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Non-perturbative topological recursion and knot invariants

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NON-PERTURBATIVE TOPOLOGICAL RECURSION AND KNOT INVARIANTS

by

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ABSTRACT

The goal of the present thesis is to study new examples, applications and computational aspects of the topological recursion formalism introduced by Eynard and Orantin. We develop efficient methods for the calculation of non-perturbative wave functions associated to spectral curves of genus one. Our results are used to test two conjectures. The first one relates perturbative knot invariants obtained from the AJ Conjecture and a state integral model to the wave function obtained from topological recursion. The second conjecture describes the structure of the quantum curve for the Weierstrass spectral curve. We are able to verify the conjectures up to some order in a formal parameter h and we state a stronger version of the conjecture in the case of the Weierstrass curve. Some of these results are based upon joint work with Greyson Potter.

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List of Abbreviations

$\omega_{0,2}$	. . . .	Fundamental bidifferential
η	. . . .	Holomorphic one-form
$\mathcal{A}, \mathcal{B}$	. . . .	Homology cycles
$\langle \tau_{a_1}^{m_1} \cdots \tau_{a_n}^{m_n} \rangle_{g,n}^{1/r}$	. . . .	Intersection number of r-spin curves
$J_K(N, q)$	. . . .	Jones polynomial of the knot K
K	. . . .	Knot or link
Λ	. . . .	Lattice in $\mathbb{CP}^n$
$\overline{\mathcal{M}}_{g,n}^{1/r}$	. . . .	Moduli space of curves with r-spin structure
Σ	. . . .	Plane curve
$\mathrm{SL}(n, \mathbb{C})$	. . . .	Special linear group of dimension n over $\mathbb{C}$
$\mathcal{S}$	. . . .	Spectral curve
$\omega_{g,n}$	. . . .	Topological recursion correlator
U	. . . .	Unknot
$\wp, \wp', \zeta$	. . . .	Weierstrass elliptic functions

Chapter 1

Introduction

The nineties saw a surge in the study of quantum field theory and its connection to enumerative geometry and knot theory. Since then, significant progress has been made which frames the current understandings between these fields. Some notable examples are two-dimensional topological gravity [Wit91] and three-dimensional Chern-Simons gauge theory [Wit89]. Roughly speaking, in quantum field theory the properties of the particles should be understood through suitable counts of their possible trajectories. In mathematics, this is embodied in the notion of moduli spaces. Regarding quantum gravity, Witten and Kontsevich formulated a rigorous approach to the moduli space of marked Riemann surfaces using random matrix theory [Kon92]. Over a decade later, the study of said matrix models led Eynard and Orantin to the discovery of a more general approach, coining for it the term *topological recursion* [EO07]. To any given spectral curve, topological recursion associates a family of *correlators* $\{\omega_{g,n}\}_{2g-2+n>0}$ that are defined recursively. Since this discovery, many examples of spectral curves and their enumerative counterparts have been found. Topological recursion can compute Kontsevich-Witten intersection numbers [Wit91], Brezin-Gross-Witten theory [DN16], Weil-Peterson volumes [Mir06], Hurwitz numbers [BM08, EMS11, ACEH18, ACEH20] and Gromov-Witten invariants [BKMP08, FLZ16]. More generally, it can be used to describe the action of the Givental-Teleman formalism on cohomological field theories [EO07, DBOSS14, Mil14]. The theory of topological recursion has led to many generalizations, including blobbed topological recursion [BS17], geometric recursion [ABO17] and singular topological recursion [BKS20]. Furthermore, when seen through the lens of matrix models, it is expected that topological recursion reconstructs an asymptotic solution to certain ordinary differential equations known as quantum curves [BE09a, BE09b].

The connection between spectral curves, quantum curves and enumerative invariants can be illustrated through the prototypical example of the Airy spectral curve

$$\mathcal{S} = \left(\mathbb{CP}^1, x(z) = \frac{z^2}{2}, y(z) = -z, \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right). \tag{1.0.1}$$

In this case, the generating function Z obtained from topological recursion is equal to the generating function of Kontsevich-Witten intersection numbers. Kontsevich proved that Z is a τ-function of the KdV hierarchy [Kon92] and showed its relation to the Airy function ψ, which satisfies the differential equation

$$\left(\frac{h^2}{2}\frac{d^2}{dx^2} - x\right) \cdot \psi = 0. \tag{1.0.2}$$

Although seemingly unrelated, the differential operator in (1.0.2) can be obtained as a quantization of the spectral curve (1.0.1). This is done by promoting the variables x and y to the corresponding differential operators x and $h\frac{d}{dx}$:

$$\frac{y^2}{2} - x \longmapsto \frac{h^2}{2}\frac{d^2}{dx^2} - x. \tag{1.0.3}$$

The operator (1.0.3) is said to be a quantum curve for the spectral curve (1.0.1). This explanatory example has given rise to several more profound questions:

- Does any given spectral curve have an enumerative geometric significance?
- Does any enumerative problem have a corresponding spectral curve?
- For a given spectral curve, what is the correct notion of the corresponding

wave function?

- Can the topological recursion formula be restated in the form of a differential operator annihilating the wave function?
- If so, can such operator be obtained after a certain quantization of the underlying spectral curve?

The past decade has seen significant advances in all of these directions. Of particular interest to us is the following result by Bouchard and Eynard [BE17]: for any given spectral curve of genus zero, the quantum curve annihilating the perturbative wave function ψ_{P} is obtained from a naive quantization of the underlying spectral curve. For curves of higher genus, unfortunately, this fails to be true. A compelling reason was given by Eynard and Marino [EM11], arguing that ψ_{P} does not satisfy certain properties expected from a wave function of a quantum field theory. The authors proposed that the correct wave function is the so-called non-perturbative wave function ψ_{NP} instead – constructed from the perturbative counterpart by adding corrections in the form of theta functions.

The non-perturbative wave function was utilized by Borot and Eynard in [BE12] to reformulate a conjectural application of topological recursion to perturbative knot invariants first introduced by Dikgraaf, Fuji and Manabe [DFM11]: topological recursion for the A-polynomial of a knot produces a formal series that is closely related to an asymptotic series obtained from the N-colored Jones polynomial for large values of N. To date, this conjecture has only been tested on two knots, both of which have associated spectral curves of genus one. Similar correspondences between topological string theory and Chern-Simons theory appear as well in the context of the Volume Conjecture [MM01, MMO+02] and the AJ Conjecture [Gar04, GL05].

The **main goal** of the present thesis is to develop new methods for the computation of non-perturbative wave-functions and quantum curves. In particular, we provide an efficient graphical computation of topological recursion, and give a systematic approach to compute non-perturbative wave-functions for spectral curves of genus one. We use our results to verify two conjectures in the literature, up to a finite order in a formal parameter h. The first conjecture is the aforementioned conjecture in [BE12] relating topological recursion and knot invariants. The second was stated by Bouchard, Chidambaram and Dauphinee in [BCD18]. It predicts the existence and properties satisfied by the quantum curve of the Weierstrass spectral curve. Based upon our calculations, in this second case we propose a conjectural closed form of the quantum curve and series expansion. Our methods rely heavily upon a graphical reformulation of the original problem. Although graph enumeration algorithms have high complexity – the growth is at least exponential with the genera and number of leaves – this novel approach significantly outperforms previous calculations in the literature.

The text is structured as follows: In Chapter 2 we review the topological recursion formalism, introducing our new interpretation using two sets of decorated graphs. Chapter 3 is dedicated exclusively to knot invariants; we discuss the relevant aspects building up to the conjectural relation to topological recursion. In Chapter 4 we lay out the details of our algorithm and the computational results. Finally, we state and prove the main theorems.

Chapter 2

Topological Recursion and Quantum Curves

The topological recursion formalism of Eynard and Orantin [CEO06, EO07] has been a prolific field of study in the recent years and has proven to have deep connections to many enumerative geometric problems. Some examples are Weil-Peterson volumes [Mir06], Hurwitz numbers [BM08, EMS11] and Gromov-Witten theory [BKMP08].

In this chapter we introduce the notions of topological recursion and quantum curves, focusing upon the results that are most relevant for our calculations in Chapter 4. We start by recalling the original recursive definition, as well as the newer formalism in terms of differential operators. We then interpret the latter as a sum over decorated graphs, an essential step that will allow us to perform effective computations. Next, we discuss the quantum curve conjecture, commenting first the perturbative case for curves of genus zero, and then centering our attention on the non-perturbative setting for curves of higher genus. Finally, we steer our discussion toward the case of spectral curves of genus one, and describe the computation of the fundamental objects required for the formalism. Throughout the discussion we will use a variable z to denote any local coordinate on a complex curve Σ.

2.1 Topological Recursion

Topological recursion associates to a given spectral curve a family of so-called correlators $\{\omega_{g,n}\}_{2g-2+n>0}$. A spectral curve

$$\mathcal{S} = (\Sigma, x, y, \omega_{0,2}) \tag{2.1.1}$$

is a tuple consisting of a smooth complex curve Σ, together with two meromorphic functions $x\colon \Sigma \to \mathbb{CP}^1$ and $y\colon \Sigma \to \mathbb{CP}^1$ and a fundamental bidifferential $\omega_{0,2}$. We denote by $\mathcal{R} = \{p_\alpha\}_{\alpha=1,\ldots,k}$ the set of ramification points of x. At this stage, we do not require the curve Σ to be connected nor compact; in particular, Σ can be a union of connected open sets around the ramification points of the map $x\colon \Sigma \to \mathbb{CP}^1$.

Definition 2.1.1 (Fundamental bidifferential)**.** A *fundamental bidifferential* $\omega_{0,2}$ is an element of $\Gamma(\Sigma \times \Sigma \setminus \Delta, T^*\Sigma \otimes T^*\Sigma)$ satisfying the following conditions:

- Symmetry: $\omega_{0,2}(z_1, z_2) = \omega_{0,2}(z_2, z_1)$
- It is holomorphic everywhere except in the diagonal $\Delta \subset \Sigma \times \Sigma$
- Along the diagonal, it has the following series expansion as $z_1 \to z_2$:

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{holomorphic part}$$

Many interesting examples of spectral curves have a global nature. We will refer to $\mathcal{S}$ as a *global spectral curve* in the cases where Σ is a compact Riemann surface. It is then convenient to choose a symplectic basis $\{\mathcal{A}_i, \mathcal{B}_i\}_{1\le i\le g}$ of $H_1(\Sigma, \mathbb{Z})$, which in turn determines the corresponding basis of holomorphic one-forms $\{da_i\}_{1\le i\le g}$ dual to $\{\mathcal{A}_i\}_{1\le i\le g}$. The non-negative integer g denotes the genus of the curve. A fundamental bidifferential $\omega_{0,2}$ is said to be *normalized* on a choice of a symplectic basis if

$$\oint_{\mathcal{A}_i} \omega_{0,2}(z_2, \cdot) \equiv 0, \forall i = 1, \ldots, g.$$

Fundamental normalized bidifferentials are unique and can be constructed explicitly by using theta functions. We discuss further properties later on in sections (2.3.2) and (2.5.3).

Topological Recursion: Residues

The topological recursion formula was introduced by B. Eynard and N. Orantin in [EO07] by drawing inspiration from spectral curves coming from matrix models. Initially it was valid only for spectral curves with simple ramification points of the map $x\colon \Sigma \to \mathbb{CP}^1$, and was later generalized to spectral curves with arbitrary ramifications [BHL+13, BE13]. For simplicity, here we only provide the recursive formula for simple ramification points, and we discuss the general case later on. From the data of the spectral curve (2.1.1), near each ramification point p_α one defines the recursion kernel as

$$K(z_1, z) = -\frac{\int_o^z \omega_{0,2}(z_1, \cdot)}{(y(z) - y(-z))dx(z)}, \tag{2.1.2}$$

where $o \in \Sigma$ denotes a choice of a generic base point in a small punctured neighborhood of p_α. Although K depends a priori on the base point o, it can be shown that the definition below is independent of that choice. The recursion kernel K should be regarded as a local section of $T\Sigma \otimes T^*\Sigma$.

In a neighborhood of each simple ramification point p_α, denote by $z \mapsto \overline{z}$ the local involution interchanging the two branches of the projection $x\colon \Sigma \to \mathbb{CP}^1$. In particular, it must satisfy $x(z) = x(\overline{z})$.

Definition 2.1.2 (Topological recursion)**.** Consider a spectral curve $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$. For any pair of non-negative integers (g, n) satisfying $\chi = 2g-2+n > 0$ and $n > 0$, the *topological recursion correlator* $\omega_{g,n}$ is a $(1, \ldots, 1)$ form on Σ^n, that is, a meromorphic section of $T^*\Sigma^{\otimes^n}$. It is defined recursively via the following formula:

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{p_\alpha\in\mathcal{R}} \operatorname*{Res}_{z=p_\alpha} K(z_1,z) \left(\omega_{g-1,n+1}(z,\overline{z},z_2,\ldots,z_n) + \sum_{\substack{g_1+g_2=g \\ I_1\amalg I_2=\{z_2,\ldots,z_n\}}} \omega_{g_1,|I_1|+1}(z,z_{I_1})\omega_{g_2,|I_2|+1}(\overline{z},z_{I_2})\right). \quad (2.1.3)$$

Each $\omega_{g,n}$ is a meromorphic symmetric form with poles only at the ramification points [Eyn11]. Since the terms in the right-hand side have Euler characteristic $\chi-1$, this formula allows one to compute the infinite sequence $\{\omega_{g,n}\}$ from the base case $\omega_{0,2}$. The $\{\omega_{g,n}\}$ correlators are often referred to in the literature as *topological recursion invariants*, as for specific choices of spectral curves they encode well-known enumerative invariants:

- Witten's r-spin intersection numbers (r-th KdV hierarchy):

$$\left(\mathbb{CP}^1, x(z) = \frac{z^r}{r}, y(z) = -z, \omega_{0,2}(z_1,z_2) = \frac{dz_1dz_2}{(z_1-z_2)^2}\right)$$

- Weil-Peterson volumes:

$$\left(\mathbb{CP}^1, x(z) = z^2, y(z) = \frac{1}{4\pi}\sin(2\pi z), \omega_{0,2}(z_1,z_2) = \frac{dz_1dz_2}{(z_1-z_2)^2}\right)$$

- Hurwitz Numbers:

$$\left(\mathbb{CP}^1, x(z) = -z+\ln(z), y(z) = z, \omega_{0,2}(z_1,z_2) = \frac{dz_1dz_2}{(z_1-z_2)^2}\right)$$

- Gromov-Witten invariants of a Calabi-Yau three fold $\mathfrak{X}$ [BKMP08]:

$$\left(\{(x,y)\in\mathbb{C}^*\times\mathbb{C}^* \mid H(x,y)=0\}, \ln(x), \ln(y), \omega_{0,2}\right),$$

  where $H(X,Y)$ denotes the mirror curve $\check{\mathfrak{X}}$ of $\mathfrak{X}$ and $\omega_{0,2}$ a corresponding normalized fundamental bidifferential.

Topological Recursion: Differential Operator

The residue formulation of topological recursion (2.1.3) fits into a more general setting in terms of Virasoro-like constraints. This was discovered first through the notion of *quantum Airy structures* for simple ramification [KS17] and later generalized to *higher Airy structures* for arbitrary ramification [BBC+19]. In both cases, the starting data is a set of *local constraints* given as differential equations, which can be shown to have a unique solution. The data of a spectral curve determines a unique set of such constraints, and the corresponding solution allows us to recover the correlators $\{\omega_{g,n}\}$. We start by recalling the main constructions and results by Borot, Bouchard, Chidambaram, Creutzig and Noshchenko in [BBC+19].

Let $\mathcal{S}$ be a spectral curve with arbitrary ramification points of $x\colon \Sigma \to \mathbb{CP}^1$. Near each ramification point $p_\alpha \in \mathcal{R}$ of order r_α, define a local coordinate ζ satisfying

$$x(p) - x(p_\alpha) = \zeta^{r_\alpha}/r_\alpha. \tag{2.1.4}$$

Note that the equation above uniquely determines ζ up to an r_α-th root of unity. The $\mathbb{C}$-valued tensors $F_{0,1}$ and ϕ are the Taylor coefficients of the following power series expansions:

$$y = \sum_{\ell \geq r_\alpha} F_{0,1}\big[\begin{smallmatrix}\alpha \\ -\ell\end{smallmatrix}\big]\zeta^{\ell - r_\alpha} \qquad \text{as} \qquad z \to z(p_\alpha) \tag{2.1.5}$$

and

$$\omega_{0,2} = \left(\frac{\delta_{\alpha_1,\alpha_2}}{(\zeta_1 - \zeta_2)^2} + \sum_{\ell_1,\ell_2>0} \phi^{\alpha_1,\alpha_2}_{\ell_1,\ell_2}\zeta_1^{\ell_1-1}\zeta_2^{\ell_2-1}\right)d\zeta_1 d\zeta_2 \qquad \text{as} \qquad z_i \to z(p_{\alpha_i}), \tag{2.1.6}$$

where it is assumed that the coefficient $F_{0,1}\Big[\begin{smallmatrix}\alpha \\ -r_\alpha - 1\end{smallmatrix}\Big]$ is non-zero. From this local data one defines the differential operator

$$D = \sum_{\alpha,\ell} \frac{F_{0,1}[\,{}^{\alpha}_{-\ell}\,] + \delta_{\ell,s_\alpha}}{\ell} \partial_{x^\alpha_\ell} + \frac{\hbar}{2} \sum_{\substack{\alpha_1,\alpha_2 \\ \ell_1,\ell_2}} \frac{\phi^{\alpha_1,\alpha_2}_{\ell_1,\ell_2}}{\ell_1 \ell_2} \partial_{x^{\alpha_1}_{\ell_1}} \partial_{x^{\alpha_2}_{\ell_2}} \tag{2.1.7}$$

and the following generating function:

Definition 2.1.3 (Perturbative generating function)**.** The *perturbative generating function Z_{P}* is defined as

$$Z_{\mathrm{P}}[x^\alpha_\ell, \hbar] := e^D \prod_{\alpha \in \mathcal{R}} Z^{(r_\alpha)} \left[x^\alpha_\ell, \hbar \right], \tag{2.1.8}$$

where $Z^{(r_\alpha)} = e^{F^{(r_\alpha)}}$ denotes the unique solution to certain $\mathcal{W}$ algebra constraints:

$$W^i_{\alpha,k} \cdot Z^{(r_\alpha)} = 0, \qquad F^{(r_\alpha)}[x^r_\ell, \hbar] =: \sum_{\substack{g \geq 0, n \geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\ell_1,\dots,\ell_n} F^{(r_\alpha)}_g[\,\ell_1 \,\dots\, \ell_n\,] \prod_{j=1}^n x^r_{\ell_i}. \tag{2.1.9}$$

These $\mathcal{W}$ algebra constraints are defined by a set of differential operators $\{W^i_{\alpha,k}\}$ of degree at most r_α in the $\{x^\alpha_\ell\}$ variables and representing the $\mathcal{W}(\mathfrak{gl}_{r_\alpha})$ algebra. The explicit form of the W's is rather complicated to write, and hence we only give the expressions for $r_\alpha = 2$ in (2.1.13), referring the reader to [BBC+19] for the more general case.

Equation 2.1.8 yields an infinite series in both $\hbar$ and $\{x^\alpha_\ell\}$. Let $F_{\mathrm{P}}[x^\alpha_\ell, \hbar]$ be the exponent of $Z_{\mathrm{P}}[x^\alpha_\ell, \hbar] = \exp\left(F_{\mathrm{P}}[x^\alpha_\ell, \hbar]\right)$. The *free energy* functions $F_{g,n}[x^\alpha_\ell]$ are homogeneous polynomials of degree n in the $\{x^\alpha_\ell\}$ variables, and together with the Taylor series coefficients $F_{g,n}\left[\begin{smallmatrix} \alpha_1 & \dots & \alpha_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix}\right] \in \mathbb{C}$ are defined as follows:

$$\begin{aligned} F_{\mathrm{P}}[x^\alpha_\ell, \hbar] &=: \sum_{\substack{g \geq 0, n \geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\substack{\alpha_1,\dots,\alpha_n \\ \ell_1,\dots,\ell_n}} F_{g,n}\left[\begin{smallmatrix} \alpha_1 & \dots & \alpha_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix}\right] \prod_{j=1}^n x^{\alpha_j}_{\ell_j} \\ &=: \sum_{\substack{g \geq 0, n \geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} F_{g,n}[x^\alpha_\ell]. \end{aligned} \tag{2.1.10}$$

The main result in [BBC+19] is the culmination point of this discussion, making a precise connection between the generating function (2.1.8) and the topological recursion formula (2.1.3). The one-forms $d\xi_\ell^\alpha$ are defined as

$$d\xi_\ell^\alpha(z) := \operatorname*{Res}_{z'=p_\alpha}\left(\int_\alpha^{z'} \omega_{0,2}(\cdot, z)\right)\frac{d\zeta(z')}{\zeta(z')^{\ell+1}}. \tag{2.1.11}$$

They are meromorphic, with a pole of order $\ell + 1$ at p_α and can be regarded as the building blocks for the correlators. Indeed, the $\{\omega_{g,n}\}$ can be written in a simple way in terms of the one-forms $\{d\xi_\ell^\alpha\}$ and the coefficients $F_{g,n}\left[\begin{smallmatrix}\alpha_1 & \dots & \alpha_n \\ \ell_1 & \dots & \ell_n\end{smallmatrix}\right]$:

Theorem 2.1.4 (Borot et al.)**.** *The correlators $\{\omega_{g,n}\}_{2g-2+n>0}$ computed by topological recursion can be decomposed as finite sums*

$$\omega_{g,n}\left(z_1, \dots, z_n\right) = \sum_{\substack{\alpha_1,\dots,\alpha_n \\ \ell_1,\dots,\ell_n>0}} F_{g,n}\left[\begin{smallmatrix}\alpha_1 & \dots & \alpha_n \\ \ell_1 & \dots & \ell_n\end{smallmatrix}\right] \bigotimes_{j=1}^{n} \mathrm{d}\xi_{\ell_j}^{\alpha_j}\left(z_j\right). \tag{2.1.12}$$

Symmetry of the tensor $F_{g,n}$ implies the symmetry of $\omega_{g,n}$.

