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The place and teaching of calculus in secondary schools

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The Place and Teaching of Calculus in Secondary Schools

Submitted by

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The Place and Teaching of Calculus

in

Secondary Schools

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I

Justification for such a Course

A. Curricula

The past two decades have seen many changes in the mathematics of secondary schools both in aim, method, and content. This movement to vitalize mathematics and make it more appreciative was begun in 1901 by Professor Perry who was at that time in charge of certain apprenticeship schools in London and who felt that the traditional mathematics of those schools meant little to the pupils in their later work. The reform movement in this country was started by Professor E. H. Moore of the University of Chicago. In 1903 associations of mathematics teachers were formed in various parts of the United States and mathematics magazines were established in order to spread the movement among classroom teachers. After twenty years of persistent effort on the part of the reformers, we can see a marked change in aim, method, and content of mathematical curricula through new types of text books, efforts of progressive teachers, and through investigations and reports of national and international committees culminating in the Reorgatization of Mathematics in Secondary Education by the National Committee on Mathematics Requirements under the auspices of the Mathematics Association of America. This noteworthy publication furnishes a national basis from which future improvement is assured.

In this movement the reformers have fixed their attention chiefly upon the junior high school period to make the mathematics of these years more meaningful and to bridge the gap between the junior and senior high school periods through the reorgatization of the mathematics of the ninth school year. Although several years are necessary to determine the most valuable content and method of organization, countless numbers of teachers have come to feel a need for breaking away from the traditional curricula and, no doubt, are generally in accord with the general aims and methods of mathematical instruction.

The reorganization of mathematics of the senior high school has taken place much more slowly because of the conservatism of secondary school teachers and because of the requirements for college entrance. Now that the College Entrance Board has abridged the requirements, more time may be given to initiative and originality in teaching. If a desire for reorganization is stimulated and only a small degree of success is apparent as yet, this "sign of the times" is a healthy one. However, many experimental[1] and public[1] high schools have reported excellent results in reorganized courses.

In the reorganization of senior high school mathematics the correlation has been between algebra and geometry, and the formula and the graph. The introduction of the elementary calculus must of necessity be the culmination of the high school course,-- a natural outgrowth of a revision of courses below. An examination of thejunior high school courses as suggested

[1] See Reorganization of Mathematics in Secondary Education

by the National Committee on Mathematics Requirements will seek and reenforce a revision of senior high school mathematics. The junior high school plans are listed below from the bulletin[1]

Plan A.

First year: Applications of arithmetic, particularly in such lines as relate to the home, to thrift, and to the various school subjects; intuitive geometry.

Second year: Algebra; applied arithmetic, particularly in such lines as relate to commercial, industrial, and social needs.

Third year: Algebra, trigonometry, demonstrative geometry.

By this plan the demonstrative geometry is introduced in the third year, and arithmetic is practically completed in the second.

PlanB.

First year: Applied arithmetic(as in plan A); intuitive geometry.

Second year: Algebra, intuitive geometry, trigonometry.

Third year: Applied arithmetic, algebra, trigonometry, demonstrative geometry.

By this plan trigonometry is taken up in two years, and the arithmetic is transferred from the second year to the third.

Plan C.

First year: Applied arithmetic, intuitive geometry, algebra.

Second year: Algebra, intuitive geometry.

Third year: Trigonometry, demonstrative geometry, applied arithmetic.

By this plan algebra is confined chiefly to the first two years.

Plan D.

First year: Applied arithmetic, intuitive geometry

Second year: Intuitive geometry, algebra.

[1] See Reorganization of Mathematics in Secondary Education p.29

Third year: Algebra, trigonometry, applied arithmetic.

By this plan algebra is confined chiefly to the first two years.

Plan E.

First year: Intuitive geometry, simple formulas, elementary principles of statistics, arithmetic.

Second year: intuitive geometry, algebra, arithmetic.

Third year: Geometry, numerical trigonometry, arithmetic.

It seems to have been the tendency in this reform movement to eliminate from the junior high school all material which has no direct bearing upon the pupil's later mathematical career, or such material that is meaningless to the child at that particular stage of his career.

The following plans for reorganizing the mathematics curricula of senior high schools are suggested by the committee. As in the previous outline, no one plan is considered superior to the other.

Plan A.

First year (tenth school year): Plane demonstrative geometry, algebra.

Second year (eleventh school year): Statistics, trigonometry, algebra.

Third year (twelfth school year): Calculus, other elective.

Plan B.

First year: Plane geometry, solid.

Second year: Algebra, trigonometry, statistics.

Third year: Calculus, other elective.

Plan C.

First year: Plane geometry, trigonometry.

Second year: Solid geometry, algebra, statistics.

Third year: Calculus, other elective.

Plan D.

First year: Algebra, statistics, trigonometry.

Second year: Plane geometry, solid geometry.

Third year: Calculus, other elective.

Those who have had any influence in bringing about these changes in the curricula of the junior and senior high schools feel that it is better to substitute the simpler parts of higher mathematics for the more difficult parts of arithmetic, algebra, and plane and solid geometry. The student who does not go on to college has a richer and more useful knowledge of mathematics and the one who does go on has a broader outlook upon subsequent mathematical courses.

