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Effective Galois descent for motives: the K3 case

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EFFECTIVE GALOIS DESCENT FOR MOTIVES: THE K3 CASE

by

ANGUS MCANDREW

B.Sc., The University of Melbourne, 2011
M.Sc., The University of Melbourne, 2013
M.Phil., The University of Melbourne, 2015

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Approved by

First Reader

Jared Weinstein, PhD
Associate Professor of Mathematics

Second Reader

Robert Pollack, PhD
Professor of Mathematics

Third Reader

Glenn Stevens, PhD
Professor of Mathematics

Fourth Reader

Siu-Cheong Lau, PhD
Professor of Mathematics

The first step is to make sure all the experts fall asleep.

Larry Washington

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EFFECTIVE GALOIS DESCENT FOR MOTIVES: THE K3 CASE

ANGUS MCANDREW

Boston University, Graduate School of Arts & Sciences, 2022

Major Professor: Jared Weinstein, PhD
Associate Professor of Mathematics

ABSTRACT

A theorem of Grothendieck tells us that if the Galois action on the Tate module of an abelian variety factors through a smaller field, then the abelian variety, up to isogeny and finite extension of the base, is itself defined over the smaller field. Inspired by this, we give a Galois descent datum for a motive H over a field by asking that the Galois action on an ℓ-adic realisation factor through a smaller field. We conjecture that this descent datum is effective, that is if a motive H satisfies the above criterion, then it must itself descend to the smaller field.

We prove this conjecture for K3 surfaces, under some hypotheses. The proof is based on Madapusi-Pera's extension of the Kuga-Satake construction to arbitrary fields.

Contents

List of Abbreviations

$\mathbb{C}$	....	Complex numbers
$\operatorname{End} A$	....	Ring of endomorphisms of an abelian variety A
$\mathbb{F}_q$	....	Finite field of size q
$\operatorname{Gal}(K/F)$	....	Galois group of the extension K/F
G_F	....	Absolute Galois group of F
$H^i_\bullet(X, \mathcal{F})$	....	ith cohomology group of X for theory $\bullet$
$[K : F]$	....	Degree of the field extension K/F
K^s	....	Separable closure of K
$M_n(R)$	....	$n \times n$ matrices over the ring R
$\mathcal{O}_X$	....	Structure sheaf of the scheme X
$\mathbb{Q}$	....	Rational numbers
$R^q f_*$	....	Right derived functor of the pushforward along f
$\operatorname{Spec} R$	....	Spectrum of the ring R
$X \otimes_K L$	....	Base change $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$
ζ_n	....	Primitive nth root of unity
$\mathbb{Z}$	....	Integers
$\mathbb{Z}_{(p)}$	....	Localisation of the integers at the ideal (p)

Chapter 1

Introduction

1.1 Elliptic curves and complex multiplication

It is a classical problem in algebraic geometry to ask for the *field of definition* of a given variety. Given a field K and a variety X/K, we can construct for any extension L/K the base change $X \otimes L$. Conversely, given X/L and a subfield $K \subseteq L$, one may ask whether X *descends* to K. That is, does there exist some X_0/K and an isomorphism $X_0 \otimes K \xrightarrow{\sim} X$.

We begin with the story for *elliptic curves*.

Definition 1.1.1. An *elliptic curve* over a field K is a pair (E, O), where E is a smooth projective curve of genus 1 over K, and $O \in E(K)$ is a K-point.

Example 1.1.2. If $\operatorname{char} K \neq 2, 3$, (dehomogenised) elliptic curves can always be written in the form $y^2 = x^3 + ax + b$, for $a, b \in K$ such that the *discriminant* $\Delta(E) = -16(4a^3 + 27b^2) \neq 0$, with the distinguished point O "at infinity" (see (Silverman, 2009), Remark 1.3). Consider for example $E : y^2 = x^3 + \pi$ over $\mathbb{C}$. For the field of definition, immediately we can deduce that in fact we have $E_0 : y^2 = x^3 + \pi$ over $\mathbb{Q}(\pi)$ and $E_0 \otimes \mathbb{C} \xrightarrow{\sim} E$. However, we can do better (see Example 1.1.7 below).

Elliptic curves admit the structure of an algebraic group, that is, it admits a multiplication and inversion morphism which are morphisms of algebraic varieties. In fact this operation is commutative, so we get morphisms $E \to E$ corresponding to multiplication by n:

$$[n] : E \longrightarrow E$$
$$P \longmapsto \underbrace{P + \ldots + P}_{n \text{ times}}$$

In general, the maps of interest between elliptic curves E, E' are *isogenies*, which are nonconstant maps $f : E \to E'$ respecting the group law. These are necessarily surjective with finite kernel. We also consider the space of endomorphisms $\mathrm{End}(E)$, which means the morphisms $f : E \to E$ of curves respecting the group law, in particular we include the zero morphism. A classical result is the following.

Theorem 1.1.3. *The ring* $\mathrm{End}(E)$ *is isomorphic to one of*

1. $\mathbb{Z}$*,*
2. *an order* $\mathcal{O}$ *in a quadratic imaginary field, or*
3. *an order* $\mathcal{O}$ *in a quaternion algebra (only occurs in positive characteristic).*

Proof. See (Silverman, 2009), Corollary 9.4. □

Definition 1.1.4. If there exists an extension L/K such that $\mathrm{End}(E \otimes L) \neq \mathbb{Z}$, i.e. case 2 or 3 above, we say E has *complex multiplication*, or *CM*.

Remark 1.1.5. Writing $\mathrm{End}^0(E) = \mathrm{End}(E) \otimes \mathbb{Q}$, the possibilities are now $\mathrm{End}^0(E) = \mathbb{Q}$, F, or B, where F is an imaginary quadratic field and B is a quaternion algebra. So we could equivalently say E has CM if $\mathrm{End}^0(E \otimes L) \neq \mathbb{Q}$.

The following is a classical theorem for CM elliptic curves in characteristic 0.

Theorem 1.1.6. *Let* $F/\mathbb{Q}$ *be a quadratic imaginary field with Hilbert class field* H_F*. Let* $E/\mathbb{C}$ *be an elliptic curve such that* $\mathrm{End}(E)$ *is an order in* F*. Then there exists an elliptic curve* E_0/H_F *with an isomorphism* $E_0 \otimes \mathbb{C} \xrightarrow{\sim} E$*.*

Proof. See (Silverman, 1994), Theorem 4.1. □

One key takeaway is that in chararacteristic 0, if an elliptic curve has CM, then it must descend to a number field.

Example 1.1.7. Returning to $E : y^2 = x^3 + \pi$ over $\mathbb{Q}(\pi)$, we see that $E \otimes \mathbb{C}$ admits CM, since it contains the automorphism $f : (x, y) \mapsto (\zeta_6^2 x, \zeta_6^3 y)$ and $f \notin \mathbb{Z}$. Hence we expect that, after extending the base field if necessary, we should be able to descend it to a number field. Indeed, letting $E_0/\mathbb{Q} : y^2 = x^3 + 1$, we have an isomorphism

$$\varphi : E_0 \otimes \mathbb{Q}(\pi^{1/6}) \xrightarrow{\sim} E \otimes \mathbb{Q}(\pi^{1/6}) \tag{1.1}$$

$$(x, y) \longmapsto (\pi^{1/3} x, \pi^{1/2} y). \tag{1.2}$$

Thus we can say that in this case, up to finite extension, $E/\mathbb{Q}(\pi)$ descends to $\mathbb{Q}$.

These results extend immediately to the case of positive characteristic.

Theorem 1.1.8. *Let K be a field with $\operatorname{char} K = p \neq 0$. Let E/K be an elliptic curve with CM. Then there exists $q = p^n$ and an elliptic curve $E_0/\mathbb{F}_q$ with an isomorphism $E_0 \otimes K^s \xrightarrow{\sim} E \otimes K^s$.*

Proof. See (Mumford, 1970), Theorem on page 217. □

We might package the theorems in characteristic 0 and p into a single statement.

An elliptic curve with CM descends to a finite extension of the prime field.

1.2 Abelian varieties and complex multiplication

As discussed above, one of the key features of elliptic curves is that they have the structure of an algebraic group. In fact, among projective curves, having genus 1 and a rational point is equivalent to have a group structure. When one wishes to generalise a theorem for elliptic curves, a natural place is to look at the higher-dimensional analogues, known as *abelian varieties*.

Definition 1.2.1. An *abelian variety* over a field K is a connected projective algebraic group A.

Remark 1.2.2. Happily for the name, the group structure of an abelian variety is indeed commutative (see (Mumford, 1970), Corollary 2 on page 44). We similarly have multiplication by n maps $[n] : A \to A$. We also have a theory of *isogenies*

$f : A \to A'$, which are morphisms of algebraic groups which surjective with finite kernel.

We would like to study the endomorphism rings of these abelian varieties to develop an analogous notion of complex multiplication.

Example 1.2.3. One simple way to construct abelian varieties is by taking products $A = E_1^{r_1} \times E_2^{r_2} \times \ldots \times E_n^{r_n}$ of elliptic curves. If there are no maps between E_i and E_j for $i \neq j$, we find

$$\mathrm{End}(A) = \prod_{i=1}^{n} \mathrm{End}(E_i^{r_i}) = \prod_{i=1}^{n} M_{r_i}(\mathrm{End}(E_i)). \tag{1.3}$$

To make sense of CM for abelian varieties, we have to account for the fact that the endomorphism ring may factor in this way.

We say an abelian variety A is *simple* if there does not exist $B \subseteq A$ such that $0 \neq B \neq A$. Further, every abelian variety is isogenous to a product of simple abelian varieties.

Definition 1.2.4. We say a simple abelian variety A/K of dimension g has *complex multiplication*, or *CM*, if there exists an extension L/K and a field F with $[F : \mathbb{Q}] = 2g$ and an embedding $F \hookrightarrow \mathrm{End}^0(A \otimes L)$. In general, we say an abelian variety A/K has *complex multiplication*, or *CM*, if there exists an extension L/K and an isogeny $A \otimes L \to \prod A_i^{r_i}$ where A_i are simple abelian varieties with CM.

The results above for elliptic curves have been generalised to abelian varieties due to Grothendieck.

Theorem 1.2.5. *Let K be a field with prime field k. Let A/K be an abelian variety with CM. Then there exists an abelian variety A_0/k^s and an isogeny $A_0 \otimes K^s \to A \otimes K^s$.*

Remark 1.2.6. The presence of the base change to K^s is to account for the descent occuring after a potential finite extension of K.

The proof is found in (Oort, 1973) and will be fully discussed in Chapter 2.

Remark 1.2.7. An alternative proof has been given in (Yu, 2004) by showing that, up to isogeny, one can enforce a strong form of complex multiplication. Given this one may show that the locus of points on the moduli space with this structure is in fact a finite scheme. Thus it must descend to a finite extension of the prime field.

To give an idea of it, we need another ingredient in our theory of abelian varieties, a particular Galois representation attached to A.

Definition 1.2.8. Let A/K be an abelian variety. Write $A[n] = \ker[n]$ for the n-torsion. The *ℓ-adic Tate module* of A is $T_\ell = \varprojlim_n A[\ell^n]$. This is a representation of the Galois group $G_K = \mathrm{Gal}(K^s/K)$.

Further, to make remarks about a Galois group, we need some condition on the field extension K/k to ensure the Galois theory is well-behaved.

Definition 1.2.9. Let K/k be a field extension. A *transcendence basis* for K/k is a set $B \subseteq K$ such that K is algebraic over $k(B)$. We say K/k is *separable* if there exists a transcendence basis B such that $K/k(B)$ is separable. We say K/k is *regular* is it is separable and k is algebraically closed in K.

We can now say more about the theorem of Grothendieck. In particular, it can be broken down into two steps.

Proposition 1.2.10. *Let K/k be a regular extension, and A/K an abelian variety with CM. Let $\rho : G_K \to \mathrm{Aut}(T_\ell)$ be the Galois representation on the Tate module. Then $\rho(\mathrm{Gal}(K^s/k^sK))$ is finite.*

This is the first suggestion that the abelian variety might descend to a smaller field. We may pass to a finite extension K'/K such that $\rho(\mathrm{Gal}(K^s/k^sK') = 1$, which indicates that the Galois action is coming entirely from the extension k^s/k. One might call this criterion on the Galois action a *descent datum*. The question is then to determine whether this datum is *effective*, that is, if an abelian variety satisfies this criterion, does it indeed descend to a smaller field?

Theorem 1.2.11. *Let K/k be a regular extension, and A/K an abelian variety of dimension g with $\rho : G_K \to \mathrm{Aut}(T_\ell)$ the Galois representation on the Tate module. Assume $\rho(\mathrm{Gal}(K^s/k^sK)) = 1$. Then there exists an abelian variety A_0/k and an isogeny $A_0 \otimes K \to A$.*

Remark 1.2.12. Note that the theorem does not provide an isomorphism, but merely an isogeny. In fact, there are three cases in which we in fact have an isomorphism $A_0 \otimes K^s \xrightarrow{\sim} A \otimes K^s$ (see (Oort, 1973), 1.3):

- A is an elliptic curve
- K is characteristic 0
- K is characteristic $p \neq 0$, and $\#A[p](\overline{K}) \geq p^{g-1}$

The final case includes the situation when $\#A[p](\overline{K}) = p^g$, in which case we say A is *ordinary*.

1.3 Motives and cohomology

Grothendieck's theorem extends the descent theorem for elliptic curves to the setting of abelian varieties. One may wish to extend further to general varieties X/K. In general, such varieties may not admit any nontrivial endomorphisms, and thus not admit a natural notion of CM.

Remark 1.3.1. In characteristic 0 there is a notion for a general variety by associating to it a Hodge structure and asking that its Mumford-Tate group be abelian. See (Milne, 1994a), §4 for further details.

However, Theorem 1.2.11 suggests a viewpoint in which we generalise not CM, but rather a certain descent datum in a Galois action. For a smooth projective variety X/K, the analogue of the Tate module is given by the ℓ-adic cohomology groups $H^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$. These carry an action of the Galois group G_K. So one might study whether the Galois action on the vector space $W = \oplus_{i=0}^{2\dim X} H^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$ factors through a smaller field.