In the original paper [BBC+19], Bouchard et al. consider a broader class of ramification points labeled by a pair (r, s), where s is the index of the first non-zero coefficient $F_{0,1}[\begin{smallmatrix}\alpha \\ -s\end{smallmatrix}]$. In fact, we should stress that the generating function Z_{P} if well-defined only if $r\mathrm{mod}(s) = \pm 1$. Since all the examples in this thesis are of the type $s = r + 1$, we will limit our discussions to this case. We choose to simplify the notation by remembering only the value of r.

From a computational standpoint, Equation (2.1.8) is extremely useful, since it allows to pre-compute the building blocks $Z^{(r)}$ once and for all for each type of ramification. Additionally, for spectral curves with an equal number of ramification points of each order, one can compute the correlators $\{\omega_{g,n}\}$ leaving the coefficients $F_{0,1}$ and ϕ indetermined and later plugging in for the specific values.

In particular, it follows from this discussion that in the case where the spectral curve is the r-Airy spectral curve

$$\left(\mathbb{CP}^1, x = \frac{z^r}{r}, y = -z, \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right),$$

the generating function $Z_{\mathrm{P}}[x_\ell, \hbar]$ coincides with the building block $Z^{(r)}[x_\ell, \hbar]$ after setting $F_{0,1}[\,-\ell\,] = -\delta_{\ell,r+1}$ and $\phi_{\ell_1,\ell_2} \equiv 0$ accordingly, which gives $D \equiv 0$.

$\mathcal{W}$ algebras, Virasoro relations and r-spin Curves

The building blocks (2.1.9) yield the generating function of r-spin intersection numbers (2.1.14), and hence a τ-function of the r-th KdV hierarchy. They are determined by the $\mathcal{W}(\mathfrak{gl}_r)$ algebra constraints with corresponding operators $\{W^i_{\alpha,k}\}$. In the case $r = 2$, this is equivalent to the famous Kontsevich-Witten theorem, and the differential operators $\{W^i_{\alpha,k}\}$ are proportional to the well-known Virasoro representation operators. More concretely, for $k \geq 0$ we have that $W^1_{2,k} = \hbar\partial_{2k}$ and $W^2_{2,k} = 2L_{k-1}$, where

$$\begin{aligned} L_{-1} &= -\frac{x_1^2}{4\hbar} + \frac{1}{2}\sum_{p_1 \geq 1} (-1)^{p_1} (p_1 + 2) x_{p_1+2} \partial_{p_1} + \frac{\partial_1}{2}, \\ L_0 &= \frac{1}{2}\sum_{p_1 \geq 1} (-1)^{p_1} p_1 x_{p_1} \partial_{p_1} - \frac{1}{16} + \frac{\partial_3}{2}, \\ L_{k\geq 1} &= \frac{\hbar}{4}\sum_{\substack{p_1+p_2=2(k-1) \\ p_1,p_2\geq 0}} (-1)^{p_1} \partial_{p_1}\partial_{p_2} + \frac{1}{2}\sum_{\substack{p_1-p_2=2(k-1) \\ p_1,p_2\geq 0}} (-1)^{p_1} p_2 x_{p_2} \partial_{p_1} + \frac{\partial_{2k+1}}{2}. \end{aligned} \tag{2.1.13}$$

One can check that these operators satisfy the Virasoro relations

$$[L_n, L_m] = (m - n) L_{m+n}.$$

A beautiful consequence of (2.1.13) is that the intersection numbers defined below in (2.1.14) are rational numbers. We will refer back to this discussion in the proof of Theorem 2.2.3.

In the case where the index of ramification r is greater than two, the solution $Z^{(r)}\left[x_\ell^\alpha, \hbar\right]$ agrees with the more general generating function of r-spin intersection numbers [Wit93, JKV01]:

$$Z^{(r)}\left[t_m^a, \hbar\right] = \exp\left(\sum_{\substack{g\geq 0, n>0\\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\substack{a_1,\ldots,a_n\\ m_1,\ldots,m_n}} \langle \tau_{a_1}^{m_1} \cdots \tau_{a_n}^{m_n}\rangle_{g,n}^{1/r} \prod_{i=1}^n t_{m_i}^{a_i}\right),$$

which is often written in terms of the scaled variables $t_a^m = (ra+m+1)!^{(r)} x_{ra+m+1}$.

Roughly speaking, an r-spin structure on a smooth curve Σ with n marked points $x_1, \ldots, x_n$ is the data of a line bundle $\mathcal{L}$ together with an isomorphism

$$\mathcal{L}^{\otimes r} \to K\left(-\sum_{i=1}^n m_i x_i\right),$$

where K is the cotangent line bundle of Σ. The integers $m_i \in \{0, \ldots r-2\}$ are chosen so that $2g - 2 - \sum m_i$ is divisible by r. The space of r-spin structures has a natural compactification $\overline{\mathcal{M}}^{1/r}_{g;m_1,\ldots,m_n}$ and a cohomology class $c_W(m_1, \ldots, m_n)$ called the Witten's r-spin class. Consider the cotangent space $T^*\Sigma\mid_{x_i}$ at the marked point x_i. As $(\Sigma; x_1, \ldots, x_n)$ varies in $\overline{\mathcal{M}}^{1/r}_{g;m_1,\ldots,m_n}$ it defines a complex line bundle L_i. Let $\psi_i = c(L_i)$ be its first Chern class.

Definition 2.1.5 (Intersection numbers)**.** The *r-spin intersection numbers* are defined as the following integrals:

$$\langle \tau_{a_1}^{m_1} \cdots \tau_{a_n}^{m_n}\rangle_g^{1/r} = \frac{1}{r^g} \int_{\overline{\mathcal{M}}^{1/r}_{g;m_1,\ldots,m_n}} \prod_{i=1}^n \psi_i^{a_i} c_W(m_1, \ldots, m_n). \tag{2.1.14}$$

Only for certain values of $(a_1, m_1), \ldots, (a_n, m_n)$ will the cohomology degree agree with the dimension of the space. More concretely, the numbers (2.1.14) are set to 0 unless the following dimensional condition is satisfied:

$$\sum_{i=1}^{n}\left(a_i + \frac{m_i}{r}\right) = 2\left(1+\frac{1}{r}\right)(g-1) + n. \tag{2.1.15}$$

2.1.1 Topological Recursion: Graph Sums

The action of the differential operator (2.1.8) yields an infinite series in $\hbar$ and x_ℓ^α's. For reasons that will become apparent later, it is advantageous to regard it as a sum over isomorphism classes of graphs with decorations. For fixed g and n, we show that there are finitely many graphs whose weights are monomials $\hbar^g x_{\ell_1}^{\alpha_1} \cdots x_{\ell_n}^{\alpha_n}$ of degree g in $\hbar$ and degree n in the variables $\{x_\ell^\alpha\}$.

In order to define the graphs we first introduce the *dilaton shift*. Consider a ramification point $\alpha \in \mathcal{R}$ of order r. The dilaton shift corresponds to a translation of the variable x_{r+1}^α and plays an important role in the theory: if the graphs are defined without this first step then the result yields an infinite sum order by order in h instead. The index x_{r+1}^α corresponds to $(a, m) = (1, 0)$ in the t_a^m variable. Applying the dilaton shift

$$\exp\left(\frac{F_{0,1}[\,{}^{\alpha}_{-r-1}\,] + 1}{r+1}\partial_{x_{r+1}^\alpha}\right) = \exp\left((F_{0,1}[\,{}^{\alpha}_{-r-1}\,] + 1)\partial_{t_1^0}\right),$$

and using the dilaton equation $\langle \tau_1^0 \cdot \prod_{i=1}^n \tau_{a_i}^{m_i}\rangle_g^{1/r} = (2g-2+n)\langle \prod_{i=1}^n \tau_{a_i}^{m_i}\rangle_g^{1/r}$ [Wit93], which follows from the $\mathcal{W}$ algebra constraints, we have that:

$$Z^{(r)}[t^a_m,\hbar] = \exp\left((F_{0,1}[{}^{\alpha}_{-r-1}]+1)\partial_{t^0_1}\right)\exp\left(\sum_{\substack{g\geq 0,n>0\\2g-2+n>0}}\frac{\hbar^{g-1}}{n!}\sum_{\substack{a_1,\ldots,a_n\\m_1,\ldots,m_n}}\langle\tau^{m_1}_{a_1}\cdots\tau^{m_n}_{a_n}\rangle^{1/r}_g\prod_{i=1}^n t^{a_i}_{m_i}\right)$$

$$= \exp\left(\sum_{\substack{g\geq 0,n>0\\2g-2+n>0}}\frac{\hbar^{g-1}}{n!}\sum_{\substack{a_1,\ldots,a_n\\m_1,\ldots,m_n}}\langle\tau^{m_1}_{a_1}\cdots\tau^{m_n}_{a_n}\rangle^{1/r}_g\prod_{i=1}^n(t^{m_i}_{a_i}+\delta^{m_i,0}_{a_i,1}(F_{0,1}[{}^{\alpha}_{-r-1}]+1))\right)$$

$$= \exp\left(\sum_{\substack{g\geq 0,n>0\\2g-2+n>0}}\frac{\hbar^{g-1}}{n!}\sum_{\substack{a_1,\ldots,a_n\\m_1,\ldots,m_n}}\prod_{i=1}^n t^{m_i}_{a_i}\sum_{\ell\geq 0}\langle(\tau^0_1)^\ell\tau^{m_1}_{a_1}\cdots\tau^{m_n}_{a_n}\rangle^{1/r}_g\frac{(F_{0,1}[{}^{\alpha}_{-r-1}]+1)^\ell}{\ell!}\right)$$

$$= \exp\left(\sum_{\substack{g\geq 0,n>0\\2g-2+n>0}}\frac{\hbar^{g-1}}{n!}\sum_{\substack{a_1,\ldots,a_n\\m_1,\ldots,m_n}}\prod_{i=1}^n t^{m_i}_{a_i}\sum_{\ell\geq 0}\langle\tau^{m_1}_{a_1}\cdots\tau^{m_n}_{a_n}\rangle^{1/r}_g(F_{0,1}[{}^{\alpha}_{-r-1}]+1)^\ell\binom{2g-3+n+\ell}{\ell}\right)$$

$$= \exp\left(\sum_{\substack{g\geq 0,n>0\\2g-2+n>0}}\frac{\hbar^{g-1}}{n!}\left(\frac{-1}{F_{0,1}[{}^{\alpha}_{-r-1}]}\right)^{2g-2+n}\sum_{\substack{a_1,\ldots,a_n\\m_1,\ldots,m_n}}\langle\tau^{m_1}_{a_1}\cdots\tau^{m_n}_{a_n}\rangle^{1/r}_g\prod_{i=1} t^{m_i}_{a_i}\right). \tag{2.1.16}$$

Note that $F_{0,1}[{}^{\alpha}_{-r-1}]\neq 0$ by assumption. The above observation allows us to introduce a set of decorated graphs that are analogous to the differential operator (2.1.7).

Perturbative Graphs

We now introduce the set $\mathcal{G}$ of *perturbative graphs*. This will play an important role in the computations of the correlators $\{\omega_{g,n}\}$ in Chapter 4. Consider a connected graph $\Gamma=(V,E,L\amalg\tilde{L})$, where V, E and $L\amalg\tilde{L}$ denote the set of edges, vertices and leaves respectively. The set of leaves is partitioned into *ordinary leaves* L and *dilaton leaves* $\tilde{L}$. Denote by H the set of *half edges*, that is, the set of all internal edges and leaves, where the internal edges come with a choice of adjacent vertex. Let $\mathfrak{a}\colon H\to V$ be the adjacency map that assigns to each half edge the corresponding vertex. At each vertex $v \in V$, we denote the sets of adjacent leaves (of each type) and internal edges by $n_v = \{l \in L, \mathfrak{a}(l) = v\}$, $\tilde{n}_v = \{l \in \tilde{L}, \mathfrak{a}(l) = v\}$ and $m_v = \{l \in H \setminus L \amalg \tilde{L}, \mathfrak{a}(l) = v\}$ respectively. Moreover, each graph $\Gamma \in \mathcal{G}$ comes with the following additional structure:

- Vertex labels:
  - Genus: $g \colon V \to \mathbb{Z}_{\geq 0}$
  - Ramification point: $\alpha \colon V \to \{1 \ldots, k\}$
- Half-edge labels:
$$\ell \colon H \to \mathbb{Z}_{\geq 1}$$
- Dilaton condition at each vertex $v \in V$:
$$\ell(l) > r_{\alpha(v)} + 1, \forall l \in \tilde{n}_v$$
- Genus $g \colon \mathcal{G} \to \mathbb{Z}_{\geq 0}$:
$$g(\Gamma) = \sum_{v \in V} g(v) + b_1(\Gamma),$$
  where $b_1(\Gamma)$ denotes the first Betti number of the graph.
- Stability: each vertex $v \in V$ must satisfy the conditions
$$2g(v) - 2 + |n_v| + |m_v| > 0, \qquad |n_v| + |m_v| > 0.$$

The *Euler characteristic* χ of a graph Γ is always positive and defined as

$$\chi(\Gamma) = 2g(\Gamma) - 2 + \sum_{v \in V} |n_v| = \sum_{v \in V} (2g(v) - 2 + |n_v| + |m_v|).$$

The second equality follows from standard graph decomposition. We will refer to the following non-negative integers as the *Euler characteristic of a vertex*:

$$\chi_v = 2g(v) - 2 + |n_v| + |m_v|.$$

The data of a spectral curve $\mathcal{S}$ can be used to define a weight function on $\mathcal{G}$ with values in the polynomial ring $\mathbb{C}[\{x^\alpha_\ell\}]$:

$$w\colon \mathcal{G} \to \mathbb{C}[\{x^\alpha_\ell\}].$$

Definition 2.1.6 (Perturbative graph weights)**.** Let $\mathcal{S}$ be a spectral curve with set of arbitrary ramifications $\mathcal{R}$. Let $F_{0,1}$ and ϕ be the corresponding tensors defined in (2.1.5) and (2.1.6). For any $\Gamma \in \mathcal{G}$ we define its *weight* by

$$w(\Gamma) = \prod_{v\in V} \left(\frac{-1}{F_{0,1}\left[\begin{smallmatrix}\alpha(v)\\ -r_{\alpha(v)}-1\end{smallmatrix}\right]} \right)^{\chi_v+|\tilde{n}_v|} \prod_{i\in L} x_{\ell_i}^{\alpha(\mathfrak{l}(i))} \prod_{j\in \tilde{L}} F_{0,1}\left[\begin{smallmatrix}\alpha(\mathfrak{a}(j))\\ -\ell_j\end{smallmatrix}\right] \prod_{e\in E} \phi^{\alpha(\mathfrak{a}(e_1)),\alpha(\mathfrak{a}(e_2))}_{\ell(e_1),\ell(e_2)} \prod_{v\in V} I_v. \tag{2.1.17}$$

In particular, each vertex is labeled by a ramification point $\alpha(v) \in \mathcal{R}$ with corresponding index r_α, and integers $\ell_1, \dots, \ell_n$ given by the labels on the leaves and internal edges. The vertex weights are thus given by

$$I_v = F^{(r_\alpha)}_{g(v)}\left[\,\ell_1 \,\dots\, \ell_n\,\right].$$

Example 2.1.7. In Figures 2·1 and 2·2 we depict one of the graphs contributing to the correlator $\omega_{g_1+g_2+1,2}$ together with the corresponding weights associated to each vertex, leaf and internal edge.

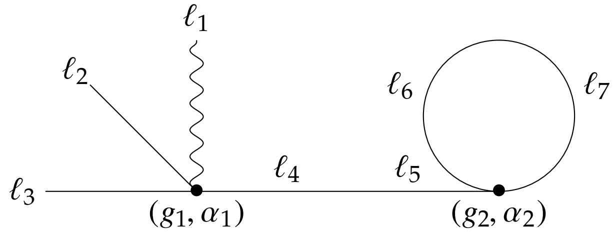


Figure 2·1: Decorated graph in $\Gamma \in \mathcal{G}$

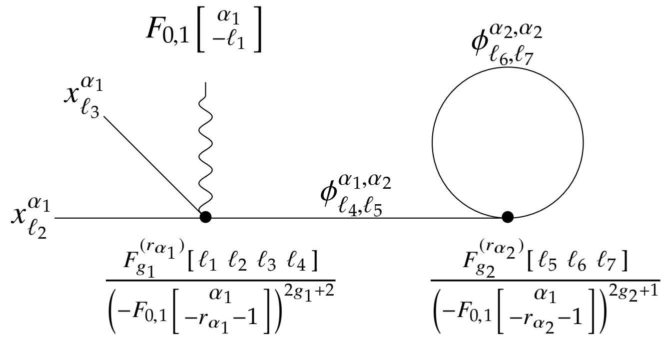


Figure 2·2: Weights corresponding to the graph in Figure 2·1

Proposition 2.1.8 (Independence of local coordinate)**.** *The weights $w(\Gamma)$ defined in (2.1.17) do not depend on the choice of local coordinate ζ near the ramification points.*

Proof. Equation (2.1.4) defines ζ up to a power of an r_α-th root of unity β. Let $\tilde{F}_{0,1}[\begin{smallmatrix}\alpha\\-\ell\end{smallmatrix}]$, $\tilde{\phi}^{\alpha_1,\alpha_2}_{\ell_1,\ell_2}$ and $d\tilde{\xi}^\alpha_\ell$ be the new quantities computed in the coordinate $\tilde{\zeta} = \beta\zeta$ and $\tilde{w}$ the new weights. One can check from the definitions (2.1.11), (2.1.5) and (2.1.6) that they satisfy

$$F_{0,1}[\begin{smallmatrix}\alpha\\-\ell\end{smallmatrix}] = \beta^\ell \tilde{F}_{0,1}[\begin{smallmatrix}\alpha\\-\ell\end{smallmatrix}], \qquad \phi^{\alpha_1,\alpha_2}_{\ell_1,\ell_2} = \beta^{\ell_1+\ell_2}\tilde{\phi}^{\alpha_1,\alpha_2}_{\ell_1,\ell_2} \qquad \text{and} \qquad d\xi^\alpha_\ell = \beta^\ell d\tilde{\xi}^\alpha_\ell.$$

Let be $v \in V$ be any vertex and set $r = r_{\alpha(v)}$. From 4.1.1 we deduce

$$(r+1)\chi_v = \sum_{l\in n_v\cup m_v} \ell(l),$$

and hence we have that

$$\sum_{l\in n_v\cup\tilde{n}_v\cup m_v} \ell(l) = (r+1)\chi_v + (r+1)\tilde{n}_v.$$

Hence, the weights associated to the vertex v remain invariant:

$$\tilde{w}(v) = \frac{\beta^{(r+1)\chi_v+(r+1)\tilde{n}_v}}{\beta^{\chi_v+\tilde{n}_v}} w(\Gamma) = (\beta^r)^{(\chi_v+\tilde{n})} w(\Gamma) = w(\Gamma).$$

□

A given a spectral curve $\mathcal{S}$ determines unique class of graphs $\mathcal{G}$ and a weight function w. This perspective is a reformulation of the differentiation rules in (2.1.8).

The above discussion allows us to recover the perturbative generating function Z_{P}, and hence the correlators $\{\omega_{g,n}\}$, purely from the $\mathcal{W}$ algebra constraints and local data at the ramification points of $\mathcal{S}$.

Proposition 2.1.9. *The exponent F_P of the perturbative function Z_P can be written as the following sum over graphs:*

$$F_P[x_\ell^\alpha, \hbar] = \sum_{\Gamma \in \mathcal{G}} \hbar^{g(\Gamma)-1} \frac{w(\Gamma)}{|Aut(\Gamma)|}.$$

To see that the function $F_{\mathrm{P}}[x_\ell^\alpha, \hbar]$ is truly a power series in $\hbar^{-1}\mathbb{C}[\![\{x_\ell^\alpha\}, \hbar]\!]$ we first write it as

$$F_{\mathrm{P}}[x_\ell^\alpha, \hbar] = \sum_{g,n} \frac{\hbar^{g-1}}{n!} \sum_{\substack{\Gamma \in \mathcal{G} \\ g(\Gamma)=g \\ \sum_{v\in\Gamma} n_v = n}} \frac{w(\Gamma)}{|\mathrm{Aut}(\Gamma)|} = \sum_{\substack{2g-2+n>0 \\ g\geq 0 \\ n>0}} \frac{\hbar^{g-1}}{n!} F_{g,n}[x_\ell^\alpha]. \tag{2.1.18}$$

Then, proposition 4.1.2 ensures that only finitely many graphs contribute to each $F_{g,n}$ and hence they are polynomial expressions in the $\{x_\ell^\alpha\}$ variables. Finally, Theorem (2.1.12) yields:

Corollary 2.1.10. *Denote by ϱ the linear symmetrizing map on $\mathbb{C}[\{x_\ell^\alpha\}]$ acting on monomial as*

$$\varrho(x_{\ell_1}^{\alpha_1} \cdots x_{\ell_n}^{\alpha_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} d\xi_{\ell_1}^{\alpha_1}(z_{\sigma_1}) \cdots d\xi_{\ell_n}^{\alpha_n}(z_{\sigma_n}).$$

The correlators $\{\omega_{g,n}\}_{2g-2+n>0}$ can be written in terms of graphs as

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\substack{\Gamma \in \mathcal{G} \\ g(\Gamma)=g \\ \sum_{v\in\Gamma} n_v = n}} \frac{\varrho(w(\Gamma))}{|Aut(\Gamma)|}. \tag{2.1.19}$$

2.2 Wave Functions and Quantum Curves

Closely related to the generating function Z_{P} is the so-called perturbative wave function ψ_{P}. The motivation for its definition comes from the determinantal formulae in matrix models [BE09a, BE09b]. The function ψ_{P} is expected to satisfy an ordinary differential equation that is a quantization of the underlying spectral curve $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$. Assuming that $P(x, y)$ is the defining polynomial of Σ, its quantization is obtained after promoting the variables $(x, y) \mapsto (\hat{x}, \hat{y}) = \left(x, h\frac{d}{dx}\right)$ and considering additional h corrections:

$$\hat{P}\left(\hat{x}, \hat{y}, h\right) = P(\hat{x}, \hat{y}) + \sum_{n\geq 1} h^n P_n(\hat{x}, \hat{y}). \tag{2.2.1}$$

Typically, one requires $\hat{P}$ to be a polynomial in $\hat{y}$, with coefficients that are possibly power series in x and h.

Remark 2.2.1. In the context of quantum curves and wave functions, we will use a formal parameter h. In our formalism, it is related to the perturbative parameter $\hbar$ introduced in the previous section by $\hbar = h^2$.