Before justifying further the place of the elementary calculus in the senior high school, we might profit by a survey of the mathematical curricula of foreign countries that correspond to our high schools. From a careful study of authoritative reports[1] we may draw these general conclusions:

Tenth school year(15-16 years of age approximately:
Realschule of Russia teaches plane analytics and elements of the infinitesimal calculus; the Realgymnasium and Oberrealschule of Germany, analytic geometry.

Eleventh school year (16-17 years of age) : Introduction to analytic geometry in the Gymnasium in Austria is given and a more difficult course in the Realschule. In the latter school easy differential and integral calculus is introduced to simplify or make more intensive the knowledge of physics. In the Oberrealschule in Germany analytic geometry appears;in the curricula of the three main schools of Austria, of the Lycees ofFrance, of the Realschule and Gymnasium of Hungary and of the Realgymnasium of Sweden.

Twelfth school year(17-18 years of age): Analytic geometry is taught in the Gymnasium schools of Holland and Hungary; analy-

[1] International Committee on Teaching of Mathematics; Place of Elementary Calculus in the Senior High School by Rosenburger.

tic geometry and the calculus in Austria, Belgium, Denmark, England, France, Germany, Roumania, Sweden, and Switzerland. The work in these two subjects range from simple introductory course to coursesequivalent to our first course in college.

From these reports of the International Committee on the Teaching of Mathematics, it seems that pupils abroad are by the end of the twelfth school year about two years ahead of our American pupils because of a more profitable arrangement of mathematics curricula in their schools. It may be contended that these foreign schools have a select group to work with but let it be remembered that any course in mathematics beyond the ninth or tenth school year, of necessity, calls for a select group,-- those who are preparing for college and others who are mathematically inclined.

B. Teachers.

In making a revision of our curricula in mathematics, we must consider the training of our teachers. Even junior high school teachers should have the minimum requirements of senior high school teachers in order to obtain a proper appreciative attitude toward this elementary content. The department of mathematics of any reputable college, like any other department, has a definite course of study for students who major in that department and in addition offer special courses for prospective teachers. The Department of Mathematics at Brown University[1], for instance, has layed down a minimum course of study for prospective teachers of mathematics who wish the backing of the department in entering their career:

[1]Bureau of Education Bulletin No. 7, 1917 by Raymond C. Archibald, Associate Professor of Mathematics. Brown University.

Plane trigonometry	3	semester hours
Higher algebra	3	
Solid geometry	3	
Plane analytic geometry	4	
Differential and Integral Calculus	8	
Teacher's Course in algebra	6	
Teacher's Course in geometry	6	
Total	33	

Since there are more teachers to select from than formerly and since teachers nowadays are more inclined to seek improvement and advancement through attendance atsummer sessions, it is very probable that there will be a sufficient supply of qualified teachers soon and no doubt there are a sufficient number now in the strong high schools. Moreover, it is needless to say that our mathematics teachers must have sufficient training in education courses also.

It might be worth while to examine the preparation of mathematics teachers of some of the stronger high schools. The following information has been culled from Chapter XII of the Reorganization of Mathematics in Secondary School Mathematics:

Cass Technical High School, Detroit, Michigan; thirteen full time mathematics teachers who average 31.5 semester hours in mathematics and 13.8 in education.

English High School, Boston; seven full time mathematics teachers, five part time, all of whom have training beyond the master's degree.

Horace Mann School, New York City; four mathematics teachers with training beyond the master's degree.

The Rochester, New York , system of training its junior high school teachers of mathematics should be an inspiration to all progressive school systems. First of all the city has a supervisor of junior high school mathematics and then it proceeds to train elementary school teachers in content and method by extension courses or Saturday Institutes. These junior high school teachers have all had courses in intermediate and advanced algebra, plane and solid geometry, trigonometry, and the elements of the calculusgiven at the University of Rochester. A similar method of selecting senior high school mathematics teachers would surely make for improvement here, also.

C. Importance of Calculus

Now that we have made a place for the elementary calculus and believe that we have teachers available, for the larger high schools, at any rate, we may be called upon to justify its place in the curriculum.

Of course, all the mathematics the the average person needs to know may be found in the first nine grades. However, higher mathematics has contributed so much of practical and cultural value to civilization that the very existance of our social fabric depends upon it. Dr. David Eugene Smith[1] tells us that,"if by some chance every trace of mathematical material were removed from the world, every mill in the whole world would slow down and every large concern would close until it could replace its accounts, its statistical material, its formulas for work, its measures, tables, and its computing machinery------every ship in the seven seas would be stricken with

[1] Teachers College Record, May 1917.

blindness. Not a rivet would be driven in a skyscraper in New York City------ Wall Street would close its portals; the engineering world would awaken tomorrow morning to a living death; the mines would shut down, and the trade world would relapse to the conditions of barter as in the days of saveragery."

