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Prefactorization and vertex algebras associated to holomorphic fibrations, the toroidal algebra, and averages of unlabeled networks

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PREFACTORIZATION AND VERTEX ALGEBRAS ASSOCIATED TO HOLOMORPHIC FIBRATIONS, THE TOROIDAL ALGEBRA, AND AVERAGES OF UNLABELED NETWORKS

by

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What if sometimes there is no choice
about what to love?

David Foster Wallace

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ABSTRACT

This thesis consists of two distinct parts. The first concerns prefactorization and vertex algebras associated to holomorphic fibrations and the second describes a notion of averaging on the space of unlabeled networks. Factorization algebras provide a geometric encoding of the algebra of observables and their symmetries in perturbative quantum field theory. Vertex algebras provide a concrete algebraic realization of the symmetry algebra of two dimensional conformal field theories. A theorem of Costello-Gwilliam connects these two worlds. Given a translation invariant holomorphic (pre)factorization algebra on the complex plane, one can associate a unique vertex algebra. The aim of the first part of this thesis is to construct a prefactorization algebra associated to a holomorphic fibration and describe the corresponding vertex algebra. Specializing to the case in which the fiber is a torus, we recover a vertex algebra naturally associated to an $(n+1)$-toroidal algebra, a multi-loop generaliztion of Kac-Moody algebras. The second part of this dissertation concerns averages of unlabeled, undirected networks with edge weights. It is becoming increasingly common to see large collections of network data objects, and as a result there is a need to develop basic statistical tools. We introduce a space parameterizing such networks, characterize some relevant topological and geometric properties, and use these properties to establish the asymptotic behavior of a generalized notion of an empirical mean. The lack of vertex labeling necessitates working with a quotient space in which we mod out permutations of labels, resulting in a nontrivial geometry which has implications on the types of probabilistic and statistical results that may be obtained and the techniques needed to obtain them.

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List of Abbreviations

BV	....	Batalin-Vilkovisky
CE	....	Chevalley-Eilenberg
$\mathbb{C}$	....	Complex numbers
DVS	....	Differentiable Vector Spaces
$\mathbb{R}$	....	Real numbers
$\mathcal{U}_d$	....	Space of unlabeled networks on d vertices
$V(\widehat{\mathfrak{g}}_R)$	....	Vertex algebra associated to generalied Kac-Moody algebra $\widehat{\mathfrak{g}}_R$
$V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$	....	Vertex algebra associated to prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$

Chapter 1

Introduction

The first part of this thesis is devoted to a general construction of a class of prefactorization algebras in the formalism developed by Kevin Costello and Owen Gwilliam in [13]. Our construction takes as inputs a locally trivial holomorphic fibration $\pi : E \to X$ with fiber F, a Lie algebra $(\mathfrak{g}, \langle , \rangle)$ with invariant bilinear form, and produces a prefactorization algebra $\mathcal{F}_{\mathfrak{g},\pi}$ on X. As will be explained below, $\mathcal{F}_{\mathfrak{g},\pi}$ can be viewed as a factorization envelope with coefficients in the algebra $H^*(F, \mathcal{O}_F)$. When $X = \mathbb{C}$, and F is a smooth complex affine variety, a result of Costello-Gwilliam extracts from $\mathcal{F}_{\mathfrak{g},\pi}$ (or more precisely, a certain dense subalgebra) a vertex algebra $V(\mathcal{F}_{\mathfrak{g},\pi})$. $V(\mathcal{F}_{\mathfrak{g},\pi})$ can be described as an induced module for a central extension $\widehat{\mathfrak{g}}_F$ of the Lie algebra

$$\mathfrak{g} \otimes H^0(F, \mathcal{O}_F)[z, z^{-1}] \tag{1.1}$$

(here $H^0(F, \mathcal{O})$ denotes the $\mathbb{C}$-algebra of regular functions on F), that is universal when $\mathfrak{g}$ is simple. When F is a point and $\pi : E \mapsto X$ the identity map, $\widehat{\mathfrak{g}}_F$ is isomorphic to the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ associated to $(\mathfrak{g}, \langle , \rangle)$, and the corresponding vertex algebra is the affine vacuum module. When F is a torus $(\mathbb{C}^*)^n$, $\widehat{\mathfrak{g}}_F$ is the $(n+1)$–toroidal algebra, and we thus obtain a natural vertex realization of toroidal algebras. The remainder of this section introduces the main ingredients in our construction.

1.1 (Pre)factorization algebras

The formalism of (pre)factorization algebras was developed by Kevin Costello and Owen Gwilliam in [13] to describe the algebraic structure of observables in quantum field theory as well as their symmetries. Roughly speaking, a prefactorization algebra $\mathcal{F}$ on a manifold X assigns to each open subset $U \subset X$ a cochain complex $\mathcal{F}(U)$, and to each inclusion

$$U_1 \sqcup U_2 \sqcup \cdots \sqcup U_n \subset V$$

of disjoint open subsets U_i of V, a map

$$m_V^{U_1,\cdots,U_n} : \mathcal{F}(U_1) \otimes \cdots\cdots \mathcal{F}(U_n) \mapsto \mathcal{F}(V) \tag{1.2}$$

subject to some natural compatibility conditions. If $\mathcal{F}$ is the prefactorization algebra of observables in a quantum field theory, the cohomology groups $H^i(\mathcal{F}(U))$ can be interpreted as the observables of the theory on U as well as their (higher) symmetries. This structure is reminiscent of a multiplicative co-sheaf, and just as in the theory of sheaves/co-sheaves a gluing axiom distinguishes factorization algebras from mere prefactorization algebras.

An important source of prefactorization algebras is the *factorization envelope* construction, which proceeds starting with a fine sheaf $\mathcal{L}$ of dg or L_∞ algebras on X. Denoting by $\mathcal{L}_c$ the cosheaf of compactly supported sections of $\mathcal{L}$, we have maps

$$\oplus_{i=1}^n \mathcal{L}_c(U_i) \cong \mathcal{L}_c(U_1 \cup \cdots \cup U_n) \mapsto \mathcal{L}_c(V) \tag{1.3}$$

for disjoint opens $U_i \subset V$, where the map on the right is extension by 0. Applying the functor $\mathcal{C}_*^{Lie}$ of Chevalley chains to (1.3) yields maps

$$\otimes_{i=1}^n \mathcal{C}_*^{Lie}(\mathcal{L}_c(U_i)) \mapsto \mathcal{C}_*^{Lie}(\mathcal{L}_c(V))$$

The argument just sketched shows that the assignment

$$U \mapsto \mathcal{C}_*^{Lie}(\mathcal{L}_c(U)) \tag{1.4}$$

defines a prefactorization algebra. It is called the *factorization envelope of* $\mathcal{L}$ and denoted $\mathbf{U}\mathcal{L}$.

Costello-Gwilliam showed that there is a close relationship between a certain class of prefactorization algebras on $X = \mathbb{C}$ and vertex algebras. More precisely, if $\mathcal{F}$ satisfies a type of holomorphic translation-invariance and is equivariant with respect to the natural S^1 action on $\mathbb{C}$ by rotations, then the vector space

$$V(\mathcal{F}) := \bigoplus_{l \in \mathbb{Z}} H^*(\mathcal{F}^{(l)}(\mathbb{C})) \tag{1.5}$$

has the structure of a vertex algebra, where $\mathcal{F}^{(l)}(\mathbb{C})$ denotes the l-th eigenspace of S^1 in $\mathcal{F}(\mathbb{C})$.

1.2 Affine Kac-Moody algebras, toroidal algebras, and associated vertex algebras

Affine Kac-Moody algebras are a class of infinite-dimensional Lie algebras which play a central role in representation theory and conformal field theory. Given a finite-dimensional complex simple Lie algebra $\mathfrak{g}$, the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$ is the universal central extension of the loop algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ by a one-dimensional center $\mathbb{C}\mathbf{k}$. I.e., there is a short exact sequence

$$0 \to \mathbb{C}\mathbf{k} \to \widehat{\mathfrak{g}} \to \mathfrak{g}[z, z^{-1}] \to 0 \tag{1.6}$$

As a complex vector space $\widehat{\mathfrak{g}} = \mathfrak{g}[z, z^{-1}] \oplus \mathbb{C}\mathbf{k}$, with bracket

$$[J \otimes f(z), J' \otimes g(z)] = [J, J'] \otimes f(z)g(z) + \mathbf{k}\langle J, J'\rangle Res_{z=0} f dg,$$

where $J, J' \in \mathfrak{g}$, and $\langle , \rangle$ denotes the Killing form. In physics, affine algebras appear as symmetries of conformal field theories (CFT's) such as the WZW model, and tools adapted from CFT, notably vertex algebras, play a crucial role in studying their representation theory. In particular, for each $K \in \mathbb{C}$, the *vacuum module*

$$V_K(\widehat{\mathfrak{g}}) = \mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}} \mathbb{C}_K$$

(where $\widehat{\mathfrak{g}}_+ = \mathfrak{g}[z] \oplus \mathbb{C}\mathbf{k}$, and $\mathbb{C}_K$ denotes its one-dimensional representation on which the first summand acts trivially and $\mathbf{k}$ acts by K) has the structure of a vertex algebra. The representation theory of $\widehat{\mathfrak{g}}$ and $V_K(\widehat{\mathfrak{g}})$ are inextricably linked - $V_K(\widehat{\mathfrak{g}})$ picks out interesting categories of representations of $\widehat{\mathfrak{g}}$, and provides computational tools for studying these.

One may consider a generalization of the universal central extension (1.6) where $\mathbb{C}[z, z^{-1}]$ is replaced by an arbitrary commutative $\mathbb{C}$–algebra R. That is, one begins instead with the Lie algebra $\mathfrak{g}_R = \mathfrak{g} \otimes R$ with Lie bracket defined by

$$[J \otimes r, J' \otimes s] = [J, J'] \otimes rs.$$

It is shown in [26] that $\mathfrak{g}_R$ has a central extension of the form

$$0 \to \Omega^1_R/\mathrm{d}R \to \widehat{\mathfrak{g}}_R \to \mathfrak{g}_R \to 0.$$

where Ω^1_R denotes the module of Kahler differentials, and $d : R \to \Omega^1_R$ is the universal derivation. Thus

$$\widehat{\mathfrak{g}}_R \cong \mathfrak{g} \otimes R \oplus \Omega^1_R/\mathrm{d}R,$$

as a vector space, with bracket

$$[J \otimes r, J' \otimes s] = [X, Y] \otimes rs + \overline{\langle X, Y \rangle r ds},$$

where $\overline{\omega}$ denotes the class of $\omega \in \Omega^1_R$ in $\Omega^1_R/\mathrm{d}R$. When $\mathfrak{g}$ is simple, the central extension $\widehat{\mathfrak{g}}_R$ is universal.

An important example occurs when $R = \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ - the ring of algebraic functions on the n–torus. In this case $\widehat{\mathfrak{g}}_R$ is a higher loop generalization of $\widehat{\mathfrak{g}}$ called the n–toroidal algebra. We note that for $n > 1$, the central term $\Omega^1_R/\mathrm{d}R$ is infinite-dimensional over $\mathbb{C}$.

It is a natural question whether one can associate to $\widehat{\mathfrak{g}}_R$, and in particular to toroidal algebras, a natural vertex algebra $V(\widehat{\mathfrak{g}}_R)$ along the same lines that $V_k(\mathfrak{g})$ is associated to $\widehat{\mathfrak{g}}$. In this paper, we provide an affirmative answer in the case when $R = A[z, z^{-1}]$ for some $\mathbb{C}$–algebra A, and give a geometric construction of $V(\widehat{\mathfrak{g}}_R)$ in terms of prefactorization algebras. Other connections between toroidal algebras and vertex algebras have been explored in the works [5, 20, 31, 35].

1.3 Prefactorization algebras from holomorphic fibrations

In this paper we construct prefactorization algebras starting with two pieces of data:

- A locally trivial holomorphic fibration $\pi : E \to X$ of complex manifolds with fiber F.
- A Lie algebra $(\mathfrak{g}, \langle , \rangle)$ with invariant bilinear form.

We begin with a sheaf of DGLA's

$$\mathfrak{g}_\pi = (\mathfrak{g} \otimes \pi_* \Omega^{0,*}_E, \overline{\partial})$$

on X with bracket

$$[J \otimes \alpha, J' \otimes \beta] = [J, J'] \otimes \alpha \wedge \beta, \quad J, J' \in \mathfrak{g}, \alpha, \beta \in \pi_* \Omega^{0,*}_E.$$

$\mathfrak{g}_\pi$ has an L_∞ central extension $\widehat{\mathfrak{g}}_\pi$ whose underlying complex of sheaves is of the form

$$\widehat{\mathfrak{g}}_\pi = \mathfrak{g}_\pi \oplus \mathcal{K}_\pi,$$

with $\mathcal{K}_\pi$ a certain three-term complex. Our prefactorization algebra is

$$\mathcal{F}_{\pi,\mathfrak{g}} := \mathcal{C}_*^{Lie}(\widehat{\mathfrak{g}}_{\pi,c}),$$

where $\widehat{\mathfrak{g}}_{\pi,c}$ denotes the cosheaf of sections with compact support. This is an instance of the factorization envelope construction (1.4) described above.

When F is a smooth affine complex variety, and $E = X \times F$ is a trivial fibration, we may pass to a somewhat "smaller" prefactorization envelope $\mathcal{G}^{alg}_{\pi,\mathfrak{g}}$ built starting with the sheaf of DGLA's

$$(\mathfrak{g} \otimes H^0(F, \mathcal{O}) \otimes \Omega^{0,*}_X, \overline{\partial})$$

There is a map

$$\mathcal{G}^{alg}_{\pi,\mathfrak{g}} \to \mathcal{F}_{\pi,\mathfrak{g}} \tag{1.7}$$

induced by the map of DGLA's

$$(\mathfrak{g} \otimes H^0(F, \mathcal{O}) \otimes \Omega^{0,*}_X, \overline{\partial}) \to (\mathfrak{g} \otimes \pi_* \Omega^{0,*}_E, \overline{\partial})$$

$\mathcal{G}^{alg}_{\pi,\mathfrak{g}}$ is more manageable from technical standpoint, as it avoids certain analytic complications involving completions of Dolbeault cohomology of products. We note that the map 1.7 depends on a choice of trivialization of E.

When $X = \mathbb{C}$ (and E is necessarily trivial), we may apply Costello-Gwilliam's result above to the prefactorization algebra $\mathcal{G}^{alg}_{\pi,\mathfrak{g}}$, recovering a vertex algebra $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ as in 1.5. It has the following simple description. Let $\widehat{\mathfrak{g}}_F$ denote universal central extension of the Lie algebra $\mathfrak{g} \otimes H^0(F, \mathcal{O}_F)[z, z^{-1}]$ (i.e. this is $\widehat{\mathfrak{g}}_R$ above, where $R =$

$H^0(F, \mathcal{O}_F)[z, z^{-1}]$), and let $\widehat{\mathfrak{g}}_F^+$ be the sub-algebra corresponding to the non-negative powers of z. Then as a representation of $\widehat{\mathfrak{g}}_F$

$$V(\mathcal{G}_{\mathfrak{g},\pi}^{alg}) \simeq \mathrm{Ind}_{\widehat{\mathfrak{g}}_F^+}^{\widehat{\mathfrak{g}}} \mathbb{C}, \tag{1.8}$$

where $\mathbb{C}$ is the trivial representation of $\widehat{\mathfrak{g}}_F^+$.

We may view these results as follows. When X is a (arbitrary) Riemann surface, and $p \in X$ a point, we may choose a coordinate z centered at p, and a local trivialization of E near p. The cohomology prefactorization algebra $H^*(\mathcal{F}_{\pi,\mathfrak{g}})$ is then locally modeled by the vertex algebra $V(\widehat{\mathfrak{g}}_F)$ via the dense inclusion (1.7).

In section (2.1) we recall universal central extensions, the construction of $\widehat{\mathfrak{g}}_R$, and vertex algebras. We also show how to associate to the algebra $R = A[z, z^{-1}]$, where A is a commutative $\mathbb{C}$–algebra a vertex algebra generalizing the affine vacuum module. Our later geometric construction will be a special case of this. In section (2.2) we recall some basic facts about prefactorization algebras. The construction of $\mathcal{F}_{\mathfrak{g},\pi}$ and the related prefactorization algebra $\mathcal{G}_{\mathfrak{g},\pi}^{alg}$ happens in section (2.3). Finally, in section (2.4) we consider the special case when $X = \mathbb{C}$, and relate $\mathcal{G}_{\mathfrak{g},\pi}^{alg}$ to the vertex algebra $V(\widehat{\mathfrak{g}}_F)$.[1]

1.4 Unlabeled Networks

Over the past 15-20 years, as the field of network science has exploded with activity, the majority of attention has been focused upon the analysis of (usually large) *individual* networks. See [24, 28, 36], for example. While it is unlikely that the analysis of individual networks will become any less important in the near future, it is likely that in the context of the modern era of 'big data' there will soon be an equal need for the analysis of (possibly large) *collections* of (sub)networks, i.e., collections of network

[1]The work in chapter 2 appears in the paper [37].

data objects.

We are already seeing evidence of this emerging trend. For example, the analysis of massive online social networks like Facebook can be facilitated by local analyses, such as through extraction of ego-networks (e.g., [23]). Similarly, the 1000 Functional Connectomes Project, launched a few years ago in imitation of the data-sharing model long-common in computational biology, makes available a large number of fMRI functional connectivity networks for use and study in the context of computational neuroscience (e.g, [?biswal2010toward]). It would seem, therefore, that in the near future networks of small to moderate size will themselves become standard, high-level data objects.

Faced with databases in which networks are the fundamental unit of data, it will be necessary to have in place a network-based analogue of the 'Statistics 101' tool box, extending standard tools for scalar and vector data to network data objects. The extension of such classical tools to network-based datasets, however, is not immediate, since networks are not inherently Euclidean objects. Rather, formally they are combinatorial objects, defined simply through two sets, of vertices and edges, respectively, possibly with an additional corresponding set of weights. Nevertheless, initial work in this area demonstrates that networks can be associated with certain natural Euclidean subspaces and furthermore demonstrates that through a combination of tools from geometry, probability theory, and statistical shape analysis it should be possible to develop a comprehensive, mathematically rigorous, and computationally feasible framework for producing the desired analogues of classical tools.

For example, in the recent work [22] we have characterized the geometry of the space of all labeled, undirected networks with edge weights, i.e., consisting of graphs $G = (V, E, W)$, for weights $w_{ij} = w_{ji} \geq 0$, where equality with zero holds if and only if $\{i, j\} \notin E$. This characterization allowed us in turn to establish a central limit theorem for an appropriate notion of a network empirical mean, as well as analogues of classical one- and two-sample hypothesis testing procedures. Other results of this type include additional work on asymptotics for network empirical means [38] and regression modeling with a network response variable, where for the latter there have been both frequentist [12] and Bayesian [19] proposals. Work in this area continues at a quick pace – see, for example, [2] which proposes a classification model based on network-valued inputs and [18] which proposes a nonparametric Bayes model for distributions on populations of networks. Earlier efforts in this space have focused on the specific case of trees. Contributions of this nature include work on central limit theorems in the space of phylogenetic trees [4, 11] and work by Marron and colleagues [3, 40] in the context of so-called object-oriented data analysis with trees.

To the best of our knowledge, all such work to date pertains to the case of *labeled* networks: that is, to networks in which the vertices V have unique labels, e.g., $V = \{1, \ldots, d\}$. In fact, *unlabeled* networks have received decidedly less attention in the network science literature as a whole but nevertheless arise in various important settings. The quintessential example of how such networks may arise in practice arguably is that of the study of ego-centric network structure in social network analysis. There, traditionally, individuals ('egos') are surveyed for a list of other individuals ('alters') with whom they share a certain relationship (e.g., friendship, colleague, etc.) and only common patterns across networks in the structure of the relationships among the individuals within each network are of interest. This leads to analyses that either ignore vertex labels or for which vertex labels are simply not available (e.g., through de-identification). See [33], for example.

In this thesis, the focus is on averages of unlabeled, undirected networks with edge weights. Adopting a perspective similar to that in our previous work [22], we (i) characterize a certain notion of the space of all such networks, (ii) describe key topological and geometric properties of this space relevant to doing probability and statistics thereupon, and (iii) use these properties to establish the asymptotic behavior of a generalized notion of an empirical mean under sampling from a distribution supported on this space. In particular, adopting the notion of a Fréchet mean, we establish a corresponding strong law of large numbers and a central limit theorem. In contrast to [22], where the corresponding space of networks was found to form a smooth manifold, here the lack of vertex labeling necessitates working with a quotient space modding out permutations of labels. As a result, we have only an orbifold – a more general geometric structure – which in turn is found to have important implications on the types of probabilistic and statistical results that may be obtained and the techniques needed to obtain them.

The nature of our work is in the spirit of statistics on manifolds and statistical shape analysis, which employs the geometry of manifolds or shape spaces for defining Fréchet means and developing large sample theory of their sample counterparts for inference. See [7] for a rather comprehensive treatment on the subject. Our approach to studying the entire family of networks subject to an equivalence relation under a group action, via forming the associated quotient or moduli space, is a common theme in modern geometry, including gauge theory [16], symplectic topology [34], and algebraic geometry [14, 39]. The appearance of orbifolds, often much more complicated than in our case, is quite common. Finally, there is a large literature on graph limits, for which substantial work has been done on analysis of appropriate spaces of networks (e.g., [32]). But the focus therein typically is, by definition, on the case of a single network asymptotically increasing in size. Here, the focus is on asymptotics in many networks, with the dimension fixed.

The organization of this chapter is as follows. In Section 3.1 we present our characterization of the space of unlabeled networks. Results from our investigation of the asymptotic behavior of the Fréchet empirical mean are then provided in Section 3.2. While a strong law of large numbers is found to emerge under quite general conditions, establishing just when conditions dictated by the current state of the art for central limit theorems on manifolds hold turns out to be a decidedly more subtle exercise. This latter is the focus of Section 3.3. The Appendices discuss implementation issues for the theoretical results in the chapter.[2]

[2]This introduction, §1.4, appears in the [29] verbatim, and is not my work. The work in chapter 3 appears verbatim in the paper [29]. I am solely responsible for the work appearing in the appendices, code, and example 3.3.1.

Chapter 2

Prefactorization and vertex algebras associated to holomorphic fibrations, and the toroidal algebra

2.1 Lie algebras and vertex algebras

2.1.1 Conventions

An L_∞ structure on a graded vector space $\mathfrak{h}$ is the data of a square 0, cohomological degree 1 coderivation $\widetilde{d}$ of

$$\mathcal{C}_*^{Lie}(\mathfrak{h}) := \mathrm{Sym}(\mathfrak{h}[1])$$

We may write $\widetilde{d} = \sum_{m=1}^{\infty} l_m$, where

$$l_m : \mathfrak{h}^{\otimes m} \to \mathfrak{h}[m-2]$$

and the l_m are extended to $\mathcal{C}_*^{Lie}(\mathfrak{h})$ as co-derivations. An ordinary Lie algebra corresponds to the case where $l_m = 0$ for $m \neq 2$, and l_2 is the Lie bracket, and a DGLA to the case where $l_m = 0$ for $m > 2$. We note that $(\mathfrak{h}, l_1)$ has the structure of a cochain complex, and $H^*(\mathfrak{h}, l_1)$ the structure of a graded Lie algebra. We refer to $\mathcal{C}_*^{Lie}(\mathfrak{h}, \widetilde{d})$ as the *complex of Chevalley chains* (even though it's cohomologically graded), and its dual

$$\mathcal{C}^{*,Lie}(\mathfrak{h}) = \mathrm{Sym}(\mathfrak{h}^*[-1], \widetilde{d}^*)$$

as the *complex of Chevalley cochains*. For more on L_∞ algebras, we refer the reader to [13].

2.1.2 Central extensions

Given a complex Lie algebra $\mathfrak{g}$ with invariant bilinear form $\langle \bullet, \bullet \rangle$, and a commutative $\mathbb{C}$-algebra R, $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$ carries a natural complex Lie algebra structure with bracket

$$[J \otimes r, J' \otimes s] = [J, J'] \otimes rs.$$

It is shown by Kassel [26] that there exists a universal central extension of the form

$$0 \to H_2^{\mathrm{Lie}}(\mathfrak{g}_R) \to \widehat{\mathfrak{g}}_R \to \mathfrak{g}_R \to 0.$$

Furthermore, when $\mathfrak{g}$ is simple and $\langle \bullet, \bullet \rangle$ the Killing form, there is an isomorphism of the Lie algebra homology $H_2(\mathfrak{g}_R) \cong \Omega^1_{R/\mathbb{C}}/\mathrm{d}R$ where $\Omega^1_{R/\mathbb{C}}$ is the R-module of Kähler differentials of $R/\mathbb{C}$ and $\mathrm{d} : R \to \Omega^1_R$ is the universal derivation. The bracket on

$$\widehat{\mathfrak{g}}_R \cong \mathfrak{g} \otimes R \oplus \Omega^1_R/\mathrm{d}R, \tag{2.1}$$

is given by

$$\begin{aligned} [J \otimes r, J' \otimes s] &= [J, J'] \otimes rs + \overline{\langle J, J' \rangle r ds} \\ &= [J, J'] \otimes rs + \frac{1}{2}\overline{\langle J, J' \rangle (rds - sdr)} \end{aligned}$$

where $\overline{\omega}$ denotes the class of $\omega \in \Omega^1_R$ in $\Omega^1_R/\mathrm{d}R$. We will find the second form of the central cocycle more convenient to use.

