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BOSTON UNIVERSITY  
GRADUATE SCHOOL OF ARTS AND SCIENCES

Dissertation

**LIQUIDATION UNDER DYNAMIC PRICE IMPACT**

by

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Submitted in partial fulfillment of the  
requirements for the degree of  
Doctor of Philosophy

2016

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I dedicate this dissertation to my mother and father, Manaleh and Hadi.

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# LIQUIDATION UNDER DYNAMIC PRICE IMPACT

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Boston University, Graduate School of Arts and Sciences, 2016

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## ABSTRACT

In order to liquidate a large position in an asset, investors face a tradeoff between price volatility and market impact. The classical approach to this problem is to model volatility via a Brownian motion, and separate price impact into its permanent and temporary components. In this thesis, we consider two variations of the Chriss-Almgren model for temporary price impact. The first model investigates the infinite-horizon optimal liquidation problem in a market with float-dependent, nonlinear temporary price impact. The value function of the investors basket and the optimal strategy are characterized in terms of classical solutions of nonlinear parabolic partial differential equations. Depending on the price impact parameters, liquidation may require finite or infinite time. The second model considers time-varying market depth, in that intense trading increases temporary price-impact, which otherwise reverts to a long-term level. We find the optimal execution policy in a finite horizon for an investor with constant risk aversion, and derive the solution using calculus of variation techniques. Although the model potentially allows for price manipulation strategies, these policies are never optimal. We study the non time-constrained case as a limit to the finite-horizon case and explain the solution through a quasi-linear PDE.

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# Chapter 1

## Introduction

### 1.1 Literature review

Financial institutions have a substantial investment position in different assets. Due to external regulatory issues, from time to time these institutions may face to be required to liquidate a substantial position in a short time. As an example, after a downgrade in an asset's credit rating a mutual fund is no longer able to hold that asset and has to liquidate his entire position in that asset. The liquidation strategy of a large position has a significant impact on its execution price. An optimal policy needs to trade off three conflicting goals. First, when a large position is earmarked for liquidation, it is essentially considered as a large source or risk that lacks sufficient reward, to be sold as quickly as possible. Second, in a market with limited depth intense selling leads to more unfavorable prices, calling for slow liquidation. Third, as the market absorbs the liquidating position, market depth can decrease because of asymmetric information or predatory trading.

Prices are established in the exchange, through the interaction and balance between supply and demand. On one side, there are market participants who are willing to buy at certain prices and other participant (who could be the same as the buyers) that are willing to sell at certain prices. These bid and offer orders (limit orders) are submitted to the exchange and recorded continuously in time in the limit orderbook. In this environment, the *observed price* in the market for a trader who wants to buy is the smallest ask quote in the book and the greatest available bid quote for a trader who wants to sell. The consistent rebalances in bid-ask orders following by the activity of the other market participants is

called *volatility* of the price. This usually is treated as an exogenous factor<sup>1</sup>. In order to make a transaction, each order needs to be price matched with order from the other side of the orderbook. Hence, the execution of limit orders is uncertain and can't be an option for a trader with time constraint. The alternative for such traders are market orders. After submitting a market order of size  $h$  any available limit order on the other side of the book with the highest price will match, if the number of shares requested in the market order be higher than the number of available share on the highest price, the second highest price will be used to fill the rest of the order. Hence, after submission of a market order, the *execution price* can be less than the observed price.

In facing such environment, the trader has to maximize the outcome of his trading activity by picking the right time and size of market orders optimally. This difference between observed price and execution price is called *price impact*. This acts endogenously and explains how after investor's market order the bid-ask orders region reshapes and affects the future price.

In the past three decades the optimal execution with limited depth has developed in several directions, starting from the seminal work of Kyle (1985) on a linear temporary impact. Almgren and Chriss (1999, 2001), building on the intuition in Stoll (1989), distinguish between temporary and permanent price impact and study the mean-variance optimal liquidation with linear impact, in which the lower risk of early liquidation trades off the costs of a higher selling rate. Huberman and Stanzl (2004) prove that, to exclude arbitrage profits through price-manipulation, permanent impact must be linear. Gatheral (2010) finds necessary conditions to exclude price manipulation in a large class of price-impact models. Perold and Salomon Jr (1991) and the empirical work of Obizhaeva (2011) suggest that the linearity of temporary price impact is unrealistic. Also Almgren (2003) studies optimal liquidation for nonlinear, power-type impact. Schied and Schneborn (2009) formulate the problem in the continuous time and find the optimal liquidation strategy for infinite horizon under linear temporary impact with variable risk aversion, and Schöneborn (2011) extends

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<sup>1</sup>In some models like Almgren (2003) the volatility increases as a function of trading intensity

the analysis to the nonlinear case and for several risky assets. Another way of addressing the execution problem is through transient impact models. In this type of models, a trade not only affects its own execution price but also its effect will be carried over into the future trades. Bouchaud et al. (2004) used the trades and quote data from the Paris stock market and measured this decay through the autocorrelation of trade signs. They argued that market impact is temporary and that it decays as a power-law. In a more extensive work, Gatheral (2010) addressed a larger variety of decay functions and found the required conditions to address the issue of price manipulation. This idea got a great recognition in the context of limit orderbook through the work of Obizhaeva and Wang (2013) where they explained price impact as a result of the dynamic of the limit-order book. In this context, Alfonsi et al. (2010) consider books with more flexible shapes, showing their connection with nonlinear price impact. Predoiu et al. (2011) further generalize the shape of the order book, including both discrete and continuous components. While these can accommodate a rich dynamics of the limit-order book, the corresponding liquidation problems are rarely tractable, in contrast to the price-impact models in reduced form, which are amenable to stochastic control techniques, as in Schied and Schneborn (2009).

In this thesis, we investigate the interplay between risk and price impact in deriving the optimal liquidation policy, under two new price impact models. In both models, we use the Bachelier model, to describe the asset price in the absence of investor's trading activity. For market impact, we use the Almgren and Chriss (1999) framework and divide the market impact into two parts of permanent and temporary impacts, where former shows how the increase in the number of sold shares moves the price in a long run, and the latter demonstrates the effect in the execution price through intense trading. The novelty of our model in the reduced form literature is that the temporary price impact is dynamic.

In the first model, temporary price impact can be a function of past trades, as to account for the limited ability of the market to fully absorb a trade, and possibly for the size of the trader. We also allow for temporary impact to increase in the size of the residual position, or *float dependence*, to reflect the potential worsening of liquidity in the late

stages of liquidation. A priori, this feature encourages a quicker and earlier liquidation to preempt deterioration in liquidity, but it also implies slower trading rate at the late stages of liquidation. A posteriori, we find the former effect to be modest and the second one to be significant, which suggests that the responses of liquidation strategies to anticipated and unanticipated changes in liquidity are similar. This analysis extends previous work of Almgren and Chriss (1999) and Schied and Schneborn (2009) on linear impact, and Almgren (2003) on nonlinear impact, in which market depth is held constant throughout liquidation.

In the second model, the temporary impact evolves dynamically according to the past trades. This change accounts for the fact that the market's depth varies subject to the intensity of submitted orders, where high intensity trading tends to reduce the market depth while in the absent of trading, it recovers its equilibrium position throughout time. We find the optimal execution policy and show that unlike some choices in the old models involving resilience, our model doesn't allow for transaction-triggered price manipulation strategies, i.e policies involving intermediate buying from the same stock in a liquidation problem could not be optimal.

The rest of this chapter is dedicated to introducing the relevant models to the work in this thesis. In particular, the model of Almgren and Chriss (1999) which is fundamental to the construction of our own models are presented in details. Moreover, the results of Schied and Schneborn (2009) which we generalized in our first model is explained briefly. Finally, for the purpose of comparison and giving an intuition to our second model, we finish this chapter by presenting some results on transient impact model.

## 1.2 Almgren-Chriss Model

### 1.2.1 Discreet path approach

In the simplest form, the finite horizon problem can be addressed via discretization. We divide the liquidation interval  $[0, T]$  to  $N$  subintervals of equal size  $\tau = \frac{T}{N}$ , then the

investor has to choose optimally a sequence of quantities  $\{n_i\}$ , representing the number of sold shares within each interval  $[t_{i-1}, t_i]$  for  $i = 0, \dots, N - 1$  where  $t_i = i\tau$  such that  $\sum_{i=1}^N n_i = x_0$ , where  $x_0$  is the initial number shares he has to sell. Equivalently, we can use the sequence  $\{x_i\}$  to denote the number of shares he's holding after the  $i$ -th period.

In each time step price evolves according to two factors of volatility and price impact. In reflecting these two factors, first we use a Brownian motion<sup>2</sup>  $B_t$  in order to reflect the effect of other traders in the outcome price in the absent of the insider. Let  $Z_1, Z_2, \dots, Z_N$  be normal I.I.D. with mean zero and standard deviation one. Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}_{i=1}^N, P)$  be the filtered probability space on which  $Z_i$  are defined

$$S_i = S_{i-1} + \sigma\sqrt{\tau}Z_i$$

For price impact, we use the Almgren-Chriss model<sup>3</sup> and divide the price impact into two parts. First, a temporary impact reflects the temporary imbalances in supply and demand caused by the very last trading of the investor and moving it away from the equilibrium. Second, a permanent impact which shows the change in the equilibrium due to trading which will last for the life of liquidation. We denote these two functions respectively by  $\text{Per}(\cdot)$  and  $\text{Temp}(\cdot)$ . Let's assume the initial price of the interested security price is  $S_0$ . After each time step the observed security price evolves as follow

$$S_i = S_{i-1} + \sigma\sqrt{\tau}Z_i - \tau\text{Per}\left(\frac{n_k}{\tau}\right) \quad (1.1)$$

and the execution price as

$$\tilde{S}_i = S_i - \text{Temp}\left(\frac{n_k}{\tau}\right) \quad (1.2)$$

---

<sup>2</sup>one could argue to use other form like geometric Brownian motion see Bertsimas and Lo (1998)

<sup>3</sup>Although older versions of this model appeared earlier in Bertsimas and Lo (1998) and Madhavan (2000), it commonly named after Almgren and Chriss

As a result, the revenue followed by the trading policy  $\{n_i\}$  is

$$R_T(X) = \sum_{i=1}^N n_i \tilde{S}_i = X S_0 + \sum_{i=1}^N (\sigma \sqrt{\tau} Z_i - \tau \text{Per}(\frac{n_i}{\tau}) x_i) - \sum_{i=1}^N n_i \text{Temp}(\frac{n_i}{\tau}) \quad (1.3)$$

Liquidation process usually is required to be completed in a substantially short time, for this reason one could ignore the effect of interest rate and omit discounting factor in evaluating the policies. The investor could be only interested in maximizing the expected value of trading revenue. In that case, the cost due to volatility of the price will be ignored in calculation. However, each investor have certain behavior against uncertainty risk. In order to measure the risk, we use a risk-aversion parameter  $A \geq 0$ , and address the optimal execution policy for an investor who wants to maximize his mean-variance utility function by minimizing his mean-variance utility cost

$$U(x) = E(x) + A V(x) \quad (1.4)$$

where the expected cost  $E(x)$  and the variance cost  $V(x)$  are given by

$$E(x) = \sum_{i=1}^N \tau x_i \text{Per}(\frac{n_i}{\tau}) + \sum_{i=1}^N n_i \text{Temp}(\frac{n_i}{\tau}) \quad (1.5)$$

$$\text{Var}(x) = \sum_{i=1}^N \tau \sigma^2 x_i^2 \quad (1.6)$$

As a first model, in addition to the permanent price impact we use a linear function for the temporary impact function. Let the constant  $\lambda$  be a linear representation of temporary price impact. A single trade of size  $n$  induces the following impacts

$$\text{Per}(\frac{n}{\tau}) = \gamma \frac{n}{\tau}$$

$$\text{Temp}(\frac{n}{\tau}) = \lambda \frac{n}{\tau}$$

Hence, the expectation of impact cost becomes

$$E(x) = \frac{\tilde{\lambda}}{\tau} \sum (x_i - x_{i-1})^2,$$

where

$$\tilde{\lambda} = \lambda - \frac{1}{2}\gamma\tau. \quad (1.7)$$

The expected utility function  $U(x)$ , constructed through  $V(x)$  via equation (1.6) and  $E(x)$  from above is strictly convex function of the control parameters  $x_1, \dots, x_{N-1}$  as long as  $\tilde{\lambda} > 0$ . Hence, there exists a unique global minimum which could be obtained by setting the partial derivatives

$$\frac{\partial U}{\partial x_i} = 2\tau \left\{ A\sigma^2 x_i - \tilde{\lambda} \frac{x_{i-1} - 2x_i + x_{i+1}}{\tau^2} \right\}$$

equal to zero. This is equivalent to have

$$\frac{1}{\tau^2} (x_{i-1} - 2x_i + x_{i+1}) = \tilde{\kappa}^2 x_i \quad (1.8)$$

where

$$\tilde{\kappa}^2 = \frac{A\sigma^2}{\lambda(1 - \frac{\gamma\tau}{2\lambda})}$$

Note that equation (1.8) is a linear difference equation whose solution may be written as a combination of the exponentials  $\exp(\pm\kappa t_i)$ , where  $\kappa$  satisfies

$$\frac{2}{\tau^2} (\cosh(\kappa\tau) - 1) = \tilde{\kappa}^2 \quad (1.9)$$

In the two extreme cases, the optimal policy for minimizing only the expected cost is to sell with constant rate  $\frac{X}{N}$ . On the other far end, in order to only minimize the variance the entire position has to be liquidated in the first step.

### 1.2.2 Price manipulation

As mentioned in the introduction, the goal in this thesis is to study the optimal execution problem under new variations of permanent-temporary price impact models. We challenge the validity of the current models for temporary price impact, however, we accept the linearity assumption for the permanent price impact. The next definition and its following theorem explain the validity of this assumption.

