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Fluctuation noise in single and multi-collector vacuum tubes in the space-charge-limited condition

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FLUCTUATION NOISE IN SINGLE AND MULTI-COLLECTOR VACUUM TUBES IN THE SPACE-CHARGE-LIMITED CONDITION

by

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TABLE OF CONTENTS

LIST OF FIGURES AND CHARTS

Abstract

This paper presents an exposition of the significant theories of fluctuation noise for vacuum tubes in the space-charge-limited condition. The theories presented are for low frequencies, that is, frequencies where transit time effects are not significant. The importance of noise in the general communications problem is discussed in Part I. In Part II the historical development of fluctuation noise theories is presented commencing with work of W. Schottky on the temperature-limited diode. The practical importance of the noise phenomenon to the tube manufacturers is pointed out. The historical development of thermionic emission theory is briefly traced beginning with C. Child's famous three-halves power law. The basic problem in the determination of space-charge conditions is next discussed. This is followed by excerpts from fluctuation noise investigations in 1925 by A. Hull and N. Williams. The progress of noise investigations from 1931 - 1940 is traced. Special reference is made to the work of B. J. Thompson, D. O. Williams, and W. A. Harris in the development of practical engineering formulae. The achievements of the Radiation Laboratory of M. I. T. from 1941 - 1950 are noted.

Part III presents a derivation of the original Shottky equation for temperature limited diodes:

$$\overline{I^2} = 2K\overline{I}\Delta F$$

where: $\overline{I^2}$ = amps r.m.s. fluctuation current

K = electron charge in coulombs

$\overline{I}$ = average D.C. anode current

ΔF = effective frequency bandwidth in cycles per second.

This equation is derived by use of Fourier series and probability theory.

Part IV considers the space-charge picture in detail. The mechanism of thermionic emission is discussed in terms of the theory of work function and Richardson's equations. Reasons for preferring the T^2 version of Richardson's equations are given. The nature of the temperature - limited condition is next considered and is followed by a description of the space-charge-limited case. The reasons for noise reduction in space-charge-limited diodes is discussed by way of reference to the experimental work of A. Hull and N. Williams. The opinions of other investigators on the cause of noise reduction are also noted. Part V discusses the various experimental confirmations of the Maxwell distribution law. Statistical mechanics is used to show how the Fermi-Dirac distribution law reduces to the Maxwell distribution law at emission temperatures. The T. C. Fry method of analysis of space-charge-potential distribution in diodes is next presented. This analysis takes into account the initial Maxwell velocity distribution of emitted electrons and enables a determination of space-charge-potential for all diode space-charge conditions. A fluctuation noise theory for multi-element tubes (tetrodes, pentodes, etc.) is presented in Part VI. The D. O. North theory of current division is given. This is a comprehensive and quantitative theory showing how excess noise is caused by a division of tube current among the collector electrodes. Several conclusions resulting from this theory are given.

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16. Pearson, G. L., Physics, Vol. 6., 1935.

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18. Rack, A. J., BSTJ Vol. 17, No. 4., Oct. 1938.

19. Richardson, O.W., "Emission of Electricity From Hot Bodies", 2nd Ed., Longmans, Green, & Co., New York, 1921.

20. Thompson, B. J., R.C.A. Review, January 1940.

21. Schottky, W., Ann. Physik, 57, 1918.

22. Smoluchowski, M., Ann. Physik 25, 1908.

23. Zartman, I. F., Phys. Rev., 37, 1931.

I. INTRODUCTION

Noise is one of the basic problems in radio communications and radar. This phenomenon places a practical limit upon the usefulness of any particular communications scheme. Signals buried by noises have no practical or economic value. Consequently, in the past thirty years engineers and physicists have devoted much time to a study of the noise problem. Time has brought a greater insight into the nature of the many types of noise present in communications systems. As a result of these extensive researches, some types of man-made noises can be minimized or completely eliminated. There is, however, one general type of noise which cannot be eliminated even theoretically. This noise is due to the atomic nature of matter. In communications the atomicity of matter produces an undesired fluctuation phenomenon generally called "fluctuation noise". This includes thermal noise, shot noise, magnetic fluctuation noise, and many other similarly described phenomena. Science, as yet, has not found a way to turn the noise phenomenon into an advantage, which would be the ideal solution to the problem. The efforts applied against the noise problem have been in the direction of determining its nature and devising methods of minimizing and eliminating it. Consequently, the very nature of noise in general has presented itself in the past and still presents itself as a problem ranging completely across both physics and engineering.

There are other types of noise which are not of atomic origin such as vacuum tube flicker effect, contact and breakdown noise, dirt and grain size noise, ignition noise, and a few others. These noises are primarily a function of the mechanical construction of devices and are, consequently, a problem in design engineering. It is,however, with the determination of the space-charge-limited shot effect phenomenon that this paper deals. In practice this noise is not unlike thermal noise, and might be justifiably considered as a kind of man-made version of thermal noise, as the shot effect does not exist in nature.

II. HISTORICAL SUMMARY

The vacuum tube fluctuation phenomenon of the shot effect was first analyzed in 1918 by W. Schottky[1]. This investigation had been preceded by the fluctuation theories of Smoluchowski[2] and others for gases. Schottky named this phenomenon "shot effect" because this fluctuation noise, created by the electrons comprising the tube current, reminded him of the noise created by a hail of shot striking a target. He asserted that because the tube current consisted of electrons each independently emitted, each carrying a finite charge, that the fluctuation current would be proportional to the electronic charge, the average tube current, and the effective frequency band width under consideration. With considerable insight he further predicted that this noise would be measurable, if sufficiently high amplification were available. A few years later, when the technique of building cascaded amplifier stages had been sufficiently perfected, Schottky's prediction was completely vindicated. However, it was a decade after Schottky's original pronouncement that this phenomenon became of considerable practical importance.

With the introduction in the late 1920's of multi-element tubes (tetrodes, pentodes, and later, hexodes,

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1. Schottky, W., "Ann. Physik", 57, 541-567, 1918
2. Smoluchowski, M. V., "Ann. Physik", 25, 205, 1908

and heptodes) the practical importance of the vacuum tube noise phenomenon of the shot effect became evident, for it was noticed at once that these multi-element tubes are substantially noisier than diodes or triodes. Like thermal noise the shot effect places a limit upon the amount of amplification available from any given amplifier. Strictly speaking, it is now known that the presence of noise does not limit the detectability of a signal. When very small signals are in the presence of noise of comparable or greater magnitude, a large number of readings of a signal indicating device (such as a mirror galvonometer or meter) will indicate a change in the average position of the indicator. Theoretically, if an unlimited number of readings could be taken the presence of any signal, no matter how small, could be detected. From experiments with radar oscilloscopes Lawson and Uhlenbeck[3] have shown that an observer visually viewing an oscilloscope screen can make a good guess that a signal of very small magnitude is present even though the noise is much greater than the signal.

Because of both the practical limitations and implications of this fluctuation phenomenon, the interest of research workers in the laboratories of the tube manufacturers as well as researchers in the academic field was aroused. Possessing the necessary extensive plant facilities, staff, and equipment required for vacuum tube investigation, industrial researchers have made notable contributions to an understanding of this problem. The shot noise phenomenon, however, has been one of considerable controversy, and it is only within the past decade that agreement has been reached on certain theoretical

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3. Lawson, J., and Uhlenbeck, G., "Threshold Signals", Vol. 24 Radiation Lab. Series, McGraw Hill Book Co., New York, 1950, p. 150

aspects of the matter. The problem is primarily one in analysis of space charge conditions in the vacuum tube.

