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An application of Markov chains

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AN APPLICATION OF MARKOV CHAINS

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I. Introduction

Probability problems in which a time parameter is involved are known as "stochastic processes". The problems vary in complexity from those simple situations in which a system is considered which has probabilities of being in several various states at each of a few discrete times, the probabilities being independent of the previous states of the system, to those complex situations involving consideration through a continuous time interval of a system whose probabilities of entering particular states at a particular time are functions of the entire past history of the system.

The simplest stochastic processes which involve some dependence on the past history of the system are those in which the system changes state at certain discrete times and the probabilities of entering each state at a particular time are functions solely of the state which the system was in at the previous time. Such processes are known as Markov chains.

A class of problems which may be fully or partially defined by appropriate Markov chains are those known as queuing problems. In these problems, an "input" of "customers" arrives with some stated distribution. These customers are served by a known number of "servers", the distribution of the serving times also being known. Since the number of servers is limited, a queue of customers waiting for service may build up. Quantities such as the optimum number of servers, the distribution of queue lengths and the expected length of the queue, and the distribution and expected lengths of the waiting time of the customers are the unknown quantities which it is desired to investigate.

A typical application of the queuing problem which is presently of considerable interest is that of a missile tracking radar system. Such a system has a limited number of trackers (servers) which must track each missile (customer) for a sufficient length of time to provide accurate trajectory information to permit destruction of the incoming missile by defensive weapons. If the waiting time between the entrance of the missiles into the radar's coverage and the availability of a tracker to commence tracking is too great, the missiles will reach their target before destruction of them can be achieved. The point of interest is thus to determine the radar system characteristics which will result in a very high probability of the waiting time being smaller than this critical value.

This paper describes the characteristics and algebra of Markov chains and the methods of applying the Markov chain techniques to queuing problems. The radar tracking problem is discussed as an example of the practical considerations encountered in suitably defining a queuing problem so that it may be characterized as a relatively simple Markov chain and in applying the various techniques to obtain useful results from this Markov chain.

II. Stochastic Processes

Suppose that n chance variables, ..., X_1, X_2, ..., X_n, ..., are selected from an infinite chain. Associated with each variable is a particular time, $t_1, t_2, \ldots, t_n$. Such a system is known as a "stochastic process". The term "stochastic process" is sometimes applied indiscriminately to all probability phenomena, but it is here reserved specifically for those probability situations which involve a time parameter. The stochastic process is thus represented by a system which may be in various possible conditions (known as states) at each time under consideration.

The mathematical complexity of a stochastic process depends upon two factors. The first of these is the number of values which may be taken by the chance variables, or in other words the number of states which the system may occupy. The highest complexity in this regard results from consideration of processes where the distribution functions of the chance variables are continuous. Such processes are fully characterized only by defining an infinite number of system states. Finite solutions are therefore impossible and hence matrix techniques cannot be used. The methods of attacking such problems involve the use of differential and difference equations and the repeated use of recurrence relations. Processes involving finite states are usually compatible with solution by matrix methods. The complexity of such processes depends directly upon the number of states involved, since each additional state adds another row and column to the characterizing matrix and the matrices thus become increasingly unwieldy.

The second factor which helps dictate how complex a stochastic process may be is the dependence between chance variables. The simplest processes have no dependence at all and the probability of the system entering any state is independent of any previous condition of the system. Such processes resolve into ordinary probability problems which may be characterized by simple sets of equations. The simplest processes which maintain dependence between random variables are those in which the probabilities of the system entering various states depend solely upon the state of the system in the immediately preceding time. For the discrete case this may be defined in terms of distribution functions by

$$f(x_{t+1} \mid x_t, x_{t-1}, \ldots) = f(x_{t+1} \mid x_t) \qquad \text{(Eq. 1)}$$

Such a process is called a Markov process, or in the discrete case a Markov chain. The Markov chain is defined by the matrix of all its transition probabilities, p_{ij}, each of which is the probability of the system entering state i if it presently is in state j. Solution of Markov chains involves the solution of matrix equations.

Many stochastic processes exist in which the dependence between chance variables is much more complicated than in the case of Markov chains. Some of these situations result in insoluble problems. Many others may theoretically be soluble, but in practice the algebra is too voluminous. Some stochastic processes which are not Markovian as they are stated, may be considered to be Markov chains if the definition of the system states is made with greater precision (as functions of additional variables not specified in the original statement of the states). However the resulting

Markov chains are usually too large for easy solution. Sometimes a portion of the problem may be used to set up a reduced Markov chain, called an "imbedded" Markov chain. This results in a simpler solution, but parts of the problem are missing and this may include some of the desired information. A more useful tool for solving those complicated stochastic processes where a solution is possible is the use of difference or differential equations to find various recurrence relationships.

III. Markov Chains

A. The Stochastic Matrix

In the Markov chain, a separate distinct probability is associated with the transition from every possible state to itself or any other state. These transition probabilities may be written as p_{ij}, this being the probability of transition from state j to state i. The transition probabilities may be written in the form of a stochastic matrix

$$P = \begin{array}{cc} & \begin{matrix} 1 & 2 & \cdot & n \leftarrow \text{present states} \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \cdot \\ n \end{matrix} & \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \end{array} \qquad \text{(Eq. 2)}$$

possible states
after next
transition

The sum of the transition probabilities in each column represents the total probability of transition from a particular present state. This total probability is always one[1] so that the sum of the terms in any column of the matrix is equal to one. Since probabilities are by definition non-negative, all of the elements of the matrix are non-negative. Furthermore, since the number of possible present states is the same as the number of possible future states, the matrix is square. These three characteristics define a stochastic matrix.