**Definition 1** *A roundtrip strategy is an order execution strategy  $X$  with  $X_0 = X_T$ . A price manipulation strategy is a strategy that involve roundtrip strategy  $X$  with strictly positive expected revenues,*

$$E[R_T(X)] > 0$$

**Theorem 1** *The Almgren-Chriss model does not admit price manipulation for all  $T > 0$ , if the permanent impact is a linear function. i.e.  $\text{Per}(\frac{n}{\tau}) = \gamma \frac{n}{\tau}$  where  $\gamma$  is a constant.*

**Remark 1** *The inverse is not correct. A nonlinear permanent impact function could be enough for preventing price manipulation strategies, if the temporary price impact function be more sensitive to trading intensity and makes the roundtrip policies unfavorable.*

### 1.2.3 Continuous path approach

In order to step into a continuous model, one could simply look at the limiting behavior of the discreet model. Taking the limit of  $N \rightarrow \infty$  and  $\tau \rightarrow 0$  yield to have  $\tilde{\lambda} \rightarrow \lambda$  and  $\tilde{\kappa} \rightarrow \kappa$ . Therefore from the equation (1.9), the optimal path becomes

$$x(t) = \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} X$$

Now let's discuss the Almgren-Chriss model in continuous time framework with a general function for temporary price impact. For the control variable, instead of the number of sold shares in each step, we use the intensity of selling i.e.  $\xi_t = -x'(t)$ . In addition to the end point behavior  $\int_0^T \xi_t dt = x_0$ , we need to assume that the liquidation paths are almost

surely differentiable. Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in R}, P)$  be the filtered probability space on which the Brownian motion  $B_t$  is defined. Similar to equations (1.1) and (1.2) the observed price and the effective price become

$$\tilde{S}_t = S_0 + \sigma \int_0^t B(s)ds - \int_0^t g(\xi(s))ds \quad (1.10)$$

$$S_t = \tilde{S}_t - h(\xi(t)) \quad (1.11)$$

Here, we used the functions  $g(\cdot)$  and  $h(\cdot)$ , respectively for continuous version of the discrete permanent and temporary price impact functions  $\text{Per}(\cdot)$  and  $\text{Temp}(\cdot)$ . Under this set up, for the expected cost and the variance are given by

$$\mathbb{E}[x] = \int_0^T x(t)g(\xi(t)) + \xi(t)h(\xi(t))dt$$

$$\text{Var}[x] = \int_0^T \sigma^2 x^2(t)dt$$

And therefore the minimization question for  $U(x)$  in (1.4) will be a standard problem in calculus of variations

$$\min_{x(t)} \int_0^T F(x(t), x'(t))dt$$

with

$$F(x, p) = xg(-p) - ph(-p) + A\sigma^2 x^2$$

Then the optimal solution solve the Euler-Lagrange equation

$$\frac{d}{dt} \{F(x, p) + pF_p(x, p)\} = 0$$

Which for linear permanent impact function reduces to solving

$$B(\xi(t)) - B(\xi(T)) = A\sigma^2 x^2 \quad (1.12)$$

where

$$B(p) = p^2 \frac{d}{dp} (h(p)) \quad (1.13)$$

As an example for power law price impact functions  $h(p) = \lambda p^\alpha$

$$x(t) = \begin{cases} \left(1 + \frac{1-\alpha}{1+\alpha} \frac{t}{T^*}\right)^{-\frac{1+\alpha}{1-\alpha}} & 0 < \alpha < 1 \\ \exp\left(-\frac{t}{T^*}\right) & \alpha = 1 \\ \left(1 - \frac{\alpha-1}{\alpha+1} \frac{t}{T^*}\right)^{\frac{\alpha+1}{\alpha-1}} & \alpha > 1 \end{cases}$$

where

$$T^* = \left(\frac{\alpha \lambda X_0^{\alpha-1}}{A \sigma^2}\right)^{\frac{1}{\alpha+1}}$$

As we mention from Huberman and Stanzl (2004) results in Theorem 1, the linearity assumption for permanent price impact is essential. In fact, one could interpret the permanent impact as the memory of the market and therefore tying it to the number of sold shares by linearity, makes this assumption natural. However, surprisingly we observe that with linearity, the effect of permanent price impact vanishes completely in determining the optimal strategy via equations (1.12) and (1.13). This unpleasant results is the intuition behind the model, we propose in the next two chapters.

Before ending this section, it's important to mention that now that we use integral frequently in continuous framework, we always need to add integrability to our assumptions. We use the term *Admissibility* for this matter.

### 1.3 Schied-Schoneborn model

The problem of finding optimal execution policy can be regarded either as static or dynamic decision making problem. In the former one, the investor could decide  $\xi_t$  at the starting time, and in the latter one, the investor decides  $\xi_t$  using all the available information a time  $t$ , i.e.  $\xi_t \in F_t$ . In the previous examples, the optimal policies we found were all deterministic, i.e. the new information doesn't improve the optimal policy, unless the

estimation of the parameters changes throughout the liquidation. This feature is mainly due to the choice of mean-variance utility function. In order to fix this issue, one could either consider an investor with non-constant risk aversion or look on other form of objective measure from portfolio choice theory.<sup>4</sup>

**Definition 2** *Risk aversion function, measures the behavior of the investor against risk as a function of his wealth. It is defined via*

$$A(R) = -\frac{u_{RR}(R)}{u_R(R)}$$

For technical reasons later in the proofs, we are required to add the following boundedness assumption on the risk aversion

$$0 < A_{\min} \leq A(R) \leq A_{\max} < \infty$$

Notice the reason we denote the risk aversion by  $A$ , which is similar to mean-variance utility function notation, is that in the case of constant risk aversion function the two concepts are identical.

In this section, we use a utility function with arbitrary risk aversion function, and the objective is to maximize the expected-utility of his final cash flow

$$\max E[u(R_T(X))]$$

We will see that in this framework, an investor with non constant risk aversion function choose a dynamic decision making behavior. For instance, an investor with high proportion of risky asset could be more sensitive to change in the price. Hence, within this framework, the unpredicted future movement of the price changes the behavior of investor; investor might react actively or passively to the favorable or unfavorable movement of the risky asset's price.

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<sup>4</sup>such as, terminal wealth, consumption utility, long run and value at risk measures

With this discussion, it's clear that in the first step we need to distinguish between the risky and riskless part of the portfolio. Let's denote the number of riskless asset held by investor at time  $t$  via  $r_t$  and by  $X_t$  number of risky asset he holds and wants to liquidate. The nominal total portfolio value of the investor at time  $t$

$$\tilde{R}_t = r_t + X_t P_t$$

We've already seen that in the case on linear price impact, the cost associated with the permanent price impact is fixed and inevitable, hence after subtracting the cost associated with permanent impact, we define the adjusted nominal value as follows

$$R_t = r_t + X_t P_t - \frac{1}{2} \gamma X_t^2$$

Due to the Markov nature of the underlying model, we assume that the value function  $\max E[u(R_T(X))]$  can be represented by function  $\nu$  which takes  $X$ ,  $R$  and the remaining time  $\tau = T - t$  as inputs. This type of assumption, requires us to give a verification argument to prove that such function is indeed the true value function.

$$\nu(X, R, \tau)$$

Following any admissible policy  $\xi(t) = -\dot{X}(t)$  in a market with a temporary impact function like  $f(\xi)$ , the portfolio value evolves as follow

$$dR = \sigma X dB_t - \xi f(\xi) dt \tag{1.14}$$

The value function  $\nu(X, R, \tau)$  has the following dynamics

$$d\nu = \nu_R dR + \nu_X dX + \nu_\tau d\tau + \frac{1}{2} \nu_{RR} d\langle R, R \rangle$$

after substituting (1.14) in the last equation, the optimal policy can be obtained by solving the following Hamilton-Jacobi-Bellman equation

$$\nu_t = \frac{1}{2}\sigma^2 X^2 \nu_{RR} + \nu_R(-f(c)c) + \nu_X c \quad (1.15)$$

with boundary condition

$$\lim_{\tau \rightarrow 0} \nu(X, R, \tau) = \begin{cases} u(R) & x = 0 \\ -\infty & x \neq 0 \end{cases}$$

The initial condition reflects the constraint for finishing the liquidation by terminal time, i.e.  $\int_0^T \xi(t)dt = x_0$  by penalizing the liquidations that have not been completed in time. Solving this problem for a general utility functions is difficult. Nevertheless, it can be discussed in two special cases.

In the case of constant risk aversion utility function, this will be equivalent to the mean-variance utility function which we already discussed in the previous section.

Another way of simplifying the (1.15) is to consider an infinite horizon problem. By dropping the time constraint, the HJB equation of  $\nu(X, R)$  takes a simpler form

$$\min_c \left\{ -\frac{1}{2}\sigma^2 x^2 \nu_{RR} + \lambda \nu_R c^2 + \nu_x c \right\} = 0$$

equipped with boundary condition  $\nu(0, r) = u(r)$ . Thanks to the risk component of value function, the investor will still tend to liquidate his position, and we don't need to worry about any penalizing like the general case.

Since the model we present in the first chapter and its proof extend this result of Scheid-Schoneborn, we skip the proof and suffice to present their results. The optimal

policy solution to this HJB equation is given by

$$c(X, R) = \tilde{c}(X^2, R)X$$

Where  $\tilde{c}$  is solution to PDE

$$\tilde{c}_Y = -\frac{3}{2}\lambda\tilde{c}\tilde{c}_R + \frac{\sigma^2}{4\tilde{c}}\tilde{c}_{RR}$$

with initial value

$$\tilde{c}(0, R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}}$$

The value function will be obtained as  $\tilde{\omega}(X^2, R)$  where  $\omega(X, R) = \tilde{\omega}$  is the solution to the following ODE

$$\tilde{\omega}_Y = -\lambda\tilde{c}\tilde{\omega}_R$$

with initial value

$$\tilde{\omega}(0, R) = u(R)$$

As mentioned earlier, at the end, a verification argument is required to prove that  $\omega$  the solution to the HJB equation is indeed equal to the expected return of the liquidation policy  $\max E[u(R_T(X))]$ .

#### 1.4 Transient price impact model

We end this chapter, by introducing the transient price impact model. The models we introduce in the next chapters are inspired by this model as another alternative for improving the temporary-permanent price impact model. The philosophy behind this model is that the Markov environment employed in the previous models is crucial in solving the execution problem, however, in a perfect model this assumption need to be dropped. Consider the following price dynamic

$$S_t = S_0 + \sigma B_t + \int_0^t h(\xi(s))G(t-s)ds \tag{1.16}$$

where  $h$  is the transient price impact function and  $G$  a positive and decreasing function which demonstrate the reduction in the effect of past trades over time. Under this model, the trading cost is

$$C[x] = \int_0^T \xi(t) \left( \int_0^t h(\xi(s)) G(t-s) ds \right) dt$$

It worths to mention that in fact the model of Obizhaeva Wang (2013) as one of the first examples of this model with linear transient price impact and exponential decay function, was based on the study of such dynamic for the shape of limit orderbook and then its connection to the transient model on the level of price impact was observed. In fact studying the optimal execution problem on the level of dynamic of shape of limit orderbook gives more insight to the observed price dynamic and enables us to explain market price impact realistically.

One can consider the Almgren-Chriss model as a special case of this model. By using a linear function for  $h$  in the current set up of (3.48), the temporary price impact imposed by  $\xi_t dt$  could be regarded as  $\xi(G(0) - \lim_{t \rightarrow 0^+} G(t)) dt$ , and the permanent impact is  $\xi \lim_{t \rightarrow \infty} G(t) dt$ .

Although this model is quite powerful, it could cause some irregularity issue. Here we outline the study of Gatheral (2010) on the choices for the price impact and decay functions which has identified a vast majority of the choices for this model which admit price manipulation.

**Proposition 1 (Gatheral)** *In a transient price impact model if the decay function  $G(t)$  be finite and continuous at  $t = 0$  and the transient price impact function  $h : R \rightarrow R$  be nonlinear. Then the model admits price manipulation.*

In addition to price manipulation, Alfonsi et al. (2012) introduced another form of arbitrage as follow, which limits the choices for decay and price impact functions even further. In the following we present the definition of this form of arbitrage and accompany it with a proposition to demonstrate a set of models which admit it.

**Definition 3 (Transaction-triggered price manipulation)** *A market impact model admits transaction-triggered price manipulation if the expected revenues of a sell (buy) program can be increased by intermediate buy (sell) trades. That is, there exists  $X_0, T > 0$ , and a corresponding order execution strategy  $\xi$  for which*

$$E[R_T(\xi^*)] > \sup\{E(R_T(\xi)) \mid \xi \text{ is a monoton order execution strategy for } X_0 \text{ and } T\}$$

**Proposition 2** *The transient price impact model (1.16) admits transaction-triggered price manipulation, if there exist  $s, t > 0$ , such that  $G(0) - G(s) < G(t) - G(t + s)$*

Although despite all these restriction, the transient price impact model can host a great flexibility in allowing a great number of models, yet unlike reduced forms of price impact model like Almgren-Chriss's, the corresponding liquidation problems are rarely tractable. In this thesis, we tried to address this issue by introducing two new form of generalization for the Almgren-Chriss model. These new models allow us to use the older machinery to derive the solution to the optimization problem while keeping some feature of transient price impact model.

## Chapter 2

# Float dependent model

### 2.1 Introduction

In this chapter we derive optimal liquidation policies in a model with permanent and temporary, potentially nonlinear impact. We also allow for temporary impact to increase in the size of the residual position, or *float dependence*, to reflect the potential worsening of liquidity in the late stages of liquidation. A priori, this feature encourages a quicker and earlier liquidation to preempt deterioration in liquidity, but it also implies slower trading rate at the late stages of liquidation. A posteriori, we find the former effect to be modest and the second one to be significant, which suggests that the responses of liquidation strategies to anticipated and unanticipated changes in liquidity are similar. This analysis extends previous work of Almgren and Chriss (1999) and Schied and Schneborn (2009) on linear impact, and Almgren (2003) on nonlinear impact, in which market depth is held constant throughout liquidation.

After introducing the price-impact model, admissible policies, and the value function, section 2 presents the main result and motivates it with a heuristic argument. Section 3 contains all the proofs, based on a transformation that tracks the trader's risk aversion, and leads to a solution to the Hamilton-Jacobi-Bellman(HJB) equation. Finally, we show that the proposed function is indeed the value function of the liquidation problem.

