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Abstract

Shear wave elastography measures shear wave speed in soft tissues for diagnostic purposes. In [1], shear wave speed was shown to depend on prestrain, but not necessarily prestress, in a perfused canine liver. We model this phenomenon by examining incremental waves in a pressurized poroelastic medium with incompressible phases. The analysis suggests novel restrictions on the strain energy functions for soft tissues.

Keywords: incremental waves, poroelasticity, elastography, strain energy function

1 Introduction

Elastic shear wave propagation in soft tissues is of current interest in the field of elastography. Shear wave speed serves as a surrogate for tissue modulus, and is used for diagnosis of various pathologies. Therefore, understanding those parameters which can lead to variations in shear wave speed *in vivo* is necessary. Here the focus is on dependence of shear wave speed on pore pressure, when modeling the soft tissue as a poroelastic solid.

We present a mathematical model for the experimental observations of [1]. In that experiment, a canine liver was re-perfused and submerged in a bath. Pore pressure within the liver was controlled by maintaining pressure in the portal vein. The bath was maintained in two conditions: a constrained condition in which the container was completely sealed; an unconstrained condition in which the container was open to the atmosphere. In each condition, the pressure in the portal vein was controlled, and the shear wave speed was measured as a function of portal vein pressure.

Below we model the liver as a nonlinear poroelastic continuum. We derive a quasistatic solution for the pressurized equilibrium configuration of the medium in both the constrained and unconstrained cases. Then we consider small amplitude shear waves superimposed on the pressurized configuration, and derive a formula for the shear wave speed.

2 Nonlinear poroelasticity model

We use equations governing a poroelastic continuum with incompressible phases, following the beautiful paper [2]. We assume the intrinsic mass density of the fluid and solid phases to be equal:

$$\rho = \rho_f = \rho_s = \text{const} \tag{1}$$

The constitutive assumptions on stress are as follows:

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}^s + \boldsymbol{\sigma}^f && (2)\\ \boldsymbol{\sigma}^f &= -\phi^f p_f \boldsymbol{I} && (3)\\ \boldsymbol{\sigma}^s &= \boldsymbol{\sigma}^{elastic} - \phi^s p_f \boldsymbol{I} && (4) \end{aligned}$$

Equation (4) assumes that the intrinsic pressure in the solid phase is equal to the intrinsic pressure in the fluid phase. This is consistent with the incompressibility assumption. The elastic stress in the solid matrix, $\boldsymbol{\sigma}^{elastic}$, is given in terms of the derivative of a strain energy function.

2.1 Problem formulation

We consider a poroelastic material submerged in a fluid bath. The fluid-solid interface between the poroelastic solid and fluid bath is denoted Γ_{fs}. On an interior boundary of the poroelastic solid, Γ_p, the pore fluid is assumed to be in contact with a second reservoir, in which the pressure can be controlled. Thus the boundary conditions are assumed to be:

$$\begin{aligned} \mathbf{t}^s &= -p_{bath}\boldsymbol{n} && \text{on } \Gamma_{fs} && (5)\\ \boldsymbol{q}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma_{fs}\setminus\Gamma_p && (6)\\ p_f &= p_0 && \text{on } \Gamma_p && (7) \end{aligned}$$

Here we consider p_0 to be a given constant.

Conservation of fluid volume in the incompressible fluid bath and equation (6) give:

$$\int_{\Omega^s} J\,dV = \Delta V_f. \tag{8}$$

Here, $J = \lambda^3$ is the relative volume change of the poroelastic solid.

We consider two different conditions on the fluid bath. In one case, the fluid bath surrounding the poroelastic solid is in a sealed container. In this, the constrained case, $\Delta V_f = 0$. In a second case, the unconstrained case, the container is unsealed and open to the atmosphere, and hence $p_b = 0$ in this case.

2.2 Solution for quasistatic expansion

For a given (positive) p_0, the quasistatic fields for both the constrained and unconstrained cases are given as follows.

Deformation:

$$\boldsymbol{x} = \lambda \boldsymbol{X} \tag{9}$$

Mixture stress:

$$\boldsymbol{\sigma} = \sigma_o(\lambda)\boldsymbol{I} - p_f \boldsymbol{I} \tag{10}$$

Boundary conditions:

$$\Gamma_p : \quad p_f = p_0 \tag{11}$$

$$\Gamma_{fs} : \quad \sigma_o(\lambda) - p_f = -p_{bath} \tag{12}$$

$$\Delta \text{ Volume} : \quad (\lambda^3 - 1)V_0 = \Delta V_f \tag{13}$$

Now we consider the two cases. For the constrained case, $\Delta V_f = 0 \Longrightarrow \lambda = 1$. Hence:

$$p_f = p_{bath} = p_0. \tag{14}$$

For the unconstrained case, $p_{bath} = 0$, thus we find:

$$\sigma_0(\lambda) = +p_f = +p_0. \tag{15}$$

We emphasize the point that in the unconstrained case, the elastic matrix is in isotropic *tension*, which counteracts the isotropic pressure in the pore fluid. The stretch of the solid matrix, $\lambda > 1$, is determined by the solution of equation (15), for any given strain energy function W.

3 Incremental waves superposed on dilatation

To consider incremental waves superposed on the quasistatic deformation, we now consider ($\nabla \boldsymbol{u} \ll 1$):

$$\boldsymbol{x} = \lambda \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{x}, t) \tag{16}$$

Then, using the results of [3] leads to the following expression for the shear wave speed:

$$\frac{c^2}{c_0^2} = \frac{(W_1(\lambda) + W_2(\lambda)\lambda^2)}{\lambda^3(W_1(1) + W_2(1))} \tag{17}$$

Here, $W_\alpha = \frac{\partial W}{\partial I_\alpha}$, where W is the strain energy function, and I_α is an invariant of the Cauchy-Green strain tensor [3].

4 Discussion

Equation (17) is our main result, and has several implications regarding the behavior of the shear wave speed and its dependence on the strain energy function.

- The shear wave speed depends on λ, independent of the pore pressure p_f. *In this case, the wave speed is independent of the applied pressure.*

- For the constrained case, $\lambda \equiv 1$. In this case, the shear wave speed is unchanged by a changing pore pressure.

- For the unconstrained case, $\lambda > 1$, and hence the shear wave speed changes with increasing pore pressure.

- For W_1, W_2 positive constants, then shear wave speed is *decreasing* with increasing λ. The shear wave speed is inversely dependent on λ for the further special case $W_1 = 0$.

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