B. Higher Transition Probabilities

An important consideration in most Markov chain problems is the effect on the system of a number of transitions defined by the transition matrix P. If the initial condition of the system is described by the vector v, the condition after one transition is Pv and after n transitions is PPP...Pv or $P^n v$. It is possible to obtain these required higher powers of P by performing the requisite number of multiplications by P. However this is a tedious process at best and in the limiting case P^∞ it is impossible. Some transformation of the matrix P which will simplify the determination of its higher powers is therefore desirable. The development of such a transformation requires some investigation of the theory of stochastic matrices.

1. W. Feller, An Introduction to Probability Theory and Its Applications, (New York, John Wiley & Sons, 1950), vol. 1, page 22.
2. Ibid, p. 22.

C. Modal Matrices

It is well known[3] that each matrix u has an associated characteristic matrix $f(\lambda)$ defined by

$$f(\lambda) = \lambda I - u \qquad \text{(Eq. 3)}$$

where λ represents a set of scalars which are the roots of the system of equations

$$|\lambda I - u| = 0 \qquad \text{(Eq. 4)}$$

The bars indicate the determinant of the enclosed function. For an nth order matrix, there are n roots of this system of equations, $\lambda_1, \lambda_2, \ldots, \lambda_n$. These are the characteristic roots of the matrix. Associated with each of these roots λ_s, are two characteristic vectors, one a row vector K_s, and the other a column vector k_s. Vectors having the direction of the characteristic vectors have their direction unchanged under transformation by the matrix u. The characteristic vectors may be simply obtained from the equations

$$\begin{bmatrix} K_{s1} & K_{s2} & \cdots & K_{sn} \end{bmatrix} \begin{bmatrix} u \end{bmatrix} = \lambda_s \begin{bmatrix} K_{s1} & K_{s2} & \cdots & K_{sn} \end{bmatrix} \qquad \text{(Eq. 5)}$$

3. R. A. Frazer, W. S. Duncan, A. R. Collar, Elementary Matrices (Cambridge, The University Press, 1938)

$$[u]\begin{bmatrix} k_{1s} \\ k_{2s} \\ \cdots \\ k_{ns} \end{bmatrix} = \lambda_s \begin{bmatrix} k_{1s} \\ k_{2s} \\ \cdots \\ k_{ns} \end{bmatrix} \qquad \text{(Eq. 6)}$$

These vectors, however, represent proportional relationships only and may be multiplied by any desired scalar factor without altering the relationships.

Assume now that the characteristic roots are all distinct. In this case, the n columns k_s are distinct and may be written as a matrix k. Likewise the rows K_s may be written as a matrix K. These matrices, which are known as the modal matrices, are thus

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \qquad \text{(Eq. 7)}$$

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \qquad \text{(Eq. 8)}$$

Using these modal matrices, the equations 5 and 6 for every characteristic root may be combined into two matrix equations. These are:

$$uk = \begin{bmatrix} \lambda_1 k_{11} & \lambda_2 k_{12} & \cdots & \lambda_n k_{1n} \\ \lambda_1 k_{21} & \lambda_2 k_{22} & \cdots & \lambda_n k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1 k_{n1} & \lambda_2 k_{n2} & \cdots & \lambda_n k_{nn} \end{bmatrix} \qquad \text{(Eq. 9)}$$

$$Ku = \begin{bmatrix} \lambda_1 K_{11} & \lambda_1 K_{12} & \cdots & \lambda_1 K_{1n} \\ \lambda_2 K_{21} & \lambda_2 K_{22} & \cdots & \lambda_2 K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_n K_{n1} & \lambda_n K_{n2} & \cdots & \lambda_n K_{nn} \end{bmatrix} \qquad \text{(Eq. 10)}$$

A matrix of the characteristic roots may be defined as:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \text{(Eq. 11)}$$

Equations 9 and 10 may then be written

$$uk = k\Lambda \qquad \text{(Eq. 12)}$$

$$Ku = \Lambda K \qquad \text{(Eq. 13)}$$

D. The $k\Lambda K$ Transformation

Equations 12 and 13 may easily be operated upon to obtain an equivalent for u. They become

$$u = k\Lambda k^{-1} \quad \text{(Eq. 14)}$$

$$u = K^{-1}\Lambda K \quad \text{(Eq. 15)}$$

These transformations are of the type desired to permit simplification of the procedure for obtaining higher powers of the stochastic matrix. However, a simpler transformation would be one of similar form but in which it would be unnecessary to derive an inverse matrix. To determine such a transformation, use is made of the unique properties of the matrix product Kk. From equation 12 a multiplication of both sides by K gives

$$Kuk = Kk\Lambda \quad \text{(Eq. 16)}$$

and from equation 13 a multiplication of both sides by k gives

$$Kuk = \Lambda Kk \quad \text{(Eq. 17)}$$

Combining equations 16 and 17 results in

$$Kk\Lambda = \Lambda Kk \quad \text{(Eq. 18)}$$

Writing this in expanded form yields matrices of the form

$$\begin{bmatrix} \lambda_1 A_{11} & \lambda_2 A_{12} & \cdots & \lambda_n A_{1n} \\ \lambda_1 A_{21} & \lambda_2 A_{22} & \cdots & \lambda_n A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1 A_{n1} & \lambda_2 A_{n2} & \cdots & \lambda_n A_{nn} \end{bmatrix} \qquad \begin{bmatrix} \lambda_1 A_{11} & \lambda_1 A_{12} & \cdots & \lambda_1 A_{1n} \\ \lambda_2 A_{21} & \lambda_2 A_{22} & \cdots & \lambda_2 A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_n A_{n1} & \lambda_n A_{n2} & \cdots & \lambda_n A_{nn} \end{bmatrix}$$

(Eq. 19)