## 2.2 Model

### 2.2.1 The Market

Consider a large investor who needs to liquidate his position in a risky asset. We suppose that the investor can only trade between his risky asset and the risk free asset. We assume that the price of the risky asset follows the Bachelier model

$$P_t = P_0 + \sigma B_t, \quad (2.1)$$

where  $B_t$  is the standard Brownian motion, the positive constant  $\sigma$  is the volatility of the unaffected stock price and  $P_0$  is the initial price. With the possibility of getting negative value for the price, this model might seem to be unrealistic. However, since in reality the liquidation process usually occur in a window of couple days and as we will see our optimal policy almost liquidate the entire position in a very short time, hence the possibility of facing negative prices during the largest part of the position is neglectable.<sup>1</sup>

The investor chooses a trading strategy that we describe by  $X_t$ , the number of shares he is holding at time  $t$ . Assuming  $X_t$  to be absolutely continuous, its derivative  $\dot{X}_t$  exists almost everywhere and

$$X_t = X_0 + \int_0^t \dot{X}_s ds. \quad (2.2)$$

Due to insufficient liquidity in the market, the investor's trading rate  $\dot{X}_t$  moves the market price. To describe the market impact, we use a model which extends the model of Almgren and Chriss (1999). This includes a permanent price impact which affects the price independent of the trading strategy of the investor, we denote it by  $\text{Per}(X_t)$ . Moreover, there exist a temporary price impact which only affects the infinitesimal orders relative to

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<sup>1</sup>Bertsimas and Lo (1998) formulated the liquidation problem by using the geometric Brownian motion for the unaffected price to rule negative prices.

the insensitivity of trade and vanishes instantaneously, we denote it by  $\text{Tem}(X, \dot{X})$ . From Theorem 1 of Huberman and Stanzl (2004), in order to rule out any price manipulation strategies the permanent price impact has to be a linear function of trading rate. We use the constant  $\gamma$  to demonstrate the permanent price impact parameter. For temporary impact we use the non-linear market impact model of Schöneborn (2011) with  $\alpha > 0$ , and generalize it by a float-dependent factor  $q(X)$  which is bounded away from zero

$$\text{Tem}(X_t, \dot{X}_t) = \lambda \dot{X}_t^\alpha q(X_t)$$

The float-dependent factor allows the market price impact to vary in the course of liquidation. To model potentially worsening market depth, we focus on decreasing functions for  $q(x)$  which are bounded, but results remain valid for any  $q(x)$  that allows the existence of an admissible path (Definition 4). Later on, we study in detail  $q(x)$  of power type to obtain closed-form solutions. With  $q(x) = 1$  this model recovers Schöneborn (2011).

In summary, under the above assumptions the effective execution price is

$$P_t = P_0 + \sigma B_t + \gamma(X_t - X_0) + \lambda q(X_t) |\dot{X}_t|^{\alpha-1} \dot{X}_t \quad (2.3)$$

### 2.2.2 Optimal Policy

The investor wants to maximize the expected utility of his final cash position. Holding initially  $r$  units of cash and  $X_0$  units of risky asset, his cash position at time  $t$  under the policy parametrized by  $\xi(t) := -\dot{X}(t)$  is

$$\mathcal{R}_t(\xi) = r + \int_0^t \xi_s P_s ds$$

Using equation (2.3) and integration by parts

$$\mathcal{R}_t(\xi) = r + P_0 X_0 - \frac{\gamma}{2} X_0^2 + \sigma \int_0^t X_s(\xi) dB_s - \lambda \int_0^t \xi_s^{1+\alpha} q(X_s(\xi)) ds$$

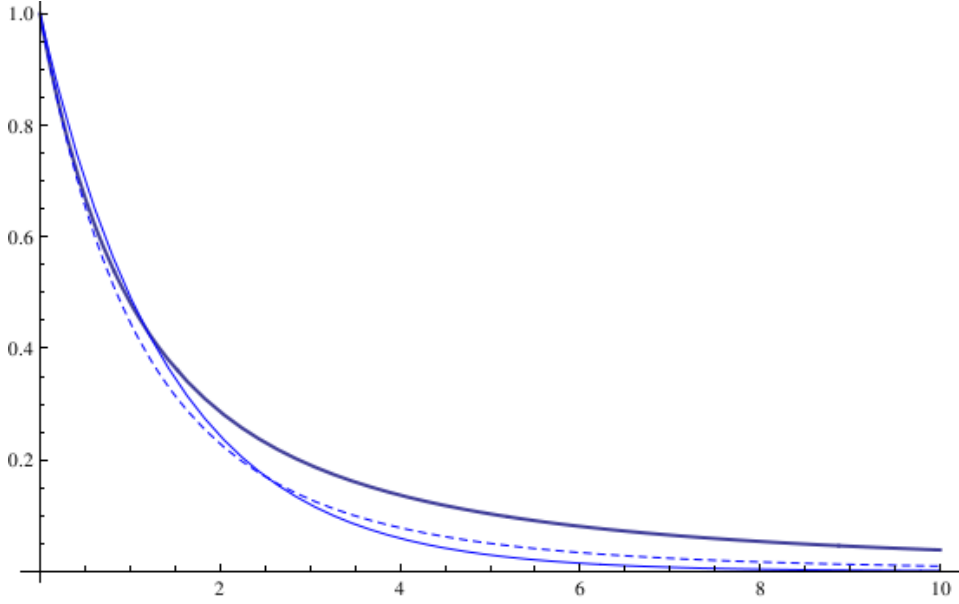


Figure 2.1: Optimal policy for the fixed parameters are  $\sigma = X_0 = \lambda_0 = 1$  and  $\beta = 0$ . The bold curve represent the case where  $\alpha = 0.3$ , the dashed curve represent the case where  $\alpha = 0.6$  and the thin line represent Schied's model with  $\alpha = 1$

$$-\left(P_0 X_t(\xi) + \frac{\gamma}{2}[X_t(\xi)^2 - 2X_0 X_t(\xi)] + \sigma X_t(\xi) B_t\right) \quad (2.4)$$

**Definition 4** Denote by  $\mathcal{X}$  the set of admissible trading strategies, all the progressively measurable processes  $\xi(t) := -\dot{X}(t)$  for which

- i)  $\int_0^t q(X_s) |\xi_s|^{1+\alpha} ds < \infty$  for all  $t > 0$ .
- ii)  $X_t(\omega)$  is bounded uniformly in  $t$  and  $\omega$  with upper and lower bounds that may depend on the choice of  $\xi$ .

Moreover, we denote by  $\mathcal{X}_1$  the set of liquidation strategies, all the admissible trading strategies which furthermore satisfy

- iii)  $\mathbb{E}[\int_0^\infty (X_s^\xi)^2 ds] < \infty$
- iv)  $\lim_{t \rightarrow \infty} (X_t^\xi)^2 t \ln \ln t = 0$

As an example, a liquidation policy which decreases the numbers of asset exponentially belongs to  $\mathcal{X}_1$  and consequently to  $\mathcal{X}$ . In fact we'll see that the optimal policy that we obtain later on in Theorem 2 is of this type.

These technical restrictions are inconsequential for applications. The first assumption ensures that the investor's loss is bounded over a finite period of time, while the second condition requires that the number of shares held is bounded.

The last two assumptions guarantee that  $\mathcal{R}_t$  defined above converges  $\mathbb{P}$ -a.s. as  $t$  goes to infinity. Therefore, for a liquidation policy  $\xi \in \mathcal{X}_1$  the final cash position is

$$\mathcal{R}_\infty^\xi := \lim_{t \rightarrow \infty} \mathcal{R}_t(\xi) \stackrel{a.s.}{=} r + P_0 X_0 - \frac{\gamma}{2} X_0^2 + \sigma \int_0^\infty X_s(\xi) dB_s - \lambda \int_0^\infty |\xi_s|^{1+\alpha} q(X_t(\xi)) ds \quad (2.5)$$

The investor wants to maximize the expected utility of his final cash position  $\mathbb{E}[u(\mathcal{R}_\infty^\xi)]$ . This plausibly depends on the number of shares and the current portfolio value

$$R_t = r + P_0 X_0 - \frac{\gamma}{2} X_0^2 + \sigma \int_0^t X_s(\xi) dB_s - \lambda \int_0^t |\xi_s|^{1+\alpha} q(X_t(\xi)) ds \quad (2.6)$$

which consists of the investor's current cash position obtained through the policy  $\xi$  and the nominal value of his position in the risky asset at the current market price minus the loss from permanent impact. If the trader sells all the risky assets, the portfolio value becomes equal to the cash position. Thus, defining the value function for  $\mathcal{X}_1$  boils down to maximizing the expected utility of the final portfolio value

$$\nu_1(X_0, R_0) := \sup_{\xi \in \mathcal{X}_1} \mathbb{E}[u(\mathcal{R}_\infty^\xi)] \quad (2.7)$$

### 2.2.3 Statement of Main Result

#### Theorem 2

i) The value function  $\nu$  in the optimal execution problem is the classical solution of the following Hamilton-Jacobi-Bellman equation

$$\min_c \left[ -\frac{1}{2}\sigma^2 X^2 \nu_{RR} + \lambda \nu_R |c|^{1+\alpha} q(X) + \nu_X c \right] = 0 \quad (2.8)$$

with the boundary condition

$$\nu(0, R) = u(R) \quad \forall R \in \mathbb{R} \quad (2.9)$$

ii) The a.s. unique optimal policy is

$$\hat{\xi} := c(X^{\hat{\xi}}, R^{\hat{\xi}}) = \left( \frac{-\nu_X}{\lambda(\alpha+1)q(X)\nu_R} \right)^{\frac{1}{\alpha}} \quad (2.10)$$

iii) The optimal strategy can be written as

$$c(X, R) := \tilde{c}(Q(X), R)^{\frac{1}{\alpha}} X^{\frac{2}{\alpha+1}} q(X)^{\frac{-1}{\alpha+1}} \quad (2.11)$$

where

$$Q(X) := \int_0^X (\alpha+1) z^{\frac{2\alpha}{\alpha+1}} q(z)^{\frac{1}{\alpha+1}} dz \quad (2.12)$$

and  $\tilde{c}$  is the unique solution to the PDE

$$\tilde{c}_Y = -\frac{2\alpha+1}{\alpha+1} \lambda \tilde{c} \tilde{c}_R + \frac{\sigma^2}{2(\alpha+1) \tilde{c}^{\frac{1}{\alpha}}} \tilde{c}_{RR} \quad (2.13)$$

with initial condition

$$\tilde{c}(0, R) = \left( \frac{A(R)\sigma^2}{2\alpha\lambda} \right)^{\frac{\alpha}{\alpha+1}}. \quad (2.14)$$

**Corollary 1** For an investor with constant risk aversion  $A$ , the optimal liquidation policy

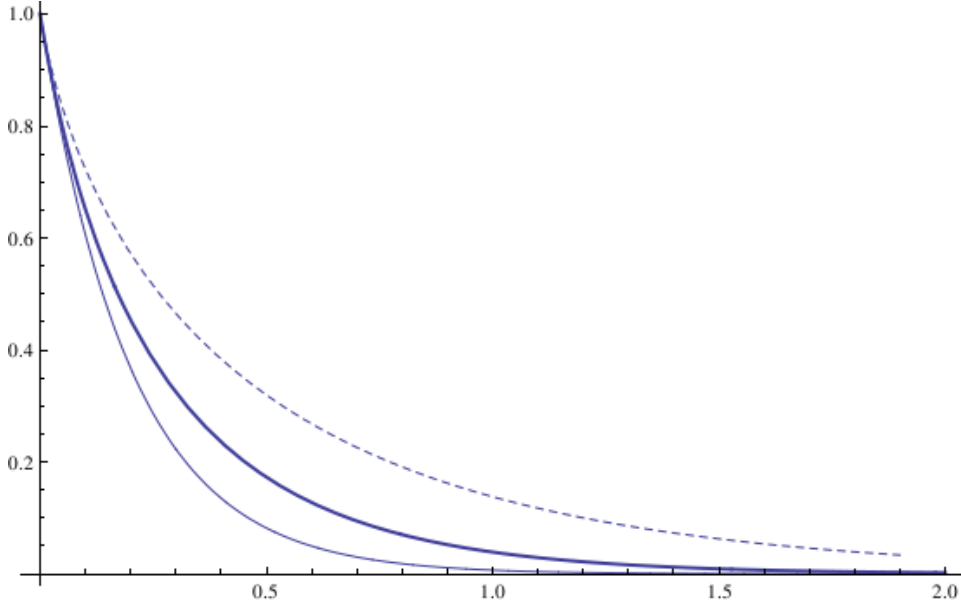


Figure 2.2: Optimal policy when  $q(X) = \delta(X_0 - X) + 1$ . Fixed parameters are  $\sigma = X_0 = A = 1$  and  $\lambda = 0.02$ . The bold curve represents the case where  $\delta = 2$ , the dashed curve represents the case where  $\delta = 10$  and the thin curve represents Schied's model with  $\delta = 0$

is deterministic and given by

$$\xi(X, R) = \left( \frac{A\sigma^2 X^2}{2\alpha\lambda q(X)} \right)^{\frac{1}{\alpha+1}}$$

**Corollary 2** For the power specification  $q(X) = X^\beta$ , under optimal liquidation policy an investor with constant risk aversion is able to liquidate his position completely in a finite time if and only if  $\alpha + \beta > 1$ . In this case, the liquidation terminates at time

$$\frac{\alpha + 1}{\alpha + \beta - 1} \left( \frac{2\lambda\alpha}{A\sigma^2} \right)^{\frac{1}{\alpha+1}} X_0^{\frac{\alpha+\beta-1}{\alpha+1}}. \quad (2.15)$$

### 2.2.4 Heuristic Argument

This section contains an informal derivation of the main results. Under the trading strategy  $c$  the dynamics of  $X$  and  $R$  are given by

$$\begin{aligned} dX &= -cdt \\ dR &= \sigma X dB - \lambda |c|^{\alpha+1} q(X) dt \end{aligned}$$

Denote by  $\nu(X, R)$  a value function which depends on the portfolio value and numbers of risky asset. Ito's formula yields,

$$\begin{aligned} d\nu(X, R) &= \nu_R dR + \nu_X dX + \frac{1}{2} \nu_{RR} d\langle R, R \rangle \\ &= \left( -\lambda \nu_R |c|^{\alpha+1} q(X) - \nu_X c + \frac{1}{2} \sigma^2 X^2 \nu_{RR} \right) dt + \sigma X \nu_R dB \end{aligned}$$

By the martingale optimality principle of stochastic control, the process  $\nu(X_t, R_t)$  must be a supermartingale for any choice of  $c_t$ . Therefore, the drift of  $\nu(X_t, R_t)$  can not be positive, and will become zero for the optimal policy. Hence

$$\max_c \left\{ -\lambda \nu_R |c|^{\alpha+1} q(X) c - \nu_X c + \frac{1}{2} \sigma^2 X^2 \nu_{RR} \right\} = 0$$

and the optimality is achieved at its unique maximizer

$$c = \left[ \frac{-\nu_X}{\lambda(\alpha+1)q(X)\nu_R} \right]^{\frac{1}{\alpha+1}}$$

## 2.3 Proof of Results

This section contains the proofs of all the results of the previous section. We use the same type of arguments as Schied and Schneborn (2009). In the first part, we show that a smooth solution to the HJB equation exists. This is achieved by first, obtaining a solution to the PDE (2.13) of the transformed optimal strategy  $\tilde{c}$ , then solving a transport equation with

coefficient  $\tilde{c}$ , and ultimately use its solution to build a solution to the HJB equation. In the last subsection, we finish the proof by presenting a verification argument, we introduce a modified value function and use it to show that the solution to the HJB equation is indeed equal to the original value function.