Example 2.1.1. Let $n \geq 0$ be an integer. An important class of examples is obtained by taking

$$R := \mathbb{C}[t_0^{\pm 1}, \cdots, t_n^{\pm -1}].$$

This is the algebra of functions on the $(n+1)$-dimensional algebraic torus.

When $n = 0$, the vector space $\Omega^1_R/\mathrm{d}R$ is one-dimensional with an explicit isomorphism given by the residue

$$\mathrm{Res} : \Omega^1_R/\mathrm{d}R \xrightarrow{\cong} \mathbb{C}.$$

The resulting Lie algebra $\widehat{\mathfrak{g}}_R$ is the ordinary affine Kac-Moody algebra usually denoted by $\widehat{\mathfrak{g}}$. For $n \geq 1$ the vector space $\Omega^1_R/\mathrm{d}R$ is infinite dimensional. Indeed, let us denote $k_i = t_i^{-1}\mathrm{d}t_i$. The space $\Omega^1_R/\mathrm{d}R$ is generated over the ring $\mathbb{C}[t_0^{\pm 1}, \ldots, t_n^{\pm 1}]$ by the symbols $k_0, \ldots, k_n$ subject to the relation

$$\sum_{i=0}^{n} m_i t_0^{m_0} \cdots t_n^{m_n} k_i = 0$$

where $(m_0, \ldots, m_n)$ is any n-tuple of integers. The Lie algebra $\widehat{\mathfrak{g}}_R$ is called the $(n+1)$-*toroidal* Lie algebra associated to $\mathfrak{g}$.

It will be useful for us to have an L_∞-model for the Lie algebra $\widehat{\mathfrak{g}}_R$. This model amounts to replacing the vector space $\Omega^1_R/\mathrm{d}R$ appearing as the central term by the cochain complex

$$\mathcal{K}_R = \mathrm{Ker}(\mathrm{d})[2] \to R[1] \xrightarrow{\mathrm{d}} \Omega^1_R.$$

Just as the Lie algebra $\widehat{\mathfrak{g}}_R$ is a central extension of $\mathfrak{g}_R = \mathfrak{g} \otimes R$, the L_∞ model we wish to construct is a central extension of $\mathfrak{g}_R = \mathfrak{g} \otimes R$ by the cochain complex $\mathcal{K}_R$.

The central extension is determined by a cocycle $\phi \in C^*(\mathfrak{g}_R, \mathcal{K}_R)$ of total degree two. The cocycle is of the form $\phi = \phi^{(0)} + \phi^{(1)}$ where

$$\begin{array}{cccc} \phi^{(1)} & : & (\mathfrak{g}_R)^{\otimes 2} & \to & \Omega^1_R \\ & & (J \otimes r) \otimes (J' \otimes s) & \mapsto & \frac{1}{2}\langle J, J' \rangle (r\mathrm{d}s - s\mathrm{d}r) \end{array}$$

and

$$\begin{array}{cccc} \phi^{(0)} & : & (\mathfrak{g}_R)^{\otimes 3} & \to & R \\ & & (J \otimes r) \otimes (J' \otimes s) \otimes (J'' \otimes t) & \mapsto & \frac{1}{2}\langle [J, J'], J'' \rangle rst \end{array}$$

Lemma 2.1.2. *The functional ϕ defines a cocycle in $C^*(\mathfrak{g}_R, \mathcal{K}_R)$ of total degree two.*

Proof. The differential in the cochain complex $C^*(\mathfrak{g}_R, \mathcal{K}_R)$ is of the form $\mathrm{d} + \mathrm{d}_{CE}$ where d is the de Rham differential defining the complex $\mathcal{K}_R$, and d_{CE} is the Chevalley-Eilenberg differential encoding the Lie bracket of $\mathfrak{g}_R$. It is immediate that $\mathrm{d}\phi^{(1)} = 0$,

$\mathrm{d}_{CE}\phi^{(0)} = 0$ by the Jacobi identity for $\mathfrak{g}$ and invariance of $\langle \bullet, \bullet \rangle$, and $\mathrm{d}\phi^{(0)} + \mathrm{d}_{CE}\phi^{(1)} = 0$ by direct calculation. Thus $(\mathrm{d}_{CE} + \mathrm{d})\phi = 0$ as desired. □

The cocycle ϕ defines an L_∞ central extension

$$\mathcal{K}_R \to \tilde{\mathfrak{g}}_R \to \mathfrak{g}_R.$$

As a vector space, $\widetilde{\mathfrak{g}}_R = \mathfrak{g}_R \oplus \mathcal{K}_R$, and the L_∞ operations are defined by $\ell_1 = \mathrm{d}, \ell_2 = [\cdot, \cdot]_{\mathfrak{g}_R} + \phi^{(1)}$, and $\ell_3 = \phi^{(0)}$. The following is immediate from our definitions:

Lemma 2.1.3. *There is an isomorphism of Lie algebras $H^*(\widetilde{\mathfrak{g}}_{\mathbb{R}}, \ell_1) = \widehat{\mathfrak{g}}_{\mathbb{R}}$.*

Proof. The cohomology of $\tilde{\mathfrak{g}}_R$ is concentrated in degree zero, and isomorphic to

$$\mathfrak{g}_R \oplus H^0(\mathcal{K}_R) = \mathfrak{g}_R \oplus \Omega^1_R/\mathrm{d}R$$

as a vector space. Our definition of ϕ implies that

$$\phi^{(0)}((J \otimes r) \otimes (J' \otimes s)) = \frac{1}{2}\langle J, J' \rangle (rds - sdr) = rds \ mod \ dR$$

so that the resulting Lie bracket is the same as that of $\widehat{\mathfrak{g}}_R$. □

2.1.3 Vertex algebras

We proceed to briefly recall the basics of vertex algebras and discuss an important class of examples, which will later be constructed geometrically via factorization algebras. We refer the reader to [21, 25] for details.

Definition 2.1.4. A vertex algebra $(V, |0\rangle, T, Y)$ is a complex vector space V along with the following data:

- A vacuum vector $|0\rangle \in V$.
- A linear map $T : V \to V$ (the translation operator).
- A linear map $Y(-, z) : V \to \mathrm{End}(V)[\![z^{\pm 1}]\!]$ (the vertex operator). We write $Y(v, z) = \sum_{n \in \mathbb{Z}} A^v_n z^{-n}$ where $A^v_n \in \mathrm{End}(V)$.

satisfying the following axioms:

- For all $v, v' \in V$ there exists an $N \gg 0$ such that $A_n^v v' = 0$ for all $n > N$. (This says that $Y(v,z)$ is a *field* for all v).
- (vacuum axiom) $Y(|0\rangle, z) = \mathrm{id}_V$ and $Y(v,z)|0\ \in v + zV[\![z]\!]$ for all $v \in V$.
- (translation) $[T, Y(v,z)] = \partial_z Y(v,z)$ for all $v \in V$. Moreover $T|0\rangle = 0$.
- (locality) For all $v, v' \in V$, there exists $N \gg 0$ such that

$$(z-w)^N [Y(v,z), Y(v',w)] = 0$$

  in $\mathrm{End}(V)[\![z^{\pm 1}, w^{\pm 1}]\!]$.

In order to prove that a given $(V, |0\rangle, T, Y)$ forms a vertex algebra, the following "reconstruction" or "extension" theorem is very useful. It shows that any collection of local fields generates a vertex algebra in a suitable sense.

Theorem 2.1.5 ([21], [15]). *Let V be a complex vector space equipped with: an element $|0\rangle \in V$, a linear map $T : V \to V$, a set of vectors $\{a^s\}_{s\in S} \subset V$ indexed by a set S, and fields $A^s(z) = \sum_{n\in\mathbb{Z}} A_n^s z^{-n-1}$ for each $s \in S$ such that:*

- *For all $s \in S$, $A^s(z)|0\rangle \in a^s + zV[\![z]\!]$;*
- *$T|0\rangle = 0$ and $[T, A^s(z)] = \partial_z A^s(z)$;*
- *$A^s(z)$ are mutually local;*
- *and V is spanned by $\{A_{j_1}^{s_1} \cdots A_{j_m}^{s_m}|0\ \}$ as the j_i's range over negative integers.*

Then, the data $(V, |0\rangle, T, Y)$ defines a unique vertex algebra satisfying

$$Y(a^s, z) = A^s(z).$$

Remark 2.1.6. The version stated above appears in [15], and is slightly more general than the version stated in [21].

2.1.4 The vertex algebras $V(\widehat{\mathfrak{g}})$ and $V(\widehat{\mathfrak{g}}_{\mathbb{R}})$

A number of vertex algebras are constructed from vacuum representations of affine Lie algebras and their generalizations. We proceed to review the vertex algebra structure on the affine Kac-Moody vacuum module $V(\widehat{\mathfrak{g}})$ and extend the construction to vacuum representations of $\widehat{\mathfrak{g}}_R$, where $R = A[t, t^{-1}]$ for some $\mathbb{C}$-algebra A.

$V(\widehat{\mathfrak{g}})$

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ be the affine Kac-Moody algebra, $\widehat{\mathfrak{g}}^+ = \mathfrak{g}[t]$ denote the positive sub-algebra, and $\mathbb{C}$ denote the trivial representation of $\widehat{\mathfrak{g}}^+$. For $J \in \mathfrak{g}$, denote $J \otimes t^n$ by J_n, and $1 \in \mathbb{C}$ by $|0\rangle$

It is well-known (see for instance [21]) that the induced vacuum representation

$$V(\widehat{\mathfrak{g}}) := \mathrm{Ind}_{\widehat{\mathfrak{g}}^+}^{\widehat{\mathfrak{g}}} \mathbb{C} := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t])} \mathbb{C}$$

has a $\mathbb{C}[\mathbf{k}]$-linear vertex algebra structure, which is generated, in the sense of the above reconstruction theorem, by the fields

$$J^i(z) := Y(J^i_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} J^i_n z^{-n-1},$$

where $\{J^i\}$ is a basis for $\mathfrak{g}$. These satisfy the commutation relations

$$[J^i(z), J^k(w)] = [J^i, J^k](w)\delta(z-w) + \langle J^i, J^k \rangle \mathbf{k} \partial_w \delta(z-w)$$

where

$$\delta(z-w) = \sum_m z^m w^{-m-1}$$

The translation operator T is determined by the properties

$$T|0\rangle = 0, [T, J^i_n] = -nJ^i_{n-1}.$$

Remark 2.1.7. The construction above produces a generic version of the affine Kac-Moody vacuum module, in the sense that $\mathbf{k}$ is not specialized to be a complex number. In the vertex algebra literature one typically specifies a level $K \in \mathbb{C}$, and defines

$$V_K(\widehat{\mathfrak{g}}) := \mathrm{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}\mathbf{k}}^{\widehat{\mathfrak{g}}}\,\mathbb{C},$$

where $\mathbb{C}$ denotes the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{k}$ on which the first factor acts by 0 and $\mathbf{k}$ acts by K. We have an isomorphism

$$V(\widehat{\mathfrak{g}})/I \simeq V_K(\mathfrak{g})$$

where I is the vertex ideal generated by $K|0\rangle - \mathbf{k}|0\rangle$. $V(\mathfrak{g})$ can therefore be viewed as a family of vertex algebras over $\mathrm{spec}(\mathbb{C}[\mathbf{k}])$, with fiber $V_K(\mathfrak{g})$ at $\mathbf{k} = K$.

$V(\widehat{\mathfrak{g}}_{\mathbb{R}})$

In this section we generalize the construction of the affine Kac-Moody vacuum module above to the Lie algebra $\widehat{\mathfrak{g}}_R$, for $R = A[t, t^{-1}]$, where A is a commutative $\mathbb{C}$–algebra. The construction specializes to $V(\widehat{\mathfrak{g}})$ for $A = \mathbb{C}$.

Let A be a commutative $\mathbb{C}$-algebra, $R = A[t, t^{-1}] := A \otimes \mathbb{C}[t, t^{-1}]$, and $\widehat{\mathfrak{g}}_R$ the Lie algebra (2.1). We have a Lie subalgebra

$$\widehat{\mathfrak{g}}_R^+ := \mathfrak{g} \otimes A[t] \oplus \Omega^1_{A[t]}/dA[t] \hookrightarrow \widehat{\mathfrak{g}}_R.$$

Let

$$V(\widehat{\mathfrak{g}}_R) := \mathrm{Ind}_{\widehat{\mathfrak{g}}_R^+}^{\widehat{\mathfrak{g}}_R}\,\mathbb{C} \tag{2.2}$$

where $\mathbb{C}$ denotes the trivial representation of $\widehat{\mathfrak{g}}_R^+$. Our goal is to define the structure of a vertex algebra on $V(\widehat{\mathfrak{g}}_R)$.

The vacuum vector is simply $|0\rangle := 1 \in \mathbb{C}$. The fields of the vertex algebra split into three classes and are defined as follows.

$$J_u(z) := Y(J \otimes ut^{-1}|0\rangle, z) := \sum_{n\in\mathbb{Z}} (J \otimes ut^n) z^{-n-1}, \tag{2.3}$$

$$K_{u\frac{dt}{t}} := Y(t^{-1}udt|0\rangle, z) := \sum_{n\in\mathbb{Z}} (ut^{n-1}dt) z^{-n}, \tag{2.4}$$

$$K_{t^{-1}\omega} := Y(t^{-1}\omega|0\rangle, z) := \sum_{n\in\mathbb{Z}} (t^n\omega) z^{-n-1} \tag{2.5}$$

where $J \in \mathfrak{g}, u \in A, \omega \in \Omega^1_A$.

The commutation relations between these fields are easily checked to be

$$[J^1_u(z), J^2_v(w)] = \left([J^1, J^2]_{uv}(w) + \langle J^1, J^2\rangle K_{t^{-1}udv}(w)\right)\delta(z-w) + \langle J^1, J^2\rangle K_{uv\frac{dt}{t}}(w)\partial_w\delta(z-w)$$

with all other commutators 0.

The operator T, corresponding to the Lie derivative $L_{-\partial_t}$, is defined by

$$T|0\rangle = 0,\ [T, J^i \otimes ft^n] = -nJ^i \otimes ft^{n-1},\ [T, ft^n dt] = -nft^{n-1}dt,\ [T, t^n\omega] = -nt^{n-1}\omega$$

Theorem 2.1.8. *The above field assignments, together with T equip $V(\widehat{\mathfrak{g}}_{\mathbb{R}})$ with the structure of a vertex algebra.*

Proof. We begin by checking that the field assignment above is well-defined. This amounts to verifying that $Y(d(t^{-1}u)|0\rangle, z) = 0$. We have

$$\begin{aligned} Y(d(t^{-1}u)|0\rangle, z) &= Y(t^{-1}du|0\rangle, z) - Y(ft^{-2}u|0\rangle, z) \\ &= Y(t{-}1du|0\rangle, z) - Y([T, t^{-1}udt]|0\rangle, z) \\ &= Y(t^{-1}du|0\rangle, z) - \partial_z Y(t^{-1}udt|0\rangle, z) \\ &= \sum_n (t^n du + nt^{n-1}u) z^{-n-1} = \sum_n d(t^n u) z^{-n-1} = 0 \end{aligned}$$

The result now follows by applying the reconstruction theorem 2.1.5 to $V(\widehat{\mathfrak{g}}_{\mathbb{R}})$ and the fields $\{J_u(z), K_{u\frac{dt}{t}}, K_{t^{-1}\omega}\}$ for $J \in \mathfrak{g}, f \in A, \omega \in \Omega^1_A$. ☐

2.2 (Pre)-factorization algebras and examples

In this section we recall basic notions pertaining to pre-factorization algebras. We refer the reader to [13] for details.

Let X be a smooth manifold, and $\mathbf{C}^{\otimes}$ a symmetric monoidal category.

Definition 2.2.1. A *prefactorization algebra* $\mathcal{F}$ on X with values in $\mathbf{C}^{\otimes}$ consists of the following data:

- for each open $U \subset M$, an object $\mathcal{F}(U) \in \mathbf{C}^{\otimes}$,
- for each finite collection of pairwise disjoint opens $U_1, \ldots, U_n$ and an open V containing every U_i, a morphism

$$m_V^{U_1,\cdots,U_n} : \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V), \tag{2.6}$$

and satisfying the following conditions:

- composition is associative, so that the triangle

$$\begin{array}{ccc} \bigotimes_i \bigotimes_j \mathcal{F}(T_{ij}) & \longrightarrow & \bigotimes_i \mathcal{F}(U_i) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(V) & \end{array}$$

  commutes for any disjoint collection $\{U_i\}$ contained in V, and disjoint collections $\{T_{ij}\}_j \subset U_i$

- the morphisms $m_V^{U_1,\cdots,U_n}$ are equivariant under permutation of labels, so that the triangle

$$\begin{array}{ccc} \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) & \xrightarrow{\simeq} & \mathcal{F}(U_{\sigma(1)}) \otimes \cdots \otimes \mathcal{F}(U_{\sigma(n)}) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}(V) & \end{array}$$

  commutes for any $\sigma \in S_n$.

In this chapter, we will take the target category $\mathbf{C}^{\otimes}$ to be Vect, dg-Vect, or their smooth enhancements DVS described below.

A *factorization algebra* is a prefactorization algebra satisfying a descent (or gluing) axiom with respect to a class of special covers called *Weiss covers*. In this chapter, we will not be concerned with verifying that our constructions satisfy this additional property, and refer the interested reader to [13] for details.

Example 2.2.2 ([13]). Given an associative algebra over $\mathbb{C}$, one can construct a prefactorization algebra $\mathcal{F}_A$ in Vect on $\mathbb{R}$ by declaring $\mathcal{F}_A(I) = A$ for a connected open interval $I \subset \mathbb{R}$, and defining the structure maps in terms of the multiplication on A. For instance, if $I = (a, b), J = (c, d), K = (e, f)$, with $e < a < b < c < d < f$, the structure map is

$$\mathcal{F}_A(I) \otimes \mathcal{F}_A(J) \mapsto \mathcal{F}_A(K)$$
$$a \otimes b \mapsto ab$$

$\mathcal{F}_A$ has the property that it is *locally constant*, in the sense that if $I \subset I'$ are connected intervals, then $\mathcal{F}_A(I) \mapsto \mathcal{F}_A(I')$ is an isomorphism. It is shown in [13] that locally constant prefactorization algebras on $\mathbb{R}$ in Vect correspond precisely to associative algebras.

Prefactorization algebras can be pushed forward under smooth maps as follows. Suppose $f : X \mapsto Y$ is a smooth map of smooth manifolds, and $\mathcal{F}$ a prefactorization algebra on X. One then defines the prefactorization algebra $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$$

The structure maps of $f_*\mathcal{F}$ are defined in the obvious way.

If $\mathcal{F}, \mathcal{G}$ are prefactorization algebras on X with values in $\mathbf{C}^{\otimes}$, then a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is the data of maps

$$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U) \in \mathrm{Hom}_{\mathbf{C}^{\otimes}}(\mathcal{F}(U), \mathcal{G}(U))$$

for each open $U \subset X$, compatible with all structure maps (2.6).

2.2.1 (Pre)factorization envelopes

Our construction of factorization algebras from holomorphic fibrations is an instance of the *factorization envelope* construction, which we proceed to briefly review following [13].

Let $\mathcal{L}$ be a fine sheaf of L_∞ algebras, and $\mathcal{L}_c$ its associated cosheaf of sections with compact support. The *factorization envelope of* $\mathcal{L}$, $\mathbf{U}\mathcal{L}$ is the complex of Chevalley chains of $\mathcal{L}_c$. In other words, for each open $U \subset X$

$$\mathbf{U}\mathcal{L}(U) := \mathcal{C}_*^{Lie}(\mathcal{L}_c(U)) \tag{2.7}$$

The factorization structure maps are given explicitly as follows. Let $U_1, \cdots, U_k$ be disjoint open subsets of an open $V \subset X$. The cosheaf $\mathcal{L}_c$ induces a map of L_∞–algebras

$$\oplus_{i=1}^k \mathcal{L}_c(U_i) \mapsto \mathcal{L}_c(V)$$

Applying the Chevalley chains functor (which sends sums to tensor products) to this sequence yields maps

$$\otimes_{i=1}^k \mathcal{C}_*^{Lie}(\mathcal{L}_c(U_i)) \mapsto \mathcal{C}_*^{Lie}(\mathcal{L}_c(V)).$$

The following is proved in [13]

Theorem 2.2.3. *If $\mathcal{L}$ is a fine cosheaf of L_∞ algebras, then $\mathbf{U}\mathcal{L}$ is a prefactorization algebra in* dg-Vect*. The cohomology $H^*(\mathbf{U}\mathcal{L})$ is a prefactorization algebra in* Vect*.*

Example 2.2.4 ([13]). Let $X = \mathbb{R}$, and $\mathfrak{g}_{dR} := (\mathfrak{g} \otimes \Omega^*_{\mathbb{R}}, d_{dR})$ the sheaf of DGLA's on $\mathbb{R}$ obtained by tensoring $\mathfrak{g}$ with the de Rham complex. The factorization envelope $\mathbf{U}(\mathfrak{g}_{dR})$ is locally constant, and the cohomology factorization algebra $H^*(\mathbf{U}(\mathfrak{g} \otimes \Omega^*))$ is a locally constant factorization algebra in Vect, corresponding to the enveloping algebra $\mathcal{U}(\mathfrak{g})$ as in Example (2.2.2).

Example 2.2.5 ([13]). Let $X = \mathbb{C}^n$, and $(\mathfrak{g} \otimes \Omega^{0,*}_{\mathbb{C}^n}, \overline{\partial})$ the sheaf of DGLA's on $\mathbb{C}^n$ obtained by tensoring $\mathfrak{g}$ with the Dolbeault complex of forms of type $(0, q), q \geq 0$. As explained below, when $n = 1$, the factorization algebra $\mathbf{U}(\mathfrak{g} \otimes \Omega^{0,*}_{\mathbb{C}}, \overline{\partial})$ allows one to recover the affine vertex algebra $V(\widehat{\mathfrak{g}})$ (at level 0).

2.2.2 Differentiable vector spaces

The prefactorization algebras considered in this chapter typically assign to each open subset $U \subset X$ a cochain complex of infinite-dimensional vector spaces. This is apparent already in the example 2.2.4 above, where the graded components of $\mathbf{U}(\mathfrak{g}_{dR}) = \mathbf{U}(\mathfrak{g} \otimes \Omega^*)(U)$ for $U \subset \mathbb{R}$ are tensors in $\mathfrak{g} \otimes \Omega^*(U)_c$. The structure maps (2.6) are thus multilinear maps between such complexes. In order to formulate the notion of translation-invariance for prefactorization algebras in the next section, we will have to discuss what it means for these to depend smoothly on the positions of the open sets $U_i \subset X$. This raises some functional-analytic issues, which in turn complicate homological algebra involving these objects.

In [13] these technical issues are resolved by introducing the category DVS of *Differentiable Vector Spaces* together with certain sub-categories. DVS provides a flexible framework within which one can discuss smooth families of smooth maps between infinite-dimensional cochain complexes parametrized by auxilliary manifolds, and carry out homological constructions. We briefly sketch this category below, and refer to [13] for all details.

Definition 2.2.6. Let C^∞ denote the sheaf of rings on the site of smooth manifolds sending each manifold M to the ring of smooth functions $C^\infty(M)$, and assigning to each smooth map $f : M \to N$ the pullback $f^* : C^\infty(N) \to C^\infty(M)$. A C^∞-*module* $\mathcal{F}$ is a sheaf of modules over C^∞. In other words, $\mathcal{F}$ assigns to each M a $C^\infty(M)$-module $\mathcal{F}(M)$, and to $f : M \to N$ a pullback map $\mathcal{F}(f) : \mathcal{F}(N) \to \mathcal{F}(M)$ of $C^\infty(N)$-modules. A *differentiable vector space* is a C^∞-module equipped with a flat connection. Explicitly, this means a flat connection

$$\nabla : \mathcal{F}(M) \to \mathcal{F}(M) \otimes_{C^\infty(M)} \Omega^1(M)$$

for each manifold M, compatible with pullbacks. The objects of the category DVS are differentiable vector spaces, and the morphisms $Hom_{DVS}(\mathcal{F}, \mathcal{G})$ maps of C^∞-modules intertwining the connections.

Any locally convex topological vector space V gives rise to a differentiable vector space as follows. There is a good notion of a smooth map from any manifold M to V introduced by Kriegl and Michor (see [30]), and we denote by $C^\infty(M,V)$ the space of such. The space $C^\infty(M,V)$ is naturally a $C^\infty(M)$-module, and carries a natural flat connection whose horizontal sections are constant maps $M \to V$. The assignment $M \to C^\infty(M,V)$ thus produces an object of DVS. Multi-linear maps

$$\mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_r \mapsto \mathcal{G} \quad \mathcal{F}_i, \mathcal{G} \in \mathrm{DVS} \tag{2.8}$$

equip DVS with the structure of a multi-category (or equivalently, a colored operad) by inserting the output of a multilinear map into another. We denote the space of such maps by $\mathrm{DVS}(\mathcal{F}_1, \cdots, \mathcal{F}_r | \mathcal{G})$.