A relative host of workers have dealt with electron emission, dating from the work of O. Richardson[4], C.Childs[5], and I. Langmuir[6], in the first decade of the 20th Century. By 1911 Childs had deduced his famous three-halves power law for current in a diode. Richardson, however, took the first important steps toward a clarification of the mechanism of thermionic emission from a theoretical standpoint. By application of the principles of thermodynamics he derived a relationship between emission current and temperature of the metal, known as Richardson's equation. By 1920 much was known about the temperature-current characteristics of simple vacuum tubes, but the mechanism of space charge remained obscure. Indeed, the very existence of a pure electron emission from incandescent solids had been seriously questioned as late as 1918 by many well-known physicists[7].

In 1921, however, T. C. Fry[8] of the then American Telephone and Telegraph Co. sought to clarify this problem

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4. Richardson, O. W., "Emission of Electricity From Hot Bodies", 2nd ed., Longmans, Green, and Co., New York, 1921.

5. Child, C. D., "Discharge From Hot CaO", Phys. Rev., Series I, 32, 1911, 498-500

6. Langmuir, I., "The Effects of Space and Residual Gases on Thermionic Currents in High Vacuum", Phys. Rev., Series II, 2, 1913, 461-470

7. Dushman, S., "Thermionic Emission", Rev. Mod. Phys., 2, 1930, 383

8. Fry, T. C., "Thermionic Current Between Parallel Planes; Velocities of Emission Distributed According to Maxwell's Law", Phys. Rev., 17, 1921, 441-452

and published a method for the general solution of voltage distribution in space-charge-limited cases. Adding to Fry's method of analysis was the work of Langmuir[9], published early in 1923. In the meantime other investigators were devising means of experimentally measuring the predicted values of shot-noise voltage given by Schottky. Two prominent American workers interested in shot noise effects in the 1920's were Hull and Williams[10] of the General Electric Company who successfully measured the predicted value of shot noise for the temperature-limited condition. However, some peculiar things happened when the space charge limited condition was encountered. To quote Hull and Williams:

> "Preliminary measurements showed that the presence of space charge caused a marked reduction of shot disturbance. This fact was first discovered and called to our attention by Mr. W. L. Carlson of the Radio Department of the General Electric Company. We at first believed that the decrease in shot noise reported by Mr. Carlson was only apparent and was accounted for, as it is to a large extent, by the high effective resistence of the tuned circuit when a space charge limited tube (low resistance is connected in multiple with it. The measurements reported in 17 show, however, that there may be a real reduction of shot effect of several fold due to space charge."

As far as can be gleaned from an examination of the literature, this is the first instance where a reduction in the shot noise was thought to be caused by the space charge. Since 1925 this space-charge reduction has become

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9. Langmuir, "Effect of Space Charge and Initial Velocities on the Potential Distribution and Thermionic Current Between Parallel Plane Electrodes", Phys. Rev. 21, 1923.

10. Hull, A., and Williams, N., "Determination of Elementary Charge E from Measurements of Shot Effect", Phys. Rev., 25, 166, 1925.

an accepted theory.

During the early 1930's investigators such as F. B. Llewellyn[11] made substantial contributions to the analysis of shot noise with related circuits. G.L. Pearson[12] performed noteworthy experimental work on space-charge-limited shot-noise measurements. A. J. Rack[13], a theorist, gave an analysis in 1938 of transit time effects on the shot noise formula. A real milestone was reached in 1940 with the joint publication by B. J. Thompson, D.A. North, and W. A. Harris[14] of the results of ten years work done in the laboratories of the Radio Corporation of America. Showing great caution by subjecting all theoretical conclusions to repeated experimental check, this group produced a very useful set of formulae for applied problems. These formulae have been used with considerable success in intermediate and radio frequency amplifiers.

During the years 1941-1950 many investigations of noise in general were instituted with special reference to the radar problem. These covered a wide variety of topics involving investigations in many different laboratories in the United States. Of significant contribution

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11. Llewellyn, F. B., "A Study of Noise in Vacuum Tubes and Attached Circuits", Proc. I.R.E., Vol. 18, pp. 243-245, 1930

12. Pearson, G. L., "Shot Effect and Thermal Agitation in Space Charge Limited Current", Physics, Vol. 6, pp. 6-9, 1935

13. Rack, A. J., "Effect of Space Charge and Transit Time on Shot Noise in Diodes", B.S.T.J., Vol. 17, No. 4, p. 592, Oct. 1938

14. Thompson, B. S., North, D. O., Harris, W. A., "Fluctuations in Space-Charge Limited Currents at Moderately High Frequencies", Parts I-V, R.C.A. Review, commencing Jan. 1940

were the achievements of the Radiation Laboratory of M.I.T.[15] While little seems to have been done concerning shot noise in vacuum tubes, much has been discovered which may lead to a quicker solution of shot-noise transit-time effects, which, no doubt, will soon be of considerable importance as the higher frequencies are used[16]. As previously pointed out, many experimental and theoretical investigations have been devoted to the question of how the Schottky formula should be modified in the space-charge-limited condition. One can say that this problem has now been solved, except for the high frequency region where the effect of transit time is an added factor[17].

III. THE SCHOTTKY EQUATION

Because electrons are randomly emitted and possess a discrete charge the number of electrons emitted in equal time intervals will fluctuate around an average value taken over a long period of time. Each emitted electron may be considered as a current impulse. The total anode current consists, therefore, of the random superposition of millions of impulses. For the purpose of analysis consider a single electron in flight between cathode and anode having a response function very similar to the Dirac delta function, as the transit time is very short[18].

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15. Cheatham, T., and Tuller, W., "Results of Transient Analysis of Impulse Noise in F.M. Receivers", Res. Lab. of Electronics, M.I.T., Tech. Rep. No. 28, Jan. 20, 1947
16. Duval, G., "The Effects of Transit Angle on Shot Noise in Vacuum Tubes", Res. Lab. of Electronics, M.I.T., Tech. Rep. No. 82, Sept. 8, 1948
17. Lawson, J. and Uhlenbeck, G., Op. cit. p. 83
18. Lawson, J. and Uhlenbeck, G., ibid, p. 79

The area under the impulse curve is equal to the charge on the electron. e.

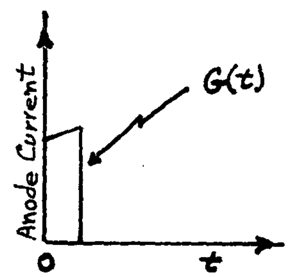

Fig 1. Anode Current Due a Single Electron G(t)

In an actual tube the transit time might be 10^{-8} seconds, and the following analysis is accurate only for frequencies where the transit time is not important.
The Fourier integral of this impulse is :

$$G(t) = \int_0^{\infty} S(\omega) \cos[\omega t + \phi(\omega)]\, d\omega$$

as the transit time is negligible, $\phi(\omega) = 0$

$$S(\omega) = e = \int_0^{\Delta t} G(t)\, dt$$

where: e = charge on the electron.

$$G(t) = \int_0^{\infty} \int_0^{\Delta t} G(t) \cos \omega t \, d\omega$$

Consider the more general condition of a long interval T. This interval may be thought of as the entire time of observation in the experiment under consideration.