### 2.3.1 Properties of the differential equation

To use some auxiliary results from PDE, we need to add some assumptions on the utility function  $u(R)$ . We summarize all of the assumptions on  $u(R)$  in the following

**Assumption 1**  $u(R)$  is a function in  $C^6$  and  $\lim_{R \rightarrow \infty} u(R) = 0$ . Moreover, its absolute risk aversion  $A(R) := -\frac{u_{RR}(R)}{u_R(R)}$  is bounded away from 0 and  $\infty$ , i.e.,

$$0 < A_{\min} = \inf_R A(R) \leq A(R) \leq \sup_R A(R) = A_{\max} < \infty \quad (2.16)$$

**Theorem 3** *The parabolic partial differential equation*

$$f_t - \frac{d}{dx} a(x, t, f, f_x) + b(x, t, f, f_x) = 0 \quad (2.17)$$

with initial condition

$$f(0, x) = \psi_0(x)$$

has a smooth solution in  $C^{2,4}$ , if all of the following conditions are satisfied:

- i)  $\psi_0(x)$  is smooth  $C^4$  and bounded
- ii)  $a$  and  $b$  are smooth (respectively,  $C^3$  and  $C^2$ )
- iii) There are constants  $b_1, b_2 \geq 0$  such that for all  $x$  and  $u$

$$\left( b(x, t, u, 0) - \frac{\partial a}{\partial x}(x, t, u, 0) \right) u \geq -b_1 u^2 - b_2 \quad (2.18)$$

- iv) For all  $M > 0$ , there are constants  $\mu_M \geq \nu_M > 0$  such that for all  $x, t, u$  and  $p$  that

are bounded in module by  $M$

$$\mu_M \geq \frac{\partial a}{\partial p}(x, t, u, p) \geq \nu_M \quad (2.19)$$

and

$$\mu_M(1 + |p|)^2 \geq \left( |a| + \left| \frac{\partial a}{\partial u} \right| \right) (1 + |p|) + \left| \frac{\partial a}{\partial x} \right| + |b| \quad (2.20)$$

**Proof:** The theorem is a direct consequence of Theorem 8.1 in Chapter V of Ladyzhenskaya et al. (1968). In the following, we outline the last step of its proof because we shall use it for the proof of the subsequent propositions.

The conditions of the theorem guarantee the existence of solutions  $f_N$  of (2.17) on the strip  $\mathbb{R}_0^+ \times [-N, N]$  with boundary conditions

$$f_N(0, x) = \psi_0(x)$$

and

$$f_N(t, \pm N) = \psi_0(\pm N)$$

These solutions converge smoothly in  $\mathcal{C}^{2,4}$  as  $N$  tends to infinity, i.e.,  $\lim_{N \rightarrow \infty} f_N = f$ .  $\square$

**Proposition 3** *There exists a smooth  $\mathcal{C}^{2,4}$  solution of*

$$\tilde{c}_Y = -\frac{2\alpha + 1}{\alpha + 1} \lambda \tilde{c} \tilde{c}_R + \frac{\sigma^2}{2(\alpha + 1) \tilde{c}^{\frac{1}{\alpha}}} \tilde{c}_{RR} \quad (2.21)$$

with initial value

$$\tilde{c}(0, R) = \left( \frac{A(R)\sigma^2}{2\lambda\alpha} \right)^{\frac{\alpha}{\alpha+1}} \quad (2.22)$$

which satisfies

$$\inf_{(Y,R) \in \mathbb{R}_0^+ \times \mathbb{R}} \tilde{c}(Y, R) = \inf_{(Y,R) \in \mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{min} = \left( \frac{A_{min}\sigma^2}{2\lambda\alpha} \right)^{\frac{\alpha}{\alpha+1}} \quad (2.23)$$

$$\sup_{(Y,R) \in \mathbb{R}_0^+ \times \mathbb{R}} \tilde{c}(Y, R) = \sup_{(Y,R) \in \mathbb{R}} \tilde{c}(0, R) =: \tilde{c}_{max} = \left( \frac{A_{max}\sigma^2}{2\lambda\alpha} \right)^{\frac{\alpha}{\alpha+1}} \quad (2.24)$$

**Proof:** We want to apply the previous results, but unlike the first three conditions, the last condition of Theorem 3 is not trivial. We first build a function  $f$  and forced it to satisfy all the conditions. Then using Theorem 3 we show its solution satisfies (2.21).

Let's define the smooth nonnegative functions  $h_1(u)$ ,  $h_2(u)$  and  $h_3(u)$  which are bounded away from zero and infinity and for  $\tilde{c}_{min} \leq u \leq \tilde{c}_{max}$  satisfy the following equations

$$h_1(u) = \frac{\sigma^2}{2(\alpha+1)} \frac{1}{u^{\frac{1}{\alpha}}} \quad (2.25)$$

$$h_2(u) = \frac{2\alpha+1}{\alpha+1} \lambda u \quad (2.26)$$

$$h_3(u) = \frac{\sigma^2}{2(\alpha+1)} \frac{1}{u^{1+\frac{1}{\alpha}}} \quad (2.27)$$

Now the following functions satisfy the conditions of Lemma 3

$$a(x, t, u, p) := \frac{\sigma^2}{2(\alpha+1)} \frac{p}{u^{\frac{1}{\alpha}}} \quad (2.28)$$

$$b(x, t, u, p) := \frac{2\alpha+1}{\alpha+1} \lambda u p + \frac{\sigma^2}{2(\alpha+1)} \frac{p^2}{u^{1+\frac{1}{\alpha}}} \quad (2.29)$$

$$\psi_0(x) := \left( \frac{A(R)\sigma^2}{2\lambda\alpha} \right)^{\frac{\alpha}{\alpha+1}} \quad (2.30)$$

And therefore the following PDE has a smooth solution in  $\mathcal{C}^{2,4}$

$$f_t = -h_2(f)f_x + h_1(f)f_{xx} \quad \text{where} \quad f(0, x) = \psi_0(x)$$

Now we show that the function  $f$  is also a solution to our desired PDE, by relabeling from  $c(Y, R)$  to  $f(x, t)$ .

$$\tilde{c}_Y = -\frac{2\alpha + 1}{\alpha + 1} \lambda \tilde{c} \tilde{c}_R + \frac{\sigma^2}{2(\alpha + 1) \tilde{c}^{\frac{1}{\alpha}}} \tilde{c}_{RR} \quad (2.31)$$

This boils down to show that  $\tilde{c}_{\min} \leq f(x, t) \leq \tilde{c}_{\max}$ . First, assume there is a point  $(t_0, x_0)$  such that  $f(t_0, x_0) > \tilde{c}_{\max}$ . Then there are  $N > 0$  and  $\gamma > 0$  such that for the new function  $\tilde{f}_N := f_N(t, x)e^{-\gamma t}$  (where  $f_N$  is the function constructed in the proof of Theorem 3)

$$\tilde{f}_N := f_N(t_0, x_0)e^{-\gamma t_0} > \tilde{c}_{\max}$$

By construction, this function will attain its maximum at a point  $(t_1, x_1)$  which does not lay on the boundary at  $\{0\} \times [-N, N]$  and  $[0, t_0] \times \{N, -N\}$ . Therefore

$$\tilde{f}_{N,t}(t_1, x_1) \geq 0 \quad (2.32)$$

$$\tilde{f}_{N,x}(t_1, x_1) = 0 \quad (2.33)$$

$$\tilde{f}_{N,xx}(t_1, x_1) \leq 0 \quad (2.34)$$

On the other hand

$$\tilde{f}_{N,t} = e^{-\gamma t} f_{N,t} - \gamma e^{-\gamma t} f_N \quad (2.35)$$

$$= -e^{-\gamma t} h_2(f_N) f_{N,x} + e^{-\gamma t} h_1(f_N) f_{N,xx} - \gamma e^{-\gamma t} f_N \quad (2.36)$$

$$= -h_2(f_N) \tilde{f}_{N,x} + h_1(f_N) \tilde{f}_{N,xx} - \gamma \tilde{f}_N \quad (2.37)$$

Therefore, from equations (2.33) and (2.34) we have

$$\tilde{f}_N(t_1, x_1) \leq 0$$

But this is a contradiction, since

$$\tilde{f}_N(t_1, x_1) \geq \tilde{f}_N(t_0, x_0) \geq \tilde{c}_{\max} \geq 0 \quad (2.38)$$

By a similar argument, we could reach to a contradiction for the other case, where

$$f(t_0, x_0) < \tilde{c}_{\min}$$

□

**Proposition 4** *There exists a  $\mathcal{C}^{2,4}$  solution  $\tilde{\omega} : \mathbb{R}_0^+ \times \mathbb{R} \rightarrow \mathbb{R}$  to the transport equation*

$$\tilde{\omega}_Y = -\lambda \tilde{c} \tilde{\omega}_R \quad (2.39)$$

*with initial value*

$$\tilde{\omega}(0, R) = u(R) \quad (2.40)$$

*the solution is increasing in  $R$ , decreasing in  $Y$ , and also satisfies*

$$0 \geq \tilde{\omega}(Y, R) \geq u(R - \lambda \tilde{c}_{\max} Y) \quad (2.41)$$

**Proof:** The proof uses the method of characteristics. Consider the function

$$P : (Y, S) \in \mathbb{R}_0^+ \times \mathbb{R} \mapsto P(Y, S) \in \mathbb{R}$$

satisfying the ODE

$$P_Y(Y, S) = \lambda \tilde{c}(Y, P(Y, S)) \quad (2.42)$$

with initial value condition  $P(0, S) = S$ . Since  $\tilde{c}$  is smooth and bounded, for each fixed  $S$  a unique solution to the ODE (2.42) exists. From Cauchy-Lipschitz theorem on ODEs for

every  $Y$ ,  $P(Y, \cdot)$  is a  $\mathcal{C}^4$ -diffeomorphism mapping  $\mathbb{R}$  onto  $\mathbb{R}$ , getting the same regularity as  $\tilde{c}$ , i.e.,  $P$  belongs to  $\mathcal{C}^{2,4}$ . Define

$$\tilde{\omega}(Y, R) = u(S) \quad \text{iff} \quad P(Y, S) = R$$

Then  $\tilde{\omega}$  is a  $\mathcal{C}^{2,4}$  function satisfying the initial value condition, and by definition

$$0 = \frac{d}{dY} \tilde{\omega}(Y, P(Y, S)) \tag{2.43}$$

$$= \tilde{\omega}_R(Y, P(Y, S)) P_Y(Y, S) + \tilde{\omega}_Y(Y, P(Y, S)) \tag{2.44}$$

$$= \tilde{\omega}_R(Y, P(Y, S)) \lambda \tilde{c}(Y, P(Y, S)) + \tilde{\omega}_Y(Y, P(Y, S)) \tag{2.45}$$

$$\tag{2.46}$$

Therefore  $\tilde{\omega}$  satisfies the desired partial differential equation. Since  $\tilde{c} \leq \tilde{c}_{\max}$ , by construction  $P(Y, S) \leq S + \lambda \tilde{c}_{\max} Y$ , and hence  $\tilde{\omega}(Y, R) \geq u(R - \lambda \tilde{c}_{\max} Y)$ .

Using positivity of  $\tilde{c}$ , the family of solutions of the ODE above do not cross. This yields the monotonicity statement in the proposition.  $\square$

**Lemma 1** *The following equality holds*

$$\frac{\tilde{\omega}_{RR} \sigma^2}{2\tilde{\omega}_R \lambda} + \alpha \tilde{c}^{\frac{\alpha+1}{\alpha}} = 0 \tag{2.47}$$

**Proof:** Consider the following linear PDE with initial condition  $f(0, R) = 0$ .

$$f_Y = -\lambda \tilde{c} f_R + \lambda \tilde{c}_R f \tag{2.48}$$

From the assumptions on  $u$ , the functions  $\tilde{c}$  and  $\tilde{c}_R$  are smooth and hence locally Lipschitz. Therefore, the PDE has a unique solution which turns out to be the constant zero function. On the other hand, we can show that the function  $f(Y, R) := \alpha \tilde{c}^{\frac{\alpha+1}{\alpha}} + \frac{\tilde{\omega}_{RR} \sigma^2}{2\tilde{\omega}_R \lambda}$  is also a solution to this PDE. After proving this, the uniqueness of the solution implies the result.

First, using the previous lemma and the fact that  $\tilde{\omega}$  is  $\mathcal{C}^{2,4}$ , we have

$$\frac{\partial}{\partial Y} \left( \frac{\tilde{\omega}_{RR}\sigma^2}{2\lambda\tilde{\omega}_R} \right) = \frac{\sigma^2}{2\lambda} \left( \frac{\tilde{\omega}_{RRY}\tilde{\omega}_R - \tilde{\omega}_{RR}\tilde{\omega}_{RY}}{\tilde{\omega}_R^2} \right) \quad (2.49)$$

$$= \frac{\sigma^2}{2\lambda} \left( \frac{\tilde{\omega}_{YRR}\tilde{\omega}_R - \tilde{\omega}_{RR}\tilde{\omega}_{YR}}{\tilde{\omega}_R^2} \right) \quad (2.50)$$

$$= \frac{\sigma^2}{2\lambda} \left( \frac{-(\lambda\tilde{c}\tilde{\omega}_R)_{RR}}{\tilde{\omega}_R} + \frac{\tilde{\omega}_{RR}(\lambda\tilde{c}\tilde{\omega}_R)_R}{\tilde{\omega}_R^2} \right) \quad (2.51)$$

$$= -\sigma^2 \frac{\tilde{c}\tilde{\omega}_{RRR} + 2\tilde{c}_R\tilde{\omega}_{RR} + \tilde{c}_{RR}\tilde{\omega}_R}{2\tilde{\omega}_R} + \sigma^2 \frac{\tilde{\omega}_{RR}(\tilde{c}_R\tilde{\omega}_R + \tilde{c}\tilde{\omega}_{RR})}{2\tilde{\omega}_R^2} \quad (2.52)$$

$$= -\sigma^2 \tilde{c}_R \frac{\tilde{\omega}_{RR}}{2\tilde{\omega}_R} - \frac{\sigma^2}{2} \tilde{c}_{RR} + \left( \frac{\tilde{\omega}_{RR}^2}{\tilde{\omega}_R^2} - \frac{\tilde{\omega}_{RRR}}{\tilde{\omega}_R} \right) \frac{\sigma^2 \tilde{c}}{2} \quad (2.53)$$

$$= -\sigma^2 \tilde{c}_R \frac{\tilde{\omega}_{RR}}{2\tilde{\omega}_R} - \frac{\sigma^2}{2} \tilde{c}_{RR} - \sigma^2 \tilde{c} \frac{\partial}{\partial R} \frac{\tilde{\omega}_{RR}}{2\tilde{\omega}_R} \quad (2.54)$$