For this equality to be true, each pair of corresponding elements must be equal. The diagonal elements of both matrices are identical, so they present no problem. For each other pair of elements an equation of the nature $ax = bx$ must hold, where the a and b cannot both be zero since each is a different characteristic root and repeated roots have been disallowed. Therefore, only the trivial solution $x = 0$ exists. This means that every other element except the diagonal elements of the matrix Kk must be zero. The transformation Kk is therefore

$$Kk = \begin{bmatrix} K_{11}k_{11}+K_{12}k_{21}+\cdots+K_{1n}k_{n1} & 0 & \cdots & 0 \\ 0 & K_{21}k_{12}+K_{22}k_{22}+\cdots+K_{2n}k_{n2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & K_{n1}k_{1n}+K_{n2}k_{2n}+\cdots+K_{nn}k_{nn} \end{bmatrix}$$

(Eq. 20)

Each element of equation 20 consists only of terms in K and k associated with a particular characteristic root. Therefore all these terms can be multiplied by an appropriate scalar without altering any equalities or without affecting the other terms. It is possible in this way to choose normalization factors that will make each element on the diagonal of the matrix equal to one. Equation 20 may then be written

$$Kk = I \qquad \text{(Eq. 21)}$$

provided that the normalization factors are correctly chosen.

Next consider equation 14. This is unaltered if it is written

$$u = k\Lambda I k^{-1} \qquad \text{(Eq. 22)}$$

Inserting the equality of equation 21 produces the following results:

$$\begin{aligned} u &= k\Lambda K k k^{-1} \\ &= k\Lambda K I \qquad \text{(Eq. 23)} \\ &= k\Lambda K \end{aligned}$$

E. Higher Powers of u

Consider the matrix u^m. Using the transformation of equation 23 yields

$$u^m = k\Lambda K\; k\Lambda K \;\ldots\; k\Lambda K \qquad \text{(Eq. 24)}$$

where there are m sets of $k\Lambda K$'s. However we have already shown Kk to be equal to the identity matrix so that

$$u^m = k\Lambda^m K \qquad \text{(Eq. 25)}$$

This is the desired transformation, since the higher powers of Λ are easily seen to be

$$\Lambda^m = \begin{bmatrix} \lambda_1^{\,m} & 0 & \cdots & 0 \\ 0 & \lambda_2^{\,m} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n^{\,m} \end{bmatrix}. \qquad \text{(Eq. 26)}$$

F. Characteristic Roots of the Stochastic Matrix

Although equation 25 furnishes a simple method of determining the higher powers of a matrix, it still does not supply any information about the limiting case of the stochastic matrix P^{∞}. To obtain a clear idea of what the limiting situation involves, the limitations upon the characteristic roots of the original matrix P must be determined. The first important property of these roots is described by the following theorem:

The stochastic matrix always has the characteristic root one. The proof of this theorem is as follows: Consider the row vector

$$x = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \tag{Eq. 27}$$

Because the sum of the columns of the stochastic matrix is one, the product xP becomes

$$xP = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} = x \tag{Eq. 28}$$

Equation 5 for the stochastic matrix is

$$K_s P = \lambda_s K_s \tag{Eq. 29}$$

The correspondence between equations 28 and 29 is obvious and it is thus evident that x is a characteristic vector of the stochastic matrix P and that its associated characteristic root is one. The theorem is thus proved.

The second unique limitation upon the characteristic roots of a stochastic matrix is described by the following theorem:

No other root of the stochastic matrix can exceed one in absolute value.

The proof of this theorem is as follows: The product of the characteristic row vector K_s and the P matrix is

$$K_sP = \left[K_{s1}p_{11} + K_{s2}p_{21} + \cdots + K_{sn}p_{n1},\ K_{s1}p_{12} + K_{s2}p_{22} + \cdots + K_{sn}p_{n2},\ \cdots,\ K_{s1}p_{1n} + K_{s2}p_{2n} + \cdots + K_{sn}p_{nn}\right] \qquad \text{(Eq. 30)}$$

Let the subscript m denote the element of a vector having the maximum absolute value, so that, for example, K_{sm} is the element of the vector K_s having maximum absolute value. Each of the other elements of the vector K_s must lie within a circle in the complex plane whose radius is the absolute value of K_{sm}. Hence

$$\left|K_{s1}p_{1j} + K_{s2}p_{2j} + \cdots + K_{sn}p_{nj}\right| \leq \left|K_{sm}\right|p_{1j} + \left|K_{sm}\right|p_{2j} + \cdots + \left|K_{sm}\right|p_{nj}$$
$$= \left|K_{sm}\right|\left(p_{1j} + p_{2j} + \cdots + p_{nj}\right) = \left|K_{sm}\right| \qquad \text{(Eq. 31)}$$

Applying equation 5 yields

$$\left|K_{sm}\right| = \left|K_{s1}p_{1j} + K_{s2}p_{2j} + \cdots + K_{sn}p_{nj}\right| = \left|\lambda_s K_{sm}\right| = \left|\lambda_s\right|\left|K_{sm}\right| \qquad \text{(Eq. 32)}$$

This inequality is true only when

$$|\lambda_s| \leqq 1 \qquad \text{(Eq. 33)}$$

Thus the theorem is proved.