Definition 2.2.2 (Perturbative wave function)**.** Given a spectral curve $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$, the *perturbative wave function*, introduced in [EO07], is a function on Σ which depends on the choice of a base point $o \in \Sigma$ defined as

$$\psi_{\mathrm{P}}(z) := \exp\left(\sum_{2g-2+n>0} \frac{h^{2g-2+n}}{n!} \underbrace{\int_o^z \cdots \int_o^z}_{n} \omega_{g,n}(z_1, \ldots, z_n)\right)$$
$$\times \exp\left(\frac{1}{h}\int_o^z y(z)dx(z) + \frac{1}{2}\int_o^z\int_o^z \left(\omega_{0,2}(z_1, z_2) - \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}\right)\right). \tag{2.2.2}$$

Since, in general, the integrals of meromorphic forms will depend on the path of integration, the function ψ_{P} is only defined in a neighborhood of the chosen base point. Given a spectral curve with defining polynomial $P(x, y)$, the Quantum Curve Conjecture – in the perturbative setting – asks whether there exists a differential operator $\hat{P}$ of the form (2.2.1) that annihilates the corresponding perturbative wave function ψ_{P}:

$$\hat{P}\left(\hat{x}, \hat{y}, h\right) \cdot \psi_{\mathrm{P}}(x) = 0. \tag{2.2.3}$$

Such an operator $\hat{P}$ satisfying the above equation is referred to as a *quantum curve*. A positive answer to this question was given in [BE17] for a big class of genus zero spectral curves. Other concrete examples in genus zero can be found in the literature [Zho12, ALS16]. In the case of the Airy spectral curve, we can show that the Quantum Curve Conjecture follows directly from the Virasoro constraints (2.1.13) and the definition of wave function. Although this is a well known result, our proof using differential operators appears to be new. The main advantage of this approach is that it can be adapted to the non-perturbative setting (2.3.6).

Quantum Curve for the Airy spectral curve

Recall the Airy spectral curve:

$$\left(\mathbb{CP}^1, x(z) = \frac{z^2}{2}, y(z) = -z, \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right).$$

Its underlying curve is compact of genus zero ($\mathbb{CP}^1$), it has one simple ramification point at 0, and the functions x and y satisfy the polynomial equation $\frac{y^2}{2} - x = 0$. The local variable ζ near 0 coincides with the parametrizing variable $z \in \mathbb{CP}^1$. In this case, the Quantum Curve Conjecture can be proved by taking an appropriate combination of the specialized $\{W^i_{\alpha,k}\}$ operators. We make the choice of base point $z = \infty$. Following (2.2.2) and (2.1.12) we write the perturbative wave function as

$$\psi_{\mathrm{P}}(z) = e^{\frac{1}{\hbar}S_0(z) + S_1(z)} Z_{\mathrm{P}}[x_\ell, \hbar = h^2]\Big|_{x_\ell = h\int_\infty^z d\xi_\ell},$$

where $S_0'(z) = -z^2$ and $S_1'(z) = -\frac{1}{2z}$. We evaluate the one-forms $d\xi_\ell(z) := d\xi^0_\ell(z)$:

$$d\xi_\ell(z) = \operatorname*{Res}_{z'=0}\left(\int_0^{z'} \omega_{0,2}(\cdot, z)\right)\frac{dz'}{z'^{\ell+1}} = \operatorname*{Res}_{z'=0}\left(\int_0^{z'} \frac{dz_1}{(z_1 - z)^2}\right) dz \frac{dz'}{z'^{\ell+1}}$$

$$= \operatorname*{Res}_{z'=0}\left(\sum_{k=1} \frac{z'^k}{z^{k+1}}\right)\frac{dz'}{z'^{\ell+1}} = \frac{dz}{z^{\ell+1}}. \tag{2.2.4}$$

It follows from the chain rule, after specializing the variables $x_\alpha := x^0_\alpha$ that

$$x_i = h\int_\infty^z d\xi_i(z) = h\int_\infty^z \frac{dz}{z^{i+1}} = \frac{-h}{iz^i}, \qquad \frac{\partial x_i}{\partial z} = \frac{h}{z^{i+1}},$$

$$\frac{d}{dz} = \sum_{i\geq 1} \frac{\partial x_i}{\partial z}\partial_i = h\sum_{i\geq 1}\frac{1}{z^{i+1}}\partial_i, \qquad \partial_i = \frac{\partial}{\partial x_i} \text{ for } i \geq 1.$$

Then, we are able to find a concrete linear combination of the $\{W^i_k := W^i_{0,k}\}$ operators defined in (2.1.13) that agrees with the Airy differential equation. Note that since $Z^{(r)}\,[t^a_m, \hbar]$ is already independent of the variables x^α_ℓ with even ℓ, the operators $\{W^1_k\}$ are redundant.

Theorem 2.2.3 (G.). *We define the differential operator Q as the following linear combination of $\{W^i_k\}$ operators:*

$$Q(z, \partial_z, h) := -\sum_{k\geq 0} \frac{1}{z^{2k+2}} W^2_k.$$

Then we have that

$$\left(\frac{h^2}{2}\frac{d^2}{dx^2} - x\right) = e^{\left(\frac{1}{h}S_0(z)+S_1(z)\right)} Q(z, \partial_z, h) e^{\left(-\frac{1}{h}S_0(z)-S_1(z)\right)},$$

where

$$S_0'(z) = -z^2 \text{ and } S_1'(z) = -\frac{1}{2z}.$$

Proof. First we must rewrtite the operators (2.1.13) in terms of the variable z:

$$\begin{aligned}
W^2_0 &= -\frac{h^2}{2z^2} + h^2\sum_{p_1\geq 1}(-1)^{p_1}(p_1+2)\frac{-h}{(p_1+2)z^{p_1+2}}\partial_{p_1} + h^2\partial_1 \\
&= -\frac{h^2}{2z^2} + \frac{h^2}{z}\frac{d}{dz} + h^2\partial_1 = z^2h^2\left(-\frac{1}{2z^4} + \frac{1}{z^3}\frac{d}{dz} + \frac{1}{z^2}\partial_1\right) \\
W^2_1 &= h^2\sum_{p_1\geq 1}(-1)^{p_1}p_1\frac{-h}{p_1 z^{p_1}}\partial_{p_1} - \frac{h^2}{8} + h^2\partial_3 \\
&= -\frac{h^2}{8} + h^2 z\frac{d}{dz} + h^2\partial_3 = z^4h^2\left(-\frac{1}{8z^4} + \frac{1}{z^3}\frac{d}{dz} + \frac{1}{z^4}\partial_3\right)
\end{aligned}$$

$$W^2_{k\geq 2} = \frac{h^4}{2}\sum_{\substack{p_1+p_2=2(k-1)\\ p_1,p_2\geq 0}}(-1)^{p_1}\partial_{p_1}\partial_{p_2} + h^2\sum_{\substack{p_1-p_2=2(k-1)\\ p_1,p_2\geq 0}}(-1)^{p_1}p_2\frac{-h}{p_2 z^{p_2}}\partial_{p_1} + h^2\partial_{2k+1}$$
$$= -\frac{h^4}{2}\sum_{\substack{p_1+p_2=2(k-1)\\ p_1,p_2\geq 0}}\partial_{p_1}\partial_{p_2} + h^2\sum_{\substack{p_1-p_2=2(k-1)\\ p_1,p_2\geq 0}}\frac{h}{z^{p_2}}\partial_{p_1} + h^2\partial_{2k+1}$$
$$= z^{2k+2}h^2\left(-\frac{1}{2}\sum h^2\frac{\partial_{p_1}\partial_{p_2}}{z^{p_1+p_2+4}} + \sum_{p_1\geq 2(k-1)}\frac{h}{z^{p_1+4}}\partial_{p_1} + \frac{1}{z^{2k+2}}\partial_{2k+1}\right)$$

Second, we expand Q as follows:

$$\sum_{k\geq 0}\frac{1}{h^2z^{2k+2}}W^2_k = -\frac{1}{2z^4} + \frac{1}{z^3}\frac{d}{dz} + \frac{1}{z^2}\partial_1 + \left(-\frac{1}{8z^4} + \frac{1}{z^3}\frac{d}{dz} + \frac{1}{z^4}\partial_3\right)$$
$$+\sum_{k\geq 2}\left(-\frac{1}{2}\sum_{p_1+p_2=2(k-1)}h^2\frac{\partial_{p_1}\partial_{p_2}}{z^{p_1+p_2+4}} + \sum_{p_1\geq 2(k-1)}\frac{h}{z^{p_1+4}}\partial_{p_1} + \frac{1}{z^{2k+2}}\partial_{2k+1}\right)$$
$$= -\frac{5}{8z^4} + \frac{2}{z^3}\frac{d}{dz} + \sum_{k\geq 1}\frac{1}{z^{k+1}}\partial_k - \frac{h^2}{2}\sum_{p_1,p_2\geq 1}\frac{\partial_{p_1}\partial_{p_2}}{z^{p_1+p_2+4}} + h\sum_{p_1\geq 1}\frac{p_1-1}{2}\frac{1}{z^{p_1+4}}\partial_{p_1}$$
$$= -\frac{5}{8z^4} + \frac{2}{z^3}\frac{d}{dz} + \frac{1}{h}\frac{d}{dz} - \frac{1}{2z^2}\left(h^2\sum_{p_1,p_2\geq 1}\frac{\partial_{p_1}\partial_{p_2}}{z^{p_1+1}z^{p_2+1}} - h\sum_{p_1\geq 1}\frac{p_1-1}{z^{p_1+2}}\partial_{p_1}\right)$$
$$= -\frac{5}{8z^4} + \frac{2}{z^3}\frac{d}{dz} + \frac{1}{h}\frac{d}{dz} - \frac{1}{2z^2}\frac{d^2}{dz^2} - h\sum_{p_1\geq 1}\frac{1}{z^{p_1+4}}\partial_{p_1}$$
$$= -\frac{5}{8z^4} + \left(\frac{1}{z^3}+\frac{1}{h}\right)\frac{d}{dz} - \frac{1}{2z^2}\frac{d^2}{dz^2} = -\frac{1}{2z^2}\left(\frac{d^2}{dz^2} - 2\left(\frac{1}{z}+\frac{z^2}{h}\right)\frac{d}{dz} + \frac{5}{4z^2}\right).$$

Therefore, on one hand we have:

$$Q(z,\partial_z,h) = \frac{h^2}{2z^2}\left(\frac{d^2}{dz^2} - 2\left(\frac{1}{z}+\frac{z^2}{h}\right)\frac{d}{dz} + \frac{5}{4z^2}\right).$$

On the other, using the fact that $x(z) = \frac{z^2}{2}$ we may rewrite the operator:

$$\left(\frac{h^2}{2}\frac{d^2}{dx^2} - x\right) = \left(\frac{h^2}{2z}\left(-\frac{1}{z^2}\frac{d}{dz} + \frac{1}{z}\frac{d^2}{dz^2}\right) - \frac{z^2}{2}\right) = \frac{h^2}{2z^2}\left(\frac{d^2}{dz^2} - \frac{1}{z}\frac{d}{dz}\right) - \frac{z^2}{2}.$$

Finally, after conjugation, the two expressions agree:

$$
\begin{aligned}
e^{\left(-\frac{1}{h}S_0(z)-S_1(z)\right)} & \left(\frac{h^2}{2z^2}\left(\frac{d^2}{dz^2}-\frac{1}{z}\frac{d}{dz}\right)-\frac{z^2}{2}\right)e^{\left(\frac{1}{h}S_0(z)+S_1(z)\right)} \\
=&\frac{h^2}{2z^2}\left(\frac{S_0''(z)}{h}+S_1''(z)+\left(\frac{S_0'(z)}{h}+S_1'(z)\right)^2+2\left(\frac{S_0'(z)}{h}+S_1'(z)\right)\frac{d}{dz}+\frac{d^2}{dz^2}\right. \\
&\left.-\frac{1}{z}\left(\frac{S_0'(z)}{h}+S_1'(z)\right)-\frac{1}{z}\frac{d}{dz}\right)-\frac{z^2}{2} \\
=&\frac{h^2}{2z^2}\left(\frac{5}{4z^2}+\frac{z^4}{h^2}+2\left(-\frac{z^2}{h}-\frac{1}{2z}\right)\frac{d}{dz}+\frac{d^2}{dz^2}-\frac{1}{z}\frac{d}{dz}\right)-\frac{z^2}{2} \\
=&\frac{h^2}{2z^2}\left(\frac{d^2}{dz^2}-2\left(\frac{1}{z}+\frac{z^2}{h}\right)\frac{d}{dz}+\frac{5}{4z^2}\right) \\
=&Q(z,\partial_z,h).
\end{aligned}
$$

□

Since, by definition the W_k^i operators annihilate the generating function Z_{P} of the Airy spectral curve, it follows that Q annihilates ψ_{P}, and hence we can recover the following well-known result:

Corollary 2.2.4. *The following differential operator is a quantum curve for the Airy spectral curve:*

$$\frac{h^2}{2}\frac{d^2}{dx^2}-x.$$

2.3 Non-Perturbative Wave Function

It was argued by Borot, Eynard and Marino in [EM11, BE12] that a wave function should satisfy background independence and have modularity properties. More specifically, it should not depend on the classical solution chosen to quantize the theory, nor upon the choice of symplectic basis of $H_1(\Sigma,\mathbb{Z})$. For compact spectral curves of genus $g>0$ the perturbative wave function ψ_{P} does not satisfy these properties. The same authors introduce the notion of non-perturbative wave function, obtained by manually introducing corrections in the form of theta functions.

The non-perturbative topological recursion is hence defined only for global spectral curves. In this section we briefly recall their construction and give a new interpretation in terms of perturbative graph sums and corresponding differential operators. For this setting, the starting data is a genus $g > 0$ spectral curve together with a choice of symplectic basis and a normalized fundamental bidifferential. A symplectic basis $\{(\mathcal{A}_j, \mathcal{B}_j)\}$ of $H_1(\Sigma, \mathbb{Z})$ is characterized by

$$\mathcal{A}_i \cap \mathcal{A}_j = 0 \qquad \mathcal{B}_i \cap \mathcal{B}_j = 0 \qquad \mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j} \qquad \forall i, j = 1, \ldots, g. \tag{2.3.1}$$

There are also g holomorphic one-forms, which we denote $da_1, \ldots, da_g$, that form a basis of $H^1(\Sigma)$ dual to the $\mathcal{A}$ cycles:

$$\oint_{\mathcal{A}_i} da_j = \delta_{ij}.$$

The coefficients of the period matrix τ are defined as

$$\oint_{\mathcal{B}_j} da_i =: \tau_{ij}.$$

One can prove that τ is symmetric using Riemann's Bilinear Identities. Choosing an arbitrary base point $q \in \Sigma$, the Abel-Jacobi map $a\colon \Sigma \to \mathbb{C}^g/\Lambda$ is defined via integration

$$x \mapsto \left(\int_q^x da_1, \ldots, \int_q^x da_g \right),$$

where $\Lambda \subset \mathbb{C}^g$ denotes a lattice. The non-perturbative corrections require the following quantities:

- Filling fractions $\{\epsilon_i := \frac{1}{2\pi i} \oint_{\mathcal{A}_i} y dx\}_{1 \le i \le g}$
- $\zeta_h := \left(\frac{1}{2\pi i h} \oint_{\mathcal{B}_1 - \tau \mathcal{A}_1} y dx, \ldots, \frac{1}{2\pi i h} \oint_{\mathcal{B}_g - \tau \mathcal{A}_g} y dx \right)$

Next, we recall the definitions and main properties of theta functions and half characteristics.

Definition 2.3.1. Let τ be the period matrix of a Riemann surface of genus g. The *theta function* associated to τ is a function of g variables defined as

$$\theta(\mathbf{z};\tau) = \sum_{n\in\mathbb{Z}^g} \exp 2\pi i \left(\frac{1}{2}n^t\tau n + n^t\mathbf{z}\right).$$

A *half-characteristic* is a vector $\nu + \tau\mu$ where $2\nu, 2\mu \in \mathbb{Z}^g$. Then the *theta function with half-characteristic* $\nu + \tau\mu$ and defined as

$$\begin{aligned}\theta\left[\begin{smallmatrix}\nu\\ \mu\end{smallmatrix}\right](\mathbf{z};\tau) &= \sum_{n\in\mathbb{Z}^g} \exp\left(i\pi(n^t+\mu^t)\tau(p+\mu) + 2\pi i(\mathbf{z}^t+\nu^t)(n+\mu)\right)\\ &= \exp\left(\pi i\mu^t\tau\mu + 2\pi i\mathbf{z}^t\mu + 2\pi i\nu^t\mu\right)\theta(\mathbf{z}+\nu+\tau\mu;\tau).\end{aligned}$$

Theta functions satisfy the following properties:

- Symmetry:

$$\theta(\mathbf{z};\tau) = \theta(-\mathbf{z};\tau)$$

- Quasi-periodicity:

$$\theta(\mathbf{z}+\lambda'+\tau\lambda;\tau) = \exp 2\pi i\left(-\lambda^t\mathbf{z} - \frac{1}{2}\lambda^t\tau\lambda\right)\theta(\mathbf{z};\tau)$$

- Heat equation:

$$\frac{\partial\theta(\mathbf{z};\tau)}{\partial\tau_{jk}} = \frac{1}{2\pi i}\frac{\partial^2\theta(\mathbf{z};\tau)}{\partial z_j\partial z_k}$$

$$\frac{\partial\theta(\mathbf{z};\tau)}{\partial\tau_{jj}} = \frac{1}{4\pi i}\frac{\partial^2\theta(\mathbf{z};\tau)}{\partial z_j^2}$$

Following [Fay73], certain fundamental objects on Riemann surfaces can be written in terms of theta functions. Of particular interest in our case is the explicit expression of the normalized fundamental bidifferential in terms of theta functions

$$\omega_{0,2}(z_1, z_2) = d_{z_1} d_{z_2} \theta(a(z_1) - a(z_2); \tau). \tag{2.3.2}$$

The non-perturbative wave function is usually defined in the literature as the *Schlesinger transform* of the corresponding non-perturbative generating function. Here we chose to give an equivalent graphical interpretation instead and we refer to the Appendix for the original definitions. The non-perturbative graphs are build upon their perturbative counterparts and hence we should not think of them as analogous, but rather complementary.

2.3.1 Non-Perturbative Graph Sum

The set $\mathcal{F}$ of non-perturbative graphs is defined as follows. Consider the set of connected bipartite graphs $\Gamma = (V, E, L)$, with the set of vertices partitioned into the sets of white vertices V_{W} and black vertices V_{B}. Bipartite means that every internal edge in E is adjacent to exactly one black vertex and one white vertex. Denote by n_v and m_v the number of adjacent leafs and internal edges at a vertex $v \in V$. Moreover, the graphs in $\mathcal{F}$ have the following additional structure:

- Black vertex labels:

$$g \colon V_{\mathrm{B}} \to \mathbb{Z}_{\geq 0}$$

- White vertices have no leaves: $n_v = 0$ if $v \in V_{\mathrm{W}}$
- Euler characteristic computed *using only* the black vertices:

$$\chi(\Gamma) = \sum_{v \in V_{\mathrm{B}}} (2g(v) - 2 + m_v + n_v)$$

In our diagrams, we will depict the internal edges by sawed lines to distinguish these graphs from the ones in Section 2.1.1. Consider a set of correlators $\{\omega_{g,n}\}$

obtained from topological recursion on a global spectral curve $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$. Let $o \in \Sigma$ be a choice of a base point and let Θ be a theta function with a choice of a half-characteristic (ν, μ):

$$\Theta(z) := \theta \left[\begin{smallmatrix} \nu \\ \mu \end{smallmatrix} \right] (z; \tau)$$

To simplify the notation, we define its logarithm σ through the equation

$$\Theta(z) = e^{\sigma(z)}.$$

Define the following functions on Σ:

$$G_g^{n,(m)}(z) := \frac{1}{(2\pi i)^m} \underbrace{\int_o^z \cdot \cdot \int_o^z}_{n} \underbrace{\oint_{\mathcal{B}} \cdot \cdot \oint_{\mathcal{B}}}_{m} \omega_{g,n+m}.$$

Definition 2.3.2 (Non-perturbative graph weights)**.** The data of a global spectral curve $\mathcal{S}$, together with a choice of basepoint $o \in \Sigma$ and a choice of theta function Θ determines the following weight function on $\mathcal{F}$:

$$w(\Gamma) = \prod_{v \in V_{\mathrm{B}}} G_{g(v)}^{n_v,(m_v)}(z) \prod_{v \in V_{\mathrm{W}}} \sigma^{(m_v)}(\eta_0),$$

where the derivatives of σ are evaluated at $\eta_0 = \zeta_h + a(z) - a(o)$.

Example 2.3.3. Figure 2·3 depicts the weights assigned to the white and black vertices. In Figure 2·4, we give an example of a non-perturbative graph of Euler characteristic $\chi = 6$ together with the corresponding weights and automorphism factor.

Definition 2.3.4 (Non-perturbative wave function)**.** The *non-perturbative wave function* ψ_{NP} is defined as

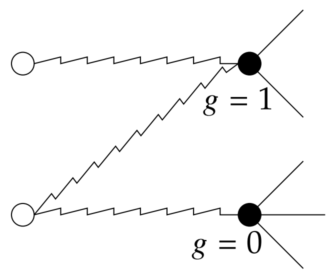
$$\psi_{\mathrm{NP}}(z) := e^{S(z)},$$

where $S(z)$ is given by the following graph sum:

$$S(z) := \sum_{\chi \geq 2} h^{\chi - 1} S_\chi(z) = \sum_{\Gamma \in \mathcal{F}} h^{\chi(\Gamma)} \frac{w(\Gamma)}{|\mathrm{Aut}(\Gamma)|}.$$

$\sigma^{(m)}(\eta)$ $\qquad\qquad$ $G_g^{n,(m)}(z)$

Figure 2·3: Non-perturbative weights



$\frac{1}{2!}\frac{1}{3!}G_1^{2,(2)}(z)G_0^{3,(1)}(z)\sigma^{(1)}(\eta_0)\sigma^{(2)}(\eta_0)$

Figure 2·4: Graph with $\chi = 6$ with corresponding weight times its automorphism factor

More generally, one can define the $[k|k]$-*kernels* $\psi_{\mathrm{NP}}^{[k|k]}$ by choosing k base points $(o_1, \ldots, o_k)$ instead, and slightly modifying the definitions of the G's:

$$G_g^{n,(m)}(z) = \frac{1}{(2\pi i)^m} \underbrace{\int_{o_1,\ldots,o_k}^{z_1,\ldots,z_k} \cdot\cdot \int_{o_1,\ldots,o_k}^{z_1,\ldots,z_k}}_{n} \underbrace{\oint_{\mathcal{B}} \cdot\cdot \oint_{\mathcal{B}}}_{m} \omega_{g,n+m},$$

where we use the notation

$$\int_{o_1,\ldots,o_k}^{z_1,\ldots,z_k} := \int_{o_1}^{z_1} + \ldots + \int_{o_k}^{z_k} .$$

In this case the theta functions are evaluated at $\eta_0 = \zeta_h + \sum_{i=1}^{n}(a(z_i) - a(o_i))$.