On the other hand, from Proposition 4

$$\frac{\partial}{\partial Y} (\alpha \tilde{c}^{\frac{\alpha+1}{\alpha}}) = (\alpha+1) \tilde{c}^{\frac{1}{\alpha}} \tilde{c}_Y = (\alpha+1) \tilde{c}^{\frac{1}{\alpha}} \left( -\frac{2\alpha+1}{\alpha+1} \lambda \tilde{c}_R + \frac{\sigma^2}{2(\alpha+1)\tilde{c}^{\frac{1}{\alpha}}} \tilde{c}_{RR} \right) \quad (2.55)$$

Now we can easily check that

$$\frac{\partial}{\partial Y} \left( \alpha \tilde{c}^{\frac{\alpha+1}{\alpha}} + \frac{\tilde{\omega}_{RR}\sigma^2}{2\tilde{\omega}_R\lambda} \right) = -\lambda \tilde{c} \frac{\partial}{\partial R} \left( \alpha \tilde{c}^{\frac{\alpha+1}{\alpha}} + \frac{\tilde{\omega}_{RR}\sigma^2}{2\tilde{\omega}_R\lambda} \right) - \lambda \tilde{c}_R \left( \alpha \tilde{c}^{\frac{\alpha+1}{\alpha}} + \frac{\tilde{\omega}_{RR}\sigma^2}{2\tilde{\omega}_R\lambda} \right) \quad (2.56)$$

Hence, the function  $f(Y, R) := \alpha \tilde{c}^{\frac{\alpha+1}{\alpha}} + \frac{\tilde{\omega}_{RR}\sigma^2}{2\tilde{\omega}_R\lambda}$  satisfies the PDE (2.48)  $\square$

**Proposition 5** *With  $Q(X)$  defined in equation (2.12), the value function  $\omega(X, R) := \tilde{\omega}(Q(X), R)$  solves the HJB equation*

$$\min_c \left[ -\frac{1}{2} \sigma^2 X^2 \nu_{RR} + \lambda \nu_R |c|^{1+\alpha} q(X) + \nu_X c \right] = 0 \quad (2.57)$$

**Proof:** By Proposition 4  $\tilde{\omega}_R > 0$ . Hence the function

$$\Theta(c) = -\frac{1}{2}\sigma^2 X^2 \omega_{RR} + \lambda \omega_R |c|^{1+\alpha} q(X) + \omega_X c \quad (2.58)$$

is convex and attains its minimum at

$$c^* = \left( \frac{-\omega_X}{(1+\alpha)\lambda q(X)\omega_R} \right)^{\frac{1}{\alpha}} \quad (2.59)$$

Therefore, we need to show that  $\Theta(c^*) = 0$ . (We drop the absolute value because from Proposition 4,  $\omega_X < 0$ ), we have

$$\Theta(c^*) = -\frac{1}{2}\sigma^2 X^2 \omega_{RR} + \lambda q(X) \omega_R |c^*|^{1+\alpha} + \omega_X c^* \quad (2.60)$$

$$= -\frac{1}{2}\sigma^2 X^2 \omega_{RR} + \left( \frac{-\omega_X}{(1+\alpha)\lambda q(X)\omega_R} \lambda \omega_R q(X) + \omega_X \right) c^* \quad (2.61)$$

$$= -\frac{1}{2}\sigma^2 X^2 \omega_{RR} + \frac{\alpha}{1+\alpha} \omega_X \left( \frac{-\omega_X}{(1+\alpha)\lambda q(X)\omega_R} \right)^{\frac{1}{\alpha}} \quad (2.62)$$

$$= -\frac{1}{2}\sigma^2 X^2 \omega_{RR} - \alpha \lambda q(X)^{-\frac{1}{\alpha}} \omega_R \left( \frac{-\omega_X}{(1+\alpha)\lambda \omega_R} \right)^{1+\frac{1}{\alpha}} \quad (2.63)$$

$$= -\lambda X^2 \omega_R \left( \frac{\omega_{RR} \sigma^2}{2\omega_R \lambda} + \frac{\alpha}{X^2 q(X)^{\frac{1}{\alpha}}} \left( \frac{-\omega_X}{(1+\alpha)\lambda \omega_R} \right)^{1+\frac{1}{\alpha}} \right) \quad (2.64)$$

From the definition of  $\omega(X, R)$

$$\omega_X = (\alpha + 1) X^{\frac{2\alpha}{\alpha+1}} q(X)^{\frac{1}{\alpha+1}} \tilde{\omega}_X$$

$$\omega_R = \tilde{\omega}_R$$

Therefore (2.64) can be written as

$$-X^2 \lambda \tilde{\omega}_R \left( \frac{\tilde{\omega}_{RR} \sigma^2}{2\tilde{\omega}_R \lambda} + \alpha \left( \frac{-\tilde{\omega}_X}{\lambda \tilde{\omega}_R} \right)^{\frac{\alpha+1}{\alpha}} \right)$$

Which by Lemma 4 is equal to zero.

□

**Remark 2** Using Proposition 4, we can rewrite the optimal policy (2.59) as

$$c(X, R) = \left[ \frac{X^2}{q(X)} \right]^{\frac{1}{\alpha+1}} \tilde{c}(Q(X), R)^{\frac{1}{\alpha}} \quad (2.65)$$

The boundedness result on  $\tilde{c}$  in Proposition 3 and even weaker assumptions than being bounded away from zero and infinity for  $q(X)$ , could insure that this policy belongs to  $\mathcal{X}_1$ . (One just needs to insure that all the admissibility requirements in Definition 4 are satisfied.)

Before starting the verification arguments, we complete our package of boundedness tools with the following lemma:

**Lemma 2** There are positive constants  $a_0, a_1, a_2, a_3$  and  $a_4$  such that

$$u(R) \geq \omega(X, R) \geq u(R) \exp(a_0 Q(X)) \quad (2.66)$$

$$0 \leq \omega_R(X, R) \leq a_1 + a_2 \exp(-a_3 R + a_4 Q(X)) \quad (2.67)$$

for all  $(X, R) \in \mathbb{R}_0^+ \times \mathbb{R}$ .

**Proof:** From monotonicity of  $\tilde{\omega}$  with respect to  $X$  and  $R$ , and its boundary condition via Proposition 4, the left-hand side of the both inequalities follow.

For the right-hand side of (2.66), since  $A(R)$  is bounded from above by  $A_{\max}$

$$u(R - \Delta) \geq u(R) \exp(A_{\max} \Delta) \quad (2.68)$$

Now by (2.41), we can establish the other part of the first inequality with  $a_0 = \lambda c_{\max} A_{\max}$  as follows

$$\omega(X, R) = \tilde{\omega}(Q(X), R) \geq u(R - \lambda \tilde{c}_{\max} Q(X)) \geq u(R) \exp(\lambda \tilde{c}_{\max} A_{\max} Q(X))$$

For the second inequality, first note that  $\omega_R = \tilde{\omega}_R$ . From Lemma 1, we have

$$-\frac{\tilde{\omega}_{RR}}{\tilde{\omega}_R} < \frac{2\alpha\lambda\tilde{c}_{\max}^{\frac{\alpha+1}{\alpha}}}{\sigma^2} = A_{\max} \quad (2.69)$$

Hence

$$\int_R^{R_1} -\frac{\tilde{\omega}_{RR}(Y, R_2)}{\tilde{\omega}_R(Y, R_2)} dR_2 \leq \int_R^{R_1} A_{\max} dR_2 \quad (2.70)$$

$$-\ln \tilde{\omega}_R(Y, R_2) \Big|_R^{R_1} \leq A_{\max}(R_1 - R) \quad (2.71)$$

$$\tilde{\omega}_R(Y, R_1) \geq \tilde{\omega}_R(Y, R)e^{-A_{\max}(R_1 - R)} \quad (2.72)$$

$$\int_R^{R_3} \tilde{\omega}_R(Y, R_1) dR_1 \geq \int_R^{R_3} \tilde{\omega}_R(Y, R)e^{-A_{\max}(R_1 - R)} dR_1 \quad (2.73)$$

$$\tilde{\omega}(Y, R_3) - \tilde{\omega}(Y, R) \geq \frac{\tilde{\omega}_R(Y, R)}{A_{\max}}(1 - e^{-A_{\max}(R_3 - R)}) \quad (2.74)$$

Then by (2.41)  $\lim_{R_3 \rightarrow \infty} \tilde{\omega}(Y, R_3) = 0$  and we have

$$0 \geq \tilde{\omega}(Y, R) + \frac{\tilde{\omega}_R(Y, R)}{A_{\max}}$$

Thus

$$\tilde{\omega}_R(Y, R) \leq -\tilde{\omega}(Y, R)A_{\max} \leq -u(R - \lambda\tilde{c}_{\max}Y)A_{\max}$$

And since  $u(R)$  is bounded by an exponential function, we can obtain the desired bound for  $\tilde{\omega}_R$ .  $\square$

### 2.3.2 Verification

We now connect the PDE results in previous section with the optimal stochastic control problem in the section 2.2.2. In order to make the connection between these two, we need to define a new value function. We assume that the investor trades the risky asset in order

to maximize the asymptotic expected utility of the portfolio value, i.e.

$$\nu_2(X_0, R_0) := \sup_{\xi \in \mathcal{X}} \lim_{t \rightarrow \infty} \mathbb{E}[u(R_t^\xi)] \quad (2.75)$$

Note that our assumptions on the admissible strategies in maximization of asymptotic portfolio value are even weaker than those for optimal liquidation (the supremum in (2.75) is taken over a larger set). In particular, we don't require  $R_t^\xi$  or  $X_t^\xi$  to converge at infinity. Moreover, through out this section we frequently use the second property in Definition 4 for admissibility. This equipped with usual assumptions on  $q(x)$  are crucial in simplifying the upcoming proofs. For any admissible strategy  $\xi \in \mathcal{X}$ , define the sequence of stopping times

$$\tau_k^\xi := \inf \left\{ t \geq 0 \mid \int_0^t q(X_s^\xi) |\xi_s|^{1+\alpha} ds \geq k \right\} \quad \text{where } k \in \mathbb{N} \quad (2.76)$$

Admissibility of  $\xi$  guarantees that  $P(\lim \tau_k = \infty) = 1$ . We proceed by first showing that  $u(R_t^\xi)$  and  $\omega(X_t^\xi, R_t^\xi)$  fulfill local supermartingale inequalities. Thereafter, we show that  $\omega(X_0, R_0) \geq \lim_{t \rightarrow \infty} \mathbb{E}[u(R_t^\xi)]$  with equality for  $\xi = \hat{\xi}$ . The next lemma in particular justifies our definition of  $\nu_2$  in (2.75)

**Lemma 3** *For any admissible strategy  $\xi$  the expected utility  $\mathbb{E}[u(R_t^\xi)]$  is decreasing in  $t$ , moreover  $\mathbb{E}[u(R_{t \wedge \tau_k^\xi}^\xi)] \geq \mathbb{E}[u(R_t^\xi)]$ .*

**Proof:** Since  $R_t^\xi - R_0$  is the difference of the true martingale  $\int_0^t \sigma X_s^\xi dB_s$  and the increasing process  $\lambda \int_0^t |\xi_s|^{1+\alpha} q(X_s^\xi) ds$ , it satisfies the inequality  $\mathbb{E}[R_t^\xi | \mathcal{F}_s] \leq R_s^\xi$  for  $s \leq t$  (even though because of lack of integrability, it may fail to be a supermartingale). Hence, from Jensen's inequality for supermartingales  $\mathbb{E}[u(R_t^\xi)]$  is decreasing.

For the second assertion, let  $n \geq k$  and  $\tau_n := \tau_n^\xi$ . By adding the martingale  $\int_{t \wedge \tau_k}^{t \wedge \tau_n} X_s^\xi dB_s$

to  $R_{t \wedge \tau_k^\xi}^\xi$  we can deduce once again through Jensen's inequality

$$\mathbb{E}[u(R_{t \wedge \tau_k^\xi}^\xi)] \geq \mathbb{E}\left[u\left(R_0 + \sigma \int_0^{t \wedge \tau_n} X_s^\xi dB_s - \lambda \int_0^{t \wedge \tau_k} |\xi_s|^{1+\alpha} q(X_s^\xi) ds\right)\right] \quad (2.77)$$

By Levy's characterization theorem for Brownian motions, the stochastic integral in the right-hand side could be regarded as a time changed standard Brownian motion

$$W_{\int_0^t X_s^2 ds} := B_t \quad (2.78)$$

With  $R_0$  constant and the other terms being bounded due to the stopping time  $\tau_k$ , and  $u(R)$  being bounded from below by an exponential function, after sending  $n$  to infinity by dominated convergence theorem the right-hand side decreases to

$$\mathbb{E}\left[u\left(R_0 + \sigma \int_0^t X_s^\xi dB_s - \lambda \int_0^{t \wedge \tau_k} |\xi_s|^{1+\alpha} q(X_s^\xi) ds\right)\right] \quad (2.79)$$

Which is clearly equal or larger than  $\mathbb{E}[u(R_t^\xi)]$ .  $\square$

**Lemma 4** *For any admissible strategy  $\xi$ ,  $\omega(X^\xi, R^\xi)$  is a local supermartingale with localizing sequence  $(\tau_k^\xi)$ .*

**Proof:** For  $T > t \geq 0$ , Ito's formula yields

$$\omega(X_T^\xi, R_T^\xi) - \omega(X_t^\xi, R_t^\xi) = \quad (2.80)$$

$$\int_t^T \omega_R(X_s^\xi, R_s^\xi) \sigma X_s^\xi dB_s - \int_t^T \left[ \lambda q(X_s^\xi) |\xi_s|^{1+\alpha} \omega_R + \xi_s \omega_X - \frac{1}{2} (\sigma X_s^\xi)^2 \omega_{RR} \right] (X_s^\xi, R_s^\xi) ds \quad (2.81)$$

By Proposition 5 the latter integral is nonnegative and we obtain

$$\omega(X_t^\xi, R_t^\xi) \geq \omega(X_T^\xi, R_T^\xi) - \int_t^T \omega_R(X_s^\xi, R_s^\xi) \sigma X_s^\xi dB_s \quad (2.82)$$

Using an argument similar to the one in Lemma 3, there exist a constant  $C_1$  such that for  $s \leq t \wedge \tau_k$

$$R_s^\xi \geq -C_1(1 + \sup_{q \leq t} |W_q|)$$

Hence, for the localizing sequence  $(\tau_k) := (\tau_k^\xi)$  which we defined in (2.76), from inequality (2.67) in Lemma 2 for any  $s \leq t \wedge \tau_k$  we have

$$0 \leq \omega_R(X_s^\xi, R_s^\xi) \leq a_1 + a_2 \exp\left(a_3 C_1 \left(1 + \sup_{q \leq t} |B_q|\right) + a_4 Q(X_s^\xi)\right) \quad (2.83)$$