G. Choice of Normalization for the $k\Lambda K$ Transformation

To obtain the transformation of equation 23, a particular normalization of the K and k vectors was required. So far, how to obtain this normalization has not been specified. Determining the normalization first requires the following theorem:

The sum of the elements of every column of the modal matrix k associated with a stochastic matrix with unlike roots except the column corresponding to the characteristic root one, or the sum of any row of the modal matrix K associated with a stochastic matrix with unlike roots except the row associated with the characteristic root one is equal to zero. In symbols:

$$\sum_{i=1}^{n} K_{ij} = 0 \qquad \text{for } j \neq m \text{ where } \lambda_m = 1 \qquad \text{(Eq. 34)}$$

$$\sum_{i=1}^{n} K_{ji}^{j} = 0 \qquad \text{for } j \neq m \text{ where } \lambda_m = 1 \qquad \text{(Eq. 35)}$$

The proof of this theorem is as follows: The relation between each pair of corresponding elements of equation 9, where the stochastic matrix is considered is

$$p_{i1}k_{1j} + p_{i2}k_{2j} + \cdots + p_{in}k_{nj} = \lambda_j k_{ij} \qquad \text{(Eq. 36)}$$

The column sums are thus

$$\sum_{i=1}^{n} \lambda_j k_{ij} = \sum_{i=1}^{n} (p_{i1}k_{1j} + p_{i2}k_{2j} + \cdots + p_{in}k_{nj}) \qquad \text{(Eq. 37)}$$

$$= k_{1j}\sum_{i=1}^{n} p_{i1} + k_{2j}\sum_{i=1}^{n} p_{i2} + \cdots + k_{nj}\sum_{i=1}^{n} p_{in}$$

But for the stochastic matrix by definition

$$\sum_{i=1}^{n} p_{ij} = 1 \qquad \text{(Eq. 38)}$$

for every j. Therefore

$$\sum_{i=1}^{n} \lambda_j k_{ij} = k_{1j} + k_{2j} + \cdots + k_{nj} = \sum_{i=1}^{n} k_{ij} \qquad \text{(Eq. 39)}$$

Thus

$$\lambda_j \sum_{i=1}^{n} k_{ij} = \sum_{i=1}^{n} k_{ij} \qquad \text{(Eq. 40)}$$

where $\lambda_j \neq 1$. This implies

$$\sum_{i=1}^{n} k_{ij} = 0 \qquad \text{(Eq. 41)}$$

The proof for the rows of K will not be given since the reasoning is identical.

Equation 23 may be written in the form

$$P = \sum_{i=1}^{n} \lambda_i \begin{bmatrix} k_{1i}K_{i1} & k_{1i}K_{i2} & \cdots & k_{1i}K_{in} \\ k_{2i}K_{i1} & k_{2i}K_{i2} & \cdots & k_{2i}K_{in} \\ \cdots & \cdots & \cdots & \cdots \\ k_{ni}K_{i1} & k_{ni}K_{i2} & \cdots & k_{ni}K_{in} \end{bmatrix} \qquad \text{(Eq. 42)}$$

The fact that the sum of the elements in each column is equal to one may be written as

$$\sum_{s=1}^{n} \left[\lambda_s K_{sj} \sum_{i=1}^{n} k_{is} \right] = 1 \qquad \text{(Eq. 43)}$$

However it has just been shown that the term represented by the last summation is zero except when $\lambda_s = 1$. Equation 43 thus simplifies to

$$K_{1j} \sum_{i=1}^{n} k_{i1} = 1 \qquad \text{(Eq. 44)}$$

It has already been shown in equation 28 that the row vector having every component one is the characteristic vector associated with the characteristic root $\lambda_1 = 1$. Substituting this for K_1 in equation 44 shows that the column vector k_1 sums to one. It will be shown later that this normalization for k_1 is that for which k_1 is the limiting distribution, and hence a stochastic vector. Thus a suitable normalization for the first row of the matrix K and the first column of the matrix k has been obtained. Now consider the situation where two matrices are obtained where w has its columns proportional to the columns of k, but only the first column has been normalized and v has its rows proportional to the rows of K, but only its first row has been normalized. In other words

$$w = \begin{bmatrix} k_{11} & a_2k_{12} & \cdots & a_nk_{1n} \\ k_{21} & a_2k_{22} & \cdots & a_nk_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & a_2k_{n2} & \cdots & a_nk_{nn} \end{bmatrix} \qquad \text{(Eq. 45)}$$

$$v = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ b_2K_{21} & b_2K_{22} & \cdots & b_2K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_nK_{n1} & b_nK_{n2} & \cdots & b_nK_{nn} \end{bmatrix} \qquad \text{(Eq. 46)}$$

These are precisely the matrices which would be obtained from equations

5 and 6 with the first column of w and the first row of v normalized as described above. Equation 42 written in terms of w and v becomes

$$P = \sum_{i=1}^{n} \frac{\lambda_i}{a_i b_i} \begin{bmatrix} w_{1i}v_{i1} & w_{1i}v_{i2} & \cdots & w_{1i}v_{in} \\ w_{2i}v_{i1} & w_{2i}v_{i2} & \cdots & w_{2i}v_{in} \\ \cdots & \cdots & \cdots & \cdots \\ w_{ni}v_{i1} & w_{ni}v_{i2} & \cdots & w_{ni}v_{in} \end{bmatrix} \qquad \text{(Eq. 47)}$$

The sum of the terms in each major diagonal is

$$w_{1i}v_{i1} + w_{2i}v_{i2} + \cdots + w_{ni}v_{in} = a_i b_i \left[k_{1i}K_{i1} + k_{2i}K_{i2} + \cdots + k_{ni}K_{in}\right] \qquad \text{(Eq. 48)}$$

The term in the brackets is identical with one of the elements of equation 20 and each of these elements is equal to one. Therefore

$$a_i b_i = \text{sum of the diagonal elements of } w_i v_i \qquad \text{(Eq. 49)}$$

In deriving an expression for a higher power of P, P^m, the simplest way to write the expression is usually as a series of terms, each having the form of the matrix of equation 47. Since the λ_i^m is outside the matrix, the proper normalizing constant for each term may be obtained by summing the diagonal elements of the associated matrix and taking the reciprocal of this sum.