The practical value of mathematics may be readily appreciated but the cultural value depends upon the method of instruction. Its socializing value rests in the pupil's proper attitude toward the specialist upon whose work our social life depends. Even though calculus should not be of any practical value to any particular child, he should not be denied the privilege of enjoying its pursuit but should have as good a right to develop that ability as any other ability. No one would think of denying one the privilege of developing one's musical ability; then why should one who enjoys mathematics for its own sake, not have the same opportunity to develop his own peculiar gift? Neither may be of any practical value or yield any financial returns, but are both not truth and beauty in different attire?

Now what advantage has the calculus over other branches of mathematics which may be taken during the senior year of the high school course? Let us consider other possibilities. There is analytic geometry but it has no applications; higher algebra including permutations and combinations, theory of equations and theory of determinants does not lead anywhere; projecting geometry although very interesting has no applications. Now an introductory course in the elementary calculus would not only prove exceedingly fascinating to the student fond of mathematics but it would give him a notion of the calculus as a powerful tool on which much of our material civilization depends and would serve as an introduction to later college courses or as a basis for future study under self-instruction. Such a course would also open up the field of pure mathematics and would be readily appreciated as the connecting link between pure and applied mathematics.

II

Subject Matter and Presentation

A. Psychological vs. Logical Method

We have already noted that it is the tendency in all subjects generally to eliminate the more difficult parts of the elementary matter and substitute the easier parts of the more advanced. This arrangement is successful only when we relate the work to the child's experience. Since the mental development of the child follows in general the mental development of the race, the calculus must be presented largely in the way in which it has developed in the minds of men. It will be remembered that the new junior high school mathematics has been successful or unseccessful in proportion as the arrangement of content and method of presentation has been in keeping with the child's experience and point of view. In other words, the psychological rather than the logical method must prevail.

With this background in mind, the calculus may be developed as a continuation of algebra, geometry, and trigonometry or simultaneously with them,-- a unifying agency, as it were, of these three provinces.

In the first place the pupil should be taught ideas, not notations and definitions, and must be shown relationships in concrete cases before he can think in abstract terms. Gradually then, he will form the habit of thinking of relationships between related quantities.

B. Fundamental Notions.

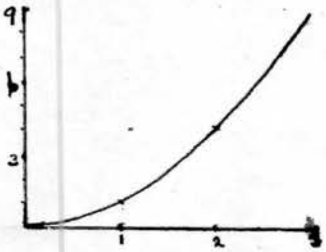
Should we scan our mathematics curricula of the junior and senior high schools and consider our methods of presentation, we would find that we have instilled in our students, consciously or unconsciously, the fundamental ideas of the calculus if we have had the thoroughly appreciative point of view. Besides the function concept, which unifies all mathematics, other ideas incidentally pre-calculus, occur frequently in our regular mathematics courses. The concept of the division of a quantity indefinitely, and the idea of a variable approaching, reaching, or passing through a limit has been readily seen in many instances in demonstrative geometry. For example, inscribed and circumscribed polygons approach the circumference of the circle as a limit, if their sides are indefinitely increased; a variable chord of a circle passes through the diameter as its maximum and becomes a point at its minimum; the tangent is the limiting position of the secant; the lower and upper limits of a variable angle formed by the intersection of two tangents, a tangent and a secant, or two secants, are 0° which the variable reaches and 180° which it passes through.

Graphic methods also furnish a good basis for future work. Here the idea of dependence of one quantity upon another is constantly made use of and is readily grasped by pupils if presented in the right way. This function concept is very important in solving problems in algebra. The pupil may often see that some relationship exists but often does not realize that he must first find a basis of relationship or starting point, so to speak, upon which the other quantities depend. If this methodof attack is put into the student's hands, the difficulties of problem solving are greatly lessened and the function concept becomes more vital and meaningful.

C. Calculus Methods

a. Differentiation

1. Related Quantities. As an introduction to calculus methods it would be well to begin with a discussion of related quantities and their graphic representation. For example, let us take the formula for the area of a square $A=x^2$. Let us show this relationship by means of a graph. This shows pictorially



that for every value of x, we get a definite value for A.

2. Differences. Now suppose we have a square sheet of metal x in. on a side. Let it be heated so that it expands slightly and becomes (x+h) in. on a side. In the figure below, the original

area is x^2, the increase is $2hx+h^2$. Now suppose we apply this example numerically, let x = 4, h = .006,

$2hx + h^2 = 2\cdot4\cdot.006 + .006$

Since the h^2 in this case is so much smaller than the other term, it may be omitted without appreciable error. We may then call a quantity containing h a small quantity[1] of the first order, one containing h^2 a small quantity of the second order, etc. Then if h is a very small fraction of x, second and higher orders may be omitted.

It might be well to work this example algebraically, too.

x^2 = Old area

$(x + h)^2 = x^2 + 2hx + h^2$ = New area

The increase in area = new area - old = $2hx + h^2$, or omitting the second order, difference in area is

$2hx = 2x(\text{diff } x)$; where diff x means simply the difference between the two values of x.