Similarly as in the perturbative case, $\psi_{\mathrm{NP}}^{[k|k]}$ is defined locally near the base points

$o_1, \ldots, o_k$. For completeness, we provide the differential operator form of the non-perturbative wave function as well. By further conjugating the expression (2.1.8), this allows us to write a set of operators annihilating the non-perturbative wave function ψ_{NP}.

Definition 2.3.5. Let $\mathcal{S}$ be a given global spectral curve. Let D denote the differential operator of (2.1.8) corresponding to $\mathcal{S}$. The *non-perturbative operator* δD is defined as follows:

$$\delta D := \sum_{\alpha,\ell} \frac{h}{2\pi i} \oint_{\mathcal{B}} d\xi_\ell^\alpha \partial_{x_\ell^\alpha} \partial_\eta. \tag{2.3.3}$$

The *non-perturbative generating function* is defined from Z_{P}, the functions σ and the non-perturbative operator δD:

$$\begin{aligned} Z_{\mathrm{NP}}[x_\ell^\alpha, \eta, h] &:= e^{D+\delta D} e^{\sigma(\eta)} \prod_{\alpha \in \mathcal{R}} Z^{(r_\alpha)}[x_\ell^\alpha, h^2] \\ &= e^{\delta D} e^{\sigma(\eta)} Z_{\mathrm{P}}[x_\ell^\alpha, h^2]. \end{aligned}$$

The non-perturbative wave-function ψ can then be obtained specializing the values of η and the $\{x_\ell^\alpha\}$ variables:

$$\psi_{\mathrm{NP}}(z) = Z_{\mathrm{NP}}[x_\ell^\alpha, \eta, h] \Big|_{\substack{x_\ell^\alpha = h \int_o^z d\xi_\ell^\alpha \\ \eta = \eta_0}}.$$

Similarly, the $[n|n]$-kernels are obtained by evaluating $\{x_\ell^\alpha\}$ at the appropriate integrals:

$$\psi_{\mathrm{NP}}^{[n|n]}(z_1, \ldots, z_n) = Z_{\mathrm{NP}}[x_\ell^\alpha, \eta, h] \Big|_{\substack{x_\ell^\alpha = h \int_{o_1, \ldots, o_n}^{z_1, \ldots, z_n} d\xi_\ell^\alpha \\ \eta = \eta_0}}.$$

Remark 2.3.6. We do not know in which cases a certain combination of these operators, after specializing the $\{x_\ell^\alpha\}$ variables, can be used to produce a quantum curve in a similar to Theorem 2.2.3. We believe it would be an interesting line to pursue.

Quantization Condition

Only in some cases will the perturbative wave function ψ_{NP} be series in h. The main impediment is the term ζ_h in the argument of the theta functions. We say that a spectral curve satisfies the *quantization condition* if there exists a formal series expansion in h. This can be achieved, for example, if the functions $S_\chi(z)$ do not depend on ζ_h. Trivially, the quantization condition is satisfied if $\zeta_h = 0$. Although this is seemingly a strong requirement, all of the spectral curves we study in Chapter 4 fall into this category. In these cases, the non-perturbative wave function is then a formal series in h:

$$\psi_{\mathrm{NP}}(z) = \exp\left(\frac{1}{h}\sum_{\chi\geq 2} h^{\chi} S_\chi(z)\right).$$

With these ingredients we can formally state the quantum curve conjecture in the non-perturbative case. It is unclear to us when this was first formally stated; we refer the reader to [BCD18].

Conjecture 2.3.7 (Quantum Curve Conjecture, Non-Perturbative Case)**.** *Let $\mathcal{S}$ be a global spectral curve of genus $g > 1$ defined as the zero locus in $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ of a polynomial $P(x, y)$. Suppose that $\mathcal{S}$ satisfies the quantization condition. Then there exists a quantum curve $\hat{P}(\hat{x}, \hat{y}, h)$ that annihilates the non-perturbative wave function ψ_{NP} of $\mathcal{S}$, for a suitable choice of half characteristic (ν, μ) and a base point $o \in \Sigma$.*

2.4 Spectral Curves of Genus One

All of the computations in Chapter 4 are carried out for spectral curves of genus one and hence this case deserves special attention. The Weierstrass-$\wp$ functions are useful to write the fundamental bidifferential $\omega_{0,2}$ and the correlators $\omega_{g,n}$ in a simple form. Suppose that the underlying Riemann surface Σ of a given spectral curve has genus one. Let $(\mathcal{A}, \mathcal{B})$ be a choice of symplectic basis for $H_1(\Sigma, \mathbb{Z})$ and let $o \in \Sigma$ be any base point. Integration of the unique holomorphic one-form $\eta \in H^1(\Sigma)$

gives the isomorphism

$$a\colon \Sigma \to \mathbb{C}/\Lambda, \qquad a(p) = \frac{1}{\varpi_1}\int_o^p \eta,$$

where $\Lambda \subset \mathbb{C}$ is a lattice. The form η may be normalized by $\varpi_1 = \oint_{\mathcal{A}} \eta$ so that $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$. In this case we have that $a_*\mathcal{A} = [0, 1]$ and $a_*\mathcal{B} = [0, \tau]$. The lattice Λ determines the Weierstrass-$\wp$ function

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

which is a meromorphic function of $z \in \mathbb{C}/\Lambda$. It has a unique double pole at $z = 0$ with the following series expansion:

$$\wp(z;\tau) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \cdots . \tag{2.4.1}$$

The numbers g_2 and g_3 are the corresponding elliptic invariants. The Weierstrass function $\wp$ and its derivative $\wp'$ generate the field of meromorphic functions on Σ and satisfy the following polynomial relation:

$$\wp'(z,\tau)^2 = 4\wp(z,\tau)^3 - g_2\wp(z,\tau) - g_3.$$

The Weierstrass-ζ function is related to $\wp$ by $\zeta'(z;\tau) = -\wp(z;\tau)$. For the rest of this section we assume that τ is fixed and thus it is dropped from the notation. From (2.4.1), it is clear that the bidifferential

$$\omega_{0,2}(z_1, z_2) = (\wp(z_1 - z_2) + \kappa)dz_1dz_2 \tag{2.4.2}$$

satisfies the properties of a fundamental bidifferential (2.1.1) for any constant κ. If one chooses

$$\kappa = G_2 \coloneqq \sum_{\omega\in\Lambda^*} \frac{1}{\omega^{2n}},$$

then $\omega_{0,2}$ satisfies the normalization conditions on the choice of cycles $\mathcal{A} = [0,1]$ and $\mathcal{B} = [0,\tau]$:

$$\begin{aligned}\oint\limits_{z_1\in\mathcal{A}} \omega_{0,2}(z_1,z_2) &= \left(\int_0^1 (\wp(z_1-z_2)+G_2)dz_1\right)dz_2 = \left(-\zeta(1-z_2)+\zeta(-z_2)+G_2\right)dz_2 \\ &= \left(-2\zeta\left(\frac{1}{2}\right)+G_2\right)dz_2 = (-G_2+G_2)dz_2 = 0, \end{aligned} \tag{2.4.3}$$

$$\begin{aligned}\oint\limits_{z_1\in\mathcal{B}} \omega_{0,2}(z_1,z_2) &= \left(\int_0^\tau (\wp(z_1-z_2)+G_2)dz_1\right)dz_2 = \left(-\zeta(\tau-z_2)+\zeta(-z_2)+\tau G_2\right)dz_2 \\ &= \left(-2\zeta\left(\frac{\tau}{2}\right)+\tau G_2\right)dz_2 = 2\pi i dz_2. \end{aligned} \tag{2.4.4}$$

The elliptic coordinate z is also useful when explicitly computing the one-forms $d\xi_\ell^\alpha$'s. Considering the local coordinate ζ near a ramification point α, it follows from definition (2.1.11) that

$$d\xi_\ell^\alpha(z) = \frac{1}{\ell!}\frac{d^{\ell-1}}{d\zeta^{\ell-1}}\left(\wp(z-z(\zeta))\frac{dz}{d\zeta}\right)\Bigg|_{\zeta=0} dz,$$

where we write $z(\zeta) = a(\zeta)$. Making use of the unique holomorphic one-form η we can explicitly relate ζ and z:

$$a_*\eta = a_*\eta(x,y)dx = dz,$$

$$\frac{dz}{d\zeta} = \frac{dz}{dx}\frac{dx}{d\zeta} = \eta(x(\zeta),y(\zeta))\zeta^{r-1}.$$

Another advantage in the case of genus one is that the θ functions have a nice relation to the Weierstrass-ζ function. More specifically, in our calculations we use the relation below, which an instance of the more general expression (2.3.2).

$$\frac{d}{dz}\ln\left(\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z;\tau)\right) = \zeta(z;\tau) - G_2 z.$$

This discussion allows us to write the correlators $\{\omega_{g,n}\}$ in a simple way for all spectral curves of genus one.

Corollary 2.4.1. *The correlators of any genus one spectral curve can be written in the coordinates $z \in \mathbb{C}/\Lambda$ as*

$$\omega_{g,n} = \sum_{\substack{I\amalg J=\{1,\ldots,n\}\\ \mathbf{p}=(p_1,\ldots,p_n)}} C(I,J,\mathbf{p}) \prod_{i\in I} \wp(z_i-\alpha_j)^{p_i} \prod_{j\in J} \wp'(z_j-\alpha_j)^{p_j}\, dz_1 \ldots dz_n,$$

where $p_i \geq 0$ and only finitely many coefficients $C \in \mathbb{C}$ are non-zero.

Since integrals of powers of $\wp(z)$ can be computed recursively, this is a useful method by which one may compute both the perturbative and non-perturbative expressions.

Results involving non-perturbative topological recursion generating functions have recently appeared for genus one curves [Iwa20] and for hyperelliptic curves [MO20, EGF21]. It would be nice to clarify the precise correspondence between these results and ours.

2.5 Fundamental Objects

The constructions of holomorphic forms and fundamental bidifferentials are outlined in this section. The zero locus of a bivariate polynomial $P(x, y)$ defines a complex curve in $\mathbb{C}^2$:

$$\Sigma = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\}, \tag{2.5.1}$$

For the purpose of non-perturbative topological recursion, the first thing one must do is compactify Σ. This can be done by regarding it as a zero locus in either $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ with respect to the corresponding standard charts $[x\colon y\colon 1]$ and $([x\colon 1],[y\colon 1])$. The next thing would be to choose two meromorphic functions, which we declare to be the coordinate projection maps x and y. Note that there is a birational map between $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by $\tau\colon [x\colon y\colon z] \mapsto ([x\colon z],[y\colon z])$. It induces a birational morphism between a complex curve $\Sigma \subset \mathbb{CP}^2$ and its image $\tau(\Sigma)$. Moreover, this map is the identity in the affine patches $\{[x\colon y\colon 1]\} \subset \mathbb{CP}^2$ and $\{([x\colon 1],[y\colon 1])\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. Given a bivariate polynomial $P(x,y)$, it is equivalent to think of its zero locus as a complex curve inside either $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$, since any birational resolution $\pi : \tilde{\Sigma} \to \Sigma$ will induce a resolution $\pi \circ \tau\colon \tilde{\Sigma} \to \tau(\Sigma)$. It is a classical result that $x \circ \pi$ and $y \circ \pi$ generate the field of meromorphic functions on $\tilde{\Sigma}$. We hence make the following definition:

Definition 2.5.1. Let $\pi\colon \tilde{\Sigma} \to \Sigma$ be a resolution of singularities and denote by g the genus of the smooth curve $\tilde{\Sigma}$. Let $\omega_{0,2}$ be the normalized fundamental bidifferential for a choice of a symplectic basis $\{\mathcal{A}_i, \mathcal{B}_i\}_{1\le i\le g}$ on $\tilde{\Sigma}$. Assuming that the ramification points of $x \circ \pi$ are away from the singular points of Σ, we define the *spectral curve of to* $P(x,y)$ to be

$$\left(\tilde{\Sigma}, x \circ \pi, y \circ \pi, \omega_{0,2}\right).$$

Calculating the genus, holomorphic one-forms and fundamental bidifferentials for smooth curves is fairly straight forward. In the following discussion we describe these procedures and extend them to the singular case. A point $p = (x_0, y_0) \in \Sigma$ is said to be singular if $P_x(x_0,y_0) = 0 = P_y(x_0,y_0)$. The order m_p of the singularity is the lowest degree of the monomials in $P(x+x_0, y+y_0)$. Let $P^{m_p}_{(x_0,y_0)}(x,y)$ be the degree m_p part. If it factors in m_p distinct linear factors, then the singularity is said to be ordinary.

Let

$$X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \;\cdots\; X_{n-1} \xleftarrow{\phi_n} X_n$$

be a sequence of holomorphic maps between complex surfaces $X_0, \ldots, X_n$ such that ϕ_{i+1} is the blowup of finitely many points $p_{ik} \in X_k$ and such that the points blown up by ϕ_{i+2} lie on the exceptional divisors $E_{ik} = \phi_{i+1}^{-1}(p_{ik}) \subset X_k$. If one blows up at exactly the singular points of a curve $\Sigma \in X_0$, such a sequence is finite and provides a resolution of singularities. For any given such sequence, the set $\{p_{ik}\}$ is called the set of *infinitely near points*.

Definition 2.5.2 (Multiplicity at infinitely near points)**.** A curve $\Sigma_0 \subset X_0$ has multiplicity of at least ν_{ik} at the *infinitely near points* p_{ik} if

- Σ_0 has multiplicity at least ν_{0k} at $p_{0k} \in \Sigma_0$,
- $\Sigma_1 = \phi_1^{-1}(\Sigma_0) - \sum \nu_{0k} E_{0k}$ has multiplicity at least ν_{ik} at the infinitely near points p_{ik} with $i \geq 1$.

Definition 2.5.3 (Adjoint)**.** Let Σ be a complex curve. Another complex curve Σ' is said to be *adjoint* to Σ if Σ' has, at all infinitely near singular points p_{ij} of Σ of multiplicity ν_{ij}, multiplicity at least $\nu_{ij} - 1$.

$$P(x, y) = \sum_{(i,j) \in \mathbb{Z}^2} P_{ij} x^i y^j .$$

Definition 2.5.4 (Newton Polygon)**.** Let $P(x, y)$ be a polynomial of the form (2.5.1). The set $N(P) = \{(i, j) \in \mathbb{C}^2 \mid P_{i,j} \neq 0\}$ is the *Newton polytope* of P. Its convex hull is the so-called *Newton polygon* of P. Denote by $\mathring{N}(P)$ the set of interior points of $N(P)$ with integer coordinates.

The combinatorial structure of the Newton polygon characterizes the space of holomorphic differentials of curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$. If Σ is smooth, then we have that

$$H^1(\Sigma, \mathbb{C}) = \left\{ \frac{Q(x, y) dx}{P_y(x, y)} \middle| Q(x, y) = \sum_{(u,v) \in \mathring{N}(P)} Q_{u,v} x^{u-1} y^{v-1} \right\} .$$

As explained in [Eyn18], the choice of interior points guarantees a finite behaviour at the points at infinity. In the smooth case, if P is generic of degree d, the count of interior points of the polygon agrees with the genus-degree formula for curves in $\mathbb{CP}^2$:

$$g = \binom{d-1}{2}.$$

For curves with singularities, $Q(x, y)$ must satisfy the appropriate tangency conditions at the singular points to ensure holomorphicity.

Theorem 2.5.5 (Holomorphic forms, singular case)**.** *Let Σ be a complex plane curve with polynomial equation $P(x, y) = 0$. The space of holomorphic one-forms on a smooth resolution $\pi\colon \tilde{\Sigma} \to \Sigma$ is characterized by*

$$H^1(\tilde{\Sigma}, \mathbb{C}) = \left\{ \frac{Q(x,y)dx}{P_y(x,y)} \middle| Q(x,y) = \sum_{(u,v)\in \mathring{N}(P)} Q_{u,v} x^{u-1} y^{v-1}, \{Q(x,y) = 0\} \text{ is adjoint to } \Sigma \right\}.$$

The proof can be found in standard literature. The local tangency condition is explained in [BK86].

Fundamental Bidifferentials

A combinatorial construction of the fundamental bidifferential $\omega_{0,2}$ is given in [Eyn18] for curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with at worst double points. Using a similar argument as in the case of holomorphic forms, we generalize it to singular curves of any type. We start with the Theorem for the smooth case:

Theorem 2.5.6 (Fundamental bidifferential, smooth case)**.** *Let Σ be a smooth complex plane curve with polynomial equation $P(x, y) = 0$. Define the polynomial Q from the Newton polygon $N(P)$ as follows:*

$$\begin{aligned} Q\left(x_1, y_1, x_2, y_2\right) = \sum_{(i,j)\in N (i',j')\in N} P_{i,j} P_{i',j'} \sum_{(u,v)\in \mathbb{Z}^2 \cap \text{ triangle } (i,j),(i',j'),(i,j')} |u-i|\,|v-j'| \\ \Big(\delta_{(u,v)\notin \mathring{N} \cup [(i,j),(i',j')]} x_1^{u-1} y_1^{v-1} x_2^{i+i'-u-1} y_2^{j+j'-v-1} \\ + \delta_{(u,v)\notin \mathring{N} \text{ and } (i+i'-u, j+j'-v)\in N} x_2^{u-1} y_2^{v-1} x_1^{i+i'-u-1} y_1^{j+j'-v-1} \\ + \frac{1}{2} \delta_{(u,v)\in [(i,j),(i',j')]} x_1^{u-1} y_1^{v-1} x_2^{i+i'-u-1} y_2^{j+j'-v-1} \Big) \end{aligned} \tag{2.5.2}$$

Then the following expression is a fundamental bidifferential on Σ, and any other fundamental bidifferential can be obtained by adding a linear combination of products of holomorphic forms.

$$\omega_{0,2}(x_1, y_1, x_2, y_2) = \frac{\frac{P(x_1,y_2)P(x_2,y_1)}{(x_1-x_2)^2(y_1-y_2)^2} + Q(x_1, y_1, x_2, y_2)}{P_y(x_1, y_1)P_y(x_2, y_2)} dx_1 dx_2 \tag{2.5.3}$$

Since any other bidifferential differs from (2.5.3) by a symmetric combination of products of holomorphic forms, it follows from the above Theorem that the space of fundamental bidifferentials is an affine space with vector space of dimension $\binom{g+1}{2}$. The normalization conditions in the $\mathcal{A}$ cycles give the same number of constraints, and hence the normalized bidifferential is unique. In the case where Σ is singular, we use a similar argument to (2.5.5):

Theorem 2.5.7 (Fundamental bidifferential, singular case). *Let Σ be a complex curve with polynomial equation $P(x, y) = 0$. Consider the polynomial*

$$Q_\kappa\,(x_1, y_1, x_2, y_2) = \sum_{(i,j),(i',j')\in \mathring{N}\times\mathring{N}} \kappa_{ij,i'j'} x_1^{i-1} y_1^{j-1} x_2^{i'-1} y_2^{j'-1}, \qquad \kappa_{ij,i'j'} = \kappa_{i'j',ij}.$$

Then there exists a choice of coefficients κ such that the following is a fundamental bidifferential on a resolution $\tilde{\Sigma} \to \Sigma$:

$$\omega_{0,2}(x_1, y_1, x_2, y_2) = \frac{\frac{P(x_1,y_2)P(x_2,y_1)}{(x_1-x_2)^2(y_1-y_2)^2} + Q(x_1, y_1, x_2, y_2) + Q_\kappa(x_1, y_1, x_2, y_2)}{P_y(x_1, y_1)P_y(x_2, y_2)} dx_1 dx_2. \tag{2.5.4}$$

Proof. Let $R(x_1, y_1, x_2, y_2)$ be the numerator of (2.5.4). Since $P(x, y)$ vanishes identically on Σ, $R(x_1, y_1, x_2, y_2)(x_1 - x_2)^2$ can be written as a polynomial after cancelling the factor $(y_1 - y_2)^2$ from the expression

$$\frac{P(x_1, y_2)P(x_2, y_1)}{(x_1 - x_2)^2(y_1 - y_2)^2} = \frac{(P(x_1, y_2) - P(x_2, y_2))\,(P(x_2, y_1) - P(x_2, y_2))}{(x_1 - x_2)^2(y_1 - y_2)^2}.$$

Call this polynomial $S(x_1, y_1, x_2, y_2)$. The addition of Q doesn't affect the pole structure along the diagonal $\Delta \subset \Sigma \times \Sigma$. We must then choose κ so that, fixing $(x_1, y_1) = (x_0, y_0)$, the bidifferential $\omega_{0,2}$ is holomorphic at all generic points (x_2, y_2)

away from (x_0, y_0). This is again done by imposing that

$$S(x_2, y_2) = S(x_0, y_0, x_2, y_2)$$

is adjoint to $P(x_2, y_2)$ modulo $P(x_0, y_0)$. □

Example 2.5.8. The A-polynomial of the figure eight knot is given by

$$A_{4_1}(x, \ell) = \ell^2 x^2 - \ell(x^4 - x^3 - 2x^2 - x + 1) + x^2. \tag{2.5.5}$$

It has two ordinary singular points of order two at $(\pm 1, \mp 1)$. Since the Newton polygon (4·2) has three interior points, the genus is $g = 3 - 2 = 1$. The holomorphic one-form can be obtained by solving the equations below for the coefficients $\{c_0, c_1, c_2\}$:

$$Q(x, \ell) = Q_{1,1} + Q_{2,1}x + Q_{3,1}x^2, \qquad Q(1, -1) = 0, \qquad Q(-1, 1) = 0,$$

which implies $Q(x, \ell) = Q_{1,1}(1 - x^2)$ and hence

$$\eta(x, \ell) = \frac{(1 - x^2)dx}{2\ell x^2 - (x^4 - x^3 - 2x^2 - x + 1)}.$$

For the fundamental bidifferential, we first compute the numerator

$$S(x_1, \ell_1, x_2, \ell_2) = S_1 + S_2$$

where

$$\begin{aligned} S_1(x_1, \ell_1, x_2, \ell_2) &\coloneqq Q(x_1, \ell_1, x_2, \ell_2) \\ &= Q_{1,1} + (x_1 + x_2)Q_{1,2} + (x_1^2 + x_2^2)Q_{1,3} + x_1x_2Q_{2,2} + (x_1^2x_2 + x_1x_2^2)Q_{2,3} + x_1^2x_2^2Q_{3,3}, \end{aligned}$$

and

$$S_2(x_1, \ell_1, x_2, \ell_2) \coloneqq \frac{(A_{4_1}(x_1, \ell_2) - A_{4_1}(x_1, \ell_1))(A_{4_1}(x_2, \ell_1) - A_{4_1}(x_2, \ell_2))}{(\ell_1 - \ell_2)^2} \tag{2.5.6}$$

$$= \left(x_1^4 - x_1^3 - x_1^2\ell_1 - x_1^2\ell_2 - 2x_1^2 - x_1 + 1\right)\left(x_2^4 - x_2^3 - x_2^2\ell_1 - x_2^2\ell_2 - 2x_2^2 - x_2 + 1\right). \tag{2.5.7}$$

Imposing the tangency conditions at the two ordinary singular points

$$\begin{cases} f(1, -1, x_2, y_2) = 0 \mod A_{4_1}(x_2, y_2) \\ f(-1, 1, x_2, y_2) = 0 \mod A_{4_1}(x_2, y_2) \end{cases}$$

we find that

$$Q_{1,2} = -1, \qquad Q_{1,3} = -Q_{1,1}, \qquad Q_{2,2} = 1,$$

$$Q_{2,3} = -1, \qquad Q_{3,3} = -Q_{1,1}.$$

There is one free parameter $Q_{1,1}$ corresponding to the fact that the curve has genus $g = 1$. We give the explicit expression of $\omega_{0,2}$ for the figure eight in (4.4.3) for a specific choice of normalization.