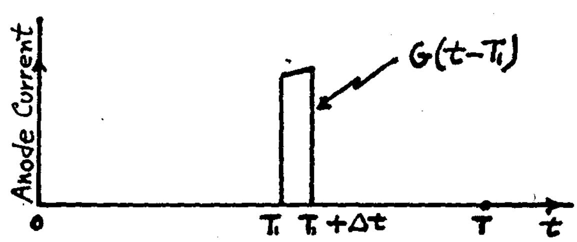

Fig 2. Anode Current Due a Single Electron $G(t - T_1)$

Expressing as a Fourier series:

$$G(t-T_1) = \frac{a_0}{2} + \sum_{q=1}^{\infty}\left(a_q \cos\frac{2\pi q t}{T} + b_q \sin\frac{2\pi q t}{T}\right)$$

$$a_0 = \frac{2}{T}\int_0^T G(t-T_1)\,dt = \frac{2e}{T}$$

$$a_q = \frac{2}{T}\int_0^T G(t-T_1)\,dt \cos\frac{2\pi q t}{T} = \frac{2e}{T}\cos\frac{2\pi q T_1}{T}$$

$$b_q = \frac{2}{T}\int_0^T G(t-T_1)\,dt \sin\frac{2\pi q t}{T} = \frac{2e}{T}\sin\frac{2\pi q T_1}{T}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$G(t-T_1) = \frac{e}{T} + \frac{2e}{T}\sum_{q=1}^{\infty}\cos\left[\frac{2\pi q(t-T_1)}{T}\right]$$

Let the superposition of all the impulses be denoted by $I(t)$. Then, consider some specific Fourier component of $I(t)$:

$$C\cos\left(\frac{2\pi q t}{T} - \phi\right) \qquad \text{(A)}$$

Each $G(t-T_1)$ contributes a component to the above expression of: $\frac{2e}{T}\cos\frac{2\pi q(t-T_1)}{T}$ randomly phased. These components must be added vectorially. The phases depend on the value of T_1 and repeat for values of T_1 at distances $\frac{T}{q}$ apart on the time axis. Then, ϕ will range from 0 to 2π. For any particular range of between ϕ and $\phi + d\phi$, there are in time T, q small intervals of length:

$$dT_1 = \frac{T\,d\phi}{2\pi q}$$

Any $G(t - T_1)$ with its T_1 in these intervals will contribute a component in this phase range.

Let: N = average number of G(t)'s per second

n = number having phases in the interval between ϕ and $\phi + d\phi$

$\bar{n}$ = ensemble average number of G(t)'s having phases in the interval between ϕ and $\phi + d\phi$

Then:

$$\bar{n} = Nq\frac{T\,d\phi}{2\pi q} = \frac{NT\,d\phi}{2\pi}$$

and the average value of the sum of elementary Fourier components of frequency $\frac{q}{T}$ in any particular phase range ϕ to $\phi + d\phi$ is:

$$\frac{2e}{T}\,\frac{NT}{2\pi}d\phi\,\mathrm{Cos}\left(\frac{2\pi q t}{T}-\phi\right) = \frac{Ne}{\pi}\,\mathrm{Cos}\left(\frac{2\pi q t}{T}-\phi\right)d\phi$$

The vector sum of all these average resultants is zero and has the graphical form of a closed polygon. Therefore, it is necessary to consider actual values of n instead of the ensemble average $\bar{n}$. Each vector is subject to a certain amount of fluctuation about its average length <u>k</u> $d\phi$. Number these vectors V from V_1, V_2 ... V_s, and consider the x and y components, remembering that V = C of equation (A)

$$V_x = V_{1x} + V_{2x} + \cdots + V_{sx} = V_1\,\mathrm{Cos}\,\phi_1 + V_2\,\mathrm{Cos}\,\phi_2 + \cdots + V_s\,\mathrm{Cos}\,\phi_s$$

$$V_x = \frac{2e}{T}\,\mathrm{Cos}\,\phi_1 n_1 + \frac{2e}{T}\,\mathrm{Cos}\,\phi_2 n_2 + \cdots + \frac{2e}{T}\,\mathrm{Cos}\,\phi_s n_s$$

$$V_y = V_{1y} + V_{2y} + \cdots + V_{sy} = V_1\,\mathrm{Sin}\,\phi_1 + V_2\,\mathrm{Sin}\,\phi_2 + \cdots + V_s\,\mathrm{Sin}\,\phi_s$$

$$V_y = \frac{2e}{T}\,\mathrm{Sin}\,\phi_1 n_1 + \frac{2e}{T}\,\mathrm{Sin}\,\phi_2 n_2 + \cdots + \frac{2e}{T}\,\mathrm{Sin}\,\phi_s n_s$$

Because the average values of V_x and V_y are zero it is the deviations that are of interest.

Let: V_ℓ = the deviation of V_x

V_m = the deviation of V_y

It can be shown that if two or more probability distributions are superimposed, the average value of the squares of the deviations of the new distribution is equal to the sum of the average values of the squares of the deviations of the original distributions. Or in general if n is a function of the chance variables n_1 and n_2, and if n_1 and n_2 are large numbers, then the statement above can be generalized to the following[19]:

$$n = An_1 + Bn_2$$
$$\bar{n} = A\bar{n}_1 + B\bar{n}_2$$

If two or more probability distributions are superimposed, the average value of the new distribution is the sum of the average values of the original distributions.

$$\therefore \overline{D^2} = A_1^2\,\overline{D_1^2} + A_2^2\,\overline{D_2^2} + \ldots + A_s^2\,\overline{D_s^2}$$

$$\overline{V_\ell^2} = \frac{4e}{T^2}\left(\overline{D_1^2}\cos^2\phi_1 + \overline{D_2^2}\cos^2\phi_2 + \ldots + \overline{D_s^2}\cos^2\phi_s\right)$$

When large numbers are involved, such as in a Gaussian distribution, percentage deviations from the average become extremely small, when the number of trials is large. That is:

$$\overline{D^2} = \bar{n}$$

This is called the "law of large numbers."

19. Uspensky, J. V., "Introduction to Mathematical Probability", McGraw Hill Book Co., 1937, p. 173

$$\overline{V_\ell^2} = \frac{4e^2}{T}(\overline{n}_1 \cos^2\phi_1 + \overline{n}_2 \cos\phi_2 + \ldots + \overline{n}_s \cos^2\phi_s)$$

$$\overline{V_\ell^2} = \frac{4e^2}{T^2}\frac{NT}{2\pi}\int_0^{2\pi}\cos^2\phi \, d\phi = \frac{2e^2N}{T}$$

Similarly:

$$\overline{V_m^2} = \frac{2e^2N}{T}$$

and $\overline{V^2} = \overline{V_\ell^2} + \overline{V_m^2} = \frac{2e^2N}{T} + \frac{2e^2N}{T} = \frac{4e^2N}{T}$

$$C\cos\left(\frac{2\pi qt}{T} - \phi\right) = V\cos\left(\frac{2\pi qt}{T} - \phi\right)$$

$$C^2\cos^2\left(\frac{2\pi qt}{T} - \phi\right) = V^2\cos^2\left(\frac{2\pi qt}{T} - \phi\right)$$

$$\overline{\cos^2\phi} = \tfrac{1}{2}$$

$$C^2\cos^2\left(\frac{2\pi qt}{T} - \phi\right) = \frac{4e^2N}{T}\cdot\frac{1}{2} = \frac{2e^2N}{T}$$

$$\overline{I} = eN;\ F = \frac{q}{T};\ \Delta q = T\Delta F$$

where: F = frequency

$$\overline{I^2} = \frac{2e\overline{I}}{T}T\Delta F = 2e\overline{I}\Delta F \qquad \text{(Schottky Equation)}$$

where: e = charge on the electron in coulombs
$\overline{I}$ = average DC anode current in amps
ΔF = frequency bandwidth in cycles per second
$\overline{I^2}$ = amps r.m.s. fluctuation current

In practice $T\Delta F$ is large enough so that $\Delta q \gg 1$. The Schottky equation is applicable only to the temperature limited condition.

For the remainder of this paper, the Schottky equation will be written as:

$$\overline{I^2} = 2K\overline{I}\Delta F$$

where: K = charge on the electron in coulomb
$\overline{I^2}$ = amps r.m.s. fluctuation current.