Let us take another example. The formula which expresses the relationship between the length of a pendulum and the time of swing is $L = 39\ T^2$ approximately. Find the relationship between the increase in length and the increase in the time of swing.

$$\text{New } L = 39(T + h)^2 = 39\ T^2 + 78\ Th + 39\ h^2.$$

$$\text{Old } L = 39T^2$$

$$\text{Diff } L = 78Th + \text{second order term}$$

$$= 78\ T\ (\text{Diff } T)$$

To apply this numerically, suppose a clock pendulum makes a swing in .999 sec. instead of one second, making a gain of nearly 1 1/2 min. a day. How much should the pendulum be lengthened?

[1] The foundation of this work on the calculus of approximations rests upon Brewster's Common Sense of the Calculus.

$T = .999$ and Diff $T = 1.000 - .999$

Diff $L = 78 \cdot 1 \cdot .078$, which is about 1/13 in.

Then the clock will keep time if the bob is lowered 1/13in.

It will be noticed that the above results are not exact because of the omission of the second order temm. This term is not only snall, but small in comparison with the first order term.

$\frac{\text{2nd order}}{\text{1st order}} = \frac{39h^2}{78Th} = \frac{h}{2T} = \frac{.001}{.999} = \frac{1}{999}$ which in practice is a very small fraction.

It should also be impressed upon the student that the data is obtained from measurement and is therefore at best subject to personal errors.

The student should be given other examples of similar character and should be encouraged to make up examples of their own, for in so doing the calculus ideas become more significant.

Here are some suggestions:

(1) What is the increase in the area of a circle when the radius is increased slightly?

(2) Find the increase in the area of a right triangle when the altitude receives a small increase.

(3) Find the increase in the volume of a sphere when the radius is increased slightly.

After the student has grasped these elementary ideas through experience, as it were, he will welcome more compact notations and general rules. We might introduce at this point the symbol ΔA for difference in area, Δx for difference in length of side, ΔL for Diff L (which means the difference between the two values of the variable), etc. The pupil might compare the results of all his examples and see if he can discover any general rule which applies to all results.

From the above examples

$A = x^2$ $\qquad A = 2hx\,\Delta x$

$L = 39\ T^2$ $\qquad L = 78T\,\Delta L$

$A = \pi r^2$ $\qquad A = 2\pi r\,\Delta r.$

$A = 1/2\ bh$ $\qquad A = 1/2\ bh\,\Delta h$

$V = 4/3\,\pi r^3$ $\qquad V = 4\pi r^2\,\Delta r$

the student is now able to write down at sight the differential (shorter name for Diff A, etc.) since the differential of any term is found by multiplying by the index, diminishing the index by one, and multiplying by Δx.

The student should now have considerable practice in manipulation and should be encouraged to bring in original problems for which he will have to search through his previous mathematical equipment.

3. Applications.

Suppose we consider another type of problem,-- a speed problem. Every one knows the ordinary conception of speed or rate as the distance per unit of time (hours, minutes, seconds, etc.) or algebraically speed = $\frac{\text{distance}}{\text{time}}$. Thus if a car goes 35 mi. in 2 hrs., it goes at the rate of 35/2 or 17.5 mi. per hr. provided the speed is constant. However, this result is only an average rate and is accurate for any given interval only when the speed has been constant. For the most part, such would not be the case. One might be going at the rate of 50 mi. in one given interval and 20 mi. in another.

Example: Suppose you are in an automobile going at the rate of 45 mi. per hour, which is equivalent to 66ft. per second.

If the speedometer needle remains steady, you are going at the rate of 660 ft. in the next 10 sec., 330 ft. in 5 sec., and 66 ft. in one sec., etc. Since the needle fluctuates slowly any of these results will be approximately true. However, the shorter the interval of time, the less likely is the needle to move perceptibly. Thus the shorter the distance, the more accurate the result.

Let S = total distance car has gone in t sec.

Then ΔS = distance it goes in Δt sec.

If Δt is taken short enough , the speed V feet per sec. will be practically constant for the given time and

$\Delta S = V \cdot \Delta t$ or $V = \frac{\Delta S}{\Delta t}$ very nearly.

Example 2. The formula for any body falling in space is given by the formula $S = 1/2\ gt^2$ where S is the distance fallen, t the time in seconds, g the acceleration of gravity 32 ft/sec.