Chapter 3

Perturbative Knot Invariants

The relation between Chern-Simons gauge theory and topological open string theory has been studied in the last decades through the Volume conjecture and AJ-conjecture. Another correspondence appears in the computation of partition functions in perturbative Chern-Simons theory. It is believed that such partition function can be computed through different methods: the state integral model, the AJ Conjecture, an asymptotic expansion of the Jones polynomial and topological recursion on the A-polynomial. In this section we review some important concepts in knot theory, focusing on the specific properties that are relevant to topological recursion.

3.1 Colored Jones Polynomials

The Jones polynomial is a link invariant that was discovered by Vaughan Jones in 1984 [Jon85]. A decade later, Edward Witten showed that it appears naturally in Chern-Simons theory as the expectation value of observables called Wilson loops [Wit89]. Three-dimensional Chern-Simons theory is one of the most well-studies examples of a topological quantum field theory. It depends on the choice of a compact connected 3-manifold M, a simple Lie group G (which can be compact or complex) and a principal G bundle $E \to M$. The space of configurations is given by connection 1-forms A with values in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The action is defined as

$$S = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA - \frac{2}{3} A \wedge A \wedge A\right)$$

and the corresponding partition function is of the form

$$Z = \int DAe^{iS[A]}.$$

The classical solutions to this system are given by the equations of motion

$$0 = \frac{\delta S}{\delta A} = \frac{k}{2\pi} F,$$

which is satisfied only if A is flat. Classical solutions thus correspond to gauge equivalence classes of flat connections of principal G-bundles on M. These are determined entirely by holonomies around noncontractible cycles on the base M. They are in one-to-one correspondence with equivalence classes of homomorphisms from the fundamental group of M on G up to conjugation in G. The most important observables in Chern-Simons theory are Wilson loops. Given an oriented loop $K \subset M$ and an irreducible representation R of G, the Wilson loop is defined as the expectation value of

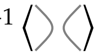
$$W_R^K(A) = \mathrm{Tr}_R\left(P\exp\left(\oint_K A\right)\right).$$

3.1.1 The Jones Polynomial

The Jones polynomial $J_K(q)$ of a knot K is an oriented knot invariant and can be defined directly using skein relations. It is a Laurent polynomial in $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Definition 3.1.1 (Bracket polynomial)**.** The bracket polynomial of a knot K, denoted by $\langle K \rangle$, is defined from the following rules:

- $\langle U \rangle = 1$, where U denotes the unknot.

-  $= A$  $+ A^{-1}$ 
- $\langle K \amalg U \rangle = (-A^2 - A^{-2})\langle K \rangle$

Definition 3.1.2 (Writhe)**.** The writhe of an oriented knot projection is the number of positive crossings minus the number of negative crossings. The writhe is not a knot invariant, since it is not invariant under Reidemeister type I moves.

Definition 3.1.3 (Jones polynomial)**.** The Jones polynomial of an oriented link K is the Laurent polynomial in $q^{-1/2}$

$$J(K) = (-A^3)^{-w(K)}\langle K \rangle \mid_{A=q^{-1/4}} .$$

It is easy to see that this definition is in fact invariant under Reidemeister type I moves. Assume K' is obtained from K via such move, then

$$J(K') = (-A^3)^{-w(K')}\langle K' \rangle = (-A^3)^{-w(K)-1}(-A)^3\langle K' \rangle = (-A^3)^{-w(K)}\langle K \rangle = J(K).$$

The Lie Group $SU(2)$ and the Colored Jones Polynomial

The colored Jones polynomial $J_K(R;q)$ is a generalization of the ordinary Jones polynomial. The *coloring* refers to a choice of a representation R of $SU(2)$. When R is the irreducible N-dimensional representation, $J_K(R;q)$ is denoted by $J_K(N;q)$. Defining $J_K(1;q) = 1$ and $J_K(2;q) = J_K(q)$, the colored Jones polynomial for any representation R of $SU(2)$ is defined via the following rules:

$$J_K(\oplus_i R_i;q) = \sum_i J_K(R_i;q)$$

$$J_{K^n}(R;q) = J_K(R^{\otimes n};q)$$

The representation theory of $SU(2)$ allows to compute any $J_K(N;q)$ from the Jones polynomial of K and its cablings K^m. For example, denoting by $\mathbf{N}$ the N-dimensional irreducible representation, the relations $\mathbf{2}^{\otimes 2} = \mathbf{1} \oplus \mathbf{3}$ and $\mathbf{2}^{\otimes 3} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4}$ imply

$$J_K(3;q) = J_{K^2}(q) - 1,$$

$$J_K(4;q) = J_{K^3}(q) - 2J_K(q).$$

Equivalently, $J_K(N,q)$ can be defined as a linear combination given by the Jones-Wenzl projection. There is a generalization of the Jones polynomial called HOMFLY polynomial that can be realized as a Chern-Simons theory for $G = \mathrm{SU}(2)$. However, its definition is not relevant for the rest of this discussion.

3.2 AJ-Conjecture and Perturbative Knot Invariants

The Volume Conjecture

The relation between the Jones and the A-polynomials of a knot goes back to the statement of the Volume Conjecture [MM01, MMO+02]:

Conjecture 3.2.1 (Murakami, Murakami, Okamoto, Takata, Yokota)**.** *Consider the N-colored Jones polynomial of a knot K with its variable specialized to $q = e^{2\pi i/N}$. In the large-N regime we have*

$$\lim_{N\to\infty} \frac{2\pi}{N} \log\left| J_K\left(N; q = e^{\frac{2\pi i}{N}}\right)\right| = Vol(S^3 \setminus K),$$

where Vol denotes the hyperbolic volume of the complement $S^3 \setminus K$.

A more general version was introduced in [Guk05]. It extends the range of values of the N-colored Jones polynomial and it is known as the Generalized Volume Conjecture:

Conjecture 3.2.2 (Gukov)**.** *Consider a pair (u,v) such that $(e^v, -e^{iu})$ is in the zero locus of the A-polynomial of a knot K. Then its N-colored Jones polynomial has the following asymptotic behaviour:*

$$v = -\frac{d}{du} \lim_{\substack{N,k\to\infty \\ N/k=u}} \frac{1}{k} \log J_K\left(N; e^{\frac{2\pi i}{k}}\right).$$

An alternative formulation can be used to define the partition function $\mathcal{J}_{\mathrm{CS}}$ as a certain asymptotic limit:

Conjecture 3.2.3 (Gukov)**.** *For any knot K, in the regime $N \to \infty$, $h \to \infty$ and $N \cdot h = u$, the colored Jones polynomial has an asymptotic expansion of the form*

$$J_K(N,q) \sim \mathcal{J}_{CS} = \exp\left(\frac{1}{h}\sum_{\chi \geq 0} h^{\chi} S_{\chi}\right)$$

The A-Polynomial

The A-polynomial is an invariant of a three-dimensional manifold M with a single boundary torus $\partial M \cong \mathbb{T}^2$ introduced by Cooper et al. in [CCG+94]. Such manifold can be obtained by considering the complement of a knot K in the three-sphere $M = S^3 \setminus K$. Consider the peripheral group $\pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$ generated by loops in the boundary torus. Denote by $\langle \mu, \lambda \rangle$ a choice of an oriented basis of $\pi_1(\partial M)$. In the case when M is a knot complement, we refer to these generators by *meridian* μ and *parallel* λ loops respectively: μ describes a loop around a strand of the knot and λ runs parallel to the knot with winding number zero. Although this choice is unique up to inversion of both μ and λ, the following construction is independent of that choice.

Let $R(\pi_1(M))$ be the $\mathrm{SL}(2,\mathbb{C})$ representation variety of $\pi_1(M)$

$$R\left(\pi_1(M)\right) := \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2,\mathbb{C})),$$

and denote by $R_U \subset R(\pi_1(M))$ the subvariety of upper triangular representations. Define the map ϵ on R_U by selecting the top left eigenvalues of $\rho(\mu)$ and $\rho(\lambda)$ respectively: $\epsilon(\rho) = (m, \ell)$. It can be seen that the closure in $\mathbb{C}^2$ of the image of

$$\epsilon \colon R_U \to \mathbb{C}^* \times \mathbb{C}^*$$

is a union of irreducible algebraic varieties $X_1 \cup \ldots \cup X_s = \Sigma_K$ of dimension one.

Definition 3.2.4 (A-Polynomial)**.** The *A-polynomial* $A_M(m, \ell)$ is the product of the defining polynomials of the irreducible varieties $X_1, \dots, X_s$:

$$A_M(m,\ell) = \prod_{i=1}^{s} A_K^{(i)}(m,\ell), \qquad X_i \cong \{(m,\ell) \in \mathbb{C}^2 \mid A_K^{(i)}(m,\ell) = 0\}.$$

Note that since the μ and λ commute in $\pi_1(\partial M)$, any given representation $\rho \in R(\pi_1(M))$ can be conjugated to one such that

$$\rho(\mu) = \begin{pmatrix} m & * \\ 0 & 1/m \end{pmatrix}, \qquad \rho(\lambda) = \begin{pmatrix} \ell & * \\ 0 & 1/\ell \end{pmatrix}.$$

The polynomial $A_M(m,\ell)$ is unique up to normalization, is independent on the choice of μ and λ and its zero locus is invariant under the involution $(m,\ell) \mapsto (1/m, 1/\ell)$ [CCG+94]. In general it is not irreducible; its *geometric component* is defined as the irreducible component containing the unique discrete faithful representation. The factor $(\ell - 1)$ is present in any A-polynomial, and it is often referred to as the *abelian component*. It can be shown that $A_K(m,\ell)$ contains only even powers of m. In Chapter 4 we choose to write it in terms of the variable $x = m^2$. For the purposes of non-perturbative topological recursion, we require Σ_K to be compact. We thus consider its compactification in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Remark 3.2.5. A knot K is said to be *hyperbolic* if the complement in the three dimensional sphere $S^3 \setminus K$ admits a complete hyperbolic structure. Every knot is either hyperbolic, a torus knot or a satellite knot.

The AJ Conjecture relates the A-polynomial to the N-colored Jones polynomial of a knot K. The starting point is that the N-colored Jones polynomial $J_K(N;q)$ of a knot K is *q-holonomic*, that is, it satisfies a finite recursion relation

$$\sum_{j=0}^{d} a_j(q^{\frac{N}{2}}, q) J_K(N+j;q) = 0, \qquad \forall N \in \mathbb{N},$$

where the a_j's are rational functions with integer coefficients [GL05]. Regarding m and ℓ as operators acting by multiplication by $q^{\frac{N}{2}}$ and shifting $N \mapsto N+1$

respectively, we may rewrite the condition above in the operator form:

$$\hat{A}(m,\ell)\cdot J_K(N;q)=0,\quad \hat{A}(m,\ell)=\sum_{j=0}^{d}a_j(m,q)\ell^j. \tag{3.2.1}$$

Note that $\hat{A}(m,\ell)$ is polynomial in ℓ and possibly a rational function in m. After clearing the denominators, the AJ Conjecture predicts that in the classical limit $\hat{A}$ should agree with the A polynomial of the knot K.

Conjecture 3.2.6 (Garoufalidis and Le)**.** *Let $J_K(N;q)$ be the N-colored Jones polynomial of a knot K. Consider the ring of Laurent polynomials in ℓ with coefficients rational functions of m*

$$\mathcal{T}=\left\{\sum_{k\in\mathbb{Z}}a_k(m)\ell^k \mid \quad a_k(m)\in\mathcal{R}(m)\right\}$$

with the commutation relation $\ell m=e^{\frac{N}{2}}m\ell$. The ideal

$$\mathcal{J}=\{a\in\mathcal{T}\mid a\cdot J_K(N;q)=0,\forall N\}$$

annihilating $J_K(N;q)$ is principal, and thus generated by a polynomial with minimal ℓ degree $\hat{A}(m,\ell)$. The A-polynomial $A_K(m,\ell)$ of K coincides with the classical limit $q\to 0$ of $\hat{A}(m,\ell)$ up to a multiplicative polynomial factor independent of ℓ.

Remark 3.2.7. This AJ Conjecture is known to hold for most 2-bridge knots and some 3-strand pretzel knots. Some 2-bridge knots include $\mathbf{4}_1,\mathbf{8}_{18}$ and $\mathbf{9}_{35}$. Some knots that do not fall into this category are $\mathbf{9}_{48},\mathbf{10}_{139}$.

Perturbative Knot Invariants

One may extend the validity of (3.2.1) in the large N limit to a formal parameter u, with $N\cdot h=u$, and look for solutions in the form of generating series. Define $\hat{m}$ and $\hat{\ell}$ to be the operators satisfying

$$\hat{m}f(u)=e^u f(u),\qquad \hat{\ell}f(u)=f(u+h),\qquad \hat{m}\hat{\ell}=q^{\frac{1}{2}}\hat{m}\hat{\ell},\qquad q=e^{2h}.$$

One should think of $u = N \cdot h$ as a formal continuous variable, while restricting N to integer values. The new equations read

$$\hat{A}(\hat{m}, \hat{\ell}) \cdot \mathcal{J}_{\mathrm{AJ}}(u, h) = 0, \quad \mathcal{J}_{\mathrm{AJ}}(u, h) =: \exp\left(\frac{1}{h}\sum_{\mathcal{X} \geq 0} h^{\mathcal{X}} S_{\mathcal{X}}(u)\right). \tag{3.2.2}$$

Physicists believe that there are several equivalent descriptions of the perturbative partition function of Chern-Simons gauge theory [DGLZ09]. In particular, the function $\mathcal{J}_{\mathrm{AJ}}$ of a knot K should coincide with the partition function $\mathcal{J}_{\mathrm{CS}}$ on $M = S^3 \setminus K$ evaluated via an asymptotic expansion of the Jones polynomial, as well as with a the state-integral model $\mathcal{J}_H$ developed by Hikami [Hik01, Hik07]. This three-fold equivalence is remarkably useful from a computational standpoint: combining the different approaches one can dramatically increase the computational power. We do not know any precise formulation of these statements other than the discussion in [DGLZ09]; in fact, whereas Chern-Simons theory may be defined for any compact oriented three-dimensional manifold M, the function $\mathcal{J}_H$ requires M to be hyperbolic with finite volume, and similarly $\mathcal{J}_{\mathrm{AJ}}$ and $\mathcal{J}_{\mathrm{CS}}$ are defined only for knot complements inside the three dimensional sphere $S^3 \setminus K$. Therefore, the following equivalence must be understood in each appropriate context:

$$\mathcal{J}_{\mathrm{AJ}} \sim \mathcal{J}_H \sim \mathcal{J}_{\mathrm{CS}}. \tag{3.2.3}$$

Example: Figure Eight Knot

Although the N-colored Jones polynomial can be computed explicitly for finite N values, its asymptotic properties as $N \to \infty$ are difficult to study. For some knots, closed forms for J_K are known. In the case of the unknot U, for example, we have

$$J_U(N; q) = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}. \tag{3.2.4}$$

Another knot for which its N-colored Jones polynomial is explicitly known is the figure eight knot, denoted by $\mathbf{4}_1$. The explicit calculation appears in [MY04]:

$$J_{\mathbf{4}_1}(N;q) = \sum_{n=0}^{N-1} \prod_{k=1}^{n} \left(q^{\frac{N-k}{2}} - q^{-\frac{N-k}{2}}\right)\left(q^{\frac{N+k}{2}} - q^{-\frac{N+k}{2}}\right) \tag{3.2.5}$$

In this case the corresponding Kashaev invariant is

$$\langle \mathbf{4}_1 \rangle_N = J_{\mathbf{4}_1}(N; q = e^{2\pi i/N}) = \sum_{n=0}^{N-1} \prod_{k=1}^{n} (1-q^k)(1-q^{-k}). \tag{3.2.6}$$

The $\hat{A}_{4_1}$ operator was explicitly computed in [Gar04] using the Mathematica package `q-Zeil`. In the $(\hat{m}, \hat{\ell})$ variables it reads:

$$\hat{A}_{\mathbf{4}_1}(\hat{\ell}, \hat{m}) = \sum_{j=0}^{3} a_j(\hat{m}, q)\hat{\ell}^j, \tag{3.2.7}$$

with coefficients

$$a_0(\hat{m}, q) = \frac{q\hat{m}^2}{\left(1+q\hat{m}^2\right)\left(-1+q\hat{m}^4\right)} \tag{3.2.8}$$

$$a_1(\hat{m}, q) = \frac{1 + \left(q^2-2q\right)\hat{m}^2 - \left(q^3-q^2+q\right)\hat{m}^4 - \left(2q^3-q^2\right)\hat{m}^6 + q^4\hat{m}^8}{q^{1/2}\hat{m}^2\left(1+q^2\hat{m}^2-q\hat{m}^4-q^3\hat{m}^6\right)} \tag{3.2.9}$$

$$a_2(\hat{m}, q) = -\frac{1 - \left(2q^2-q\right)\hat{m}^2 - \left(q^5-q^4+q^3\right)\hat{m}^4 + \left(q^7-2q^6\right)\hat{m}^6 + q^8\hat{m}^8}{q\hat{m}^2\left(1+q\hat{m}^2-q^5\hat{m}^4-q^6\hat{m}^6\right)} \tag{3.2.10}$$

$$a_3(\hat{m}, q) = -\frac{q^4\hat{m}^2}{q^{1/2}\left(1+q^2\hat{m}^2\right)\left(-1+q^5\hat{m}^4\right)}. \tag{3.2.11}$$

In this case one can explicitly check the AJ Conjecture:

$$\hat{A}_{\mathbf{4}_1}(\hat{m}, \hat{\ell}) \underset{\hbar\to 0}{\longrightarrow} \frac{A_{\mathbf{4}_1}(m, \ell)}{m^2\left(m^2-1\right)\left(m^2+1\right)^2}. \tag{3.2.12}$$

Finally, solving Equation 3.2.2 one can recursively evaluate the functions S_χ in the expansion $\mathcal{J}_{\mathrm{AJ}}$. The explicit calculations can be found in [DGLZ09]. We use these to compare our calculations for the figure eight knot in Section 4.4.1.

Example: Once Punctured Torus Bundle L^2R

Consider the once punctured torus bundle over S^1

$$M = (\mathbb{T}^2 \setminus \{0\}) \times [0,1]/(x,0) \sim (\phi(x),1),$$

with monodromy $\phi \in \mathrm{SL}(2,\mathbb{C})$. Thurston's hyperbolization theorem implies that if ϕ has two distinct real eigenvalues, then M admits a hyperbolic metric with finite volume. This is equivalent to ϕ admitting a decomposition of the form $\phi = L^{s_1}R^{t_1}\cdots L^{s_k}R^{t_k}$, where

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

If $\phi = LR$, M corresponds to the figure eight complement in the previous example. Consider instead $\phi = L^2R$. In this case M is isomorphic to a knot complement in $\mathbb{RP}^3$ [BPZ87]. Hikami [Hik07] describes how to decompose M into three ideal hyperbolic tetrahedra and, using a state model, obtains the partition function

$$\mathcal{J}_H(u,h) = \frac{\sqrt{2}}{4\pi h}\int_C dp_1 dp_2 \frac{\Phi_h\left(-p_1 - 2u + i\pi + h\right)}{\Phi_h\left(-p_1 - p_2 - 2u - i\pi - h\right)\Phi_h\left(2p_1 + p_2 + 2u - i\pi - h\right)}$$
$$\times \exp\left[-\frac{2}{h}\left(u\left(u + p_1 + p_2\right) + \frac{1}{2}\left(p_1 + \frac{1}{2}p_2\right)^2 - \frac{\pi^2}{12} - \frac{h^2}{12} - \frac{\pi}{4}h\right) - u\right],$$

where Φ_h denotes the quantum dilogarithm function. Expanding the integral above around the appropriate critical point one obtains

$$\mathcal{J}_H(u,h) = \frac{e^{u+\frac{1}{h}V(u)}}{2\sqrt{2}\pi h}\int_C dp_1 dp_2 e^{-\frac{b_{11}(u)p_1^2 + b_{22}(u)p_2^2 + b_{12}(u)p_1p_2}{2h}}$$
$$\times \exp\left[\frac{1}{h}\sum_{i+j=3}^{\infty}\Upsilon_{i,j,-1}p_1^i p_2^j + \sum_{i,j=0}^{\infty}\sum_{k=0}^{\infty}\Upsilon_{i,j,k}p_1^i p_2^j h^k\right], \tag{3.2.13}$$

where Υ are functions of u, and the potential $V(u)$ is explicitly given by

$$\begin{aligned} V(u) =& \mathrm{Li}_2\left(\frac{1}{m^2x}\right) - \mathrm{Li}_2\left(\frac{1}{m^2xy}\right) - \mathrm{Li}_2\left(m^2x^2y\right) \\ & - \log\left(m^2\right)\log\left(m^2x^2y^2\right) - 2\left[\log\left(xy^{1/2}\right)\right]^2 + \frac{\pi^2}{6}. \end{aligned}$$

Writing $\mathcal{J}_H(u,h) = \exp\left(\frac{1}{h}\sum_{\chi\geq 0} h^{\chi} S_{\chi}(u)\right)$ one can compute the S_{χ} via Feynman rules corresponding to the Gaussian. The explicit calculations can be found in [DFM11]; we will compare these results to our calculations in Section 4.4.2.

