{-y}{\gamma} + d_1\Delta y \\ z_t &= -z + \frac{y}{1+y} - \frac{y}{\gamma(1+y)^2} + d_2\Delta z + \frac{(d_1-d_2)}{(1+y)^2}\Delta y + \frac{2d_2}{(1+y)^3}(\nabla y)^2. \end{aligned} \quad (5.1.14)$$

Here, $(\nabla y)^2 = \Sigma_{j=1}^N \left(\frac{\partial y}{\partial x_j}\right)^2$ and $\Delta y = \Sigma_{j=1}^N \frac{\partial^2 y}{\partial x_j^2}$.

By direct substitution, one finds that $\mathcal{M} = \{(y,z)|z = \frac{y}{1+y}\}$ is also a one-dimensional invariant manifold of (5.1.14) for any N, any $\gamma > 0$, and all $d_1, d_2 \geq 0$. In addition, the reduced equation on $\mathcal{M}$ is again given by the linear, decoupled equation for y in (5.1.14), with z slaved to y via the formula defining $\mathcal{M}$. In the case of general N, the exact solution for $y(x,t)$ is

$$y(x,t) = e^{-\frac{1}{\gamma}t}\frac{1}{(4\pi d_1 t)^{N/2}}\int\limits_{\mathbb{R}^N} e^{-\frac{|x-x'|^2}{4d_1 t}} y(x',0)dx'$$

with general initial condition $y(x,0)$.

Furthermore, $\mathcal{M}$ is normally attracting also for (5.1.14), as may be shown following the procedure used above in the case of $N = 1$. In particular, by writing $z = (y/1+y)+w$, one finds that w, the component of z in the orthogonal complement to the tangent bundle of $\mathcal{M}$, satisfies the same equation $w_t = -w + d_2\Delta w$, now in $\mathbb{R}^N$, and the solutions of this equation decay as e^{-t}. Hence, there is the same geometric decomposition of all solutions of (5.1.14) near $\mathcal{M}$, as well.

5.2 Second example: A modified Michaelis-Menten-Henri mechanism with diffusion and cross-diffusion

In this section, we analyze a multi-scale R-D system with modified Michaelis-Menten-Henri kinetics,

$$\begin{aligned} y_t &= -\varepsilon y + \varepsilon(y+a-b)z + d_1 y_{xx} \\ z_t &= y - (y+a)z + d_2 z_{xx} - \frac{\varepsilon aby}{(y+a)^3} + \frac{a(d_1-d_2)y_{xx}}{(y+a)^2} + \frac{2ad_2 y_x^2}{(y+a)^3}. \end{aligned} \tag{5.2.1}$$

Here, the kinetics terms consist of the classical Michaelis-Menten-Henri model with quadratic nonlinearities (Edelstein-Keshet, 2005; Heineken et al., 1967; Michaelis and Menten, 1913), plus an additional kinetics term, $-\frac{\varepsilon aby}{(y+a)^3}$, which is chosen based on the strategy of (Davis and Skodje, 1999), so that the kinetics ODE ((5.2.1) with

$d_1, d_2 = 0$) has a closed-from, one-dimensional, invariant manifold. Also, the diffusion terms in this system consist of classical diffusion of both species, as well as the spatial derivative terms in the z-component representing cross-diffusion, $\frac{a(d_1-d_2)y_{xx}}{(y+a)^2}$, and the 'gradient squared', $\frac{2ad_2y_x^2}{(y+a)^3}$.

Remark. The original MMH model with classical diffusivities (see for example (Yannacopoulos et al., 1995; Kalachev et al., 2007; Kaper and Kaper, 2002)) is

$$\begin{aligned} y_t &= -\varepsilon y + \varepsilon(y+a-b)z + d_1 y_{xx} \\ z_t &= y - (y+a)z + d_2 z_{xx}. \end{aligned} \tag{5.2.2}$$

5.2.1 A closed-form, one-dimensional, invariant manifold for (5.2.1)

The PDE (5.2.1) has been developed so that it has a closed-form, one-dimensional, invariant manifold,

$$\mathcal{M} = \left\{ (y,z) | \;\; z(y) = \frac{y}{y+a} \right\}, \tag{5.2.3}$$

similar to the manifold in the first example PDE. One may verify by direct substitution that $\mathcal{M}$ satisfies the invariance equation of (5.2.1) for all $d_1, d_2, \varepsilon \geq 0$, as follows. In compact form, the invariance equation is

$$z'(y) y_t = z_t.$$

Then, translated in terms of the components of the vector field, the invariance equation is

$$z'(y)[-\varepsilon y + \varepsilon(y+a-b)z + d_1 y_{xx}] = y - (y+a)z + d_2 z_{xx} - \frac{\varepsilon aby}{(y+a)^3} + \frac{a(d_1-d_2)y_{xx}}{(y+a)^2} + \frac{2ad_2y_x^2}{(y+a)^3}. \tag{5.2.4}$$

Now, with $z(y) = \frac{y}{y+a}$, one has $z'(y) = \frac{a}{(y+a)^2}$ and $z_{xx} = \frac{y_{xx}}{(y+a)^2} - \frac{2y_x^2}{(y+a)^3}$, by the chain rule. Therefore, $z(y) = \frac{y}{y+a}$ satisfies the invariance equation.

5.2.2 The reduced scalar PDE governing the dynamics of (5.2.1) on $\mathcal{M}$

In this section, we analyze the reduced scalar PDE which governs the dynamics of (5.2.1) on the one-dimensional invariant manifold $\mathcal{M}$. This example illustrates the method developed for the general class of multi-scale R-D systems studied in this article, in which the kinetics terms in both of the components are nonlinear and fully coupled.

To derive the reduced PDE, we substitute $z = \frac{y}{y+a}$ into the equation for y and simplify,

$$y_t = -\frac{\varepsilon b y}{a} + \frac{\varepsilon b y^2}{a(y+a)} + d_1 y_{xx}. \tag{5.2.5}$$

This is the reduced scalar PDE which governs the dynamics of the full PDE (5.2.1) on $\mathcal{M}$. Moreover, the $z(x,t)$ component of solutions on $\mathcal{M}$ is slaved to $y(x,t)$ via the formula defining $\mathcal{M}$.

Therefore, the numbers of dependent variables and of nonlinear differential equations have each been reduced from two in the full system (5.2.1) to one in the system (5.2.5) on $\mathcal{M}$. Moreover, to provide a clear picture of the reduced dynamics on $\mathcal{M}$, the linear and nonlinear terms in the reduced kinetics have been separated in (5.2.5). We especially note the effective rate constant on the linear term.

5.2.3 The invariant manifold $\mathcal{M}$ of (5.2.1) is exponentially attracting

In this section, we show that manifold $\mathcal{M} = \{(y,z) | z = \frac{y}{y+a}\}$ is exponentially attracting in system (5.2.1). Just as for the first example (5.1.1), we introduce a new dependent variable $w(x,t)$ here for (5.2.1) via

$$z(x,t) = \frac{y(x,t)}{y(x,t)+a} + w(x,t). \tag{5.2.6}$$

By substituting (5.2.6) into (5.2.1), we see that w satisfies

$$w_t = -(y+a)w + d_2 w_{xx}. \tag{5.2.7}$$

Let $u(x,t) = e^{at}w(x,t)$. Then, u satisfies

$$u_t = -yu + d_2 u_{xx}, \qquad u(x,0) = w(x,0). \tag{5.2.8}$$

This is a second order parabolic PDE of form $u_t + Lu = 0$, where $Lu = -d_2 u_{xx} + y(x,t)u$.

We consider the solution on the domain $[0, X] \times (0, \infty)$, with initial condition $u(x, 0) = w(x, 0)$ and zero boundary condition. Since $y(x,t) \geq 0$, according to the strong maximum principle for second order parabolic PDE, u attains its maximum value over the boundary, $u(x,t) \leq w(x,0)$. Hence, we obtain

$$\|w(x,t)\| \leq e^{-at} \|w(x,0)\|. \tag{5.2.9}$$

Therefore, solutions are attracted to the one-dimensional invariant manifold $\mathcal{M}$ exponentially fast.

In numerical solutions of the system (5.2.1), one may also directly solve the PDE (5.2.7) numerically for $w(x,t)$, using the numerical values of $y(x,t)$ as input, and thereby explicitly monitor the decay toward the invariant manifold $\mathcal{M}$ during the simulation of the system.

5.2.4 Geometric decomposition of general solutions of (5.2.1)

By combining the results (5.2.3), (5.2.5), and (5.2.9), from the previous sections, we arrive at the geometric decomposition of general solutions of (5.2.1). Consider arbitrary non-negative initial data $(y(x,0), z(x,0))$, bounded in norm. Let $w(x,0)$ now be defined by $w(x,0) = z(x,0) - \frac{y(x,0)}{y(x,0)+a}$. The solution $(y(x,t), z(x,t))$ through this initial data may (by invariance of $\mathcal{M}$) be written as

$$(y(x,t), z(x,t)) = \left(y(x,t), \frac{y(x,t)}{y(x,t)+a} + w(x,t) \right) \tag{5.2.10}$$

where $y(x,t)$ is the solution of (5.2.5) with the same initial data, and $w(x,t)$ is the solution of (5.2.7). Hence, also here, there is an invariant family of fast stable fibers $\mathcal{F}_{b_t(x)}$, with base points $b_0(x) = \left(y_0(x), \frac{y_0(x)}{y_0(x)+a} \right)$ on $\mathcal{M}$, such that the image of $b_0(x)$ after time t is

$$b_t(x) = \left(y(x,t), \frac{y(x,t)}{y(x,t)+a} \right). \tag{5.2.11}$$

Moreover, for any initial condition $\left(y_0(x), \frac{y_0(x)}{y_0(x)+a} \right) + (0, w(x,0))$ on the fiber $\mathcal{F}_{b_0(x)}$, the solution at later times t is on $\mathcal{F}_{b_t(x)}$ and is given by $b_t(x) + (0, w(x,t))$.

Therefore, one has precisely the same type of geometric decomposition of general solutions of (5.2.1) into two components, as in the first example, even though (5.2.1) is nonlinear and fully coupled. The first component is a one-dimensional component which governs the slow evolution of the base points of the fibers on $\mathcal{M}$. The other component is the infinite-dimensional component normal to $\mathcal{M}$, which governs the exponential decay of w to zero, so that z decays toward $\mathcal{M}$ along the invariant family of fast stable fibers.

Remark. In developing (5.2.1), we have chosen to modify the kinetics in the z-component of the MMH mechanism. Alternatively, one may add $\frac{\varepsilon by}{y+a}$ to the y-component of (5.2.2), and $\mathcal{M}$ is again invariant.

5.3 Dynamics on the fast stable fibers of (5.2.1)

In this section, we use the geometric decomposition derived in the previous section to carry out a quantitative analysis of the dynamics on the fast stable fibers of (5.2.1). We begin with $\mathcal{O}(\varepsilon)$ diffusivities, so that the time scale of diffusion is comparable to that of the slower reaction kinetics. The asymptotic expansions of the fast stable fibers provide useful quantitative information about the decay of solutions to $\mathcal{M}$. Then, we examine other interesting cases in which the diffusivities are $\mathcal{O}(\sqrt{\varepsilon})$ and $\mathcal{O}(1)$, respectively. In these cases, the time scale of diffusion is between those of the fast and slow kinetics, and comparable to that of the fast kinetics, respectively. We show that there are major qualitative changes in the mode of decay and the rate of decay along the fibers in the case of $\mathcal{O}(1)$ diffusivities, as compared to the other two cases.

5.3.1 Dynamics on the fast stable fibers of (5.2.1) for $\mathcal{O}(\varepsilon)$ diffusivities

In this section, we show that the dynamics along the fast stable fibers consists of exponential decay in time, with a point-dependent rate given by $-(y_0(x) + a)$ for each x. Set $d_1 = \varepsilon\tilde{d}_1$ and $d_2 = \varepsilon\tilde{d}_2$. Expand the general solutions, as follows:

$$y(x,t) = y_0(x,t) + \varepsilon y_1(x,t) + \mathcal{O}(\varepsilon^2), \quad z(x,t) = z_0(x,t) + \varepsilon z_1(x,t) + \mathcal{O}(\varepsilon^2). \quad (5.3.1)$$

Also, set $y_0(x,0) = y(x,0)$ and $z_0(x,0) = z(x,0)$, so that $y_i(x,0) = 0$ and $z_i(x,0) = 0$, for $i = 1, 2, \ldots$, without loss of generality. Then, we substitute the expansions (5.3.1) into (5.2.1) and equate the coefficients on like powers of ε.

At $\mathcal{O}(1)$, the fast stable fibers are given by

$$y_{0t} = 0, \qquad z_{0t} = y_0 - (y_0 + a)z_0. \quad (5.3.2)$$

Hence,

$$\begin{aligned} y_0(x,t) &= y_0(x,0) := y_0(x), \\ z_0(x,t) &= \frac{y_0(x)}{y_0(x)+a} + e^{-(y_0(x)+a)t}\left(z_0(x,0) - \frac{y_0(x)}{y_0(x)+a}\right). \end{aligned} \tag{5.3.3}$$

Therefore, $z_0(x,t)$ decays exponentially, at an x-dependent rate, $-(y_0(x)+a)$, toward $\frac{y_0(x)}{y_0(x)+a}$, which is the $\mathcal{O}(1)$ term of the closed-form expression for the invariant manifold $\mathcal{M} = \{(y,z)|z = \frac{y}{y+a}\}$. One may also see this by substituting the leading order asymptotics (5.3.3) for $y(x,t)$ into the equation (5.2.7) for w, and by recalling that the diffusion term is higher order.