In fact, in practice it is often the case that the Galois action operates quite differently on different submodules of W. We could ask instead whether there was a reasonable notion of an object that we might say would capture these submodules of the cohomology. For this we have the notion of a *motive*. These will be made precise in Chapter 2. For the moment, a *motive* is an object in a $\mathbb{Q}$-linear category through which all cohomology theories factor. In particular, we have for each smooth projective variety X objects $h(X)$ and $h^i(X)$, which map to $\oplus_{i=0}^{2\dim X} H^i_\bullet(X_{K^s})$ and $H^i_\bullet(X)$ respectively, under a cohomology theory $\bullet$. The image of a motive H under a cohomology theory is known as a *realisation*.

With this terminology in hand we arrive at the following conjectural generalisation of Theorem 1.2.11.

Conjecture 1.3.2. *Let K/k be a regular extension and let H/K be a motive with Galois representatation $\rho : G_K \to \operatorname{Aut} H_\ell$ on its ℓ-adic realisation. Assume*

$$\rho(\operatorname{Gal}(K^s/k^s K)) = 1. \tag{1.4}$$

Then there exists a motive H_0/k^s and an isomorphism $H_0 \otimes K^s \xrightarrow{\sim} H \otimes K^s$.

Example 1.3.3. For an abelian variety A we can consider either the motive $h(A)$ or $h^1(A)$. In fact, for abelian varieties we have $h^i(A) = h^1(A)^{\wedge i}$ (see (Milne, 1986), Theorem 15.1(b)), so it is sufficient to consider $h^1(A)$. Further, we have that $H^1_{\text{ét}}(A_{K^s}, \mathbb{Q}_\ell) = (T_\ell A[\frac{1}{\ell}])^\vee$ as G_K-representations (see (Milne, 1986), Theorem 15.1(a)). Finally, since the category of motives is $\mathbb{Q}$-linear, an isogeny of abelian varieties will induce an isomorphism of motives.

Thus, for $H = h^1(A)$ for A an abelian variety, we let $H_0 = h^1(A_0)$, where A_0 is as in Theorem 1.2.11. So, for the subcategory generated by $h^1(A)$ for abelian varieties A, the conjecture holds.

1.4 K3 surfaces and Shimura varieties

Our goal now is to find examples of motives H for which we can prove cases of Conjecture 1.3.2. For this we consider another generalisation of elliptic curves known as

K3 surfaces. Among smooth projective curves, having a group structure is equivalent to being elliptic, i.e. genus 1. One way to see this is to consider the *canonical bundle* ω, which has degree $2g-2$. The group structure forces that the canonical bundle be trivial, thus $2g-2=0$. By the same token, all abelian varieties have trivial canonical bundle. A smooth projective variety with trivial canonical bundle is known as a *Calabi-Yau variety*.

Due to the existence of abelian varieties, we have examples of Calabi-Yau varieties in each dimension. In the case of dimension 1, i.e. curves, the Calabi-Yau varieties are precisely the genus 1 curves. In dimension 2 we have abelian surfaces, but now there are new examples as well. K3 surfaces are precisely the 2 dimensional Calabi-Yau varieties which are *simply connected*.

Definition 1.4.1. A K3 surface X/K is a smooth projective surface with trivial canonical bundle, i.e. $\omega_X \cong \mathcal{O}_X$, and $H^1(X, \mathcal{O}_X) = 0$.

Example 1.4.2. We will give a complete introduction to the theory in chapter 3.

Abelian surfaces A are close to K3, since they are also Calabi-Yau. To construct a K3 surface from them one may first take the quotient $A/[-1]$. This is not a smooth surface, due to the sixteen 2-torsion points on A, but if we blow up at these points we get a smooth surface $\mathrm{Km}(A)$, which is in fact a K3 surface (see (Huybrechts, 2016), Example 1.3).

The surface $\mathrm{Km}(A)$ is known as the *Kummer surface* associated to A.

A striking feature of K3 surfaces is their cohomology. As bare $\mathbb{Q}_\ell$-vector spaces, we have

$$H^0_{\text{ét}}(X, \mathbb{Q}_\ell) = H^4_{\text{ét}}(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell, \qquad H^1_{\text{ét}}(X, \mathbb{Q}_\ell) = H^3_{\text{ét}}(X, \mathbb{Q}_\ell) = 0, \tag{1.5}$$

$$\text{and} \quad H^2_{\text{ét}}(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell^{\oplus 22}. \tag{1.6}$$

The Galois action is easy to understand on most components.

In particular, letting $\rho_i : G_K \to \mathrm{Aut}(H^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$ and K/k a regular extension, it is automatic that $\rho_i(\mathrm{Gal}(K^s/k^sK) = 1$ for $i \neq 2$ (this uses the fact that

$H^4_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(2)$, i.e. the action is through a power of the cyclotomic character). Hence it is sufficient to study the Galois action on $H^2_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$.

Our main theorem is that Conjecture 1.3.2 holds for K3 surfaces, with some hypotheses. Precisely, we can say the following.

Theorem 1.4.3. *Let K/k be a regular extension and let X/K be a K3 surface. Let $\rho : G_K \to \operatorname{Aut}(H^2_{\acute{e}t}(X_{K^s}, \mathbb{Q}_\ell))$ denote the Galois action on H^2 and assume $\rho(\operatorname{Gal}(K^s/k^sK)) = 1$. Further, assume either (1) K is characteristic* 0, *or (2) X is* ordinary*. Then there exists a K3 surface X_0/k^s and an isomorphism $X_0 \otimes K^s \xrightarrow{\sim} X \otimes K^s$.*

There is a method that associates to a K3 surface an abelian variety, known as the *Kuga-Satake abelian variety.* Classically this is a construction that was applied to a complex K3 surface and outputs a complex abelian variety (see (Huybrechts, 2016), Chapter 4). In (Deligne, 1972), this construction was descended to number fields. In (Madapusi Pera, 2015), this was further extended to any field of characteristic $p \neq 2$.

Precisely, what is constructed is a certain diagram

$$\begin{array}{ccc} & \mathcal{S}(\operatorname{GSpin}(L_d)) & \longrightarrow \mathcal{S}(\operatorname{GSp}(\operatorname{Cl}^+(L_d), \psi_\delta)) \\ & \downarrow & \\ \tilde{M}_{2d,\gamma} \longrightarrow & \mathcal{S}(\operatorname{SO}(L_d)) & \end{array}$$

where:

- $\tilde{M}$ is a moduli space of "γ-oriented" K3 surfaces (quasi-polarised of degree $2d$).
- $\mathcal{S}(\cdot)$ is the integral model of a Shimura variety $\operatorname{Sh}(\cdot)$.
- All morphisms are finite étale over their image.

We will discuss this diagram in full in Chapter 4.

In Chapter 5 we will give the proof of Theorem 1.4.3. Case (1) was sketched above. For cases (2) and (3), the strategy is to begin with a K3 surface over K, from which we may construct a K-point in $\tilde{M}_{2d,\gamma}$. After finite extension of K, this may be mapped to a K-point in the Shimura viarety $\mathcal{S}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta))$, which has a moduli interpretation in terms of abelian varieties. Taking the section of the universal abelian variety at that point gives the Kuga-Satake abelian variety. We show the Galois descent criterion passes to the abelian variety, and with the conditions provided we find that the K-point was in fact a k^s-point. Finally, one sees that the same is true for the point on $\tilde{M}_{2d,\gamma}$ and the theorem follows.

Chapter 2

Abelian varieties and motives

The goal of this chapter is to review the necessary theory of abelian varieties to discuss the proof of Grothendieck's theorem. Some primary references include (Milne, 1986), (Mumford, 1970), (Conrad, 2006), and (Oort, 1973).

2.1 Basic definitions

Abelian varieties are generalisations of elliptic curves. Classically, they were studied as certain complex tori equipped with a Hermitian pairing (called a *polarisation* on their first homology (see (Mumford, 1970), Chapter 1). We will take as starting point the modern algebro geometric perspective.

Definition 2.1.1. Let S be a scheme. A *group scheme G over S* is an S-scheme $G \to S$ with an identity section $e : S \to G$, multiplication morphism $\mu : G \times_S G \to G$ and inversion morphism $\iota : G \to G$ satisfying the usual group axioms.

An *abelian scheme A over S* is a proper, smooth group scheme $A \to S$ whose geometric fibres are connected of some fixed dimension g.

Definition 2.1.2. If $S = \operatorname{Spec} K$ for K a field, we say G as above is a *group variety* and A as above is an *abelian variety.*

One can show that abelian varieties are projective and the multiplication morphism is commutative (see (Milne, 1986)).

Definition 2.1.3. A *homomorphism* of abelian schemes is a morphism of schemes that commutes with the identity section (note that this is sufficient to commute with multiplication and inversion ((Milne, 1986)). A *isogeny* of abelian schemes is a surjective homomorphism with finite kernel.

If there exists an isogeny $A \to B$, we say A and B are *isogenous*. This is an equivalence relation.

Since they are commutative group schemes, further examples of isogenies are the *multiplication by n* morphisms $[n] : A \to A$.

Proposition 2.1.4. *For A an abelian scheme, $A[n]$ is a finite flat group scheme of rank n^{2g}.*

In particular, say A/K is an abelian variety. Then one has that

$$\#A[p](\bar{K}) \cong \begin{cases} p^{2g}, & \text{if } p \neq \operatorname{char} K, \\ p^h, \ 0 \leq h \leq g & \text{if } p = \operatorname{char} K, \end{cases} \tag{2.1}$$

In the case that $\operatorname{char} K = p$, the quantity h is called the *p-rank* of A. If $h = g$, we say A is *ordinary*. If $h = 0$, we say A is *supersingular*.

Take an inverse limit gives the following.

Definition 2.1.5. The *ℓ-adic Tate module* of A/S is given by

$$T_\ell A = \varprojlim_n A[\ell^n]. \tag{2.2}$$

and the *rational* Tate module $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

If ℓ is invertible on S, then $V_\ell A$ is an étale $\mathbb{Q}_\ell$-local system on S. In particular, if $S = \operatorname{Spec} K$ then $V_\ell A$ is a $\mathbb{Q}_\ell$-module with an action of the Galois group $\operatorname{Gal}(K^s/K)$.

To say more about the structure of this action we introduce another source of examples of isogenies. Given an abelian variety A/K, one may construct a variety $A^\vee$, known as the *dual abelian variety* (see (Milne, 1986), §I.8). The dual $A^\vee$ classifies line bundles on A, in particular $A^\vee(K) = \operatorname{Pic}^0(A)$. For $a \in A$, denote the translation by a morphism as $t_a : A \to A$.

Proposition 2.1.6. *Let L be an ample line bundle on A. Then the map*

$$\lambda_L : A \longrightarrow A^\vee \tag{2.3}$$
$$a \longmapsto t_a^* L \otimes L^{-1} \tag{2.4}$$

is an isogeny.

We call a map of the form $\lambda_L : A \to A^\vee$ a *polarisation*. If λ_L is an isomorphism we call it a *principal polarisation*.

Proposition 2.1.7. *Let A/K be an abelian variety of dimension g with a choice of principal polarisation λ. The Galois representation $\rho : \mathrm{Gal}(K^s/K) \to \mathrm{Aut}(V_\ell A)$ has image in a symplectic group $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$. Let ω denote the symplectic similitude character of $\mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$. Then the composition $\omega \circ \rho = \chi_\ell$, the ℓ-adic cyclotomic character.*

Proof. There exists a Galois-equivariant pairing $e : A[n] \times A^\vee[n] \to \mu_n$ known as the *Weil pairing*. With λ one can define a pairing on $A[n]$ by $e(a, \lambda(b))$ which is bilinear, alternating, and nondegenerate. Thus we have the symplectic group of linear automorphisms which are equivariant for the pairing, which hence contains the image of ρ.

Composition with the similitude character gives the action of $\mathrm{Gal}(K^s/K)$ on $\varprojlim_n \mu_{\ell^n}$, which is precisely the ℓ-adic cyclotomic character. □

For an abelian scheme $f : A \to S$ and a constant sheaf F on A we can consider the local system $R^q f_{\text{ét},*} F$ on S. Taking the inverse limit gives the ℓ-adic local system $R^q f_{\text{ét},*}\mathbb{Z}_\ell = \varprojlim_n R^q f_{\text{ét},*}\mathbb{Z}/\ell^n\mathbb{Z}$ and $R^q f_{\text{ét},*}\mathbb{Q}_\ell = (R^q f_{\text{ét},*}\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. In particular, if $S = \operatorname{Spec} K$, we recover the ℓ-adic cohomology $H^q_{\text{ét}}(A, \mathbb{Q}_\ell)$.

Theorem 2.1.8. *We have isomorphims of $\mathrm{Gal}(K^s/K)$-modules*

$$H^1_{\acute{e}t}(A_{K^s}, \mathbb{Q}_\ell) \xrightarrow{\sim} (V_\ell A)^\vee, \tag{2.5}$$
$$H^q_{\acute{e}t}(A_{K^s}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^1(A_{K^s}, \mathbb{Q}_\ell)^{\wedge q}. \tag{2.6}$$

Proof. See (Milne, 1986), Theorem 15.1. □

Recall that a *regular* extension of fields is an extension K/k which is separated and k is algebraically closed in K. The goal for this chapter is to prove the following theorem, which appears in (Oort, 1973).

Theorem 2.1.9. *Let K/k be a regular extension. Let A/K be an abelian variety with $\rho : \mathrm{Gal}(K^s/K) \to \mathrm{Aut}(V_\ell A)$ the Galois representation on the Tate module. Assume $\rho(\mathrm{Gal}(K^s/k^sK)) = 1$. Then there exists an abelian variety A_0/k and an isogeny $A_0 \otimes K \to A$.*

2.2 The K/k-trace

To construct A_0 in Theorem 2.1.9, we require the theory of the K/k-*trace* and the Lang-Néron-Mordell-Weil theorem. This section follows the excellent expositions in (Conrad, 2006) and (Lang, 1959).

Many results of this section apply to a slightly larger class field extensions than just the regular ones.

Definition 2.2.1. An extension of fields K/k is *primary* if the algebraic closure of k in K is purely inseparable over k.

Henceforth in this section we assume K/k is always a primary extension.

Imprecisely, the K/k-trace of an abelian variety A/K is the "largest" isogeny factor of A that descends to k.

Definition 2.2.2. Let K/k be a field extension and A/K be an abelian variety. Let $\mathcal{C}_A$ be the category of pairs (B, f) where B/k is an abelian variety and f is a K-morphism $f : B \otimes K \to A$. A K/k-*trace* of A is a final object in this category.