3.3 Topological Recursion and the AJ-Conjecture

The A-polynomial of a knot K can be regarded as the input spectral curve of topological recursion.

Definition 3.3.1 (A-polynomial spectral curves)**.** Let K be a hyperbolic knot. Denote by $A_K(x,\ell)$ the geometric component of its A-polynomial. The A-polynomial spectral curve of K is defined to be

$$\left(\{A_K(x,\ell)=0\}, \ln(m), \ln(\ell), \omega_{0,2}\right),$$

where $m^2 = x$ and for a choice of a bidifferential $\omega_{0,2}$ on a smooth resolution.

It follows from the previous discussion that A has an involution $\iota(m,\ell) = (1/m, 1/\ell)$. In the case of genus one spectral curves, we can show that the quantization condition $\zeta_h = 0$ is satisfied when ι coincides with the elliptic involution $z \mapsto -z$ in the corresponding elliptic curve $\mathbb{C}/\Lambda$. This follows directly from the definition in Section 2.3:

$$\int_{\mathcal{B}-\tau\mathcal{A}} \ln(\ell)d\ln(m) = \int_{\iota_*(\mathcal{B}-\tau\mathcal{A})} \iota^*\ln(\ell)d\ln(m) = \int_{-(\mathcal{B}-\tau\mathcal{A})} \ln(1/\ell)d\ln(1/m) = -\int_{\mathcal{B}-\tau\mathcal{A}} \ln(\ell)d\ln(m).$$

Proposition 3.3.2. *The involution on the geometric components of the knots* 4_1*,* L^2R*,* 9_{35}*,*

9_{48} *and* 10_{139} *coincides with the elliptic involution of their corresponding elliptic models. Hence, they satisfy the quantization condition* $\zeta_h = 0$.

Proof. By direct computation. For each component of genus one, we find the explicit expression of the holomorphic one-form η and check that the involution $(m, \ell) \mapsto (1/m, 1/\ell)$ acts by $\iota^*(\eta) = -\eta$. □

The involution ι is also used by Borot and Eynard in the formulation of their conjectures. The appropriate function that should coincide with the Chern-Simons partition function is the non-perturbative [2|2]-kernel $\psi_{\mathrm{NP}}^{[2|2]}(p, \iota(p))$, for a choice of base points $(o, \iota(o))$. Adapted to our framework, Conjecture 5.6 in [BE12] can be restated in two versions as follows:

Conjecture 3.3.3. *Let* K *be a hyperbolic knot. Consider the Chern-Simons partition function* $\mathcal{J}_{AJ} = \exp\left(\sum_{\chi \geq 0} h^{\chi-1} S_\chi\right)$ *computed from the AJ Conjecture (3.2.2). Let* $\psi_{NP}^{[2|2]}$ *be the non-perturbative wave function corresponding to its A-polynomial spectral curve* $A_K(m, \ell)$*. In the regime* $N \to \infty$, $h \to 0$ *with* $Nh = u$ *fixed and* $m = e^u$*, the following asymptotic expansions agree*

$$\mathcal{J}_{AJ}(u, h) = C_h e^{\frac{1}{h} S_0 + S_1} \left(\psi_{NP}^{[2|2]}(m, 1/m)\right)^{1/2},$$

for a suitable choice of base point o*, up to some constant* C_h *depending on* h*. This must be understood as an all-order* h *expansion where the coefficients are meromorphic functions on the underlying spectral curve.*

Conjecture 3.3.4. *Let* M *be a hyperbolic manifold with toroidal boundary. Consider the Chern-Simons partition function* $\mathcal{J}_H = \exp\left(\sum_{\chi \geq 0} h^{\chi-1} S_\chi\right)$ *computed from the Hikami state integral model as in (3.2.13). Let* $\psi_{NP}^{[2|2]}$ *be the non-perturbative wave function corresponding to its A-polynomial spectral curve* $A_M(m, \ell)$*. Then the asymptotic expansions*

$$\mathcal{J}_H(u, h) = C_h e^{\frac{1}{h} S_0 + S_1} \left(\psi_{NP}^{[2|2]}(m, 1/m)\right)^{1/2},$$

agree for a suitable choice of base point o*, up to some constant* C_h *depending on* h*.*

This is one of the main conjectures we study in Section 4. It presents yet another way of obtaining a generating series for Chern-Simons theory, in this case using topological recursion. For the knots $\mathbf{4}_1$ and L^2R, the calculations using the state-integral model $\mathcal{J}_{\mathrm{H}}$ can be found in the paper by Dijkgraaf, Fuji and Manabe [DFM11], where the authors give the explicit form for the functions $S_{\mathcal{X}}(u)$ up to $\mathcal{X} = 4$ for $\mathbf{4}_1$, and $\mathcal{X} = 5$ for L^2R. For the figure eight knot $\mathbf{4}_1$, the computations of $\mathcal{J}_{\mathrm{AJ}}$ can be found in [DGLZ09], where the authors make use of the explicit form of the AJ Conjecture (3.2.8)-(3.2.11) to evaluate the functions $S_{\mathcal{X}}(u)$ up to $\mathcal{X} = 8$. A version of Conjecture 3.3.4 was proposed initially in [DFM11], considering the perturbative wave function instead. However, the authors found discrepancies that had to be fixed a posteriori by adding corrections order by order in h. The discrepancy was solved by Borot and Eynard in [BE12] by considering the non-perturbative version instead. The authors verified that topological recursion on the knots $\mathbf{4}_1$ and L^2R agrees with the previous calculations of $\mathcal{J}_{\mathrm{AJ}}$ and $\mathcal{J}_H$ up to orders h^3 and h^2 respectively.

Chapter 4

Computations and Examples

4.1 Computational Aspects

By definition, the topological recursion correlators $\{\omega_{g,n}\}$ are given by a recursive formula which involves residues about the ramification points of the spectral curve. At first glance, calculating the correlators would seem to be as simple as implementing (2.1.3) into a computer program (e.g. Sage or Mathematica) and hoping for a timely output. Unfortunately, this approach quickly becomes prohibitively slow for the spectral curves of interest as the Euler characteristic $\chi = 2g-2+n$ increases.

A more sophisticated approach is thus necessary in order to make further progress. We break up the calculation of the correlators in two steps. The first step is to calculate the free energies $F_{g,n}$ from Equation (2.1.10) by summing over (weights of) perturbative graphs, up to an automorphism factor, as explained in Section 2.1.1. The vertices of the graphs use information about the local structure of the corresponding singularity (r-spin intersection numbers, in our examples) while the edges contain information from the bidifferential $\omega_{0,2}$. Then, after specializing the x_ℓ^α variables as in Corollary (2.1.10), one obtains the correlators $\{\omega_{g,n}\}$.

An important virtue of this approach is that the first step yields a universal formula for $F_{g,n}$, valid for all spectral curves with a fixed number of ramifications points of each type. For example, formula for $F_{g,n}$ with the values ϕ (2.1.6) and $F_{0,1}$ (2.1.5) left unevaluated could be applied to any spectral curve with, for example, 3 ramification points of order 2 and one of order 3. Then plugging in the values for the coefficients in the expansions of $\omega_{0,2}$ and of y. This is fortunate since the first step is, by far, the most time consuming of the two steps. Furthermore, this computation can be implemented purely combinatorially without any need for a calculus and, is therefore, much faster than calculating residues.

Unfortunately, while the direct implementation of this second algorithm is significantly faster than the recursive formula in terms of residues, the complexity of the algorithm still grows too quickly as χ increases. This is because the usual way that one obtains all of the graphs in the calculation of $F_{g,n}$ is the way physicists obtain Feynman diagrams of genus g with n leaves. This boils down to applying differential operators to a generating function and then setting the variables to zero. The problem with this approach is that there will be many terms arising from the generating function approach which correspond to the same graph and the number of terms that correspond to a given graph grows very quickly with the complexity of the graph. What is necessary, then, is a new algorithm to generate all the terms with fewer redundant graphs. Such an algorithm is described in a paper by Maggiolo and Pagani [MP11], a version of which was implemented in Python by Greyson Potter. This code runs significantly faster than other methods of calculating the TR correlators and, in joint work, we were able to debug and utilize the code to run part of the computations in this thesis.

To describe in more detail our algorithm, we list below the steps to compute the correlators $\{\omega_{g,n}\}$. For this part, the starting data is a (local or global) spectral curve $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$ and a maximum Euler characteristic χ.

- Compute the set of ramification points $\mathcal{R}$, and the corresponding degrees of ramification $\{r_\alpha\}_{\alpha\in\mathcal{R}}$
- Solve local constraints for each $\alpha \in \mathcal{R}$ to obtain the generating functions $Z^{(r_\alpha)}$'s
- For all the pairs (g, n), $g \geq 0$, $n \geq 1$ such that $\chi \geq 2g - 2 + n > 0$, generate the

corresponding perturbative graphs of genus g, number of leaves $n+n_{\max}$ and vertices labeled by $\mathcal{R}$

- Generate all the possible labelings of the half edges according to the dimensional condition on the coefficients $F_g^{(r_\alpha)}$ at each vertex
- Compute the automorphism factor of each graph
- Sum over all the weights with corresponding automorphism factors, and symmetrize the x_ℓ^α variables to write it in terms of the $d\xi_\ell^\alpha$ forms

The rest of this section is aimed to discuss specific properties of the perturbative graphs that can be used to optimize its generating algorithm. Recall that the vertices of the graphs are weighted by quantities $F_g^{(r)}$, which satisfy the dimensional condition (2.1.15):

$$\sum_{i=1}^{n}(r\alpha_i+m_i+1)=2\,(r+1)\,(g-1)+rn+n=(r+1)(2g-2+n). \tag{4.1.1}$$

Proposition 4.1.1 (Bound on the dilaton leaves)**.** *Let $\Gamma\in\mathcal{G}$ be any graph such that $w(\Gamma)\neq 0$. Let $v\in\Gamma$ be a vertex of order $r=r_{\alpha(v)}$. The number of dilaton leaves adjacent to v satisfies $\tilde{n}_v\leq\tilde{n}_v^{max}$ where*

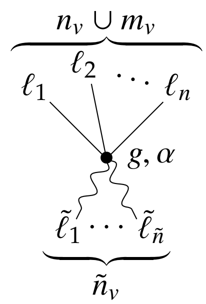
$$\tilde{n}_v^{max}=\begin{cases}3g(v)-3+|n_v|, & \chi=2\\ (r+1)\chi_v-1, & \chi\neq 2, odd\\ (r+1)\chi_v-2, & \chi\neq 2, even.\end{cases}$$

Proof. The Euler characteristic of v is by definition

$$\chi_v=2g(v)-2+|n_v|+|m_v|,$$

and the dimensional condition is

$$(r+1)\chi_v+(r+1)|\tilde{n}_v|=\sum_{l\in n_v\cup\tilde{n}_v\cup m_v}\ell(l).$$



From (2.1.17) it follows that all dilaton leaves have index $\tilde{\ell}_i > r+1$. The minimum number of ordinary leaves is 1 if χ is odd, and 2 if χ is even. The minimum valency of an ordinary leaf is 1. Therefore, it follows that

$$(r+1)\chi_v + (r+1)\tilde{n}_v^{\max} = 2 + (r+2)\tilde{n}_v^{\max}, \text{ if } \chi_v \text{ is even},$$

$$(r+1)\chi_v + (r+1)\tilde{n}_v^{\max} = 1 + (r+2)\tilde{n}_v^{\max}, \text{ if } \chi_v \text{ is odd}.$$

and hence at each vertex

$$\tilde{n}_v^{\max} = (r+1)\chi_v - 2, \text{ if } \chi_v \text{ is even},$$

$$\tilde{n}_v^{\max} = (r+1)\chi_v - 1, \text{ if } \chi_v \text{ is odd}.$$

Note that the case $r = 2$ is slightly different since the index $r+2 = 4$ does not appear in the x_ℓ^α variables. The smallest valency of the dilaton leaves is 5 instead. □

Proposition 4.1.2. *For fixed g and n, the number of graphs $\Gamma \in \mathcal{G}$ of genus g and n ordinary leaves such that $w(\Gamma) \neq 0$ is finite.*

Proof. It follows from the fact that there are finitely many isomorphism classes of graphs of a given Euler characteristic and that the maximum total number of dilaton leaves at each vertex is bounded. □

In particular, Proposition 4.1.2 implies that the functions $F_{g,n}[x_\ell^\alpha]$ defined in (2.1.18) are polynomials. We are also interested in bounding the amount of necessary input data, namely the values of the tensors $F_{0,1}$ and $\phi_{\ell_1,\ell_2}^{\alpha_1,\alpha_2}$, the one-forms $d\xi_\ell^\alpha$ and the r-spin intersection numbers. Assuming we want to compute the non-perturbative wave function up to $\hbar^{\chi+1}$, we need:

Proposition 4.1.3 (Expansions for the graph sum). *To compute the forms $\omega_{g,n}$ up to order $\hbar^{\chi+1}$, where the Euler characteristic $\chi = 2g-2+n$, one needs the values of*

- $F_{0,1}\left[\begin{smallmatrix}\alpha\\ -\tilde{\ell}\end{smallmatrix}\right], \tilde{\ell} \leq \begin{cases}(r+1)\chi + r - 1, & \text{if } \chi_v \text{ is even}\\ (r+1)\chi + r, & \text{if } \chi_v \text{ is odd}\end{cases}$

- $d\xi^{\alpha}_{\ell}, \qquad \ell \leq \begin{cases}(r+1)\chi - 1, & \text{if } \chi_v \text{ is even}\\ (r+1)\chi, & \text{if } \chi_v \text{ is odd}\end{cases}$

- $\phi^{\alpha_1,\alpha_2}_{\ell_1,\ell_2}, \ell_1+\ell_2 \leq \begin{cases}(r+1)\chi - 3, & \text{if } \chi_v \text{ is even}\\ (r+1)\chi - 2, & \text{if } \chi_v \text{ is odd}\end{cases}$

Proof. Consider all graphs of fixed Euler characteristic χ. According to (4.1.1), the maximum dimension of a vertex occurs when it has maximal Euler characteristic $\sum \ell + \sum \tilde{\ell} = (r+1)\chi$, where (ℓ, n) and $(\tilde{\ell}, \tilde{n})$ are the valences and number of the ordinary and dilaton leaves respectively. Again all dilaton leaves have $\tilde{\ell} > r+1$, and hence the maximum value of ℓ occurs when $\tilde{n} = 0$. The minimum number of leaves is 1 when χ is odd, and 2 if even. This makes $\ell_{\max} = (r+1)\chi - \delta_{\chi \mathrm{mod}(2),0}$. Considering now the case with at least one dilaton leaf, the dimensional condition $(r+1)\chi + (r+1)\tilde{n}$ implies that the maximum valence of single dilaton leaf will occur when there is exactly one and a minimum number of ordinary leaves of valence 1. Hence $\tilde{\ell}_{\max} = (r+1)\chi + (r+1) - 1 - \delta_{\chi \mathrm{mod}(2),0} = (r+1)\chi + r - \delta_{\chi \mathrm{mod}(2),0}$. Finally, for the internal edges, consider a vertex with minimum ordinary leaves and a single loop (4·1). The dimension of the vertex is $(r+1)(\chi - 2 + 2) = (r+1)\chi$, and hence again the maximum valency for a half internal edge is $(r+1)\chi - 1 - 1 - \delta_{\chi \mathrm{mod}(2),0} = (r+1)\chi - 2 - \delta_{\chi \mathrm{mod}(2),0}$. ☐

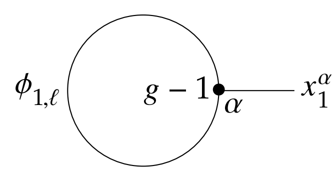


Figure 4·1: Maximum subscript index of ϕ

Corollary 4.1.4. *To compute the correlators $\{\omega_{g,n}\}$ up to Euler characteristic χ, one needs the values of $F_g^{(r)}[\,\ell_1 \dots \ell_n\,]$ up to Euler characteristic $\chi + \tilde{n}^{max}$.*

4.2 The Algorithm

The input of our algorithm is a plane curve Σ defined as the zero locus of bivariate polynomial $P(x, y)$ in either $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$. If the curve is singular, then we work on a smooth resolution $\pi\colon \tilde{\Sigma} \to \Sigma$ as explained earlier in (2.5.1). The output has three parts. First, the correlators $\{\omega_{g,n}\}$. Second, the functions $G_n^{g,(d)}$ and finally the coefficient functions S_χ of the h-series expansion of the suitable wave function or $[k|k]$-kernel. We break up the algorithm in the following steps:

- Compute Weierstrass model and elliptic invariants. This can be done using standard software like Sage or Magma.
- Find the possible singularities and genus of Σ, find the branchpoints of the map $x\colon \Sigma \to \mathbb{CP}^1$ and compute the expressions for the holomorphic one-form η and the fundamental bidifferential $\omega_{0,2}$
- Compute the values of the $F_g^{(r)}[\, \ell_1 \dots \ell_n \,]$. Since this only depends on the order of ramification, it may be computed once and for all using the corresponding $\mathcal{W}$ algebra constraints
- Generate the perturbative graphs and sum over the weights to obtain the polynomials $F_{g,n}$
- Evaluate the theta functions Θ and the one-forms $d\xi_\ell^\alpha$, together with the corresponding cycle integrals and indefinite integrals to find the functions $G_n^{g,(d)}$
- Generate the non-perturbative graphs and sum over the weights to obtain the functions S_χ
- Simplify the expressions

4.2.1 Benchmarks

Since we are interested in the performance of our algorithm, we provide several benchmarks and running times of our computations. Our initial attempts to compute the correlators $\{\omega_{g,n}\}$ using the recursive formula (2.1.3) fell short of our expectations. With the graph sum method, we are able to compute on average about 2 to 4 additional orders in h. In particular, this implies finding the explicit form of the correlators $\{\omega_{g,n}\}$ up to a fixed Euler characteristic $\chi = 2g - 2 + n$. The bottleneck of our algorithm is the generation of the perturbative graphs; in Table (4.1) we provide the running times for the $F_{g,n}$'s with $\chi \leq 5$ and four simple ramification points. Within a fixed χ, we observe that the running times are significantly longer in higher genus.

The residue formula (2.1.3) seemingly has the advantage that the successive correlators can be computed recursively from the ones with lower χ. Nevertheless, each step involves computing residues of excessively large expressions. Our naive implementation in Mathematica became prohibitively slow for $\chi > 3$.

Table 4.1: Run times for $F_{g,n}$ with four simple ramifications

(g,n)	Run time (s)	(g,n)	Run time (s)
(0, 3)	0.003	(0, 6)	2.088
(1, 1)	0.003	(1, 4)	17.84
(0, 4)	0.011	(2, 2)	52.57
(1, 2)	0.078	(0, 7)	48.47
(0, 5)	0.103	(1, 5)	554.8
(1, 3)	0.593	(2, 3)	2121
(2, 1)	1.249	(3, 1)	2404

Although being significantly advantageous, the graphical method has its own limitations. In similar enumeration problems, the growth rates are typically exponential with the genus and number of leaves. In our case, the number of graphs further increases with the labels, the number of ramification points, and the ramification indices. In Tables (4.2) and (4.4) we collect the data of the number of graphs with unlabeled edges that appear in $F_{g,n}[x_\ell^\alpha]$. We compare one simple ramification to four simple ramifications. Since many of these graphs will be isomorphic, we compare it to Table (4.4), were we list the isomorphism classes of unlabeled graphs instead. Currently our algorithm cannot count the number of isomorphism classes of graphs with labeled half-edges. Instead, we provide the total number of monomials in $F_{g,n}[x_\ell^\alpha]$. Note that two non-isomorphic graphs of the same genus and labelings on the ordinary half-edges would contribute to the coefficient of the same monomial.

Table 4.2: Total number of graphs in $\mathcal{G}$ of $\chi \leq 3$, with labeled vertices and unlabeled half-edges for the cases of one and four simple ramifications

	1			4		
$n\backslash g$	0	1	2	0	1	2
1	-	4	175	-	16	2584
2	-	28	1594	-	220	45232
3	6	198	17520	24	2404	697640
4	56	2224	-	248	34819	-
5	512	36562	-	2528	525160	-
6	7684	-	-	38160	-	-
7	175828	-	-	806336	-	-

Finally, we provide the count of isomorphism classes of non-perturbative graphs in Table 4.5. As it can be seen, the number of non-perturbative graphs is significantly smaller compared to the perturbative case, and moreover it does not depend on the number or order of the ramification points.