The multicategory DVS allows us to formulate the notion of a smooth family of multilinear operations parametrized by an auxiliary manifold M. For $\mathcal{F} \in \mathrm{DVS}$, one first defines the mapping space $\mathbf{C}^\infty(\mathbf{M}, \mathcal{F}) \in \mathrm{DVS}$ as the differentiable vector space given by the assignment $N \mapsto \mathcal{F}(N \times M)$. As explained in [13], it has a natural flat connection along N.

Definition 2.2.7. Let $\mathcal{F}_1, \cdots, \mathcal{F}_r, \mathcal{G} \in \mathrm{DVS}$. A smooth family of multilinear operations $\mathcal{F}_1 \times \cdots \mathcal{F}_r \mapsto \mathcal{G}$ parametrized by a manifold M is by definition an element of

$$\mathrm{DVS}(\mathcal{F}_1, \cdots, \mathcal{F}_r | \mathbf{C}^\infty(\mathbf{M}, \mathcal{G}))$$

where $\mathbf{C}^\infty(\mathbf{M}, \mathcal{G})$ is as explained in the preceding paragraph.

DVS has several good properties. Among these are:

- DVS is complete and co-complete.
- DVS is a Grothendieck Abelian Category.

The second property ensures that all standard constructions in homological algebra behave well in DVS. This is in contrast to the category of topological vector spaces, which is not even Abelian. As the authors explain in [13], this is because DVS has essentially been defined as the category of sheaves on a site.

Finally, we review some examples of differentiable vector spaces which will be useful to us.

Example 2.2.8. The following is an important example from [13]. Suppose $p : W \to X$ is a vector bundle over the manifold X. Then $V = \Gamma(X, W)$ is naturally a Frechet space, and so locally convex. $C^\infty(M, V)$ is then identified with $\Gamma(M \times X, \pi_X^* E)$, where $\pi_X : M \times X \to X$ is the projection on X. In particular, taking W to be the trivial bundle, we have $C^\infty(M, C^\infty(X)) = C^\infty(M \times X)$. The same line of reasoning shows that the space of compactly supported sections of W, $V' = \Gamma_c(X, W)$ has a DVS structure.

Example 2.2.9. The following generalization of the previous example will be useful in Sections 2.3, 2.4. Let $\pi : E \to X$ be a smooth map, and $p : W \to E$ a vector bundle on E. Denote by $\mathcal{W}$ the sheaf of smooth sections of W on E. Then $V = \Gamma(X, \pi_* \mathcal{W})$ yields a differentiable vector space, with $C^\infty(M, V) = \Gamma(M \times X, \tilde{\pi}_* \widetilde{\mathcal{W}})$, where $\tilde{\pi} : E \times M \to X \times M$ is defined by $\tilde{\pi}(e, m) = (\pi(e), m)$, and $\widetilde{\mathcal{W}}$ denotes the sheaf of sections of $\pi_E^* W$ on $E \times M$, with $\pi_E : E \times M \to E$ the projection on the first factor. The connection is determined by the condition that the horizontal sections are those constant in the M direction. When $E = X$ and $\pi = id_X$, this example reduces to the previous one. We may similarly equip $V' = \Gamma_c(X, \pi_* \mathcal{W})$ with a DVS structure.

Example 2.2.10. Suppose that $\mathcal{F} \in$ DVS, and V is any vector space (note that we don't specify a topology on V). Then the assignment $M \mapsto \mathcal{F}(M) \otimes_{\mathbb{R}} V$, with the connection acting trivially on the V factor, yields an object of DVS which we denote $\mathcal{F}_V$, When V is finite-dimensional, this amounts to a finite direct sum of $\mathcal{F}$.

Monoidal structures on DVS

To discuss prefactorization algebras with values in DVS, we must specify a symmetric monoidal structure, which is used in defining the structure maps 2.6. Certain subtleties arise on this point, typical of the issues one encounters when trying to define tensor products of inifinite-dimensional topological vector spaces. We restrict ourselves to a few brief remarks, and refer the interested reader to [13] for details.

- Given $\mathcal{F}, \mathcal{G} in$ DVS, we can define $\mathcal{F} \otimes \mathcal{G}$ as the sheafification of the presheaf $X \mapsto \mathcal{F}(X) \otimes_{C^\infty(X)} \mathcal{G}(X)$, equipped with the flat connection $\nabla^{\mathcal{F}} \otimes Id + Id \otimes \nabla^{\mathcal{G}}$. When $\mathcal{F} = C^\infty(M), \mathcal{G} = C^\infty(N)$, and $X = pt$ is a point, this yields $\mathcal{F} \otimes \mathcal{G}(pt) = C^\infty(M) \otimes_{\mathbb{R}} C^\infty(N)$. We call this symmetric monoidal structure the *naive tensor product* in DVS.

- The naive tensor product has certain shortcomings. Most importantly, it does not represent the space of multilinear maps (2.8). In order to remedy this situation, a certain completed tensor product $\hat{\otimes}_\beta$ has to be introduced. This operation is only defined on a certain sub-category of DVS however. In the last example, we would obtain $\mathcal{F} \hat{\otimes}_\beta \mathcal{G}(pt) = C^\infty(M \times N)$. We will refer to this operation as the *completed tensor product*.

Using $\hat{\otimes}_\beta$ rather than $\otimes$ is important if one wishes to obtain a factorization, rather than merely a prefactorization algebra. As we work with prefactorization algebras in this chapter, the naive tensor product is adequate, and will be the symmetric monoidal structure on DVS throughout.

2.2.3 Translation-invariant (pre)factorization algebras

Our construction in Section 2.3, when applied to the trivial fibration $E = F \times \mathbb{C}^n \mapsto \mathbb{C}^n$, produces a prefactorization algebra which is *holomorphically translation-invariant*. This property will be used when extracting a vertex algebra in Section 2.4 in the case $n = 1$. We proceed to briefly review this notion and refer the interested reader to [13] for details.

Discrete translation-invariance

Suppose now that $\mathcal{F}$ is prefactorization algebra on $\mathbb{C}^n$ in the category of complex vector spaces. $\mathbb{C}^n$ acts on itself by translations. For an open subset $U \subset \mathbb{C}^n$ and

$x \in \mathbb{C}^n$, let

$$\tau_x U := \{y \in \mathbb{C}^n | y - x \in U\}$$

Clearly, $\tau_x(\tau_y U) = \tau_{x+y} U$. We say that $\mathcal{F}$ is *discretely translation-invariant* if we are given isomorphisms

$$\phi_x : \mathcal{F}(U) \to \mathcal{F}(\tau_x U) \tag{2.9}$$

for each $x \in \mathbb{C}^n$ compatible with composition and the structure maps of $\mathcal{F}$. We refer to section 4.8 of [13] for details.

Example 2.2.11. For any Lie algebra $\mathfrak{g}$, $\mathbf{U}(\mathfrak{g} \otimes \Omega^{0,*}_{\mathbb{C}^n})$ in Example 2.2.5 is discretely translation-invariant.

Smooth and holomorphic translation-invariance

A refined version of translation-invariance expresses the fact that the maps ϕ_x, and hence the structure maps $m_V^{U_1,\cdots,U_n}$ depend smoothly/holomorphically on the positions of the open sets U_i. This notion is operadic in flavor.

For $z \in \mathbb{C}^n$ and $r > 0$ let $\mathrm{PD}(z, r)$ denote the polydisk

$$\mathrm{PD}_r(z) = \{w \in \mathbb{C}^n || w_i - z_i| < r, 1 \leq i \leq n\}$$

and let

$$\mathrm{PD}(r_1, \cdots, r_k | s) \subset (\mathbb{C}^n)^k$$

denote the open subset $(z_1, \cdots, z_k) \in (\mathbb{C}^n)^k$ such that the polydisks $\mathrm{PD}_{r_i}(z_i)$ have disjoint closures and are all contained in $\mathrm{PD}_s(0)$. The collections $\mathrm{PD}(r_1, \cdots, r_k|s)$ form an $\mathbb{R}_{>0}$–colored operad in the category of complex manifolds under insertions of polydisks.

Suppose that $\mathcal{F}$ is a discretely translation-invariant prefactorization algebra on $\mathbb{C}^n$ with values in DVS. We may then identify $\mathcal{F}(\mathrm{PD}_r(z)) \simeq \mathcal{F}(\mathrm{PD}_r(z'))$ for any two $z, z' \in \mathbb{C}^n$ using the isomorphisms (2.9), and denote the corresponding complex simply by $\mathcal{F}_r$. For each $p \in \mathrm{PD}(r_1, \cdots, r_k|s)$, we have a multilinear map

$$m[p] : \mathcal{F}_{r_1} \times \cdots \times \mathcal{F}_{r_k} \mapsto \mathcal{F}_s \tag{2.10}$$

As explained in Section 2.2.2, we say that $m[p]$ *depends smoothly on* p if

$$m \in \mathrm{DVS}(\mathcal{F}_{r_1}, \cdots, \mathcal{F}_{r_k}, \mathbf{C}^\infty(\mathrm{PD}(\mathbf{r_1}, \cdots, \mathbf{r_k}|\mathbf{s}), \mathcal{F}_\mathbf{s})).$$

To formulate the definition of smooth translation-invariance, we will need the notion of a derivation of a prefactorization algebra.

Definition 2.2.12 ([13])**.** A degree k derivation of a prefactorization algebra $\mathcal{F}$ is a collection of maps $D_U : \mathcal{F}(U) \mapsto \mathcal{F}(U)$ of cohomological degree k for each open subset $U \subset M$, with the property that for any finite collection $U_1, \cdots, U_n \subset V$ of disjoint opens and elements $\alpha_i \in \mathcal{F}(U_i)$, the following version of the Leibniz rule holds

$$D_V m_V^{U_1,\cdots,U_n}(\alpha_1, \cdots, \alpha_n) = \sum_i (-1)^{k(|\alpha_1|+\cdots+|\alpha_{i-1})|} m_V^{U_1,\cdots,U_n}(\alpha_1, \cdots, \alpha_{i-1}, D_{U_i}\alpha_i, \cdots, \alpha_n)$$

As shown in [13], the derivations of $\mathcal{F}$ form a DGLA, with bracket $[D, D']_U = [D_U, D'U]$ and differential d given by $dD_U = [d_U, D_U]$, where d_U is the differential on $\mathcal{F}(U)$.

The notion of smoothly translation-invariant prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ can now be formulated as follows:

Definition 2.2.13 ([13])**.** A prefactorization algebra $\mathcal{F}$ on $\mathbb{C}^n$ with values in DVS is (smoothly) translation-invariant if:

1. $\mathcal{F}$ is discretely translation-invariant.
2. The maps (2.10) are smooth as functions of $p \in \mathrm{PD}(r_1, \cdots, r_k|s)$
3. $\mathcal{F}$ carries an action of the complex Abelian Lie algebra $\mathbb{C}^n$ by derivations compatible with differentiating $m[p]$.

We can further refine the notion of translation-invariance to consider the holomorphic structure. We say that $\mathcal{F}$ is *holomorphically translation invariant* if

- $\mathcal{F}$ is smoothly translation invariant.
- There exist degree -1 derivations $\eta_i : \mathcal{F} \to \mathcal{F}$ such that
  - $[d, \eta_i] = \frac{\partial}{\partial_{\bar{z}_i}}$ (as derivations of $\mathcal{F}$)
  - $[\eta_i, \eta_j] = [\eta_i, \frac{\partial}{\partial_{\bar{z}_j}}] = 0$

  for $i = 1, \cdots, n$, and where d is the differential on $\mathcal{F}$.

This condition means that anti-holomorphic vector fields act homotopically trivially on $\mathcal{F}$.

As explained in [13], if $\mathcal{F}$ is a holomorphically translation-invariant prefactorization algebra, then upon passing to cohomology, the induced structure maps

$$m[p] : H^*(\mathcal{F}_{r_1}) \times \cdots \times H^*(\mathcal{F}_{r_k}) \mapsto H^*(\mathcal{F}_s) \tag{2.11}$$

are holomorphic as functions of $p \in \mathrm{PD}(r_1, \cdots, r_k|s)$. In other words, m can be viewed as a map

$$m : H^*(\mathcal{F}_{r_1}) \times \cdots \times H^*(\mathcal{F}_{r_k}) \mapsto \mathrm{Hol}(\mathrm{PD}(r_1, \cdots, r_k|s), H^*(\mathcal{F}_s)) \tag{2.12}$$

where Hol denotes the space of holomorphic maps in DVS.

Example 2.2.14. For any Lie algebra $\mathfrak{g}$, the holomorphic factorization envelope $\mathbf{U}(\mathfrak{g} \otimes \Omega^{0,*}_{\mathbb{C}^n})$ of Example 2.2.5 is holomorphically translation-invariant, with $\eta_i = \frac{d}{d(d\bar{z}_i)}$

2.3 Factorization algebras from holomorphic fibrations

In this section, we describe our main construction of prefactorization algebras from locally trivial holomorphic fibrations.

Our starting point is the following data:

- Complex manifolds F, X.
- $(\mathfrak{g}, \langle , \rangle)$ a Lie algebra with an invariant bilinear form.

- A locally trivial holomorphic fibration $\pi : E \to X$ with fiber F (i.e. locally isomorphic to $X \times F$)

We begin by constructing a sheaf $\widehat{\mathfrak{g}}_E$ of L_∞ algebras on the total space E of the fibration. Let

$$\mathfrak{g}_E = \mathfrak{g} \otimes \Omega^{0,*}_E$$

$\mathfrak{g}_E$ is a sheaf of DGLA's on E with bracket

$$[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes \alpha \wedge \beta$$

and differential $\overline{\partial}$. Let K_E be the complex

$$K_E := \mathbb{C}[2] \to \Omega^{0,*}_E[1] \xrightarrow{\partial} \Omega^{1,*}_E.$$

where the first arrow is an inclusion, and the second is given by ∂. K_E can be viewed as a double complex, with $\overline{\partial}$ acting as vertical differential on $\Omega^{p,*}_E$, $p = 0, 1$. Let $\mathcal{K}_E = \mathrm{Tot}(K_E)$ - the totalization of this double complex.

Let

$$\phi^{(1)} : (\mathfrak{g}_E)^{\otimes 2} \to \Omega^{1,*}_E$$

$$\phi^{(1)}((J \otimes \alpha) \otimes (J' \otimes \beta)) = \frac{1}{2}\langle J, J'\rangle(\alpha \wedge \partial\beta - (-1)^{|\alpha|}\partial\alpha \wedge \beta), \quad \alpha, \beta \in \Omega^{0,*}_E$$

and

$$\phi^{(0)} : (\mathfrak{g}_E)^{\otimes 3} \to \Omega^{0,*}_E[1]$$

$$\phi^{(0)}((J \otimes \alpha) \otimes (J' \otimes \beta) \otimes (J'' \otimes \gamma)) = \frac{1}{2}\langle [J, J'], J''\rangle \alpha \wedge \beta \wedge \gamma$$

We may view $\phi = \phi^{(0)} + \phi^{(1)}$ as an cochain in the cohomological Chevalley-Eilenberg complex

$$\mathcal{C}^*(\mathfrak{g}_E, \mathcal{K}_E)$$

of total degree 2. A short calculation shows the following:

Lemma 2.3.1. *ϕ defines a cocycle in $\mathcal{C}^*(\mathfrak{g}_\pi, \mathcal{K}_E)$ of total degree 2.*

We can now use the cocycle ϕ to equip

$$\widehat{\mathfrak{g}}_E := \mathfrak{g}_E \oplus \mathcal{K}_E$$

with the structure of an L_∞ central extension of $\mathfrak{g}_E$. Explicitly, this means that $\widetilde{d} = d + d_{CE} + \phi$ is a differential of cohomological degree 1 on the complex of sheaves on E

$$\mathcal{C}_*^{Lie}(\widehat{\mathfrak{g}}_E) := \mathrm{Sym}(\widehat{\mathfrak{g}}_E[1]),$$

where

- $d = \overline{\partial} + d_{\mathcal{K}_E}$ is the differential on $\widehat{\mathfrak{g}}_E$ (with the summands acting on $\mathfrak{g}_E, \mathcal{K}_E$ respectively)
- d_{CE} is the Chevalley-Eilenberg differential of $\mathfrak{g}_E$
- ϕ is extended to $\mathrm{Sym}(\widehat{\mathfrak{g}}_E[1])$ as a co-derivation.

2.3.1 The prefactorization algebra $\mathcal{F}_{\mathfrak{g},\pi}$

We proceed to construct a prefactorization algebra on X - the base of the fibration $\pi : E \to X$. For an open subset $U \subset X$, let

$$\widehat{\mathfrak{g}}_\pi^c(U) = \Gamma_c(U, \pi_*(\widehat{\mathfrak{g}}_E)) \tag{2.13}$$

In other words, $\widehat{\mathfrak{g}}_\pi^c$ is the cosheaf of sections with compact support of $\pi_*(\widehat{\mathfrak{g}}_E)$. $\widehat{\mathfrak{g}}_\pi^c$ has the structure of a complex of pre-cosheaves of L_∞ algebras on X.

Let

$$\mathcal{F}_{\mathfrak{g},\pi} := \mathcal{C}_*^{Lie}(\widehat{\mathfrak{g}}_\pi^c) \tag{2.14}$$

be the corresponding homology Chevalley-Eilenberg complex, which for each open $U \subset X$ assigns the complex

$$\mathcal{F}_{\mathfrak{g},\pi}(U) = \mathcal{C}_*^{Lie}(\widehat{\mathfrak{g}}_\pi^c(U)) \tag{2.15}$$

$\widehat{\mathfrak{g}}_\pi^c$ is equipped with a DVS structure as in Example 2.2.9, and therefore so is $\mathcal{F}_{\mathfrak{g},\pi}$, being constructed from (naive) tensor products of $\widehat{\mathfrak{g}}_\pi^c$.

Proposition 2.3.2. *$\mathcal{F}_{\mathfrak{g},\pi}$ has the structure of a pre-factorization algebra on X valued in $dg-\mathrm{DVS}$. When $X = \mathbb{C}^n, n \geq 1$, $\mathcal{F}_{\mathfrak{g},\pi}$ is holomorphically translation-invariant.*

Proof. We begin by noting that if $\pi : E \to X$ is a locally trivial fibration and W is a smooth vector bundle on E, then $\pi_*\widetilde{W}$ is fine. The smooth translation-invariance of $\mathcal{F}_{\mathfrak{g},\pi}$ is established just as in the example of the free scalar field in Section 4.8 of [13]. Finally, $\eta_i = \frac{d}{d(d\bar{z}_i))}$ are degree (-1) derivations satisfying the conditions in the definition of holomorphic translation-invariance. □

2.3.2 The prefactorization algebra $\mathcal{G}_{\mathfrak{g},\pi}$

In this section we discuss some prefactorization algebras closely related to $\mathcal{F}_{\mathfrak{g},\pi}$, which are more convenient from a computational standpoint. While the definition of $\mathcal{F}_{\mathfrak{g},\pi}$ is reasonably simple, explicit calculations of $H^*(\mathcal{F}_{\mathfrak{g},\pi}(U))$ for an open subset $U \subset X$ require the $\overline{\partial}$-cohomology of the complex $\Gamma_c(U, \pi_*(\widehat{\mathfrak{g}}_E))$ in (2.13). This complex involves forms with compact support along the base X and arbitrary support along the fiber F, and its $\overline{\partial}$-cohomology even when E is a trivial fibration is a certain completion of $H_c^{0,*}(U) \otimes H^{0,*}(F)$ whose explicit description involves non-trivial analytic issues, due to the failure of naive Kunneth-type theorems for Dolbeault cohomology.

Let us first suppose that $E = X \times F$ is a trivial fibration. We have a map of co-sheaves

$$\Omega_{X,c}^{p,q} \otimes \Gamma(F, \Omega_F^{p',q'}) \to (\pi_*\Omega_E^{p+p',q+q'})_c$$

where the subscript c denotes sections with compact support. Explicitly, for an open subset $U \subset X$, this map is

$$\Omega^{p,q}_{X,c}(U) \otimes \Gamma(F, \Omega^{p',q'}_F) \to \Gamma_c(U, \pi_* \Omega^{p+p',q+q'}_E)$$
$$\alpha \otimes \beta \to \alpha \wedge \beta$$

It is injective provided all three factors are non-zero.

We may now define a sub-cosheaf $\widehat{\mathfrak{g}}^{\#c}_\pi$ of $\widehat{\mathfrak{g}}^c_\pi$ by

$$\widehat{\mathfrak{g}}^{\#c}_\pi := \mathfrak{g} \otimes \Omega^{0,*}_{X,c} \otimes \Gamma(F, \Omega^{0,*}_F) \oplus \mathcal{K}^{\#}_\pi$$

where

$$\mathcal{K}^{\#}_\pi := \mathrm{Tot}(\Omega^{0,*}_{X,c} \otimes \Gamma(F, \Omega^{0,*}_F) \xrightarrow{\partial} \Omega^{1,*}_{X,c} \otimes \Gamma(F, \Omega^{0,*}_F) \oplus \Omega^{0,*}_{X,c} \otimes \Gamma(F, \Omega^{1,0}_F))$$

where $\overline{\partial}$ acts "vertically" within each term. $\widehat{\mathfrak{g}}^{\#c}_\pi$ may be equipped with a $dg-$DVS structure as in Example 2.2.10.

The L_∞ structure on $\widehat{\mathfrak{g}}^c_\pi$ induces one on the sub-complex $\widehat{\mathfrak{g}}^{\#c}_\pi$. The advantage of $\widehat{\mathfrak{g}}^{\#c}_\pi$ lies in the fact that it's constructed from ordinary (algebraic) tensor products of complexes whose cohomology is easy to describe. We now define the prefactorization algebra $\mathcal{G}_{\mathfrak{g},\pi}$ on X by

$$\mathcal{G}_{\mathfrak{g},\pi} := \mathcal{C}^{Lie}_*(\widehat{\mathfrak{g}}^{\#c}_\pi)$$

The arguments of the previous section show:

Proposition 2.3.3. *$\mathcal{G}_{\mathfrak{g},\pi}$ has the structure of a pre-factorization algebra on X. When $X = \mathbb{C}^n, n \geq 1$, $\mathcal{G}_{\mathfrak{g},\pi}$ is holomorphically translation-invariant.*

Suppose now that $\pi : E \to X$ is a general locally trivial fibration. We may then construct a map

$$\mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi}$$

locally, depending on certain choices. Suppose $x \in X$, and $z_1, \cdots z_n$ a system of holomorphic coordinates centered at x. These identify a neighorhood U of x with a subset of $\mathbb{C}^n$. If we choose a trivialization $E|_U \simeq X \times F$, we may proceed as above to construct a map of DGLA's $\widehat{\mathfrak{g}}_\pi^{\#c} \to \widehat{\mathfrak{g}}_\pi^c$, inducing a map $\mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi}$. This map depends on both the choice of local coordinate and trivialization.

$\mathcal{G}_{\mathfrak{g},\pi}^{alg}$

When the fiber F is a smooth complex affine variety and $X = \mathbb{C}^n$, we may further refine $\mathcal{G}_{\mathfrak{g},\pi}$ to obtain a prefactorization algebra $\mathcal{G}_{\mathfrak{g},\pi}^{alg}$ with stronger finiteness properties, by considering the algebraic rather than analytic cohomology of $\mathcal{O}_F$. This variation will be important in the next section, when we make contact with vertex algebras. Let $\mathcal{O}_F^{alg}$ denote the sheaf of algebraic regular functions on F, and $\Omega^{1,alg}$ the sheaf of Kahler differentials. Since F is affine, hence Stein, we have

$$H^0(F, \mathcal{O}_F^{alg}) \subset H^0(F, \mathcal{O}_F) \hookrightarrow (\Omega_F^{0,*}, \overline{\partial})$$
$$H^0(F, \Omega_F^{1,alg}) \subset H^0(F, \Omega_F^1) \hookrightarrow (\Omega_F^{1,*}, \overline{\partial})$$

We define

$$\widehat{\mathfrak{g}}_\pi^{\#c,alg} := \mathfrak{g} \otimes \Omega_{X,c}^{0,*} \otimes H^0(F, \mathcal{O}_F^{alg}) \oplus \mathcal{K}_\pi^{\#,alg}$$

where

$$\mathcal{K}_\pi^{\#,alg} := \mathrm{Tot}(\Omega_{X,c}^{0,*} \otimes H^0(F, \mathcal{O}_F^{alg}) \xrightarrow{\partial} \Omega_{X,c}^{1,*} \otimes H^0(F, \mathcal{O}_F^{alg}) \oplus \Omega_{X,c}^{0,*} \otimes H^0(F, \Omega_F^{1,alg}))$$

and $\overline{\partial}$ acts vertically on $\Omega_{X,c}^{*,*}$. Now take $\mathcal{G}_{\mathfrak{g},\pi}^{alg} := \mathcal{C}_*^{Lie}(\widehat{\mathfrak{g}}_\pi^{\#c})$, with the structure maps induced from those of $\mathcal{G}_{\mathfrak{g},\pi}$. $\widehat{\mathfrak{g}}_\pi^{\#c,alg}$ may be equipped with the DVS structure of Example 2.2.10, yielding a prefactorization algebra in $dg - \mathrm{DVS}$.