IV. THE SPACE CHARGE PICTURE

A. Thermionic Emission

The mechanism of emission is best understood by considering the properties of an electron gas in the interior of the emitter. The free electrons move about with a random distribution of velocities, their mean kinetic energy, however, being proportional to the absolute temperature. At ordinary room temperatures a metal does not lose electrons in appreciable quantities. It is an experimental fact that a metal does not become positively charged when standing idle and well insulated. This is because any free electrons trying to leave the surface are restrained by the attraction of positively charged nuclei at the boundary of the metal. The amount of energy required to liberate an electron from this restraining force is called the work function, W, of the metal. From experimental evidence W seems to depend slightly on temperature but, as yet, there is no conclusive experimental proof. When an electron with kinetic energy greater than W happens to move toward the surface from a point just inside, it is able to escape, thereby loosing an amount W of kinetic energy in the process. The higher the temperature the greater is the number of electrons with the necessary energy for escape. The emission, therefore, increases with temperature.

From the kinetic theory of gases Richardson calculated the rate at which electrons are emitted as a function of temperature. By assuming the Maxwell distribution he showed that if W, the work function, is constant, the emission-current density J must have the form:

$$J = A'T^{1/2} e^{-\frac{b'}{T}}$$

where: T is the absolute temperature and A' and b' are constants for any particular metal. If the variation in W with temperature is taken into account, the emission equation is:

$$J = AT^2 e^{-\frac{b}{T}}$$

It is difficult to distinguish between these two equations experimentally within the range of temperatures available in the laboratory. The last equation, however, is superior theoretically because it is derivable from the more exact Fermi-Dirac statistics rather than from kinetic theory and thermodynamics. As the Fermi-Dirac distribution law was not discovered until later, Richardson assumed that the Maxwellian law for gas molecules was applicable to an electron gas. This gas consisted of the free electrons sufficiently separated from each other and sufficiently small to be treated as a perfect gas. He also made the assumption that the number of free electrons per cubic centimeter inside the metal was independent of temperature and so obtained the equation containing $T^{\frac{1}{2}}$. Later, Richardson assumed that the number of free electrons increased as the three halves power of the absolute temperature and thus derived the T^2 form of the equation. From thermodynamics, rather than from kinetic theory, Dushman also derived the T^2 form of equation. In view of the fact that the Fermi-Dirac law also gives the T^2 form the latter equation is considered more satisfactory.

B. The Temperature-Limited Condition

In the usual high vacuum diode if the filament temperature is fixed, and the anode temperature is raised, a point will be reached where any further increase in anode voltage will not result in an increase in tube current. This is true if one ignores the Schottky effect, which lowers the emitter surface work function as the anode voltage is raised. The Schottky effect, therefore, means that the anode current actually rises slightly with anode voltage beyond the usual temperature-limited point. Then, in the usual temperature-limited condition all available electrons pass to the anode. The quantitative determination of the magnitude of shot effect is simplest when the tube is operated in the temperature-limited condition. Conversely, in the space-charge-limited case the problem is quite complex and involves extensive manipulations of space-charge equations. When a diode is in a temperature-limited condition, the mean-square anode fluctuation current given by Schottky and derived in Part III is:

$\overline{I^2} = 2K\overline{I}\Delta F$ Actually, the above equation also holds for negative grid triodes in the temperature-limited condition or for any tube in which the entire emission current goes to one collector electrode, such as in a photoelectric cell. However, the derivation assumes that the frequency is low enough so that electron-transit time is not important. When the electron-transit time becomes significant, the value of K must be revised.

C. The Space-Charge-Limited Case

In the high-vacuum tube with any given plate voltage the repulsion force between electrons in transit between the cathode and the anode sets the upper limit to the magnitude of current that will be conducted for any given geometric configuration. If the anode voltage of a diode is held fixed, and the filament temperature is increased, the tube current will increase according to Richardson's law, but, ultimately, any further increase in cathode temperature will not result in any appreciable increase in tube current. This limitation is caused by a charged cloud of electrons in front of the surface of the emitter. This charged cloud, or space charge, will tend to slow down electrons emitted from the cathode. At the same time the electric field due to the anode voltage tends to accelerate them. The net result is that the space charge changes the potential in the space until, if the cathode temperature is raised high enough, the electric field at the cathode is reversed. The force on the electrons is then in the direction of the cathode instead of toward the anode. However, because of the Maxwellian distribution of velocities some electrons have sufficient kinetic energy to penetrate the space-charge and become part of the anode current. Others are turned back to the cathode. The anode current is, therefore, smaller than the emitted current and, thus, the current is said to be limited by the electron-space-charge.

D. Cause of Space Charge Shot Noise Reduction

As mentioned on page 5 , the temperature-limited condition reduces the magnitude of shot noise. This reduction is now believed to be caused by the following: In the region between the cathode and the point of maximum negative space charge voltage E_m (this is usually called the alpha region), the electrons are flowing in opposite directions; those which have been turned back to the cathode passing by those newly emitted electrons heading for the space charge. The present (partial) explanation for noise reduction is that only fluctuations due to the difference between the two currents is effective in producing fluctuation noise at the anode. See page 26 .

As a matter of historical interest, the table below is an extract from the work of Hull and Williams[20] published in 1925. It has been previously pointed out that these two investigators were the first to make quantitative measurements of the reduced space-charge shot effect. The following data was taken with a UV 199 tube:

Filament Current	Filament Temp.°K	Emission ma	Shot Noise in Volts Observed	Calculated	Ratio Obs/calc
140	1675	1.0	67	71.7	0.93
150	1750	2.0	71	87.7	0.82
152	1765	2.5	51	83.8	0.61
160	1805	3.0	38	77.2	0.49
167	1850	3.5	28	73.0	0.39
170	1867	4.0	13.6	75.0	0.18
172	1940	5.0	15.9	80.0	0.20

Chart I. Shot Noise Voltage Vs. Filament Temperature

20. Hull and Williams, op. cit., p. 169

With reference to these observations the following is quoted from Hull and Williams:

> "The data obtained illustrates well the important fact which we wish to emphasize, namely, that when thermionic current is limited by space-charge, the shot effect is only a small fraction of that to be expected from independently moving electrons. In the transition region between full space charge and full temperature limitation, the shot voltage approaches more nearly its theoretical value the more the current is depressed (in per cent) below its space charge value. As the cathode temperature was gradually raised, the shot voltage fell from the essentially full theoretical value to 18 per cent of theoretical".

In the same publication they attempted to explain the phenomenon as follows:

> "Preliminary with the two electrode tubes showed no shot effect when the current was limited by space charge, so far as could be detected with the degree of amplification used for previous (temperature limited) measurements. Calculation showed that this was to be expected on account of the very low a.c. resistance (5,000 to 10,000 ohms) of a space charge limited tube. This produced a high effective series resistance of a tuned circuit to which it was multiply connected, and so reduced the amplitude of oscillation produced in this circuit by shot effect. With increased amplification these shot oscillations were easily detected, however, though still too small for convenient measurement."

Other investigators have since taken sharp exception to the complete validity of the above explanation of the cause of reduced shot noise. B. J. Thompson[21] has said:

> "It has been erroneously stated that the shot-effect currents are the same in the

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21. Thompson, B. J., op. cit., Jan. 1940, p. 275

> presence of space charge as with temperature limitation of emission, and that the reduction in shot effect voltage results only from the shunting effect of the lower anode resistance in the case of space-charge limitation. On the other hand experimental results have frequently been published in which the varying shunt effect of the anode resistance was ignored."

A more recent point of view by Goldman[22] explains reduced shot noise in the space-charge-limited condition in terms of the concept of coherence between the various currents caused by changes in the magnitude of E_m, the maximum space-charge voltage. Goldman says:

> "The fact that the anode current fluctuations in the presence of space charge are less than indicated by $2K\overline{I}\Delta f$ is therefore evidence that there is a certain amount of coherence between the impulses of anode current due to the individual electrons."

From this viewpoint fluctuations in emission tend to be offset by a compensating action of the space charge which serves to partially nullify the fluctuations.