Using Doob's martingale inequality, all the moments of  $\sup_{q \leq t} |B_q|$  are integrable. Moreover, admissibility of  $\xi$  guarantees the existence of a bound for other terms in (2.83). Therefore, the integrand term in (2.82) is  $L^2$  integrable and as a result the stochastic integral in (2.82) is a local martingale. This proves that  $\omega(X^\xi, R^\xi)$  is a local supermartingale.  $\square$

**Lemma 5**  $\hat{\xi}$  defined via equation (2.65) as

$$\hat{\xi}_t = c(X_t^{\hat{\xi}}, R_t^{\hat{\xi}})$$

is admissible for optimal liquidation and maximization of asymptotic portfolio value problems, i.e.  $\xi \in \mathcal{X}_1$ . Furthermore,  $\omega(X^{\hat{\xi}}, R^{\hat{\xi}})$  is a martingale and

$$\omega(X_0, R_0) = \lim_{t \rightarrow \infty} \mathbb{E}[u(R_t^{\hat{\xi}})] \leq \nu_2(X_0, R_0) \quad (2.84)$$

**Proof:** Admissibility of  $\hat{\xi}$  follows from Remark 2. Therefore,  $\int_0^\infty |\hat{\xi}_t|^{1+\alpha} dt < K$  for some constant  $K$ . The choice of  $\xi = \hat{\xi}$  vanishes the second integral in (2.80), with  $\tau_K^{\hat{\xi}} = \infty$ , this yields the martingale property of  $\omega(X_t^{\hat{\xi}}, R_t^{\hat{\xi}})$ . Moreover, by taking the infinite limit of time in equation (2.66)

$$\lim_{t \rightarrow \infty} u(R_t^{\hat{\xi}}) \geq \lim_{t \rightarrow \infty} \omega(X_t^{\hat{\xi}}, R_t^{\hat{\xi}}) \geq \lim_{t \rightarrow \infty} u(R_t^{\hat{\xi}}) \exp(a_0 Q(X_t^{\hat{\xi}})) \quad (2.85)$$

From Remark 2  $\lim_{t \rightarrow \infty} X_t^{\hat{\xi}} = 0$  and hence, the left and the right-hand side of (2.85) are equal. This proves the left-hand side of (2.84). The other part follows from the definition of  $\nu_2$  in (2.75). □

**Proposition 6** *In the case of asymptotic maximization of the portfolio value, we have  $\nu_2 = \omega$  and  $\hat{\xi}$  is the a.s. unique optimal strategy.*

**Proof:** From Lemma 5, we already have  $\omega \leq \nu_2$ . For the other side, let  $\xi$  be any admissible strategy such that

$$\lim_{t \rightarrow \infty} \mathbb{E}[u(R_t^\xi)] > -\infty$$

For all  $k, t$  and  $\xi$ , with  $(\tau_k) := (\tau_k^\xi)$ , Lemma 4 and inequality (2.66) yield

$$\omega(X_0, R_0) \geq \mathbb{E}[\omega(X_{t \wedge \tau_k}^\xi, R_{t \wedge \tau_k}^\xi)] \geq \mathbb{E}[u(R_{t \wedge \tau_k}^\xi) \exp(a_0 Q(X_{t \wedge \tau_k}^\xi))]$$

As in the proof Lemma 3, one could show

$$\liminf_{k \rightarrow \infty} \mathbb{E}[u(R_{t \wedge \tau_k}^\xi) \exp(a_0 Q(X_{t \wedge \tau_k}^\xi))] \geq \liminf_{k \rightarrow \infty} \mathbb{E}[u(R_t^\xi) \exp(a_0 Q(X_{t \wedge \tau_k}^\xi))] = \mathbb{E}[u(R_t^\xi) \exp(a_0 Q(X_t^\xi))]$$

And hence,

$$\omega(X_0, R_0) \geq \mathbb{E}[u(R_t^\xi)] + \mathbb{E}[u(R_t^\xi)(\exp(a_0 Q(X_t^\xi)) - 1)]$$

We claim that the second argument on the right attains values arbitrarily close to zero.

Accepting this claim for a moment, we get

$$\omega(X_0, R_0) \geq \lim_{t \rightarrow \infty} \mathbb{E}[u(R_t^\xi)]$$

Then, taking the supremum over all the admissible strategies  $\xi$  gives  $\omega \geq \nu_2$ . The optimality of  $\hat{\xi}$  follows from Lemma 5 and uniqueness from the fact that  $c$  is the unique solution to the HJB equation (2.57).

Before addressing the claim, we prove some boundedness results. From Assumption 1, we have

$$A_{\min}u_R \leq -u_{RR} \leq A_{\max}u_R$$

By integration and using the end point behavior of  $u(R)$ , we conclude

$$A_{\min}u \leq -u_R \leq A_{\max}u \quad (2.86)$$

This gives the required bound for  $u_R$ , which we need later to conclude the  $L^2$  property of the stochastic integral (2.89). Moreover, by substituting  $\frac{u_{RR}}{A(R)}$  for the middle term in the previous inequality, we can conclude that for  $a_5 := A_{\min}A_{\max}$

$$0 \geq u(R) \geq a_5 u_{RR}(R) \quad (2.87)$$

Now we can prove the claim. From Lemma 3, for all  $k, t$  and  $(\tau) := (\tau_k^\xi)$  we have

$$-\infty < \lim_{s \rightarrow \infty} \mathbb{E}[u(R_s^\xi)] \leq \mathbb{E}[u(R_t^\xi)] \leq \mathbb{E}[u(R_{t \wedge \tau_k}^\xi)] \quad (2.88)$$

$$= u(R_0) + \mathbb{E} \left[ \int_0^{t \wedge \tau_k} u_R(R_s^\xi) \sigma X_s^\xi dB_s \right] \quad (2.89)$$

$$- \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \lambda |\xi_s|^{1+\alpha} q(X_s^\xi) u_R(R_s^\xi) - \frac{1}{2} (\sigma X_s^\xi)^2 u_{RR}(R_s^\xi) ds \right] \quad (2.90)$$

$$= u(R_0) - \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \lambda |\xi_s|^{1+\alpha} q(X_s^\xi) u_R(R_s^\xi) - \frac{1}{2} (\sigma X_s^\xi)^2 u_{RR}(R_s^\xi) ds \right] \quad (2.91)$$

Sending  $k$  and  $t$  to infinity yields

$$\int_0^\infty \mathbb{E}[(X_s^\xi)^2 u_{RR}(R_s^\xi)] ds > -\infty \quad (2.92)$$

On the other hand, using admissibility of  $\xi$ , there is a constant  $a_6$  such that

$$\exp(a_0 Q(X_t^\xi)) - 1 \leq a_6 a_0 (X_t^\xi)^2$$

Hence, by using (2.87)

$$0 \geq \mathbb{E}[u(R_t^\xi)(\exp(a_0 Q(X_t^\xi)) - 1)] \geq \mathbb{E}[a_0 a_5 a_6 u_{RR}(R_t^\xi)(X_t^\xi)^2]$$

Therefore by (2.92) the right-hand side of the above equation attains values arbitrarily close to zero.  $\square$

**Proposition 7** *In the case of the optimal liquidation problem, we have  $\nu_1 = \omega$  and the a.s. unique optimal strategy is given by  $\hat{\xi}$  respectively c.*

**Proof:** For any strategy  $\xi$  that is admissible for optimal liquidation, the martingale  $\sigma \int X_s dB_s$  is uniformly integrable due to the third requirement in Definition 4. Therefore, similar as in the proof of Lemma 3, it follows that  $\mathbb{E}[u(R_t^\xi)] \geq \mathbb{E}[u(R_\infty^\xi)]$ . Hence Proposition 6 yields

$$\mathbb{E}[u(R_\infty^\xi)] = \lim_{t \rightarrow \infty} \mathbb{E}[u(R_t^\xi)] \leq \nu_2(X_0, R_0) \leq \omega(X_0, R_0)$$

Taking the supremum over all admissible strategies  $\xi$ , we have  $\nu_1 \leq \omega$ . The converse inequality follows from the left-hand side of (2.66) and Lemma 5.  $\square$

## Chapter 3

# Dynamic impact

### 3.1 Introduction

In this chapter we derive optimal liquidation policies in a model with dynamic price impact. In this model we allow temporary price impact to evolve according to the past trades, with increase in intense trading and decaying overtime in its absent. This change accounts for the fact that the market's depth varies subject to the intensity of submitted orders, where high intensity trading tends to reduce the market depth while in the absent of trading, it recovers its equilibrium position throughout time. This model generalizes the Almgren and Chriss (1999) model in a different dimension by incorporating the idea of resilience in the market in describing the dynamic of temporary price impact. We find the optimal execution policy and show that unlike some choices in the old models involving resilience, our model doesn't allow for Transaction-triggered price manipulation strategies, i.e policies involving intermediate buying from the same stock in a liquidation problem could not be optimal. We study the problem only for the case of a buyer investor. The results could be applied to a seller investor as well. The rest of this paper is organized as follows. In the next section, after introducing our price impact model, we discuss the admissibility criterion and present the main result for the investor under finite time horizon constrained case. We prove the related results using Calculus of Variations techniques to find necessary conditions for the minimizing path and identify the unique solution to this minimization problem. In the last section, we study the non-time constrained case as a limiting solution to the finite horizon case and investigate its connection to the result of Schied and Schneborn (2009).

## 3.2 Model

### 3.2.1 The Market

We consider the liquidation problem, for a trader who wants to sell a specific number of shares denoted by  $x_0$  in a given time  $T$ . The liquidation strategy of the investor can be described by  $x(t)$ , representing the number of shares hold by him at time  $t \in [0, T]$ .

Due to our form of penalization for intense trading, we can rule out noncontinuous liquidation strategy. Hence, from the start we can only focus on continuous paths. Moreover, we go one step further and assume that the trading policy  $x(t)$  is absolutely continuous. This not only will help us in ruling out unrealistic case like Cantor function for trading policy function, but also through existence of the derivative function almost surely, as an alternative we can represent the investor's trading strategy by  $c(t) = -x'(t)$  and refer to his trading intensity.

$$x(t) = x_0 - \int_0^t c(s)ds \quad (3.1)$$

The investor's objective is to liquidate all the shares, i.e.  $x(T) = 0$  i.e.  $\int_0^T c(t)dt = x_0$ . Due to insufficient liquidity in the market, the investor's trading moves the market price against him. We use the Almgren-Chriss framework to demonstrate this trading impact. This model consists of two component. A permanent impact which affects the price through the number of sold shares and independently of the investor's trading intensity. We follow the price manipulation theorem of Huberman and Stanzl (2004) which requires the permanent price impact to be a linear function. We use the constant  $\gamma$  to demonstrate it. Furthermore, there exist a temporary price impact which affects the infinitesimal orders relative to the intensity of trading and vanishes instantaneously. Unlike the older model of Schied and Schneborn (2009), where temporary price impact  $\lambda$  is a constant, we consider the following

dynamic for the temporary price impact

$$\lambda' = -\alpha\lambda + \beta c \quad (3.2)$$

Where the positive coefficients  $\alpha$  and  $\beta$  stand for the resilience in the market and the linear drift which capture the effect of the past trades on the temporary price impact respectively.

In addition to the large investor's impact, the price process  $P$  is driven by a Brownian motion with volatility  $\sigma$ . Hence, the execution price is given by

$$P(t) = P_0 + \sigma B(t) + \gamma(x(t) - x(0)) - \lambda(t)x'(t) \quad (3.3)$$

Where  $\lambda(t)$  with initial value  $\lambda_0$ , is given by

$$\lambda(t) = e^{-\alpha t} \left( \lambda_0 + \int_0^t c(s)e^{\alpha s} ds \right) \quad (3.4)$$

**Remark 3** *With Bachelier model for price and existence of permanent and temporary price impact, the execution price could take negative value. In order to simplify the notation, we don't restrict our assumption as much as we can, however we always assume that the price is positive.*

### 3.2.2 Optimal Policy

Consider a Von-Neumann-Morgenstern investor with constant absolute risk aversion. In other words, for some constants  $A, B > 0$  his utility function is given by  $u(r) = -Be^{-Ar}$ . The investor's objective is to maximize the expected utility of his terminal revenue over the set of all admissible trading strategies.

**Definition 5** *We parametrize the trading strategies with progressively measurable processes  $\xi(t) = -x'(t)$ . We call a process admissible if  $c \in L^2[0, T]$ . Furthermore, due to the relationship between  $c$  and  $\lambda$  from equation (3.4) we require that for the resulting process*

$$\lambda_\xi(t) \geq 0$$

The assumption on  $\lambda$  is crucial for the existence of the solution. By allowing negative values for  $\lambda$  one could simply gain infinite wealth by extreme back and forth trading in a very short time while keeping  $\lambda < 0$ . After excluding this possibility, the other assumption excludes all the strategies with infinite loss. The process  $c_t$  determines the other two processes  $x_\xi(t)$  and  $\lambda_\xi(t)$  which are adapted to the same filtration, uniquely throughout equations (3.1) and (3.4).

The resulting cash position for admissible policy  $\xi$  is given by

$$\begin{aligned} \mathcal{R}(t) &= \int_0^t \xi_s P_s ds = P_0 x_0 - \frac{\gamma}{2} x_0^2 + \sigma \int_0^t x_\xi(s) dB(s) - \int_0^t \lambda_\xi(s) \xi^2(s) ds \\ &\quad - \left( P_0 x_\xi(t) + \frac{\gamma}{2} [x_\xi^2(t) - 2x_0 x_\xi(t)] + \sigma x_\xi(t) B(t) \right) \end{aligned} \quad (3.5)$$

Since price is positive, in order to maximize this revenue, the investor is required to liquidate his entire position by time  $T$ , i.e.  $x_\xi(t) = 0$ . Hence, the resulting cash flow of such trading strategy  $\xi$  is given by

$$R_T(\xi) = P_0 x_0 - \frac{\gamma}{2} x_0^2 + \int_0^T \sigma x_\xi(s) dB(s) - \int_0^T \lambda_\xi(s) c_\xi^2(s) ds \quad (3.6)$$

By admissibility assumption  $c \in L^2$ , the stochastic integral in (3.6) is martingale. Hence, the terminal wealth has a Gaussian distribution and the expected value of the investor's utility function is

$$E[-B e^{-A \mathcal{R}_\xi(T)}] = -B \exp\left(-A(E[\mathcal{R}_\xi(T)] + \frac{1}{2} A \text{Var}[\mathcal{R}_\xi(T)])\right)$$

This means that the maximization problem is equivalent to minimization problem of the trading cost. Consisting of two components, the price impact cost and volatility cost

$$\int_0^T \lambda_\xi(s) c_\xi^2(s) ds + \int_0^T \frac{1}{2} A \sigma^2 x_\xi^2(s) dB(s) \quad (3.7)$$

The challenge for the investor is to find a balance in the tradeoff between these two component.