Proposition 2.3.4. *Suppose that F is a smooth complex affine variety and $X = \mathbb{C}^n$. Then $\mathcal{G}_{\mathfrak{g},\pi}^{alg}$ has the structure of a holomorphically translation-invariant pre-factorization algebra valued in $dg - \mathrm{DVS}$.*

The reasoning at the end of the previous section shows that for a general locally trivial fibration $\pi : E \to X$, a choice of point $x \in X$, local coordinates $z_1, \cdots, z_n$ on $x \in U \subset X$, and trivialization of $E|_U$, yields prefactorization algebra maps

$$\mathcal{G}^{alg}_{\mathfrak{g},\pi} \to \mathcal{G}_{\mathfrak{g},\pi} \to \mathcal{F}_{\mathfrak{g},\pi}|_U$$

2.4 $X = \mathbb{C}$ and vertex algebras

In [13], it is shown that prefactorization algebras on $X = \mathbb{C}$ which are holomorphically translation-invariant and S^1-equivariant for the natural action by rotations are closely related to vertex algebras. More precisely, given such a prefactorization algebra $\mathcal{F}$, $V(\mathcal{F}) = \bigoplus_l H^*(\mathcal{F}^{(l)}(\mathbb{C}))$, the direct sum of S^1-eigenspaces in $H^*(\mathcal{F}(\mathbb{C}))$ has a vertex algebra structure. We begin by reviewing this correspondence following [13], and then apply it to the case of the prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$, where $\pi : \mathbb{C} \times F \mapsto \mathbb{C}$ is the trivial fibration on $\mathbb{C}$ with fiber a smooth complex affine variety F. We show that resulting vertex algebra is isomorphic to $V(\widehat{\mathfrak{g}}_R)$ where $R = H^*(F, \mathcal{O}^{alg}_F)$ from Section 2.1.4. As a special case, when $F = (\mathbb{C}^*)^k$, we recover a toroidal vertex algebra.

2.4.1 Prefactorization algebras on $\mathbb{C}$ and vertex algebras

We review here the correspondence between prefactorization algebras on $\mathbb{C}$ and vertex algebras established in [13], where we refer the reader for details. Recall that S^1 acts on $\mathbb{C}$ by rotations via $z \mapsto \exp(i\theta)z$. Suppose that $\mathcal{F}$ is a prefactorization algebra on $\mathbb{C}$ that is holomorphically translation-invariant and S^1-equivariant. Let $\mathcal{F}(r) := \mathcal{F}(D(0,r))$ be the complex assigned by $\mathcal{F}$ to a disk of radius r (we allow here $r = \infty$, in which case $D(0,\infty) = \mathbb{C}$), and $\mathcal{F}^{(l)}(r) \subset \mathcal{F}(r)$ be the lth eigenspace for the S^1-action. The following theorem from [13] establishes a bridge between prefactorization and vertex algebras:

Theorem 2.4.1 (Theorem 5.2.2.1 [13]). *Let $\mathcal{F}$ be a unital S^1-equivariant holomorphically translation invariant prefactorization algebra on $\mathbb{C}$. Suppose*

- *The action of S^1 on $\mathcal{F}(r)$ extends smoothly to an action of the algebra of distributions on S^1.*
- *For $r < r'$ the map*
$$\mathcal{F}^{(l)}(r) \to \mathcal{F}^{(l)}(r')$$
is a quasi-isomorphism.
- *The cohomology $\mathrm{H}^*(\mathcal{F}^{(l)}(r))$ vanishes for $l \gg 0$.*
- *For each l and $r > 0$ we require that $\mathrm{H}^*(\mathcal{F}^{(l)}(r))$ is isomorphic to a countable sequential colimit of finite dimensional vector spaces.*

Then $V(\mathcal{F}) := \bigoplus_l \mathrm{H}^*(\mathcal{F}^{(l)}(r))$ (which is independent of r by assumption) has the structure of a vertex algebra.

We briefly sketch how the vertex algebra structure on $V(\mathcal{F})$ can be extracted from the prefactorization structure on $\mathcal{F}$.

- Polydisks in one dimension are simply disks, and we denote $\mathrm{PD}(r_1, \cdots, r_k|s)$ by $\mathrm{Discs}(r_1, \cdots, r_k|s)$. If $r_i' < r_i$, we obtain an inclusion
$$\mathrm{Discs}(r_1, \cdots, r_k|s) \subset \mathrm{Discs}(r_1', \cdots, r_k'|s) \tag{2.16}$$
In the limit $\lim_{r_i \to 0}$, these spaces approach Conf_k, the configuration space of k distinct points in $\mathbb{C}$.
- The structure maps 2.12 are compatible with the maps $\mathcal{F}_{r_i'} \mapsto \mathcal{F}_{r_i}$ and the inclusions 2.16, and one may take $\lim_{r_i \to 0}$, $s = \infty$, obtaining maps
$$m : (\lim_{r \to 0} H^*(\mathcal{F}_r))^{\otimes k} \to \mathrm{Hol}(\mathrm{Conf}_k, H^*(\mathcal{F}(\mathbb{C})) \tag{2.17}$$

- We set $k = 2$, and fix one of the points to be the origin. There is a natural map $V(\mathcal{F}) \mapsto \lim_{r\to 0} H^*(\mathcal{F}_r)$, as well as projections $H^*(\mathcal{F}(\mathbb{C})) \to H^*(\mathcal{F}^{(l)}(\mathbb{C}))$. Pre and post-composing by these in 2.17, yields a map

$$\overline{m_{0,z}} : V(\mathcal{F}) \otimes V(\mathcal{F}) \to \prod_l \mathrm{Hol}(\mathbb{C}^\times, V(\mathcal{F})_l) \tag{2.18}$$

  where $V(\mathcal{F})_l = H^*(\mathcal{F}^{(l)}(\mathbb{C}))$. Laurent expanding $\overline{m_{0,z}}$ we obtain

$$\overline{m_{0,z}} : V(\mathcal{F}) \otimes V(\mathcal{F}) \to \prod_l V(\mathcal{F})_l[[z, z^{-1}]]$$

  whose image can be shown to lie in $V(\mathcal{F})((z))$. The vertex operator can now be defined by

$$Y : V(\mathcal{F}) \to \mathrm{End}(V(\mathcal{F}))[[z, z^{-1}]]$$
$$Y(v, z)v' = m_{0,z}(v', v)$$

- Holomorphic translation invariance yields an action of ∂_z

$$\partial_z : \mathcal{F}^{(l)}(r) \to \mathcal{F}^{(l-1)}(r).$$

  which descends to $H^*(\mathcal{F}^l(r))$. This induces the translation operator $T : V(\mathcal{F}) \to V(\mathcal{F})$.

- The vacuum vector is obtained from the unit in $\mathcal{F}(\emptyset)$.

2.4.2 The main theorem

Our goal in this section is to prove the following theorem

Theorem 2.4.2. *Let F be a smooth complex affine variety, and $\pi : \mathbb{C} \times F \to \mathbb{C}$ the trivial fibration with fiber F. Then*

1. *The prefactorization algebra $\mathcal{G}^{alg}_{\mathfrak{g},\pi}$ satisfies the hypotheses of Theorem 2.4.1*

2. The vertex algebra $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ is isomorphic to $V(\widehat{\mathfrak{g}}_{\mathbb{R}})$, with $R = H^0(F, \mathcal{O}^{alg}_F)[t, t^{-1}]$

Throughout this section, R will denote the algebra $H^0(F, \mathcal{O}^{alg}_F)[t, t^{-1}]$. We will denote $H^0(F, \mathcal{O}^{alg}_F)$ simply by $\mathbb{C}[F]$, so $R = \mathbb{C}[F][t, t^{-1}]$. Recall that

$$\begin{aligned}\widehat{\mathfrak{g}}_R &= \mathfrak{g} \otimes \mathbb{C}[F][t, t^{-1}] \oplus \Omega^1_{\mathbb{C}[F][t,t^{-1}]}/d(\mathbb{C}[F][t, t^{-1}]) \\ &= \mathfrak{g} \otimes \mathbb{C}[F][t, t^{-1}] \oplus \frac{\mathbb{C}[t, t^{-1}] \otimes \Omega^1_{\mathbb{C}[F]} \oplus \mathbb{C}[F] \otimes \Omega^1_{\mathbb{C}[t,t^{-1}]}}{\langle t^k du + kt^{k-1}udt \rangle}\end{aligned}$$

Recollections on Dolbeault cohomology and preliminary computations

In this section we recall some facts regarding ordinary and compactly supported Dolbeault cohomology and apply these to compute $H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(U))$ over opens $U \subset \mathbb{C}$. These results will be used in proving Theorem 2.4.2.

Recall that Stein manifolds are complex analytic analogues of smooth affine varieties over $\mathbb{C}$. In particular, $\mathbb{C}$ and all open subsets $U \subset \mathbb{C}$ are Stein. We recall the following basic result:

Theorem 2.4.3 (Cartan's Theorem B)**.** *Let X be a Stein manifold. Then*

$$H^k(\Omega^{p,*}(X), \overline{\partial}) = \begin{cases} 0 & k \neq 0 \\ \Omega^p_X & k = 0 \end{cases}$$

where Ω^p_X denotes the space of holomorphic p-forms on Z.

On a complex manifold X of dimension n, there is a non-degenerate pairing between ordinary and compactly supported forms

$$\begin{aligned}\Omega^{p,q}_{X,c} \otimes \Omega^{n-p,n-q}_X &\to \mathbb{C} \\ \alpha \otimes \beta &\to \int_X \alpha \wedge \beta\end{aligned}$$

thus compactly supported forms yield continuous linear functionals on forms. At the level Dolbeault cohomology, one obtains the following corollary to Threorem 2.4.3

noted by Serre:

Corollary 2.4.4. *Let X be a Stein manifold. Then*

$$H^k(\Omega_c^{p,*}(X), \overline{\partial}) = \begin{cases} 0 & k \neq dim(X) \\ (\Omega_X^{n-p}(X))^\vee & k = n = dim(X) \end{cases}$$

where $(\Omega_X^{n-p}(X))^\vee$ denotes the continuous dual to the space of holomorphic $n-p$ forms with respect to the Frechet topology.

With $X = \mathbb{C}$, F a complex affine variety, and the trivial fibration $\pi : X \times F \to X$, the cosheaf $\widehat{\mathfrak{g}}_\pi^{\#c,alg}$ on $\mathbb{C}$ has the form

$$\widehat{\mathfrak{g}}_\pi^{\#c,alg} := \mathfrak{g} \otimes \mathbb{C}[F] \otimes \Omega_{X,c}^{0,*} \oplus \mathcal{K}_\pi^{\#,alg}$$

where $\mathcal{K}_\pi^{\#,alg}$ is the total complex of the following double complex

$$\begin{array}{ccc} \Omega_c^{0,1} \otimes \mathbb{C}[F] & \xrightarrow{\partial + d} & \Omega_c^{1,1} \otimes \mathbb{C}[F] \oplus \Omega_c^{0,1} \otimes \Omega^1_{\mathbb{C}[F]} \\ \uparrow \overline{\partial} & & \overline{\partial} \uparrow \\ \Omega_c^{0,0} \otimes \mathbb{C}[F] & \xrightarrow{\partial + d} & \Omega_c^{1,0} \otimes \mathbb{C}[F] \oplus \Omega_c^{0,0} \otimes \Omega^1_{\mathbb{C}[F]} \end{array}$$

in which $\Omega_c^{p,q}$ denote cosheaves of compactly supported forms on $\mathbb{C}$, and $\Omega^1_{\mathbb{C}[F]}$ is the space of *algebraic* 1-forms (i.e. Kahler differentials) on F.

$$\mathcal{G}_{\mathfrak{g},\pi}^{alg} = \mathcal{C}_*^{Lie}(\widehat{\mathfrak{g}}_\pi^{\#c,alg}) = \mathrm{Sym}(\widehat{\mathfrak{g}}_\pi^{\#c,alg}[1], \widetilde{d})$$

where the co-derivation $\widetilde{d}$ may be decomposed as $\widetilde{d} = d_1 + d_2 + d_3$, with

$$d_i : \mathrm{Sym}^i(\widehat{\mathfrak{g}}_\pi^{\#c,alg}[1]) \to \widehat{\mathfrak{g}}_\pi^{\#c,alg}[1]$$

of degree 1. We have

$$d_1 = \overline{\partial} + d_{\mathcal{K}_\pi^{\#,alg}}$$

The complex $\mathcal{C}_*^{Lie}(\widehat{\mathfrak{g}}_\pi^{\#c,alg})$ has an increasing filtration by symmetric degree, leading to a spectral sequence whose E_0 page is $\mathrm{Sym}(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}[1], d_1)$, as d_2 and d_3 lower symmetric degree. We have

$$H^*(\mathrm{Sym}(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}[1], d_1)) = \mathrm{Sym}(H^*(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}[1], d_1))$$

Now $(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}, d_1)$ is the direct sum of the complexes $(\mathfrak{g} \otimes \mathbb{C}[F] \otimes \Omega^{0,*}_c, \overline{\partial})$ and $\mathcal{K}^{\#,alg}_{\pi}$. Applying Theorem 2.4.3 on an open Stein subset $U \subset \mathbb{C}$, the cohomology of the first is

$$\mathfrak{g} \otimes \mathbb{C}[F] \otimes (\Omega^1(U))^\vee.$$

Similarly, by first computing the $\overline{\partial}$ cohomology in $\mathcal{K}^{\#,alg}_{\pi}$, we have

$$H^*(\mathcal{K}^{\#,alg}_{\pi}(U)) = \mathrm{Coker}\Big((\Omega^1_X(U))^\vee \otimes \mathbb{C}[F]) \overset{1\otimes\partial+\partial^\vee\otimes 1}{\rightarrow} (\Omega^1_X(U))^\vee \otimes \Omega^1_{\mathbb{C}[F]} \oplus (\mathcal{O}(U))^\vee \otimes \mathbb{C}[F]\Big) \quad (2.19)$$

where $\partial^\vee$ denotes the transpose of $\partial : \Omega^{n-1}_X(U) \mapsto \Omega^n_X(U)$. We have the following:

Lemma 2.4.5. *Let $U \subset \mathbb{C}$ be an open subset. Then*

$$H^*(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}(U), d_1) = \mathfrak{g} \otimes (\Omega^1_X(U))^\vee \otimes \mathbb{C}[F] \bigoplus \mathrm{Coker}\Big((\Omega^1_X(U))^\vee \otimes \mathbb{C}[F]) \overset{1\otimes\partial+\partial^\vee\otimes 1}{\rightarrow} (\Omega^1_X(U))^\vee \otimes \Omega^1_{\mathbb{C}[F]} \oplus (\mathcal{O}(U))^\vee \otimes \mathbb{C}[F]\Big)$$

where $\partial^\vee$ denotes the transpose of $\partial : \mathcal{O}_X(U) \mapsto \Omega^1_X(U)$.

It follows that $H^*(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}[1], d_1)$ is concentrated in degree 0. Since $\widetilde{d}$ has cohomological degree 1, this means that the spectral sequence computing $H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(U))$ collapses at E_1, and we have an isomorphism of vector spaces

$$H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(U)) \simeq \mathrm{Sym}(H^*(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}(U)[1], d_1))$$

Suppose now that $U \subset \mathbb{C}$ is a disk or annulus centered at 0. We have a non-degenerate residue pairing

$$\Omega^1(\mathbb{C}^*) \otimes \mathcal{O}(\mathbb{C}^*) \to \mathbb{C}$$
$$\omega \otimes f \to \mathrm{Res}_{z=0} f\omega$$

which yields maps $\Omega^1(\mathbb{C}^*) \to (\mathcal{O}(U))^\vee$ and $\mathcal{O}(\mathbb{C}^*) \to (\Omega^1(U))^\vee$.

Lemma 2.4.6. *Let z be a global coordinate on $\mathbb{C}$, and $U = D(0,r)$ a disk. There is an isomorphism of vector spaces*

$$V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \simeq \mathrm{Sym}(\widehat{\mathfrak{g}}_S/\widehat{\mathfrak{g}}^+_S) \simeq U(\widehat{\mathfrak{g}}_S) \otimes_{U(\widehat{\mathfrak{g}}^+_S} \mathbb{C} \tag{2.20}$$

where $S = \mathbb{C}[F][z, z^{-1}]$.

Proof. We introduce the vector spaces

$$\begin{aligned} S^+ &= \mathbb{C}[F][z] \\ S^- &= \mathbb{C}[F] \otimes z^{-1}\mathbb{C}[z^{-1}] \\ \Omega^1_{S^+} &= \Omega^1_{\mathbb{C}[F]} \otimes \mathbb{C}[z] \oplus \mathbb{C}[z]dz \otimes \mathbb{C}[F] \\ \Omega^1_{S^-} &= \Omega^1_{\mathbb{C}[F]} \otimes z^{-1}\mathbb{C}[z^{-1}] \oplus z^{-1}\mathbb{C}[z^{-1}]dz \otimes \mathbb{C}[F] \end{aligned}$$

We have $S = S^+ \oplus S^-$ and $\Omega^1_S = \Omega^1_{S^+} \oplus \Omega^1_{S^-}$ as vector spaces, and these decompositions are moreover compatible with the differential, in the sense that $d(S^\pm) \in \Omega^1_{S^\pm}$. Hence

$$\Omega^1_S/dS \simeq \Omega^1_{S^+}/dS^+ \oplus \Omega^1_{S^-}/dS^-$$

which implies that as vector spaces

$$\widehat{\mathfrak{g}}_S/\widehat{\mathfrak{g}}^+_S \simeq \mathfrak{g} \otimes S^- \oplus \Omega^1_{S^-}/dS^-$$

We note that the S^1 action on $\mathbb{C}$ extends naturally to $\widehat{\mathfrak{g}}^{\#c,alg}_\pi(D(0,r))$ and $\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r))$, and that the differential $\widetilde{d}$ on the latter is S^1-equivariant. Using the residue pairing to identify $\mathbb{C}[z, z^1]$ with a subspace of $(\Omega^1(U))^\vee$ and $\mathbb{C}[z, z^{-1}]dz$ with a subspace of $(\mathcal{O}(D(0,r)))^\vee$, we obtain that

$$H^*(\widehat{\mathfrak{g}}^{\#c,alg}_\pi(D(0,r)), d_1)^{(l)} \simeq \mathfrak{g} \otimes \mathbb{C}[F] \otimes \{z^l\} \oplus \left(\Omega^1_{\mathbb{C}[F]} \otimes \{z^l\} \oplus \mathbb{C}[F] \otimes \{z^{l-1}dz\}\right) / \mathrm{im}(\widetilde{d})$$

when $l \geq 0$ and 0 otherwise. Therefore, as vector spaces

$$\begin{aligned} V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) &= \mathrm{Sym}\left(\bigoplus_{l\geq 0} H^*(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}(D(0,r)),d_1)^{(l)}\right) \\ &= \mathrm{Sym}(\mathfrak{g}\otimes S^- \oplus \Omega^1_{S^-}/dS^-) \\ &= \mathrm{Sym}(\widehat{\mathfrak{g}}_S/\widehat{\mathfrak{g}}_S^+) \end{aligned}$$

□

Verifying the hypotheses of Theorem 2.4.1

We proceed to verify the hypotheses of Theorem 2.4.1, establishing part (1) of Theorem 2.4.2 above.

- The first hypothesis is verified as in Section 5.4.2 of [13]
- The second and third hypotheses follow from Lemma 2.4.6, from which it follows in particular that $H^*((\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))^{(l)})$ is non-zero only if $l \leq 0$
- The last hypothesis requires some attention. We note that it is necessary to ensure that the structure map $m_{z,0}$ possesses a Laurent expansion. By Lemma 2.4.6 $H^*((\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))^{(l)})$ may be identified with the elements of weight l in

  $$\mathrm{Sym}\left(\mathfrak{g}\otimes S^- \oplus \Omega^1_{S^-}/dS^-\right)$$

  where z and dz have S^1 - weight 1. We begin by showing that $\mathbb{C}[F]$ and $\Omega^1_{\mathbb{C}[F]}$ are naturally a sequential colimit of finite-dimensional vector spaces. This can be done as follows. Embed $F \subset \mathbb{A}^N = \mathrm{Spec}\,\mathbb{C}[x_1,\cdots,x_N]$. This induces an increasing filtration $F^k\mathbb{C}[F]$, $k \geq 0$, where $F^k\mathbb{C}[F]$ is spanned by the images of polynomials of degree $\leq k$ in $x_1,\cdots,x_N$. $\mathbb{C}[F]$ and by the same reasoning $\Omega^1_{\mathbb{C}[F]}$ can therefore be expressed as a countable union of finite-dimensional vector spaces. This induces a filtration on $\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}$ compatible with the DVS structure, which in turn induces one on $H^*((\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))^{(l)})$.

Constructing the isomorphism

We proceed to prove part (2) of Theorem 2.4.2. The proof is a variation on the approach taken in Williams with respect to the Virasoro factorization algebra, and involves three main steps:

1. Showing that $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ has the structure of a $\widehat{\mathfrak{g}}_R$-module.

2. Showing that $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \simeq V(\widehat{\mathfrak{g}}_R)$ as $\widehat{\mathfrak{g}}_R$-modules.

3. Checking that the vertex algebra structures agree by using the reconstruction theorem 2.1.5.

Let $\rho : \mathbb{C}^\times \to \mathbb{R}_{>0}$ be the map $\rho(z) = z\overline{z} = |z|^2$. As explained in example **??**, the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{g}}_R)$, being an associative algebra, defines a prefactorization algebra on $\mathbb{R}_{>0}$ which we denote $\mathcal{AU}(\widehat{\mathfrak{g}}_R)$.

Lemma 2.4.7. *There is a homomorphism $\phi : \mathcal{AU}(\widehat{\mathfrak{g}}_R) \to \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ of prefactorization algebras on $\mathbb{R}_{>0}$.*

Proof.

- It is shown in [13] that a map of prefactorization algebras on $\mathbb{R}_{>0}$ is determined by the maps ϕ_I on connected open intervals. For each open interval $I \subset \mathbb{R}_{>0}$, $A_I = \rho^{-1}(I)$ is an annulus. We choose for each such a bump function $f_I : A_I \to R$ having the properties

  - f is a function of $r^2 = z\overline{z}$ only.
  - $f \geq 0$ and f is supported in A_I.
  - $\int_A f dz d\overline{z} = 1$.

  ϕ_I is uniquely determined by where it sends the generators of $\widehat{\mathfrak{g}}_R$. We define ϕ_I by the assignments:

$$\phi_I(J \otimes ut^k) = -[J \otimes uz^{k+1} f_I d\overline{z}]$$
$$\phi_I(t^k\omega) = [z^{k+1} f_I \omega \wedge d\overline{z}]$$
$$\phi_I(t^k u dt) = [uz^{k+1} f_I dz d\overline{z}]$$

where $J \in \mathfrak{g}, u \in \mathbb{C}[F], \omega \in \Omega^1_{\mathbb{C}[F]}$, and $[\alpha] \in H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(\rho^{-1}(I)))$ denotes the cohomology class of α. The elements on the right are clearly closed for the differential $\widetilde{d}$, and the corresponding cohomology classes are easily seen to be independent of the choice of the function f.

- We first check that ϕ_I is well-defined, which amounts to verifying that

$$\phi_I(d(ut^k)) = \phi_I(t^k du + kt^{k-1}udt) = [0] \in H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(\rho^{-1}(I))$$

for each $u \in \mathbb{C}[F], k \in \mathbb{Z}$. We have

$$\phi_I(t^k du + kt^{k-1}udt) = -[z^{k+1}f_I d\overline{z}du] - [kz^k uf_I dzd\overline{z}]$$

We note that this cohomology class lies in the image of the natural map

$$\Omega^{1,1}_c(\rho^{-1}(I))\otimes\mathbb{C}[F]\oplus\Omega^{0,1}_c(\rho^{-1}(I))\otimes\Omega^1_{\mathbb{C}[F]} \subset (\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}(\rho^{-1}(I)))^{cl} \to H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(\rho^{-1}(I)) \tag{2.21}$$

where $(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}(\rho^{-1}(I)))^{cl}$ denotes the subspace of sections closed for the differential d_1, and therefore $\widetilde{d}$. The projection 2.21 may be factored by first taking the quotient by $\mathrm{im}(\overline{\partial})$, in other words as

$$\Omega^{1,1}_c(\rho^{-1}(I)) \otimes \mathbb{C}[F] \oplus \Omega^{0,1}_c(\rho^{-1}(I)) \otimes \Omega^1_{\mathbb{C}[F]} \to$$
$$(\Omega^1_X(U))^\vee \otimes \Omega^1_{\mathbb{C}[F]} \oplus (\mathcal{O}(U))^\vee \otimes \mathbb{C}[F] \to H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(\rho^{-1}(I))$$

We calculate

$$\begin{aligned}
-[z^{k+1}f_I dud\overline{z}] &- [kz^k uf_I dzd\overline{z}] + [d_{\mathcal{K}^{\#,alg}_{\pi}}(z^{k+1}uf_I d\overline{z})] \\
&= -[z^{k+1}f_I dud\overline{z}] - [kz^k uf_I dzd\overline{z}] + \\
&= [z^{k+1}f_I dud\overline{z} + (k+1)z^k uf_I dzd\overline{z} + uz^{k+1}\partial f \wedge d\overline{z}] \\
&= +[u(z^k f_I dzd\overline{z} + z^{k+1}\partial f \wedge d\overline{z})]
\end{aligned}$$

It therefore suffices to show that $[z^k f_I dzd\overline{z} + z^{k+1}\partial_z f_I \wedge d\overline{z}] = [0] \in (\mathcal{O}_X(U))^\vee$, or equivalently, that

$$\int_U z^m \wedge (z^k f_I dzd\overline{z} + z^{k+1}\partial f \wedge d\overline{z}) = 0 \ \ \forall m \in \mathbb{Z}$$

This follows from the identities

$$\int_U z^a f_I dz d\overline{z} = \delta_{a,0} \quad , \int_U z^b \partial f \wedge d\overline{z} = -\delta_{b,1} \tag{2.22}$$

which can be established integrating by parts.