For convenience in certain types of calculations, it is usual to express the magnitude of shot effect in the space-charge-limited case as follows:

$$\overline{I^2} = \Gamma^2 2K\overline{I}\Delta F$$

where: Γ^2 is a positive constant less than 1 which takes into account the space-charge effects. For the temperature-limited case Γ^2 obviously equals 1. Of course, the real problem in the space-charge-limited case is to determine Γ^2. This is a complicated and lengthy procedure. It required over 10 years to determine reliable formulas for Γ^2 for multi-element

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22. Goldman, S., "Frequency Analysis, Modulation and Noise", McGraw Hill, 1948, p. 359

tubes. Consequently, only the more significant and useful methods will be outlined in the next section.

V. SPACE-CHARGE EQUATIONS

A. Maxwell Vs. Fermi-Dirac

If certain simplifying assumptions are made, the complications of space-charge manipulations are greatly reduced. This is, of course, one of the standard methodologies of science and was used effectively by Childs in the derivation of his three-halves power law. Childs made the following assumptions: the electrodes are infinite, plane, parallel, equipotential surfaces, a space-charge-limited condition exists consisting of charges all of the same sign (in this case electrons), the charges start from rest at the cathode, and a condition of equilibrium exists. These assumptions enabled a rather neat derivation for tube current as a function of anode potential for a fixed geometric configuration.

The main reason for the complexity in the space-charge-limited shot-noise derivation is that it is not possible to ignore the fact that the electrons are emitted with an initial velocity distribution of Maxwellian type, a conclusion reached by Richardson and Schottky from theoretical considerations. Childs' assumption that the electrons all start from rest at the cathode completely ignores the role of the emitter surface work function, W, which requires that the electrons reach the emitter surface with some initial velocity. This initial velocity cannot be ignored in determining shot effects, as it leads to conclusions in gross conflict with experimental measurements.

That the emission has a Maxwellian velocity distribution at emitter temperatures has been confirmed experimentally. Germer[23] of the Bell Telephone Co. experimentally re-affirmed this conclusion in 1925 by carefully testing the Richardson equation. He made very careful measurements for eight different emitter temperatures from 1440°K to 2475°K. After carefully correcting for contact potentials, the results showed that the tube current varied with temperature in just the manner calculated on the assumption that the electrons leave the emitter with a Maxwellian velocity distribution. A slightly more recent confirmation of the Maxwell law in another way was that of Zartman[24] who studied the deposition of evaporated bismuth atoms on the inside of a revolving cylinder and obtained a spectrum in good agreement with Maxwell's law. Thus, experimentally, it is possible to assert that the high velocity side of the distribution law has been verified with good precision over a rather large range. This is not to ignore the more general Fermi-Dirac law of statistical mechanics but simply to imply that the emission based on Fermi-Dirac statistics reduces to the Maxwell law at high temperatures. At low temperatures the Maxwell law is not correct. Thus, in a sense, the Maxwell law is just a special case of the Fermi-Dirac distribution law. It is noted that the Maxwell law is an equilibrium condition only, and its form depends on this circumstance. Hence, in non-equilibrium processes, such as heat conduction,

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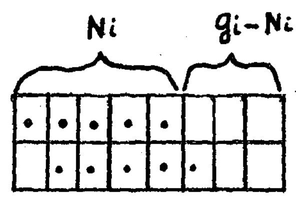
23. Germer, L.H., Phys. Rev., 25, 795, 1925
24. Zartman, I.F., Phys. Rev., 37, 383, 1931

diffusion, and viscosity, and in the case of ions or electrons in electrical fields, especially in discharges, the distribution is not Maxwellian[25]. It will now be of interest to see why the Maxwell distribution may be considered as a special case of the Fermi-Dirac statistics.

In brief the Fermi-Dirac statistics postulates the existence of different energy levels in phase space composed of little cells of h^3 in volume where h is Planck's constant. There are g_i cells and N_i particles. Each cell g_i may contain only one or zero particles according to the Pauli Exclusion Principle. Therefore, the total number of particles is less than the total number of cells. For example:

Energy level 2.
Energy level 1.



In this example:

$g_i = 8$
$N_i = 5$

In statistical mechanics there is a most useful concept called thermodynamic probability. The thermodynamic probability W of a given state is the number of different distributions which specify the state. Statistical mechanics assumes that the equilibrium state is the most probable state. Therefore, for the equilibrium condition, W or $\log W$ will be a maximum. The relationship between entropy S and thermodynamic probability W is: $S = K \log W$ where K is Boltzman's constant.

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25. Loeb, L. B., "Atomic Structure", Wiley and Sons, New York, 1938, p. 345

In the Fermi-Dirac statistics the total number distributions that can be achieved in general is:

$$W = \prod_i \frac{g_i!}{N_i!\,(g_i - N_i)!} \qquad \text{(summing over all states)}$$

$$\therefore \lg W = \sum_i \left[\lg g_i! - \lg N_i! - \lg (g_i - N_i)! \right]$$

Applying the "super" Sterling approximation for large numbers:

$$\lg N! \cong N \lg N - N$$

$$\lg W = \sum_i \left[g_i \lg g_i - g_i - N_i \lg N_i + N_i - (g_i - N_i)\lg(g_i - N_i) - g_i + N_i \right]$$

For a maximum, the first variation must vanish. Differentiating with respect to N_i :

$$\delta \lg W = \sum_i \delta N_i \left[-1 + \lg N_i + 1 + \lg(g_i - N_i) + 1 \right] = 0$$

Also, the following constraints on the system exist.

a) The total number of particles N_i is fixed:

$$N = \sum N_i$$

$$dN = \sum \delta N_i = 0$$

b) The total energy is fixed:

$$E = \sum E_i N_i$$

$$dE = \sum E_i \delta N_i = 0$$

In the equations below introduce one Lagrange multiplier for each constraint. Multiply the second equation by λ and the third by $-\beta$.

$$\sum_i \delta N_i \left[\lg N_i + 1 + \lg (g_i - N_i) \right] = 0$$

$$\lambda \sum_i \delta N_i = 0$$

$$-\beta \sum_i E_i \delta N_i = 0$$

adding: $\sum_i \delta N_i \left[\lambda + 1 - \beta E_i + \lg N_i + \lg(g_i - N_i) \right] = 0$

let $\lg A = \lambda + 1$

$$\lg A - \beta E_i + \lg N_i + \lg (g_i - N_i) = 0$$

$$\frac{A(g_i - N_i)}{N_i} = e^{\beta E_i}$$

$$\frac{g_i}{N_i} = 1 + \frac{e^{\beta E_i}}{A}$$

$$N_i = \frac{g_i}{\frac{1}{A}e^{\beta\epsilon_i}+1} = \frac{g_i}{\frac{1}{A}e^{\beta\epsilon_i}+1}$$ Fermi-Dirac Distribution Law

It can be shown that $\frac{1}{A} = e^{-\frac{\mu}{KT}}$ where: μ = the energy value of the highest occupied level in the metal.

At emission temperatures $E_i-\mu$ = Work function of the metal.

$$N_i = \frac{g_i}{e^{\frac{E_i-\mu}{KT}}+1} \quad \text{and} \quad \frac{E_i-\mu}{KT} \gg 1; \quad e^{\frac{E_i-\mu}{KT}} \gg 1$$

$$N_i = g_i\, e^{-\frac{E_i-\mu}{KT}}$$

This is the same form as the Maxwell Distribution.

B. Method of T. C. Fry

Determination of the reduced-shot effect equation in the space-charge-limited case began with the work of T. C. Fry[26]. He independently solved the problem of space-charge-density distribution, although P.S. Epstein of Germany had formulated a similar solution two years earlier in 1919. As pointed out before, many experimental and theoretical investigations have been devoted to the question of how the Schottky formula should be modified. One can say that the problem has now, except for the high frequency region, been solved at least in the most commonly used types of tubes. Solution of the problem where transit time is important has proven to be quite elusive. The transit time condition enormously complicates the problem when the traditional approach is employed. This section presents the method used and published by Fry in 1921 in setting up the space-

26. Fry, T. C., op. cit., p. 442

charge equations. As the complete derivation is very long and complicated, only the significant points will be presented.