It is not immediately apparent that such a final object should exist. If it does, then it is unique up to unique isomorphism by virtue of the universal property. To prove existence, it is convenient to first introduce a related construction.

Definition 2.2.3. Let K/k be a field extension and A/K be an abelian variety. Let $\mathcal{C}_A^\vee$ be the category of pairs (C, g) where C/k is an abelian variety and g is a K-morphism $g : A \to C \otimes K$. A K/k-*image* of A is a final object in this category.

Theorem 2.2.4. *The K/k-image of A exists and is unique, we denote it* $\mathrm{Im}_{K/k} A$.

Before we prove this we require one further input.

Theorem 2.2.5. *Let K/k be a primary extension and A/k an abelian variety. If $B \subseteq A \otimes K$ is an abelian subvariety, then there exists $B_0 \subseteq A$ an abelian subvariety such that $B_0 \otimes K = B$.*

Proof. Recall first Chow's theorem (see (Chow, 1955)) that for $C, D/k$ abelian varieties and K/k primary, we have a bijection

$$\mathrm{Hom}_k(C, D) \xrightarrow{\sim} \mathrm{Hom}_K(C \otimes K, D \otimes K). \tag{2.7}$$

Now, we have an isogeny decomposition $A \sim \prod A_i^{e_i}$ where the A_i are simple and non-isogenous. Then B is the image of a map $\varphi_K : \prod A_{i,K}^{f_i} \to A_K$ for $f_i \leq e_i$. By Chow's theorem this descends to a map $\varphi : \prod A_i^{f_i} \to A$. Thus the desired subvariety B_0/k is the image of φ. □

Proof of Theorem 2.2.4. It is sufficient to consider the case that $g : A \to C \otimes K$ is surjective, and is thus determined by $\ker g \subseteq A$. Give two such pairs $g : A \to C \otimes K$, $g' : A \to C' \otimes K$, we construct $(g, g') : A \to (C \times C') \otimes K$. The image is some subvariety which, by Theorem 2.2.5, is defined over k. Thus the collection of kernels $\ker g$ is closed under finite intersection. Applying the descending chain condition on A gives the desired initial object. □

Corollary 2.2.6. *The K/k-trace of A exists and is unique, we denote it* $\mathrm{Tr}_{K/k} A$.

Proof. Set $\mathrm{Tr}_{K/k} A = (\mathrm{Im}_{K/k} A^\vee)^\vee$. □

We note a useful property of this morphism.

Lemma 2.2.7. *Let K/k be a regular extension, A/K be an abelian variety, and $\mathrm{Tr}_{K/k} A = (B, \tau)$ be the K/k-trace. Then τ is purely inseparable.*

Proof. This is (Lang, 1959) VIII.3 Corollary 2. □

2.3 Grothendieck's theorem

The theory of the K/k-trace leads to the following theorem, Lang-Néron's extension of the Mordell-Weil theorem.

Theorem 2.3.1. *Let K/k be a regular extension and A/K an abelian variety. Then $A(K)/\operatorname{Tr}_{K/k}(A)(k)$ is finitely generated.*

Proof. See Theorem 7.1 in (Conrad, 2006) for full details.

The strategy is to reduce to the case k is algebraically closed and $K = k(X)$ is the function field of a curve X/k. There exists a dense open U in X such that A extends to an abelian scheme $\mathcal{A} \to U$. From here one has a weak Lang-Néron-Mordell-Weil theorem by considering the Kummer sequence of sheaves of U

$$0 \to \mathcal{A}[m] \to \mathcal{A} \xrightarrow{m} \mathcal{A} \to 0. \tag{2.8}$$

Note that $\mathcal{A}[m]$ is a locally constant constructible sheaf of $\mathbb{Z}/m\mathbb{Z}$-modules on U. This gives an injection $A(K)/mA(K) \hookrightarrow H^1_{\text{ét}}(U, \mathcal{A}[m])$. Applying Poincare duality relates this to compactly support cohomology, which is finite by (Milne, 1980) Theorem VI.2.1. In particular we can conclude $A(K)/mA(K)$ is finite.

Similar to the case of number fields, one has a theory of heights to complete the full statement. □

We may now prove the main theorem of this chapter.

Proof of Theorem 2.1.9. Recall we have K/k a regular extension and A/K an abelian variety such that $\rho(\operatorname{Gal}(K^s/k^sK)) = 1$, where ρ is the Galois representation on the Tate module. Let $A_0 = \operatorname{Tr}_{K/k} A$ with the homomorphism $\tau : A_0 \otimes K \to A$.

We now consider now the regular extension k^sK/k^s. Applying Theorem 2.3.1 we have $A(k^sK)/\tau(A_0(k^s))$ is finitely generated. Thus the ℓ^n-torsion $A_0[\ell^n](k^s)$ is finite index in $A[\ell^n](k^sK)$ independent of n. By assumption on ρ we have $A[\ell^n](k^sK) = A[\ell^n](K^s)$. Taking the direct limit of n we find the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-modules $A_0[\ell^\infty](k^s)$ and $A[\ell^\infty](K^s)$ have the same rank. Thus we find $\dim B = \dim A$ and thus τ is an isogeny, as required. □

For our purposes the following cases will be important.

Proposition 2.3.2. *Consider the cases (1) $\operatorname{char} K = 0$ or (2) $\operatorname{char} K = p \neq 0$ and A is ordinary. Then the isogeny $A_0 \otimes K \to A$ is an isomorphism.*

Proof. Let $(A_0, \tau) = \operatorname{Tr}_{K/k} A$. We have that τ is an isogeny. Further, since K/k is regular, by Lemma 2.2.7, we know that τ is purely inseparable.

In case (1), since char $K = 0$, there are no nontrivial purely inseparable extensions. Hence τ is degree one and thus an isomorphism.

In case (2), since A is ordinary, we have the Serre-Tate theory of canonical lifts, giving $\tilde{A}_0$ and $\tilde{A}$ over $F \hookrightarrow F'$ fields of characteristic 0. Further we have a bijection

$$\mathrm{Hom}(A_0 \otimes K, A) \xrightarrow{\sim} \mathrm{Hom}(A_0 \otimes F', A). \tag{2.9}$$

By case (1) we know that on the right the maps in characteristic 0 are isomorphisms, thus the same for the map in our case (2). □

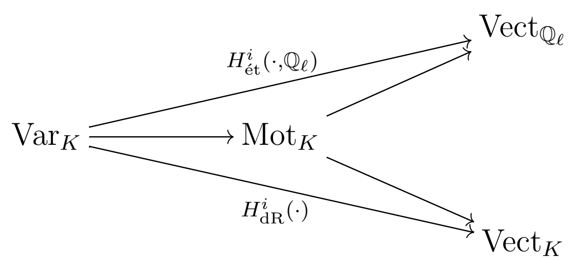
2.4 Motives

To generalise Theorem 2.1.9 we recall that by Theorem 2.1.8 the Tate module of an abelian variety is naturally isomorphism to (the dual of) the first ℓ-adic étale cohomology group. Thus, beginning with a variety X/K with $\rho_i : \mathrm{Gal}(K^s/K) \to H^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$ the Galois representation on the ℓ-adic cohomology, we might conjecture X descends to k if $\rho(\mathrm{Gal}(K^s/k^sK)) = 1$ where $\rho = \oplus\rho_i : \mathrm{Gal}(K^s/K) \to \oplus_i H^i_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$.

However, in practice each Galois representation ρ_i may behave somewhat differently. Even further, the groups $H^i_{\text{ét}}$ are often decomposable as Galois modules. Thus, one may find that in fact only some submodule of cohomology has the desired Galois descent property. The object that allows us to access this is a *motive*. For our purposes, the following definition will generally sufficient.

Definition 2.4.1. Let Var_K be the category of smooth, projective varieties over K. A *category of motives* is any $\mathbb{Q}$-linear category through which all Weil cohomology functors factor.

However, for concreteness we will describe one such category Mot_K, the category of *Chow motives* over K. So in particular it sits in the following diagram



(2.10)

A downside is that for general K it is not known to be sufficiently rich, in particular may not be sufficient to single out individual cohomology groups H^i. See (Milne, 2013) for full details.

Definition 2.4.2. The category of Chow Motives, Mot_K, is defined by

$$\begin{array}{rl} \textbf{Objects:} & h(X,e,m) \\ \textbf{Morphisms:} & \mathrm{Hom}(h(X,e,m),h(Y,f,n)) = f \circ C^{\dim X+n-m}_{\sim}(X \times Y)_{\mathbb{Q}} \circ e \end{array}$$

where:

- X, Y are smooth, projective varieties.
- m, n are integers.
- $C^{\dim X+n-m}_{\sim}(X \times Y)_{\mathbb{Q}}$ is the group of codimension $\dim X + n - m$ cycles on $X \times Y$ up to rational equivalence, tensored with $\mathbb{Q}$ over $\mathbb{Z}$.
- e, f are idempotents in $C^{\dim X}_{\sim}(X \times X)_{\mathbb{Q}}, C^{\dim Y}_{\sim}(Y \times Y)_{\mathbb{Q}}$, respectively.

The idea is to use such cycles as homomorphisms to capture all maps on cohomology induced by correspondences. The role of the idempotents is to single out parts of cohomology beyond the full $H^*(X)$. The addition of the integer m allows for duals to exist.

Given such a motive, it admits *realisations* in the various cohomology theories, by $h(X,e,m) \mapsto e^*H^*(X) \otimes H^2(\mathbb{P}^1)^{\otimes m}$.

Example 2.4.3. If $\Delta_X \subseteq X \times X$ is the diagonal, then we define $h(X) = h(X, \Delta_X, 0)$. Under any realisation this gives the full cohomology group $H^*(X)$.

If X is defined over a finite field $\mathbb{F}_q$ and $\ell \nmid q$, let g_i be the characteristic polynomial of Frobenius on $H^i(\overline{X}, \mathbb{Q}_\ell)$. Find a polynomial G that is $1 \pmod{g_i}$ and $0 \pmod{g_j}$ for $j \neq i$, this is possible due to the Riemann Hypothesis. Let Γ_F denote the graph of Frobenius on $X \times X$. Then $h(X, G(\Gamma_F), 0)$ under any realisation gives the cohomology group $H^i(X)$.

Over a general field K, the existence of a cycle which cuts out precisely $H^i(X)$ is an open conjecture, known as the Künneth type Standard Conjecture. Nonetheless, for a variety X we will refer to $h^i(X)$ for the (conjectural) motive whose realisation is $H^i(X)$. We now are in position to discuss Galois descent. For an abelian variety A, we have that $V_\ell A \cong H^1_{\text{ét}}(\overline{A}, \mathbb{Q}_\ell)^\vee$. Hence, for a general motive, we are lead to the following.

Definition 2.4.4. Let K/k be a regular field extension and H/K a motive. Let H_ℓ be the ℓ-adic realisation, equipped with Galois representation $\rho : \mathrm{Gal}_K \to \mathrm{Aut}(H_\ell)$. We say a motive H/K satisfies *K/k-Galois-ℓ-descent* if $\rho(\mathrm{Gal}(K^s/k^sK))$ is trivial.

Given this, we can now state the main conjecture, i.e. that this descent criterion is, up to finite extension, *effective*.

Conjecture 2.4.5. *Let H/K be a motive satisfying K/k-Galois-ℓ-descent. Then there exists a motive H_0/k^s and an isomorphism $H_0 \otimes_{k^s} K^s \to H \otimes_K K^s$.*

Remark 2.4.6. Since the category of motives is $\mathbb{Q}$-linear and the morphisms are induced by correspondences, this can be viewed as analogous to saying that this criterion for a variety Y/K predicts a variety Y_0/k^s and an algebraic correspondence $Y_0 \otimes_{k^s} K^s \to Y \otimes_K K^s$.

Proposition 2.4.7. *Conjecture 2.4.5 holds for the sub-tensor-category of* Mot_K *generated by $h^1(A)$ for all abelian varieties A.*

Proof. We have the Galois representation $\rho : \mathrm{Gal}(K^s/K) \to \mathrm{Aut}(H^1_{\text{ét}}(A_{K^s}, \mathbb{Q}_\ell))$ satisfying $\rho(\mathrm{Gal}(K^s/k^sK)) = 1$. By Theorem 2.1.8 we have $V_\ell A \cong H^1_{\text{ét}}(A_{K^s}, \mathbb{Q}_\ell)^\vee$ as Galois representations. So, since $\rho^\vee(\mathrm{Gal}(K^s/k^sK)) = 1$ we can apply Theorem 2.1.9. Thus A is isogenous to some A_0/k^s. Since Mot_K is $\mathbb{Q}$-linear, all maps induced by isogenies are invertible and hence isomorphisms. Thus in particular $h^1(A_0) \otimes K \cong h^1(A)$, as required. $\square$

Chapter 3

K3 surfaces

This chapter covers K3 surfaces, a natural testing ground for Conjecture 2.4.5 beyond abelian varieties. The main reference is (Huybrechts, 2016).

3.1 Basic definitions

Elliptic curves are the only smooth projective curves which admit a group law. Hence it is natural to generalise theorems for elliptic curves to the case of abelian varieties. Another feature uniquely specifying elliptic curves among smooth projective curves is that they have trivial canonical bundle.

Generalising this to higher dimensions give the notion of a *Calabi-Yau* variety. Examples of these are given by abelian varieties in each dimension, however these do not constitute all examples. In dimension 2 we have *Calabi-Yau surfaces*, which are divided into two classes - abelian surfaces and *K3 surfaces*.

Definition 3.1.1. A K3 surface X/K is a smooth projective surface over K such that the canonical bundle is trivial, i.e. $\omega_X \cong \mathcal{O}_X$, and $H^1(X, \mathcal{O}_X) = 0$.

Example 3.1.2. Any smooth quartic X in $\mathbb{P}^3$ is a K3 surface. To see this, start with the short exact sequence of sheaves on $\mathbb{P}^3$

$$0 \to \mathcal{O}(-4) \to \mathcal{O} \to \mathcal{O}_X \to 0, \tag{3.1}$$

the long exact sequence for which allows us to conclude that $H^1(X, \mathcal{O}_X) = 0$. Applying the adjunction formula computes

$$\omega_X \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}(4))|_X \cong \mathcal{O}_{\mathbb{P}^3}|_X \cong \mathcal{O}_X. \tag{3.2}$$

This can be generalised to any complete intersection of type $(d_1, \ldots, d_n)$ in $\mathbb{P}^{n+2}$ under the condition that $\sum_{i=1}^n d_i = n+3$.