H. The Limiting Transition Matrix

It is often of interest to obtain the limiting transition matrix P^{∞}. If equation 25 is written in the same form as equation 42, it becomes

$$P^m = \sum_{i=1}^{n} \lambda_i^m \begin{bmatrix} k_{1i}K_{i1} & k_{1i}K_{i2} & \cdots & k_{1i}K_{in} \\ k_{2i}K_{i1} & k_{2i}K_{i2} & \cdots & k_{2i}K_{in} \\ \cdots & \cdots & \cdots & \cdots \\ k_{ni}K_{i1} & k_{ni}K_{i2} & \cdots & k_{ni}K_{in} \end{bmatrix} \quad \text{(Eq. 50)}$$

Since the λ_i's are all less than one for $i \neq 1$, the following is true

$$\lim_{m \to \infty} \lambda_i^m = 0 \qquad \text{for } i \neq 1 \qquad \text{(Eq. 51)}$$

Thus all of the terms of equation 50 are zero except that associated with $\lambda_1 = 1$. The infinite transition matrix is therefore

$$P^\infty = \begin{bmatrix} k_{11}K_{11} & k_{11}K_{12} & \cdots & k_{11}K_{1n} \\ k_{21}K_{11} & k_{21}K_{12} & \cdots & k_{21}K_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1}K_{11} & k_{n1}K_{12} & \cdots & k_{n1}K_{1n} \end{bmatrix} \qquad \text{(Eq. 52)}$$

It has already been specified that the normalization is such that $K_{1j} = 1$ for every j. Therefore

$$P^\infty = \begin{bmatrix} k_{11} & k_{11} & \cdots & k_{11} \\ k_{21} & k_{21} & \cdots & k_{21} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n1} & \cdots & k_{n1} \end{bmatrix} \qquad \text{(Eq. 53)}$$

I. The Limiting Distribution

Consider the effect of applying P^∞ to a vector representing the initial probabilities x. Then if x is a stochastic vector,

$x_1 + x_2 + \dots + x_n = 1$ and

$$P^\infty x = \begin{bmatrix} k_{11} & k_{11} & \cdots & k_{11} \\ k_{21} & k_{21} & \cdots & k_{21} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n1} & k_{n1} & \cdots & k_{n1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} k_{11} \\ k_{21} \\ \cdots \\ k_{n1} \end{bmatrix} \qquad \text{(Eq. 54)}$$

The resulting vector, which will be referred to as k_1, represents the limiting or stationary distribution of the system. It is evident that this corresponds to the characteristic vector associated with the characteristic root one of the matrix P and that it is independent of the initial conditions.

J. Classification of Markov Chains

The $k\Lambda K$ transformation technique for finding the higher powers of P, the properties of the roots of the stochastic matrix, and the proof of a limiting distribution were derived under the assumption that all of the characteristic roots of the stochastic matrix under consideration were distinct and that only the root one had absolute value one. It is important to know when this condition is met, since only then can the material derived in the preceding sections be used with justification. Several properties of the stochastic matrix are evident upon inspection. These will be defined below.

A Markov chain is said to contain a closed set if there is a set of states from which it is impossible to reach any state outside the set. In other words each probability $p_{jk} = 0$ if k is within the closed set and j is outside it. If a Markov chain contains no closed sets other than the complete set of states of the entire system, it is said to be irreducible.

A typical system characterized by a Markov chain will initially be in some state E_i. Various transitions will occur and eventually the system may again return to the state E_i, in which case the process begins again exactly as it began the first time. The return to the state E_i or to any other state in which the system has existed before is a recurrent event which has an associated recurrence time distribution $(f_i^{(n)})$ which is the probability that the first return to E_i will be at time n. If the f_i's sum to one, the system is certain to return to E_i and the state E_i is called a persistent state. If the sum of the f_i's is less than one, the system may never return to the state E_i. In this case E_i is called a transient state.

Persistent states may be classified in accordance with the characteristics of their recurrence time. If the mean recurrence time is infinite, the state is known as a null state. If return to the state E_i is impossible except in steps of t, 2t, 3t, ..., the state is called a periodic state, with period t. If the state is not periodic, but has a mean recurrence time less than infinity, it is known as an ergodic state.[4]

4. W. Feller, pp. 351-353

The techniques which have been developed for obtaining the limiting distribution were obtained under the conditions that there are no repeated characteristic roots and that only a single root has the absolute value one. It can be shown that repeated roots having absolute values less than one will not affect the limiting distribution and thus the techniques are still valid. All finite ergodic Markov chains fall into one of these two categories[5]. The periodic cases, which have no unique limiting distribution, are characterized by a number of characteristic roots having absolute values of one.

5. M. S. Bartlett, An Introduction to Stochastic Processes, (Cambridge, The University Press, 1956), p. 33.

IV. Queuing Problems

A. General Discussion

A queuing or waiting line problem arises when there is an input of customers which must be served by a given number of servers. If over any time period more customers arrive than can be served, a waiting line builds up. Queuing systems of this sort may be completely defined by specifying

(1) input

(2) queue-discipline

(3) service mechanism.

Specification of the input implies stating the number of customers entering the system within a certain time interval and the arrival times of these customers. This information may either be in terms of probability distributions or in specific statements of numbers of customers and arrival times. The two most commonly treated inputs are the regular, in which one customer arrives regularly after every time interval Δt, and the Poissonian, in which customer arrivals have a probability distribution

$$P_j\left(\frac{v}{a}\right) = \frac{1}{j!}\left(\frac{v}{a}\right)^j e^{-\frac{v}{a}} \qquad \text{(Eq. 56)}$$

where $P_j\left(\frac{v}{a}\right)$ is the probability that j customers will arrive over a time interval v where a is the mean arrival time.