Fry divided the space-charge region into two subregions alpha and beta as shown:

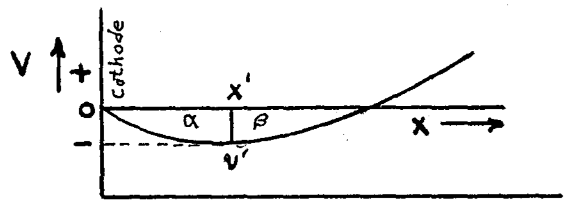

Fig. 3. Space-Charge Potential Vs. Distance

V = potential at any arbitrary distance X from the cathode.

V' = potential of the potential minimum or virtual cathode

X' = position of the potential minimum V'

v_0 = emission velocity of any particular electron

v' = emission velocity of any particular electron at X', V'

v_0' = critical emission velocity (to overcome)

v = electron velocity at any arbitrary distance

α = region between 0 and V'

β = region between V' and the anode

Let the Maxwell distribution function representing the number of electrons emitted per unit area with a particular velocity v_0 by symbolically represented by $n(v_0)$.

Then:

$$N_{\text{Total no/unit area}} = \int_0^\infty n(v_0)\, dv_0 \qquad \text{(integrating over all velocities)}$$

In a current where all the electrons travel at the same speed v, the space-charge is given by:

$$\rho = \frac{e\,[n(v_0)]}{v}$$

where: ρ = space-charge density
$n(v_0)$ = number of electrons emitted per unit time
e = electron charge

Hence, where velocities differ the space-charge equation is given by:

$$\rho = e\int \frac{n(v_0)}{v}\,dv_0$$

the integration being performed over all velocities which are of such magnitude that the electrons pass any particular point. If this point lies in the β region, all electrons which pass completely through it have emission velocities greater than $\sqrt{2V'\frac{e}{m}}$, that is, high enough to pass the region of adverse gradient of the potential minima. Hence, denoting this critical emission velocity by v_0'

$$\rho = \int_{v_0'}^{\infty} \frac{n(v_0)}{v}\,dv_0$$

This expresses the charge density in the β region. The space charge at any arbitrary point X (to the left of the virtual cathode) is obviously not so simply expressed, since it includes electrons which get thru the vertical cathode (β electrons), and those which are turned back, (α electrons). In the α region there are two unequally dense streams of electrons passing in opposite directions, each of which contribute to the space-charge.

If V is the potential at any point X in the α region, these electrons having a velocity less than $\sqrt{2V\frac{e}{m}}$ do not reach X, and those having a velocity greater than $\sqrt{2V'\frac{e}{m}}$ pass the V', virtual cathode, and do not return. Hence: $v_0' = \sqrt{2V'\frac{e}{m}}$

$$\rho_\alpha = 2e\int_{\sqrt{2V\frac{e}{m}}}^{\sqrt{2V'\frac{e}{m}}} \frac{n(v_0)\,dv_0}{v} = e\int_{\sqrt{2V'\frac{e}{m}}}^{\infty} \frac{n(v_0)\,dv_0}{v} \qquad (2)$$

These formulas assume that all electrons are shot out normally to the cathode. That is, v_0 is the normal component. The current to the cathode is:

$$i = e\int_{\sqrt{2V'\frac{e}{m}}}^{\infty} n(v_0)\,dv_0 \qquad (3)$$

The only other equations necessary to determine the solution of the problem are the equations of energy.

$$v^2 = v_0^2 - 2V\frac{e}{m} \qquad (4)$$

and Poisson's equation in one dimension:

$$\frac{\partial^2 V}{\partial x^2} = -\frac{4\pi\rho}{K} \quad \text{where: } K = \text{dielectric constant} \qquad (5)$$

<u>In the β region</u> substituting for ρ :

$$\frac{d^2V}{dx^2} = -\frac{4\pi e}{K}\int_{v_0'}^{\infty} \frac{n(v_0)\,dv_0}{v}$$

Then, to integrate the above equation multiply both sides by $2\frac{dV}{dx}$ and integrating the right hand sign under the integral sign where: $\frac{d}{dx}\left(\frac{dV}{dx}\right)^2 = 2\frac{dV}{dx}\frac{d^2V}{dx^2}$

$$\left(\frac{dV}{dx}\right)^2 = \frac{4\pi e}{K}\int_{v_0'}^{\infty} \frac{n(v_0)\,dv_0}{\sqrt{v_0^2 - 2V\frac{e}{m}}} \int 2\frac{dV}{dx}\,dx$$

$$\left(\frac{dV}{dX}\right)^2 = -\frac{8\pi e}{K}\int_{v_0'}^{\infty} n(v_0)\,dv_0 \int_{V'(x)}^{V(x)} \frac{dV}{\sqrt{v_0^2 - 2V\frac{e}{m}}}$$

$$\left(\frac{dV}{dx}\right)^2 = -\frac{8\pi e m}{e K}\int_{v_0'}^{\infty} n(v_0)\,dv_0 \sqrt{v_0^2 - 2V\frac{e}{m}}\,\Bigg|_{V'(x)=v'}^{V(x)=v}$$

$$\left(\frac{dV}{dX}\right)^2 = \frac{8\pi m}{K}\int_{v_0'}^{\infty} n(v_0)\,dv_0\, v\,\Bigg|_{v'}^{v} = \frac{8\pi m}{K}\int_{v_0'}^{\infty} n(v_0)\,dv_0 (v - v')$$

$$\left(\frac{dV}{dX}\right)^2 = \frac{8\pi m}{K}\int_{v_0'}^{\infty} n(v_0)(v - v')\,dv_0 \qquad (6)$$

This result applies to the β region only.

Using equations (2) (4) and (5) in exactly the same way a similar result can be obtained for the region provided that the limits of integration are properly dealt with when the order of integration is changed:

$$\left(\frac{dV}{dX}\right)^2 = \frac{8\pi m}{K}\left[\int_{v_0'}^{\infty} n(v_0)(v - v')\,dv_0 + 2\int_{\sqrt{2V\frac{e}{m}}}^{v_0'} v\, n(v_0)\,dv_0\right] \qquad (7)$$

It has been shown experimentally that for high temperatures the velocity distribution function for thermionic emission is essentially Maxwellian. That is:

$$n(v_0) = \frac{\pi}{2}\frac{N v_0}{\overline{v_0}^2}\, e^{-\left(\sqrt{\frac{\pi v_0}{2 \overline{v_0}}}\right)^2}$$

where: $\overline{v_0}$ = the average emission velocity.

$$\overline{v_0} = \frac{1}{N}\int_0^{\infty} n(v_0)\, v_0\,dv_0 \qquad (8)$$

It is somewhat simpler to use v instead of v_0 as the variable of integration in (6) and (7) when this is done and $n(v_0)$ is given the value of (8) then:

$$\left(\frac{dV}{dX}\right)^2 = \frac{4\pi N m}{K\,\bar{v}_0^2}\left[\int_0^\infty v^2 e^{-\frac{\pi}{4\bar{v}_0^2}\left(v^2+2V\frac{e}{m}\right)}dv - \int_{\sqrt{\frac{2e}{m}(V'-V)}}^\infty v v' e^{-\frac{\pi}{4\bar{v}_0^2}\left(v^2+2V\frac{e}{m}\right)}dv \pm \int_0^{\sqrt{\frac{2e}{m}(V'-V)}} v^2 e^{\frac{\pi}{4\bar{v}_0^2}\left(v^2+2V\frac{e}{m}\right)}dv\right] \qquad (9)$$

The upper sign is to be used in the alpha region and the lower in the Beta region. The integrals involved in the above expression may be expressed in known types, letting:

$$\xi = e^{-\frac{\pi e V'}{4\bar{v}_0^2 m}}\sqrt{\frac{2\pi^2 e^2 N}{k m \bar{v}_0^2}}\,(x-x')$$

$$\eta = \frac{\pi e}{2\bar{v}_0^2 m}(V-V')$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$$

then equation (9) may be expressed as:

$$\left(\frac{d\eta}{d\xi}\right)^2 = e^\eta - 1 \pm\left(e^\eta \operatorname{erf}\sqrt{\eta} - \frac{2}{\sqrt{\pi}}\sqrt{\eta}\right) \qquad (10)$$

or:

$$\xi = \int_0^\eta \frac{d\eta}{\sqrt{\phi(\eta)}}$$

Concerning these equations, Fry says:

> "It is the introduction of these quantities ξ and η that the generality of the solution which we obtain must be attributed. They are both dimentionless, but, as the defining equations show, they are proportional to distance and potential difference respectively, both being measured from the potential minimum. Hence, for any one state of the system the curve which represents η as a function of ξ will also represent V as a function of X."