For further examples, we have the following construction, beginning with an abelian surface.

Proposition 3.1.3. *Let A/K be an abelian surface, where* $\operatorname{char} K \neq 2$.. *Let* $\operatorname{Km}(A)$ *be the surface created by blowing up $A/[-1]$ at the 16 nonsmooth points corresponding to the* 2*-torsion points of A. Then* $\operatorname{Km}(A)$ *is a K3 surface.*

We call such a surface a *Kummer surface.*

Remark 3.1.4. When $\operatorname{char} K = 2$ there will be fewer than 16 nonsmooth points of $A/[-1]$. However, the singularities will generally be more complicated. For a discussion of this case see (Shioda, 1974).

Proof of Proposition 3.1.3. Let $X = \operatorname{Km}(A)$. If we first blow up A at the 2-torsion points to get $\tilde{A}$ we have the diagram

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X = \operatorname{Km}(A) & \longrightarrow & A/[-1] \end{array} \tag{3.3}$$

Note that $\omega_A \cong \mathcal{O}_A$. The canonical bundle formula (see (Hartshorne, 1977), Proposition V.3.3) shows that $\omega_{\tilde{A}} \cong \mathcal{O}(\sum E_i)$ where E_i are the exceptional divisors of the blowup. On the other hand the canonical bundle formula for ramified covers gives $\omega_{\tilde{A}} \cong \pi^*\omega_X \otimes \mathcal{O}(\sum E_i)$. Thus we find $\pi^*\omega_X \cong \mathcal{O}_{\tilde{A}}$. Since π is degree 2 we can find a line bundle $\mathcal{L}$ on X such that $\pi^*\mathcal{L}^{\otimes 2} \cong \mathcal{O}(2\sum E_i)$ which gives an isomorphism $\pi_*\mathcal{O}_{\tilde{A}} \cong \mathcal{O}_X \oplus \mathcal{L}^\vee$. Finally we note that the following composition is an isomorphism.

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X \oplus \mathcal{L}^\vee \cong \pi_*\mathcal{O}_{\tilde{A}} \cong \pi_*\pi^*\omega_X \cong \pi_*\mathcal{O}_{\tilde{A}} \otimes \omega_X \tag{3.4}$$

$$\cong (\mathcal{O}_X \oplus \mathcal{L}^\vee) \otimes \omega_X \cong \omega_X \oplus (\mathcal{L}^\vee \otimes \omega_X) \to \omega_X \tag{3.5}$$

We have an injection $H^1(X, \mathcal{O}_X) \to H^1(\tilde{A}, \mathcal{O}_{\tilde{A}}) = H^1(A, \mathcal{O}_A)$ which is contained in the subspace fixed by $[-1]$. In particular we find $H^1(X, \mathcal{O}_X) = 0$. □

Our goal now is to describe the cohomology of a K3 surface. Immediately from the definition we see $\dim H^0(X, \mathcal{O}_X) = 1$ and $\dim H^1(X, \mathcal{O}_X) = 0$. Applying Serre duality, since $\omega_X \cong \mathcal{O}_X$ we find

$$H^2(X, \mathcal{O}_X) = H^0(X, \omega_X \otimes \mathcal{O}_X^\vee) = H^0(X, \mathcal{O}_X). \tag{3.6}$$

Thus $\chi(X, \mathcal{O}_X) = 1 - 0 + 1 = 2$.

To go further we need some technical machinery.

Theorem 3.1.5 (Rieman-Roch for line bundles on surfaces)**.** *Let $\mathcal{L}$ be line bundle on a surface X. Then*

$$\chi(X, \mathcal{L}) = \frac{1}{2}(\mathcal{L}.\mathcal{L} \otimes \omega_X^\vee) + \chi(X, \mathcal{O}_X), \tag{3.7}$$

where $(L.N)$ is the intersection pairing for line bundles on X (see (Hartshorne, 1977), V.1).

In the case of K3 surfaces, this formula reduces to $\chi(X, \mathcal{L}) = \frac{1}{2}(\mathcal{L}.\mathcal{L}) + 2$.

There is a deeper Riemann-Roch theorem, for which we now introduce some new concepts. This treatment follows (Hartshorne, 1977), Appendix A.

Definition 3.1.6. Let X be a variety. We denote $A^r(X)$ the group of codimension r cycles on X up to rational equivalence. The *Chow ring of X* is the graded ring $A(X) = \bigoplus A^r(X)$, with multiplication the intersection pairing.

For $\mathcal{F}$ be a locally free sheaf of rank r on a variety X we can form the projective bundle $\pi : \mathbb{P}(\mathcal{F}) \to X$. We have the class $\xi \in A^1(\mathbb{P}(\mathcal{F}))$ corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. Then $\pi^* : A(X) \to A(\mathbb{P}(\mathcal{F}))$ expresses $A(\mathbb{P}(\mathcal{F}))$ as a free $A(X)$-module generated by $1, \xi, \ldots, \xi^{r-1}$.

Definition 3.1.7. Let $\mathcal{F}$ be a locally free sheaf of rank r on X a smooth projective variety. For $i = 0, \ldots, r$ the *ith Chern class* $c_i(\mathcal{F}) \in A^i(X)$ is determined by the conditions $c_0(\mathcal{F}) = 1$ and

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathcal{F}).\xi^{r-i} = 0 \tag{3.8}$$

in $A^r(\mathbb{P}(\mathcal{F}))$.

From this one can construct the *Chern polynomial*, *exponential Chern character*, and *Todd class* as

$$c_t(\mathcal{F}) = c_0(\mathcal{F}) + c_1(\mathcal{F})t + \ldots + c_r(\mathcal{F})t^r = \prod_{i=1}^{r}(1 + a_i t) \tag{3.9}$$

$$\mathrm{ch}(\mathcal{F}) = \exp(a_1) + \ldots + \exp(a_r) \tag{3.10}$$

$$\mathrm{td}(\mathcal{F}) = \prod_{i=1}^{r} \frac{a_i}{1 - \exp(-a_i)}, \tag{3.11}$$

where the latter two are viewed as power series. Note they only have finitely many nonzero terms, since $A^r(X) = 0$ for $r > \dim X$.

Example 3.1.8. For the trivial bundle $\mathcal{O}_X$, we have $\mathrm{ch}(\mathcal{O}_X) = 1$.

In the case of a sheaf $\mathcal{F}$ on a surface X we have

$$\mathrm{ch}(\mathcal{F}) = r + c_1(\mathcal{F}) + \frac{1}{2}(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})) \tag{3.12}$$

$$\mathrm{td}(\mathcal{F}) = 1 + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}(c_1(\mathcal{F})^2 + c_2(\mathcal{F})). \tag{3.13}$$

We are now in position to state the following theorem.

Theorem 3.1.9 (Hirzebruch-Riemann-Roch)**.** *Let $\mathcal{F}$ be a locally free sheaf of rank r on X a smooth projective variety of dimension n. Then*

$$\chi(X, \mathcal{F}) = \deg(\mathrm{ch}(\mathcal{F}).\, \mathrm{td}(\Omega_X^\vee))_n, \tag{3.14}$$

where $(\cdot)_n$ means take the degree n component in $A(X) \otimes \mathbb{Q}$.

Remark 3.1.10. When applied to $\mathcal{F} = \mathcal{O}_X$ for X a surface, we find

$$\chi(X, \mathcal{O}_X) = \deg\left(1.\left(1 + \frac{1}{2}c_1(\Omega_X^\vee) + \frac{1}{12}\left(c_1(\Omega_X^\vee)^2 + c_1(\Omega_X^\vee)\right)\right)\right)_2 \tag{3.15}$$

$$= \frac{1}{12}\left(c_1(\Omega_X^\vee)^2 + c_2(\Omega_X^\vee)\right), \tag{3.16}$$

which is known as *Noether's formula.*

In the case of X a K3 surface, we have $c_1(\Omega_X^\vee) = c_1(\mathcal{O}_X) = 0$. Thus $\mathrm{ch}(\Omega_X^\vee) = 2 - c_2(\Omega_X^\vee)$ and $\mathrm{td}(\Omega_X^\vee) = 1 + \frac{1}{12}c_2(\Omega_X^\vee)$.

We may now turn to the computation of the dimensions $h^{p,q} = \dim H^q(X, \Omega_X^p)$. These are captured by the *Hodge diamond* of X.

Proposition 3.1.11. *Let X be a K3 surface. The Hodge diamond is*

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{3.17}$$

Proof. We have already computed $h^{0,0} = h^{0,2} = 1$ and $h^{0,1} = 0$. The diagram has horizontal and vertical symmetries, thus it remains to compute $h^{1,1} = \dim H^1(X, \Omega_X)$.

Noether's formula for a K3 surface gives

$$2 = \chi(X, \mathcal{O}_X) = \frac{1}{12}c_2(\Omega_X^\vee), \tag{3.18}$$

so $c_2(\Omega_X^\vee) = 24$. Now, applying Theorem 3.1.9 to $\mathcal{F} = \Omega_X^\vee$ we find

$$\chi(X, \Omega_X^\vee) = \deg\left((2 - c_2(\Omega_X^\vee)).\left(1 + \frac{1}{12}c_2(\Omega_X^\vee)\right)\right)_2 = 4 - c_2(\Omega_X^\vee) = -20. \tag{3.19}$$

Thus

$$\begin{aligned} h^{1,1} &= \dim H^1(X, \Omega_X) = \dim H^0(X, \Omega_X) + \dim H^2(X, \Omega_X) - \chi(X, \Omega_X) &(3.20)\\ &= -\chi(X, \Omega_X) = -\chi(X, \Omega_X^\vee) = 20. &(3.21) \end{aligned}$$

□

In particular, we find that for X/K and $\ell \neq \mathrm{char}\, K$ we have as a $\mathbb{Z}_\ell$-module, $H^2_{\text{ét}}(X, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell^{22}$. The cup product equips this lattice with a quadratic pairing. One can compute that, as quadratic lattices, we have $H^2_{\text{ét}}(X, \mathbb{Z}_\ell) \cong L \otimes \mathbb{Z}_\ell$, where L is the *K3 lattice*, given by $L = U^{\oplus 3} \oplus E_8^{\oplus 2}$, where U is the hyperbolic lattice. That is, $U \cong \mathbb{Z}^2$ with basis vectors e, f satisfying $e^2 = f^2 = 0$ and $\langle e, f \rangle = 1$.

A *polarisation* of X is a class ξ of ample line bundle on X. A polarisation determines a primitive cohomology subgroup $PH^2_\xi(X,\mathbb{Z}_\ell) = \langle\xi\rangle^\perp \subseteq H^2_\ell(X,\mathbb{Z}_\ell)$. As quadratic lattices we have $H^2_{\text{ét}}(X,\mathbb{Z}_\ell) \cong L_d \otimes \mathbb{Z}_\ell$. Here d is the degree of ξ and $L_d = \langle e - df\rangle^\perp \subseteq L$, where e, f are basis elements of one copy of U.

3.2 The main theorem

To state our main theorem we need to introduce a class of K3 surfaces. We follow (Huybrechts, 2016), Chapter 18. Given a K3 surface X/K we have the formal Brauer group $\widetilde{\mathrm{Br}}_X$ as defined in (Artin and Mazur, 1977). This is a smooth one-dimensional formal group over K. In particular, If $\mathrm{char}\, K = p \neq 0$ then $\widetilde{\mathrm{Br}}_X$ is characterised in terms of its height.

Definition 3.2.1. Let K be a field of characteristic $p \neq 0$ and X/K a K3 surface. The *height of* X, denoted $h(X)$, is defined as the height of the formal Brauer group of X.

Proposition 3.2.2. *Let K be a field of characteristic $p \neq 0$ and X/K a K3 surface. Then $h(X) = \infty$ or $h(X) = 1, \dots, 10$.*

In the case $h(X) = \infty$, i.e. $\widetilde{\mathrm{Br}}_X = \hat{\mathbb{G}}_a$, we say X is *supersingular*. In the case $h(X) = 1$, i.e. $\widetilde{\mathrm{Br}}_X = \hat{\mathbb{G}}_m$, we say X is *ordinary*.

Remark 3.2.3. One may characterise the ordinary K3 surfaces X in terms of the crystalline cohomology $H^2_{\text{cris}}(X)$. This is equipped with a filtration $F^\bullet$ and a *conjugate* filtration $F^\bullet_{\text{con}}$. Then X is ordinary if $F^2H^2_{\text{cris}} \cap F^2_{\text{con}}H^2_{\text{cris}} = 0$.

We may now state the main theorem.

Theorem 3.2.4. *Let K/k be a regular extension and X/K a K3 surface. Let $\ell \neq \mathrm{char}\, K$ and $\rho : G_K \to \mathrm{Aut}(H^2(X_{K^s}, \mathbb{Q}_\ell))$ be the Galois representation on the ℓ-adic cohomology. Assume $\rho(\mathrm{Gal}(K^s/k^sK)) = 1$ and either (1) K is charactersitic 0, or (2) X is ordinary in odd characteristic. Then there exists X_0/k^s and an isomorphism $\varphi : X_0 \otimes_{k^s} K^s \to X \otimes_K K^s$.*

Remark 3.2.5. Precisely, this is saying that the K/k-Galois-ℓ-descent criterion is effective for K3 surfaces satisfying the above hypotheses. Or equivalently that Conjecture 2.4.5 holds for $h(X)$ where X is a K3 surface satisfying the above hypotheses.

We note at this stage the we expect a simplified proof in the Kummer case, using a direct argument. At time of writing there is a representation-theoretic obstruction to this proof, and thus it does not appear in this thesis.