- Consider three disjoint open intervals $I_1, I_2, I_3 \subset \mathbb{R}_{>0}$, such that I_{i+1} is located to the right of I_i, all contained in a larger interval I. Their inverse images under ρ correspond to three nested annuli A_{I_i} inside a larger annulus A_I. We have structure maps

$$\bullet_{i,i+1} : \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I_i) \otimes \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I_{i+1}) \to \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I) \;\; i = 1, 2$$

To show that ϕ is a prefactorization algebra homomorphism, we have to check that for $X, Y \in \widehat{\mathfrak{g}}_R$,

$$\phi_{I_1}(X) \bullet_{1,2} \phi_{I_2}(Y) - \phi_{I_2}(Y) \bullet_{2,3} \phi_{I_3}(X) = \phi_I([X, Y])$$

Let

$$F_m(z, \overline{z}) = z^m \int_0^{z\overline{z}} (f_{I_1}(s) - f_{I_3}(s)) ds$$

Then on A_{I_2}, $F_m = z^m$, and moreover,

$$\begin{aligned}
\overline{\partial} F_m(z, \overline{z}) &= z^m \overline{\partial}\Big(\int_0^{z\overline{z}} (f_{I_1}(s) - f_{I_3}(s)) ds\Big) \\
&= z^m \frac{\partial(z\overline{z})}{\partial \overline{z}} \frac{\partial}{\partial(z\overline{z})}\Big(\int_0^{z\overline{z}} (f_{I_1}(s) - f_{I_3}(s)) ds\Big) d\overline{z} \\
&= z^{m+1}(f_{I_1}(z\overline{z}) - f_3(z\overline{z})) d\overline{z}
\end{aligned}$$

Let $J_1, J_2 \in \mathfrak{g},\ u, v \in \mathbb{C}[F]$. Then

$$\begin{aligned}
&\phi_{I_1}(J_1 ut^k) \bullet_{1,2} \phi_{I_2}(J_2 vt^l) - \phi_{I_2}(J_2 vt^l) \bullet_{2,3} \phi_{I_3}(J_1 ut^k) - \phi_{I_2}([J_1 ut^k, J_2 vt^l]) \\
&\quad = \phi_{I_1}(J_1 ut^k) \bullet_{1,2} \phi_{I_2}(J_2 vt^l) - \phi_{I_2}(J_2 vt^l) \bullet_{2,3} \phi_{I_3}(J_1 ut^k) \\
&\qquad - \phi_{I_2}\left([J_1, J_2]uvt^{k+l} + \frac{1}{2}\langle J_1, J_2\rangle (ut^k d(vt^l) - vt^l d(ut^k))\right) \\
&\quad = \big([J_1 uz^{k+1} f_{I_1} d\overline{z}] \cdot [J_2 vz^{l+1} f_{I_2} d\overline{z}] - [J_2 vz^{l+1} f_{I_2} d\overline{z}] \cdot [J_1 uz^{k+1} f_{I_3} d\overline{z}]\big) + \\
&\qquad + [[J_1, J_2] uvz^{k+l+1} f_{I_2} d\overline{z}] + \frac{1}{2}\langle J_1, J_2\rangle [z^{k+l+1} f_{I_2} \\
&\quad = (udv - vdu)d\overline{z} + (l-k)uvz^{k+l} f_{I_2} dz d\overline{z}]
\end{aligned}$$

We also have

$$\begin{aligned}
&\widetilde{d}\big([J_1 uF_k] \cdot [J_2 vz^{l+1} f_{I_2} d\overline{z}]\big) \\
&\quad = \big([J_1 uz^{k+1} f_{I_1} d\overline{z}] \cdot [J_2 vz^{l+1} f_{I_2} d\overline{z}] - [J_2 vz^{l+1} f_{I_2} d\overline{z}] \cdot [J_1 uz^{k+1} f_{I_3} d\overline{z}]\big) \\
&\quad = +[[J_1, J_2] uvz^{k+l+1} f_{I_2} d\overline{z}] \\
&\qquad + \frac{1}{2}\langle J_1, J_2\rangle [uF_k \partial(vz^{l+1} f_{I_2} d\overline{z}) - \partial(F_k u) \wedge (z^{l+1} v f_{I_2} d\overline{z})] \\
&\quad = \big([J_1 uz^{k+1} f_{I_1} d\overline{z}] \cdot [J_2 vz^{l+1} f_{I_2} d\overline{z}] - [J_2 vz^{l+1} f_{I_2} d\overline{z}] \cdot [J_1 uz^{k+1} f_{I_3} d\overline{z}]\big) \\
&\quad = +[[J_1, J_2] uvz^{k+l+1} f_{I_2} d\overline{z}] \\
&\qquad + \frac{1}{2}\langle J_1, J_2\rangle [z^{k+l+1}(udv - vdu) f_{I_2} d\overline{z} \\
&\quad = +uv((l-k+1) f_{I_2} z^{k+l} dz d\overline{z} - z^{k+l+1} \partial f_{I_2} \wedge d\overline{z})]
\end{aligned}$$

where we have used the fact that over the support of f_{I_2}, $F_k = z^k$. Using the identities 2.22, we obtain

$$[(l-k+1)z^{k+l} f_{I_2} dz d\overline{z} - z^{k+l+1} \partial f_{I_2} \wedge d\overline{z}] = [(l-k) z^{k+l} f_{I_2} dz d\overline{z}].$$

It follows that

$$\begin{aligned}
&\phi_{I_1}(J_1 ut^k) \bullet_{1,2} \phi_{I_2}(J_2 vt^l) - \phi_{I_2}(J_2 vt^l) \bullet_{2,3} \\
&\qquad \phi_{I_3}(J_1 ut^k) - \phi_{I_2}([J_1 ut^k, J_2 vt^l]) = 0 \in \rho_* H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi})(I) \qquad (2.23)
\end{aligned}$$

proving the lemma.

□

The homomorphism ϕ of Proposition (2.4.7) equips $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ with the structure of a $\widehat{\mathfrak{g}}_R$-module. Let us fix $0 < r < r' < R$ We have the following commutative diagram:

$$\begin{array}{ccc} H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r))) \otimes H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(A(r',R))) & \xrightarrow{m} & H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R)) \\ {\scriptstyle \iota\otimes\phi}\uparrow & & {\scriptstyle \iota}\uparrow \\ V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \otimes \mathcal{A}\mathcal{U}(\widehat{\mathfrak{g}}_R) & \dashrightarrow & V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \end{array}$$

where ι denotes the inclusion of $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \subset H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r)))$ (for any r), and m is the prefactorization structure map. As explained in [13], the existence of the dotted arrow (i.e. the fact that the $\mathcal{U}(\widehat{\mathfrak{g}}_R)$-action preserves the subspace $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$) follows from the fact that the structure map m is S^1-equivariant.

In concrete terms, the action of $X \in \widehat{\mathfrak{g}}_R$ on $v \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ is given as follows: we may represent v by a closed chain $\widetilde{v} \in \mathcal{C}^{Lie}_*(\widehat{\mathfrak{g}}^{\#c,alg}_{\pi}(D(0,r))$ - then $X \cdot v$ is represented by $\phi_{(r',R)}(X) \cdot \widetilde{v}$.

Lemma 2.4.8. *There is an isomorphism of $\widehat{\mathfrak{g}}_R$-modules*

$$\eta : V(\widehat{\mathfrak{g}}_R) \to V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$$

which sends $|0\rangle \in V(\widehat{\mathfrak{g}}_R)$ to $1 \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$.

Proof. Let $h(z,\overline{z}) = \int_0^{z\overline{z}} f(s)ds$. By the chain rule, we have that

$$\overline{\partial}(z^n h(z,\overline{z})) = z^{n+1} f(z\overline{z}) d\overline{z}$$

Thus in $H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R))$, we have in the notation of ??, for $k \geq 0$

$$\begin{aligned} \phi_{(r',R)}(Jut^k) &= [Juz^{k+1} f(z\overline{z}) d\overline{z}] = \widetilde{d}(Juz^k h(z,\overline{z})) \\ \phi_{(r',R)}(t^k udv) &= [z^{k+1} f(z\overline{z}) ud\overline{z}dv] = \widetilde{d}(z^k h(z,\overline{z}) udv) \\ \phi_{(r',R)}(ut^k dt) &= [uz^{k+1} f(z\overline{z}) d\overline{z}dz] = \widetilde{d}(uz^k h(z,\overline{z}) dz) \end{aligned}$$

In other words, if $X \in \widehat{\mathfrak{g}}_R^+$, then $\phi_{(r',R)}(X) = 0 \in H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R))$. This shows that the vector $1 \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ is annihilated by $\widehat{\mathfrak{g}}_R^+$. It follows that there exists a unique map of $\widehat{\mathfrak{g}}_R$-modules $\eta : V(\widehat{\mathfrak{g}}_R) \to V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ sending $|0\rangle \to 1$. It remains to show this is an isomorphism, which can be done as in **??** for the affine and Virasoro algebra, so we will be brief. $V(\widehat{\mathfrak{g}}_R)$ and $V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$ have the structure of filtered $\mathcal{U}(\widehat{\mathfrak{g}}_R)$-modules, where in each case the filtration is induced by symmetric degree. It is straightforward to verify that η induces an isomorphism at the level of associated graded modules, proving the result.

□

To complete the proof of Theorem 2.4.2, we check that η induces an isomorphism of vertex algebras. Suppose that $z \in A((r',R))$. Recall that the operation

$$Y : V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \otimes V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \to V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})((z))$$

is induced from the diagram

$$\begin{array}{ccc} V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) \otimes V(\mathcal{G}^{alg}_{\mathfrak{g},\pi}) & & \\ \big\downarrow{\scriptstyle \iota\otimes\iota_z} & & \\ H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(z,\epsilon))) \otimes H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,r))) & \xrightarrow{m_{z,0}} & H^*(\mathcal{G}^{alg}_{\mathfrak{g},\pi}(D(0,R))) \end{array}$$

as the Laurent expansion of the map $m_{z,0} \circ \iota \otimes \iota_z$. By the Reconstruction Theorem 2.1.5, it suffices to show that the generating field assignments agree, that is we need to verify that for $v \in V(\mathcal{G}^{alg}_{\mathfrak{g},\pi})$,

$$m_{z,0}(\iota_z(\eta(Jut^{-1} \cdot |0\rangle)), \iota(v)) = \sum_{n\in\mathbb{Z}} (\phi(Jut^n) \cdot v) z^{-n-1}$$

$$m_{z,0}(\iota_z(\eta(ut^{-1}dt \cdot |0\rangle)), \iota(v)) = \sum_{n\in\mathbb{Z}} (\phi(ut^{n-1}dt) \cdot v) z^{-n}$$

$$m_{z,0}(\iota_z(\eta(t^{-1}\omega \cdot |0\rangle)), \iota(v)) = \sum_{n\in\mathbb{Z}} (\phi(t^n\omega) \cdot v) z^{-n-1}$$

$\iota_z(\eta(Jut^{-1}\cdot|0\rangle))$ may be identified with the element $Ju\psi_z \in \mathfrak{g}\otimes\mathbb{C}[F]\otimes(\Omega^1(D(z,\epsilon)))^\vee$, where $\psi_z \in (\Omega^1(D(z,\epsilon)))^\vee$ is defined by

$$\psi_z(h(w)dw) = \frac{1}{2\pi i}\oint_{C(z,\delta)} \frac{h(w)dw}{w-z}$$

By the residue theorem, for $h(w)dw \in \Omega^1(A(r,R))$, we may switch contours, to write

$$\begin{aligned}\oint_{C(z,\delta)} \frac{h(w)dw}{w-z} &= \oint_{C(0,R-\delta)} \frac{h(w)dw}{w-z} - \oint_{C(0,r'+\delta)} \frac{h(w)dw}{w-z} \\ &= \sum_{n\geq 0}(\oint_{C(0,R-\delta)} w^{-n-1}h(w)dw)z^n + \sum_{n<0}(\oint_{C(0,r'+\delta)} w^{-n-1}h(w)dw)z^n\end{aligned}$$

where in the second line we have expanded $\frac{1}{w-z}$ into a geometric series in the domains $|w|>|z|$ and $|w|<|z|$ respectively. Using the fact that

$$Res_0 h(w)w^{-n-1}dw = \int_{A(r',R)} h(w)w^{-n}f_{(r,R)}dwd\overline{w}$$

and $\phi(Jut^{-n-1})\cdot v = [Juz^{-n}f_{(r',R)}d\overline{z}]\cdot v$ we obtain the first identity. Similarly, we may identify $\iota_z(\eta(ut^{-1}dt\cdot|0\rangle))$ with the element $u\xi_z \in \mathbb{C}[F]\otimes(\mathcal{O}(D(z,\epsilon)))^\vee$, where

$$\xi_z(h(w)) = h(z) = \frac{1}{2\pi i}\oint_{C(z,\delta)} \frac{h(w)dw}{w-z}$$

and $\iota_z(\eta(t^{-1}\omega\cdot|0\rangle))$ with $\omega\psi_z \in \Omega^1_{\mathbb{C}[F]}\otimes(\Omega^1(D(z,\epsilon)))^\vee$. Expanding these in contour integrals centered at 0, and identifying the coefficients with appropriate elements in the image of ϕ as above proves the remaining two identities.

Chapter 3

Averages of Unlabeled Networks

3.1 The space of unlabeled networks

Our ultimate focus in this chapter is on a certain well-defined notion of an 'average' on elements drawn randomly from a 'space' of unlabeled networks and on the statistical behavior of such averages. Accordingly, we need to establish and understand the relevant topology and geometry of this space. We do so by associating labeled networks with vectors and mapping those to unlabeled networks through the use of equivalence classes in an appropriate quotient space. In this section we provide relevant definitions, characterization, and illustrations of this space of unlabeled networks.

3.1.1 The topological space of unlabeled networks

Let $G = (V, E)$ be a labeled, undirected graph/network with weighted edges and with d vertices/nodes. We always think of E as having $D := \binom{d}{2}$ elements, where some of the edge weights can be zero. We think of the edge weight between vertices i and j as the strength of some unspecified relationship between i and j.

Let Σ_d be the group of permutations of $\{1, 2, \ldots, d\}$. A permutation $\sigma \in \Sigma_d$ of the d vertex labels technically produces a new graph σG, but with no new information. To define σG precisely, note that the weight function $w_G : E \to \mathbb{R}_{\geq 0}$ can be thought of as a symmetric function $w_G : V \times V \to \mathbb{R}_{\geq 0}$, with $w_G(i, j)$ the weight of the edge joining vertex i and vertex j in G. Therefore the action of Σ_d on w_G is given by

$$(\sigma \cdot w_G)(i,j) = w_G(\sigma^{-1}(i), \sigma^{-1}(j)).$$

(The inverse guarantees that $(\sigma\tau) \cdot w_G = \sigma \cdot (\tau \cdot w)_G$.) Note that for general G, not all permutations of the entries of w_G are of the form $\sigma \cdot w_G$, as w_G may have $[d(d-1)/2]!$ distinct permutations and Σ_d has $d!$ elements

In summary, σG is defined to be the graph on d vertices with weight function $\sigma \cdot w_G : E \to \mathbb{R}_{\geq 0}$. Let $\mathcal{G} = \mathcal{G}_d$ be the set of all labeled graphs with d vertices. Then the quotient space

$$\mathcal{U}_d = \mathcal{G}_d / \Sigma_d$$

is the space of unlabeled graphs, the object we want to study. This means that an unlabeled network $[G] \in \mathcal{U}_d$ is an equivalence class

$$[G] := \{\sigma \cdot G : \sigma \in \Sigma_d\}.$$

As we now explain, $\mathcal{G}_d$ looks like an explicit subset of $\mathbb{R}^d$, and so is easy to picture. In contrast, the quotient space $\mathcal{U}_d$ is difficult to picture. Nevertheless, as we describe in the following paragraphs, the topology of $\mathcal{U}_d$ may be characterized through standard point-set topology techniques, with the conclusion that everything works as well as possible.

Fix an ordering of the vertices $1, ..., d$, and take the lexicographic ordering $\{(i,j) : 1 \leq i < j \leq d\}$ on the set of edges. (Thus $(i,j) < (k,\ell)$ if $i < k$ or $i = k$ and $j < \ell$.) Given this ordering, we get an injection

$$\alpha : \mathcal{G}_d \to \mathbb{R}^D, \quad \alpha(G) = (w_1(G), \ldots, w_D(G)),$$

where $w_i(G)$ is the weight of the i^{th} edge of G. The image of α is the first "octant"

$\mathcal{O}_D = \{\vec{x} = (x^1, \ldots, x^D) : x^i \geq 0\}$. For simplicity, we choose the standard Euclidean metric on $\mathcal{O}_D$. This pulls back via α to a metric on $\mathcal{G}_d$ with the desirable property that two networks are close iff their edge weights are close. Similarly, the standard topology on $\mathcal{O}_D$ (an open ball in $\mathcal{O}_D$ is the intersection of an open $\mathbb{R}^D$-Euclidean ball with $\mathcal{O}_D$) pulls back to a topology on $\mathcal{G}_D$. (This just means that $A \subset \mathcal{G}_D$ is open iff $\alpha(U)$ is open in $\mathcal{O}_D$. This makes α a homeomorphism.) Just as in $\mathbb{R}^D$, the metric and topology are compatible: a sequence of graphs/weight vectors $\vec{x}_i$ in $\mathcal{O}_D$ converges to a graph/weight vector $\vec{x}$ in the topology of $\mathcal{O}_D$ iff the distance from $\vec{x}_i$ to $\vec{x}$ goes to zero.

Via the bijection α, the action of Σ_d on $\mathcal{G}_d$ transfers to an action on $\mathcal{O}_D$. First, $\sigma \in \Sigma_d$ acts on $\{1, \ldots, D\}$ by $\sigma \cdot i = j$ if i corresponds to the edge (i_1, i_2) and j corresponds to the edge $(\sigma(i_1), \sigma(i_2))$. Then σ acts on $\mathcal{O}_D$ by $\sigma \cdot \vec{x} = (x^{\sigma^{-1}(1)}, \ldots, x^{\sigma^{-1}(D)})$. Since we've arranged the actions to be compatible with $\alpha : \mathcal{G}_d \to \mathcal{O}_D$, we get a well defined bijection $\overline{\alpha}$:

$$\mathcal{U}_d = \mathcal{G}_d/\Sigma_d \xrightarrow{\overline{\alpha}} \mathcal{O}_D/\Sigma_d, \ \overline{\alpha}[G] = [\alpha(G)].$$

From now on, we just denote $\overline{\alpha}$ by α.

To complete the topological discussion, we note that $\alpha : \mathcal{U}_d \to \mathcal{O}_D/\Sigma_d$ is a homeomorphism if we give both sides the quotient topology: for the map $q : \mathcal{G}_d \to \mathcal{U}_d$ taking a graph to its equivalence class, a set $U \subset \mathcal{U}_d$ is open iff $q^{-1}(U)$ is open in $\mathcal{O}_D$. The quotient topology on $\mathcal{O}_D/\Sigma_d$ is defined similarly.

3.1.2 Examples of quotient spaces

As a warmup, we first give a simple example of a quotient space resulting from the action of a finite group on a Euclidean space. This particular example is important in providing a relevant non-network analogy to our network-based results. We will revisit it frequently throughout the chapter.

Example 3.1.1. The group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ acts on the plane $\mathbb{R}^2$ by rotation counterclockwise by 90 degrees: specifically, for $k \in \mathbb{Z}_4$ and $z \in \mathbb{R}^2 = \mathbb{C}$,

$$k \cdot z = e^{ik\pi/2} \cdot z.$$

Thus $0 \cdot z = z, 1 \cdot z = e^{i\pi/2}z$, etc. A point in the quotient space $\mathbb{C}/\mathbb{Z}_4$ is the set $[z_0] = \{e^{ik\pi/2}z_0 : k \in \mathbb{Z}_4\}$. The set $[z_0]$ is called the orbit of z_0 under $\mathbb{Z}_4$. Note that every orbit is a four element set except for the exceptional orbit $[\vec{0}] = \{\vec{0}\}$.

The closed first quadrant $F = \{(x, y) : x \geq 0, y \geq 0\}$ is a *fundamental domain* for this action; *i.e.*, each orbit $[z_0]$ has a unique representative/element in F, except possibly for the orbits of points on the boundary $\partial F = \{(x, y) : x = 0 \text{ or } y = 0\}$ of F. Orbits of boundary points like $[(5, 0)]$ have two representatives $(5, 0), (0, 5)$ in F, while the origin of course has only one representative.

Here is a precise definition of a fundamental domain for the action of a group G on a set S:

Definition 3.1.2. $F \subset S$ is a fundamental domain for the action of G if (i) S is the union of the orbits of F ($S = \cup_{k \in G} k \cdot F$);
(ii) orbits can intersect only at boundary points ($k_1 \cdot F \cap k_2 \cdot F = \emptyset$ or $k_1 \cdot F \cap k_2 \cdot F \subset \partial(k_1 \cdot F) \cap \partial(k_2 \cdot F)$).

In this example, $G = \mathbb{Z}_4$ and $S = \mathbb{C}$. It follows that the quotient map $q : F \to \mathbb{C}/\mathbb{Z}_4$ is surjective, a homeomorphism on the interior of F (where $\mathbb{C}/\mathbb{Z}_4$ has the quotient topology), and finite-to-one on the boundary of F.

If we want to picture a set that is bijective to $\mathbb{C}/\mathbb{Z}_4$, we could take e.g. F' to be F minus the positive y-axis. This is not so helpful topologically or geometrically, as the points $[(5, 0)]$ and $[(0, 5.01)]$ have close representatives $(5, 0), (5.01, 0)$ in F, while their representatives $(5, 0)$ and $(0, 5.01)$ are not close in F'. In particular, the sequence $(10^{-k}, 5)$ does not converge in F', but the orbits $[10^{-k}, 5]$ converge to $[0, 5] = [5, 0]$ in $\mathbb{C}/\mathbb{Z}_4$. Thus F' does not give us a good picture of $\mathbb{C}/\mathbb{Z}_4$ topologically.

In summary, it is much better to keep both positive axes in F, and to consider $\mathbb{C}/\mathbb{Z}_4$ as (in bijection with) F with the boundary points $(a, 0)$ and $(0, a)$ "glued

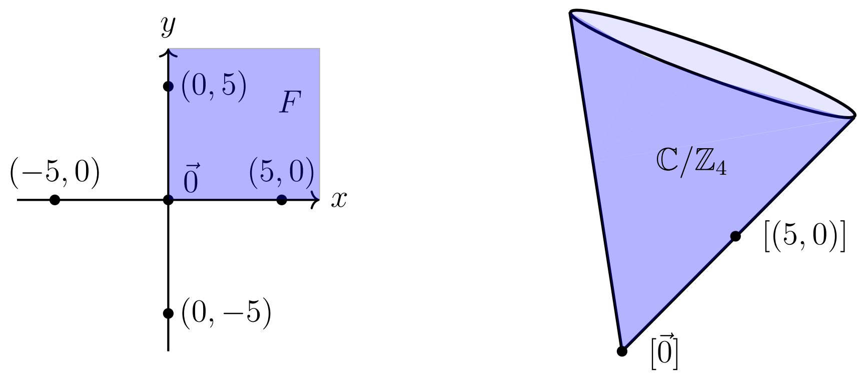

Figure 3·1: In the figure on the left, F is a fundamental domain F for the action of $\mathbb{Z}_4$ on $\mathbb{C}$. The four point orbit of $(5, 0)$ and the one point orbit of $\vec{0}$ are shown. In the figure on the right, the quotient space $\mathbb{C}/\mathbb{Z}_4$ is drawn as a hollow cone given by taking F and gluing $(x, 0)$ to $(0, x)$.

together." More precisely, we have a bijection

$$\beta : \mathcal{F} := \frac{F}{(a, 0) \sim (0, a)} \to \mathbb{C}/\mathbb{Z}_4,$$

where the denominator indicates that the two point set $\{(a, 0), (0, a)\}$ $(a \neq 0)$ is one point of $\mathcal{F}$, while all other points of F correspond to a single point in $\mathcal{F}$. At the price of this gluing, we now have that β is a homeomorphism: in particular, $\lim_{i\to\infty} x_i = x$ in $\mathcal{F}$ iff $\lim_{i\to\infty} \beta(x_i) = \beta(x)$ in $\mathbb{C}/\mathbb{Z}_4$. (Technical remark: $\mathcal{F}$ gets the quotient topology from the standard topology on F and the obvious surjection $q : F \to \mathcal{F}$.)

Although this seems a little involved, it is quite easy to perform the gluing in $\mathcal{F}$ in rubber sheet topology: stretching the interior of F to allow the gluing of the two axes shows that $\mathcal{F}$ and hence $\mathbb{C}/\mathbb{Z}_4$ is a hollow cone. See Figure 3·1.

Example 3.1.3. We discuss the case of a network with three vertices. This is a deceptively easy case, as $3 = d = D$ implies that every permutation of the D edge weights comes from a permutation in Σ_d. In higher dimensions, the details are more complicated.