Next, at $\mathcal{O}(\varepsilon)$, we find that the rate of decay is also exponential in time with the same x-dependence. However, the coefficients depend also on the Laplacians of y and z due to the small diffusivities. The equations are

$$\begin{aligned} y_{1t} &= -y_0 + (y_0+a-b)z_0 + \tilde{d}_1 y_{0xx}, \\ z_{1t} &= y_1 - (y_0+a)z_1 - y_1 z_0 - \frac{aby_0}{(y_0+a)^3} + \tilde{d}_2 z_{0xx} + \frac{a(\tilde{d}_1 - \tilde{d}_2)y_{0xx}}{(y_0+a)^2} + \frac{2a\tilde{d}_2(y_{0x})^2}{(y_0+a)^3}. \end{aligned} \tag{5.3.4}$$

The solution for $y_1(x,t)$ is

$$y_1(x,t) = \left[1 - e^{-(y_0+a)t}\right]\left(\frac{y_0+a-b}{y_0+a}\right)\left(z_0(x,0) - \frac{y_0}{y_0+a}\right) + t\left(\frac{-by_0}{y_0+a} + \tilde{d}_1 y_{0xx}\right). \tag{5.3.5}$$

Similarly, though it requires more effort, the solution for $z(x,t)$ is

$$\begin{aligned} &z_1(x,t) \\ =&\frac{ay_1(x,t)}{(y_0+a)^2} + \tilde{d}_2 t e^{-(y_0+a)t}\left(z_0(x,0) - \frac{y_0}{y_0+a}\right)_{xx} \\ &- e^{-(y_0+a)t}\left(z_0(x,0) - \frac{y_0}{y_0+a}\right)\left[\left(\frac{y_0+a-b}{y_0+a}\right)\left(\frac{a}{y_0+a} + \left(z_0(x,0) - \frac{y_0}{y_0+a}\right)\right)t\right. \\ &\left.+\frac{t^2}{2}\left(\frac{-by_0}{y_0+a} + \tilde{d}_1 y_{0xx}\right) - (1 - e^{-(y_0+a)t})\left(z_0(x,0) - \frac{y_0}{y_0+a}\right)\left(\frac{y_0+a-b}{(y_0+a)^2}\right)\right], \end{aligned}$$

where we used $z_1(x,0) = 0$. Therefore, one sees that $z_1(x,t)$ decays exponentially to $\frac{ay_1(x,t)}{(y_0(x)+a)^2}$, which is the $\mathcal{O}(\epsilon)$ term in the closed-form expression for the invariant manifold $\mathcal{M} = \{(y,z)|z(x,t) = \frac{y(x,t)}{y(x,t)+a}\}$.

Together, the exponentially decaying components of $z_0(x,t)$ and $z_1(x,t)$ give the essential analytical information about the point-dependent exponential decay toward $\mathcal{M}$ along the fast stable fibers.

5.3.2 Point-dependent exponential decay along the fast stable fibers of (5.2.1) for $\mathcal{O}(\sqrt{\varepsilon})$ diffusivities

In this section, we show that also for intermediate diffusivities the dynamics along the fast stable fibers consists of point-dependent exponential decay in time. Set $d_1 = \sqrt{\varepsilon}\hat{d}_1$ and $d_2 = \sqrt{\varepsilon}\hat{d}_2$. Expand the general solutions, as follows:

$$\begin{aligned} y(x,t) &= y_0(x,t) + \sqrt{\varepsilon}y_1(x,t) + \varepsilon y_2(x,t) + \mathcal{O}(\varepsilon^{3/2}), \\ z(x,t) &= z_0(x,t) + \sqrt{\varepsilon}z_1(x,t) + \varepsilon z_2(x,t) + \mathcal{O}(\varepsilon^{3/2}). \end{aligned} \tag{5.3.6}$$

Also, set $y_0(x,0) = y(x,0)$ and $z_0(x,0) = z(x,0)$, so that $y_i(x,0) = 0$ and $z_i(x,0) = 0$, for $i = 1, 2, \ldots$, without loss of generality. Then, we substitute the expansions (5.3.6) into (5.2.1) and equate the coefficients on like powers of $\sqrt{\varepsilon}$.

At $\mathcal{O}(1)$, the equations and solutions for the leading order terms are the same as in the previous section, recall (5.3.3). one again sees from the w-equation (5.2.7) that there is the same x-dependent rate of decay of w toward zero, to leading order, and hence of z toward $\mathcal{M}$. This leading order decay result is the same as that obtained above with $\mathcal{O}(\varepsilon)$ diffusivities.

Then, at $\mathcal{O}(\sqrt{\varepsilon})$, the coefficients in the expansions are

$$\begin{aligned} y_1(x,t) =& t\hat{d}_1 y_{0_{xx}} \\ z_1(x,t) =& \frac{ay_1(x,t)}{(y_0+a)^2} + te^{-(y_0(x)+a)t}\tilde{d}_2\left(z_0(x,0) - \frac{y_0(x)}{y_0(x)+a}\right)_{xx} \\ &- \frac{t^2}{2}e^{-(y_0(x)+a)t}\tilde{d}_1\left(z_0(x,0) - \frac{y_0(x)}{y_0(x)+a}\right)y_{0_{xx}} \end{aligned}$$

where we used $z_1(x,0) = 0$. Hence, the first order corrections are determined purely by the diffusion terms.

5.3.3 Diffusive dynamics on the fast stable fibers of (5.2.1) for $\mathcal{O}(1)$ diffusivities

For $\mathcal{O}(1)$ diffusivities, there is a major qualitative change, and hence also a significant quantitative change, in the dynamics along the invariant family of fast stable fibers. This change is seen in both the mode of decay and the rate of decay to $\mathcal{M}$. There are two contributing factors. First, the diffusion term $d_2 w_{xx}$ in the w equation (5.2.7) is now $\mathcal{O}(1)$, and no longer higher order as it is for the smaller diffusivities analyzed above. Hence, the mode of decay of w is no longer purely exponential in time with the point-depemndent rate fixed at $-(y_0(x) + a)$ for each point x. Rather, the mode of decay is determined by the combination of the diffusion and kinetics. Second, to leading order, $y_0(x,t)$ satisfies $y_{0t} = d_1 y_{0xx}$, instead of being constant in time at (the spatially-dependent value) $y_0(x)$, as in the previous two cases. Hence, the coefficient, $-(y_0(x,t) + a)$, on the linear decay term in the heat equation (5.2.7) for w is itself the solution of a heat equation, and the decay rate approaches $-a$ asymptotically at all points x, instead of being a fixed x-dependent quantity $(-(y_0(x) + a))$, as it is for smaller diffusivities.

5.4 General systems of multi-scale R-D equations

The results for the closed-form, low-dimensional, invariant manifolds of the R-D systems (5.1.1) and (5.2.1) for $x \in \mathbb{R}^N$ $(N \geq 1)$ may be extended to a general class of n-species R-D equations, with $y(x,t) \in \mathbb{R}^\ell$, $z(x,t) \in \mathbb{R}^m$, and $\ell + m = n$.

For general $\ell, m \geq 1$, we start with R-D systems with classical diffusion terms of the form

$$y_t = \epsilon f(y,z,\varepsilon) + d_1\Delta y$$
$$z_t = g_1(y,z) + \epsilon g_2(y,z) + d_2\Delta z. \tag{5.4.1}$$

Here, f, g_1, and g_2 are C^3 functions. We assume that there exists a compact set $K \in \mathbb{R}^\ell$ and a C^3 function $z_0(y)$ for all $y \in K$ such that $g_1(y, z_0(y)) = 0$ and the graph of $(y, z_0(y))$ is a normally attracting, ℓ-dimensional, invariant manifold of the leading order kinetics ODE $z_t = g_1(y,z)$ for each fixed $y \in K$, where K is a compact set.

Then, the main R-D system we study is

$$y_t = \epsilon f(y,z,\varepsilon) + d_1\Delta y$$
$$z_t = g_1(y,z) + \varepsilon g_2(y,z) + d_2\Delta z + (z_0)_t - \epsilon g_2(y,z_0) - d_2\Delta z_0. \tag{5.4.2}$$

Here, the terms $(z_0)_t - \epsilon g_2(y,z_0) - d_2\Delta z_0$ have been added to the z-component of the vector field to make the manifold $\mathcal{M} = \{(y,z)|z = z_0(y)\}$ the closed-form, ℓ-dimensional, invariant manifold of (5.4.2). Indeed, with z set equal to z_0, the second equation in (5.4.2) is trivially satisfied, since $g_1(y, z_0(y)) = 0$.

For systems (5.4.2), the invariant manifold given by $z_0(y)$ is exponentially attracting. Let

$$z(x,t) = z_0(y(x,t)) + w(x,t) \tag{5.4.3}$$

and substitute this into (5.4.2) to find that w satisfies

$$w_t = g_1(y, z_0 + w) + \epsilon\left(g_2(y, z_0 + w) - g_2(y, z_0)\right) + d_2\Delta w. \tag{5.4.4}$$

Using the Taylor expansions of $g_1(y, z_0 + w)$ and $g_2(y, z_0 + w)$ at (y, z_0), we see that

w satisfies

$$w_t = d_2 \Delta w + [D_z g_1(y, z_0) + \epsilon D_z g_2(y, z_0)]\, w + \left[D_z^2 g_1(y, z_0) + \epsilon D_z^2 g_2(y, z_0)\right] w^2 + \mathcal{O}(w^3). \tag{5.4.5}$$

Now, since $\mathcal{M}$ is normally attracting for the leading order ODE system $z_t = g_1(y, z)$ for all $y \in K$, we know that there exists a $\sigma > 0$ such that $(g_1)_z(y, z_0(y)) < -\sigma$ for all $y \in K$. Hence, by taking ε sufficiently small, we see that the coefficient on the linear term is bounded away from zero for all $y \in K$. Therefore, locally, $w = 0$ is a stable stationary state.

Moreover, one has the same type of geometric decomposition of general solutions now into ℓ-dimensional components which undergo slow evolution along $\mathcal{M}$ (the evolution of the base points) and infinite-dimensional components which undergo fast exponential contraction in the normal directions (the fast stable fibers).

Finally, the first and second example PDEs in Sections 5.1 and 5.2 illustrate these general systems. For (5.2.1), $f = -y + (y + a - b)z$, $g_1 = y - (y + a)z$, and $g_2 = 0$, as in the original MMH PDE (5.2.2). Hence, on $z_0(y) = y/(y + a)$, one has $g_{1_{yy}} = 0$, $g_{1_{yz}} = -1$, $g_{1_{zz}} = 0$, $(z_0)_t = -\frac{\varepsilon aby}{(y+a)^3} + \frac{ad_1 y_{xx}}{(y+a)^2}$, and $(z_0)_{xx} = \frac{a y_{xx}}{(y+a)^2} - \frac{2a y_x^2}{(y+a)^3}$, so that one recovers (5.2.1).

Remark. It is useful to further examine the three terms added to (5.4.2). For simplicity, we do so in the particular case $\ell, m, N = 1$. From the defining condition, $g_1(y, z_0(y)) = 0$, one obtains

$$z_0' = -(g_{1_z})^{-1}(g_{1_y}), \tag{5.4.6}$$

for all $y \in K$, by differentiating on y. Hence, by chain rule, one also has

$$\begin{aligned}
(z_0)_t =& -(g_{1_z})^{-1}(g_{1_y})[\varepsilon f + d_1 y_{xx}] \\
(z_0)_x =& -(g_{1_z})^{-1}(g_{1_y})y_x \\
(z_0)_{xx} =& (g_{1_z})^{-2}[g_{1_{zy}} y_x + g_{1_{zz}} z_0'(y) y_x](g_{1_y}) y_x \\
& - (g_{1_z})^{-1}[g_{1_{yy}} y_x + g_{1_{yz}} z_0'(y) y_x] y_x - (g_{1_z})^{-1}(g_{1_y}) y_{xx}.
\end{aligned}$$

Together, the first term from $(z_0)_t$ and the term $-\epsilon g_2(y, z_0)$ guarantee that $z = z_0(y)$ is an invariant manifold of the kinetics ODE ((5.4.2) with $d_1, d_2 = 0$), just as is the case for the Davis-Skodje ODE model and the MMH ODE. All of the other terms are proportional to $(d_1 - d_2)y_{xx}$, representing cross-diffusion, and to $d_2(y_x)^2$, representing the 'gradient squared'.

Chapter 6

Analysis of the Approximate Slow Invariant Manifold

The material of this chapter has been presented in (Wu and Kaper, 2017).

6.1 Overview of the ASIM Method for Reaction-Diffusion Equations

For system (1.2.1) with an equilibrium point at which $\boldsymbol{F}$ vanishes, it is assumed here for the analysis of the ASIM method that $\boldsymbol{F}(\mathbf{0}) = \mathbf{0}$, which may be achieved by a translation of the coordinate system if necessary. We let $\boldsymbol{J} = D_{\boldsymbol{u}}\boldsymbol{F}$ be the Jacobian matrix, and let $\boldsymbol{V}$ be the eigenvector matrix of $\boldsymbol{J}$. Let $\widetilde{\boldsymbol{V}} = \boldsymbol{V}^{-1}$. In the reaction kinetics, the time scales are determined by the eigenvalues of $\boldsymbol{J}$. For problems in which model reduction may be achieved, the eigenvalues split into two sets, the slow eigenvalues $\lambda_1, \ldots, \lambda_m$ and the fast eigenvalues $\lambda_{m+1}, \ldots, \lambda_{m+n}$, with a spectral gap in between the two sets. For the analysis, we order the eigenvalues as follows:

$$Re(\lambda_{m+n}) \leq \ldots \leq Re(\lambda_{m+1}) < Re(\lambda_m) \leq \ldots \leq Re(\lambda_1) < 0. \tag{6.1.1}$$

Here, the spectral gap condition is that

$$\left| \frac{Re(\lambda_m)}{Re(\lambda_{m+1})} \right| \ll 1. \tag{6.1.2}$$

The spectral gap naturally induces a splitting of the eigenvector matrix $\boldsymbol{V}$ into block form,

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_s & \boldsymbol{V}_f \end{bmatrix}, \tag{6.1.3}$$

where $\boldsymbol{V}_s$ is an $(m+n) \times m$ matrix and $\boldsymbol{V}_f$ is an $(m+n) \times n$ matrix. We also represent the column vectors of $\boldsymbol{V}$ by $\boldsymbol{v}_i,\ i = 1, \ldots, m+n$. $\widetilde{\boldsymbol{V}}$ can also be written in block form as

$$\widetilde{\boldsymbol{V}} = \begin{bmatrix} \widetilde{\boldsymbol{V}_s} \\ \widetilde{\boldsymbol{V}_f} \end{bmatrix}, \tag{6.1.4}$$

where $\widetilde{\boldsymbol{V}_s}$ is $m \times (m+n)$ and $\widetilde{\boldsymbol{V}_f}$ is $n \times (m+n)$. The row vectors of $\widetilde{\boldsymbol{V}}$ may be represented by $\widetilde{\boldsymbol{v}}_i,\ i = 1, \ldots, m+n$. By definition, $\widetilde{\boldsymbol{V}_s} \perp \boldsymbol{V}_f$ and $\widetilde{\boldsymbol{V}_f} \perp \boldsymbol{V}_s$. These orthogonality properties are crucial for the projection step in the ASIM method. Moreover, we know that $\boldsymbol{J} = \boldsymbol{V}\boldsymbol{\Lambda}\widetilde{\boldsymbol{V}}$, where $\boldsymbol{\Lambda}$ is the diagonal matrix with the eigenvalues on the diagonal, with their real parts ordered from least negative to most negative. It is sometimes also useful to decompose $\boldsymbol{\Lambda}$ into block diagonal form,

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_s & 0 \\ 0 & \boldsymbol{\Lambda}_f \end{bmatrix}, \tag{6.1.5}$$

where $\boldsymbol{\Lambda}_s$ is $m \times m$, $\boldsymbol{\Lambda}_f$ is $n \times n$.