3.3 The Clifford algebra and GSpin

Let L be a quadratic lattice of rank n with quadratic form q over a commutative ring R in which 2 is not a zero divisor. We have the Clifford algebra

$$\mathrm{Cl}(L) = (\oplus_{n\geq 0} L^{\otimes n})/(v \otimes v - q(v)). \tag{3.22}$$

This comes with a $\mathbb{Z}/2\mathbb{Z}$ grading, with even and odd parts denoted $C^+(L)$ and $C^-(L)$, respectively. We define the group $\mathrm{GSpin}(L)$ as

$$\mathrm{GSpin}(L) = \{v \in \mathrm{Cl}^+(L)^\times \mid vLv^{-1} = L\}. \tag{3.23}$$

We will use the same notation $\mathrm{GSpin}(L)$ for an algebraic group over R such that $\mathrm{GSpin}(L)(R') = \mathrm{GSpin}(L \otimes R')$. We get an orthogonal representation of $\mathrm{GSpin}(L)$ on L via conjugation, on which one can see the scalars act trivially. This gives the following short exact sequence, expressing $\mathrm{GSpin}(L)$ as a central extension of $\mathrm{SO}(L)$.

$$1 \to \mathbb{G}_m \to \mathrm{GSpin}(L) \xrightarrow{\mathrm{ad}} \mathrm{SO}(L) \to 1 \tag{3.24}$$

There is an anti-involution $c \mapsto c^*$ on $\mathrm{Cl}(L)$ induced by reversing the order of simple tensors. We have the *spinor norm*

$$\nu : \mathrm{GSpin}(L) \longrightarrow \mathbb{G}_m \tag{3.25}$$
$$x \longmapsto x^* x. \tag{3.26}$$

Note that restricting to $\ker \mathrm{ad} \cong \mathbb{G}_m$ gives $\nu(x) = x^2$.

Further, on $\mathrm{Cl}(L)$ we have an R-linear reduced trace map $\mathrm{Trd} : \mathrm{Cl}(L) \to R$ such that $(x, y) \mapsto \mathrm{Trd}(xy)$ is a non-degenerate symmetric bilnear form on $\mathrm{Cl}(L)$.

Lemma 3.3.1. *For $\delta \in \mathrm{Cl}(L)^\times$ such that $\delta^* = -\delta$, the form $\psi_\delta(x, y) = \mathrm{Trd}(x\delta y^*)$ defines an R-valued symplectic form on $\mathrm{Cl}(L)$. For $g \in \mathrm{GSpin}(L)$ we have*

$$\psi_\delta(gx, gy) = \nu(g)\psi_\delta(x, y). \tag{3.27}$$

Proof. To see that ψ_δ is alternating and hence symplectic we compute

$$\mathrm{Trd}(x\delta x^*) = \mathrm{Trd}((x\delta x^*)^*) = \mathrm{Trd}(x\delta^* x^*) = -\mathrm{Trd}(x\delta x^*),$$

noting that $\mathrm{Trd}(x) = \mathrm{Trd}(x^*)$.

For the second statement, we compute

$$\psi_\delta(gx, gy) = \mathrm{Trd}((gx)\delta(gy)^*) = \mathrm{Trd}(g(x\delta y^*)g^*) \tag{3.28}$$
$$= \mathrm{Trd}(g(x\delta y^*)\nu(g)g^{-1}) \tag{3.29}$$
$$= \nu(g)\,\mathrm{Trd}(x\delta y^*) = \nu(g)\psi_\delta(x, y), \tag{3.30}$$

noting that if $g \in \mathrm{GSpin}(L)$ then $g^* = \nu(g)g^{-1}$ and Trd is conjugation-invariant. □

Remark 3.3.2.

- The above gives a map $\mathrm{GSpin}(L) \hookrightarrow \mathrm{GSp}(\mathrm{Cl}(L), \psi_\delta)$ under which the spinor norm corresponds to the similitude character.
- There is an isomorphism $\mathrm{Cl}(L) \cong \mathrm{End}_{\mathrm{Cl}(L)}(\mathrm{Cl}(L))$ of $\mathrm{GSpin}(L)$-representations, where it acts on the right by right multiplication and on the left by conjugation. The inclusion $L \subseteq \mathrm{Cl}(L)$ is $\mathrm{GSpin}(L)$-equivariant only with respect to the conjugation action.

Let $\langle , \rangle_q$ be the pairing associated to the quadratic form q. We have a pairing $\langle \varphi_1, \varphi_2 \rangle = \frac{1}{2^n}\mathrm{Tr}(\varphi_1 \circ \varphi_2)$ on $\mathrm{End}_R(\mathrm{Cl}(L))$ which is symmetric, non-degenerate and restricts to $\langle , \rangle$ on L. Choose an R-basis $e_1, \dots, e_n$ of L and let $A \in \mathrm{Aut}(L) \cong \mathrm{GL}_n(R)$ be the matrix with inverse $(\langle e_i, e_j \rangle_q)$. Define an element $\pi \in \mathrm{End}_R(\mathrm{End}_R(\mathrm{Cl}(L)))$ by

$$\pi : \mathrm{End}_R(\mathrm{Cl}(L)) \longrightarrow \mathrm{End}_R(\mathrm{Cl}(L)) \tag{3.31}$$

$$\varphi \longmapsto \sum_i \langle \varphi, e_1 \rangle A e_i. \tag{3.32}$$

Lemma 3.3.3. *1. π is idempotent with image $L \subseteq \mathrm{End}_R(\mathrm{Cl}(L))$.*

2. π is the unique projector onto L satisfying.

$$\ker \pi = \{\varphi \in \mathrm{End}_R(\mathrm{Cl}(L)) \mid \langle \varphi, v \rangle = 0 \text{ for all } v \in L\}. \tag{3.33}$$

3. $\mathrm{GSpin}(L)$ is the stabiliser of π.

Proof. See (Madapusi Pera, 2016), Lemma 1.4. □

Note that property 2 allows us to conclude that π is independent of the choice of basis for L. We may equivalently describe π as an element of $\mathrm{Cl}(L)^{\otimes 2} \otimes (\mathrm{Cl}(L)^\vee)^{\otimes 2}$.

3.4 Hodge structures and Kuga-Satake

The classical Kuga-Satake construction applies to K3 surfaces $X/\mathbb{C}$. In this case we have the singular cohomology $H^2(X, \mathbb{Z})$, which has a Hodge structure determined by Proposition 3.1.11. We introduce a general class of Hodge structures of a similar shape to this.

Definition 3.4.1. A *K3 Hodge structure* L is of type $\{(2,0),(1,1),(0,2)\}$ such that $\dim L^{2,0} = \dim L^{0,2} = 1$.

Example 3.4.2. If X is a complex K3 surface or complex abelian surface then $H^2(X, \mathbb{Z})$ is a K3 Hodge structure. For any sub-Hodge structure $V \subseteq L^{1,1}$ then $V^\perp$ is a K3 Hodge structure. In particular $PH^2(X, \mathbb{Z})$ is a K3 Hodge structure, as is the transcendental lattice $T(X) = \mathrm{Pic}(X)^\perp$.

Under these circumstances, $\mathrm{Cl}^+(L_{\mathbb{R}})$ can be equipped with a complex structure, or equivalently a weight 1 Hodge structure, i.e. of type $\{(1,0),(0,1)\}$ (see (Huybrechts, 2016), 4.2.1). If we denote the Hodge structure on L as $\rho : \mathbb{S} \to \mathrm{SO}(L_{\mathbb{R}})$, where $\mathbb{S}$ is the Deligne torus, then this weight 1 structure can be viewed as exhibiting a lift as in the following diagram.

$$\begin{array}{ccccc} & & \mathrm{GSpin}(L_{\mathbb{R}}) & \longrightarrow & \mathrm{GSp}(\mathrm{Cl}^+(L_{\mathbb{R}})) \\ & \overset{\tilde{\rho}}{\nearrow} & \downarrow{\scriptstyle \mathrm{ad}} & & \\ \mathbb{S} & \xrightarrow[\rho(1)]{} & \mathrm{SO}(L_{\mathbb{R}}(1)) & & \end{array}$$

This allows us to form a complex torus $\mathrm{KS}(L) = \mathrm{Cl}^+(L_{\mathbb{R}}) / \mathrm{Cl}^+(L)$.

Proposition 3.4.3. *Let L be a K3 Hodge structure with polarisation q. Consider $f_1, f_2 \in V$ such that $(f_1, f_2) = 0$ and $q(f_i) > 0$. Then there exists a choice of sign such that*

$$Q : \mathrm{Cl}^+(L) \times \mathrm{Cl}^+(L) \longrightarrow \mathbb{Q}(-1) \tag{3.34}$$

$$(v, w) \longmapsto \pm \operatorname{tr}(f_1 \cdot f_2 \cdot v^* \cdot w) \tag{3.35}$$

is a polarisation for the weight one Hodge structure on $\mathrm{Cl}^+(L)$.

Proof. This is Proposition 2.5 in (Huybrechts, 2016). The fact that it is a symmetric nondegenerate pairing and a morphism of Hodge structures can be checked directly. One can further compute that $Q(v, \rho(i)w)$ is definite, and ensures it is positive definite by choice of sign. □

We may then apply Riemann's theorem.

Theorem 3.4.4. *Let $\mathcal{C}$ be the category of Hodge structures of type $\{(1,0),(0,1)\}$ which are polarisable. There is an equivalence of categories*

$$\mathrm{AbVar}_{\mathbb{C}} \longrightarrow \mathcal{C} \tag{3.36}$$

$$A \longmapsto H_1(A, \mathbb{Z}). \tag{3.37}$$

Thus we may attach an abelian variety to a polarisable K3 Hodge structure.

The intersection pairing does not define a polarisation on the Hodge structure $H^2(X, \mathbb{Z})$, since it has signature $(3, 19)$. To form a polarisation one must change the sign of the class ξ of an ample line bundle. Alternatively, one may pass to the primitive cohomology, on which the intersection pairing itself is a polarisation.

Definition 3.4.5. Given a K3 surface $X/\mathbb{C}$ the above construction applied to $L = PH^2(X, \mathbb{Z})$ gives the *Kuga-Satake abelian variety* $A = \mathrm{KS}(X)$.

In (Deligne, 1972) it is proven that if K is a number field and X/K is a K3 surface, then there exists a finite extension K'/K and an abelian variety A/K' such that $A \otimes \mathbb{C} \cong \mathrm{KS}(X \otimes \mathbb{C})$.

This construction will be a central in our proof of Theorem 3.2.4. To apply this over general fields, we will require the theory of Shimura varieties.

Chapter 4

Shimura varieties

For basic properties of Shimura varieties, we refer to (Milne, 2005) and (Lan, 2017). The material on integral models largely follows (Kisin, 2010), (Madapusi Pera, 2015), and (Madapusi Pera, 2016).

4.1 Basic definitions

The story of Shimura varieties typically begins with *modular curves*. Given a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ we have a Riemann surface $\Gamma \setminus \mathcal{H}$, where $\mathcal{H}$ is the upper half plane and Γ acts by fractional linear transformations. This is called a modular curve.

It has many important properties and applications.

- There exists a number field F and an algebraic curve Y_Γ/F such that $Y_\Gamma(\mathbb{C}) \cong \Gamma \setminus \mathcal{H}$.

- There is an integer N such that $\Gamma \supseteq \Gamma(N)$, where

$$1 \longrightarrow \Gamma(N) \longrightarrow \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 1.$$

  The minimal such N is called the *level*.

  Let p be a prime not dividing N and $\mathfrak{p}|p$ a prime in F above it. Let $\mathcal{O}_\mathfrak{p}$ be the localisation of the ring of integers of F at $\mathfrak{p}$. Then there exists a smooth regular scheme $\mathcal{Y}_\Gamma \to \mathcal{O}_\mathfrak{p}$ such that $\mathcal{Y}_\Gamma \otimes F \cong Y_\Gamma$. It satisfies an *extension property*.

That is, for every smooth regular scheme $S \to \mathcal{O}_\mathfrak{p}$, every morphism $S \otimes L \to Y_\Gamma$ extends uniquely to a morphism $S \to \mathcal{Y}_\Gamma$.

- $\mathcal{Y}_\Gamma$ represents the moduli problem

$$\mathcal{O}_\mathfrak{p}\text{-Sch} \longrightarrow \text{Set} \tag{4.1}$$

$$S \longmapsto \{(E/S, \varphi)\}, \tag{4.2}$$

where E is an elliptic scheme over S and φ is a Γ-level structure.

- One may form an alternative *adelic* construction. Let $X = \mathbb{C} \setminus \mathbb{R} = \mathcal{H} \sqcup (-\mathcal{H})$. Then we have

$$\mathrm{GL}_2(\mathbb{Q}) \setminus X \times \mathrm{GL}_2(\mathbb{A}_f)/K \cong \coprod_{g \in \mathcal{C}} \Gamma_g \setminus \mathcal{H}, \tag{4.3}$$

where $\Gamma_g = gKg^{-1} \cap \mathrm{GL}_2(\mathbb{Q})$. In particular, classic modular curves are precisely the connected components of this adelic construction.

This last perspective is the one we will generalise to other groups G. Recall that the reductive group $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ over $\mathbb{R}$ is the *Deligne torus*. For any group G we let $G^{\mathrm{ad}} = G/Z$ be the quotient of G by its center Z.

Definition 4.1.1. A *Shimura datum* is a pair (G, X) where $G/\mathbb{Q}$ is a reductive group and X is a conjugacy class of morphisms $h : \mathbb{S} \to G_\mathbb{R}$ satisfying the following properties.

1. For all $h \in X$, the Hodge structure on $\mathrm{Ad} \circ h : \mathbb{S} \to \mathrm{Lie}(G_\mathbb{R})$ is of type { (-1,1),(0,0),(1,-1) }.

2. For all $h \in X$, the group

$$\{g \in G^{\mathrm{ad}}_\mathbb{R}(\mathbb{C}) \mid g = \mathrm{ad}(h(i))(\bar{g})\} \tag{4.4}$$

is compact. We say $\mathrm{ad}(h(i))$ is a *Cartan involution* of $G^{\mathrm{ad}}_\mathbb{R}$.

3. G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Example 4.1.2. Consider the $\mathrm{GL}_2(\mathbb{R})$-conjugacy class $X = \left\{ h_0 : a+bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\}$. One can check that indeed the pair (GL_2, X) is a Shimura datum. Note that we have a bijection $X \to \mathbb{C} \setminus \mathbb{R}, h_0 \mapsto i$ that respects the $\mathrm{GL}_2(\mathbb{R})$-actions.

We then wish to construct a variety from the pair (G, X) which generalises the modular curves above.

Proposition 4.1.3. *Let (G, X) be a Shimura datum and $K \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. If K is sufficiently small, then the double quotient*

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q}) \setminus X \times G(\mathbb{A}_f)/K \tag{4.5}$$

is a variety over $\mathbb{C}$.

We call $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ the *Shimura variety of level K* attached to the pair (G, X).