We describe the quotient space $\mathcal{U}_3$ of unlabeled graphs directly. However, it is easier to picture $\mathcal{U}_3$ by finding a fundamental domain F inside $\mathcal{O}_3$ for the action of Σ_3. As in the previous example, F is a closed set such that the quotient map $q|_F : F \to \mathcal{U}_3$ is a continuous surjection, a homeomorphism from the interior of F to its image, and a finite-to-one map on the boundary ∂F of F. Thus F represents $\mathcal{U}_3$ bijectively except for some gluings on the boundary. This is illustrated in Figure 2, where $F = \{(x, y, z) : x \geq y \geq z \geq 0\}$. Again, the case $d = 3$ is deceptively easy, as F is a bijection even on ∂F.

The situation is more complicated for graphs with 4 (or more) vertices. For $d = 4$, if we label the edges as $(1, 2), \ldots, (3, 4)$, then the weight vectors $(1, 1, 1, 0, 0, 0)$ and $(1, 1, 0, 1, 0, 0)$ have the same distributions of ones and zeros, but correspond to binary graphs which are not in the same orbit of Σ_4. In particular, the region $\{(x_1, \ldots, x_6) : x_1 \geq x_2 \geq \ldots \geq x_6\}$ is not a fundamental domain for the action of Σ_4.

While a fundamental domain is harder to find in high dimensions (see Section 4), the overall structure of $\mathcal{U}_d$ for general d is similar to the $d = 3$ case, with just increased notation.

Theorem 3.1.4. *The space of unlabeled graphs $\mathcal{U}_d = \mathcal{G}/\Sigma_d = \mathcal{O}_D/\Sigma_d$ is a stratified space.*

By Lange, $\mathbb{R}^D/\Sigma_d$ is PL or Lipschitz homeomorphic to $\mathbb{R}^{D-1} \times \mathbb{R}_{\geq 0}$, but the proof does not give a cell decomposition of $\mathbb{R}^D/\Sigma_d$, much less of $\mathcal{U}_d$. We do have information about the topology of $\mathcal{U}_d$: it can be shown that $\mathcal{U}_d$ is contractible, and more surprisingly, that the natural slice of $\mathcal{U}_d$ given by the hyperplane $\sum_{i=1}^{D} x_i = 1$ is contractible. The practical implication of these results is that the usual topological invariants of $\mathcal{U}_d$ and its slice (the fundamental group, the homology/cohomology groups) provide no information.

3.2 Network averages and their asymptotic behavior

In this section we define the mean of a distribution Q on the space of networks and investigate the asymptotic behavior of the empirical (or sample) mean network based

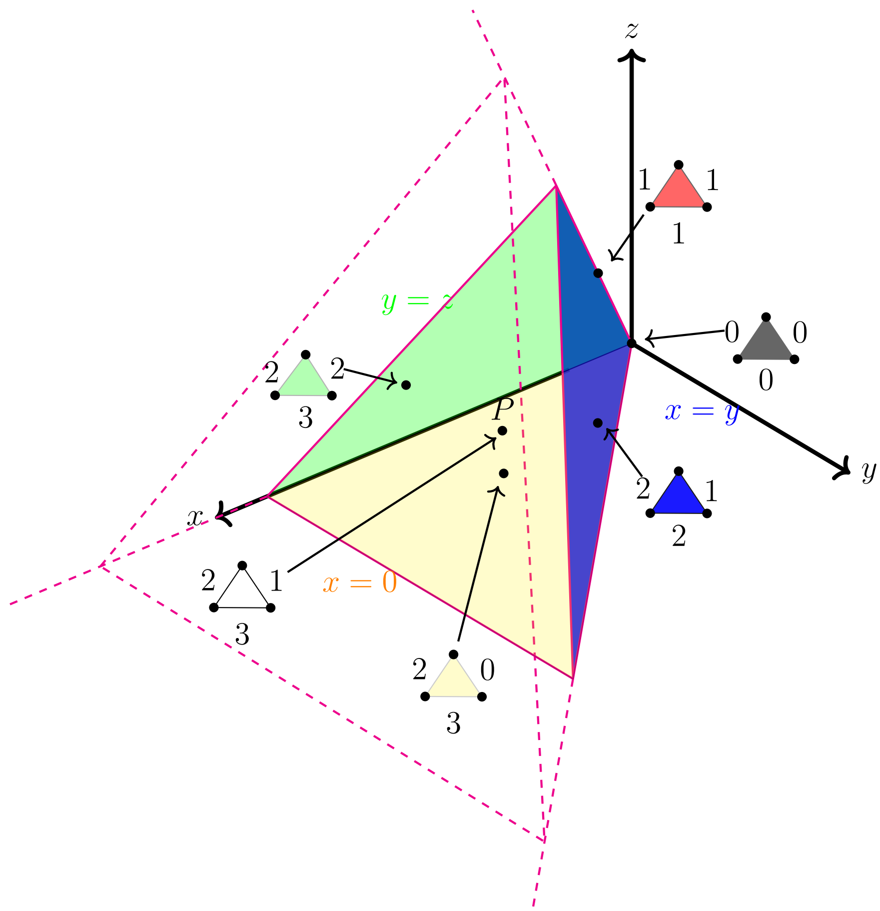

Figure 3·2: As explained in Section 4.1, the infinite solid cone, which is the region $\{x \geq y \geq z \geq 0\}$, is a fundamental domain F for unlabeled networks with three nodes. With the convention that the bottom side of the triangle has weight x, the left side has weight y, and the right side has weight z, the network with edge weights $1, 2, 3$ corresponds to the point P in the interior of the cone. Other networks shown are color coded to correspond to points on faces or edges of the cone.

on an i.i.d sample of networks from Q. Statistical inference can be carried out based on the asymptotic distribution of the empirical mean. We illustrate with an example from hypothesis testing. The results of the previous section, characterizing the topology and geometry of the space of unlabeled networks, are essential for achieving our goals in this section.

3.2.1 Network averages through Fréchet means

Let Q be some distribution on a general metric space (M, ρ). One can define the Fréchet function $f(p)$ on M as

$$f(p) = \int_M \rho^2(p, z) Q(dz) \ (p \in M). \tag{3.1}$$

If f is finite on M and has a unique minimizer

$$\mu = \text{argmin}_p f(p), \tag{3.2}$$

then μ is called the *Fréchet mean* of Q (with respect to the metric ρ). Otherwise, the minimizers of the Fréchet function form a *Fréchet mean set* C_Q. Given an i.i.d sample $X_1, \ldots, X_n \sim Q$ on M, the empirical Fréchet mean can be defined by replacing Q with the empirical distribution $Q_n = \dfrac{1}{n} \sum_{i=1}^n \delta_{X_i}(\cdot)$, that is,

$$\mu_n = \text{argmin}_p \frac{1}{n} \sum_{i=1}^n \rho^2(p, X_i). \tag{3.3}$$

When M is a manifold, one can equip M with a metric space structure through an embedding into some Euclidean space or employing a Riemannian structure of M. Respectively, ρ can be taken to be the Euclidean distance after embedding (extrinsic distance) or the geodesic distance (intrinsic distance), giving rise to extrinsic and intrinsic means. Asymptotic theory for extrinsic and intrinsic analysis has been developed in [7, 9, 10], and applied to many manifolds of interest (see e.g., [6], [17]).

Now take $M = \mathcal{U}_d$, the space of unlabeled networks with d nodes, our space of interest, and let Q be a distribution on $\mathcal{U}_d$. Given an i.i.d sample $X_1, \dots, X_n$ from Q, in order to define the Fréchet mean μ of Q and empirical Fréchet mean μ_n of Q_n, one needs an appropriate choice of distance on $\mathcal{U}_d$. Given the quotient space structure characterized in the previous section, i.e., $\mathcal{U}_d = \mathcal{G}_d/\Sigma_d$, a natural choice for the distance ρ is the *Procrustean distance* d_P, where

$$d_P([\vec{x}], [\vec{y}]) := \min_{\sigma_1, \sigma_2 \in \Sigma_d} d_E(\sigma_1 \cdot \vec{x}, \sigma_2 \cdot \vec{y}), \tag{3.4}$$

for unlabeled networks $[\vec{x}], [\vec{y}] \in \mathcal{U}_d$, with $\vec{x}$ denoting the vectorized representation of a representative network x. We recall that $\mathcal{G}_d$ is the set of all labeled graphs with d vertices and Σ_d is the group of permutations of $\{1, 2, \dots, d\}$.

In order to carry out statistical inference based on μ_n, defined with respect to the distance (3.4), some natural and fundamental questions related to μ and μ_n need to be addressed, which we aim to do in the following subsections. Here are some of the most crucial ones:

1. (Consistency.) What are the consistency properties of the network empirical mean μ_n, i.e., is μ_n a consistent estimator of the population Fréchet mean μ? Can we establish some notion of a law of large numbers for μ_n?

2. (Uniqueness of Fréchet mean.) This question is concerned with establishing general conditions on Q for uniqueness of the Fréchet mean μ. In general this a challenging task – indeed, the lack of general uniqueness conditions for Fréchet means is still one of the main hurdles for carrying out intrinsic analysis on manifolds [27]. To date the most general results in the literature for generic manifolds [1] force the support of Q to be a small geodesic ball to guarantee uniqueness of the intrinsic Fréchet mean. We address this question for the space

of unlabeled networks in Section 3.3.

3. (CLT.) Once conditions for uniqueness of μ are provided, the next key question is whether one can derive the limiting distribution for μ_n for purposes of statistical inference, e.g., proving a central limit type of theorem for μ_n, which in turn might be used for hypothesis testing.

We first illustrate the difficult nature of these problems (in particular for question 2 above) through the example $C = \mathbb{R}^2/\mathbb{Z}_4$ in Section 3.1, by explicitly constructing a distribution on C that has non-unique Fréchet means.

Example 3.2.1 (Example $C = \mathbb{R}^2/\mathbb{Z}_4$ continued)**.** As in Example 3.1.1 in Section 3.1, the quotient $C = \mathbb{R}^2/\mathbb{Z}_4$ is a cone. Working in polar coordinates and taking $F_0 = [0, \infty) \times [-\pi/4, \pi/4)$ to be a fundamental domain, we consider probability distributions of the form $\nu(r, \theta) = \frac{1}{Z} R(r)\chi(\theta)$, where $Z = \int_{F_0} R(r)\chi(\theta) dr\ d\theta$.

We can explicitly compute the Fréchet function $f(x)$ with respect to ν. For $x = (r, \theta) \in F_0$, $F_\theta = [0, \infty) \times [-\pi/4 + \theta, \pi/4 + \theta)$ is a fundamental domain. For $y \in F_\theta$, $d_P([x], [y]) = d_E(x, y)$. Then

$$f(x) = \int\limits_{F_\theta} ||x - y||^2 \nu(y) dy = \frac{1}{Z} \left(c_1\chi_1 r^2 - 2c_2\chi_2(\theta) r + c_3\chi_1 \right),$$

where

$$c_k = \int\limits_0^\infty r^k R(r) dr \text{ for } k = 1, 2, 3$$

$$\chi_1(\theta) = \int\limits_{\theta-\pi/4}^{\theta+\pi/4} \chi(t) dt$$

$$\chi_2(\theta) = \int\limits_{\theta-\pi/4}^{\theta+\pi/4} \chi(t) \cos(\theta - t) dt.$$

The Fréchet mean occurs when $\partial f/\partial r = \partial f/\partial\theta = 0$, which is difficult to compute in general. Consider the special case

$$\nu(r,\theta) = \frac{1}{Z}\exp(-(r-\alpha)^2) \ , \tag{3.5}$$

where α is a fixed constant, $\chi(\theta) \equiv 1$, $Z = (\pi^{3/2}/4)(1+\text{erf}(\alpha))$; this distribution for $\alpha = 15$ is plotted in Figure 3. The minimum for this f occurs at

$$r_0 = \frac{\sqrt{2}\left(2\alpha + \sqrt{\pi}e^{\alpha^2}\left(2\alpha^2+1\right)\left(\text{erf}(\alpha)+1\right)\right)}{\pi^{3/2}e^{\alpha^2}\alpha(\text{erf}(\alpha)+1)+\pi},$$

with θ arbitrary. When α is large, $r_0 \approx \frac{2\sqrt{2}}{\pi}\alpha$. For $\alpha = 15$, $r_0 \approx 13.5348$. This shows that ν has a circle's worth of Fréchet means; the θ-independence of ν implies θ-independence of the Fréchet means. One can see this in Figure 3·4 where the Fréchet function $f(x)$ is minimal and most blue on a circle of radius approximately 13.53 (corresponding to the red circle on the cone in Figure 3·3).

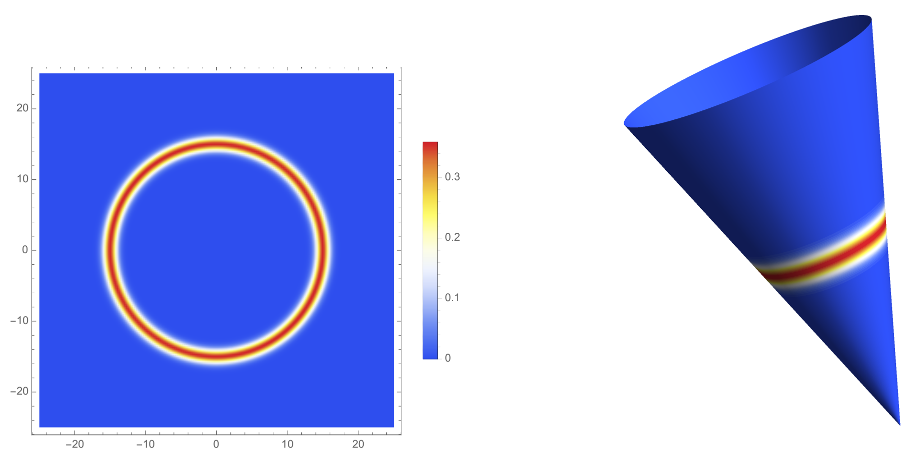

Figure 3·3: Plot of the probability distribution $\nu(r,\theta)$ in (3.5), with $\alpha = 15$. The distribution peaks in the red region where ν is large, and is small in the blue region. The left hand side shows the plot on $\mathbb{R}^2$, and the right hand side shows that plot on the cone.

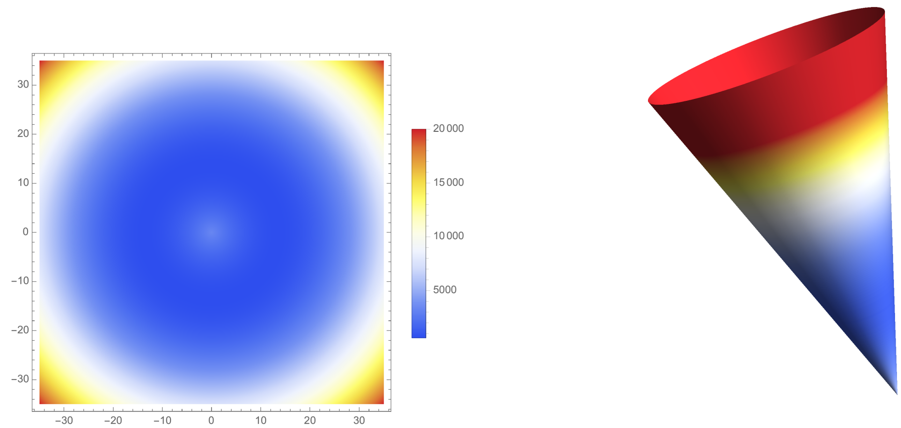

Figure 3·4: Plot of the Fréchet function $f(x)$ with $\alpha = 15$ on $\mathbb{R}^2$ and on the cone. The Fréchet mean occurs in the "most blue" region, on a circle of radius approximately 13.53.

3.2.2 A strong law of large numbers

Before establishing the limiting distribution for μ_n, a natural first step is to explore the consistency properties of μ_n.Drawing on Theorem 3.3 in [7] for general metric spaces, we have the following result.

Theorem 3.2.2. *Let Q be a distribution on $\mathcal{U}_d$, let C_Q be the set of means of Q with respect to the Procrustean distance d_P, and let C_{Q_n} be the set of empirical means with respect to a sample of unlabeled networks $X_1, \ldots, X_n$. Assume that the Fréchet function is everywhere finite. Then the following holds: (a) the Fréchet mean set C_Q is nonempty and compact; (b) for any $\epsilon > 0$, there exists a positive integer-valued random variable $N \equiv N(\epsilon)$ and a Q-null set $\Omega(\epsilon)$ such that*

$$C_{Q_n} \subseteq C_Q^\epsilon \doteq \{p \in \mathcal{U}_d : d_P(p, C_Q) < \epsilon\} \; \forall \; n \geq N \tag{3.6}$$

outside of $\Omega(\epsilon)$; (c) if C_Q is a singleton, i.e., the Fréchet mean μ is unique, then μ_n converges to μ almost surely.

Proof. We first prove that every closed and bounded subset of $M = \mathcal{U}_d$ is compact.

Let F be a fundamental domain for the action of Σ_d on $\mathcal{G}_d$, as defined in Section 4, with the associated projection $q : F \to \mathcal{U}_d$. This map is continuous and a diffeomorphism on the interior of F. Take a closed and bounded set S in $\mathcal{U}_d$. Because q is continuous, $q^{-1}(S)$ is closed. We now show that $q^{-1}(S)$ is also bounded. S is contained in a ball centered at some $[\vec{z}] \in \mathcal{U}_d$ with radius r, so $d_P([\vec{z}], [\vec{x}]) < r$ for $[\vec{x}] \in S$. Now say that the largest entry in $[\vec{z}]$ (in any ordering of the entries of $\vec{z}$) is C. If the largest entry in $\vec{x}$ (in any ordering) is greater than $C + r$, then $d_P([\vec{z}], [\vec{x}]) > (C + r) - C$, a contradiction. (This holds because under any permutation σ of the entries of $\vec{x}$, one entry in $d_E(\vec{z}, \sigma \cdot \vec{x})$ is greater than $(C + r) - C$.) Thus for any choice of $\vec{x} \in [\vec{x}] \in S$, $\|\vec{x}\| \leq \sqrt{D}(C + r)$. Thus $q^{-1}(S)$ is contained in the ball of radius $\sqrt{D}(C + r)$ centered at the origin, and thus is bounded.

Since F is a closed subset of $\mathbb{R}^D$, the closed and bounded set $q^{-1}(S)$ is compact. Since q is continuous, S is compact.

Then by Theorem 3.3 in [7], (a) and (b) follow.

Part (c) follows from [41] under the uniqueness of Fréchet mean. □

Remark 3.2.3. For every model

$$Q \in \mathcal{Q} = \{Q : C_Q \text{ is a singleton with a finite Fréchet function}\}, \tag{3.7}$$

the sample Fréchet mean μ_n is a strongly consistent estimate of μ with respect to this model.

3.2.3 A central limit theorem

The goal of this section is to derive a central limit theorem for the empirical Fréchet mean, as an important precursor for statistical inference. One of the key challenges is to establish geometric conditions on distributions on $\mathcal{U}_d$ which ensure the uniqueness of the population Fréchet mean. We discuss and address the uniqueness issue in detail in Section 4. Here, our central limit theorem assumes that the uniqueness conditions of Section 4 are met.

Let $q : \mathcal{O}_D \to \mathcal{U}_d$ be the projection from the space of labeled networks to the space of unlabeled networks.

Theorem 3.2.4. *Assume Q' has support on a compact set $K' \subset \mathcal{O}_D$ defined in Theorem 3.3.7, so that the pushdown measure $Q = q_* Q'$ supported on $K = q(K')$ has* *a unique Fréchet mean μ. Let μ_n be the empirical Fréchet mean of an i.i.d sample $X_1, \ldots, X_n \sim Q$ with respect to the distance (3.4). Let $\phi = q^{-1}$. Then we have*

$$\sqrt{n}\,(\phi(\mu_n) - \phi(\mu)) \xrightarrow{L} N(0, \Sigma), \tag{3.8}$$

where $\Sigma = \Lambda^{-1} C \Lambda^{-1}$ with the Hessian matrix

$$\Lambda = \left(\mathbb{E}[D_{r,s}\|\phi(\mu) - \phi(X_1)\|]\right)_{r,s=1,\ldots,D},$$

and C is the covariance matrix of $\{D_r\|\phi(\mu) - \phi(X_1)\|\}_{r=1,\ldots,D}$.

Here D_r denotes the partial derivative with respect to the r^{th} direction, $D_{r,s}$ denotes second partial derivatives, and $\xrightarrow{L}$ means convergence in law or distribution.

Proof. Since μ_n converges to μ almost surely under our support condition for Q', one can find a small open neighborhood U of μ inside K such as that $P(\mu_n \in U) \to 1$. Let $S = q^{-1}(U)$ which is an open subset of $\mathbb{R}^D$. Note that $\phi : U \to S$ is a homeomorphism. By Theorem 3.3.7, the projection map q is a Euclidean isometry. Therefore, for any vectorized network $\vec{x} \in S$ and $[\vec{z}] \in U$, one has

$$d_P^2([\vec{x}], [\vec{z}]) = d_P^2(\phi^{-1}(\vec{x}), [\vec{z}]) = \|\vec{x} - \vec{z}\|^2,$$

where $\vec{z} = \phi([\vec{z}])$. Thus $d_P^2(\phi^{-1}(\vec{x}), [\vec{z}])$ is twice differentiable in $\vec{x}$ for any $[\vec{z}] \in U$. Tracing through the definition of the smooth structure on U induced from the standard structure on $\mathbb{R}^D$, we see that $d_P^2([\vec{x}], [\vec{z}])$ is twice differentiable in $[\vec{x}]$. One can also verify the conditions (A5) and (A6) on the Hessian matrices of Theorem 2.2 in [8], and our Theorem follows. □

As an immediate consequence of this central limit theorem, we can define natural analogues of classical hypothesis tests. For example, consider the construction of a statistical test for two or more independent samples using the same framework. Assume that we have k independent sets of networks on d vertices, and consider the problem of testing whether or not these sets have in fact been drawn from the same population. Formally, we have independent samples $X_{ij} \sim Q_j$, for $i = 1, \ldots, n_j$ and

$j = 1, \ldots, k$. Each of these k populations has an unknown mean, denoted $\mu^{(j)}$. Then, as a direct corollary to Theorem 3.2.4, we have the following asymptotic result.

Corollary 3.2.5. Assume that the distributions $Q_1, \ldots, Q_k$ satisfy the conditions of Theorem 3.2.4. Moreover, also assume that $n_j/n \to p_j$ for every sample, with $n := \sum_j n_j$, and $0 < p_j < 1$. Then, under $H_0 : \phi(\mu^{(1)}) = \ldots = \phi(\mu^{(k)})$, we have

$$T_k := \sum_{j=1}^{k} n_j \left(\phi(\mu_{j,n_j}) - \phi(\mu_n)\right)' \widehat{\Xi}^{-1} \left(\phi(\mu_{j,n_j}) - \phi(\mu_n)\right) \longrightarrow \chi^2_{(k-1)D},$$

where μ_{j,n_j} denotes the empirical mean of the j^{th} sample, μ_n represents the grand empirical mean of the full sample, and $\widehat{\Xi} := \sum_{j=1}^{k} \widehat{\Xi}_j/n_j$ is a pooled estimate of covariance, with the $\widehat{\Xi}_j$'s denoting the individual covariance matrices estimates of each subsample.

As previously noted, this central limit theorem and such corollaries hold only if the population Fréchet mean(s) is unique. This depends crucially on the nontrivial geometry of the space of unlabeled networks. The following section deals exclusively with this issue.

3.3 Geometric requirements for uniqueness of the Fréchet mean

Underlying the central limit theorem in Theorem 3.2.4, the basic question is: which compact subsets K of $\mathcal{U}_d$ have a unique Fréchet mean? We have seen in Section 3.1 that $\mathcal{U}_d$ may be difficult to work with, while a fundamental domain $F \subset \mathcal{O}_D$ for the action of Σ_d on the space of labeled networks $\mathcal{O}_D$ seems more tractable. Indeed, finding the Fréchet mean for a distribution supported in F is a standard center of mass calculation in Euclidean space. However, it is not clear that this Fréchet mean in F projects to the Fréchet mean in the quotient space $\mathcal{U}_d = \mathcal{O}_D/\Sigma_d$, because the metric used to compute Fréchet means in $\mathcal{U}_d$ is the Procrustean distance, which may or may not equal the Euclidean distance.

In §4.1, we find a fundamental domain F by a standard procedure (Lemma 3.3.2), and find compact subsets $K' \subset F$ for which the Fréchet mean in $\mathcal{O}_d$ is guaranteed to project to the Fréchet mean of $K \subset \mathcal{U}_d$, where $K = q(K')$ is the projection of K' under the quotient/projection map $q : \mathcal{O}_D \to \mathcal{U}_d = \mathcal{O}_D/\Sigma_d$. This is the content of the main result in this section (Theorem 3.3.7). We also show that this result in our particular setting is an improvement of the best result for general Riemannian manifolds due to Afsari [1] (see Figure 6). In §4.2, we generalize Theorem 3.3.7 in two different directions: Theorem 3.3.9 allows distributions with small tails outside the compact set K above, and Theorem 3.3.10 shows that a unique Fréchet mean can be found for any compact set K after first embedding K isometrically into a larger Banach space.