$$\text{Then } S + \Delta S = 16(t + \Delta t)^2$$

$$= 16(t^2 + 2t\Delta t + \Delta t^2)$$

$$S = 16t^2$$

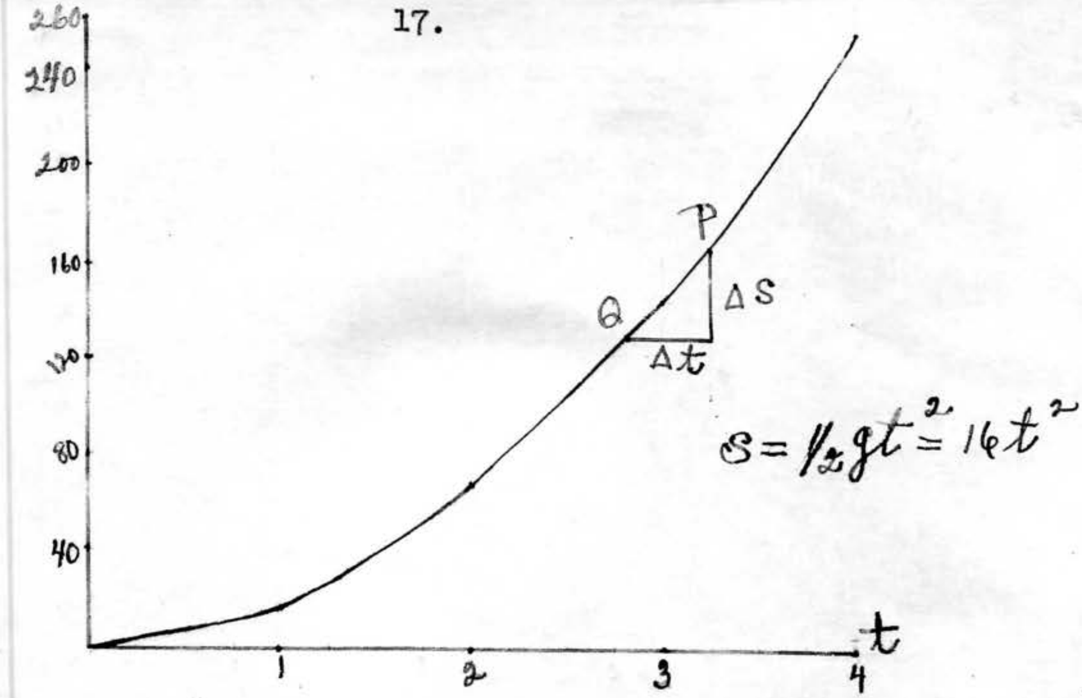
$$\text{Subtracting, } S = 32t\Delta t + 16\Delta t^2$$

The smaller the interval Δt, the smaller will $(\Delta t)^2$ become, and the more closely will $32t\Delta t$ approach the ideal value. Then

S will equal $32t\Delta t$ or

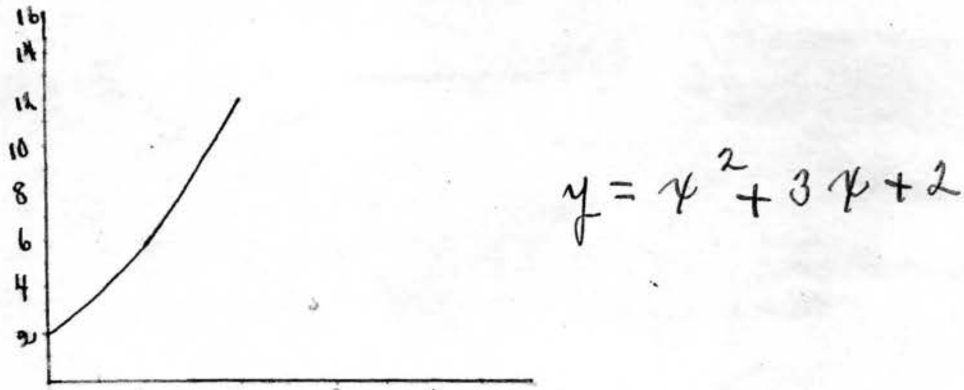
$\frac{\Delta S}{\Delta t} = 32t$ which gives the rate for any value of t.

Plotting this relationship between time and distance in this problem, we have the following curve:



Now from the figure let us consider the relationship geometrically. Take any point Q on the curve and a short interval of time Δt which will give the correspondingly short distance ΔS as shown on the graph. Considering Q P a straight line (which it is very approximately) we have $\frac{\Delta S}{\Delta t}$ which ratio is the tangent of the angle at Q, the slope or gradient of the curve at Q, or the speed or rate at this point.

4. Method of Limits. After the student has had sufficient work in the calculus of approximations, he may be given the general method of working out the slope of a curve. Let us plot the curve $y = x^2 + 3x + 2$. We wish to find a general method of getting the slope at any point.



Let us find the change in y as x changes slightly.

$$y + \Delta y = (x + \Delta x)^2 + 3(x + \Delta x) + 2$$

$$= x^2 + 2x\Delta x + (\Delta x)^2 + 3x + 3\Delta x + 2$$

Subtracting $y = x^2 \qquad\qquad + 3x \qquad + 2$

$$\Delta y = 2x\Delta x + (\Delta x)^2 + 3\Delta x$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 2x + 3 + \Delta x$$

We have seen hitherto that the smaller Δx is taken, the more accurate our results. Suppose we make use of our knowledge of limits, since Δx is a changing quantity, and let Δx approach 0. Then Δy will also approach 0 but not at the same rate, of course, unless $\Delta x = \Delta y$ (in which case $\frac{\Delta y}{\Delta x} = 1$). The difficulty here will be in making the pupil see that $\frac{\Delta y}{\Delta x}$ approaches a finite quantity (not 0) as Δx and Δy approach 0. Give him numerical work like the following until he is satisfied that the ratio of two infinitesimals is finite. From the equation $y = x^2$, $\Delta y = 2x\Delta x + (\Delta x)^2$.

x	y	Δx	Δy	$\frac{\Delta y}{\Delta x}$
.3	.09	.3	.30	$\frac{.40}{.40} = 1.000$
		.2	.16	$\frac{.16}{.20} = 0.80$
		.1	.07	$\frac{.07}{.10} = 0.70$
		.05	.0325	$\frac{.0325}{.05} = 0.65$
		.04	.0256	$\frac{.0256}{.04} = 0.64$
		.02	.0124	$\frac{.0124}{.02} = 0.62$
		.01	.0061	$\frac{.0061}{.01} = 0.61$
		.005	.003025	$\frac{.003025}{.002} = 0.602$
		.002	.001204	$\frac{.001204}{.002} = 0.602$
		.001	.000601	$\frac{.000601}{.001} = 0.601$

These show that the smaller Δx, the nearer $\frac{\Delta y}{\Delta x}$ comes to .6 which is the limit of the ratio $\frac{\Delta y}{\Delta x}$.