Remark 4.1.4. We will often omit the conjugacy class X from the notation, and simply write $\mathrm{Sh}_K(G)$.

Work beginning with Shimura and being completed in (Deligne, 1979) and (Milne, 1983) shows that these objects can be descended down from $\mathbb{C}$ to a number field.

Theorem 4.1.5. *Let (G, X) be a Shimura datum. There exists a number field F, called the* reflex field*, and a variety $\mathrm{Sh}_K(G, X)$ over F such that $\mathrm{Sh}_K(G, X) \otimes \mathbb{C} = \mathrm{Sh}_K(G, X)_{\mathbb{C}}$.*

Remark 4.1.6. In fact more is true, and these models are *canonical*. The formulation of this property can be found in (Milne, 2005), Definition 12.8.

4.2 Moduli interpretations

Recall the Shimura datum (GL_2, X) from example 4.1.2. In equation 4.3 we saw that this was related to a modular curve. As mentioned above, an important feature of modular curves is that they have moduli interpretations in terms of elliptic curves with level structure.

Example 4.2.1. Let V be a $\mathbb{Q}$-vector space of dimension $2g$ and ψ a nondegenerate alternating form on V. Such a pair (V, ψ) is called a *symplectic space.* Attach to this we have the group of *symplectic similitudes* $\mathrm{GSp}_\psi/\mathbb{Q}$, whose F-points for any $\mathbb{Q}$-algebra F are

$$\mathrm{GSp}_\psi(F) = \{g \in \mathrm{GL}(V \otimes F) \mid \psi(gu, gv) = \nu(g)\psi(u,v), u, v \in V \otimes F, \nu(g) \in F^\times\}. \tag{4.6}$$

Let $\mathcal{H}^+$ (respectively $\mathcal{H}^-$) be the set of complex structures J on $V \otimes \mathbb{R}$ such that $\psi(Ju, Jv) = \psi(u, v)$ and $\psi(u, Jv)$ is positive-definition (respectively negative-definite). These are known as *Siegel half-spaces.* Let $\mathcal{H}^\pm = \mathcal{H}^+ \sqcup \mathcal{H}^-$. Then we have an action of $\mathrm{GSp}_\psi(\mathbb{R})$ on X by $g \cdot J = gJg^{-1}$.

Proposition 4.2.2. *The pair* $(\mathrm{GSp}_\psi, \mathcal{H}^\pm)$ *defines a Shimura datum with reflex field* $\mathbb{Q}$.

To discuss the moduli interpretation we require the notions of *polarisation* and *level structure* at a prime p.

Definition 4.2.3. Let $f : A \to S$ be an abelian scheme of dimension $2g$ over S a $\mathbb{Z}_{(p)}$-scheme.

A *polarisation* is an isogeny $\lambda : A \to A^\vee$ induced by an ample line bundle. A *weak polarisation* a class of a polarisation up to multiplication by $\mathbb{Z}_{(p)}^\times$.

Let V be a free $\mathbb{Z}$-module of rank $2g$ and ψ a nondegenerate alternating form on V. Let $K_p = \mathrm{GSp}_\psi(\mathbb{Z}_p)$ and K^p be open compact in $\mathrm{GSp}_\psi(\mathbb{A}_f^p)$. Let $K = K_pK^p \subseteq \mathrm{GSp}_\psi(\mathbb{Q}_p) \times \mathrm{GSp}_\psi(\mathbb{A}_f^p) = \mathrm{GSp}_\psi(\mathbb{A}_f)$. A K*-level structure* on A is a section

$$\varepsilon \in H^0(S, \underline{\mathrm{Isom}}(\underline{V \otimes \mathbb{A}_f^p}, R^1 f_{\text{ét}*}\underline{\mathbb{A}_f^p})/K^p), \tag{4.7}$$

where $\underline{\mathrm{Isom}}(\mathcal{F}, \mathcal{G})$ is the sheaf of $\mathcal{O}_S$-modules given by

$$U \longmapsto \mathrm{Isom}_{\mathcal{O}_S(U)}(\mathbb{F}(U), \mathbb{G}(U)). \tag{4.8}$$

Theorem 4.2.4. *For sufficiently small* K*, the moduli problem*

$$\mathbb{Z}_{(p)}\text{-Sch} \longrightarrow \mathrm{Set} \tag{4.9}$$

$$S \longmapsto \{(A/S, \lambda, \varepsilon)\}, \tag{4.10}$$

where λ is a weak polarisation and ε is a K-level structure, is representable over $\mathbb{Z}_{(p)}$ by a smooth regular scheme $\mathcal{S}_K(\mathrm{GSp}_\psi, \mathcal{H}^\pm)$ such that

$$\mathcal{S}_K(\mathrm{GSp}_\psi, \mathcal{H}^\pm) \otimes \mathbb{Q} \cong \mathrm{Sh}_K(\mathrm{GSp}_\psi, \mathcal{H}^\pm). \tag{4.11}$$

This moduli interpretation provides a universal abelian scheme

$$f : \mathcal{A} \to \mathcal{S}_K(\mathrm{GSp}_\psi, \mathcal{H}^\pm). \tag{4.12}$$

From this we have a bundle $R^1 f_{\text{ét}*}\underline{\mathbb{A}}_f^p$. Similar moduli interpretations and integral models can be found for other groups, corresponding to *PEL-type* Shimura varieties (see (Lan, 2017), §5.1).

However, there are many Shimura varieties which do not fit into the above framework.

Definition 4.2.5. A *Hodge-type Shimura datum* is a Shimura datum (G, X) such that there exists an embedding $G \hookrightarrow \mathrm{GSp}_{2g}$ for some g inducing an embedding $X \hookrightarrow \mathcal{H}_g^\pm$. A *Hodge-type Shimura variety* is a Shimura variety associated with a Hodge-type Shimura datum.

For a Hodge-type Shimura variety we have a coarse moduli interpretation of the complex points $\mathcal{S}_K(G, X)(\mathbb{C})$ in terms of abelian varieties equipped with a collection of *Hodge tensors*. See (Milne, 2005), Theorem 7.4.

Definition 4.2.6. An *abelian-type Shimura datum* is a Shimura datum (G, X) such that there exists a Hodge-type Shimura datum (G_1, X_1) and a central isogeny from $[G_1, G_1]$ to $[G, G]$, which induces an isomorphism $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. An *abelian-type Shimura variety* is a Shimura variety associated with an abelian-type Shimura datum.

For an abelian-type Shimura variety we have a coarse moduli interpretation of the complex points $\mathcal{S}_K(G, X)(\mathbb{C})$ in terms of abelian motives. See (Milne, 2005), Theorem 9.4.

Below we will describe a formulation of these moduli due to Madapusi-Pera in the case of $G = \mathrm{SO}(L)$ for a certain quadratic lattice L.

4.3 The main diagram

We are aiming to prove Theorem 3.2.4, for which we will use an extension of the Kuga-Satake construction due to Madapusi-Pera in (Madapusi Pera, 2015). The goal of the remainder of this chapter is to construct the objects and morphisms in the following diagram and note their consequences for K3 surfaces.

$$\begin{array}{ccc} & \mathcal{S}(\mathrm{GSpin}(L_d)) & \longrightarrow \mathcal{S}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)) \\ & \downarrow & \\ \tilde{M}_{2d,\gamma} \longrightarrow & \mathcal{S}(\mathrm{SO}(L_d)) & \end{array} \tag{4.13}$$

where:

- $\tilde{M}$ is a moduli space of "γ-oriented" K3 surfaces (quasi-polarised of degree $2d$).
- $\mathcal{S}(\cdot)$ is the integral model of a Shimura variety $\mathrm{Sh}(\cdot)$.
- All morphisms are finite étale over their image.

Recall $L = U^{\oplus 3} \oplus E_8^{\oplus 2}$ is the K3 lattice, equipped with its quadratic form. For $d \in \mathbb{Z}$ we have $L_d = \langle e - df \rangle^\perp \subseteq L$, where $U = \langle e, f \rangle$. Let Ω be the space of oriented negative definite planes in $L_d \otimes \mathbb{R}$.

Lemma 4.3.1 ((Madapusi Pera, 2016), 3.1)**.** *The pairs* $(\mathrm{GSpin}(L_d \otimes \mathbb{Q}), \Omega)$ *and* $(\mathrm{SO}(L_d \otimes \mathbb{Q}), \Omega)$ *are Shimura data with reflex field* $\mathbb{Q}$*.*

We have the even Clifford algebra $\mathrm{Cl}^+(L_d)$. As in Lemma 3.3.1, for each $\delta \in \mathrm{Cl}^+(L_d)^\times$ there exists a symplectic form ψ_δ on $\mathrm{Cl}^+(L_d)$ and hence an embedding $\mathrm{GSpin}(L_d) \hookrightarrow \mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)$. Let $\mathcal{H}_\delta^\pm$ be the union of Siegel half-spaces attached to $(\mathrm{Cl}^+(L_d), \psi_\delta)$. Then, as is in Proposition 4.2.2, for each δ we get a Shimura datum $(\mathrm{GSp}(\mathrm{Cl}^+(L_d \otimes \mathbb{Q}), \psi_\delta), \mathcal{H}_\delta^\pm)$ with reflex field $\mathbb{Q}$.

Lemma 4.3.2 ((Madapusi Pera, 2016), Lemma 3.6)**.** *There exists a choice of δ such that the embedding $\mathrm{GSpin}(L_d) \hookrightarrow \mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)$ induces an embedding of Shimura data $(\mathrm{GSpin}(L_d \otimes \mathbb{Q}), \Omega) \hookrightarrow (\mathrm{GSp}(\mathrm{Cl}^+(L_d \otimes \mathbb{Q}), \psi_\delta), \mathcal{H}_\delta^\pm)$.*

In particular, $(\mathrm{GSpin}(L_d \otimes \mathbb{Q}), \Omega)$ is of Hodge type.

The adjoint representation $\mathrm{GSpin}(L_d) \to \mathrm{SO}(L_d)$ also induces a morphism of Shimura varieties, giving the diagram

$$\begin{array}{ccc} \mathrm{Sh}(\mathrm{GSpin}(L_d)) & \longrightarrow & \mathrm{Sh}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)) \\ \downarrow & & \\ \mathrm{Sh}(\mathrm{SO}(L_d)) & & \end{array} \tag{4.14}$$

We now seek to extend this diagram past objects over $\mathbb{Q}$.

4.4 Integral Models

Fix a prime p. By Theorem 4.2.4 we an integral model $\mathcal{S}_K(\mathrm{GSp}(\mathrm{Cl}^+(L_d)), \psi_\delta)$ over $\mathbb{Z}_{(p)}$ for a sufficiently small K. From here we will set $K_p = \mathrm{GSpin}(L_d)(\mathbb{Z}_p)$, $K_{0,p} = \mathrm{SO}(L_d)(\mathbb{Z}_p)$, and $\mathcal{K}_p = \mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)(\mathbb{Z}_p)$. Further set $K = K_p K^p$, $K_0 = K_{0,p} K_0^p$ and $\mathcal{K} = \mathcal{K}_p \mathcal{K}^p$ to be a compatible (with respect to 4.15) choice of open compact subgroup of each of $\mathrm{GSpin}(L_d)(\mathbb{A}_f)$, $\mathrm{SO}(L_d)(\mathbb{A}_f)$ and $\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)(\mathbb{A}_f)$ simultaneously.

To define integral models we work locally at p and introduce a few technical notions.

Definition 4.4.1. A regular local $\mathbb{Z}_{(p)}$-algebra R with maximal ideal $\mathfrak{m}$ is *quasi-healthy* if it is faithfully flat over $\mathbb{Z}_{(p)}$ and every abelian scheme over $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ extends uniquely to an abelian scheme over $\operatorname{Spec} R$.

A regular $\mathbb{Z}_{(p)}$-scheme X is *healthy* if it is faithfully flat over $\mathbb{Z}_{(p)}$ and if, for every open subscheme $U \subseteq X$ containing $X \otimes \mathbb{Q}$ and all generic points of $X \otimes \mathbb{F}_p$, every abelian scheme over U extends uniquely to an abelian scheme over X. It is *locally healthy* if, for every point $x \in X \otimes \mathbb{F}_p$ of codimension at least 2, the complete local ring $\hat{\mathcal{O}}_{X,x}$ is quasi-healthy.

To construct an integral model which is *canonical*, one asks that it satisfy a certain extension property. The following formulation appears in (Madapusi Pera, 2016), Definition 4.2. A similar definition appears in (Kisin, 2010), 2.3.7.

Definition 4.4.2. A scheme X over $\mathbb{Z}_{(p)}$ satisfies the extension property if, for any regular, locally healthy $\mathbb{Z}_{(p)}$-scheme S, any map $S \otimes \mathbb{Q} \to X$ extends to a map $S \to X$.

The setup of (Madapusi Pera, 2016) applies in fact to the full pro-scheme $\mathcal{S}_{K_p} = \varprojlim_{K'^p \subseteq K^p} \mathcal{S}_K$ taken over K with fixed level K_p at p. We may now give the definition of what it means to have a canonical intergral model.

Definition 4.4.3. A model $\mathcal{S}_{K_p}$ for Sh_{K_p} over $\mathbb{Z}_{(p)}$ is an *integral canonical model* if it is regular, locally healthy, and has the extension property.

We recover an integral model at finite level by $\mathcal{S}_K = \mathcal{S}_{K_p}/K$.

We now have the main result on integral models for the Shimura varieties in question.

Theorem 4.4.4 ((Madapusi Pera, 2016) Theorem 4.4, (Kisin, 2010) 2.3.8, (Vasiu, 1999) 3.4.14)**.** *Let $\mathcal{S}_{K_p}(\mathrm{GSpin}(L_d))$ be the normalisation of the Zariski closure of $\mathrm{Sh}_{K_p}(\mathrm{GSpin}(L_d))$ in $\mathcal{S}_{K_p}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta))$. Then $\mathcal{S}_{K_p}(\mathrm{GSpin}(L_d))$ is a smooth integral canonical model for $\mathrm{Sh}_{K_p}(\mathrm{GSpin}(L_d))$. Further, the finite Galois cover $\mathrm{Sh}_{K_p}(\mathrm{GSpin}(L_d)) \to \mathrm{Sh}_{K_{0,p}}(\mathrm{SO}(L_d))$ extends to $\mathcal{S}_{K_p}(\mathrm{GSpin}(L_d)) \to \mathcal{S}_{K_{0,p}}(\mathrm{SO}(L_d))$, where $\mathcal{S}_{K_{0,p}}(\mathrm{SO}(L_d))$ is a smooth integral canonical model for $\mathrm{Sh}_{K_p}(\mathrm{SO}(L_d))$.*

Thus far we have a diagram

$$\begin{array}{ccc} \mathcal{S}(\mathrm{GSpin}(L_d)) & \longrightarrow & \mathcal{S}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)) \\ \downarrow & & \\ \mathcal{S}(\mathrm{SO}(L_d)) & & \end{array} \tag{4.15}$$

What remains is to relate this to K3 surfaces and abelian varieties.