In order to present an overview of the ASIM method, let $\boldsymbol{w} = \widetilde{\boldsymbol{V}}\boldsymbol{u}$. Then, $w_i = \widetilde{\boldsymbol{v}}_i \boldsymbol{u},\ i = 1, \ldots, m+n$. Also, define $\boldsymbol{G}$ to be the nonlinear part of $\boldsymbol{F}$. Hence,

$$\boldsymbol{F} = \boldsymbol{J}\boldsymbol{u} + \boldsymbol{G}, \quad \widetilde{\boldsymbol{V}}\boldsymbol{F} = \boldsymbol{\Lambda}\boldsymbol{w} + \widetilde{\boldsymbol{V}}\boldsymbol{G}. \tag{6.1.6}$$

Therefore, system (1.2.1) can be written in component form as

$$\frac{1}{\lambda_i}\left(\frac{\partial w_i}{\partial t} + \widetilde{\boldsymbol{v}}_i \sum_{j=1}^{m+n} \frac{\partial \boldsymbol{v}_j}{\partial t} w_j\right) = w_i + \frac{1}{\lambda_i}(\widetilde{\boldsymbol{v}}_i \boldsymbol{G}) - \frac{1}{\lambda_i}\left(\widetilde{\boldsymbol{v}}_i \frac{\partial \boldsymbol{h}}{\partial x}\right),\ i = 1, \ldots, m+n. \tag{6.1.7}$$

In the ASIM Method, the faster processes are neglected. In particular, since the eigenvalues of J are ordered as in (6.1.1), we see that $\mathrm{Re}(1/\lambda_{m+n}), \ldots, \mathrm{Re}(1/\lambda_{m+1})$ are all much less than $\mathrm{Re}(1/\lambda_m)$. Hence, for $i = m+1, \ldots, m+n$, the left hand side of (6.1.7) is much smaller than it is for $i = 1, \ldots m$, and it may be set to zero. Hence, for $i = m+1, \ldots, m+n$, (6.1.7) is approximated in (Singh et al., 2002) by the reduced equation,

$$w_i + \frac{1}{\lambda_i}(\widetilde{\boldsymbol{v}}_i \boldsymbol{G}) - \frac{1}{\lambda_i}\left(\widetilde{\boldsymbol{v}}_i \frac{\partial \boldsymbol{h}}{\partial x}\right) = 0, \;\; i = m+1, \ldots, m+n. \tag{6.1.8}$$

Condition (6.1.8), which is an elliptic PDE, may be written in matrix-vector form as

$$\widetilde{\boldsymbol{V}}_f \boldsymbol{F} - \widetilde{\boldsymbol{V}}_f \frac{\partial \boldsymbol{h}}{\partial x} = \boldsymbol{0}, \tag{6.1.9}$$

where we used equation (6.1.6).

The central idea of the ASIM method is that the solution of the elliptic PDE (6.1.9) approximates the nonlinear slow mode of system (1.2.1). It is an infinite-dimensional approximate manifold, and it is labeled as the Approximate Slow Invariant Manifold (ASIM). The faster processes equilibrate to it, and on the long time scale both the reaction and the reactive flow components of the full system (1.2.1) occur near it.

The ASIM method also derives an effective reduced dynamical equation for the system on the approximate manifold by projecting the full system (1.2.1) onto the orthogonal complement of the fast subspace,

$$\widetilde{\boldsymbol{V}}_s \frac{\partial \boldsymbol{u}}{\partial t} = \widetilde{\boldsymbol{V}}_s \boldsymbol{F} - \widetilde{\boldsymbol{V}}_s \frac{\partial \boldsymbol{h}}{\partial x}. \tag{6.1.10}$$

Here, all of the terms are evaluated on the ASIM defined by (6.1.9). Due to the projection by $\widetilde{\boldsymbol{V}}_s$ onto the orthogonal complement of the fast subspace, the system of PDEs (6.1.10) has m components, $u_i, i = 1, \ldots, m$. Hence, it is of significantly lower dimension than the full system (1.2.1). Furthermore, system (6.1.10) is substantially less stiff than the full system, because those components that are determined to be fast by the Jacobian of the reaction kinetics have equilibrated.

In summary, the ASIM Method consists of the following two PDEs:

$$\begin{cases} \mathbf{0} = \widetilde{\boldsymbol{V}}_f \boldsymbol{F} - \widetilde{\boldsymbol{V}}_f \dfrac{\partial \boldsymbol{h}}{\partial x} \\ \widetilde{\boldsymbol{V}}_s \dfrac{\partial \boldsymbol{u}}{\partial t} = \widetilde{\boldsymbol{V}}_s \boldsymbol{F} - \widetilde{\boldsymbol{V}}_s \dfrac{\partial \boldsymbol{h}}{\partial x}. \end{cases} \tag{6.1.11}$$

The first equation in (6.1.11) defines the ASIM, which is an approximation of the nonlinear slow mode. The accuracy of the ASIM determines the accuracy of the method. The second equation defines the reduced PDE on the ASIM that governs the flow of the slow components. It approximates the exact reduced equation on the nonlinear slow mode.

Remark. As observed in (Singh et al., 2002), in the special case in which the species only undergo reaction, i.e. $\boldsymbol{h} = \mathbf{0}$, the ASIM method reduces to the ILDM method (Kaper and Kaper, 2002). In fact, from (6.1.11), one sees that the approximate manifold is defined by $\mathbf{0} = \widetilde{\boldsymbol{V}}_f \boldsymbol{F}$, which is exactly the ILDM condition.

6.2 Accuracy of the ASIM Method - Two Species Case

In this section, we analyze the accuracy of the ASIM method in the context of two-component reaction-diffusion systems of the form

$$\begin{cases} \dfrac{\partial y}{\partial t} = f(y, z, \epsilon) + d_1 \dfrac{\partial^2 y}{\partial x^2} \\ \dfrac{\partial z}{\partial t} = \epsilon^{-1} g(y, z, \epsilon) + d_2 \dfrac{\partial^2 z}{\partial x^2}. \end{cases} \tag{6.2.1}$$

This system is of the form (1.2.1) with $m = 1, n = 1$,

$$\boldsymbol{u} = \begin{bmatrix} y \\ z \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} f \\ \epsilon^{-1} g \end{bmatrix}, \text{ and } \boldsymbol{h} = \begin{bmatrix} -d_1 \frac{\partial y}{\partial x} \\ -d_2 \frac{\partial z}{\partial x} \end{bmatrix}.$$

Here, the variables y and z denote the species concentrations. Both concentrations depend on space, x, and time, t. The functions f and g represent the reaction terms, and $d_1\frac{\partial^2 y}{\partial^2 x}$ and $d_2\frac{\partial^2 z}{\partial^2 x}$ are the diffusion terms. There is a separation of scales in the reaction kinetics, with y the slow variable and z the fast one. The separation is measured by the small parameter ϵ. Throughout this analysis, we will assume, strictly for convenience, that the spatial variable x is a real number. We add that all of the analysis applies as well to the cases of spatially-periodic boundary conditions and homogeneous Dirichlet or Neumann boundary conditions on finite intervals. Moreover, for problems on bounded domains, the PDE governing the ASIM and the reduced PDE on the ASIM may be equipped with the same boundary conditions as the full model (6.2.1).

We study systems of the form (6.2.1) that have a slow manifold on which reaction and diffusion only occur on the slow (long) time scale. For $\epsilon = 0$, we assume that the reaction kinetics in (6.2.1) has a critical manifold

$$\mathcal{M}_0 = \{(y, z) : z = h_0(y), g(y, h_0(y), 0) = 0\}, \tag{6.2.2}$$

where h_0 is at least $\mathcal{C}^2$. Moreover, we assume $g_z < 0$ on $\mathcal{M}_0$, so that the critical manifold is exponentially stable, which is the natural case for systems in which model reduction is desired. The solutions on $\mathcal{M}_0$ are also spatially homogeneous solutions of (6.2.1) with $\epsilon = 0$.

With $0 < \epsilon \ll 1$, we assume that the full system (6.2.1) has nonlinear slow mode

$$\mathcal{M}_\epsilon = \{(y, z) : z = h_\epsilon(y, Dy, D^2y, D^3y, \ldots)\}, \tag{6.2.3}$$

where D^i denotes the ith order derivative with respect to x. $\mathcal{M}_\epsilon$ are the graphs of solutions, h_ϵ, of the invariance equation,

$$\begin{aligned}\epsilon^{-1}g(y,h_\epsilon,\epsilon)+d_2D^2h_\epsilon =& \frac{\partial h_\epsilon}{\partial y}\left(f(y,h_\epsilon,\epsilon)+d_1D^2y\right)+\frac{\partial h_\epsilon}{\partial(Dy)}D\left(f(y,h_\epsilon,\epsilon)+d_1D^2y\right)\\ &+\frac{\partial h_\epsilon}{\partial(D^2y)}D^2\left(f(y,h_\epsilon,\epsilon)+d_1D^2y\right)+\dots\end{aligned} \tag{6.2.4}$$

We note that the LHS is obtained from (6.2.1)(b), and the RHS is obtained by applying the chain rule to calculate $\frac{\partial z}{\partial t}$ using (6.2.1)(a). The nonlinear slow mode $\mathcal{M}_\epsilon$ is close in the L^∞ norm to $\mathcal{M}_0$ for $\epsilon>0$ sufficiently small. In order to calculate h_ϵ as the solution of the invariance equation (6.2.4), we expand h_ϵ as,

$$\begin{aligned}z&=h_\epsilon(y,Dy,D^2y,D^3y\dots)\\ &=h^{(0)}(y)+\epsilon h^{(1)}(y,Dy,D^2y)+\epsilon^2h^{(2)}(y,Dy,D^2y,D^3y,D^4y)+\mathcal{O}(\epsilon^3).\end{aligned} \tag{6.2.5}$$

Hence, by substituting this expansion into the invariance equation (6.2.4), equating coefficients in terms of like order, beginning with the terms of $\mathcal{O}(1/\epsilon)$, and letting prime, $'$, denote the derivative with respect to y, we find that the nonlinear slow mode is given by

$$h^{(0)}=h_0, \tag{6.2.6}$$

$$g_zh^{(1)}=(f+d_1D^2y)h_0'-g_\epsilon-d_2(h_0'D^2y+h_0''(Dy)^2), \tag{6.2.7}$$

$$g_zh^{(2)}=-\frac{1}{2}g_{zz}{h^{(1)}}^2-g_{z\epsilon}h^{(1)}-\frac{1}{2}g_{\epsilon\epsilon}+(f_zh^{(1)}+f_\epsilon)h_0'+(f+d_1D^2y)\frac{\partial h^{(1)}}{\partial y} \tag{6.2.8}$$

$$+\frac{\partial h^{(1)}}{\partial(Dy)}D(f+d_1D^2y)+\frac{\partial h^{(1)}}{\partial(D^2y)}D^2(f+d_1D^2y) \tag{6.2.9}$$

$$-d_2q(h^{(1)},y,Dy,D^2y,D^3y,D^4y),$$

where

$$
\begin{aligned}
&q(h^{(1)}, y, Dy, \ldots, D^4y) \\
=&\frac{\partial h^{(1)}}{\partial y}D^2y + \frac{\partial h^{(1)}}{\partial (Dy)}D^3y + \frac{\partial h^{(1)}}{\partial (D^2y)}D^4y + \frac{\partial^2 h^{(1)}}{\partial y^2}(Dy)^2 \\
&+ \frac{\partial^2 h^{(1)}}{\partial (Dy)^2}(D^2y)^2 + 2\frac{\partial^2 h^{(1)}}{\partial y \partial (Dy)}(DyD^2y) + 2\frac{\partial^2 h^{(1)}}{\partial y \partial (D^2y)}(DyD^3y).
\end{aligned}
\tag{6.2.10}
$$

The terms involving h_0 and $h^{(1)}$ and their derivatives are evaluated on the critical manifold. We note that $h^{(1)}$ has no terms that are dependent on both Dy and D^2y, and the term dependent on D^2y is linear in D^2y.

This is the exact result. At leading order, the nonlinear slow mode is given by $\mathcal{M}_0$, the critical manifold, which is the graph of $z = h_0(y)$. At $\mathcal{O}(\epsilon)$, the term $h^{(1)}$ in the expansion consists of four terms. The first is proportional to both the slope, h_0', of the critical manifold, and the slow component, $f + d_1D^2y$, of the reaction-diffusion system; the second is given by the rate of change of the fast kinetics with respect to ϵ; the third is proportional to h_0' and the Laplacian of the slow species y; and, the fourth is proportional to the curvature, h_0'', of the critical manifold. Next, at $\mathcal{O}(\epsilon^2)$, the term $h^{(2)}$ in the asymptotic expansion of the nonlinear slow mode depends on a series of terms, including the variation of the reaction kinetics with respect to ϵ; the rate of change h_0' and the first variation, $f_zh^{(1)} + f_\epsilon$, of the slow kinetics; the rate of change of $h^{(1)}$; as well as the curvature and other second derivatives of $h^{(1)}$. This exact result will be used as the point of comparison for determining the accuracy of the ASIM for system (6.2.1).