The solution to the above equation is most expeditiously performed with the aid of calculating machines. Langmuir[27] has compiled tables for the general solution which enables a plot of the potential distribution at each point between the cathode and anode to be made.

The practical use of the noise equations by design engineers requires that the space-charge-reduction factor Γ^2 be determined for the different classes of tubes such as diodes, triodes, pentodes, etc. For any particular tube this is a complex matter, and only the method will be indicated here.

Assuming that the electrons are randomly and independently emitted, the average number emitted per second, p, determines the saturation current I_s. Suppose that in a time Δt more electrons than the average number $p\Delta t$ are emitted by the cathode and let this excess be called Δn[28]. Because of the existence of a potential minimum, the number of electrons delivered at the anode in a time of the order of the transit

27. Langmuir, I., op. cit., p. 423

28. Lawsen, J., and Uhlenbeck, G., op. cit., p. 85

time τ will not exceed the average number by the same amount Δn. The excess at the anode will actually be less because the Δn electrons will have slightly lowered the potential minimum, making it more negative. This will reduce the number of electrons reaching the anode. In the reverse process when fewer electrons are emitted, the space-charge potential becomes more positive and allows more electrons to pass to the anode. Consequently, the magnitude of the emitter fluctuations is reduced by the actions of the virtual cathode and Γ^2 will be less than unity. For precise determination of Γ^2 the velocity distribution of emitted electrons must be taken into account since the effect on the potential minimum will depend strongly on the velocity of the group under consideration. Also, it is necessary to distinguish between the reflected electrons in the alpha region and the electrons with sufficient energy to cross the potential minimum into the Beta region. Or, restated:

$$\Gamma^2 = \Gamma_\alpha^2 + \Gamma_\beta$$

Usually $$\Gamma_\alpha^2 << \Gamma_\beta^2$$

In the determination of Γ^2 an important simplifying assumption is that $\frac{I}{I_s} \simeq 0$. That is, the space-charge-limited anode current is very small compared to the temperature limited current. This condition is called the complete space charge limited case, and the resulting shot noise formulas are completely valid only under this condition. For other practical space-charge conditions they are in error by a very small per cent. For most space-charge conditions Γ^2 varies from about .04 to .06 [29].

In attempting to simplify analysis for practical purpose an attempt has been made to draw an analogy between shot noise in tubes and thermal noise. According to the well known Nyquist[30] formula:

$$\overline{I^2} = 4KTg\Delta F$$

Now, assuming that the shot noise of a space-charge-limited diode can be represented as thermal noise:

$$\overline{I^2} = \Theta 4KT_c g\Delta F \quad = \quad .644\ 4KT_c g\Delta F$$

where:

g = diode conductance

T_c = cathode temperature

Θ = constant of average value 0.644

Calculation shows that for most practical purposes:

$$\Theta \cong 3\left(1-\frac{\pi}{4}\right) = .644$$

Thus, in the words of North:

> "It may then be said that the mean-square noise current generated by emission fluctuations in a space-charge-limited diode is roughly numerically equal to two thirds of the noise of thermal agitation generated by a resistance of magnitude equal to the a-c resistance of a diode possessing a temperature equal to the cathode temperature."

Quoting North further:

> "Yet, the two phenomena must not be confused in concept. For thermal agitation is known

29. North, D.O., op. cit., p. 462

30. Nyquist, H., "Thermal Agitation of Electric Charge in Conductors," Phys. Rev., 32, No. 1., p. 110-113, July 1928.

> to be a form of Brownian movement, and finds its origin in the equiportion of energy among the various mechanical and electrical degrees of freedom of a substance in thermal equilibrium .... the mechanics of the two phenomena are, therefore, distinct; the formulas alone exhibit a resemblance."

In the retarded field case where the plate voltage is negative with respect to the cathode, Γ^2 has a value of unity as in the temperature-limited case, and $\theta = \frac{1}{2}$. When used in their respective formulas, they produce identical numerical results. The three conditions are summarized below by regions[31].

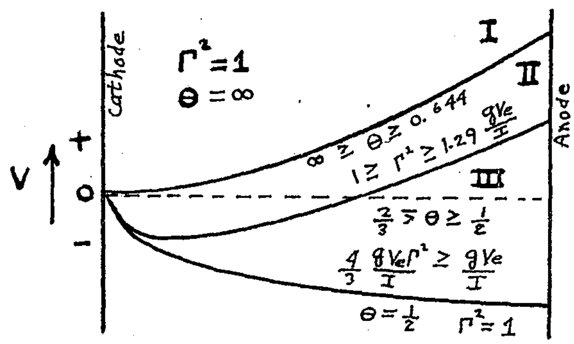

Fig. 4. Gamma and Theta For Various Space-Charge Regions

The triode in the temperature-limited case is no more complicated than the diode, and the Schottky formula applies. For the space-charge-limited case, the equation is similar in form to the diode:

$$\overline{I^2} = \frac{.644}{\beta} 4KT_c g_m \Delta F$$

where: g_m = the transconductance of the tube.

31. North, op. cit., p. 464

$$B = \left[1 + \frac{1}{\mu}\left(1 + \frac{4}{3}y\right)\right]$$

μ = electrostatic amplification factor

$$y = \frac{\text{grid-anode spacing}}{\text{gird-cathode spacing}}$$

B is usually between .5 and 1.0.

VI. SHOT NOISE IN MULTI-ELEMENT TUBES

It was pointed out in Section II that tetrodes and pentodes show a sharp increase in shot noise over diodes and triodes. This increase in noise in tetrodes was at first thought to be caused by secondary emission from the screen grid. However, tests with pentodes, which have greatly reduced secondary emission, still showed higher shot noise than diodes or triodes. Ultimately, a theory credited to D. O. North[32] shows that the excess noise is due to the division of tube current among the collector electrodes, when the tube is in the space-charge-limited condition.

The main gist of North's theory is that while in a space charge limited diode the potential variation of the virtual cathode is coherent with the change in emission current (this renders the shot-noise-reduction factor Γ), in the multicollector tube this coherence factor Γ is effectively divided among the several collector anodes present. Also, the amount of coherence varies from electrode to electrode due to geometrical and possibly electrical circuit conditions. The net effect of this condition is that the mean-square fluctuation current in each electrode is in excess of any estimate based upon a simple aportioning of the cathode current fluctuations. Goldman[33]

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32. North, D. O., "R.C.A. Review", Oct. 1940, p. 244
33. Goldman, S., op. cit., p. 365

has put the North derivation into somewhat neater form, and consequently, it will be presented instead.