The queue discipline refers to the order in which the incoming customers are served. The most common scheme is that in which customers are served in the order of their arrival, or "first-come, first-served". However almost any scheme of queue discipline is possible. For instance every nth customer might be served, or customers having some particular characteristic might be given priority. In the radar tracking problem which will be discussed in more detail later, a priority requirement might be that missiles entering the radar coverage from the direction of Russia would be tracked before any other targets.

Specification of the service mechanism requires statement of the number of servers and a definition of the serving time. The most commonly encountered serving times are the regular or fixed serving time which might be encountered in such situations as toll collection where the serving action is identical for each customer and the exponential, in which the serving times have a probability distribution of $\frac{1}{b} e^{-\frac{v}{b}}$ where b is the mean serving time. This serving time distribution is characteristic of those situations in which random factors enter into the determination of the length of service required. Other more complicated service mechanisms are also possible. For example, the service time may be dependent upon queue size. This would be typical of restaurant service, where the servers speed up their efforts when a crowded condition occurs.

In discussing the solution of queuing problems, consideration will be limited to problems in which the random variables determining the input are independently and identically distributed, the queue discipline is "first-come, first-served", and the service times are independent of each other and of the input. Problems having other than these properties are too complex to permit consideration in this paper.

Where arrivals or service represent continuous distributions in time, the queuing problem in its entirety is a rather complex stochastic process. In the special case where the input is Poissonian and the service times exponential, the complete queuing problem is Markovian and is susceptible to relatively simple solution. In other cases, it is often necessary to further complicate the problem by adding other time-dependent variables to completely define each state, both in terms of number of customers and elapsed service or inter-arrival times so that the system may be completely described in terms of a Markov chain. Many of these more complex queuing processes may be analyzed in terms of what is known as an "imbedded" Markov chain. The technique involved is to consider the state of the system only at certain discrete, well-defined times. Now if either the arrival times or the service times have distributions of an exponential nature, the required Markovian property exists and the system (defined at the discrete times only) can be represented by a Markov chain. This imbedded Markov chain provides a means of investigating the system which is much simpler than the Markov chain which might be developed by considerably augmenting the description of the system states. However, since the imbedded Markov chain does not fully characterize the system, but describes it at selected discrete times only, some of the properties of the system which it is desired to learn may not be obtainable from a particular imbedded Markov chain.

B. The Poissonian Input Single Server Queuing Process

Consider a queuing process which has a Poissonian input, a single server and an undefined service time distribution. Suppose this system is considered only at those discrete times when a customer is just leaving, his service completed. The state of the system is defined as the number of persons who are left waiting by the departing customer. This system may thus be represented by the imbedded Markov chain defined by[6]

$$P = \begin{bmatrix} q_o & q_o & 0 & 0 & \cdots \\ q_1 & q_1 & q_o & 0 & \cdots \\ q_2 & q_2 & q_1 & q_o & \cdots \\ q_3 & q_3 & q_2 & q_1 & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \end{bmatrix} \qquad \text{(Eq. 57)}$$

where q_j represents the probability that j customers will arrive during the service interval. The equation for the q_j's is easily determined. If the input is Poissonian with mean arrival time a, the probability that j customers will arrive during the interval v is $\frac{1}{j!} e^{-\frac{v}{a}} (\frac{v}{a})^j$. The probability distribution of the service times will be defined as dB(v), where this is the probability of the service time being v. The product of these probabilities is the probability of the service time being v and j customers entering within that time, and the sum of these

6. D. G. Kendall, "Stochastic Processes Occurring in the Theory of Queues and Their Analysis by the Method of the Imbedded Markov Chain", <u>The Annals of Mathematical Statistics</u>, vol. 24, no. 3, Sept 1953, p. 343.

products for all possible v's gives the total probability of j customers entering within the service interval. Thus

$$q_j = \frac{1}{j!} \int_0^{\infty} e^{-\frac{v}{a}} \left(\frac{v}{a}\right)^j dB(v) \qquad (j = 0, 1, 2, ..) \qquad \text{(Eq. 58)}$$

The traffic intensity, ρ , is defined as the ratio of the mean service time to the mean arrival time, or

$$\rho = b/a \qquad \text{(Eq. 59)}$$

It can be shown that the matrix of equation 58 is irreducible and aperiodic in all cases and that its states are ergodic when $\rho < 1$, null when $\rho = 1$ and transient when $\rho > 1$.[7] In the ergodic case, equation 6 is applicable and thus in terms of an unnormalized characteristic vector for the characteristic root one

$$Pw_1 = w_1 \qquad \text{(Eq. 60)}$$

It is then possible to obtain and normalize w_1, resulting in the limiting distribution.

Suppose a single server, Poissonian input queuing process has an exponential distribution of serving times having a mean b, such that

$$dB(v) = 1/b \quad e^{-\frac{v}{b}} \, dv \qquad \text{(Eq. 61)}$$

Equation 58 now has the form

7. Kendall, p. 344. Feller, p. 378 & p. 450.

$$q_j = \frac{1}{bj!} \int_0^{\infty} e^{-v\left(\frac{1}{a} + \frac{1}{b}\right)} \left(\frac{v}{a}\right)^j dv \qquad \text{(Eq. 62)}$$

which fortunately is easily integratable and reduces to

$$q_j = \left(\frac{\rho}{1+\rho}\right)^j \left(\frac{1}{1+\rho}\right) \qquad \text{(Eq. 63)}$$