4.5 Bundles and abelian varieties

We now describe certain bundles on these Shimura varieties.

Recall that for any Shimura datum (G, X) we have an isomorphism of complex manifolds

$$\mathrm{Sh}_K(G)(\mathbb{C}) \cong G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f))/K. \tag{4.16}$$

This allows us to construct a functor taking an algebraic representation V of G to a bundle on $\mathrm{Sh}_K(G)(\mathbb{C})$. First by taking the constant local system $V \times X \times G(\mathbb{A}_f)/K$, then quotienting by the action of $G(\mathbb{Q})$ gives a local system $\mathbf{V}_B$ on $\mathrm{Sh}_K(G)(\mathbb{C})$. We can further equip the bundle $(V \otimes \mathbb{C}) \times X$ with a filtration such that at each point $h \in X$ the filtration is induced by the homomorphism $h : \mathbb{S} \to G \otimes \mathbb{R}$ at that point. This descends to $\mathrm{Sh}_K(G)(\mathbb{C})$, giving an *algebraic variation of Hodge structures*, denoted $(\mathbf{V}_B, \mathrm{Fil}^\bullet \mathbf{V}_{\mathrm{dR},\mathbb{C}})$ (see (Madapusi Pera, 2016), 3.3).

Applying the above construction to the representation of $\mathrm{GSpin}(L_d)$ on $\mathrm{Cl}^+(L_d)$ by right multiplication gives an algebraic variation of Hodge structures $(\mathbf{H}_B, \mathrm{Fil}^\bullet \mathbf{H}_{\mathrm{dR},\mathbb{C}})$ on $\mathrm{Sh}_K(\mathrm{GSpin}(L_d))(\mathbb{C})$. This is further equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading, a right $\mathrm{Cl}^+(L_d)$-action, and, following the construction of (3.31), a tensor

$$\boldsymbol{\pi}_B \in H^0\left(\mathrm{Sh}_K(\mathrm{GSpin}(L_d))(\mathbb{C}), (\mathbf{H}_B^{\otimes 2} \otimes (\mathbf{H}_B^\vee)^{\otimes 2}) \otimes \mathbb{Q}\right), \tag{4.17}$$

where $(\mathbf{L}_B, \mathrm{Fil}^\bullet \mathbf{L}_{\mathrm{dR},\mathbb{C}})$ is the variation of Hodge structures arising from the adjoint representation of $\mathrm{GSpin}(L_d)$ into $\mathrm{SO}(L_d)$, identified with $\boldsymbol{\pi}_B((\mathbf{H}_B, \mathrm{Fil}^\bullet \mathbf{H}_{\mathrm{dR},\mathbb{C}}))$.

From Theorem 4.2.4, we have over $\mathrm{Sh}_{\mathcal{K}}(\mathrm{GSp}(\mathrm{Cl}^+(L_d)), \psi_\delta)$ the tautological tuple $(\mathcal{A}, \lambda, \varepsilon)$ where $\mathcal{A}$ is an abelian scheme, λ is a weak polarisation, and ε is a $\mathcal{K}$-level structure. Pulling this back to $\mathrm{Sh}_K(\mathrm{GSpin}(L_d))$ gives a bundle $(\mathcal{A}_{\mathbb{Q}}^{\mathrm{KS}}, \lambda_{\mathbb{Q}}^{\mathrm{KS}}, \varepsilon_{\mathbb{Q}}^{\mathrm{KS}})$, where $\mathcal{A}_{\mathbb{Q}}^{\mathrm{KS}}$ is known as the *Kuga-Satake abelian scheme*. On $\mathrm{Sh}_K(\mathrm{GSpin}(L_d))(\mathbb{C})$ the algebraic variation of Hodge structures arising from the induced $\mathrm{GSpin}(L_d)$ representation on $H^1(\mathcal{A}^{\mathrm{KS}}_{\mathbb{C}})$ is identified with $(\mathbf{H}_B, \mathrm{Fil}^\bullet \mathbf{H}_{\mathrm{dR},\mathbb{C}})$. Thus $\mathcal{A}^{\mathrm{KS}}_{\mathbb{C}}$ carries a $\mathbb{Z}/2\mathbb{Z}$-grading and $\mathrm{Cl}^+(L_d)$-action. By (Madapusi Pera, 2016), Proposition 3.11, the identification of bundles, along with the $\mathbb{Z}/2\mathbb{Z}$-grading and $\mathrm{Cl}^+(L_d)$-action, descend to $\mathcal{A}^{\mathrm{KS}}_{\mathbb{Q}}$ on $\mathrm{Sh}_K(\mathrm{GSpin}(L_d))$ over $\mathbb{Q}$.

We have the integral model $\mathcal{S}_K(\mathrm{GSpin}(L_d))$ described in Theorem 4.4.4. The polarised abelian scheme extends to $(\mathcal{A}^{\mathrm{KS}}, \lambda^{\mathrm{KS}}, \varepsilon^{\mathrm{KS}})$ over this model. Further, the $\mathbb{Z}/2\mathbb{Z}$-grading and $\mathrm{Cl}^+(L_d)$-action extend to $\mathcal{A}^{\mathrm{KS}}$ by the theory of Néron models, see (Madapusi Pera, 2016) 4.5, (Faltings and Chai, 1990) I.2.7.

Write $f : \mathcal{A}^{\mathrm{KS}} \to \mathcal{S}_K(\mathrm{GSpin}(L_d))$. This allows us to construct bundles $\mathbf{H}_\ell$ with tensor $\boldsymbol{\pi}_\ell$ on $\mathcal{S}_K(\mathrm{GSpin}(L_d))$ using $R^1 f_{\text{ét}*}\underline{\mathbb{Q}}_\ell$. Similarly, we construct $\mathbf{L}_\ell$ on $\mathcal{S}_K(\mathrm{SO}(L_d))$ by $\boldsymbol{\pi}_\ell(\mathbf{H}_\ell)$.

4.6 The period morphism

To discuss the moduli of K3 surfaces, we begin by extending Definition 3.1.1 to families. We also introduce some additional structure for our moduli problem to allow us to relate it to Shimura varieties.

Definition 4.6.1. A family of K3 surfaces $f : X \to S$ over a scheme S is a smooth and proper algebraic space such that the geometric fibres are K3 surfaces. A polarisation is a section $\xi \in \underline{\mathrm{Pic}}(X/S)(S)$ whose fibre at each geometric point is an ample line bundle. A quasi-polarisation is a section $\xi \in \underline{\mathrm{Pic}}(X/S)(S)$ whose fibre at each geometric point is a big and nef line bundle. If $\xi(s)$ is degree $2d$ for all geometric points $s \to S$, we say ξ has degree $2d$. If $\xi(s)$ is primitive for all geometric points $s \to S$, we say ξ is primitive.

We have a functor which sends each $\mathbb{Z}[1/2]$-scheme S to the groupoid of pairs $(f : X \to S, \xi)$ where $f : X \to S$ is a K3 surface and ξ is a primitive quasi-polarisation of deg $2d$. This is represented by a Deligne-Mumford stack M_{2d} of finite type over $\mathbb{Z}$. There is an open substack M°_{2d} on which ξ is a polarisation.

We have the universal object $(f : \mathcal{X} \to M_{2d}, \boldsymbol{\xi})$ over M_{2d}. We get vector bundles $\mathbf{P}^2_\ell$ over $M_{2d,\mathbb{C}}, M_{2d,\mathbb{Z}[1/2\ell]}$, respectively, arising from the second primitive cohomology of $\mathcal{X}$. That is $\mathbf{P}^2_\ell = \langle \mathrm{cl}(\boldsymbol{\xi}) \rangle^\perp$, i.e. the complement of the class of $\boldsymbol{\xi}$ in $R^2 f_{\text{ét}*}\underline{\mathbb{Q}}_\ell$.

For a morphism $T \to M_{2d}$ we may pull back the universal object $\mathcal{X}$ and vector bundle $\mathbf{P}^2_\ell$ to get $\mathbf{P}^2_T$. We may now introduce the notion of a γ-orientation.

Definition 4.6.2. Fix $\gamma \in \det(L_d)$. Let $\tilde{M}_{2d,\gamma}$ over $\mathbb{Z}[1/2]$ be the Deligne-Mumford stack such that for a scheme T, the points in $\tilde{M}_{2d}(T)$ are pairs $(T \to M_{2d}, \boldsymbol{\beta}_T)$, where $\boldsymbol{\beta}_T \in H^0(T, \det(\mathbf{P}^2_T))$ such that $\langle \boldsymbol{\beta}_T, \boldsymbol{\beta}_T \rangle$ is the constant section $\langle \gamma, \gamma \rangle$. We call $\tilde{M}_{2d,\gamma}$ the space of *γ-oriented quasi-polarised K3 surfaces of deg $2d$.*

The map that forgets $\boldsymbol{\beta}_T$ gives a degree 2 étale cover $\tilde{M}_{2d,\gamma} \to M_{2d}$. This map admits a non-canonical section.

As in (Milne, 1994b) Proposition 3.10, (Madapusi Pera, 2015) Proposition 3.3, the Shimura variety $\mathrm{Sh}(\mathrm{SO}(L_d))_\mathbb{C}$ admits a moduli interpretation in terms of certain Hodge structures. In particular, it admits a universal property which induces a map $\iota_\mathbb{C} : \tilde{M}_{2d,\gamma,\mathbb{C}} \to \mathrm{Sh}(\mathrm{SO}(L_d))_\mathbb{C}$ (Madapusi Pera, 2015) Proposition 4.2. The results (Madapusi Pera, 2015) Corollary 4.4, (Rizov, 2005) Theorem 3.16, (Maulik, 2014), Proposition 5.7, allow us to extend $\iota_\mathbb{C}$ first to $\iota_\mathbb{Q}$. Recalling that the integral model $\mathcal{S}(\mathrm{SO}(L_d))$, by Theorem 4.4.4, satisfies the extension property (see Definition 4.4.2), this results in the *period morphism*

$$\iota : \tilde{M}_{2d,\gamma} \longrightarrow \mathcal{S}(\mathrm{SO}(L_d)). \tag{4.18}$$

Remark 4.6.3. To apply Theorem 4.4.4 directly, one needs to introduce K-level structures to $\tilde{M}_{2d,\gamma}$ and check that for each level we get a map $\iota_{K,\mathbb{Q}}$. It is for these that we use the extension property.

We now recall some important properties.

Theorem 4.6.4 ((Madapusi Pera, 2015) Theorem 4.8, Corollary 4.15)**.** *The map ι is étale. For any choice of section $M_{2d} \to \tilde{M}_{2d,\gamma}$, the induced map $\iota : M^\circ_{2d} \to \mathcal{S}(\mathrm{SO}(L_d))$ is an open immersion.*

This also allows for comparisons of bundles, which will be a key ingredient for us to make statements about Galois representations.

Proposition 4.6.5 ((Madapusi Pera, 2015) 4.5)**.** *We have a compatible system of isometries of étale local systems* $\alpha_\ell : \iota^*\mathbf{L}_\ell \xrightarrow{\sim} \mathbf{P}_\ell^2$ *on* $\tilde{M}_{2d,\gamma}$.

4.7 The Kuga-Satake abelian variety

The setup above leads to the following theorem.

Theorem 4.7.1. *Let* F *be a field of characteristic* $p \neq 2$ *and* (X, ξ) *a polarised K3 surface over* F*. Passing to a finite separable extension if necessary, there exists an abelian variety* A/F*, called the* Kuga-Satake abelian variety *such that:*

1. *A has a $\mathbb{Z}/2\mathbb{Z}$-grading and an action of the Clifford algebra* $\mathrm{Cl}^+(L_d)$.
2. *For each $\ell \neq p$ there is an isomorphism of $\mathbb{Q}_\ell$-vector spaces*

$$H^1(A_{F^s}, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{Cl}^+(PH^2(X_{F^s}, \mathbb{Q}_\ell(1))).$$

3. *The image of the Galois representation* $\tilde{\rho} : G_F \to \mathrm{GSp}(H^1(A_{F^s}, \mathbb{Q}_\ell))$ *lands in*

$$\mathrm{GSpin}(PH^2(X_{F^s}, \mathbb{Q}_\ell)) \subseteq \mathrm{GSp}(H^1(A_{F^s}, \mathbb{Q}_\ell)).$$

4. *The image of the Galois representation* $\rho : G_F \to \mathrm{GL}(PH^2(X_{F^s}, \mathbb{Q}_\ell(1))$ *lands in*

$$\mathrm{SO}(PH^2(X_{F^s}, \mathbb{Q}_\ell)) \subseteq \mathrm{GL}(PH^2(X_{F^s}, \mathbb{Q}_\ell(1))).$$

Further, we have an equality $\mathrm{ad} \circ \tilde{\rho} = \rho$*, where* $\mathrm{ad} : \mathrm{GSpin} \to \mathrm{SO}$ *is the adjoint map.*

Proof. A form of this theorem appears as Theorem 4.17 in (Madapusi Pera, 2015). Properties 1 and 2 can be read off directly from that theorem.

Consider the diagram

$$\begin{array}{ccc} & \mathcal{S}(\mathrm{GSpin}(L_d)) & \longrightarrow \mathcal{S}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)) \\ & \downarrow & \\ \tilde{M}_{2d,\gamma} \longrightarrow & \mathcal{S}(\mathrm{SO}(L_d)) & \end{array} \tag{4.19}$$

over $\mathbb{Z}_{(p)}$. We begin with a F-valued point of M°_{2d}. This lifts to a point $s \in \tilde{M}_{2d,\gamma}$. After passing to a finite extension of F, we may lift $\iota(s) \in \mathcal{S}(\mathrm{SO}(L_d))$ to an F-point $\widetilde{\iota(s)} \in \mathcal{S}(\mathrm{GSpin}(L_d))$. Taking the fibre of $\mathcal{A}^{\mathrm{KS}}$ at this point gives the abelian variety $A = \mathcal{A}^{\mathrm{KS}}_{\widetilde{\iota(s)}}/K$.