Going back to the previous example, let us denote the limit of this ratio $\frac{\Delta y}{\Delta x}$ by $\frac{dy}{dx}$. Then the limit of $\frac{\Delta y}{\Delta x}$ $(\text{Lim} \frac{\Delta y}{\Delta x}) = 2x + 3$ for Δx in the right hand member disappears entirely since $\Delta x \to 0$.

The method may now be made more general by differentiating the general polynomial $y = ax^n + bx^{n-1} + \text{-----} + K$ which gives $\frac{dy}{dx} = anx^{n-1} + b(n-1)x^{n-2} + \text{---}$ as the formula for differentiating an algebraic polynomial.

The student should be given further work in manipulation. Problem solving would, no doubt, be more valuable since it makes the work seem more worth while.

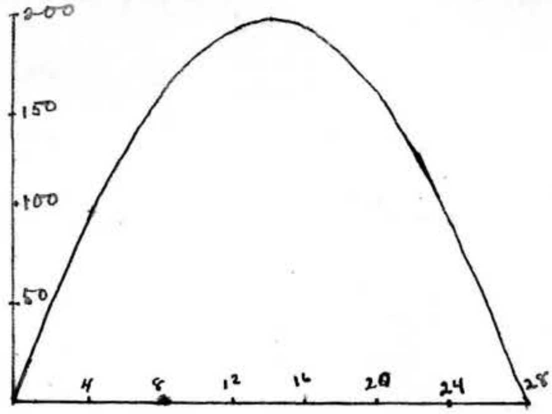
5. Applications. a. Maxima and minimum.

Maxima and minimua problems are always stimulating even though they are worked out by cumbersome methods. Suppose we have a wire 56 ft. long with which we wish to fence in a plot of ground. What length and width will give the largest area?

Length + width = 1/2 x 56 = 28. Therefore we must find two numbers whose sum is 28 and whose product is the largest possible.

4 x 24 = 96 sq. ft.
8 x 20 = 160
12 x 16 = 192
13 x 15 = 195
14 x 14 = 196
15 x 13 = 195
16 x 12 = 192

The dimensions which give the largest area are 14 x 14. Now let us work this out algebraically and graphically.



x = width

$28 - x$ = length

$x(28 - x) = A$

$28x - x^2 = A$

From the graph we find that the curve continually rises until it reaches its maximum x = 14, A = 14, and then decreases correspondingly. Let us differentiate the function and see what happens.

$dA = (28 - 2x)dx$

$\frac{dA}{dx} = 28 - x$ which on the graph is 0 at the minimum value of A and which means that the slope of the curve is 0 at this value. So setting $\frac{dA}{dx} = 0$ and solving for x we have

$28 - 2x = 0$

$x = 14$

$28 - x = 14$

Other examples which are less easily solved by arithmetic should be given here so that the student may realize the superiority of the calculus method. For instance,

(1) Find the maximum rectangle that can be inscribed in a circle.

(2) Find the volume of the largest right circular cone which can be inscribed in a sphere of diameter 15 in. ($V = 1/3 \;\; r^2h$).

(3) Divide 24 into two parts so that the product of the square of one part multiplied by the other may be a maximum.

(4) Find a general rule which will apply to any problem like the above. (Ans. Number must be divided in ratio 2:3)

Similar problems dealing with minimum values may be given and through graphic representation the student may be shown that maximum and minimum mean greatest and least in a given vicinity and that there may be more than one value for each in the given curve.

b. Rate of change.

In the equation $S = 1/2\ gt^2$, we have found that the velocity or speed is not uniform but is constantly changing.

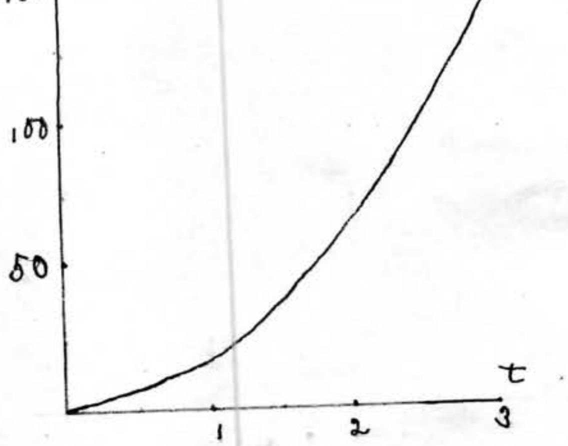
32 ft./sec. at end of first sec.