In order to apply the ASIM method, we begin with the Jacobian of the reaction kinetics of system (6.2.1),

$$
J = \begin{bmatrix} f_y & f_z \\ \epsilon^{-1}g_y & \epsilon^{-1}g_z \end{bmatrix}. \tag{6.2.11}
$$

We are interested in points on and near $\mathcal{M}_0$, and we recall that $g_z < 0$ in a neighborhood of $\mathcal{M}_0$. Hence, in a neighborhood of $\mathcal{M}_0$, the eigenvalues of J may be written as

$$\begin{aligned} \lambda_s &= f_y - \frac{f_z g_y}{g_z} + \epsilon\left(-f_y + \frac{f_z g_y}{g_z}\right)\frac{f_z g_y}{g_z^2} + \mathcal{O}(\epsilon^2), \\ \lambda_f &= \epsilon^{-1} g_z + \frac{f_z g_y}{g_z} + \epsilon\left(f_y - \frac{f_z g_y}{g_z}\right)\frac{f_z g_y}{g_z^2} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{6.2.12}$$

Here, λ_s is $\mathcal{O}(1)$ and referred to as the slow eigenvalue, and λ_f is order $\mathcal{O}(\epsilon^{-1})$ and referred to as the fast eigenvalue. The corresponding eigenvectors are

$$v_s = \begin{bmatrix} \lambda_s - \epsilon^{-1} g_z \\ \epsilon^{-1} g_y \end{bmatrix}, \; v_f = \begin{bmatrix} f_z \\ \lambda_f - f_y \end{bmatrix}. \tag{6.2.13}$$

For (6.2.1), the inverse of the Jacobian J is given by

$$\begin{bmatrix} \widetilde{v_s} \\ \widetilde{v_f} \end{bmatrix} = \begin{bmatrix} v_s & v_f \end{bmatrix}^{-1} = \frac{1}{det}\begin{bmatrix} \lambda_f - f_y & -f_z \\ -\epsilon^{-1} g_y & \lambda_s - \epsilon^{-1} g_z \end{bmatrix}, \tag{6.2.14}$$

where $det = det \begin{bmatrix} v_s & v_f \end{bmatrix} \sim -\epsilon^2 / g_z^2$ to leading order.

Applying the projections (6.1.11) at the heart of the ASIM method to system (6.2.1), where for these systems the general matrices $\widetilde{\boldsymbol{V}}_s$ and $\widetilde{\boldsymbol{V}}_f$ are the vectors $\widetilde{v_s}$ and $\widetilde{v_f}$ given by (6.2.14), we find

$$-\epsilon^{-1} g_y f + \epsilon^{-1}(\lambda_s - \epsilon^{-1} g_z) g - \epsilon^{-1} g_y d_1 D^2 y + (\lambda_s - \epsilon^{-1} g_z) d_2 D^2 z = 0, \tag{6.2.15}$$

$$(\lambda_f - f_y)\frac{\partial y}{\partial t} - f_z \frac{\partial z}{\partial t} = (\lambda_f - f_y) f - \epsilon^{-1} f_z g + (\lambda_f - f_y) d_1 D^2 y - f_z d_2 D^2 z. \tag{6.2.16}$$

Condition (6.2.15) defines the ASIM, which is an approximation to $\mathcal{M}_\epsilon$, and condition (6.2.16), evaluated on the ASIM, gives the reduced PDE on the ASIM. We now prove the following theorem, which establishes the accuracy of the ASIM:

Theorem 6.2.1. *(Two Species Case)*
The solution $z = \psi(y, Dy, D^2y, D^3y, \dots, \epsilon)$ *of equation* (6.2.15) *admits the following asymptotic expansion:*

$$\begin{aligned} z =&\psi(y, Dy, D^2y, D^3y, \dots, \epsilon) \\ =&\psi^{(0)}(y) + \epsilon\psi^{(1)}(y, Dy, D^2y) + \epsilon^2\psi^{(2)}(y, Dy, D^2y, D^3y, D^4y) + \mathcal{O}(\epsilon^3), \end{aligned} \tag{6.2.17}$$

where $\psi^{(0)}, \psi^{(1)}$ *and* $\psi^{(2)}$ *are defined by the equations*

$$\psi^{(0)} =h_0, \tag{6.2.18}$$

$$g_z\psi^{(1)} =(f + d_1D^2y)h_0' - g_\epsilon - d_2(h_0'D^2y + h_0''(Dy)^2), \tag{6.2.19}$$

$$\begin{aligned} g_z\psi^{(2)} =& -\frac{1}{2}g_{zz}{\psi^{(1)}}^2 - g_{z\epsilon}\psi^{(1)} - \frac{1}{2}g_{\epsilon\epsilon} + (f_z\psi^{(1)} + f_\epsilon)h_0' + (f + d_1D^2y)\frac{\partial\psi^{(1)}}{\partial y} \\ &+ (g_z)^{-1}[d_2(h_0''D^2y + h_0'''(Dy)^2)(f + d_1D^2y) - h_0''(f + d_1D^2y)^2] \\ &- d_2q(\psi^{(1)}, y, Dy, D^2y, D^3y, D^4y). \end{aligned} \tag{6.2.20}$$

where $q(\psi^{(1)}, y, Dy, D^2y, D^3y, D^4y)$ *is given by* (6.2.10) *with* $h^{(1)}$ *replaced by* $\psi^{(1)}$*. For these formulas, the functions* f *and* g *and their derivatives are evaluated at* $(y, h_0(y), 0)$*.*

Corollary 6.2.2. *(Two Species Case)*
A comparison of the coefficients in the expansion (6.2.18)-(6.2.20) *of the ASIM with the coefficients* (6.2.6)-(6.2.8) *in the expansion of the nonlinear slow mode* $\mathcal{M}_\epsilon$ *shows that*

$$\psi^{(0)} =h^{(0)}, \tag{6.2.21}$$

$$\psi^{(1)} =h^{(1)}, \tag{6.2.22}$$

$$\begin{aligned} \psi^{(2)} =&h^{(2)} + (g_z)^{-2}\left[d_2(h_0''D^2y + h_0'''(Dy)^2)(f + d_1D^2y) - h_0''(f + d_1D^2y)^2\right] \\ &- (g_z)^{-1}\left[\frac{\partial h^{(1)}}{\partial(Dy)}D(f + d_1D^2y) + \frac{\partial h^{(1)}}{\partial(D^2y)}D^2(f + d_1D^2y)\right]. \end{aligned} \tag{6.2.23}$$

Theorem 6.2.1 and Corollary 6.2.2 have several important consequences. First, it follows from (6.2.21) and (6.2.22) that the ASIM is accurate up to and including

$\mathcal{O}(\epsilon)$, compared with the nonlinear slow mode $\mathcal{M}_\epsilon$.

Second, it follows from (6.2.23) that the difference between the ASIM and the nonlinear slow mode is $\mathcal{O}(\epsilon^2)$, and formula (6.2.23) gives the precise dependence of the error on the various terms in system (6.2.1). In particular, at $\mathcal{O}(\epsilon^2)$, the ASIM method picks up many of the terms in $h^{(2)}$, including some that are nonlinear and proportional to the curvature of h_0. However, there are some terms from $h^{(2)}$ that are not captured by the ASIM method, and there are some terms in $\psi^{(2)}$ that are not in $h^{(2)}$. The net difference is given by (6.2.23). We may compute $\frac{\partial h^{(1)}}{\partial(Dy)}$ and $\frac{\partial h^{(1)}}{\partial(D^2y)}$ by differentiating (6.2.7) with respect to Dy and D^2y. Hence, a more explicit formula for the coefficient on the difference at $\mathcal{O}(\epsilon^2)$ is

$$\begin{aligned}\psi^{(2)} - h^{(2)} =&(g_z)^{-2}[d_2(h_0''D^2y + h_0'''(Dy)^2)(f + d_1D^2y) - h_0''(f + d_1D^2y)^2 \\ &2d_2h_0''DyD(f + d_1D^2y) - (d_1 - d_2)h_0'D^2(f + d_1D^2y)].\end{aligned} \tag{6.2.24}$$

Therefore, for general $d_1, d_2 > 0$, the difference between $\psi^{(2)}$ and $h^{(2)}$ consists of four terms: one proportional to the curvature, h_0'', of the critical manifold, and the term $f+d_1D^2y$, which represents the reaction kinetics and diffusion of the slow species; one proportional to the rate of change, h_0''', of the curvature, and also the term $f+d_1D^2y$; one proportional to the gradient of $f + d_1D^2y$; and, one proportional to the curvature of $f + d_1D^2y$ and to the difference $d_1 - d_2$ of the diffusivities. Overall, each of the terms in this error is readily available in two species reaction-diffusion systems, since they only depend on quantities known from the leading order ($\epsilon = 0$) system. Moreover, this formula permits one to see which terms may dominate the error in any given system and, hence, be useful for suggesting possible refinements of the method.

Third, in the special case in which $\mathcal{M}_0$ is a linear manifold, i.e. $h_0(y)$ is a linear function, we see from (6.2.24) that the ASIM is exact if $d_1 = d_2$.

Fourth, in the special case in which $d_1, d_2 = 0$, system (6.2.1) is a kinetics model to which the ILDM method applies. In this case, the error between the ASIM and the nonlinear slow mode at $\mathcal{O}(\epsilon^2)$ reduces to $-(g_z)^{-2}h_0''f^2$, which is exactly the curvature term identified in (Davis and Skodje, 1999; Kaper and Kaper, 2002) as the error in the ILDM method for the reaction kinetics.

Proof of Theorem 6.2.1.

To find the asymptotic expansion of the solution to the ASIM equation (6.2.15), we start by substituting λ_s from (6.2.12) into formula (6.2.15), and also multiplying (6.2.15) by $-g_z$. Keeping all of the terms at the three leading orders, namely at $\mathcal{O}(\epsilon^{-2}), \mathcal{O}(\epsilon^{-1})$, and $\mathcal{O}(1)$, we have

$$\begin{aligned} &\epsilon^{-1} g_y g_z f + \epsilon^{-1} g \left(\epsilon^{-1} g_z^2 - f_y g_z + f_z g_y + \epsilon \frac{f_z g_y}{g_z^2}(f_y g_z - f_z g_y) \right) \\ &+ \epsilon^{-1} g_y g_z d_1 D^2 y + (\epsilon^{-1} g_z^2 - f_y g_z + f_z g_y) d_2 D^2 z = 0 \end{aligned} \tag{6.2.25}$$

Assume that $z = \psi(y, Dy, D^2y, \ldots, \epsilon)$ is given by an asymptotic expansion of the form (6.2.17). Using this expansion, we see that the Laplacian term on the fast species, D^2z, may be written as

$$\begin{aligned} D^2 z = &\left[\frac{\partial^2 \psi^{(0)}}{\partial y^2}(Dy)^2 + \frac{\partial \psi^{(0)}}{\partial y} D^2 y \right] + \epsilon \left[\frac{\partial \psi^{(1)}}{\partial y} D^2 y + \frac{\partial \psi^{(1)}}{\partial (Dy)} D^3 y + \frac{\partial \psi^{(1)}}{\partial (D^2 y)} D^4 y \right. \\ &+ \frac{\partial^2 \psi^{(1)}}{\partial y^2}(Dy)^2 + \frac{\partial^2 \psi^{(1)}}{\partial (Dy)^2}(D^2 y)^2 + \frac{\partial^2 \psi^{(1)}}{\partial (D^2 y)^2}(D^3 y)^2 + 2 \frac{\partial^2 \psi^{(1)}}{\partial y \partial (Dy)} Dy D^2 y \\ &\left. + 2 \frac{\partial^2 \psi^{(1)}}{\partial y \partial (D^2 y)} Dy D^3 y + 2 \frac{\partial^2 \psi^{(1)}}{\partial (Dy) \partial (D^2 y)} D^2 y D^3 y \right] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{6.2.26}$$

Also, we Taylor expand f at $(y, \psi^{(0)}(y), 0)$,

$$f(y, \psi, \epsilon) = f + \epsilon(f_z\psi^{(1)} + f_\epsilon) + \epsilon^2 \left(f_z\psi^{(2)} + \frac{1}{2} f_{zz}(\psi^{(1)})^2 + f_{z\epsilon}\psi^{(1)} + \frac{1}{2} f_{\epsilon\epsilon} \right) + \mathcal{O}(\epsilon^3). \tag{6.2.27}$$

Here, the terms in the right hand side are evaluated at $(y, \psi^{(0)}(y), 0)$. Similar expansions hold for all derivatives of f, and for g and its derivatives.

Next, we substitute (6.2.26) and (6.2.27) into equation (6.2.25) and equate the coefficients of like powers of ϵ.

At $\mathcal{O}(\epsilon^{-2})$, equation (6.2.25) gives

$$g g_z^2|_{(y,\psi^{(0)},0)} = 0.$$

Recalling that $g_z < 0$ on $\mathcal{M}_0$, we see that this condition is satisfied if $g = 0$, and hence

$$\psi^{(0)} = h_0.$$

This establishes (6.2.18).

At $\mathcal{O}(\epsilon^{-1})$, we use $g(y, h_0(y), 0) = 0$ and (6.2.18) to derive

$$g_y g_z f + g_z^2(g_z\psi^{(1)} + g_\epsilon) + g_y g_z d_1 D^2 y + g_z^2 d_2[h_0''(Dy)^2 + h_0' D^2 y] = 0. \tag{6.2.28}$$

This formula may be simplified by noting that $g_y + g_z h_0' = 0$ for all points on $\mathcal{M}_0$. Hence, after multiplying (6.2.28) by $1/g_z^2$, we find

$$g_z\psi^{(1)} = h_0'(f + d_1 D^2 y) - g_\epsilon - d_2(h_0' D^2 y + h_0''(Dy)^2), \tag{6.2.29}$$

which is precisely (6.2.19).