The stochastic matrix defining the imbedded Markov chain of this process is thus

$$P = \frac{1}{1+\rho} \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ \frac{\rho}{1+\rho} & \frac{\rho}{1+\rho} & 1 & 0 & \cdots \\ \left(\frac{\rho}{1+\rho}\right)^2 & \left(\frac{\rho}{1+\rho}\right)^2 & \frac{\rho}{1+\rho} & 1 & \cdots \\ \left(\frac{\rho}{1+\rho}\right)^3 & \left(\frac{\rho}{1+\rho}\right)^3 & \left(\frac{\rho}{1+\rho}\right)^2 & \frac{\rho}{1+\rho} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad \text{(Eq. 64)}$$

The limiting distribution is now found by solving the set of equations

$$p_{j1}w_{11} + p_{j2}w_{21} + \cdots = w_{j1} \qquad \text{(Eq. 65)}$$

which is easily found to yield

$$w_{j1} = \rho^j \qquad \text{(Eq. 65)}$$

The limiting distribution prior to normalization is thus

$$w_1 = \begin{bmatrix} 1 \\ \rho \\ \rho^2 \\ \cdot\cdot \end{bmatrix} \qquad \text{(Eq. 66)}$$

The sum of the elements of this vector is the sum of a geometric series which, since ρ is less than unity, sums to $\frac{1}{1-\rho}$. Thus the normalization factor is $1-\rho$ and the normalized limiting distribution is

$$k_1 = \begin{bmatrix} 1-\rho \\ (1-\rho)\rho \\ (1-\rho)\rho^2 \\ (1-\rho)\rho^3 \\ \cdots \end{bmatrix} \qquad \text{(Eq. 67)}$$

It is also important to determine the probability that a customer will not have to wait any longer than a time T to have his service completed.[18] If there are no customers present when the customer arrives, the probability that his serving time will exceed T is

$$P_o(T) = e^{-T/b} \qquad \text{(Eq. 68)}$$

With one customer present, the probability that the arriving customer's time within the system will exceed T is the sum of the probability that the present customer's serving time will exceed T and the probability that the present customer's service will be finished in time T but the arriving customer's will not, or

$$P_o(T) + P_1(T) = e^{-T/b} + (T/b)\ e^{-T/b} \qquad \text{(Eq. 69)}$$

Similarly if there are n customers ahead of the arriving customer, the

18. P. M. Morse, Queues, Inventories and Maintenance, (New York, John Wiley & Sons, Inc., 1958), p. 70.

probability that he will still be in the system after time T is

$$\sum_{m=0}^{n} P_m(T) = \sum_{m=0}^{n} \frac{1}{m!}\left(\frac{T}{b}\right)^m e^{-T/b} \qquad \text{(Eq. 70)}$$

The average probability that a customer will be in the system for a longer time than T is found by multiplying equation 70 by the probability that the waiting line will contain n customers and summing over n. The resulting equation is

$$G(T) = \sum_{n=0}^{\infty} (1-\rho)\rho^n \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{T}{b}\right)^m e^{-T/b} \qquad \text{(Eq. 71)}$$

$$= e^{-(1-\rho)T/b}$$

If the service time is not exponential, the neat solutions of equation 58 do not exist and the problem of determining the correct values for the matrix becomes much more difficult. Difficulty may also be encountered in determining the values for the limiting distribution. Furthermore closed sloutions for G (T) usually do not exist so that numerical methods must be used.

C. Many Server Queuing Processes

Queuing processes where more than a single server is involved may be treated by the imbedded Markov chain technique if they have an exponential service time. In this case it is more convenient to consider the state of the system each time a customer arrives rather than each time one departs. The matrix of transition probabilities for the imbedded

Markov chain is much more complicated than for the single server case, including a matrix similar to that of equation 57 preceded by a section having as many rows as there are servers. The limiting distribution can be shown to be a vector similar to that of equation 67, its terms forming a geometric series, but with modifications to the first s-1 terms if s is the number of servers.[19]

V. The Radar Tracking Problem

Consider a radar system whose function is to detect enemy missiles and track them for a sufficient time to permit their destruction by some type of defensive weapon. For simplicity assume that the system is primarily concerned with intercontinental ballistic missiles which have trajectories which may be approximated as entering at the top of the radar's coverage volume and travelling straight down to the ground, each with the same velocity, so that each missile is within the radar coverage volume for a time T. Furthermore the radar detection is said to be 100% efficient so that all targets are detected at the limits of the radar coverage. However, the tracking capacity of the radar is limited. The problem is to determine the probability that missiles will not spend greater than time T within the system before tracking is completed.

The practical problem is to set up a mathematical model of this radar system which will not be too far from the actual situation but which will be simple enough to permit solution. The first consideration is the

19. Kendall, p. 349

distribution of the incoming missiles. Here a Poisson distribution having a mean arrival time a is a realistic description of the actual situation. The simplest solutions may be obtained if the radar system is considered to have a single tracker. The results thus obtained will be representative of the many conventional radar systems having a single antenna which must be focussed on the target being tracked and also of situations where several trackers are each restricted to a particular non-overlapping coverage volume. This situation also gives a pessimistic estimate of the situation where several trackers cover the same volume, if the results are altered by a factor representing the number of trackers. The queue discipline will be considered to be "first-come, first-served". This corresponds to the usual situation where the missile nearest the ground offers the greatest immediate threat. It would be tempting to specify a specific standard length of time as the tracking time, since it normally should take the same length of time to accurately track each missile. This leads to more complex solutions, however. The tracking time will therefore be considered to be exponential with mean b. A number of reasons could be advanced as to why variations of tracking time could occur which would make this description of tracking time correspond to the actual situation. It is sufficient to say that the most improvement in system capability which could occur over the exponential tracking time figures if the tracking time were constant would be a factor of two.[10]