64 ft./sec " " " second "

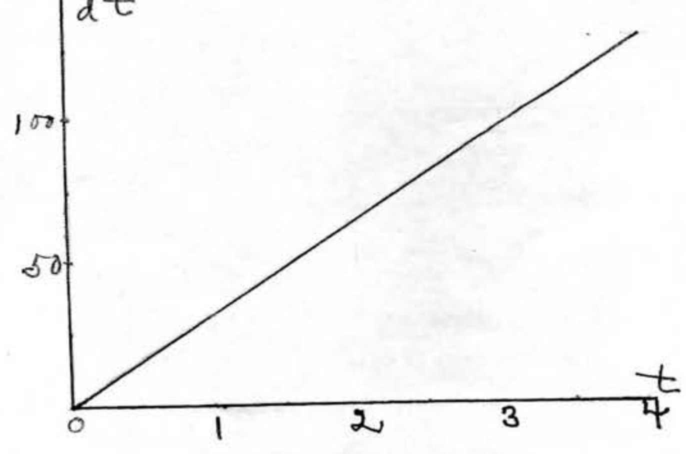
96 ft./sec. " " " third ", etc.

The rate of change then is 32 ft. per sec. or a uniform change in the speed.

Let us plot the curve $S = 1/2\ gt^2$ as we did before and beside it the curve $\frac{dS}{dt} = 32t$.



$S = \frac{1}{2}gt^2 = 16t^2$



$\frac{dS}{dt} = 32t$

In the first graph we see that the slope of the curve (which we have found to be the same as the speed) is changing at every point of the curve or at every instant of time; in the second curve, the rate curve, we see that the rate of the change in the speed is constant since the slope of the curve is constant.

Examples:

(1) The area of a circular piece of metal is expanding by heat. Compare the rates of increase of area and radius.

(2) A cube of 5 in. edge is expanding. The edge is increasing at the rate of .003 in. per min. At what rate is the volume increasing?

(3) A spherical rubber bag is pumped with air at the rate of 8.4 cu. in. per sec. Find the rate of increase of its diameter when it measures 6 ft.

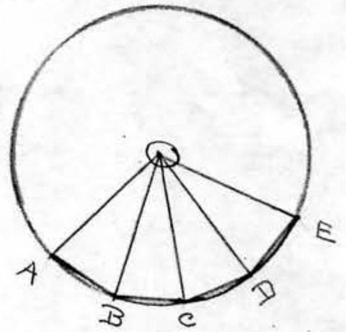
It may be possible that students will find acceleration or rate of change in velocity confusing at first. In such an event it would be advisable to postpone it for a later date,- until they take it up again in the infinitesimal calculus.

b. Integration.

1. Summation - A kind of addition.

Integration, as the students might guess from its derivation, may be defined as a "putting together" of the various parts of a whole to make it complete, i. e. it may be represented as a special kind of addition, the reverse of differentiation which we found to be a special kind of subtraction.

Let the circle O be cut into a large number of equal triangular strips. Draw AB, BC, CD, etc. to form the triangles AOB, BOC, COD, etc.



Area of AOB = 1/2 AB·p

BOC = 1/2 BC·p

COD = 1/2 CD·p, etc.

Area of all the triangles is

1/2 AB·p + 1/2 BC·p + 1/2 CD·p + --- or

1/2 p(AB + BC + CD + ---) = 1/2 perimeter of polygon ABCD ---.

Now let the sides of the polygon and consequently the number of triangles increase indefinitely. Then the sum of the areas of the triangles will approach the area of the circle as a limit and the perimeter of the polygon approaches the circumference of the circle as a limit. Hence,

Area of circle = 1/2 circumference x radius

It is quite possible that the students may have forgotten their plane geometry underlying the above discussion, in which case it would be well to review a few propositions which have to do with the theorem of limits.

2. Summation - Limit of a sum

Using our notations of differentiation, we may express this problem thus:

Let the area of each little triangle be denoted by ΔA and each little bit of perimeter by ΔP, then

$A = 1/2\ p\Delta P$

Adding together all these little ΔA's we have

Sum of ΔA's = sum of $1/2\ p\Delta P$'s. Now let the number of ΔA's increase indefinitely and we see that

Limit of sum of ΔA's = area of ⊙. Putting this idea in more convenient form,

$\text{Lim}\ \Delta A = \text{Lim}\ 1/2\ p\Delta P$ as ΔA and ΔP approach 0

$A = 1/2\ pC$ or using $\int$ a kind of S to denote the limit of the sum of the tiny triangles or elements

$\int \Delta A = 1/2\ p \int \Delta P$ or

$A = 1/2\ pC$

3. Derivation of Formula. It might be of interest at this stage to go back to our first examples in differentiation and see if we can obtain a general rule for finding the whole when a little bit or element is given.