Also, we notice that by using (6.2.29), the second order derivatives of $\psi^{(1)}$ in equation (6.2.26) satisfy

$$\frac{\partial^2\psi^{(1)}}{\partial(D^2y)^2} = 0, \quad \frac{\partial^2\psi^{(1)}}{\partial(Dy)\partial(D^2y)} = 0. \tag{6.2.30}$$

Next, at $\mathcal{O}(1)$, we also use $g = 0$ on $\mathcal{M}_0$ to find that equation (6.2.25) becomes

$$\begin{aligned}
&(g_{yz}\psi^{(1)} + g_{y\epsilon})g_z f + (g_{zz}\psi^{(1)} + g_{z\epsilon})g_y f + g_y g_z(f_z\psi^{(1)} + f_\epsilon)\\
&+(g_z\psi^{(1)} + g_\epsilon)[2g_z(g_{zz}\psi^{(1)} + g_{z\epsilon}) - f_y g_z + f_z g_y]\\
&+(g_z)^2(g_z\psi^{(2)} + \frac{1}{2}g_{zz}{\psi^{(1)}}^2 + g_{z\epsilon}\psi^{(1)} + \frac{1}{2}g_{\epsilon\epsilon})\\
&+[(g_{yz}\psi^{(1)} + g_{y\epsilon})g_z + (g_{zz}\psi^{(1)} + g_{z\epsilon})g_y]d_1 D^2y\\
&+[2g_z(g_{zz}\psi^{(1)} + g_{z\epsilon}) - f_y g_z + f_z g_y]d_2[h_0' D^2y + h_0''(Dy)^2] + (g_z)^2 d_2 q = 0.
\end{aligned} \tag{6.2.31}$$

Here, we used equation (6.2.30) to simplify the $\mathcal{O}(\epsilon)$ term in (6.2.26), and q is given by (6.2.10).

Next, we separate the terms into those independent of d_1 and d_2, those dependent on d_1, and those dependent on d_2. To simplify the first set of terms, we use (6.2.29) to substitute for the term $g_z\psi^{(1)}$, and we use the fact that $g_z h_0' = -g_y$ for all points on $\mathcal{M}_0$. The terms proportional to d_1 also simplify by using $g_z h_0' = -g_y$. Moreover, there is substantial cancellation in the terms proportional to d_2. We find

$$\begin{aligned}
&g_z^2\left(g_z\psi^{(2)} + \frac{1}{2}g_{zz}{\psi^{(1)}}^2 + g_{z\epsilon}\psi^{(1)} + \frac{1}{2}g_{\epsilon\epsilon}\right) + (g_{yz}\psi^{(1)} + g_{y\epsilon})g_z f + g_y g_z(f_z\psi^{(1)} + f_\epsilon)\\
&-f[g_y(g_{zz}\psi^{(1)} + g_{z\epsilon}) - f_y g_y - h_0' f_z g_y] + [g_z(g_{yz}\psi^{(1)} + g_{y\epsilon}) - g_y(g_{zz}\psi^{(1)} + g_{z\epsilon})\\
&+f_y g_y + h_0' f_z g_y]d_1 D^2y + g_z^2 d_2 q = 0.
\end{aligned} \tag{6.2.32}$$

The expression (6.2.32) may be further simplified. In particular, we note that by differentiating equation (6.2.29) with respect to y and z respectively, we have

$$g_{yz}\psi^{(1)} + g_z\frac{\partial\psi^{(1)}}{\partial y} = h_0' f_y + h_0'' f - g_{y\epsilon} + (d_1 - d_2)h_0'' D^2 y - d_2 h_0''' (Dy)^2, \tag{6.2.33}$$

$$g_{zz}\psi^{(1)} = f_z h_0' - g_{z\epsilon}. \tag{6.2.34}$$

Using equations (6.2.33) and (6.2.34) to replace $g_{yz}\psi^{(1)}$ and $g_{zz}\psi^{(1)}$ in (6.2.32), in both the terms independent of d_1 and d_2 and in the terms proportional to d_1, then multiplying through by $1/g_z^2$, and using again $g_z h_0' = -g_y$, we find that (6.2.32) reduces to

$$\begin{aligned} &g_z\psi^{(2)} + \frac{1}{2}g_{zz}{\psi^{(1)}}^2 + g_{z\epsilon}\psi^{(1)} + \frac{1}{2}g_{\epsilon\epsilon} - h_0'(f_z\psi^{(1)} + f_\epsilon) - \frac{\partial\psi^{(1)}}{\partial y}(f + d_1 D^2 y) \\ &-(g_z)^{-1}[d_2(h_0'' D^2 y + h_0'''(Dy)^2)(f + d_1 D^2 y) - h_0''(f + d_1 D^2 y)^2] + d_2 q = 0. \end{aligned} \tag{6.2.35}$$

Therefore, we have established (6.2.20), and this concludes the proof of Theorem 6.2.1.

□

6.3 The MMH Model

In this section, we illustrate the result of the ASIM method on the Michaelis-Menten-Henri model (Edelstein-Keshet, 2005; Heineken et al., 1967; Michaelis and Menten, 1913; Palsson, 2011) with diffusion on the slow species,

$$\begin{cases} \dfrac{\partial y}{\partial t} = -y + (y + a - b)z + d\dfrac{\partial^2 y}{\partial x^2}, \\ \dfrac{\partial z}{\partial t} = \epsilon^{-1} y - \epsilon^{-1}(y + a)z. \end{cases} \tag{6.3.1}$$

System (6.3.12) is of the general fast-slow form (6.2.1) with $f(y, z) = -y + (y + a - b)z$, $g(y, z) = y - (y + a)z$, $d_1 = d$, and $d_2 = 0$. It has nonlinear slow mode

$\mathcal{M}_\epsilon = \{(y,z) : z = h_\epsilon(y, D^2y, D^4y, \ldots)\}$, where h_ϵ has the asymptotic expansion

$$z = h_\epsilon(y, D^2y, D^4y, \ldots) = h^{(0)}(y) + \epsilon h^{(1)}(y, D^2y) + \epsilon^2 h^{(2)}(y, D^2y, D^4y) + \mathcal{O}(\epsilon^3), \quad (6.3.2)$$

with

$$h^{(0)} = \frac{y}{y+a}, \quad (6.3.3)$$

$$h^{(1)} = \frac{aby}{(y+a)^4} - \frac{a}{(y+a)^3} dD^2y, \quad (6.3.4)$$

$$h^{(2)} = \frac{aby(2ab - a^2 - ay - 3by)}{(y+a)^7} + \frac{a(a^2 + ay + 6by - 2ab)}{(y+a)^6} dD^2y - \frac{3a}{(y+a)^5} d^2(D^2y)^2 \quad (6.3.5)$$

$$+ \frac{a}{(y+a)^4} dD^2 \left(\frac{-by}{y+a} + dD^2y \right).$$

To analyze the application of the ASIM method to (6.3.12), we begin with the Jacobian,

$$J = \begin{bmatrix} -1+z & y+a-b \\ \epsilon^{-1}(1-z) & -\epsilon^{-1}(y+a) \end{bmatrix}. \quad (6.3.6)$$

Its eigenvalues are

$$\lambda_{s,f}(y,z) = -\frac{1}{2}(\epsilon^{-1}(y+a) + 1 - z) \pm \sqrt{\frac{1}{4}(\epsilon^{-1}(y+a) + 1 - z)^2 - \epsilon^{-1}b(1-z)}, \quad (6.3.7)$$

where the upper (lower) sign is associated with $\lambda_f(\lambda_s)$. The eigenvalues have asymptotic expansions

$$\lambda_s = -\frac{b(1-z)}{y+a} + \epsilon \frac{b(y+a-b)(1-z)^2}{(y+a)^3} + \mathcal{O}(\epsilon^2) \quad (6.3.8)$$

$$\lambda_f = -\epsilon^{-1}(y+a) - \frac{(y+a-b)(1-z)}{y+a} + \mathcal{O}(\epsilon). \quad (6.3.9)$$

The corresponding (non-normalized) eigenvectors are

$$v_s = \begin{bmatrix} \lambda_s + \epsilon^{-1}(y+a) \\ \epsilon^{-1}(1-z) \end{bmatrix}, \quad v_f = \begin{bmatrix} y+a-b \\ \lambda_f + 1 - z \end{bmatrix}. \quad (6.3.10)$$

Then

$$\begin{bmatrix} \widetilde{v}_s \\ \widetilde{v}_f \end{bmatrix} = \begin{bmatrix} v_s & v_f \end{bmatrix}^{-1} = \frac{1}{det} \begin{bmatrix} \lambda_f + 1 - z & -(y + a - b) \\ -\epsilon^{-1}(1 - z) & \lambda_s + \epsilon^{-1}(y + a) \end{bmatrix}, \tag{6.3.11}$$

where det is the determinant of J.

Application of the ASIM conditions (6.1.11) and (6.2.15) yields the elliptic PDE whose solution is the ASIM for system (6.3.12),

$$\epsilon^{-1}(1 - z)(y - (y + a - b)z) + \epsilon^{-1}(\lambda_s + \epsilon^{-1}(y + a))(y - (y + a)z) - \epsilon^{-1}(1 - z)dD^2y = 0. \tag{6.3.12}$$

Hence, by Theorem 6.2.1, system (6.3.12) has an ASIM, which is given by the solution of (6.3.12),

$$z = \psi(y, D^2y, \dots, \epsilon) = \psi^{(0)}(y) + \epsilon\psi^{(1)}(y, D^2y) + \epsilon^2\psi^{(2)}(y, D^2y) + \mathcal{O}(\epsilon^3), \tag{6.3.13}$$

where

$$\psi^{(0)} = \frac{y}{y + a}, \tag{6.3.14}$$

$$\psi^{(1)} = \frac{aby}{(y + a)^4} - \frac{a}{(y + a)^3}dD^2y, \tag{6.3.15}$$

$$\psi^{(2)} = \frac{aby(2ab - a^2 - ay - by)}{(y + a)^7} + \frac{a(a^2 + ay + 2by - 2ab)}{(y + a)^6}dD^2y - \frac{a}{(y + a)^5}d^2(D^2y)^2. \tag{6.3.16}$$

A comparison of the coefficients in the expansions (6.3.2) and (6.3.13) shows that the nonlinear slow mode and the ASIM agree up to and including $\mathcal{O}(\epsilon)$. The difference is $\mathcal{O}(\epsilon^2)$, as expected from Corollary 6.2.2. Moreover, the coefficient on ϵ^2 in this difference is

$$\psi^{(2)} - h^{(2)} = \frac{2a}{(y + a)^5}\left(-\frac{by}{y + a} + dD^2y\right)^2 - \frac{a}{(y + a)^4}dD^2\left(\frac{-by}{y + a} + dD^2y\right). \tag{6.3.17}$$

The first term in this error is proportional to the curvature of $\mathcal{M}_0$, $h_0'' = -\frac{2a}{(y+a)^3}$, as well as to the square of the slow component of the vector field evaluated at $z = h_0(y) = \frac{y}{y+a}$. The second term is proportional to the Laplacian of the slow component of the vector field. Also, we note that, if $d = 0$, the system (6.3.12) reduces to the classical MMH kinetics model, the ASIM result agrees with the ILDM, and the error is the same as that calculated in (Kaper and Kaper, 2002).

6.4 The Davis-Skodje Model

In this section, we will apply Theorem 6.2.1 and Corollary 6.2.2 to the Davis and Skodje model (Davis and Skodje, 1999), with diffusion of both the slow and fast species,

$$\begin{cases} \dfrac{\partial y}{\partial t} = -y + d_1 \dfrac{\partial^2 y}{\partial x^2} \\ \dfrac{\partial z}{\partial t} = -\gamma z + \dfrac{(\gamma - 1)y + \gamma y^2}{(1+y)^2} + d_2 \dfrac{\partial^2 z}{\partial x^2}, \end{cases} \tag{6.4.1}$$

where $d_1, d_2 > 0$, and $\gamma \gg 1$. This system is of the form (6.2.1), where y is the slow species, z is the fast species, and $\epsilon = 1/\gamma$. Also, $f(y,z) = -y$, and $g(y,z) = -z + \frac{y}{1+y} - \frac{y}{\gamma(1+y)^2}$. We include this as a second example, because it illustrates those aspects of the ASIM theory that arise when both species diffuse. In particular, diffusion of the fast species, $D^2 z$, gives rise to terms in both h_ϵ and ψ_ϵ that also depend on spatial derivatives of y of odd order, $Dy, D^3 y, \ldots$, whereas in the MMH example there are only terms depending on even order spatial derivatives.

For any $\gamma \gg 1$, the system (6.4.1) has nonlinear slow mode $\mathcal{M}_\gamma = \{(y,z) : z = h_\gamma(y, Dy, D^2 y, ...)\}$, which is invariant under the dynamics of equations (6.4.1).

Asymptotically,

$$z = h_\gamma(y, Dy, D^2y, ...) = h^{(0)}(y) + \frac{1}{\gamma}h^{(1)}(y, Dy, D^2y) + \frac{1}{\gamma^2}h^{(2)}(y, Dy, ..., D^4y) + \mathcal{O}\left(\frac{1}{\gamma^3}\right) \tag{6.4.2}$$

with

$$h^{(0)} = \frac{y}{1+y}, \tag{6.4.3}$$

$$h^{(1)} = -\frac{1}{(1+y)^2}d_1D^2y + d_2\left[\frac{1}{(1+y)^2}D^2y - \frac{2}{(1+y)^3}(Dy)^2\right], \tag{6.4.4}$$

$$\begin{aligned} h^{(2)} =& d_2q - \left[\frac{2}{(1+y)^3}d_1D^2y + d_2\left(\frac{6}{(1+y)^4}(Dy)^2 - \frac{2}{(1+y)^3}D^2y\right)\right](-y + d_1D^2y) \\ &+ \frac{4}{(1+y)^3}d_2DyD(-y + d_1D^2y) + \frac{1}{(1+y)^2}(d_1 - d_2)D^2(-y + d_1D^2y), \end{aligned} \tag{6.4.5}$$

where q can be computed by (6.2.10). We note that, in the absence of diffusion, system (6.4.1) reduces to the kinetics model of (Davis and Skodje, 1999), which was designed as an example for which the function $\frac{y}{1+y}$ is an exact invariant slow manifold, i.e. no higher order corrections are necessary. Here, we see that the presence of diffusivity of both species results in the nonlinear slow mode depending also on the gradient and Laplacian of y, as well as on higher spatial derivatives of y.