The radar tracking system as thus described corresponds to the sin-

10. Morse, p. 75.

gle server queuing process with Poissonian input and exponential serving time as has been analyzed. It may seem that this technique is a bit unfair, since this is the only type of system for which simple results exist, and it is hence quite obvious that the radar characteristics were described in such a way as to make these results applicable. In actual practice, however, the solution of such problems as the radar tracking problem represents an engineering effort which must be balanced in cost against the cost of additional equipment. It has been shown that the description of the radar system may be made to correspond to the simple type problem without departing excessively from what are assumed to be the actual physical characteristics. Any great effort to improve the accuracy of the results by more complex definition of the system parameters with consequent complications in the mathematics would be of little value unless some sizeable savings in cost or an appreciable improvement in the system could be seen to result from the more accurate figures. Furthermore improved system definitions are probably not possible from a purely mathematical approach. Some experimentation would be required to more accurately determine the input and service time distributions.

The desired solution to the radar tracking problem as set up is found in equation 71. If T (the total time of the missile within the radar coverage volume) is specified in multiples of the mean missile arrival time ($T = ma$) then equation 71 becomes

$$G(T) = e^{-(1-\rho)m/\rho} \qquad \text{(Eq. 72)}$$

The results in the following table are easily obtainable from equation 72.

In this situation G(T) is the probability that the missile will not be destroyed before hitting its target.

ρ	G(T) T=a	G(T) T=2a	G(T) T=3a	G(T) T=4a	T=5a G(T)
.1	1.23×10^{-4}	1.53×10^{-8}	1.88×10^{-12}	2.32×10^{-16}	2.86×10^{-20}
.2	.0183	3.35×10^{-4}	6.14×10^{-6}	1.13×10^{-7}	2.06×10^{-9}
.3	.0970	9.46×10^{-3}	9.12×10^{-4}	8.87×10^{-5}	8.72×10^{-6}
.4	.223	.0497	.0111	2.48×10^{-3}	5.54×10^{-4}
.5	.368	.135	.0497	.0183	6.74×10^{-3}
.6	.513	.264	.135	.0700	.0356
.7	.652	.424	.278	.181	.118
.8	.779	.607	.472	.365	.287
.9	.895	.801	.717	.641	.574

ρ	G(T) T=6a	G(T) T=7a	G(T) T=8a
.1	3.53×10^{-24}	4.36×10^{-28}	5.38×10^{-32}
.2	3.78×10^{-11}	6.91×10^{-13}	1.27×10^{-14}
.3	8.31×10^{-7}	8.33×10^{-8}	8.36×10^{-9}
.4	1.23×10^{-4}	2.75×10^{-5}	6.14×10^{-6}
.5	2.48×10^{-3}	9.11×10^{-4}	3.35×10^{-4}
.6	.0183	9.38×10^{-3}	4.84×10^{-3}
.7	.0757	.0497	.0321
.8	.223	.174	.135
.9	.513	.459	.411

VI. Conclusion

The algebraic handling of Markov chains has been described and applied to setting up and analyzing the imbedded Markov chains in certain queuing processes. It has been shown how a typical problem may be made to correspond to a simple queuing process within the acceptable limits of error and some typical results have been obtained.

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ABSTRACT

Probability problems in which a time parameter is involved are known as stochastic processes. The simplest time dependent stochastic processes are those in which the probabilities of a system changing to various states are solely dependent upon the present state of the system. These processes are known as Markov processes, or for the case where only discrete time intervals are considered, as Markov chains. A Markov chain may be completely defined by the matrix of its transition probabilities. This matrix is called a stochastic matrix and is characterized by the facts that it is a square matrix, that the elements of each column sum to one and that all the elements are non-negative.

An important consideration in most Markov chain problems is the effect of a number of transitions as defined by the stochastic matrix. Performing this operation requires determining the higher powers of the stochastic matrix. Two modal matrices are defined, where k is the matrix of the column characteristic vectors of the stochastic matrix and K is the matrix of the row characteristic vectors. It is shown that with proper normalization of these vectors, the stochastic matrix P is equal to $k\Lambda K$, where Λ is the matrix of the characteristic roots along the diagonal and zeroes elsewhere. The higher powers of the stochastic matrix, P^m, are then found to be equal to $k\Lambda^m K$. The stochastic matrix is found always to have a characteristic root one, and all the other roots are shown to be less than one in absolute value. The limiting transition matrix P^∞ is found to have identical columns, each consisting of the

characteristic column vector associated with the characteristic root one. The limiting distribution is the same vector and is independent of the initial conditions.

A queuing or waiting line problem arises when there is an input of customers which must be served by a given number of servers. If over any time period more customers arrive than can be served, a waiting line builds up. Queuing problems of this sort may be completely defined by specifying the input, the queue-discipline and the service mechanism. Except in special cases, these problems cannot be represented in full by Markov chains without augmenting the state descriptions with a number of additional variables, resulting in very complex Markov chains. However, if the system is defined only at discrete intervals, such as at the times when a customer is just leaving, an imbedded Markov chain may be set up. Such a chain is determined for single-server, Poissonian input queuing problems. The chain is analyzed for the case of exponential service times and the limiting distribution and the probability of a customer remaining in the system for a time longer than T are obtained. The imbedded Markov chains for many-server, exponential serving time systems having generalized inputs are discussed briefly.

A typical radar tracking problem is defined and shown to be closely correspondent to the single-server, Poissonian input and exponential service time queuing problem. In this case, the probability that the customer remains in the system for a time longer than T corresponds to the probability that an incoming enemy missile will succeed in hitting its target rather than being destroyed by defensive weapons. These probabilities for various time relations are computed and tabulated.