$A = x^2$	$A = r^2$
$dA = 2xdx$	$dA = 2\ r\ dr$
$L = 39\ T^2 dt$	$y = ax^n$
$dL = 78\ T\ dt$	$dy = anx^{n-1}$

From these formulas we find that in order to get y from dy, A from dA, etc. we increase the exponent by one and divide the coefficient by the given exponent, or

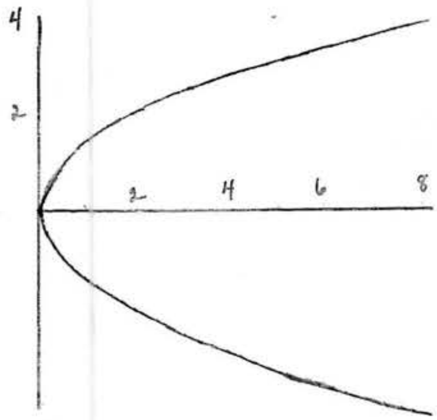
$$y = \frac{anx^{(n-1)+1}}{n} = ax^n$$

At this point the students should have practice in solving simple problems involving the above general formula.

Examples:

(1) Find the area enclosed by the curve $y^2 = 2x$ and the

x-axis. Suppose we let this area be divided up into very thin slices each in height its own y and dx its width. Each one of these tiny slices is very nearly a rectangle whose area is y dx. Let us take the limit of the sum of these rectangles as they increase indefinitely and we will



have $\int dA = \int y dx$, but

$$y = \sqrt{2}\, x^{1/2} \quad dx$$

$$\int dA = \frac{2\sqrt{2}}{3} x^{3/2}$$

Since dA is the sum of the ordinates of the curve, A, the area under the curve must be equal to the sum of the ordinates.

To make this method more general, let us take the general equation $y = f(x)$, then when we integrate as above we will have

for $\int dA = \frac{2\sqrt{2}}{3} x^{3/2} \; dx$

$$\int dA = \int f(x)\, dx \quad \text{or}$$

$$A = f(x)\, dx.$$

It would be a very good exercise for the student to derive this formula step by step as has been done in the example above. Then the student should be given or should make up similar examples for drill in these new ideas.

4. Definite Integral

Now if we wish to find only a part of the area under a curve, we may integrate between definite limits. Suppose we are required to find the area in the above curve between the points a and b, any two points chosen on the curve. Obviously, the area would be equal to the sum of the ordinates between these two points, a and b, or using the calculus notation,

$$A = \int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

The student should be shown this graphically and should be given problems involving the integration of plane figures, as the circle, ellipse, parabola, etc.

The integration of solid figures may be taken up now in the ordinary calculus way by getting an element of volume and integrating. Illustrations and examples may be found in almost any text and the simpler ones (as volume of a solid with parallel bases and volume of a solid of revolution such as the student has had in plane geometry) may prove interesting from the calculus point of view. He, of course, should also be given some problems that would be difficult or impossible to do without the calculus in order that he may appreciate the power of the calculus method.

D. The Infinitesimal Calculus.

If the student has grasped the fundamental notions of the calculus as presented in the foregoing discussion, he is now ready for a simple course in the infinitesimal calculus. It is planned that about one semester will be spent on the introductory work (providing, of course, that the student has been previously well grounded in the first three year's work as described in one of the plans suggested on pp.4 and 5; the second semester may be spent on such topics as are taken up in any first course in calculus and as time allows. Among such topics may be included the following:

1. Differentiation of Algebraic Functions

    a. Polynomial
    b. Product
    c. Quotient
    d. Power

2. Differentiation of Transcendental Functions

    a. Trigonometric Functions
    b. Logarithmic Functions

3. Integration

    a. As in introduction
    b. Applications from Geometry and Physics
    c. Length of Plane Curve
    d. Area of Surface of Revolution

4. Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

5. Differential Equations (Very simple ones)

III

Conclusion

We have seen in this discussion that the calculus is the logical outgrowth of present day tendencies in the reorganization of mathematics courses in secondary education and in the improvement of mathematical instruction; that schools abroad corresponding to our high schools have been teaching it successfully for several years; and that those public high schools and experimental schools in this country that have introduced such a course have reported success.

Since well trained teachers are becoming more numerous and competition is growing keener among them, soon there will be a supply,- for the stronger high schools at least,- competent to teach the elementary calculus in secondary schools. As for the importance of the calculus, let us reiterate that science, engineering, and industry are demanding more and more calculus and that America, the greatest manufacturing and engineering nation in the world, needs more trained mathematical minds; as a powerful mathematical tool, it shows the student the far-reaching influences of mathematics upon which the development of civilization has always been dependent; it develops a kind of thought, in its dealings with small changes in related quantities, that is useful in considering everyday problems.

The subject matter and method of presentation as outlined in this treatise need not necessarily be adhered to, for this is but one of the many possible courses for secondary schools. However, one general rule must prevail: To make the course successful, the calculus must be presented to the student as a continuation of his previous mathematical training and must be related to his experience in a common sense way.

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[1]These quotations heading two chapters from Bisacre are quite appropos of this treatise:

"Nothing can be more fatal to progress than a too confident reliance on mathematical symbols; for the student is only too apt to take the easier course and not the fact as the physically reality."(Kelvin and Tait's Natural Philosophy)

"It's not what we hae but what we do wi' what we hae that counts." (Old Scots Proverb)