The Jacobian of the right hand side of equations (6.4.1) is

$$J = \begin{bmatrix} -1 & 0 \\ \frac{\gamma-1+(\gamma+1)y}{(1+y)^3} & -\gamma \end{bmatrix}. \tag{6.4.6}$$

It has eigenvalues $\lambda_s = -1$ and $\lambda_f = -\gamma$. The corresponding eigenvectors are given by

$$v_s = \begin{bmatrix} 1 \\ \frac{\gamma-1+(\gamma+1)y}{(\gamma-1)(1+y)^3} \end{bmatrix}, \; v_f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.4.7}$$

The perpendicular vectors, which span the orthogonal complements, are

$$\begin{bmatrix} \widetilde{v_s} \\ \widetilde{v_f} \end{bmatrix} = \begin{bmatrix} v_s & v_f \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{\gamma-1+(\gamma+1)y}{(\gamma-1)(1+y)^3} & 1 \end{bmatrix}. \tag{6.4.8}$$

Therefore, the ASIM for the Davis-Skodje model, which is given by the solution of the elliptic PDE (6.2.15), is

$$z = \psi_\gamma(y, Dy, D^2y, ...) = \psi^{(0)}(y)+\frac{1}{\gamma}\psi^{(1)}(y, Dy, D^2y)+\frac{1}{\gamma^2}\psi^{(2)}(y, Dy, ..., D^4y)+\mathcal{O}\left(\frac{1}{\gamma^3}\right) \tag{6.4.9}$$

with

$$\psi^{(0)} = \frac{y}{1+y}, \tag{6.4.10}$$

$$\psi^{(1)} = -\frac{1}{(1+y)^2}d_1D^2y + d_2\left[\frac{1}{(1+y)^2}D^2y - \frac{2}{(1+y)^3}(Dy)^2\right], \tag{6.4.11}$$

$$\psi^{(2)} = d_2q - \frac{2y}{(1+y)^3}(-y + d_1D^2y), \tag{6.4.12}$$

where q is the same as the q in $h^{(2)}$.

Therefore, comparing the terms (6.4.3)-(6.4.5) in the expansion of the nonlinear slow mode with those (6.4.10)-(6.4.12) in the expansion of the ASIM, we see that the ASIM method is accurate up to and including $\mathcal{O}(\epsilon)$, as expected from Theorem 6.2.1 and Corollary 6.2.2. Also, we see that the error is $\mathcal{O}(\epsilon^2)$. In particular, the difference between the nonlinear slow mode (6.4.2) and the ASIM (6.4.9), $\psi_\gamma - h_\gamma$, at $\mathcal{O}(\frac{1}{\gamma^2})$ has coefficient

$$
\begin{aligned}
\psi^{(2)} - h^{(2)} =& \frac{2}{(1+y)^3}(-y + d_1 D^2 y)^2 \\
&+ d_2 \left(\frac{6}{(1+y)^4}(Dy)^2 - \frac{2}{(1+y)^3} D^2 y \right) (-y + d_1 D^2 y) \\
&- \frac{4}{(1+y)^3} d_2 Dy D(-y + d_1 D^2 y) + \frac{1}{(1+y)^2}(d_2 - d_1) D^2(-y + d_1 D^2 y).
\end{aligned}
\tag{6.4.13}
$$

The first three terms in the difference are proportional to the curvature of $\mathcal{M}_0$, $h_0'' = -\frac{2}{(1+y)^3}$, the rate of change of this curvature, $h_0''' = \frac{6}{(1+y)^4}$, and to the slow component of the vector field (including both reaction and diffusion) evaluated at $z = h_0(y) = \frac{y}{1+y}$. The last term is proportional to the Laplacian of the slow component of the vector field. This establishes the accuracy of the ASIM for the Davis-Skodje model.

Remark. Of the four principal terms in the $\mathcal{O}(\frac{1}{\gamma^2})$ error, (6.4.13), the third and fourth come from the fact that $h^{(2)}$ includes terms proportional to $\frac{\partial h^{(1)}}{\partial (Dy)}$ and $\frac{\partial h^{(1)}}{\partial (D^2 y)}$, which are not captured by $\psi^{(2)}$. In addition, the first term is in $\psi^{(2)}$ (arising from $h_0''(f + d_1 D^2 y)^2$), but not in $h^{(2)}$, and the second term is one of that occurs in $h^{(2)}$ (arising from $(g_z)^{-1} d_2 (h_0'' D^2 y + h_0'''(Dy)^2)(f + d_1 D^2 y)$), however, it is absent from $\psi^{(2)}$ due to a cancellation with the term $\frac{\partial \psi^{(1)}}{\partial y}(f + d_1 D^2 y)$.

Remark. In the special case in which $d_1, d_2 = 0$, system (6.4.1) reduces to the original reaction kinetics model due to Davis and Skodje (Davis and Skodje, 1999). The ASIM, (6.4.9)-(6.4.12), reduces exactly to the ILDM, as also shown in (Singh et al., 2002).

6.5 Accuracy of the ASIM Method - General Case

In this section, we analyze the accuracy of the ASIM for the general case of reaction-diffusion equations with m slow components and n fast components

$$\begin{cases} \dfrac{\partial y}{\partial t} = f(y, z, \epsilon) + D_1 \dfrac{\partial^2 y}{\partial x^2} \\ \dfrac{\partial z}{\partial t} = \epsilon^{-1} g(y, z, \epsilon) + D_2 \dfrac{\partial^2 z}{\partial x^2}. \end{cases} \tag{6.5.1}$$

This system is of the form (1.2.1) with general m and n,

$$\boldsymbol{u} = \begin{bmatrix} y \\ z \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} f \\ \epsilon^{-1} g \end{bmatrix}, \text{ and } \boldsymbol{h} = \begin{bmatrix} -D_1 \frac{\partial y}{\partial x} \\ -D_2 \frac{\partial z}{\partial x} \end{bmatrix}.$$

Here, the $\mathbb{R}^m$-valued vector y and $\mathbb{R}^n$-valued vector z denote the species concentrations. Both concentrations depend on space, x, and time, t. The functions f and g represent the reaction terms with values in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. $D_1 \frac{\partial^2 y}{\partial^2 x}$ and $D_2 \frac{\partial^2 z}{\partial^2 x}$ are the diffusion terms, where D_1 and D_2 are $m \times m$ and $n \times n$ diagonal matrices respectively. In matrix form,

$$D_1 \frac{\partial^2 y}{\partial x^2} \equiv \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_m \end{bmatrix} \begin{bmatrix} \frac{\partial^2 y_1}{\partial^2 x} \\ \frac{\partial^2 y_2}{\partial^2 x} \\ \vdots \\ \frac{\partial^2 y_m}{\partial^2 x} \end{bmatrix} = \begin{bmatrix} d_1 \frac{\partial^2 y_1}{\partial^2 x} \\ d_2 \frac{\partial^2 y_2}{\partial^2 x} \\ \vdots \\ d_m \frac{\partial^2 y_m}{\partial^2 x} \end{bmatrix}, \tag{6.5.2}$$

$$D_2 \frac{\partial^2 z}{\partial x^2} \equiv \begin{bmatrix} d_{m+1} & & & \\ & d_{m+2} & & \\ & & \ddots & \\ & & & d_{m+n} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 z_1}{\partial^2 x} \\ \frac{\partial^2 z_2}{\partial^2 x} \\ \vdots \\ \frac{\partial^2 z_n}{\partial^2 x} \end{bmatrix} = \begin{bmatrix} d_{m+1} \frac{\partial^2 z_1}{\partial^2 x} \\ d_{m+2} \frac{\partial^2 z_2}{\partial^2 x} \\ \vdots \\ d_{m+n} \frac{\partial^2 z_n}{\partial^2 x} \end{bmatrix}. \tag{6.5.3}$$

There is a separation of time scales in the reaction kinetics, with the vector y of the slow variables and the vector z of the fast variables, measured by the small parameter ϵ.

We assume that system (6.5.1) has a nonlinear slow mode on which reaction and diffusion only occur on the slow (long) time scale. For $\epsilon = 0$, we assume that the reaction kinetics in (6.5.1) has a critical manifold

$$\mathcal{M}_0 = \{(y,z) : z = h_0(y), g(y, h_0(y), 0) = 0\}, \tag{6.5.4}$$

where h_0 is a sufficiently differentiable function. Moreover, we assume that the real part of each eigenvalue of the Jacobian

$$J = \begin{bmatrix} f_y & f_z \\ \epsilon^{-1} g_y & \epsilon^{-1} g_z \end{bmatrix} \tag{6.5.5}$$

is negative, so that the critical manifold is exponentially stable. The full system (6.5.1) with $0 < \epsilon \ll 1$ has nonlinear slow mode

$$\mathcal{M}_\epsilon = \{(y,z) : z = h_\epsilon(y, Dy, D^2y, \ldots)\}, \tag{6.5.6}$$

which are close to $\mathcal{M}_0$ for $\epsilon > 0$ sufficiently small. D^i denotes the ith order derivative with respect to x. Let $\frac{dh_0}{dy} \equiv \frac{dh_0}{dy}(y)$ be a linear operator from $\mathbb{R}^m$ to $\mathbb{R}^n$, which is represented by the $n \times m$ matrix of $\frac{\partial h_{0,i}}{\partial y_j}$. Next, $\frac{d^2h_0}{dy^2} = \frac{d}{dy}(\frac{dh_0}{dy}) \equiv (\frac{d^2h_0}{dy^2})(y)$ is a bilinear map from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^n$, $(\frac{d^2h_0}{dy^2})(u,v) = ((\frac{d^2h_0}{dy^2})u)v$ for all $u, v \in \mathbb{R}^m$, and $\frac{d^3h_0}{dy^3} = \frac{d}{dy}(\frac{d^2h_0}{dy^2}) \equiv (\frac{d^3h_0}{dy^3})(y)$ is a 3-linear map from $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^n$, $(\frac{d^3h_0}{dy^3})(u,v,w) = (((\frac{d^3h_0}{dy^3})u)v)w$ for all $u, v, w \in \mathbb{R}^m$.

The function h_ϵ admits an asymptotic expansion

$$z = h_\epsilon = h^{(0)}(y) + \epsilon h^{(1)}(y, Dy, D^2y) + \epsilon^2 h^{(2)}(y, Dy, D^2y, D^3y, D^4y) + \mathcal{O}(\epsilon^3), \tag{6.5.7}$$

and the Laplacian of z, D^2z, has the following asymptotic expansion

$$D^2z = D^2z_h^{(0)} + \epsilon D^2 z_h^{(1)} + \mathcal{O}(\epsilon^2). \tag{6.5.8}$$

Here, similar to the two-species case in Section 3, we find that the terms $h^{(1)}$, $h^{(2)}$, $D^2 z_h^{(0)}$, and $D^2 z_h^{(1)}$ are given by

$$\begin{aligned}
h^{(0)} =& h_0, && (6.5.9)\\
(D_z g)h^{(1)} =& \frac{dh_0}{dy}(f + D_1 D^2 y) - g_\epsilon - D_2 \left[\frac{dh_0}{dy} D^2 y + \frac{d^2 h_0}{dy^2}(Dy, Dy)\right], && (6.5.10)\\
(D_z g)h^{(2)} =& \frac{dh_0}{dy}((D_z f)h^{(1)} + f_\epsilon) - \frac{1}{2}(D_z^2 g)(h^{(1)}, h^{(1)}) - (D_z g_\epsilon)h^{(1)} - \frac{1}{2} g_{\epsilon\epsilon}\\
&+ \frac{\partial h^{(1)}}{\partial y}(f + D_1 D^2 y) + \frac{\partial h^{(1)}}{\partial (Dy)} D(f + D_1 D^2 y)\\
&+ \frac{\partial h^{(1)}}{\partial (D^2 y)} D^2 (f + D_1 D^2 y) - D_2 D^2 z_h^{(1)}, && (6.5.11)\\
D^2 z_h^{(0)} =& \frac{dh_0}{dy} D^2 y + \frac{d^2 h_0}{dy^2}(Dy, Dy), && (6.5.12)\\
D^2 z_h^{(1)} =& \frac{\partial h^{(1)}}{\partial y} D^2 y + \frac{\partial h^{(1)}}{\partial (Dy)} D^3 y + \frac{\partial h^{(1)}}{\partial (D^2 y)} D^4 y + \frac{\partial^2 h^{(1)}}{\partial y^2}(Dy)^2 + \frac{\partial^2 h^{(1)}}{\partial (Dy)^2}(D^2 y)^2\\
&+ 2\frac{\partial^2 h^{(1)}}{\partial y \partial (Dy)} Dy D^2 y + 2\frac{\partial^2 h^{(1)}}{\partial y \partial (D^2 y)} Dy D^3 y. && (6.5.13)
\end{aligned}$$

In order to introduce the main theorem for the general case, we need a block diagonalization of the Jacobian J by the following process. First, we use the standard Schur decomposition

$$J = QTQ', \text{ where } T = \begin{bmatrix} \Lambda_f & \Lambda \\ 0 & \Lambda_s \end{bmatrix} \text{ and } Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \qquad (6.5.14)$$

Here, Λ_f is an $n \times n$ upper triangular matrix, Λ is an $n \times m$ full matrix, Λ_s is an $m \times m$ lower triangular matrix. The diagonal elements of Λ_f are the $\mathcal{O}(\epsilon^{-1})$ eigenvalues of J, and the diagonal elements of Λ_s are the $\mathcal{O}(1)$ eigenvalues of J. The eigenvalues are generated in order of descending absolute values of their real parts. Q is a unitary matrix. Q_{11} is an $m \times n$ matrix, Q_{12} is an $m \times m$ matrix, Q_{21} is an $n \times n$ matrix, Q_{22} is an $n \times m$ matrix. Q' is the transpose of Q.