3.3.1 The main result on the uniqueness of the Fréchet mean

We now discuss the construction of a fundamental domain F. By Definition 3.1.2, a fundamental domain is characterized by: (i) every weight vector x can be permuted by some $\sigma \in \Sigma_d$ to a network $\sigma \cdot x$ in F, (ii) if $\vec{w} \in \mathcal{O}_D$ has $\vec{w} \in F \cap \sigma F$ for $\sigma \in \Sigma_d, \sigma \neq \mathrm{Id}$, then $\vec{w} \in \partial F$. (As a technical note, we always consider ∂F with respect to the induced topology on $\mathcal{O}_D$ from the standard topology on $\mathbb{R}^D$.) Once F has been constructed, we are guaranteed that the projection map $q : \mathcal{O}_D \to \mathcal{U}_d,\ q(\vec{w}) = [\vec{w}]$ restricts to a surjective map $q : F \to \mathcal{U}_d$ which is a homeomorphism from the interior F^o of F to $q(F^o)$. (In fact, q is a diffeomorphism on this region by definition of the smooth structure on $q(F^o)$.)

It is convenient to center a choice of fundamental domain on a weight vector with trivial stabilizer.

Definition 3.3.1. A vector $\vec{w} = (w_1, \dots, w_D) \in \mathcal{O}_D$ is *distinct* if it has trivial stabilizer for the action of Σ_d: *i.e.*, if $\sigma \cdot \vec{w} \neq \vec{w}$ for all $\sigma \in \Sigma_d$, $\sigma \neq \mathrm{id}$.

A vector with trivial stabilizer is also called a vector with trivial automorphism

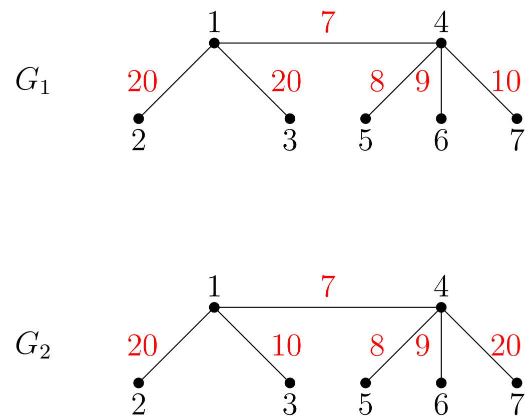

Figure 3·5: Two networks of identical connectivity with (G_1) and without (G_2) distinct weight vectors.

group. Weight vectors with all different entries are distinct, which implies that the distinct vectors are dense in $\mathcal{O}_D$.

For example, consider the two graphs G_1 and G_2 in Figure 3·5. Both share the same connectivity pattern (i.e., are isomorphic), but have different weight vectors. The weight vector $\vec{w}^1 = (20, 20, 7, \ldots)$ of G_1 satisfies $\sigma \cdot \vec{w}^1 = \vec{w}^1$ for $\sigma = (23) \in \Sigma_7$. In contrast, for all $\sigma \in \Sigma_7 \setminus \{\text{id}\}$, $\sigma \cdot \vec{w}^2 \neq \vec{w}^2$, where $\vec{w}^2$ is the weight vector of G_2, because the two 20's belong to nodes with different valences. Thus even though $\vec{w}^1, \vec{w}^2$ have the same set of weights, $\vec{w}^2$ is distinct, while $\vec{w}^1$ is not.

We now explain a standard procedure to construct a fundamental domain as one region in the Voronoi diagram of the orbit of a distinct vector. (Geometers call this a Dirichlet domain.) Let d_P be the Procrustean distance on $\mathcal{O}_D/\Sigma_d$. From now on, we just write $\vec{w}$ instead of $[\vec{w}]$ for elements of $\mathcal{U}_d$.

Lemma 3.3.2. *Fix a distinct vector $\vec{w} \in \mathcal{O}_D$. Set*

$$\begin{aligned} F = F_{\vec{w}} &= \{\vec{w}' \in \mathcal{O}_D : d_E(\vec{w}, \vec{w}') \leq d_E(\vec{w}, \sigma \cdot \vec{w}'),\ \forall \sigma \in \Sigma_d\} \\ &= \{\vec{w}' \in \mathcal{O}_D : d_E(\vec{w}, \vec{w}') = d_P(\vec{w}, \vec{w}')\}. \end{aligned}$$

Then

(i) F is a fundamental domain for the action of Σ_d on $\mathcal{G}_d$.

(ii) F is a solid cone with polyhedral cross section.

In the proof, we use the fact that $\vec{w}$ is distinct just below (3.9).

Proof. (i) First, for fixed $\vec{w}_1$, a minimum of $d_E(\vec{w}, \sigma \cdot \vec{w}_1)$ is attained, since Σ_d is finite. Thus every network (characterized by its weight vector $\vec{w}_1$) has a permutation in F.

Second, we can rewrite F as

$$F = \{\vec{w}_1 \in \mathcal{O}_D : d_E(\vec{w}, \vec{w}_1) \leq d_E(\sigma \cdot \vec{w}, \vec{w}_1),\ \forall \sigma \in \Sigma_d\}.$$

Let $\vec{w}_1 \in F \cap \sigma F$. Then $\sigma^{-1}\vec{w}_1 \in F$, and

$$\begin{aligned} d_E(\vec{w}, \vec{w}_1) &= \min_\tau d_E(\vec{w}, \tau\vec{w}_1),\ d_E(\vec{w}, \sigma^{-1}\vec{w}_1) = \min_\tau d(\vec{w}, \tau\sigma^{-1}\vec{w}_1) \\ &= \min_\tau d(\vec{w}, \tau\vec{w}_1). \end{aligned}$$

Thus

$$d_E(\vec{w}, \vec{w}_1) = d_E(\vec{w}, \sigma^{-1}\vec{w}_1) = d_E(\sigma\vec{w}, \vec{w}_1), \tag{3.9}$$

so $\vec{w}_1$ is equidistant to $\vec{w}$ and $\sigma\vec{w}$. Since $\vec{w}$ is distinct, $\sigma\vec{w} \neq \vec{w}$ for $\sigma \neq \mathrm{id}$. Thus $\vec{w}_1$ lies on a hyperplane in $\mathcal{O}_D$ defined by (3.9), and any ball around $\vec{w}'$ contains points that are closer to $\vec{w}$ than to $\sigma\vec{w}$, and points that are farther from $\vec{w}$ than from $\sigma\vec{w}$. Therefore $\vec{w}_1 \in \partial F$.

(ii) We can construct F' for the action of Σ_d on all of $\mathbb{R}^D$ by taking the set of hyperplanes H_σ of points equidistant from $\vec{w}$ and $\sigma\vec{w}$ for $\sigma \in \Sigma_d$, and taking the connected component of $\mathcal{O}_D \setminus \cup_\sigma H_\sigma$ containing $\vec{w}$. Since these hyperplanes all pass through the origin, this component is a solid cone on the origin. The boundary is given by a union of hyperplanes, so the cross section is a polyhedron. (Not all hyperplanes contribute edges to the cross sectional polygon; see the next example.) Moreover, Σ_d preserves $\mathcal{O}_D$, so a fundamental domain in $\mathcal{O}_D$ is given by $F = F' \cap \mathcal{O}_D$. The boundary planes of F are either boundary planes of F' or (subspaces of) the boundary of $\mathcal{O}_D$. □

This Dirichlet/Voronoi fundamental domain depends on a choice of $\vec{w}$. In particular, for a fixed distinct $\vec{w}_0$, we can guarantee that $\vec{w}_0$ is in the interior of F by setting $\vec{w} = \vec{w}_0$ in the lemma.

F is a solid cone cut out by at most $d! - 1 + D$ hyperplanes, where $d! - 1$ is the order of $\Sigma_d \setminus \{\mathrm{Id}\}$ and D is the number of coordinate hyperplanes. Thus this construction of F is not very practical except in low dimensions. Looking back at Figure 2, the infinite solid cone is the fundamental domain for the distinct vector $(3, 2, 1)$. See Appendix A.1 for an algorithm that computes the fundamental domain and examples for $d = 3, 4$.

We now give more information about F. The following result, although interesting, is not used below.

Proposition 3.3.3. *Let F be the fundamental domain associated to a distinct vector $\vec{w} \in \mathcal{O}_D$. All distinct vectors in F have a representative in the interior F^o of F.*

Proof. See Supplement B of [29]. □

Remark 3.3.4. Any nondistinct vector $\vec{w}'$ has an arbitrarily close distinct vector $\vec{w}$. Then $d_P(\vec{w}, \vec{w}') = d_E(\vec{w}, \vec{w}')$, and this remains true for vectors close to $\vec{w}$. Therefore, $\vec{w}'$ is in the interior F^o of $F = F_{\vec{w}}$. Thus for any nondistinct vector $\vec{w}'$, we can find a fundamental domain that contains $\vec{w}'$ in its interior.

The next lemma gives a sense of the minimal size of F. Let F_c denote the solid cone with vertex at the origin, axis $\vec{w}$, and cone angle c. Let $a = a_{\vec{w}}$ be the smallest angle between $\vec{w}$ and $\sigma \cdot \vec{w}$ for $\sigma \in \Sigma_d$, for $\sigma \neq \mathrm{id}$. Of course, $a/2$ is the smallest angle between $\vec{w}$ and a hyperplane boundary of F.

Lemma 3.3.5. *F contains the solid cone $F_{a/2}$.*

Proof. We claim that for vectors $\vec{u}, \vec{\ell}, \vec{v}$, the angles formed by them satisfy

$$\angle(\vec{u}, \vec{\ell}) + \angle(\vec{\ell}, \vec{v}) \geq \angle(\vec{u}, \vec{v}).$$

To prove this, we may assume the $\vec{u}, \vec{v}, \vec{\ell}$ are unit vectors. Let $\vec{\ell}'$ be the projection of $\vec{\ell}$ into the plane of $\vec{u}$ and $\vec{v}$. Assume that $\angle(\vec{u}, \vec{\ell}') + \angle(\vec{\ell}', \vec{v}) = \angle(\vec{u}, \vec{v})$. Then $\cos(\angle(\vec{u}, \vec{\ell}')) = \vec{u} \cdot \vec{\ell}'/|\vec{\ell}'| \geq \vec{u} \cdot \vec{\ell}'$, and $\cos(\angle(\vec{u}, \vec{\ell})) = \vec{u} \cdot \vec{\ell} = \vec{u} \cdot \vec{\ell}' + \vec{u} \cdot (\vec{\ell} - \vec{\ell}') = \vec{u} \cdot \vec{\ell}'$, since $\vec{\ell} - \vec{\ell}'$ is normal to the $(\vec{u}, \vec{v})$-plane. Thus $\angle(\vec{u}, \vec{\ell}') \leq \angle(\vec{u}, \vec{\ell})$. Similarly, $\angle(\vec{\ell}', \vec{v}) \leq \angle(\vec{\ell}, \vec{v})$. Thus $\angle(\vec{u}, \vec{\ell}) + \angle(\vec{\ell}, \vec{v}) \geq \angle(\vec{u}, \vec{\ell}') + \angle(\vec{\ell}', \vec{v}) = \angle(\vec{u}, \vec{v})$. The other possibility is that $\angle(\vec{u}, \vec{\ell'}) - \angle(\vec{\ell'}, \vec{v}) = \angle(\vec{u}, \vec{v})$ or the same with $\vec{u}$ and $\vec{v}$ switched. In this case, $\cos(\angle(\vec{u}, \vec{\ell})) = \vec{u} \cdot \vec{\ell'} \leq \cos(\angle(\vec{u}, \vec{\ell'}))$ implies $\angle(\vec{u}, \vec{\ell}) \geq \angle(\vec{u}, \vec{v})$.

Thus for a fixed permutation σ, we have $\angle(\vec{\ell}, \vec{u}) + \angle(\vec{u}, \sigma \cdot \vec{\ell}) \geq \angle(w, \sigma \cdot w) \geq a$. Therefore

$$\vec{u} \in F_{a/2} \Longrightarrow \angle(\vec{u}, \sigma \cdot \vec{\ell}) \geq a/2. \tag{3.10}$$

We may assume that $|\vec{\ell}| = |\vec{u}|$. Because $\vec{u}$ and the $\sigma \cdot \vec{\ell}$ all lie on the sphere of radius $|\vec{\ell}|$, the distances between $\vec{u}, \vec{\ell}, \sigma \cdot \vec{\ell}$ are proportional to the angles they form with the origin. Thus (3.10) implies that $d^E(\vec{u}, \sigma \cdot \vec{\ell}) \geq d^E(\vec{u}, \vec{\ell})$, so $\vec{u} \in F$. $\square$

Although the topology of the interior F^o and its image $q(F^o)$ are the same, their geometries are very different; this underlies the difference in general between the Fréchet means in F^o and $q(F^o)$.

Example 3.3.6 (Example $C = \mathbb{R}^2/\mathbb{Z}_4$ continued)**.** F^o is the open first quadrant. Let $\vec{v}, \vec{\ell} \in F^o$ cut out angles α, β with the positive x-axis, respectively. By the law of cosines, the Euclidean distance between $\vec{v}, \vec{\ell}$ is less than the distance between $\vec{v}$ and $R_{\pi/2}\vec{\ell}$, the rotation of $\vec{\ell}$ by $\pi/2$ radians counterclockwise, iff $|\alpha - \beta| < \pi/4$. Thus for $|\alpha - \beta| > \pi/4$, $d_E(\vec{v}, \vec{w}) < d_P([\vec{v}], [\vec{\ell}])$. Thus distances in F^o and C are different.

This affects the Fréchet means. Let $\nu = (4/3\pi) r dr d\theta$ be the uniform probability measure on F supported on $\{(r, \theta) \in [1, 2] \times [0, \pi/2]\}$. The Fréchet mean on F is the center of mass $(3/2, 3/2)$. The cone C has a circle action which rotates points equidistant from the vertex, and the Procrustean distance is clearly invariant under this action. This implies that if $[(r_0, \theta_0)]$ is a Fréchet mean on C, so is $[(r_0, \theta)]$ for all θ. Therefore, we can compute the Fréchet mean at $\theta = \pi/4$. Since $|(\pi/4) - \beta| \leq \pi/4$ in F, $d_E((r_0, \pi/4), (r, \theta)) = d_P((r_0, \pi/4), (r, \theta))$ for all $(r, \theta) \in F$. The previous computation gives $r_0 = 3/2$. We conclude that the Fréchet mean on C is the entire circle $r = 3/2$.

We now find a sub-cone of F such that the Fréchet mean of a compact convex set K inside this sub-cone projects to the unique Fréchet mean of the associated quotient space $K = q(K')$ inside $\mathcal{U}_d$. As explained at the beginning of this section, this allows us to derive a central limit theorem on K.

Theorem 3.3.7. *Let $K \subset \mathcal{U}_d$ be such that there exists a compact convex set $K' \subset F_{a/4}$ with $q : K' \to K$ a homeomorphism. Then the Fréchet mean μ_K of K is unique and satisfies $q(\mu_{K'}) = \mu_K$.*

See Figure 3·6 for a schematic picture. This result is discussed without a full proof in [?Jainb2016, ?Jaina2016]. Note that $\mu(K)$ can be computed by finding the center of mass $\mu_{K'}$ of K' (with respect to the pullback of a distribution Q supported in K) by standard integrals, and then projecting to $\mathcal{U}_d$.

Proof. Take $\vec{u}, \vec{v} \in F_{a/4}$ with $|\vec{u}| = |\vec{v}| = 1$. As in the previous lemma, $\angle(\vec{u}, \vec{\ell}) + \angle(\vec{\ell}, \vec{v}) \geq \angle(\vec{u}, \vec{v})$, with equality iff $\vec{u}, \vec{v}, \vec{\ell}$ are coplanar. Thus

$$\angle(\vec{u}, \vec{v}) \leq 2(a/4).$$

On the other hand, for $\sigma \neq \mathrm{id}$, we have as above

$$\angle(\vec{u}, \sigma \cdot \vec{v}) + \angle(\sigma \cdot \vec{v}, \sigma \cdot \vec{\ell}) \geq \angle(\vec{u}, \sigma \cdot \vec{\ell}) \Rightarrow \angle(\vec{u}, \sigma \cdot \vec{v}) \geq \angle(\vec{u}, \sigma \cdot \vec{\ell}) - \angle(\vec{v}, \vec{\ell}).$$

Let the plane containing $\vec{0}, \vec{u}, \sigma \cdot \vec{\ell}$ intersect ∂F at a line containing the unit vector $\vec{z}$. Then

$$\angle(\vec{u}, \sigma \cdot \vec{\ell}) = \angle(\vec{u}, \vec{z}) + \angle(\vec{z}, \sigma \cdot \vec{\ell}) \geq (a/2 - a/4) + a/2 = 3a/4.$$

Therefore

$$\angle(\vec{u}, \sigma \cdot \vec{v}) \geq 3a/4 - a/4 = a/2 \geq \angle(\vec{u}, \vec{v}).$$

Since $|\vec{u}| = |\vec{v}| = |\sigma \cdot \vec{v}| = 1$, the distance between these vectors is proportional to their angles. This implies that

$$d_E(\vec{u}, \vec{v}) = d_P([\vec{u}], [\vec{v}])$$

on $F_{a/4}$.

As a compact convex subset of Euclidean space, K' has a unique Fréchet mean. It follows that K is compact convex. $q : K' \to K$ is a homeomorphism, so for $[\vec{x}] \in K$, the Fréchet functions on K' and K satisfy

$$f([\vec{x}]) = \int_K d_P^2([\vec{x}], [\vec{y}]) q_* Q(d[\vec{y}]) = \int_{K'} d_E^2(\vec{x}, \vec{y}) Q(d\vec{y}) = f(\vec{x}).$$

Thus the Fréchet mean $\mu_{K'}$ of f on K' projects to μ_K, the unique Fréchet mean of f on K.

□

From Thm. 3.3.7, we derive the main result Thm. 3.2.4.

We now prove that Thm. 3.3.7 is a quantitative improvement over the optimal estimate for general Riemannian manifolds due to Afsari:

Theorem 3.3.8. [1, Thm. 2.1] *Let M be a complete Riemannian manifold with sectional curvatures at most Δ and injectivity radius ι_M. Set*

$$\rho_0 = \frac{1}{2}\min\left\{\iota_M, \frac{\pi}{\Delta}\right\},$$

with the convention that $\frac{\pi}{\Delta} = \infty$ if $\Delta \leq 0$. For any $\rho < \rho_0$, a geodesic ball of radius ρ in M has a unique Fréchet mean.

The injectivity radius is the supremum of $r > 0$ such that every geodesic ball of radius r is a topological ball. Afsari's theorem does not apply to $\mathcal{U}_d$, which is not a manifold. We also cannot apply this theorem to the more tractable interior F_w^o of F_w, which is diffeomorphic to $\mathcal{U}_d$ minus a set of measure zero, because F_w^o is not complete in the Euclidean metric, and has zero injectivity radius.

However, we can compare Afsari's result to Thm. 3.3.7 on (the locally complete) geodesic balls inside F_a, as these are ordinary Euclidean balls. The Euclidean metric has zero curvature, so $\pi/\Delta = \infty$ in our convention. Therefore, we need the injectivity radius of the smooth points of cone F_a. Take $\vec{v}$ lying on the cone axis. A ball $B_{\vec{v}}$ centered at $\vec{v}$ and tangent to the cone at a point P determines a right triangle $\Delta O\vec{v}P$. This ball has radius $|\vec{v}|\sin(a/2)$, so this is the injectivity radius $\iota_{B_{\vec{v}}}$ in Thm. 3.3.8. Thus Afsari's theorem applies to a ball of half this radius, denoted $B_{\vec{v}}\left(|\vec{v}|\sin(a/2)/2\right)$.

For Thm. 3.3.7, we can take any compact set K' inside $F_{a/4}$. To show that this Theorem improves the general Afsari result, we find such a K' containing $B_{\vec{v}}\left(|\vec{v}|\sin(a/2)/2\right)$. This follows if the $F_{a/4}$ cone contains the cone containing $B_{\vec{v}}\left(|\vec{v}|\sin(a/2)/2\right)$, which has cone angle $\sin^{-1}(\sin(a/2)/2)$. Since sin is increasing for $a \in (0, \pi/2)$, it suffices to show that

$$\sin(a/4) \geq \sin(a/2)/2. \tag{3.11}$$

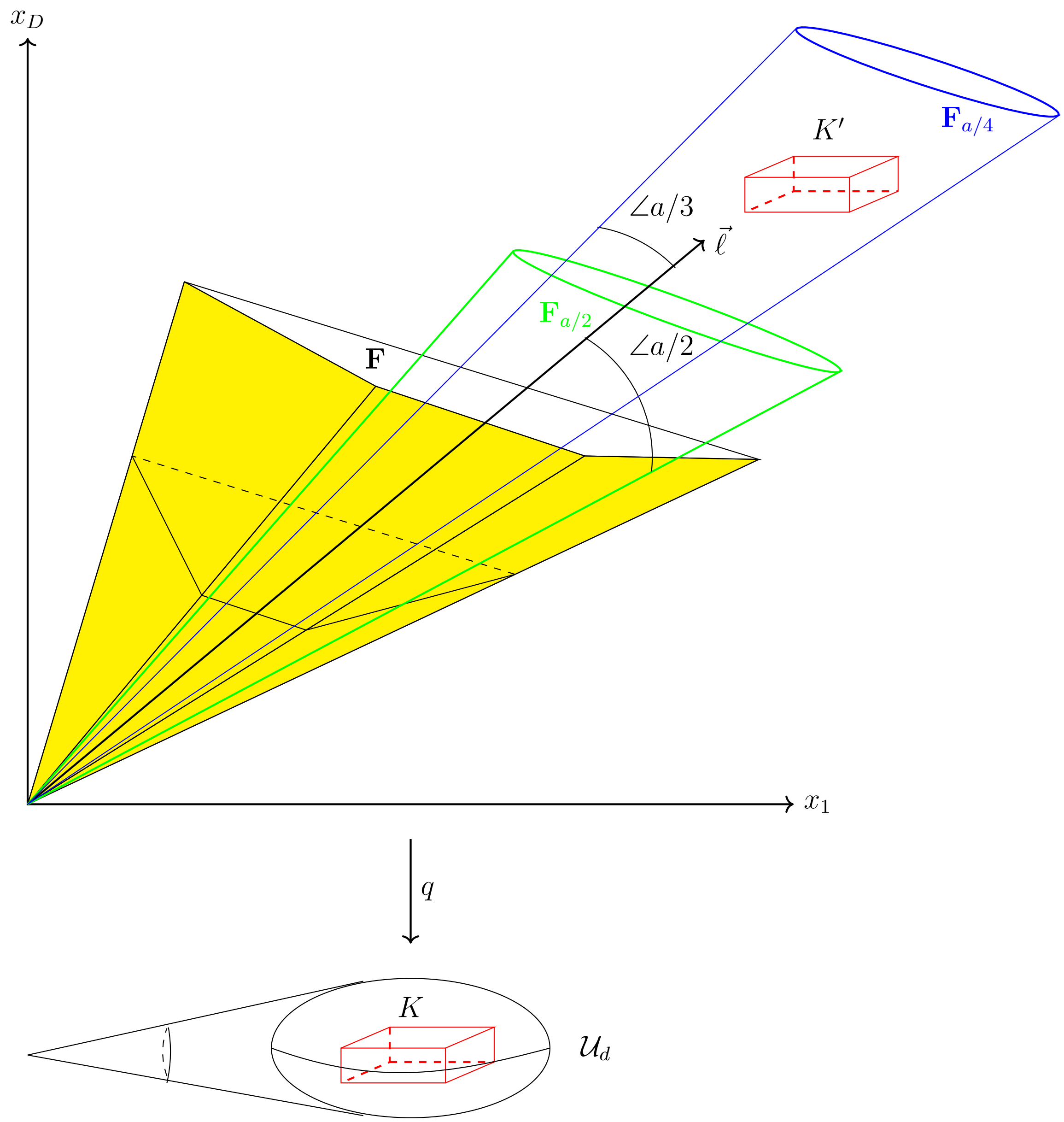

Figure 3·6: $K \subset F_{a/4}$, the blue cone. K and K' are homeomorphic via q. The Euclidean distance between points $\vec{x}, \vec{y} \in F_{a/4}$ is the same as the Procrustean distance between their orbits $[\vec{x}], [\vec{y}]$, so K' and K are actually isometric. The Fréchet means of K and K' are related by $\mu_K = q(\mu_{K'})$. In particular, the Fréchet mean of K is unique.

This follows from $\sin(2\theta) = 2\sin(\theta)\cos(\theta) \leq 2\sin(\theta)$.

As a result, $K' = q(K) \subset \mathcal{U}_d$ has a unique Fréchet mean, even though its radius is larger than the bound in Thm. 3.3.8. This just says that the Afsari bound, which is universal for all Riemannian manifolds, may have an improvement on specific manifolds. Figure 3·7 illustrates this improvement.

3.3.2 Generalizations of Thm. 3.3.7

Thm. 3.3.7 can be improved in two directions. The first improvement, Theorem 3.3.9, is a technical analytic extension which shows that the uniqueness of the Fréchet mean still holds for smooth distributions that are close to compactly supported distributions supported in the $F_{a/4}$ cone. In practical terms, this means that distributions may have small tails outside the $F_{a/4}$ cone and still have a unique Fréchet mean. The second improvement, Theorem 3.3.10 shows that any compact subset $E \subset \mathcal{U}_d$ has a unique Fréchet mean after isometrically embedding E into the Banach space of bounded functions on E in the sup norm. While this is more appealing, the new Fréchet mean may not lie in the image of E and may be hard to compute.