Simplification

Although we do not have an explicit proof, we expect the S_χ appearing in the asymptotic expansion of the wave functions to be rational functions of the spectral curve variables x and y with integer coefficients. This is the case for all of our

Table 4.3: Number of isomorphism classes of graphs in $\mathcal{G}$ of $\chi \leq 3$, with labeled vertices and unlabeled half-edges for the cases of one and four simple ramifications

	1			4		
$n\backslash g$	0	1	2	0	1	2
1	-	3	44	-	12	944
2	-	9	188	-	78	10694
3	1	25	746	4	512	117480
4	3	81	-	18	4143	-
5	6	255	-	84	35220	-
6	16	-	-	524	-	-
7	38	-	-	3580	-	-

Table 4.4: Number of distinct monomials in $F_{g,n}$ for $\chi \leq 6$. Each column represents a set of ramification points of orders $(\alpha_1, \ldots, \alpha_n)$

(g,n)	(2)	(2,2)	(2,2,2)	(2,2,2,2)	(3)	(2,3)	(2,2,3)
(0,3)	1	2	3	4	2	3	4
(0,4)	3	7	12	18	17	23	30
(0,5)	9	28	60	108	92	-	228
(0,6)	23	107	306	690	-	-	-
(0,7)	55	402	1578	4540	-	-	-
(0,8)	122	1443	7941	29606	-	-	-
(1,1)	3	6	9	12	7	10	13
(1,2)	9	25	48	78	46	74	109
(1,3)	23	104	268	540	-	-	-
(1,4)	55	434	1554	3991	-	-	-
(1,5)	122	1686	8613	28796	-	-	-
(1,6)	261	6108	45177	199598	-	-	-
(2,1)	42	182	468	948	-	-	-
(2,2)	109	856	3240	8698	-	-	-
(2,3)	248	3626	20556	73840	-	-	-
(2,4)	520	13948	118563	575818	-	-	-
(3,1)	334	4064	21690	76228	-	-	-
(3,2)	849	19773	162801	793602	-	-	-

calculations so far. There are several intuitive reasons to believe this is the case: the intersection numbers are rational numbers and in the knot theory examples, the values of the theta functions are rational as well. However, after summing over the

Table 4.5: Number of isomorphism classes of non-perturbative graphs contributing to S_X

n	Number of graphs in S_X
2	9
3	51
4	284
5	1454
6	6534
7	27064

non-perturbative graph weights, the raw outputs are large algebraic expressions that contain roots and complex numbers. Simplification of such expressions is a difficult computational task, and may require prior knowledge of each particular case. In the few cases where the simplification couldn't be done computationally in a reasonable amount of time, we give rational approximations of the numerical values instead. The rational numbers are chosen with the smallest denominator that gives an exact result in the working precision.

4.3 The Weierstrass spectral curve

The simplest family of genus one spectral curves was studied by Bouchard, Chidambaram and Dauphinee in [BCD18]. The authors considered elliptic curves parametrized by the Weierstrass-$\wp$ functions, explicitly checking the quantum curve conjecture in a particular case, up to order h^5. With our method, we are able to verify the conjecture up to h^9. This also allows us to identify a pattern and conjecture a closed form and its series expansion. We first recall the main results.

Definition 4.3.1. Let $\wp$ be the Weierstrass-$\wp$ function corresponding to the elliptic invariants (g_2, g_3) and period τ. The Weierstrass spectral curve is defined to be

$$(\mathbb{C}/\Lambda, x = \wp(z), y = \wp'(z), (\wp(z_1 - z_2) + G_2)dz_1dz_2). \tag{4.3.1}$$

The fundamental bidifferential is normalized over the choice of symplectic basis

$\mathcal{A} = [0, 1]$ and $\mathcal{B} = [0, \tau]$.

Note that this spectral curve is equivalent to the polynomial spectral curve $y^2 = 4x^3 - g_2 x - g_3$. Such spectral curve has three simple ramification points at the half periods $\{\varpi_1, \varpi_2, \varpi_1 + \varpi_2\}$. In [BCD18] it was also checked that $\zeta_\hbar = 2g_2/5\hbar$, and hence argued that the quantization condition is satisfied if $g_2 = 0$. There is only one such elliptic curve up to isomorphism, of equation

$$y^2 = 4x^3 - 4.$$

This curve has period $\varpi_1/\varpi_2 = \tau = e^{2\pi i/3}$, and the values of $\wp$ at the half-periods are the cube roots of unity. The most general quantum curve one can consider is of the form

$$\hat{P}(\hat{x}, \hat{y}, h) = h^2 \frac{d^2}{dx^2} - 4\left(x^3 - 1\right) + \sum_{i \geq 1} h^{2i} A_{2i}(x) \frac{d}{dx} + \sum_{j \geq 1} h^{2j} B_{2j}(x),$$

where $A_i(x)$ and $B_j(x)$ are polynomials in $x = \wp(z)$. Imposing that $\hat{P}(\hat{x}, \hat{y}, h)$ annihilates the non-perturbative wave function (2.3.7) is equivalent to the following equation being satisfied for all $k \geq 1$:

$$S''_{k-1} + \sum_{l=0}^{k} S'_l S'_{k-l} + B_k + \sum_{l=0}^{k} S'_{k-l} A_{l+1} = 0. \tag{4.3.2}$$

This allows us to recursively compute the coefficient functions $A_i(x)$ and $B_i(x)$. Here, the wave function ψ_{NP} is evaluated for the choice of half-characteristic $(\nu, \mu) = (\frac{1}{2}, \frac{1}{2})$ and base point $0 \in \mathbb{C}/\Lambda$. Using our algorithm we compute the correlators $\{\omega_{g,n}\}$ for $2g - 2 + n \leq 6$. After evaluating the necessary integrals, we obtain the expressions of the $G_n^{g,(d)}$'s and ultimately the S_X's for $X \leq 7$. Note that the base of the recursion $k = 1$ involves the functions S_0 and S_1, which are defined at hoc. We give the expressions below:

- $S_0(z) = \int_0^z \wp'(z)^2$
- $S_1(z) = -\frac{\log \wp'(z)}{2}$
- $S_2(z) = \frac{1}{12\wp'(z)^3}(19\wp(z)^2 - 4\wp(z)^5)$
- $S_3(z) = \frac{1}{36\wp'(z)^6}(-2\wp(z)^{10} + 18\wp(z)^7 + 159\wp(z)^4 + 230\wp(z))$
- $S_4(z) = \frac{1}{6480\wp'(z)^9}(-80\wp(z)^{15}+640\wp(z)^{12}+9400\wp(z)^9+337067\wp(z)^6+906596\wp(z)^3+ 88952)$
- $S_5(z) = \frac{1}{648\wp'(z)^{12}}(-2\wp(z)^{20} + 36\wp(z)^{17} - 6\wp(z)^{14} + 12016\wp(z)^{11} + 538839\wp(z)^8 + 2345964\wp(z)^5 + 810118\wp(z)^2)$
- $S_6(z) = \frac{1}{816480\wp'(z)^{15}}(-672\wp(z)^{25} + 21519680\wp(z)^{22} - 159722000\wp(z)^{19}+ 515348640\wp(z)^{16} - 682123900\wp(z)^{13} + 15322295767\wp(z)^{10} + 88286246520\wp(z)^7 + 63712854800\wp(z)^4 + 4159251040\wp(z))$
- $S_7(z) = \frac{1}{21870\wp'(z)^{18}}(-5\wp(z)^{30} - 22850913\wp(z)^{27} + 206235312\wp(z)^{24}- 827215278\wp(z)^{21} + 1936629852\wp(z)^{18} - 2769153528\wp(z)^{15} + 12887012988\wp(z)^{12} + 80565969228\wp(z)^9 + 98319454233\wp(z)^6 + 18580283543\wp(z)^3 + 245668068)$

These results allow us to verify the conjecture in [BCD18] up to order h^8.

Theorem 4.3.2 (G.)**.** *The quantum curve associated to the Weierstrass spectral curve $y^2 = 4x^3 - 4$ is given by the following expression up to order $O(h^8)$.*

$$\begin{aligned}\hat{P}(\hat{x}, \hat{y}, h) = h^2\frac{d^2}{dx^2} - 4(x^3 - 1) + h^2\frac{x}{2^2 3} + h^4\frac{1}{2^6 3^2}\frac{d}{dx} + h^4\frac{x^2}{2^8 3^3} + h^6\frac{x}{2^{12} 3^4}\frac{d}{dx} \\ + h^6\frac{x^3}{2^{14} 3^5} + h^8\frac{x^2}{2^{18} 3^6}\frac{d}{dx} + O(h^8).\end{aligned} \tag{4.3.3}$$

From these results we can identify clear pattern and conjecture a closed form for the quantum curve.

Conjecture 4.3.3 (G.). *The quantum curve associated to the Weierstrass spectral curve $y^2 = 4x^3 - 4$ has the following infinite expansion in h:*

$$\begin{aligned}
\hat{P}(\hat{x}, \hat{y}, h) &= h^2 \frac{d^2}{dx^2} - 4(x^3 - 1) + \sum_{i=1} h^{2i} \frac{x^i}{2^{6i-4} 3^{2i-1}} + \sum_{j=1} h^{2j+2} \frac{x^{j-1}}{2^{6j} 3^{2j}} \frac{d}{dx} \\
&= h^2 \frac{d^2}{dx^2} - 4(x^3 - 1) + \frac{48 h^2 x}{576 - h^2 x} + \frac{h^4}{576 - h^2 x} \frac{d}{dx} \\
&= h^2 \frac{d^2}{dx^2} - 4(x^3 - 1) + \frac{h^2}{576 - h^2 x} \left(48x + h^2 \frac{d}{dx} \right).
\end{aligned} \tag{4.3.4}$$

4.4 A-polynomial spectral curves

4.4.1 The Figure Eight Knot 4_1

The figure eight knot is one of the two knots studied in [BE12]. The geometric component of the A-polynomial is

$$A_{4_1}(x, \ell) = \ell^2 x^2 - \ell(x^4 - x^3 - 2x^2 - x + 1) + x^2. \tag{4.4.1}$$

Its zero locus has two ordinary singular points $(x, \ell) = (\pm 1, \mp 1)$ of order two. Since the interior of the Newton polygon (4·2) has 3 points, we deduce the genus is $g = 1$. We summarize its properties:

- Four ramification points of order two:

$$(x, \ell) = \left(\frac{3 \pm \sqrt{5}}{2}, 1 \right), \left(\frac{-1 \pm i\sqrt{3}}{2}, -1 \right)$$

- Discriminant

$$\Delta(x) = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}$$

- Weierstrass model and elliptic invariants:

$$y^2 = 4x^3 - \frac{1}{12}x + \frac{161}{216}, \qquad (g_2, g_3) = \left(\frac{1}{12}, -\frac{161}{216} \right)$$

- Holomorphic form:

$$\eta(x,\ell) = \frac{(1-x^2)}{2\ell x^2 - (x^4 - x^3 - 2x^2 - x + 1)}dx \tag{4.4.2}$$

- Fundamental Bidifferential:

$$\begin{aligned}
&\omega_{0,2}(x_1,\ell_1,x_2,\ell_2) = \\
&(12(1 - x_1 - x_2 + x_1^2x_2 + x_1x_2^2) - 19(x_1^2 + x_2^2) + 7(x_1^4 + x_2^4) + 2x_1x_2 + 38x_1^2x_2^2 \\
&+ 10(x_1^3x_2 + x_1x_2^3) + 12(x_1^3x_2^2 + x_1^2x_2^3) - 19(x_1^4x_2^2 + x_1^2x_2^4) + 2x_1^3x_2^3 + 12x_1^4x_2^4 \\
&+ 2(x_1^2x_2^3\ell_1 + x_1^3x_2^2\ell_2 - x_1^2x_2^4\ell_1 - x_1^4x_2^2\ell_2 + x_1x_2^2\ell_2 + x_1^2x_2\ell_1) - 12(x_1^2\ell_1 + x_2^2\ell_2) \\
&+ 24(x_1^2x_2^2\ell_2 + x_1^2x_2^2\ell_1 + x_1^2x_2^2\ell_1\ell_2) - 12(x_1^4x_2^3 + x_1^3x_2^4))dx_1dx_2 \\
&/12(x_1 - x_2)^2(1 - x_1 - 2x_1^2 - x_1^3 + x_1^4 - 2x_1^2\ell_1)(1 - x_2 - 2x_2^2 - x_2^3 + x_2^4 - 2x_2^2\ell_2).
\end{aligned} \tag{4.4.3}$$

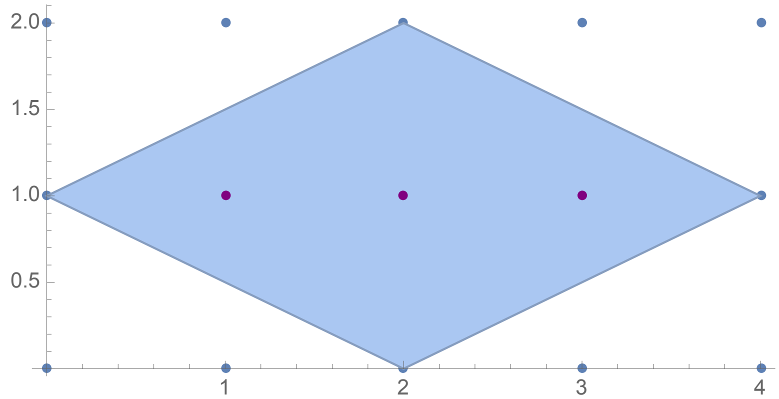


Figure 4·2: Newton Polygon of 4_1.

To compute ψ_{NP} we make the same choices of base point and half characteristics as in [BE12]. Through our more efficient algorithm we are able to verify the conjecture through order h^7, whereas previously it had only been done up to order h^4.

Theorem 4.4.1 (G., Potter)**.** *Let $\psi_{NP}^{[2|2]}$ be the non-perturbative* [2|2]*-kernel obtained from topological recursion on the A-polynomial of the figure right knot $\mathbf{4}_1$, for the choice of half-*

characteristic $(\nu, \mu) = (0, \frac{1}{2})$ *and basepoint* $o = \left(\frac{3+\sqrt{5}}{2}, 1\right)$. *Then* $(\psi_{NP}^{[2|2]})^{1/2}$ *agrees with the expansions of the state integral model* $\mathcal{J}_H$ *and the perturbative function obtained from the AJ Conjecture* $\mathcal{J}_{CS}$ *of* $\mathbf{4}_1$ *up to order* $O(h^7)$.

Proof. The results of our computations, listed below, agree with [DGLZ09, DFM11]. After obtaining the necessary correlators $\{\omega_{g,n}\}$ from the perturbative graphs, we evaluate the $G_n^{g,(d)}$ and the theta functions. Finally we sum over the corresponding non-preturbative graphs to obtain the S_X's. Although the expressions of the $\omega_{g,n}$'s are overly long, the expressions of the $G_n^{g,(d)}$ are significantly simpler. For convenience, here we provide the expressions of the S_X and write some of the corresponding $G_n^{g,(d)}$'s in the Appendix A.0.4.

- $S_2(x) = \frac{x^6-x^5-2x^4+15x^3-2x^2-x+1}{6\Delta(x)^3}$
- $S_3(x) = \frac{4x^3(x^6-x^5-2x^4+5x^3-2x^2-x+1)}{\Delta(x)^6}$
- $S_4(x) = -\frac{x}{45\Delta(x)^9}(x^{16} - 4x^{15} - 128x^{14} + 36x^{13} + 1074x^{12} - 5630x^{11} + 5782x^{10} + 7484x^9 - 18311x^8 + 7484x^7 + 5782x^6 - 5630x^5 + 1074x^4 + 36x^3 - 128x^2 - 4x + 1)$
- $S_5(x) = \frac{4x^3}{3\Delta(x)^{12}}(x^{18}+5x^{17}-35x^{16}+240x^{15}-282x^{14}-978x^{13}+3914x^{12}-3496x^{11}-4205x^{10}+9819x^9-4205x^8-3496x^7+3914x^6-978x^5-282x^4+240x^3-35x^2+5x+1)$
- $S_6(x) = \frac{2x}{945\Delta(x)^{15}}(x^{28}+2x^{27}+169x^{26}+4834x^{25}-24460x^{24}+241472x^{23}-65355x^{22}-3040056x^{21} + 13729993x^{20} - 15693080x^{19} - 36091774x^{18} + 129092600x^{17}-103336363x^{16} - 119715716x^{15} + 270785565x^{14} - 119715716x^{13} - 103336363x^{12} + 129092600x^{11} - 36091774x^{10} - 15693080x^9 + 13729993x^8 - 3040056x^7 - 65355x^6 + 241472x^5 - 24460x^4 + 4834x^3 + 169x^2 + 2x + 1)$
- $S_7(x) = \frac{4x^3}{\Delta(x)^{18}}(x^{30}+47x^{29}-176x^{28}+3373x^{27}+9683x^{26}-116636x^{25}+562249x^{24}-515145x^{23} - 3761442x^{22} + 14939871x^{21} - 15523117x^{20} - 29061458x^{19}+96455335x^{18} - 71522261x^{17} - 80929522x^{16} + 179074315x^{15} - 80929522x^{14}-71522261x^{13} + 96455335x^{12} - 29061458x^{11} - 15523117x^{10} + 1493971x^9-3761442x^8-515145x^7+562249x^6-116636x^5+9683x^4+3373x^3-176x^2+47x+1)$

□

4.4.2 The Once Punctured Torus Bundle L^2R

The once punctured torus bundle L^2R is a knot complement in $\mathbb{RP}^3$.

$$A_{L_2R}(x,\ell) = \ell^2x^2 + \ell(-x^3 + 2x^2 + 2x - 1) + x$$

It has one ordinary singular point $(x,\ell) = (1,-1)$ of order two. Since the Newton

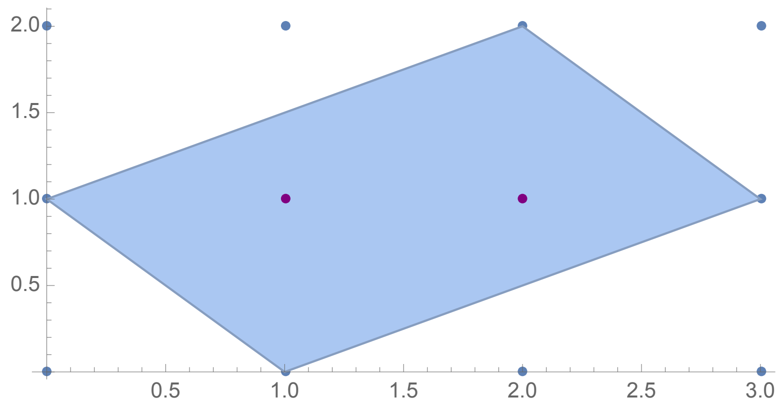


Figure 4·3: Newton Polygon of L^2R.

polygon has 2 interior points, the genus is again $g = 1$.

- Four ramification points of order two:

$$(x,\ell) = \left(\frac{1}{2} + \sqrt{2} \mp \frac{1}{2}\sqrt{5+4\sqrt{2}}, \frac{1}{2}\left(1+\sqrt{2} \pm \sqrt{2\sqrt{2}-1}\right)\right),$$

$$(x,\ell) = \left(\frac{1}{2}\left(1-2\sqrt{2}\right) \mp \frac{1}{2}i\sqrt{4\sqrt{2}-5}, \frac{1}{2}\left(1-\sqrt{2} \mp i\sqrt{1+2\sqrt{2}}\right)\right).$$

- Discriminant

$$\Delta(x) = \sqrt{x^4 - 2x^3 - 5x^2 - 2x + 1}$$

- Weierstrass model and elliptic invariants:

$$y^2 = 4x^3 + \frac{25}{12}x + \frac{253}{216}, \qquad (g_2, g_3) = \left(\frac{25}{12}, -\frac{253}{216}\right)$$

- Holomorphic form:

$$\eta(x,\ell) = \frac{1}{-2lx^2 + x^3 - 2x^2 - 2x + 1}dx \tag{4.4.4}$$

- Fundamental bidifferential:

$$\begin{aligned}
&\omega_{0,2}(x_1,\ell_1,x_2,\ell_2) = \\
&(12x_1^3x_2^3 - 12x_1^3x_2^2\ell_2 - 24x_1^3x_2^2 - 11x_1^3x_2 + 11x_1^3 - 12x_1^2x_2^3\ell_1 - 24x_1^2x_2^3 \\
&+ 24x_1^2x_2^2\ell_1\ell_2 + 24x_1^2x_2^2\ell_1 + 24x_1^2x_2^2\ell_2 + 22x_1^2x_2^2 + 24x_1^2x_2\ell_1 + 37x_1^2x_2 \\
&- 12x_1^2\ell_1 - 11x_1^2 - 11x_1x_2^3 + 24x_1x_2^2\ell_2 + 37x_1x_2^2 + 22x_1x_2 - 24x_1 + 11x_2^3 \\
&- 12x_2^2\ell_2 - 11x_2^2 - 24x_2 + 12)dx_1dx_2 \\
&/12(x_1-x_2)^2\left(x_1^3 - 2x_1^2\ell_1 - 2x_1^2 - 2x_1 + 1\right)\left(x_2^3 - 2x_2^2\ell_2 - 2x_2^2 - 2x_2 + 1\right)
\end{aligned} \tag{4.4.5}$$

Theorem 4.4.2 (G., Potter). *Let $\psi_{NP}^{[2|2]}$ be the non-perturbative* [2|2]*-kernel obtained from topological recursion on the A-polynomial of knot L^2R, for the choice of half-characteristic $(\nu,\mu) = (0,\frac{1}{2})$ and basepoint $o = \left(\frac{1}{2} + \sqrt{2} - \frac{1}{2}\sqrt{5+4\sqrt{2}}, \frac{1}{2}\left(1 + \sqrt{2} + \sqrt{2\sqrt{2}-1}\right)\right)$. Then $(\psi_{NP}^{[2|2]})^{1/2}$ agrees with the expansions of the state integral model $\mathcal{J}_H$ of the knot L^2R up to order $O(h^5)$.*

Proof. Again, we list the expressions of the S_X, and write some of the $G_n^{g,(d)}$ in the Appendix. They agree with the results in [DFM11] using the state integral model.