Next, let X be the solution of Sylvester equation

$$\Lambda_f X - X\Lambda_s = -\Lambda \tag{6.5.15}$$

for the $n \times m$ matrix X. Together with the identity matrices I_n and I_m, this matrix T forms a matrix

$$Y = \begin{bmatrix} I_n & X \\ 0 & I_m \end{bmatrix}. \tag{6.5.16}$$

which yields the following block diagonalization of J:

$$J = (QY)T_d(QY)^{-1}, \tag{6.5.17}$$

where

$$T_d = \begin{bmatrix} \Lambda_f & 0 \\ 0 & \Lambda_s \end{bmatrix}, \; QY = \begin{bmatrix} Q_{11} & Q_{11}X + Q_{12} \\ Q_{21} & Q_{21}X + Q_{22} \end{bmatrix}. \tag{6.5.18}$$

In the decomposition, QY reduces the matrix J to a block diagonal matrix, T_d, where the blocks are the fast and slow components. The first n columns of QY form an orthogonal basis for the fast subspace, and the second m columns of QY form an orthogonal basis for the slow subspace.

This decomposition yields the central matrix in the ASIM method, (6.5.1),

$$\begin{bmatrix} \widetilde{v_f} \\ \widetilde{v_s} \end{bmatrix} = (QY)^{-1} = \begin{bmatrix} Q'_{11} - XQ'_{12} & Q'_{21} - XQ'_{22} \\ Q'_{12} & Q'_{22} \end{bmatrix}. \tag{6.5.19}$$

Using this matrix, we may now directly apply the ASIM conditions (6.1.11),

$$(Q'_{11} - XQ'_{12})f + \epsilon^{-1}(Q'_{21} - XQ'_{22})g + (Q'_{11} - XQ'_{12})D_1D^2y + (Q'_{21} - XQ'_{22})D_2D^2z = 0, \tag{6.5.20}$$

$$Q'_{12}\frac{\partial y}{\partial t} + Q'_{22}\frac{\partial z}{\partial t} = Q'_{12}f + \epsilon^{-1}Q'_{22}g + Q'_{12}D_1D^2y + Q'_{22}D_2D^2z. \tag{6.5.21}$$

Here, equation (6.5.20) defines the ASIM, and equation (6.5.21) defines the approximate flow on the ASIM.

We now prove the following theorem which establishes the accuracy of the ASIM method for general systems (6.5.1):

Theorem 6.5.1. *(General Case)*
The solution $z = \psi(y, Dy, D^2y, D^3y, \ldots, \epsilon)$, *of equation* (6.5.20) *has an asymptotic expansion of the form*

$$z = \psi = \psi^{(0)}(y) + \epsilon\psi^{(1)}(y, Dy, D^2y) + \epsilon^2\psi^{(2)}(y, Dy, D^2y, D^3y, D^4y) + \mathcal{O}(\epsilon^3). \quad (6.5.22)$$

The $\mathbb{R}^n$ *-valued functions* $\psi^{(0)}, \psi^{(1)}, \psi^{(2)}$ *are defined by the equations*

$$\psi^{(0)} = h_0, \quad (6.5.23)$$

$$(D_z g)\psi^{(1)} = \frac{dh_0}{dy}(f + D_1 D^2 y) - g_\epsilon - D_2\left[\frac{dh_0}{dy}D^2y + \frac{d^2h_0}{dy^2}(Dy, Dy)\right], \quad (6.5.24)$$

$$\begin{aligned}(D_z g)\psi^{(2)} =& \frac{dh_0}{dy}[(D_z f)\psi^{(1)} + f_\epsilon] - (D_z g)^{-1}\frac{d^2h_0}{dy^2}(f, f) - \frac{1}{2}(D_z^2 g)(\psi^{(1)}, \psi^{(1)}) \\ &- (D_z g_\epsilon)\psi^{(1)} - \frac{1}{2}g_{\epsilon\epsilon} + \frac{\partial\psi^{(1)}}{\partial y}(f + D_1 D^2 y) \\ &+ (D_z g)^{-1}\left[D_2\left(\frac{d}{dy}D^2 z_\psi^{(0)}\right)(f + D_1 D^2 y) - \frac{d^2h_0}{dy^2}(2f + D_1 D^2 y, D_1 D^2 y)\right] \\ &- D_2 D^2 z_\psi^{(1)}, \end{aligned} \quad (6.5.25)$$

where the $\mathbb{R}^m$ *-valued function* f *and the* $\mathbb{R}^n$ *-valued function* g *and their derivatives are evaluated along* $\mathcal{M}_0$. *Also,* $D^2z_\psi^{(0)}$ *and* $D^2z_\psi^{(1)}$ *are the coefficients of the* $\mathcal{O}(1)$ *and* $\mathcal{O}(\epsilon)$ *terms in* D^2z, *where* z *has the asymptotic expansion* (6.5.22). *Then,* $D^2z_\psi^{(0)}$ *and* $D^2z_\psi^{(1)}$ *are given by* (6.5.12) *and* (6.5.13), *respectively, with* h *replaced by* ψ.

We notice that $D^2z_h^{(0)} = D^2z_\psi^{(0)}$ and $D^2z_h^{(1)} = D^2z_\psi^{(1)}$, because $\psi^{(0)} = h^{(0)}$ and $\psi^{(1)} = h^{(1)}$. Thus, in the following Corollary, we will write $D^2z_\psi^{(0)} \equiv D^2z^{(0)}$ and

$D^2 z_\psi^{(1)} \equiv D^2 z^{(1)}$. We now compare the asymptotic expansions of ψ and h, and establish the Corollary, which explicitly gives the accuracy of the ASIM method.

Corollary 6.5.2. *(General Case)*
A comparison of the coefficients (6.5.23)-(6.5.25) *in the expansion* (6.5.22) *of the ASIM with the coefficients* (6.5.9)-(6.5.11) *in the expansion* (6.5.7) *of the nonlinear slow mode* $\mathcal{M}_\epsilon$ *shows that*

$$\psi^{(0)} = h^{(0)}, \tag{6.5.26}$$
$$\psi^{(1)} = h^{(1)}, \tag{6.5.27}$$
$$\begin{aligned}\psi^{(2)} =& h^{(2)} - (D_z g)^{-1}\left[\frac{\partial h^{(1)}}{\partial(Dy)} D(f + D_1 D^2 y) + \frac{\partial h^{(1)}}{\partial(D^2 y)} D^2(f + D_1 D^2 y)\right] \\ &+ (D_z g)^{-2}\left[D_2\left(\frac{d}{dy} D^2 z_\psi^{(0)}\right)(f + D_1 D^2 y) - \frac{d^2 h_0}{dy^2}(f + D_1 D^2 y, f + D_1 D^2 y)\right].\end{aligned} \tag{6.5.28}$$

Theorem 6.5.1 and Corollary 6.5.2 have three important consequences. First, the ASIM is accurate up to and including $\mathcal{O}(\epsilon)$ compared with the nonlinear slow mode $\mathcal{M}_\epsilon$. Therefore, for system (6.5.1), the ASIM method is highly accurate.

Second, the leading term in the difference is $\mathcal{O}(\epsilon^2)$, with coefficient

$$\begin{aligned}\psi^{(2)} - h^{(2)} =& (D_z g)^{-2}\Bigg[D_2\left(\frac{d^2 h_0}{dy^2}(D^2 y, f + D_1 D^2 y) + \frac{d^3 h_0}{dy^3}(Dy, Dy, f + D_1 D^2 y)\right) \\ & - \frac{d^2 h_0}{dy^2}(f + D_1 D^2 y, f + D_1 D^2 y) + 2 D_2 \frac{d^2 h_0}{dy^2}(Dy, D(f + D_1 D^2 y)) \\ & - \left(\frac{d h_0}{dy} D_1 - D_2 \frac{d h_0}{dy}\right) D^2(f + D_1 D^2 y)\Bigg].\end{aligned} \tag{6.5.29}$$

As in the two-species case, the first term in the error involves the curvature of the critical manifold $\mathcal{M}_0$, $\frac{d^2 h_0}{dy^2}$, the rate of change of the curvature, $\frac{d^3 h_0}{dy^3}$, and the reaction kinetics and diffusion of the slow species, $f + D_1 D^2 y$. The second terms involves $\frac{d^2 h_0}{dy^2}$ and the square of the slow component. The third and fourth terms are proportional to the gradient and Laplacian, respectively, of the slow component of the reaction-diffusion system. These terms are precisely the generalization of the error terms in the two-species case.

Third, in the special case in which there is no diffusion, i.e. D_1 and D_2 are zero matrices, we find that the difference reduces to $-(D_z g)^{-2} \frac{d^2 h_0}{dy^2}(f, f)$, which is exactly the curvature term identified in Corollary 5.1 in reference (Kaper and Kaper, 2002).

Proof of Theorem 6.5.1

In the Schur decomposition of matrix J, let

$$Q \equiv Q(\epsilon) \equiv \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \tag{6.5.30}$$

where the blocks have the following dimensions: Q_{11} $m \times n$ matrix, Q_{12} $m \times m$ matrix, Q_{21} $n \times n$ matrix, Q_{22} $n \times m$ matrix. Given the structure of matrix J, see (6.5.5), the blocks of Q have the following asymptotic expansions:

$$Q_{11} = \epsilon Q_{11}^{(1)} + \mathcal{O}(\epsilon^2) \tag{6.5.31}$$

$$Q_{12} = I_m + \mathcal{O}(\epsilon^2) \tag{6.5.32}$$

$$Q_{21} = Q_{21}^{(0)} + \mathcal{O}(\epsilon^2) \tag{6.5.33}$$

$$Q_{22} = -\epsilon Q_{21}^{(0)} Q_{11}^{(1)\prime} + \mathcal{O}(\epsilon^2). \tag{6.5.34}$$

Here, $Q_{21}^{(0)}$ is a unitary $n \times n$ matrix, and $Q_{11}^{(1)}$ is an $m \times n$ matrix to be determined. Higher- order terms can be found in a consistent manner so that $Q(\epsilon)$ is unitary.

Next, we know from the structure of the block upper triangularization of J, (6.5.14), and its block diagonalization, (6.5.18), that the block matrices in T have the following expansions:

$$\Lambda_f \equiv \Lambda_f(\epsilon) = \epsilon^{-1}\Lambda_f^{(-1)} + \Lambda_f^{(0)} + \epsilon\Lambda_f^{(1)} + \mathcal{O}(\epsilon^2) \tag{6.5.35}$$

$$\Lambda \equiv \Lambda(\epsilon) = \epsilon^{-1}\Lambda^{(-1)} + \Lambda^{(0)} + \epsilon\Lambda^{(1)} + \mathcal{O}(\epsilon^2) \tag{6.5.36}$$

$$\Lambda_s \equiv \Lambda_s(\epsilon) = \Lambda_s^{(0)} + \epsilon\Lambda_s^{(1)} + \mathcal{O}(\epsilon) \tag{6.5.37}$$

$$X \equiv X(\epsilon) = X^{(0)} + \epsilon X^{(1)} + \mathcal{O}(\epsilon^2). \tag{6.5.38}$$

We take $z = \psi(y, Dy, D^2y, \ldots, \epsilon)$ to be given by an asymptotic expansion of the form

$$z = \psi = \psi^{(0)}(y) + \epsilon\psi^{(1)}(y, Dy, D^2y) + \epsilon^2\psi^{(2)}(y, Dy, D^2y, D^3y, D^4y) + \mathcal{O}(\epsilon^3). \tag{6.5.39}$$

We also need the Taylor expansion of the $\mathbb{R}^m$-valued function f at $(y, \psi^{(0)}(y), 0)$,

$$\begin{aligned} f(y, \psi, \epsilon) =& f + \epsilon((D_z f)\psi^{(1)} + f_\epsilon) \\ & + \epsilon^2\left((D_z f)\psi^{(2)} + \frac{1}{2}(D_z^2 f)(\psi^{(1)}, \psi^{(1)}) + (D_z f_\epsilon)\psi^{(1)} + \frac{1}{2}f_{\epsilon\epsilon}\right) + \mathcal{O}(\epsilon^3). \end{aligned} \tag{6.5.40}$$

Here, the terms in the right hand side are evaluated at $(y, \psi^{(0)}(y), 0)$. Similar expansions hold for all derivatives of f and for g and its derivatives.

To prove the theorem, we substitute the various expansions into equation (6.5.20) and equate the coefficients of like powers in ϵ in the usual manner.

$\mathcal{O}(\epsilon^{-1})$: The ASIM equation (6.5.20) gives

$$Q_{21}^{(0)\prime} g|_{(y,\psi^{(0)}(y),0)} = 0. \tag{6.5.41}$$

Since $Q_{21}^{(0)}$ is unitary, equation (6.5.41) reduces to $g|_{(y,\psi^{(0)}(y),0)} = 0$, which is satisfied when

$$\psi^{(0)} = h_0. \tag{6.5.42}$$

This establishes (6.5.23) in Theorem 6.5.1.