### 3.2.3 Statement of main results

We present all the result for finite horizon case in the following statement

**Theorem 4** *For an investor with a constant risk aversion  $A$  who is required to liquidate  $x_0$  number of shares, in a finite horizon time interval  $[0, T]$*

- i) There exist an optimal liquidation policy which maximize the expected value of his terminal utility.*
- ii) The optimal policy is given by*

$$x(t) = -\frac{\alpha u(t) + u'(t)}{\alpha \beta}$$

where  $u(t)$  is a solution to the ODE

$$-A\alpha^3\sigma^2u + A\alpha\sigma^2u'' - 6\alpha^2u'u'' + 4u''u^{(3)} + 2u'u^{(4)} = 0 \quad (3.8)$$

with initial conditions

$$u(0) = \lambda_0 + \beta x_0 \quad (3.9)$$

$$u'(0) = -\alpha \lambda_0 \quad (3.10)$$

$$u'(T) = -\alpha u(T) \quad (3.11)$$

$$\alpha^2 u'(T)^2 - u''(T)^2 - 2\alpha u'(T)u''(T) - 2u'(T)u'''(T) = 0 \quad (3.12)$$

iii) *The optimal policy is unique.*

iv) *Although the model allows for intermediate buying, one cannot improve the trading performance through such policies.*

**Theorem 5** *In infinite time horizon environment, the optimal policy is given via*

$$c(x, \lambda) = xH(x, \lambda)$$

where  $H(x, \lambda)$  is the unique solution of the quasilinear partial differential equation

$$xHH_x + (\alpha\lambda + \beta xH)H_\lambda = \frac{A\sigma^2\beta x}{4\lambda^2} + \frac{A\sigma^2}{2\lambda} - \alpha H - H^2 \quad (3.13)$$

with boundary condition

$$\lim_{x \rightarrow 0} H(x, \lambda) = \sqrt{\frac{A\sigma^2}{2\lambda} \frac{K_0(\sqrt{\frac{2A\sigma^2}{\alpha^2\lambda}})}{K_1(\sqrt{\frac{2A\sigma^2}{\alpha^2\lambda}})}}$$

where  $K_0$  and  $K_1$  are modified Bessel function.

### 3.3 Proof of Results

#### 3.3.1 Finite horizon case

**Remark 4** *Since all the cost functions are positive, after finishing the liquidation task before the terminal time, the investor could only increase his trading cost by back and forth trading. Hence, we could restrict the set of admissible policies to those which doesn't end with buying, i.e.  $c(T) \geq 0$ .*

**Proof:** [proof of Theorem 4 part (i)] We use the direct method of calculus of variations to show the existence of the solution. Consider a minimizing sequence  $\{\xi_n\}$  of admissible liquidation policies for execution cost. Since trading cost is a non-negative function, such

a sequence exists. By  $L^2$  property of admissibility there exists a subsequence  $\{\xi_{n_i}\}$  which converge weakly to an element  $\tilde{c} \in L^2$ . From weak convergence for any  $t \in [0, T]$

$$\tilde{x}(s) := \int_0^t \tilde{c}(s) ds = \lim_{n_i \rightarrow \infty} \int_0^t c_{n_i}(s) ds = x_{n_i}(s)$$

Hence, we have the pointwise convergence of  $x_{n_i}$  almost surely. In particular  $\tilde{x}(T) = x_0$ , hence  $\tilde{c}$  is also an admissible policy which liquidates all the required number of shares. The final task is to show that the liquidation cost from the limiting policy  $c$  is no more than the limit of costs. First, for the volatility cost, using Fatou's lemma for the a.s. pointwise convergence sequence of  $\{x_i\}$ , we have

$$\int_0^T A\sigma^2 x^2(t) dt \leq \lim_{n_i \rightarrow \infty} \int_0^T A\sigma^2 x_{n_i}^2(t) dt$$

For price impact cost, by weak convergence for any set  $A \in [0, T]$  we have

$$\int_0^T \mathbb{1}_A(t) c^2(t) dt \leq \liminf_{n_i \rightarrow \infty} \int_0^T \mathbb{1}_A(t) c_{n_i}^2(t) dt \quad (3.14)$$

Therefore to complete the proof we need to show that the sequence  $\{\lambda_{n_i}\}$  converges uniformly. Equivalently, we show the uniform convergence of the sequence  $\{\lambda_{n_i}(t)e^{\alpha t}\}$ . From Holder inequality

$$\begin{aligned} \lambda_i(t_1)e^{\alpha t_1} - \lambda_i(t_2)e^{\alpha t_2} &= \\ \int_0^T c_{n_i}(s)e^{\alpha s} \mathbb{1}_{(t_1, t_2)}(s) ds &\leq \left( \int_0^T c_{n_i}^2 ds \right)^{\frac{1}{2}} \left( \int_0^T (\mathbb{1}_{(t_1, t_2)}(s)e^{\alpha s})^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

With the first term being bounded due to admissibility. We conclude that  $\{\lambda_{n_i}\}$  are equicontinuous and therefore by Arzela-Ascoli theorem there exist a subsequence subsequence  $\{\xi_{m_i}\}$  of  $\{\xi_{n_i}\}$  such that  $\{\lambda_{m_j}\}$  are uniformly convergent. For each  $i \in \{m_j\}$  the

following equation holds.

$$\int_0^T \lambda_i c_i^2 ds = \int_0^T (\lambda_i - \lambda) c_i^2 + \int_0^T \lambda (c_i^2 - c^2) + \int_0^T \lambda c^2$$

After taking the limit, from uniform convergence of  $\lambda_{m_j}$  to  $\lambda$ , and  $c_i$  being bounded in  $L^2$  the first integral on the right approaches to zero. So it's sufficient to show that

$$\liminf_i \int_0^T \lambda (c_i^2 - c^2) \geq 0$$

Which from positivity of  $\lambda$  and inequality (3.14) follows.  $\square$

**Proof:** [proof of Theorem 4 part (ii)] We face a minimization problem for the functional (3.7) in the space of continuous functions with the fixed endpoints  $x(0) = x_0$  and  $x(T) = 0$ . Since  $\lambda(t)$  is a function of  $x'$  over  $[0, t]$ , we define the following control variable to use the method of calculus of variations

$$u(t) := \lambda(t) + \beta x(t)$$

From equation (3.2) we can write all the variables of the objective function in terms of  $u$

$$\lambda(t) = -\frac{u'(t)}{\alpha} \tag{3.15}$$

$$x(t) = -\frac{\alpha u(t) + u'(t)}{\alpha\beta} \tag{3.16}$$

$$c(t) = \frac{\alpha u'(t) + u''(t)}{\alpha\beta} \tag{3.17}$$

from initial condition at time zero for  $\lambda$  and  $x$ , any admissible path has to satisfy the following initial conditions

$$u(0) = \lambda_0 + \beta x_0 \tag{3.18}$$

$$u'(0) = -\alpha\lambda_0 \tag{3.19}$$

Moreover, since the liquidation is done by time  $T$ , the following terminal condition holds as well

$$u'(T) = -\alpha u(T) \quad (3.20)$$

We can now write the cost function under this new variable

$$-\int_0^T u'(t)(\alpha u'(t) + u''(t))^2 dt + \frac{1}{2}A\alpha\sigma^2 \int_0^T (\alpha u(t) + u'(t))^2 dt \quad (3.21)$$

Hence, the objective is to maximize the functional  $J[u] = \int_0^T L(u, u', u'')dt$ , where

$$L(u, u', u'') = u'(\alpha u' + u'')^2 - \frac{1}{2}A\alpha\sigma^2(\alpha u + u')^2 \quad (3.22)$$

From calculus of variation, we know that for the optimal policy, the first order variation of a candidate function  $u$  has to be zero, so for  $u$  and  $u + h$  we have

$$\Delta J = J[u + h] - J[u] = \int_0^T L(u + h, u' + h', u'' + h'') - L(u, u', u'')$$

As  $h$  goes to zero

$$\delta J = \int_0^T (L_u h + L_{u'} h' + L_{u''} h'') dt$$

Using the integration by parts for the last term we have

$$\delta J[h] = \int (L_u h + L_{u'} h' - \frac{d}{dt} L_{u''} h') dt + (L_{u''} h') \Big|_0^T$$

By using the integration by parts once again, this time for  $h'$  terms, we have

$$\delta J[h] = \int (L_u - \frac{d}{dt} L_{u'} + \frac{d^2}{dt^2} L_{u''}) h dt + (L_{u''} h') \Big|_0^T + (L_{u'} - \frac{d}{dt} L_{u''}) h \Big|_0^T$$

From calculus of variation, we know that for the optimal policy  $\delta J[h] = 0$ . Therefore

$$L_u - \frac{d}{dt}L_{u'} + \frac{d^2}{dt^2}L_{u''} = 0 \quad (3.23)$$

Also for any  $h$  in such a way that  $u + h$  be an admissible policy, we have

$$\left( L_{u''}h' \right) \Big|_0^T + \left( L_{u'} - \frac{d}{dt}L_{u''} \right) h \Big|_0^T = 0 \quad (3.24)$$

The first equation which is the Euler-Lagrange equation for  $L(u, u', u'')$  in (3.22) is the following

$$\begin{aligned} & 2 \left[ u^{(3)}(\alpha u' + u'') + u'(\alpha u^{(3)} + u^{(4)}) + 2u''(\alpha u'' + u^{(3)}) \right] - A\alpha^2\sigma^2(\alpha u + u') \\ & - \left[ 2\alpha u''(\alpha u' + u'') + 2\alpha u'(\alpha u'' + u^{(3)}) + 2(\alpha u' + u'')(\alpha u'' + u^{(3)}) - A\alpha\sigma^2(\alpha u' + u'') \right] = 0 \end{aligned}$$

Which after simplification is equal to

$$-A\alpha^3\sigma^2u + A\alpha\sigma^2u'' - 6\alpha^2u'u'' + 4u''u^{(3)} + 2u'u^{(4)} = 0 \quad (3.25)$$

On the other hand, for equation (3.24), since  $u+h$  satisfies the initial conditions (3.18), (3.19) and (3.20) as well

$$h(0) = h'(0) = 0$$

$$-\alpha h(T) = h'(T)$$

Hence, we can rewrite (3.24) as following

$$-\alpha L_{u''} + \left( L_{u'} - \frac{d}{dt}L_{u''} \right) \Big|_T = 0$$

Which gives the following condition at the terminal time  $T$

$$\alpha^2 u'^2 - u''^2 - 2\alpha u' u'' - 2u' u^{(3)} - A\alpha\sigma^2(\alpha u + u') \Big|_T = 0,$$

and via the terminal condition (3.20) is equivalent to

$$\alpha^2 u'(T)^2 - u''(T)^2 - 2\alpha u'(T)u''(T) - 2u'(T)u^{(3)}(T) = 0. \quad (3.26)$$

□

After proving the existence and the form of the solution in theorems, now we need to prove the uniqueness. First we prove the following proposition

**Lemma 6** *The solution to ODE (3.8) can be written in terms of  $x$  and  $\lambda$  in the following form*

$$2\lambda(t)\lambda''(t) + \lambda'^2(t) - 3\alpha^2\lambda^2(t) + \mathcal{K}(t) = 0 \quad (3.27)$$

where  $\mathcal{K}(t)$  given recursively for  $t \leq T$  by following

$$\mathcal{K}_\xi(t) = 2\alpha\lambda(T)(\lambda'(T) + \alpha\lambda(T)) + A\sigma^2\beta\left(x(t) + \alpha\int_t^T x(s)ds\right)$$

is a positive function.

**Proof:** Let's write (3.25) in terms of  $\lambda$  and  $x$

$$-6\alpha^2\lambda\lambda' + 4\lambda'\lambda'' + 2\lambda\lambda^{(3)} = \frac{A\sigma^2}{\alpha}(\alpha^2 u - u'')$$

The left-hand side is equal to

$$\frac{d}{dt}\left(2\lambda\lambda'' + \lambda'^2 - 3\alpha^2\lambda^2\right)$$

Therefore, for  $t \leq T$  we have

$$\left(2\lambda\lambda'' + \lambda'^2 - 3\alpha^2\lambda^2\right)\Big|_t^T = \frac{A\sigma^2}{\alpha} \int_t^T (\alpha^2u(s) - u''(s))ds$$

This gives

$$\begin{aligned} &2\lambda(t)\lambda''(t) + \lambda'^2(t) - 3\alpha^2\lambda^2(t) = \\ &(2\lambda(T)\lambda''(T) + \lambda'^2(T) - 3\alpha^2\lambda^2(T)) - \frac{A\sigma^2}{\alpha} \int_t^T (\alpha^2u(s) - u''(s))ds \end{aligned} \quad (3.28)$$

By writing the equation (3.26) in terms of  $\lambda$

$$2\lambda(T)\lambda''(T) = \alpha^2\lambda^2(T) - \lambda'^2(T) - 2\alpha\lambda(T)\lambda'(T) \quad (3.29)$$

Using this expression for the first term in the right hand side of the equation (3.28), we have

$$2\lambda(T)\lambda''(T) + \lambda'^2(T) - 3\alpha^2\lambda^2(T) = -2\alpha\beta\lambda(T)(\alpha\lambda(T) + \lambda'(T))$$

The last term in the parentheses is  $c(T)$  which by Lemma 4 is positive. On the other hand, from equations (3.16) and (3.17)

$$\int_t^T (\alpha^2u(s) - u''(s))ds = \alpha\beta \int_t^T (\alpha x(s) - x'(s))ds = \alpha\beta(x(t) + \int_t^T \alpha x(s)ds)$$

□

**Proof:** [proof of Theorem 4 part (iii)] To show that the solution to the ODE (3.8) is the unique minimizer of the functional (3.21), we need to show that

$$\delta^2 J < 0$$

The second variation of the functional  $J[u]$  is given by

$$\frac{1}{2} \int_0^T (L_{uu}h^2 + L_{u'u'}h'^2 + L_{u''u''}h''^2) dt + \int_0^T (L_{uu'}hh' + L_{u''u'}h''h' + L_{uu''}hh'') dt$$

Using integration by parts for terms in the second integral and noting that  $L_{uu''} = 0$  we can write the second order variation as

$$\frac{1}{2} \int_0^T \left[ (L_{uu} - \frac{d}{dt}L_{uu'})h^2 + (L_{u'u'} - \frac{d}{dt}L_{u''u'})h'^2 + L_{u''u''}h''^2 \right] dt + \frac{1}{2}L_{uu'}h^2 \Big|_0^T + \frac{1}{2}L_{u''u'}h'^2 \Big|_0^T \quad (3.30)$$

Since  $u$  and  $u + h$  both satisfy conditions (3.18) and (3.19) we have

$$h(0) = h'(0) = 0$$

Hence, it's sufficient to show that every term in the expression (3.30) of the second order variation is negative. For the first term in the integral have

$$L_{uu} - \frac{d}{dt}L_{uu'} = -A\alpha^3\sigma^2 < 0 \quad (3.31)$$

For the second term, from Lemma 6 we have

$$L_{u'u'} - \frac{d}{dt}L_{u''u'} = 6\alpha^2u' - 2u^{(3)} - A\sigma^2\alpha \quad (3.32)$$

which can be represented in terms of  $\lambda$  as follow

$$-6\alpha^3\lambda + 2\alpha\lambda'' - A\alpha\sigma^2 \quad (3.33)$$

Multiplying this by  $\frac{\lambda}{\alpha}$ , adding it to (3.27) and using the positivity of  $K(t)$  in Lemma 6 we conclude that (3.32) is strictly negative. For the last integrand

$$L_{u''u''} = 2u' \quad (3.34)$$

Which from equation (3.15) and positivity of  $\lambda$  is clearly negative.