Theorem 3.3.9. *Let Q be a smooth probability distribution with support inside a compact set $E \subset F_{a/4}$. There exists $\zeta > 0$ such that for every $Q' \in B^3_Q(\zeta) \cap C_c(F)$, the Fréchet function $f_{Q'}$ has a unique minimum.*

Here $B^3_Q(\zeta)$ denotes a ball of radius ζ around Q in a Sobolev 3-norm, and $C_c(F)$ denotes continuous functions with compact support in F. The proof is in Supplement C in [29].

Secondly, we prove that any compact subset E of $\mathcal{U}_d$ admits an isometric embedding in a normed space $\mathcal{B}(E)$ such that any smooth distribution supported in E has a unique Fréchet mean in $\mathcal{B}(E)$. This Fréchet mean may not lie in the image of E.

Theorem 3.3.10. *Let Q be a continuous probability distribution with support in a compact subset E of $\mathcal{U}_d$ There is an isometric embedding ι of E into $\mathcal{B}(E)$, the space of*

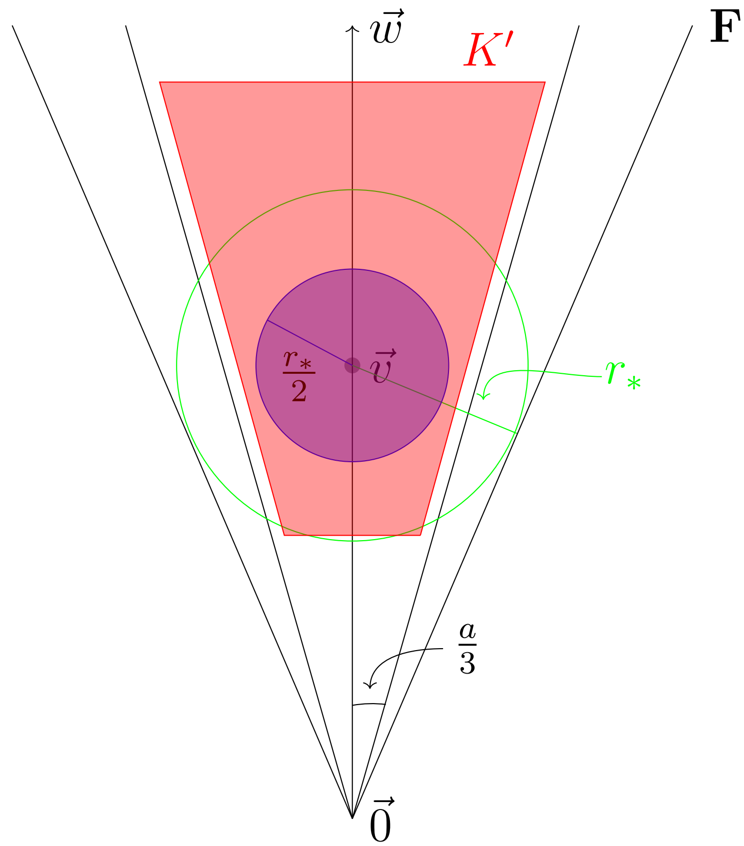

Figure 3·7: A cross-section of the fundamental domain **F** showing the improvement of Thm. 3.3.7 over the Afsari result. The Afsari work gives a unique Fréchet mean of the image in $\mathcal{U}_d$ to the ball of radius $r_*/2$, where r_* is the injectivity radius at $\vec{v}$. Thm. 3.3.7 gives a unique Fréchet mean to the image in $\mathcal{U}_d$ of the larger compact set K' inside the $a/3$ cone. The boundary of K' can extend down to $\vec{0}$, outwards as far as the walls of the $a/3$ cone, and to any finite height.

bounded functions on E in the sup norm, such that the Fréchet function f_Q extends to a function $\overline{f}_Q$ on the closed convex hull of $\iota(E)$ and such that $\overline{f}_Q$ has a unique minimum.

The proof is also in Supplement C in [29].

We end this section with an example of the difficulties of handling a graph with a non-distinct weight vector.

Example 3.3.11. As in Figure 2, take the distinct weight vector $\vec{w} = (3, 2, 1)$. The fundamental domain is $F = F_{\vec{w}} = \{x \geq y \geq z\}$. The proof of Corollary 3.3.5 shows that every $\vec{w}_1 \in F_{a/2}$ has a neighborhood U on which

$$d_P([\vec{w}_2], [\vec{w}_3]) = d_E(\vec{w}_2, \vec{w}_3), \text{ for all } \vec{w}_2, \vec{w}_3 \in U. \tag{3.12}$$

In fact, we can take $U = F_{a/2}$ (See Example B.1 for a stronger statement.)

We now show that the vector $(1, 1, 1) \in \partial F$ has no neighborhood U in F on which (3.12) holds. Note that the stabilizer group of $(1, 1, 1)$ is all of Σ_3.

Take $\vec{w}_2 = (1 + a_1, 1 + a_2, 1 + a_3)$ with $a_1 > a_2 > a_3 > 0$, and $\vec{w}_3 = (1 + b_1, 1 + b_2, 1 + b_3)$ with $b_1 > b_2 > b_3 > 0$. For small a's and b's, these vectors will be arbitrarily close to $(1, 1, 1)$. Then

$$d_E^2(\vec{w}_2, \vec{w}_3) = \sum_{i=1}^{3} (a_i - b_i)^2.$$

Take

$$b_1 \approx a_2, b_2 \approx a_3 \approx b_3 \tag{3.13}$$

and let $\sigma = (132)$. Then

$$d_P^2([\vec{w}_2], [\vec{w}_3]) < d_E^2(\vec{w}_2, \vec{w}_3) \tag{3.14}$$

if

$$\sum_{i=1}^{3} (a_i - b_i)^2 > \sum_{i=1}^{3} (a_i - b_{\sigma(i)})^2. \tag{3.15}$$

The left hand side of (3.15) is approximately $(a_1 - a_2)^2 + (a_2 - a_3)^2$, and the right hand side is approximately $(a_1 - a_3)^2$. Thus (3.15) holds if

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 > (a_1 - a_3)^2, \textit{i.e.} \text{ if } a_2^2 + 2a_1 a_3 > a_2 a_1 + a_2 a_3.$$

Since $a_2^2 > a_2 a_3$, we just need

$$2a_3 > a_2. \tag{3.16}$$

So once we choose the a's and b's to satisfy (3.13),(3.16), we obtain (3.14), which proves the claim.

We conclude that there is no neighborhood U of $(1, 1, 1)$ inside $\mathbb{R}^3$ (not just inside F) on which Euclidean distances in U agree with Procrustean distances in $q(U) \subset \mathcal{U}_3$. This illustrates the impossibility of applying [1] to the singular point $[(1, 1, 1)]$ of $\mathcal{U}_3$.

Appendix A

Appendix

A.1 Computing fundamental domains

In the two appendices, we discuss the computational difficulties of implementing the theory in Section 4. In this appendix, we show that the fundamental domain $F_{\vec{w}}$ is highly sensitive to the choice of distinct vector $\vec{w}$ as axis.

We use a Dirichlet fundamental domain for the action of Σ_d on $\mathbb{R}^D_{\geq 0}$. By Lemma 3.3.2, $F = F_{\vec{w}} = \bigcap_{\sigma \in \Sigma_n} \{z \in \mathbb{R}^D_{\geq 0} : d_E(\vec{w}, \vec{z}) \leq d_E(\vec{w}, \sigma\vec{z})\}$ for a distinct vector $\vec{w} \in \mathbb{R}^D_{\geq 0}$. This is the intersection of $d! + D - 1$ half-spaces, where the D coordinate half-spaces are given by the inequalities $z_j \geq 0$. The cone F is a convex, non-compact polyhedral region in $\mathbb{R}^D_{\geq 0}$. The $d! - 1$ half-space regions are given by the linear inequalities:

$$\sum_{j=1}^{D} (w_{\sigma(j)} - w_j) z_j \leq 0,$$

for $\sigma \neq \mathrm{Id}$.

Sage provides efficient tools for converting an input system of linear inequalities into a minimal description of the polyhedral output region; see Supplement B.

We consider the simplest nontrivial example of graphs with four vertices: $d = 4$, $D = \binom{d}{2} = 6$.

Example A.1.1. Choose the distinct vector $\vec{w} = (1, 2, 3, 4, 5, 6)$. Sage gives F_w as the intersection of 7 half-spaces or the convex hull of 7 rays [see Figure A·1].

Example A.1.2. Now choose $\vec{w} = (1, 2, 3, 4, 5, 6.1)$. $F_{\vec{w}}$ is now described as the convex hull of 79 rays, or the intersection of 18 half-spaces [see Figure A·2].

```
sage: load('fund_decomp.sage')
Please specify n: 4
sage: l=[1,2,3,4,5,6]
sage: F=fund_domain(l)[1]; F
A 6-dimensional polyhedron in QQ^6 defined as the convex hull of 1 vertex and 7 rays
sage: F.n_Hrepresentation()
7
sage: F.Hrepresentation()
(An inequality (1, 0, 0, 0, 0, 0) x + 0 >= 0,
 An inequality (0, 1, 0, 0, 0, 0) x + 0 >= 0,
 An inequality (0, 0, 1, 0, 0, 0) x + 0 >= 0,
 An inequality (0, 0, 0, 1, 0, 0) x + 0 >= 0,
 An inequality (0, -1, 1, -1, 1, 0) x + 0 >= 0,
 An inequality (0, -1, -1, 1, 1, 0) x + 0 >= 0,
 An inequality (-1, 1, 0, 0, -1, 1) x + 0 >= 0)
sage: F.Vrepresentation()
(A vertex at (0, 0, 0, 0, 0, 0),
 A ray in the direction (0, 1, 0, 0, 1, 0),
 A ray in the direction (0, 0, 1, 1, 0, 0),
 A ray in the direction (1, 0, 0, 0, 0, 1),
 A ray in the direction (0, 0, 0, 0, 0, 1),
 A ray in the direction (0, 0, 0, 1, 1, 1),
 A ray in the direction (0, 0, 0, 0, 1, 1),
 A ray in the direction (0, 0, 1, 0, 1, 1))
```

Figure A·1: Sage output for computation of a fundamental domain centered at the distinct vector $\vec{w} = (1, 2, 3, 4, 5, 6)$. $F.Hrepresentation()$ lists the half-spaces whose intersection is the fundamental domain. Note that we started with $4! + \binom{4}{2} - 1 = 29$ inequalities, and have narrowed it down to 7. $F.Vrepresentation()$ lists the 7 rays whose convex hull is the fundamental domain.

A.2 Computation of the Fréchet integral

In this Appendix, we highlight the difficulty of computing the Fréchet integral $f([\vec{x}]) = \int_{\mathcal{U}_d} d_P^2([\vec{x}], [\vec{y}])dQ'([\vec{y}])$ for all $[\vec{x}]$. (Here $Q' = q_*Q$ in the notation of Thm 3.3.7.) Even when a fundamental domain $F_{\vec{x}}$ has been explicitly determined for a fixed $\vec{x}$, so that $f([\vec{x}]) = \int_{F_{\vec{x}}} d_E^2(\vec{x}, \vec{y})dQ(\vec{y})$, we have seen in Appendix A.1 that the shape of $F_{\vec{x}}$ depends delicately on $[\vec{x}]$.

We can instead divide $\mathbb{R}^D_{\geq 0}$ into $d!$ regions $F_\sigma(\vec{x}) = \{\vec{y} \in \mathbb{R}^D_{\geq 0} : d_P([\vec{x}], [\vec{y}]) = d_E(\vec{x}, \sigma \cdot \vec{y})\}$. Note that $F_\sigma(\vec{x})$ is a fundamental domain for $\sigma \cdot \vec{x}$. Since $F_{\vec{w}} = \bigcup_{\sigma \in \Sigma_d} F_{\vec{w}} \cap F_{\sigma \cdot \vec{x}}$ for a fixed distinct vector $\vec{w}$, the Fréchet integral is given by

$$f(\vec{x}) = \int_{\mathcal{U}_d} d_P^2([\vec{x}], [\vec{y}])dQ'([\vec{y}]) = \sum_{\sigma \in \Sigma_d} \int_{F_{\vec{w}} \cap F_{\sigma \cdot \vec{x}}} d_E^2(\vec{x}, \sigma \cdot \vec{y})dQ(\vec{y}).$$

```
sage: load('fund_domain.sage')
Please specify n: 4
sage: l=[1,2,3,4,5,6.1]
sage: F=fund_domain(l)[1]; F
A 6-dimensional polyhedron in QQ^6 defined as the convex hull of 1 vertex
and 79 rays
sage: F.n_Hrepresentation()
18
sage: F.n_Vrepresentation()
80
sage: F.Vrepresentation()

(A vertex at (0, 0, 0, 0, 0, 0),
 A ray in the direction (1, 0, 0, 0, 0, 1),
 A ray in the direction (408619973699943265900067052289, 0, 0, 0, 371472703
3635828000796950588, 408619973699943265900067052289),
 A ray in the direction (876314648233908669903446388536338026325749, 0, 0
, 0, 796649680212641841246892697107500458824308, 876314648233908767193916
3170936968054434609),
 A ray in the direction (780092677063528297159301852241, 0, 78009267706353
0115920689828791, 780092677063530115920689828791, 0, 780092677063528383766
986993981),
 A ray in the direction (1, 0, 1, 1, 0, 1),
```

Figure A·2: A partial Sage output for computation of a fundamental domain centered at the distinct vector $\vec{w} = (1, 2, 3, 4, 5, 6.1)$. $F.Hrepresentation()$ lists the half-spaces whose intersection is the fundamental domain. Note that we started with $4! + \binom{4}{2} - 1 = 29$ inequalities, and have narrowed it down to 18. $F.Vrepresentation()$ lists the 79 rays whose convex hull is the fundamental domain.

Although on the right hand side we now have a sum of Euclidean integrals for each $\vec{x}$, as in Appendix A.1 it is difficult to explicitly compute $F_{\vec{w}} \cap F_{\sigma \cdot \vec{x}}$.

We illustrate this computation in the simple case $d = 3$. Already for $d = 4$, the computation becomes too lengthy for inclusion here.

Example A.2.1. $d = 3$. First we show that the fundamental domain $F_{\vec{x}}$ can be chosen to depend only on the ordering $\mathrm{ord}(\vec{x})$ (e.g. from largest to smallest, as in Figure 2) of the components of a distinct vector $\vec{x}$. The Procrustean distance between two points is $d_P([\vec{x}], [\vec{y}]) = \min_{\sigma \in \Sigma_3} d_E(\vec{x}, \sigma \cdot \vec{y})$. To minimize the Euclidean distance, we choose σ which reorders $\vec{y}$ to match the ordering of $\vec{x}$, as any other σ cannot decrease the distance. (This uses the special fact that $\Sigma_{d=3}$ is the full permutation group of the $D = 3$ set of weight vectors). Therefore, we can choose $F = F_{\vec{x}} = \{\vec{y} \in \mathbb{R}^3_{\geq 0} : d_E(\vec{x}, \vec{y}) = d_P([\vec{x}], [\vec{y}])\} = \{\vec{y} \in \mathbb{R}^3_{\geq 0} : \mathrm{ord}(\vec{y}) = \mathrm{ord}(\vec{x})\}$ to be independent of the distinct vector $\vec{x}$. The Fréchet integral for a compactly supported probability measure Q' on $\mathcal{U}_3$ (or equivalently for Q on F) equals

$$\begin{aligned} f([\vec{x}]) &= \int_F d_E(\vec{x},\vec{y})^2 dQ(\vec{y}) \\ &= \|\vec{x}\|^2 \int_F dQ(\vec{y}) - 2\vec{x}\cdot \int_F \vec{y}\; dQ(\vec{y}) + \int_F \|\vec{y}\|^2 dQ(\vec{y}) \\ &= \|\vec{x}\|^2 - 2\vec{x}\cdot \int_F \vec{y}\; dQ(\vec{y}) + B \\ &= \|\vec{x}\|^2 - 2\sum_i C_i x^i + B, \end{aligned}$$

where the second integral on the last line is the dot product of a vector and a vector valued integral, $C_i = \int_F y^i dQ(\vec{y})$, and $B = \int_F \|\vec{y}\|^2 dQ(\vec{y})$. Thus f is quadratic in $\vec{x}$ on distinct vectors. Since the distinct vectors are dense in Euclidean space, f is quadratic on all vectors. Therefore, f is strictly convex. Since we are minimizing over a convex region, f has a unique global minimum. To explicitly compute it, note that F has eight strata in varying dimensions 0, 1, 2, and 3; these are labeled $(1), \ldots, (8)$ in the table below. The restriction to each stratum is smooth away from the lower dimensional boundaries, so we can simply minimize on each open piece to find eight local minima x^*.

#	dim	Region	$x^* =$
(1)	3	$x_3 \geq 0$ and $0 < x_1 < x_2 < x_3$	(C_1, C_2, C_3)
(2)	2	$x_1 = 0$ and $0 < x_2 < x_3$	$(0, C_2, C_3)$
(3)	2	$0 < x_1 = x_2 < x_3$	$\left(\frac{1}{2}(C_1+C_2), \frac{1}{2}(C_1+C_2), C_3\right)$
(4)	2	$0 < x_1 < x_2 = x_3$	$\left(C_1, \frac{1}{2}(C_2+C_3), \frac{1}{2}(C_2+C_3)\right)$
(5)	1	$x_1 = x_2 = 0$ and $x_3 > 0$	$(0, 0, C_3)$
(6)	1	$x_1 = 0$ and $0 < x_2 = x_3$	$\left(0, \frac{1}{2}(C_2+C_3), \frac{1}{2}(C_2+C_3)\right)$
(7)	1	$0 < x_1 = x_2 = x_3$	(C', C', C')
(8)	0	$x_1 = x_2 = x_3 = 0$	(0,0,0)

Here $C' = (1/3)(C_1 + C_2 + C_3)$. The true global minimum will depend on the values of C_1, C_2, and C_3.

List of Journal Abbreviations

Adv. Appl. Math. = Advances in Applied Mathematics
Ann. Statist. = Annals of Statistics
Electon. J. Probab. = Electronic Journal of Probability
Geom. Dedicata = Geometriae Dedicata
Jpn. J. Math. = Japanese Journal of Mathematics
Princeton Univ. Press = Princeton University Press
Proc. Amer. Math. Soc., Ser. A = Proceedings of the Royal Society of London, Series A

Bibliography

[1] B. Afsari, *Riemannian L^p center of mass: existence, uniqueness, and convexity.*, Proc. Amer. Math. Soc. **139** (2011), 655–673.

[2] J. D. Arroyo Relión, D. Kessler, E. Levina, and S. F. Taylor, *Network classification with applications to brain connectomics*, ArXiv e-prints (2017jan), available at 1701.08140.

[3] B. Aydin, G. Pataki, H. Wang, E. Bullitt, and J.S. Marron, *A principal component analysis for trees*, The Annals of Applied Statistics **3** (2009), no. 4, 1597–1615.

[4] Dennis Barden, Huiling Le, and Megan Owen, *Central limit theorems for fréchet means in the space of phylogenetic trees*, Electron. J. Probab. **18** (2013), 1–25.

[5] Stephen Berman, Yuly Billig, and Jacek Szmigielski, *Vertex operator algebras and the representation theory of toroidal algebras*, Recent developments in infinite-dimensional Lie algebras and conformal field theory (Charlottesville, VA, 2000), 2002, pp. 1–26. MR1919810

[6] A. Bhattacharya, *Statistical analysis on manifolds: a nonparametric approach for inference on shape spaces*, Sankhya **70** (2008), 1–43.

[7] A. Bhattacharya and R.N. Bhattacharya, *Nonparametric inference on manifolds: With applications to shape spaces*, IMS monograph series, # 2, Cambridge University Press, 2012.

[8] R. Bhattacharya and L. Lin, *Omnibus CLTs for Fréchet means and nonparametric inference on non-Euclidean spaces*, The Proceedings of the American Mathematical Society **145** (2017), 413–428.

[9] R. N. Bhattacharya and V. Patrangenaru, *Large sample theory of intrinsic and extrinsic sample means on manifolds-I*, Ann. Statist. **31** (2003), 1–29.

[10] ______, *Large sample theory of intrinsic and extrinsic sample means on manifolds-II*, Ann. Statist. **33** (2005), 1225–1259.

[11] L.J. Billera, S. Holmes, and K. Vogtmann, *Geometry of the space of phylogenetic trees.*, Adv. Appl. Math. **27** (2001), 733–767.

[12] E.l. Cornea, H. Zhu, P. Kim, and J.G. Ibrahim, *Regression models on riemannian symmetric spaces*, Journal of the Royal Statistical Society: Series B (Statistical Methodology) (2016).

[13] Kevin Costello and Owen Gwilliam, *Factorization algebras in quantum field theory. Vol. 1*, New Mathematical Monographs, vol. 31, Cambridge University Press, Cambridge, 2017. MR3586504

[14] David A Cox and Sheldon Katz, *Mirror symmetry and algebraic geometry*, Vol. 68, American Mathematical Society Providence, RI, 1999.

[15] Alberto De Sole and Victor G. Kac, *Finite vs affine W-algebras*, Jpn. J. Math. **1** (2006), no. 1, 137–261. MR2261064

[16] Simon Kirwan Donaldson and Peter B Kronheimer, *The geometry of four-manifolds*, Oxford University Press, 1990.

[17] Ian L. Dryden, Alexey Koloydenko, and Diwei Zhou, *Non-euclidean statistics for covariance matrices, with applications to diffusion tensor imaging*, The Annals of Applied Statistics **3** (200909), no. 3, 1102–1123.

[18] D. Durante, D.B. Dunson, and J.T. Vogelstein, *Nonparametric bayes modeling of populations of networks*, Journal of the American Statistical Association (2016). (In press).

[19] Daniele Durante and David B. Dunson, *Nonparametric bayes dynamic modelling of relational data*, Biometrika **101** (2014), no. 4, 883–898, available at /oup/backfile/content_public/journal/biomet/101/4/10.1093/biomet/asu040/2/asu040.pdf.

[20] S. Eswara Rao and R. V. Moody, *Vertex representations for n-toroidal Lie algebras and a generalization of the Virasoro algebra*, Comm. Math. Phys. **159** (1994), no. 2, 239–264. MR1256988

[21] Edward Frenkel and David Ben-Zvi, *Vertex algebras and algebraic curves*, Second, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2004. MR2082709

[22] C.E. Ginestet, J. Li, P. Balanchandran, S. Rosenberg, and E.D. Kolaczyk, *Hypothesis testing for network data in functional neuroimaging*, Annals of Applied Statistics **11** (2017), no. 2, 725–750.

[23] M. Gjoka, M. Kurant, C.T. Butts, and A. Markopoulou, *Walking in Facebook: a case study of unbiased sampling of osns*, Infocom, 2010 proceedings ieee, 2010, pp. 1–9.

[24] M.O. Jackson, *Social and economic networks*, Princeton Univ Press, 2008.

[25] Victor Kac, *Vertex algebras for beginners*, Second, University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998. MR1651389

[26] Christian Kassel, *Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra*, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), 1984, pp. 265–275. MR772062

[27] W. S. Kendall, *Probability, convexity, and harmonic maps with small image. I. Uniqueness and fine existence*, Proc. London Math. Soc. (3) **61** (1990), no. 2, 371–406.

[28] E.D. Kolaczyk, *Statistical analysis of network data: Methods and models*, Springer Verlag, 2009.

[29] Rosenberg Xu Walters Kolaczyk Lin, *Averages of unlabeled networks: Geometric characterization and asymptotic behavior*, arXiv preprint arXiv:1709.02793 (2019).

[30] Andreas Kriegl and Peter W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997. MR1471480

[31] Haisheng Li, Shaobin Tan, and Qing Wang, *Toroidal vertex algebras and their modules*, J. Algebra **365** (2012), 50–82. MR2928453

[32] László Lovász, *Large Networks and Graph Limits*, Vol. 60, American Mathematical Society Providence, 2012.

[33] Christopher McCarty, *Structure in personal networks*, Journal of social structure **3** (2002), no. 1, 20.

[34] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, Second, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2012.

[35] Robert V. Moody, Senapathi Eswara Rao, and Takeo Yokonuma, *Toroidal Lie algebras and vertex representations*, Geom. Dedicata **35** (1990), no. 1-3, 283–307. MR1066569

[36] M.E.J. Newman, *Networks: An introduction*, Oxford University Press, Inc., 2010.

[37] Williams Szczesny Walters, *Toroidal prefactorization algebras associated to holomorphic fibrations and a relationship to vertex algebras*, arXiv preprint arXiv:1904.03176 (2019).

[38] R. Tang, M. Ketcha, J.T. Vogelstein, C.E. Priebe, and D.L. Sussman, *Law of large graphs*, arXiv preprint arXiv:1609.01672 (2016).

[39] Ravi Vakil, *The moduli space of curves and its tautological ring*, Notices of the AMS **50** (2003), no. 6, 647–658.

[40] H. Wang and J.S. Marron, *Object oriented data analysis: Sets of trees*, Annals of Statistics **35** (2007), no. 5, 1849–1873.

[41] H. Ziezold, *On expected figures and a strong law of large numbers for random elements in quasi-metric spaces*, Transactions of the Seventh Prague Conference on Information Theory, Statistical Functions, Random Processes and of the Eightth European Meeting of Statisticians **A** (1977), 591–602. (Tech. Univ. Prague, Prague, 1974).

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[redacted]