$\mathcal{O}(1)$: Using

$$g(y, h_0(y), 0) = 0, \tag{6.5.43}$$

equation (6.5.20) gives

$$-X^{(0)} f + Q_{21}^{(0)'}[(D_z g)\psi^{(1)} + g_\epsilon] - X^{(0)} D_1 D^2 y + Q_{21}^{(0)'} D_2 D^2 z_\psi^{(0)} = 0. \tag{6.5.44}$$

To simplify this equation, first compute $X^{(0)}$ by equating the coefficients of the $\mathcal{O}(\epsilon^{-1})$ terms in the Sylvester equation (6.5.15),

$$\Lambda_f^{(-1)} X^{(0)} = -\Lambda^{(-1)}. \tag{6.5.45}$$

The matrices $\Lambda_f^{(-1)}$ and $\Lambda^{(-1)}$ follow from the $\mathcal{O}(\epsilon^{-1})$ terms in the Schur decomposition equation (6.5.14),

$$Q_{21}^{(0)} \Lambda^{(-1)} = D_y g, \;\; Q_{21}^{(0)} \Lambda_f^{(-1)} Q_{21}^{(0)'} = D_z g. \tag{6.5.46}$$

$Q_{21}^{(0)}$ is unitary, thus invertible. Hence we obtain

$$\Lambda^{(-1)} = Q_{21}^{(0)'}(D_y g), \;\; \Lambda_f^{(-1)} = Q_{21}^{(0)'}(D_z g) Q_{21}^{(0)}. \tag{6.5.47}$$

and

$$X^{(0)} = -Q_{21}^{(0)'}(D_z g)^{-1}(D_y g). \tag{6.5.48}$$

Differentiating $g(y, h_0(y), 0) = 0$ with respect to y gives a relation between $D_z g$ and $D_y g$,

$$D_y g + (D_z g)\frac{dh_0}{dy} = 0. \tag{6.5.49}$$

Then, $X^{(0)}$ can be written as

$$X^{(0)} = Q_{21}^{(0)\prime} \frac{dh_0}{dy}. \tag{6.5.50}$$

Thus, using (6.5.50) and multiplying through by the unitary matrix $Q_{21}^{(0)}$, we see that equation (6.5.44) reduces to

$$(D_z g)\psi^{(1)} = \frac{dh_0}{dy}(f + D_1 D^2 y) - g_\epsilon - D_2 D^2 z_\psi^{(0)}. \tag{6.5.51}$$

Thus, we have established (6.5.24) in Theorem 6.5.1.

$\mathcal{O}(\epsilon)$: We use the result for $\psi^{(0)}$ from equation (6.5.42) and the result for $\psi^{(1)}$ from equation (6.5.51). Then, substitute $X^{(0)}$ from equation (6.5.50). The ASIM condition (6.5.20) gives

$$\begin{aligned}
&\left[Q_{11}^{(1)\prime} - X^{(1)} + Q_{21}^{(0)\prime}\frac{dh_0}{dy}Q_{11}^{(1)}Q_{21}^{(0)\prime}\frac{dh_0}{dy}\right] f - Q_{21}^{(0)\prime}\frac{dh_0}{dy}[(D_z f)\psi^{(1)} + f_\epsilon] \\
&+Q_{21}^{(0)\prime}\left[(D_z g)\psi^{(2)} + \frac{1}{2}(D_z^2 g)(\psi^{(1)}, \psi^{(1)}) + (D_z g_\epsilon)\psi^{(1)} + \frac{1}{2}g_{\epsilon\epsilon}\right] + \left[Q_{11}^{(1)\prime} - X^{(1)}\right. \\
&\left. + Q_{21}^{(0)\prime}\frac{dh_0}{dy}Q_{11}^{(1)}Q_{21}^{(0)\prime}\frac{dh_0}{dy}\right] D_1 D^2 y + Q_{21}^{(0)\prime} D_2 D^2 z_\psi^{(1)} = 0.
\end{aligned} \tag{6.5.52}$$

To simplify (6.5.52), we first compute the matrices $X^{(1)}$ and $Q_{11}^{(1)}$. $X^{(1)}$ is determined from the $\mathcal{O}(1)$ terms in the Sylvester equation (6.5.15),

$$\Lambda_f^{(-1)} X^{(1)} + \Lambda_f^{(0)} X^{(0)} - X^{(0)}\Lambda_s^{(0)} = -\Lambda^{(0)}. \tag{6.5.53}$$

The matrices $\Lambda_f^{(0)}$, $\Lambda_s^{(0)}$, and $\Lambda^{(0)}$ follow from the $\mathcal{O}(1)$ terms in the Schur decomposition (6.5.14). Solve a system of four equations with four unknowns $\Lambda_f^{(0)}$,$\Lambda_s^{(0)}$, $\Lambda^{(0)}$

and $Q_{11}^{(1)}$, we obtain

$$Q_{11}^{(1)} = (D_z f)(D_z g)^{-1} Q_{21}^{(0)}, \tag{6.5.54}$$

$$\Lambda_s^{(0)} = D_y f + (D_z f)\frac{dh_0}{dy}, \tag{6.5.55}$$

$$\Lambda^{(0)} = Q_{21}^{(0)'}((D_z(D_y g))\psi^{(1)} + D_y g_\epsilon - (D_z g)((D_z f)(D_z g)^{-1})'), \tag{6.5.56}$$

$$\Lambda_f^{(0)} = Q_{21}^{(0)'}\left((D_z^2 g)\psi^{(1)} + D_z g_\epsilon - (D_z g)\frac{dh_0}{dy}(D_z f)(D_z g)^{-1}\right) Q_{21}^{(0)}. \tag{6.5.57}$$

Here, we used equations (6.5.49) and (6.5.47).

Next, we compute $X^{(1)}$ from (6.5.52),

$$\begin{aligned} X^{(1)} =& Q_{21}^{(0)'}(D_z g)^{-1}\left[-((D_z(D_y g))\psi^{(1)} + D_y g_\epsilon) + (D_z g)((D_z f)(D_z g)^{-1})' - \Big((D_z^2 g)\psi^{(1)}\right. \\ & \left. + D_z g_\epsilon - (D_z g)\frac{dh_0}{dy}(D_z f)(D_z g)^{-1}\Big)\frac{dh_0}{dy} + \frac{dh_0}{dy}\left(D_y f + (D_z f)\frac{dh_0}{dy}\right)\right]. \end{aligned} \tag{6.5.58}$$

Substituting $X^{(1)}$ from (6.5.58) and $Q_{11}^{(1)}$ from (6.5.54), then multiplying through by $Q_{21}^{(0)}$, we see that equation (6.5.52) reduces to

$$\begin{aligned} &(D_z g)^{-1}\left[(D_z(D_y g))(\psi^{(1)}, f) + (D_y g_\epsilon) f + (D_z^2 g)\left(\psi^{(1)}, \frac{dh_0}{dy} f\right) + (D_z g_\epsilon)\frac{dh_0}{dy} f\right. \\ &\left. - \frac{dh_0}{dy}\left(D_y f + (D_z f)\frac{dh_0}{dy}\right) f\right] - \frac{dh_0}{dy}((D_z f)\psi^{(1)} + f_\epsilon) + (D_z g)\psi^{(2)} + \frac{1}{2}(D_z^2 g)(\psi^{(1)}, \psi^{(1)}) \\ &+ (D_z g_\epsilon)\psi^{(1)} + \frac{1}{2} g_{\epsilon\epsilon} + (D_z g)^{-1}\left[(D_z(D_y g))\psi^{(1)} + D_y g_\epsilon + (D_z^2 g)\left(\psi^{(1)}, \frac{dh_0}{dy}\right)\right. \\ &\left. + (D_z g_\epsilon)\frac{dh_0}{dy} - \frac{dh_0}{dy}\left(D_y f + (D_z f)\frac{dh_0}{dy}\right)\right] D_1 D^2 y + D_2 D^2 z_\psi^{(1)} = 0. \end{aligned} \tag{6.5.59}$$

In order to further simplify (6.5.59), we need the following results. First, we observe that differentiation of equation (6.5.51) leads to

$$
\begin{aligned}
&(D_y(D_z g))\psi^{(1)} + D_y g_\epsilon + (D_z^2 g)\left(\frac{dh_0}{dy}, \psi^{(1)}\right) + (D_z g_\epsilon)\frac{dh_0}{dy} + (D_z g)\frac{\partial \psi^{(1)}}{\partial y} \\
=&\frac{d^2 h_0}{dy^2} f + \frac{dh_0}{dy}\left(D_y f + (D_z f)\frac{dh_0}{dy}\right) + \frac{d^2 h_0}{dy^2} D_1 D^2 y - D_2\left(\frac{d}{dy} D^2 z_\psi^{(0)}\right).
\end{aligned}
\tag{6.5.60}
$$

This is a relation in the space of linear operators from $\mathbb{R}^m$ to $\mathbb{R}^n$. When applied to $f \in \mathbb{R}^m$, it gives the relation

$$
\begin{aligned}
&(D_y(D_z g))(f, \psi^{(1)}) + (D_y g_\epsilon) f + (D_z^2 g)\left(\frac{dh_0}{dy} f, \psi^{(1)}\right) + (D_z g_\epsilon)\frac{dh_0}{dy} f + (D_z g)\frac{\partial \psi^{(1)}}{\partial y} f \\
=&\frac{d^2 h_0}{dy^2}(f, f) + \frac{dh_0}{dy}\left(D_y f + (D_z f)\frac{dh_0}{dy}\right) f + \frac{d^2 h_0}{dy^2}(D_1 D^2 y, f) - D_2\left(\frac{d}{dy} D^2 z_\psi^{(0)}\right) f.
\end{aligned}
\tag{6.5.61}
$$

Second, we observe that the bilinear maps $D_z(D_y g)$ and $D_z^2 g$ satisfy the symmetry relations

$$
\begin{aligned}
&(D_z(D_y g))(u, v) = (D_y(D_z g))(v, u), \ \text{for } u \in \mathbb{R}^n, v \in \mathbb{R}^m, \\
&(D_z^2 g)(u, v) = (D_z^2 g)(v, u), \ \text{for } u, v \in \mathbb{R}^n,
\end{aligned}
$$

and use these to rewrite the first and third terms in (6.5.60) and (6.5.61).

Therefore, we find that the ASIM equation (6.5.59) reduces to

$$
\begin{aligned}
&(D_z g)^{-1}\frac{d^2 h_0}{dy^2}(f, f) - \frac{dh_0}{dy}((D_z f)\psi^{(1)} + f_\epsilon) + (D_z g)\psi^{(2)} + \frac{1}{2}(D_z^2 g)(\psi^{(1)}, \psi^{(1)}) \\
&+(D_z g_\epsilon)\psi^{(1)} + \frac{1}{2} g_{\epsilon\epsilon} - \frac{\partial \psi^{(1)}}{\partial y}(f + D_1 D^2 y) - (D_z g)^{-1}\left[D_2\left(\frac{d}{dy} D^2 z_\psi^{(0)}\right)(f + D_1 D^2 y)\right. \\
&\left. + \frac{d^2 h_0}{dy^2}(2f + D_1 D^2 y, D_1 D^2 y)\right] + D_2 D^2 z_\psi^{(1)} = 0.
\end{aligned}
\tag{6.5.62}
$$

We have established (6.5.25), and this concludes the proof of Theorem 6.5.1.

□

6.6 Turing Stability Analysis

In this section, we briefly review the Turing analysis (Edelstein-Keshet, 2005; Szili and Toth, 1993; Turing, 1952) for the stability of homogeneous states in systems of reaction-diffusion equations. In particular, we show that for reaction-diffusion systems of the form (6.2.1) there is no Turing instability that arises. This result is due to the large separation in the time scales of evolution of y and z, and it is essential for the application of model reduction methods to know that the homogeneous states are stable even in the presence of diffusion.

Without loss of generality, we carry out the analysis with the stable equilibrium located at $(y, z) = (\bar{y}, \bar{z})$,

$$\begin{cases} f(\bar{y}, \bar{z}) = 0 \\ g(\bar{y}, \bar{z}) = 0. \end{cases} \tag{6.6.1}$$

At $(\bar{y}, \bar{z})$, the eigenvalues of the Jacobian J, recall (6.2.11), have negative real parts, $f_y + \frac{1}{\epsilon} g_z < 0$, and we have $\det(J) > 0$. Let $\tilde{y} = y - \bar{y}$ and $\tilde{z} = z - \bar{z}$ be small perturbations of the homogeneous equilibrium. The linearized system on the real line is

$$\begin{bmatrix} \frac{\partial \tilde{y}}{\partial t} \\ \frac{\partial \tilde{z}}{\partial t} \end{bmatrix} = J|_{(\bar{y},\bar{z})} \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} d_1 \frac{\partial^2 \tilde{y}}{\partial x^2} \\ d_2 \frac{\partial^2 \tilde{z}}{\partial x^2} \end{bmatrix}. \tag{6.6.2}$$

Solutions of the linear system (6.6.2) are built from the basis functions $e^{\sigma t} \cos(qx)$ and $e^{\sigma t} \sin(qx)$. Any general solution of linear partial differential equations on $\mathbb{R}$ can be written as a linear combination of these modes, and for the analysis we only need to examine one set,

$$\begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} e^{\sigma t} \cos(qx). \tag{6.6.3}$$

By substituting perturbations (6.6.3) into system (6.6.2), we obtain

$$M \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ with } M = \begin{bmatrix} \sigma + d_1 q^2 & 0 \\ 0 & \sigma + d_2 q^2 \end{bmatrix} - J. \tag{6.6.4}$$

A nontrivial solution can only exist if the determinant of the coefficient matrix M is zero,

$$\sigma^2 + (d_1 q^2 + d_2 q^2 - f_y - \frac{1}{\epsilon} g_z)\sigma + (d_1 q^2 - f_y)(d_2 q^2 - \frac{1}{\epsilon} g_z) - \frac{1}{\epsilon} g_y f_z = 0. \tag{6.6.5}$$

From equation (6.6.3), we see that the homogeneous equilibrium of system (6.6.2) is stable provided that both eigenvalues σ have negative real part,

$$- d_1 q^2 - d_2 q^2 + f_y + \frac{1}{\epsilon} g_z < 0, \tag{6.6.6}$$

$$(d_1 q^2 - f_y)(d_2 q^2 - \frac{1}{\epsilon} g_z) - \frac{1}{\epsilon} g_y f_z > 0. \tag{6.6.7}$$

Condition (6.6.6) holds for all real q, since $\mathcal{M}_0$ is attracting. Condition (6.6.7) is equivalent with the following quadratic equation for q^2:

$$d_1 d_2 q^4 - (\frac{1}{\epsilon} d_1 g_z + d_2 f_y) q^2 + \frac{1}{\epsilon}(f_y g_z - g_y f_z) > 0. \tag{6.6.8}$$

Here, $\frac{1}{\epsilon} g_z \ll -1$, $f_y g_z - g_y f_z = \det(J) > 0$. So, by systematically examining the three cases in which q is $\mathcal{O}(1)$, $q \ll 1$ and $q \gg 1$, we find that condition (6.6.8) holds for all real q. In conclusion, there are no modes (6.6.3) that grow, and there is no Turing instability for our system of diffusion equations on $\mathbb{R}$.

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