For the last two terms which are evaluated at  $T$ . We have

$$L_{uu'}|_T = -A\alpha^2\sigma^3 \quad (3.35)$$

$$L_{u''u'}|_T = 2u''(T) + 4\alpha u'(T) \quad (3.36)$$

Negativity of the second term follows from Lemma after rewriting it in terms of  $\lambda$  and  $c$  using equations (3.15), (3.17) and Remark 4.

□

**Proof:** [proof of Theorem 4 part (iv)] Let's write the equation (3.27) in the following form

$$2\lambda c' + c(\lambda' - 3\alpha\lambda) + \mathcal{K}(t) = 0 \quad (3.37)$$

Suppose the solution given in part (i) suggests to the investor to buy intermediately. With the final goal being to reach  $x(T) = 0$ , there exists a point in time  $t_0$  such that  $x'(t_0) = 0$  and  $x''(t_0) < 0$ , i.e.  $c'(t_0) > 0$ . Then in equation (3.37) at time  $t_0$  the first term is positive and the second one is zero. But this is a contradiction since from Lemma 6  $K(t)$  is always positive.

□

### 3.3.2 Infinite horizon case

Now we discuss the optimal execution problem for an investor with no time constraint. For a moment, let's ignore the risk component by assuming  $A = 0$ . In this case any admissible policy could be improved by delaying the policy and hence, and hence there is no optimal policy in this case. However, under risk consideration, the problem is not trivial anymore. Before studying the problem, let's first recall the Schied-Schoneborn results, which gives an intuition for the choices we make in the next proofs.

**Theorem 6** [Schied and Schneborn (2009)] *In a model where execution price evolves as equation (3.3), under constant temporary price impact  $\lambda$ . The optimal policy of the investor*

is given by

$$c(t) = \sqrt{\frac{A\sigma^2}{2\lambda}}x(t) \quad (3.38)$$

In order to address the problem for infinite horizon, we look at the limiting behavior of the solutions in the finite horizon case. The first question is to whether or not the investor will keep delaying the liquidation till to the end. The next proposition answers to this question and furthermore provide us with the limiting behavior for we're required in the last proof.

**Proposition 8** *In the absence of time restriction for liquidation in a market with dynamic temporary price impact, liquidating the entire position in a finite time is never optimal.*

**Proof:** Let's assume that the optimal liquidation policy  $c$  is taking place in the finite time  $[0, T]$ . We show that this policy could be improved and hence contradict the optimality claim. Define the function  $F$  for  $x \neq 0$  by

$$F(x, \lambda) = \frac{c}{\frac{x}{\sqrt{\lambda}}}$$

Since the liquidation is taking place in a finite time,  $F(x, \lambda)$  can't be bounded. Hence, for any  $M$  there is an interval  $I = [t_M, T)$  such that for all  $t \in I$

$$c > M \frac{x}{\sqrt{\lambda}} \quad (3.39)$$

By looking at the cost function

$$\int \lambda c^2 + \int \frac{1}{2} A \sigma^2 x^2$$

Over interval  $I$  we can see that the cost from price impact is  $\frac{M^2}{\frac{1}{2}A\sigma^2}$  times higher than the volatility risk. We create  $\tilde{c}$  through modifying  $c$  over this interval, in such a way that it reduces the price impact much higher than it increases the risk cost. Let  $J = [t_M, \tilde{T})$  be

the interval with a length  $N$  times length of  $I$ . Define  $\tilde{c}$  as follow

$$\tilde{c}(s) = \frac{1}{N}c(t_M + \frac{s - t_M}{N}) \quad \forall s \in [t_M, \tilde{T})$$

Under this choice of liquidation policy, for the volatility cost we have

$$\int_J A\sigma^2 \tilde{x}^2 = N \int_I A\sigma^2 x^2$$

Also for price impact cost, over each interval  $I' \subset I$  and its corresponding subset  $J' \subset J$  we have

$$\int_{J'} \tilde{c}^2 = \frac{1}{N} \int_{I'} c^2$$

Now let's compare the value of  $\lambda$  and  $\tilde{\lambda}$  over these to intervals. With  $t_I \in [t_M, \tilde{T})$  and its corresponding time  $t_J = t_I + (t_I - t_M)N$ , we claim that

$$\lambda_{t_I} \geq \tilde{\lambda}_{t_J} \tag{3.40}$$

At time  $t = t_M$  these two are equal. After this point we have

$$\lambda_{t_I} = (\lambda_{t_M} + \int_0^{t_I} \beta c_s e^{\alpha s} ds) e^{-\alpha t_I}$$

$$\lambda_{t_J} = (\lambda_{t_M} + \int_0^{t_J} \beta \tilde{c}_s e^{\alpha s} ds) e^{-\alpha t_J}$$

The claim is proven after comparing the derivative of the two integrals. Hence, the liquidation cost can be reduced, which this is a contradiction to the optimality claim.  $\square$

**Theorem 7** *The optimal liquidation policy of an investor with infinite time horizon has the following dynamic.*

$$\alpha c - c' = \frac{A\sigma^2}{2} \left[ \frac{\beta}{2} \left( \frac{x}{\lambda} \right)^2 + \frac{x}{\lambda} \right] \tag{3.41}$$

**Proof:** For fixed  $\lambda(T)$  and  $\lambda'(T)$  define

$$\mathcal{A}_{(\lambda_T, \lambda'_T)}(t) = \lambda_T(\alpha\lambda_T + \lambda'_T) + \frac{A\sigma^2}{2}(\beta^2 x^2 + 2\beta\lambda x) \quad (3.42)$$

Since  $\mathcal{A}'(t) = \lambda(t)K'(t)$  one can check that the solutions to equation (3.27) are in one to one correspondence with the solutions of the following differential equation

$$\lambda'^2 = -\mathcal{K}(t) + \alpha^2\lambda^2 + \frac{\mathcal{A}(t)}{\lambda}$$

Combining these two representation, The optimality path can be represented in the following form

$$\alpha^2\lambda - \lambda'' = \frac{\mathcal{A}(t)}{2\lambda^2}$$

Which after writing the left-hand side in terms of the rate of selling is equivalent to

$$\alpha c - c' = \frac{\mathcal{A}(t)}{2\beta\lambda^2} \quad (3.43)$$

From previous proposition we know that the trader tends to use all the infinite horizon for optimal trading, therefore, the infinite horizon solution can be obtained by approaching terminal time to infinity in the finite horizon case. Hence, in the limit at infinity  $\lambda(T)$  and  $\lambda(T)'$  approach to zero and (3.42) becomes

$$\mathcal{A}(t) = \frac{A\sigma^2}{2}(\beta^2 x^2 + 2\beta\lambda x)$$

and hence from (3.43) the infinite horizon optimal policy path satisfies

$$\alpha c - c' = \frac{A\sigma^2}{2} \left[ \frac{\beta}{2} \left( \frac{x}{\lambda} \right)^2 + \frac{x}{\lambda} \right]$$

□

Given a value for  $c(0)$  this ODE (3.41) could identify the liquidation path uniquely, however, we don't have any constraint left to find this initial condition. In order to address this issue, we transform the ODE into a PDE and observe that the initial condition to the PDE appears naturally in this algebraic form.

**Proof:** [proof of Theorem 5] With no time constraint, the optimal policy  $c$  is only a function of number of shares and the current temporary price impact parameter, hence by defining it as  $c(x_t, \lambda_t)$ , we have

$$\frac{d}{dt}c(x, \lambda) = \frac{dx}{dt}c_x + \frac{d\lambda}{dt}c_\lambda$$

the ODE (3.41) can be written as following in a PDE form

$$2\alpha c + 2\alpha\lambda c_\lambda - 2\beta c c_\lambda - \frac{A\sigma^2}{2} \frac{\beta x^2}{\lambda^2} - \frac{A\sigma^2 x}{\lambda} + 2c c_x = 0$$

In order to address the singularity at  $x = 0$  and obtain the boundary condition, we rewrite the PDE for the function

$$H(x, \lambda) = \frac{c(x, \lambda)}{x} \tag{3.44}$$

we obtain

$$xH + \lambda x H_\lambda - \frac{\beta}{\alpha} x^2 H H_\lambda - \frac{A\sigma^2}{4\alpha} \frac{\beta x^2}{\lambda^2} - \frac{A\sigma^2 x}{2\alpha\lambda} + \frac{1}{\alpha} x H (H + x H_x) = 0 \tag{3.45}$$

Now the singularity of this PDE at  $x = 0$  allows us to observe its initial condition in limit at  $x = 0$  as a Riccati equation

$$H_\lambda = \frac{A\sigma^2}{2\alpha\lambda^2} - \frac{1}{\lambda} H - \frac{1}{\alpha\lambda} H^2 \tag{3.46}$$

Let's denote  $h(\lambda) = \lim_{x \rightarrow 0} H(x, \lambda)$  and use the prime notation, for derivative with respect to its argument  $\lambda$ . By using the Riccati transformation, we can write this ODE in the form

of second order differential equation

$$y'' + \frac{2}{\lambda}y' - \frac{A\sigma^2}{2\alpha^2\lambda^3}y = 0 \quad (3.47)$$

which after solving, will return us the solution to the ODE (3.46) via following transformation

$$h(\lambda) = \alpha\lambda \frac{y'(\lambda)}{y(\lambda)} \quad (3.48)$$

Let's denote  $M = \frac{2A\sigma^2}{\alpha^2} > 0$ . Then for constants  $C_1$  and  $C_2$  the solutions to (3.47) are given by

$$u(\lambda) = \frac{1}{\sqrt{\lambda}}(C_1f(\lambda) + C_2g(\lambda))$$

where  $f(\lambda) = I_1(\sqrt{M/\lambda})$  and  $g(\lambda) = K_1(\sqrt{M/\lambda})$  are modified Bessel functions. Let's assume for a moment that both  $C_1, C_2$  are nonzero.

Let's denote  $w(\lambda) = C_1f(\lambda) + C_2g(\lambda)$ . From (3.48) we have

$$h(\lambda) = \alpha\lambda \frac{-\frac{1}{2}\lambda^{-\frac{3}{2}}w(\lambda) + \lambda^{-\frac{1}{2}}w'(\lambda)}{\lambda^{-\frac{1}{2}}w(\lambda)} = \alpha\left(-\frac{1}{2} + \lambda\frac{w'}{w}\right) \quad (3.49)$$

From proposition 8, we know  $\lim_{\lambda \rightarrow 0} h(\lambda) = \infty$  and  $\lim_{\lambda \rightarrow \infty} h(\lambda) = 0$ . Hence, we for the function  $\frac{w'}{w}$  we need to have

$$\lim_{\lambda \rightarrow 0} \frac{w'}{w} = \infty \quad \lim_{\lambda \rightarrow \infty} \frac{w'}{w} \approx \frac{1}{2\lambda} \quad (3.50)$$

We can observe that for the function  $f(\lambda)$  and  $g(\lambda)$ , we have

$$\lim_{\lambda \rightarrow \infty} f(\lambda) = \lim_{\lambda \rightarrow \infty} f'(\lambda) = 0 \quad \lim_{\lambda \rightarrow 0} f(\lambda) > 0$$

$$\lim_{\lambda \rightarrow 0} g(\lambda) = \lim_{\lambda \rightarrow 0} g'(\lambda) = 0 \quad \lim_{\lambda \rightarrow \infty} g(\lambda) > 0$$

Hence, from these results we have

$$\lim_{\lambda \rightarrow \infty} \frac{w'}{w} = \lim_{\lambda \rightarrow \infty} \frac{C_1 f'(\lambda) + C_2 g'(\lambda)}{C_1 f(\lambda) + C_2 g(\lambda)} = \lim_{\lambda \rightarrow \infty} \frac{C_2 g'(\lambda)}{C_2 g(\lambda)} = \lim_{\lambda \rightarrow \infty} \frac{g'(\lambda)}{g(\lambda)}$$

$$\lim_{\lambda \rightarrow 0} \frac{w'}{w} = \lim_{\lambda \rightarrow 0} \frac{C_1 f'(\lambda) + C_2 g'(\lambda)}{C_1 f(\lambda) + C_2 g(\lambda)} = \lim_{\lambda \rightarrow 0} \frac{C_1 f'(\lambda)}{C_1 f(\lambda)} = \lim_{\lambda \rightarrow 0} \frac{f'(\lambda)}{f(\lambda)}$$

However, from the asymptotic behavior of modified Bessel functions we have

$$\lim_{\lambda \rightarrow \infty} \frac{f'(\lambda)}{f(\lambda)} = -\infty$$

which is contradiction to the boundary condition (3.50). Therefore, either  $C_1 = 0$  or  $C_2 = 0$ . By the same argument as above, we can immediately see that only with  $C_1 = 0$  the initial conditions are satisfied. After simplifying the result in equation (3.49) we obtain the required initial condition for the solving the PDE (3.45)

$$\lim_{x \rightarrow 0} H(x, \lambda) = \sqrt{\frac{A\sigma^2}{2\lambda}} \frac{K_0\left(\sqrt{\frac{2A\sigma^2}{\alpha^2\lambda}}\right)}{K_1\left(\sqrt{\frac{2A\sigma^2}{\alpha^2\lambda}}\right)}.$$

□

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## Curriculum Vitae