Let: i_c = total instantaneous emission current

i_{cs} = those electrons lying within any particular velocity ranges

I_c = the average total emission current

I_{cs} = the average emission current of electrons with a particular velocity

then, summing over all velocity ranges:

$$i_c = \sum_s i_{cs}$$

also: $I_c = \sum_s I_{cs}$

We know that $\overline{I^2} = 2K\overline{I}\Delta F$ (Schottky)

Letting: $\Delta i_c = i_c - I_c$ and: $\Delta i_{cs} = i_{cs} - I_{cs}$

(representing a sudden fluctuation at the cathode)

$$\overline{(\Delta i_c)^2} = \sum_s \overline{(\Delta i_{cs})^2} = \sum_s 2K I_{cs}\Delta F = 2K I_c \Delta F$$

This represents the non-reduced or free shot effect, since its magnitude is determined solely by the random emission processes. Now, in the space-charge-limited condition:

Let: α = electrons with insufficient velocity to pass the virtual cathode

β = electrons with sufficient velocity to pass the virtual cathode

I_t = algebraic sum of the currents to all collector electrodes

I_c = total average emission current.

$$I_c = \sum_\alpha I_{cs} + \sum_\beta I_{cs}$$

But as the alpha electrons do not get to the collector electrodes,

$$I_t = \sum_\beta I_{cs}$$

It has been pointed out previously that any fluctuations in emission of electrons of any velocity class causes a minute change in the virtual cathode potential, which in turn causes a change in the anode current. Expressing this mathematically,

$$\Delta i_t = \sum_\beta \Delta i_{cs} + \sum_\beta \Delta i_{cs} b_s + \sum_\beta \Delta i_{cs} b_s$$

$$\Delta i_t = \sum_\beta \Delta i_{cs}(1+b_s) + \sum_\alpha \Delta i_{cs} b_s$$

where:

b_s = a function of s less than unity and negative

$\Delta i_{cs} b_s$ = change in the virtual cathode potential caused by Δi_{cs}.

If two or more random noise functions are superimposed, the ensemble averages of their squares are additive. (Theorem)

$$\overline{(\Delta i_t)^2} = \sum_\beta (1+b_s)^2 \overline{(\Delta i_{cs})^2} + \sum_\alpha b_s^2 \overline{(\Delta i_{cs})^2}$$

$$\overline{(\Delta i_t)^2} = \sum_\beta (1+b_s)^2 2KI_{cs}\Delta F + \sum_\alpha b_s^2 2KI_{cs}\Delta F$$

again: $\Sigma(\Delta i_t)^2 = \Gamma^2 2KI_t \Delta F$

$$\Gamma^2 = \frac{\overline{(\Delta i_t)^2}}{2KI_c\Delta F} = \frac{\sum_\beta (1+b_s)^2 2KI_{cs}\Delta F + \sum_\alpha b_s^2 2KI_{cs}\Delta F}{2KI_t\Delta F}$$

$$\Gamma^2 = \frac{\sum_\beta (1+b_s)^2 I_{cs} + \sum_\alpha b_s^2 I_{cs}}{I_t}$$

To find Γ^2, b_s must be evaluated, and this has been done by North.

Consider any particular electrode n :

Let: i_n = instantaneous current to n^{th} electrode

I_n = average current to n^{th} electrode

In most multi-element tubes the currents to the different electrodes are practically superimposed. A shift in the potential of the virtual cathode due to current in the nth electrode will consequently change the current to each other electrode by the same amount as an equal shift caused by a current fluctuation in any other electrode. Consequently, the fluctuation current to the nth electrode is:

$$\Delta i_n = \sum_\beta \left(1 - \frac{I_n}{I_t} b_s\right) \Delta i_{cns} + \sum_\beta \frac{I_n}{I_t} b_s \Delta i_{c(-n)s} + \sum_\alpha \frac{I_n}{I_t} b_s \Delta i_{cs}$$

where: Δi_{cns} = the fluctuation current that would go to the n th electrode in the absence of space charge.

$\Delta i_{cns} \frac{I_n}{I_t} b_s$ = the fluctuation current that must be subtracted due to space-charge coherence.

$\frac{I_n}{I_t} b_s \Delta i_{c(-n)s}$ = coherence reduction due to all the other electrodes in the Beta region.

$\frac{I_n}{I_t} b_s \Delta i_{cs}$ = coherence reduction in the alpha region.

Therefore:

$$\overline{(\Delta i_n)^2} = \sum_\beta \left(1 + \frac{I_n}{I_t} b_s\right)^2 \overline{(\Delta i_{cns})^2} + \sum_\beta \overline{(\Delta i_{c(-n)s})^2} + \sum_\alpha \left(\frac{I_n}{I_t} b_s\right) \overline{(\Delta i_{cs})^2}$$

$$\overline{(\Delta i_n)^2} = \sum_\beta \left(1 + \frac{I_n}{I_t} b_s\right)^2 2K \frac{I_n}{I_t} I_{cs} \Delta F + \sum_\beta \left(\frac{I_n}{I_t} b_s\right)^2 2K \left(\frac{I_t - I_n}{I_t}\right) I_{cs} \Delta F$$

$$+ \sum_\alpha \left(\frac{I_n}{I_t} b_s\right)^2 2K I_{cs} \Delta F$$

substituting for Γ^2 in equation:

$$\overline{(\Delta i_n)^2} = \left[1 - \frac{I_n}{I_t}\left(1 - \Gamma^2\right)\right] I_n 2K \Delta F$$

letting $\Gamma_n^2 = 1 - \frac{I_n}{I_t}\left(1-\Gamma^2\right)$

$$\overline{(\Delta i_n)^2} = \Gamma_n^2 \, 2KI_n\,\Delta F$$

As a general interpretation of the above results, it may be said the division of space current between different electrodes decreases the amount by which the space charge reduces the shot noise. The principle reason for this is that in this case only part of the coherent currents due to the motion or change in potential of the virtual cathode, which reduces shot effects in diodes and triode, will now go to the electrodes with the current of which they are coherent. The above equations do not apply to tubes with aligned grids such as the beam power tube types, as the current streams in this type of tube are not essentially superimposed.

North[34] gives six important conclusions to the above equations.

1. No fluctuation current is greater than the true shot effect for the current considered.
2. The smaller the fraction of the current that an electrode collects the more nearly the noise in that current approaches true shot effect.
3. For any vanishingly small Γ the mean square fluctuation in the current collected at any electrode is equal to the product of the free shot effect for säid current and the fraction of the total current not collected at said electrode.
4. The ratio of actual noise to free shot effect in

34. North, D. O., op. cit., p. 249

a divided portion of the total current exceeds the corresponding ratio for the total current itself.

5. The noise in the current of an electrode exceeds the noise in the total current provided:

$$\frac{\Gamma^2}{1-\Gamma^2} < \frac{I_n}{I_t} < 1$$

In conventional tubes this is usually true for collector electrodes.

6. With constant Γ, the noise in a given collector electrode current is a maximum (against variations in I_n) when:

$$\frac{I_n}{I_t} = \frac{1}{2(1-\Gamma^2)}$$

In other words, provided $\Gamma^2 < \frac{1}{2}$, the noise in no collector lead should exceed:

$$\frac{1}{4(1-\Gamma^2)} I_t 2K\Delta F$$

VI. CONCLUSION

It has been shown that the basic problem in the determination of reduced-vacuum tube noise lies in an analysis of the effect of the space charge or virtual cathode on the inherent fluctuations in the tube current. This effect has been shown to be best understood in terms of the principle of coherence between the compensating actions of the virtual cathode and the fluctuation noise current. The diode space-charge-limited noise-reduction factor Γ^2 thus is shown to be a measure of the coherence effect. When the space charge intensity had been determined by the methods of T. C. Fry, it was then possible to proceed to a determination of Γ^2 by the technique of O. D. North and others. For multi-element tubes, the coherence principle had to be modified to take into account the division of current amongst the various collector electrodes. This division results in a lower coherence factor than in diodes. Consequently, it was shown that this new factor may be calculated and has been found to be in agreement with experiment.