To study the Galois representation on $H^1(A_{F^s}, \mathbb{Q}_\ell)$, we first introduce some notation. For a group scheme $G/\mathbb{Z}$ we define a subgroup $G(N) \subseteq G(\hat{\mathbb{Z}})$ by the exact sequence

$$1 \longrightarrow G(N) \longrightarrow G(\hat{\mathbb{Z}}) \longrightarrow G(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 1. \tag{4.20}$$

For a subgroup $H \subseteq G(\mathbb{A}_f)$ we set $H(N) = H \cap G(N)$.

Let $K \subseteq \mathrm{GSpin}(L_d)(\mathbb{A}_f)$ and $\mathcal{K} \subseteq \mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)(\mathbb{A}_f)$ be compatible choices of open compact subgroup, sufficiently small such that $\mathcal{S}_{\mathcal{K}}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta))$ acquires its moduli description. For N such that $p \nmid N$, consider the diagram

$$\begin{array}{ccc} \mathcal{S}_{K(N)}(\mathrm{GSpin}(L_d)) & \longrightarrow & \mathcal{S}_{\mathcal{K}(N)}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)) \\ \downarrow{\scriptstyle K/K(N)} & & \downarrow{\scriptstyle \mathcal{K}/\mathcal{K}(N)} \\ \mathcal{S}_{K}(\mathrm{GSpin}(L_d)) & \longrightarrow & \mathcal{S}_{\mathcal{K}}(\mathrm{GSp}(\mathrm{Cl}^+(L_d), \psi_\delta)) \end{array} \tag{4.21}$$

where the vertical maps are a $K/K(N)$ torsor and $\mathcal{K}/\mathcal{K}(N)$ torsor, respectively. The fibre of such a vertical map above a point s with corresponding abelian variety A parameterises bases for the N-torsion $A[N]$ which extend the given $\mathcal{K}$-level structure.

Returning to the situation of the theorem, we consider the image of the F-point $\widetilde{\iota(s)}$ in $\mathcal{S}_K(\mathrm{GSpin}(L_d))$. The action of the Galois group G_F on $\mathcal{S}_{K(N)}(\mathrm{GSpin}(L_d))$ preserves the fibre above $\widetilde{\iota(s)}$ and corresponds to the action on $A[N]$. In particular, the action on G_F on $T_\ell A = \varprojlim_n A[\ell^n]$ factors through $\varprojlim_n K/K(\ell^n) \subseteq \mathrm{GSpin}(L_d)(\mathbb{Z}_\ell)$, giving property 3 above.

Finally, by Proposition 4.6.5 we have an identification between $\rho^*\mathbf{L}_\ell$ and $\mathbf{P}^2_\ell$ as bundles on $\tilde{M}_{2d,\gamma}$. Further, we have $\mathbf{L}_\ell = \boldsymbol{\pi}_\ell(\mathbf{H}_\ell)$, in particular the finite cover $\mathcal{S}_{K_0(N)}(\mathrm{SO}(L_d)) \to \mathcal{S}_{K_0}(\mathrm{SO}(L_d))$ is a torsor for $K_0/K_0(N)$, identified with the image of $K/K(N)$ under the adjoint action, giving property 4 above. □

Chapter 5

Main Theorem

5.1 Passing the descent

This begins with the following lemma.

Lemma 5.1.1. *Let K/k be a regular extension. Let X/K be a K3 surface and let ρ be the representation $\rho : G_K \to \mathrm{GL}\,(PH^2(X_{K^s}, \mathbb{Q}_\ell))$ for $\ell \neq \mathrm{char}\,K$. Assume $\rho(\mathrm{Gal}(K^s/k^sK)) = 1$. Passing to a finite extension if necessary, let A/K be the Kuga-Satake abelian variety with $\tilde{\rho} : G_K \to \mathrm{GL}\,(H^1(A_{K^s}, \mathbb{Q}_\ell))$. Then $\tilde{\rho}(\mathrm{Gal}(K^s/k^sK')) = 1$ for some $[K' : K] \leq 2$.*

Proof. By Theorem 4.7.1 property (3), we know that $\tilde{\rho}(G_K) \subseteq \mathrm{GSpin}(L)(\mathbb{Q}_\ell)$. Combining property (4) with the short exact sequence for ad, we have the following commutative diagram.

$$\begin{array}{ccccc} & & \mathrm{Gal}(K^s/k^sK) & & \\ & \swarrow & \downarrow{\scriptstyle\tilde{\rho}} & \searrow{\scriptstyle\rho} & \\ \mathbb{G}_m(\mathbb{Q}_\ell) & \longrightarrow & \mathrm{GSpin}(L)(\mathbb{Q}_\ell) & \xrightarrow{\mathrm{ad}} & \mathrm{SO}(L)(\mathbb{Q}_\ell) \end{array}$$

Hence we see that $\mathrm{ad}(\tilde{\rho}(\mathrm{Gal}(K^s/k^sK))) = \rho(\mathrm{Gal}(K^s/k^sK)) = 1$ and thus

$$\tilde{\rho}(\mathrm{Gal}(K^s/k^sK)) = \ker \mathrm{ad} = \mathbb{G}_m. \tag{5.1}$$

Let $\mu : \mathrm{GSp}(H^1(A, \mathbb{Q}_\ell)) \to \mathbb{Q}_\ell$ be the symplectic similitude character. Then by Proposition 2.1.7 we have $\mu \circ \rho = \chi_{\mathrm{cycl}} : G_{K/k} \to \mathbb{Q}_\ell^\times$, the ℓ-adic cyclotomic character. However, by Lemma 3.3.1 we know that the symplectic similitude character coincides with the spinor norm ν on $\mathrm{GSpin}(L)(\mathbb{Q}_\ell) \subseteq \mathrm{GSp}(H^1(A, \mathbb{Q}_\ell))$. Further, on $\ker \mathrm{ad} \cong \mathbb{G}_m$ we have $\nu|_{\ker \mathrm{ad}} : x \mapsto x^2$. Thus we conclude for $g \in G_{K/k}$ that

$$\tilde{\rho}(g)^2 = \nu(\tilde{\rho}(g)) = \mu(\tilde{\rho}(g)) = \chi_{\mathrm{cycl}}(g).$$

However, note that since $g \in \mathrm{Gal}(K^s/k^sK)$ it fixes k^s which necessarily contains all roots of unity, hence $\tilde{\rho}(g)^2 = \chi_{\mathrm{cycl}}(g) = 1$. Thus the action factors through an extension $[K' : K] \leq 2$. Then $\tilde{\rho}(\mathrm{Gal}(K^s/k^sK')) = 1$, as required. □

Hence, replacing K with K', we conclude that if X satisfies K/k-Galois-ℓ-descent, the same is true for its Kuga-Satake abelian variety A. We can then apply Theorem 2.1.9, Grothendieck's result, to find an abelian variety A_0/k and an isogeny $A_0 \otimes K \to A$.

5.2 The cases of X ordinary or $\mathrm{char}\, K = 0$

We can now prove the remaining cases of the main theorem.

Proof of Theorem 3.2.4. We begin with X/K a K3 surface such that

$$\rho'(\mathrm{Gal}(K^s/k^sK)) = 1, \tag{5.2}$$

where $\rho' : G_K \to \mathrm{Aut}(H^2(X_{K^s}, \mathbb{Q}_\ell))$. Choose any polarisation ξ on X, allowng us to define PH^2, and we can conclude $\rho(\mathrm{Gal}(K^s/k^sK)) = 1$, where $\rho : G_K \to \mathrm{Aut}(PH^2(X_{K^s}, \mathbb{Q}_\ell))$. Let $2d = \deg(\xi)$. Then (X, ξ) gives a K-point $s \in M^\circ_{2d}(K)$. As in Theorem 4.7.1 we have, after finite extension of K, a point $\widetilde{\iota(s)} \in \mathcal{S}(\mathrm{GSpin}(L_d))(K)$ and the Kuga-Satake abelian variety A/K with $\tilde{\rho} : G_K \to \mathrm{Aut}(H^1(A_{K^s}, \mathbb{Q}_\ell))$. By Lemma 5.1.1, after quadratic extension of K we have $\tilde{\rho}(\mathrm{Gal}(K^s/k^sK)) = 1$. Thus, by Theorem 2.1.9, there exists an abelian variety A_0/k and an isogeny $A_0 \otimes K \to A$.

Now assume we are in case (2), i.e. $\mathrm{char}\, K = 0$. Then by Proposition 5.2, in fact we have an isomorphism $A_0 \otimes K \xrightarrow{\sim} A$. We therefore conclude $\widetilde{\iota(s)}$ was in fact a k-point, and after finite extension of k we may assume the same of s.

Now assume we are in case (3), i.e. X/K is ordinary. By (Nygaard, 1983) Proposition 2.5 we have that A/K is an ordinary abelian variety. Then by Proposition , in fact we have an isomorphism $A_0 \otimes K \xrightarrow{\sim} A$. We therefore conclude $\widetilde{\iota(s)}$ was in fact a k-point, and after finite extension of k we may assume the same of s.

Thus, having taken suitable extensions of k and K, s is a k-point and we have X_0/k with an isomorphism $X_0 \otimes K \xrightarrow{\sim} X$. □

For the purpose of this proof it was necessary to restrict to some cases of K3 surfaces. However, in light of Conjecture 2.4.5, we expect it should be possible to adapt these techniques to remove the reliance on these hypotheses. The issue is that outside our hypotheses, we cannot conclude the point $\widetilde{\iota(s)}$ is defined over the subfield k, but merely that it is *isogenous* to such a point. Thus one would need to show that this other point arose from a K3 surface X_0 defined over k which admits an algebraic correspondence from X.

The work of Yang in (Yang, 2020) proves a similar result. Recall that the abelian varieties arising on $\mathrm{Sh}_K(\mathrm{GSpin}(L_d))$ are equipped with an action of the Clifford algebra, $\mathbb{Z}/2\mathbb{Z}$ grading, and a Hodge tensor π. The notion of a *spin isogeny* is introduced, which is an isogeny of such abelian varieties which preserves the Clifford algebra action, the grading, and the tensor π. What is then shown is that if one has two K3 surfaces over a finite field $\mathbb{F}_q$ such that there exists a spin isogeny between their Kuga-Satake abelian varieties, then there exists a correspondence between those K3 surfaces.

To remove the hypotheses from our theorem, it would require first extending Yang's result to an arbitrary field of positive characteristic. The key step would then be to show that for an abelian variety arising on $\mathrm{Sh}_K(\mathrm{GSpin}(L_d))$ such that the K/k trace is an isogeny, it is further a spin isogeny.

References

Artin, M. and Mazur, B. (1977). Formal groups arising from algebraic varieties. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 10(1):87–131.

Chow, W.-L. (1955). Abelian varieties over function fields. *Transactions of the American Mathematical Society*, 78:253–275.

Conrad, B. (2006). Chow's K/k-image and K/k-trace, and the Lang-Néron theorem. *L'Enseignement Mathématique. Revue Internationale. 2e Série*, 52(1-2):37–108.

Deligne, P. (1972). La conjecture de Weil pour les surfaces $K3$. *Inventiones Mathematicae*, 15:206–226.

Deligne, P. (1979). Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 247–289. Amer. Math. Soc., Providence, R.I.

Faltings, G. and Chai, C.-L. (1990). *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin. With an appendix by David Mumford.

Hartshorne, R. (1977). *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg.

Huybrechts, D. (2016). *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge.

Kisin, M. (2010). Integral models for Shimura varieties of abelian type. *Journal of the American Mathematical Society*, 23(4):967–1012.

Lan, K.-W. (2017). An example-based introduction to shimura varieties. *Preprint.*

Lang, S. (1959). *Abelian varieties.* Interscience Tracts in Pure and Applied Mathematics, No. 7. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London.

Madapusi Pera, K. (2015). The Tate conjecture for K3 surfaces in odd characteristic. *Inventiones Mathematicae*, 201(2):625–668.

Madapusi Pera, K. (2016). Integral canonical models for spin Shimura varieties. *Compositio Mathematica*, 152(4):769–824.

Maulik, D. (2014). Supersingular K3 surfaces for large primes. *Duke Mathematical Journal*, 163(13):2357–2425. With an appendix by Andrew Snowden.

Milne, J. S. (1980). *Étale cohomology*. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J.

Milne, J. S. (1983). The action of an automorphism of **C** on a Shimura variety and its special points. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 239–265. Birkhäuser Boston, Boston, MA.

Milne, J. S. (1986). Abelian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 103–150. Springer, New York.

Milne, J. S. (1994a). Motives over finite fields. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 401–459. Amer. Math. Soc., Providence, RI.

Milne, J. S. (1994b). Shimura varieties and motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 447–523. Amer. Math. Soc., Providence, RI.

Milne, J. S. (2005). Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI.

Milne, J. S. (2013). Motives—Grothendieck's dream. In *Open problems and surveys of contemporary mathematics*, volume 6 of *Surv. Mod. Math.*, pages 325–342. Int. Press, Somerville, MA.

Mumford, D. (1970). *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London.

Nygaard, N. O. (1983). The Tate conjecture for ordinary $K3$ surfaces over finite fields. *Inventiones Mathematicae*, 74(2):213–237.

Oort, F. (1973). The isogeny class of a CM-type abelian variety is defined over a finite extension of the prime field. *Journal of Pure and Applied Algebra*, 3:399–408.

Rizov, J. (2005). Complex multiplication for k3 surfaces. *arXiv preprint math/0508018*.

Shioda, T. (1974). Kummer surfaces in characteristic 2. *Proceedings of the Japan Academy*, 50:718–722.

Silverman, J. H. (1994). *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

Silverman, J. H. (2009). *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition.

Vasiu, A. (1999). Integral canonical models of Shimura varieties of preabelian type. *Asian Journal of Mathematics*, 3(2):401–518.

Yang, Z. (2020). Isogenies between K3 surfaces over $\bar{\mathbb{F}}_p$. *International Mathematics Research Notices*.

Yu, C.-F. (2004). The isomorphism classes of abelian varieties of CM-type. *Journal of Pure and Applied Algebra*, 187(1-3):305–319.

CURRICULUM VITAE

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