- $S_2(x) = \frac{5x^6-11x^5+22x^4+105x^3+22x^2-11x+5}{24\Delta(x)^3}$
- $S_3(x) = \frac{-x^{12}+6x^{11}+67x^{10}+466x^9-298x^8-130x^7+1339x^6-130x^5-298x^4+466x^3+67x^2+6x-1}{64\Delta(x)^6}$
- $S_4(x) = \frac{x}{720\Delta(x)^9}(1 - 68x - 3770x^2 + 137x^3 - 30073x^4 - 58605x^5 + 104390x^6 + 20753x^7 - 222062x^8 + 20753x^9 + 104390x^{10} - 58605x^{11} - 30073x^{12} + 137x^{13} - 3770x^{14} - 68x^{15} + x^{16})$
- $S_5(x) = \frac{x^2}{24\Delta^{12}}(1 + 86x + 179x^2 + 3870x^3 + 7447x^4 - 7820x^5 + 51914x^6 + 60396x^7 - 183475x^8 - 25486x^9 + 311325x^{10} - 25486x^{11} - 183475x^{12} + 60396x^{13} + 51914x^{14} - 7820x^{15} + 7447x^{16} + 3870x^{17} + 179x^{18} + 86x^{19} + x^{20})$

□

4.4.3 The Knot 8_{18}

The A-polynomial is given by

$$\begin{aligned}A_{8_{18}}(x,\ell) = {} & x^{16}\ell^2 - 12x^{15}\ell^2 + 54x^{14}\ell^2 - 112x^{13}\ell^2 - 2x^{12}\ell^3 + 109x^{12}\ell^2 - 2x^{12}\ell \\ & + 12x^{11}\ell^3 - 64x^{11}\ell^2 + 12x^{11}\ell - 14x^{10}\ell^3 + 74x^{10}\ell^2 - 14x^{10}\ell - 28x^9\ell^3 \\ & - 100x^9\ell^2 - 28x^9\ell + x^8\ell^4 + 68x^8\ell^3 + 106x^8\ell^2 + 68x^8\ell + x^8 - 28x^7\ell^3 \\ & - 100x^7\ell^2 - 28x^7\ell - 14x^6\ell^3 + 74x^6\ell^2 - 14x^6\ell + 12x^5\ell^3 - 64x^5\ell^2 \\ & + 12x^5\ell - 2x^4\ell^3 + 109x^4\ell^2 - 2x^4\ell - 112x^3\ell^2 + 54x^2\ell^2 - 12x\ell^2 + \ell^2. \quad (4.4.6)\end{aligned}$$

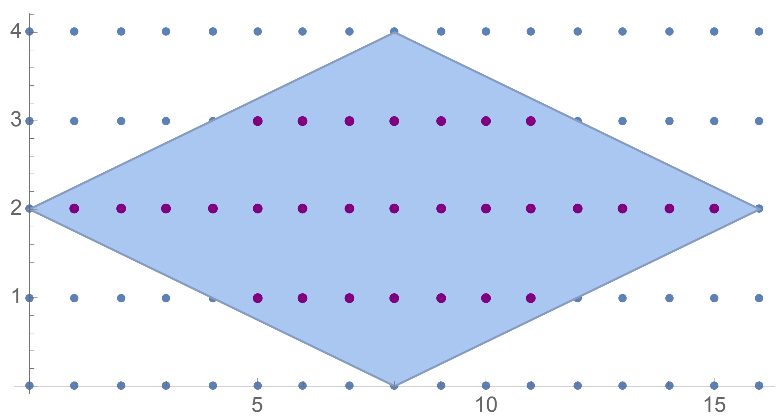


Figure 4·4: Newton Polygon of 8_{18}.

There are 12 ordinary double points: $(-1, -1 \pm 12\sqrt{2})$, $\left(\frac{1}{2}(1 \pm i\sqrt{3}), 1\right)$, four of the form $(r_1, -1)$ with r_1 a root of $S_1(x) = 1 - 3x - 3x^3 + x^4$ and four more of the form $(r_2, 1)$ with r_2 a root of $S_2(x) = 1 - x - 2x^2 - x^3 + 1$. There is an ordinary singularity of order 4 at $(1, -1)$. Moreover, there are four non-ordinary double points at $(0, 0)$, $(0, \infty)$, $(\infty, 0)$ and (∞, ∞) with delta invariant $\delta = 1$, and two non-ordinary singularities of order 2 and delta invariant $\delta = 3$ at $(2 \pm \sqrt{3}, 1)$. Counting the points interior to the

Newton polygon, the genus is

$$g = 29 - 12 - \binom{4}{2} - 4 \cdot 1 - 2 \cdot 3 = 1.$$

- Six ramification points or order two
- Weierstrass model and elliptic invariants:

$$y^2 = 4x^3 - \frac{8}{3}x + \frac{1}{27}, \qquad (g_2, g_2) = \left(-\frac{8}{3}, \frac{1}{27}\right)$$

- Holomorphic form:

$$\eta(x,\ell) = \frac{dx}{\partial_\ell A_{8_{18}}(x,\ell)}(-1+x)^4(-x^4 - 3x^5 - x^6 + \ell - 5x\ell + 4x^2\ell + x^3\ell$$
$$+ 9x^4\ell + 2x^5\ell + 9x^6\ell + x^7\ell + 4x^8\ell - 5x^9\ell$$
$$+ x^{10}\ell - x^4\ell^2 - 3x^5\ell^2 - x^6\ell^2)$$

4.4.4 The Knot 9_{35}

The A-polynomial is given by

$$A_{9_{35}}(x,\ell) = \ell^3x^9 - 2\ell^2x^9 + 3\ell^2x^8 + 12\ell^2x^7 - 19\ell^2x^6 + 6\ell^2x^5 + 9\ell^2x^4 - 13\ell^2x^3$$
$$+ 9\ell^2x^2 - 3\ell^2x + \ell^2 + \ell x^9 - 3\ell x^8 + 9\ell x^7 - 13\ell x^6 + 9\ell x^5 + 6\ell x^4$$
$$- 19\ell x^3 + 12\ell x^2 + 3\ell x - 2\ell + 1. \tag{4.4.7}$$

The zero locus has 13 singular points. The first ten are ordinary double points of the form $(x, S(x))$, where x is a root of

$$5 - 10x + 21x^2 - 5x^3 - 26x^4 + 57x^5 - 26x^6 - 5x^7 + 21x^8 - 10x^9 + 5x^{10}$$

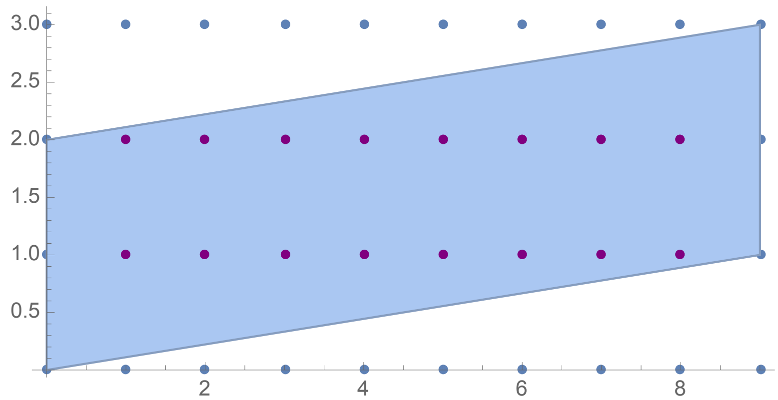


Figure 4·5: Newton Polygon of 9_{35}.

and

$$S(x) = \frac{1}{54}\Big(54 + 1481x - 11967x^2 + 25345x^3 - 14432x^4 - 27747x^5 + 58355x^6 - 40835x^7 + 1623x^8 + 18704x^9 - 15955x^{10} + 7095x^{11} - 1775x^{12}\Big).$$

Additionally, there are two additional ordinary double points at $(0, 1)$ and $(\infty, 1)$, and an ordinary singularity of order 3 at $(1, -1)$. Since there are 16 interior points in the Newton polygon, the genus is

$$g = 16 - 10 - 2 - \binom{3}{2} = 1.$$

It has the following properties:

- Four ramification points or order two and one of order three at $(-1, 1)$
- Weierstrass model and elliptic invariants:

$$y^2 = 4x^3 - \frac{8}{3}x + \frac{1}{27}, \qquad (g_2, g_3) = \left(\frac{8}{3}, -\frac{1}{27}\right)$$

- Holomorphic form:

$$\eta(x,\ell) = \frac{dx}{\partial_\ell A_{9_{35}}(x,\ell)}(x-1)(\ell x^6 - 3\ell x^5 - \ell x^4 + 4\ell x^3 - 5\ell x^2 + 2\ell x - \ell - x^6 + 2x^5 - 5x^4 + 4x^3 - x^2 - 3x + 1)$$

4.4.5 The Knot 9_{48}

The A-polynomial is given by

$$A_{9_{48}}(x,\ell) = \ell - 9\ell x - x^2 + 24\ell x^2 - 12\ell x^3 - 24\ell x^4 + 21\ell x^5 - \ell x^6 - 3\ell^2 x^6 - 3\ell x^7 - \ell^2 x^7 + 21\ell^2 x^8 - 24\ell^2 x^9 - 12\ell^2 x^{10} + 24\ell^2 x^{11} - \ell^3 x^{11} - 9\ell^2 x^{12} + \ell^2 x^{13}. \tag{4.4.8}$$

It has 11 singular points: an ordinary singularity of order 3 at $(1,-1)$ and ten ordinary double points of the form $(x, S(x))$, where x is a root of

$$1 - 8x + 24x^2 - 31x^3 - 25x^4 + 105x^5 - 25x^6 - 31x^7 + 24x^8 - 8x^9 + x^{10}$$

and

$$S(x) = \frac{1}{18}\Big(2449 - 14928x + 1979x^2 + 87302x^3 - 72777x^4 - 104639x^5 + 131972x^6 - 8847x^7 - 42320x^8 + 26515x^9 - 7560x^{10} + 836x^{11}\Big).$$

Since there are 14 interior points in the Newton polygon, the genus is

$$g = 14 - 10 - \binom{3}{2} = 1.$$

It has the following properties:

- Two ramification points or order two and one of order three at $(-1,1)$

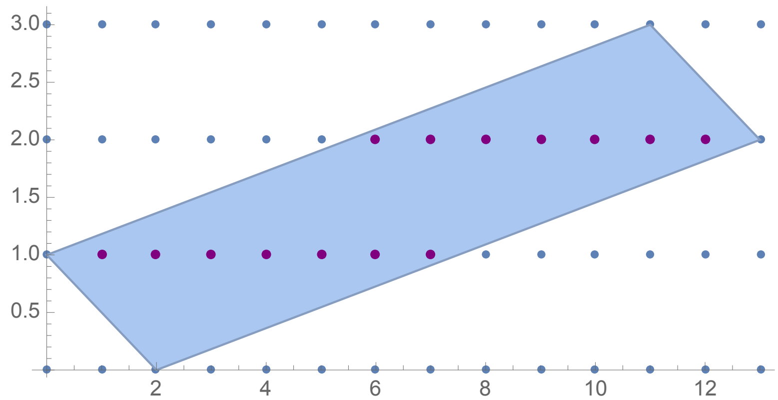


Figure 4·6: Newton Polygon of 9_{48}.

- Weierstrass model and elliptic invariants:

$$y^2 = 4x^3 - \frac{4}{3}x + \frac{19}{27}, \qquad (g_2, g_3) = \left(\frac{4}{3}, -\frac{19}{27}\right)$$

- Holomorphic form:

$$\eta(x,\ell) = \frac{dx}{\partial_\ell A_{9_{48}}(x,\ell)}(x-1)(\ell x^{10} - 6\ell x^9 + 8\ell x^8 + 2\ell x^7 - 3\ell x^6 + \ell x^5 + x^5 - 3x^4 + 2x^3 + 8x^2 - 6x + 1)$$

4.4.6 The Knot 10_{139}

The A-polynomial is given by

$$A_{10_{139}}(x,\ell) = x^{28}\ell^4 - x^{22}\ell^3 + 7x^{21}\ell^3 - 3x^{20}\ell^3 + x^{19}\ell^3 + 6x^{14}\ell^2 + x^9\ell - 3x^8\ell + 7x^7\ell - x^6\ell + 1 \tag{4.4.9}$$

It has one ordinary singular point of multiplicity 4 at $(1,-1)$. Hence the genus is

$$g = 7 - \binom{4}{2} = 1.$$

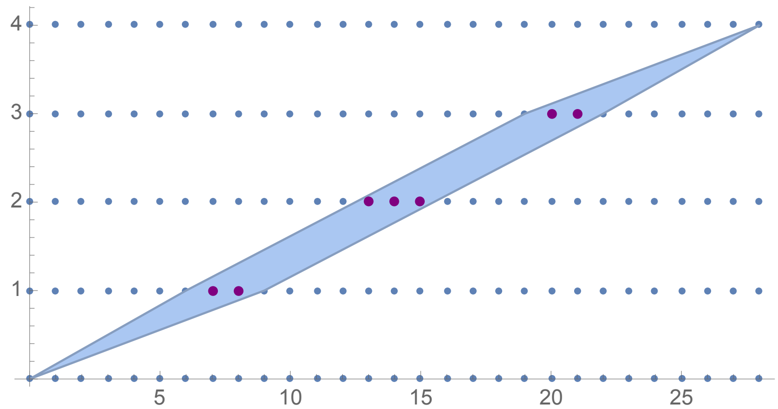


Figure 4·7: Newton Polygon of 10_{139}.

It has the following properties:

- Six ramification points or order two.
- Weierstrass model and elliptic invariants:

$$y^2 = 4x^3 - \frac{4}{3}x + \frac{35}{27}, \qquad (g_2, g_3) = \left(\frac{4}{3}, -\frac{35}{27}\right)$$

- Holomorphic form:

$$\eta(x, \ell) = \frac{(1-x)x^6\left(lx^6+1\right)\left(lx^7+1\right)}{\partial_\ell A_{10_{139}}(x, \ell)}dx$$

Appendix A

Additional formulas and proofs

A.0.1 Elliptic integrals

The non-perturbative corrections include two types of integrals of the correlators: definite integrals from a base point and integrals over cycles. The forms $\omega_{g,n}$ have been expressed in powers of $\wp(z)$ and $\wp'(z)$. Consider an elliptic curve of equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$. Then for $n \geq 3$ the following equation is satisfied:

$$\int \wp(z)^n dz = \left[\frac{\wp(z)^{n-2}\wp'(z)}{4n-2} + \frac{g_2(2n-3)}{8n-4}\int \wp(z)^{n-2}dz + \frac{g_3(n-2)}{4n-2}\int \wp(z)^{n-3}dz\right].$$

Note that any integral of the sort

$$\int \wp(z)^n \wp'(z)^m dz$$

can then be evaluated using the equation above.

A.0.2 Non-Perturbative Generating and Wave Functions

To present the original definition of the non-perturbative generating and wave functions, first we introduce the topological recursion *free energies*:

$$\mathcal{F}_g = \frac{1}{2g-2}\sum_{p_\alpha \in \mathcal{R}} \operatorname*{Res}_{z=p_\alpha}\left(\int_o^z x(z)dy(z)\right)\omega_{g,1}(z), \forall g \geq 0.$$

Definition A.0.1. Let $\mathcal{S}$ be a spectral curve, with a choice of cycles and normalized fundamental bidifferential. Let θ denote the theta function with a choice of half characteristic μ, ν and $\zeta_\hbar$ as defined in Section 2.3. Then the non-perturbative generating $\mathcal{Z}_{\text{NP}}$ function is defined as.

$$\begin{aligned}\mathcal{Z}_{\text{NP}} &= \exp\left(\sum_{g\geq 0}\hbar^{2g-2}\mathcal{F}_g\right)\\ &\times\left(\sum_{r\geq 0}\frac{1}{r!}\sum_{\substack{h_j,d_j\geq 0\\ 2h_j+d_j-2>0}}\hbar^{\sum 2h_j+d_j-2}\prod_{j=1}^{r}\left(\frac{\mathcal{F}_{h_j}^{(d_j)}}{(2\pi i)^{d_j}d_j!}\right)\right)\nabla^{(\sum d_j)}\Theta(\zeta_\hbar;\tau), \qquad \text{(A.0.1)}\end{aligned}$$

where

$$\mathcal{F}_g^{(d)} = \frac{1}{n!}\frac{1}{(2\pi i)^d d!}\underbrace{\oint_{\mathcal{B}}\cdots\oint_{\mathcal{B}}}_{d}\omega_{g,d}(z_1,\ldots,z_d), \qquad \nabla\Theta(\zeta_\hbar;\tau) = \left(\frac{d}{dz}\Theta(z;\tau)\right)\Bigg|_{z=\zeta_\hbar}.$$

The non-perturbative wave function is defined as the Schlesinger transformation of the non-perturbative generating function:

$$\psi_{\text{NP}}(p,o) = \frac{\mathcal{Z}_{\text{NP}}\left[ydx \to ydx + \hbar\int_o^p \omega_{0,2}\right]}{\mathcal{Z}_{\text{NP}}[ydx]}$$

A.0.3 Holomorphic differentials

By applying a linear change of coordinates, without loss of generality we may assume that any given singularity is located at the origin $(0,0)$. The adjoint condition is local about the singular points, hence for the following proposition we work on an open set $U \subset \mathbb{C}^2$ containing the origin.

Proposition A.0.2. *Let Σ be the zero locus in U of a polynomial $P(x,y)$. Suppose Σ has an ordinary singularity of multiplicity m at $(0,0)$. Let Σ_A be an adjoint curve to Σ in U with defining polynomial $A(x,y)$. Denote by $\pi\colon\tilde{\Sigma}\to\Sigma$ the smooth curve obtained after blowing up U at $(0,0)$. Then the form on $\tilde{\Sigma}$ below is holomorphic at $\pi^{-1}((0,0))$:*

$$\eta(x,y) = \frac{A(x,y)}{P_y(x,y)}dx,$$

where x and y must be understood as the functions $x\circ\pi$ and $y\circ\pi$ on $\tilde{\Sigma}$.

Proof. Recall that Σ_A being adjoint to Σ means that A has multiplicity $m-1$ at $(0,0)$. Write

$$P(x,y) = P^m(x,y) + \ldots + P^d(x,y)$$

and

$$A(x,y) = A^{m-1}(x,y) + \ldots + A^f(x,y),$$

where P^ℓ (vs A^ℓ) are homogeneous polynomials of degree ℓ. In the blow-up chart U_0 where $(x,y) = (u_0, u_0v_0)$ we have

$$\frac{1}{u_0^m}P(u_0,u_0v_0) = P^m(1,v_0) + \ldots + P^d(1,v_0)u_0^{d-m}.$$

The zeros of $A_m(1,v_0)$ are the points the exceptional divisor $u_0 = 0$. The assumption that $(0,0)$ is an ordinary singularity assures that $A_m(1,v_0)$ has no repeated factors. It takes a moment to realize that $A_m(1,v_0)$ can either be of degree m or $m-1$.

- If $A_m(1,v_0)$ is of degree m, then the points in $\pi^{-1}((0,0))$ are exactly its m roots. We must show η is smooth at all of these points. Pulling back the expression of η we have

$$\begin{aligned}\tilde{\eta} := \pi^*\eta(x,y) = \eta(u_0,u_0v_0) &= \frac{u_0^{m-1}\left(A^{m-1}(1,v_0) + \ldots + A^f(1,v_0)u_0^{f-m+1}\right)}{u_0^{m-1}\left(P_y^m(1,v_0) + \ldots + P_y^d(1,v_0)u_0^{d-m+1}\right)}du_0 \\ &= \frac{A^{m-1}(1,v_0) + \ldots + A^f(1,v_0)u_0^{f-m+1}}{P_y^m(1,v_0) + \ldots + P_y^d(1,v_0)u_0^{d-m+1}}du_0.\end{aligned}$$

  Therefore, at the points in the exceptional divisor $(u_0,v_0) = (0,v_{0i})$, we have

$$\tilde{\eta}(0,v_{0i}) = \frac{A^{m-1}(1,v_{0i})}{P_y^m(1,v_{0i})}du_0.$$

  One again, since P^m has no repeated factors, $P_y^m(1,v_{0i}) \neq 0$ and therefore η is holomorphic at these points.

- If $A_m(1,v_0)$ is of degree $m-1$, then there is an additional point $q \in \pi^{-1}((0,0))$ that is not in U_0. In this case, consider the second chart U_1, where $(x,y) = (u_1v_1, v_1)$. We must show that η is holomorphic at the point q, which corresponds to $(u_1, v_1) = (0,0)$. Using the equation of the curve we first rewrite

$$\eta(x,y) = -\frac{A(x,y)}{P_x(x,y)}dy.$$

$$\begin{aligned}\tilde{\eta} := \pi^*\eta(x,y) = \eta(u_1 v_1, u_1) &= \frac{u_1^{m-1}\left(A^{m-1}(v_1,1) + \ldots + A^f(v_1,1)u_1^{f-m+1}\right)}{u_1^{m-1}\left(P_x^m(v_1,1) + \ldots + P_x^d(v_1,1)u_1^{d-m+1}\right)}du_1 \\ &= \frac{A^{m-1}(v_1,1) + \ldots + A^f(v_1,1)u_1^{f-m+1}}{P_x^m(v_1,1) + \ldots + P_x^d(v_1,1)u_1^{d-m+1}}du_1.\end{aligned}$$

Again, since P^m has no repeated factors we have that $P_x^m(0,1) \neq 0$.

This shows that η is holomorphic at all the points $\pi^{-1}(0,0) \subset \tilde{\Sigma}$. □

A.0.4 Expressions for the $G_n^{g,(d)}$.

We provide the expressions of the functions $G_n^{g,(d)}$. For the figure eight not $\mathbf{4}_1$, let

$$\Delta(x) := \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}.$$

The G's of Euler characteristic $\chi = 4$ read:

- $G_0^{0,(6)} = \frac{1122304}{307546875}$
- $G_2^{0,(4)} = \frac{2}{20503125\Delta(x)^8}(-234752x^{16} + 2564703x^{15} - 8808220x^{14} + 6090610x^{13} + 5770000x^{12} + 72992081x^{11} - 163015744x^{10} - 24209375x^9 + 281797720x^8 - 24209375x^7 - 163015744x^6 + 72992081x^5 + 5770000x^4 + 6090610x^3 - 8808220x^2 + 2564703x - 234752)$
- $G_4^{0,(2)} = \frac{1}{1366875\Delta(x)^{10}}(12736x^{20} - 263667x^{19} + 1499890x^{18} - 1315655x^{17} - 5030352x^{16} - 1733042x^{15} + 166645716x^{14} - 374143470x^{13} + 99288000x^{12} + 1029625271x^{11} - 1679582106x^{10} + 1029625271x^9 + 99288000x^8 - 374143470x^7 + 166645716x^6 - 1733042x^5 - 5030352x^4 - 1315655x^3 + 1499890x^2 - 263667x + 12736)$

- $G_6^{0,(0)} = \frac{1}{360\Delta(x)^{12}}(3x^{23} - 12x^{22} + 28x^{21} + 1320x^{20} - 5106x^{19} + 6924x^{18} + 38643x^{17} - 152016x^{16} + 222629x^{15} + 33156x^{14} - 560514x^{13} + 862392x^{12} - 560514x^{11} + 33156x^{10} + 222629x^9 - 152016x^8 + 38643x^7 + 6924x^6 - 5106x^5 + 1320x^4 + 28x^3 - 12x^2 + 3x)$

- $G_0^{1,(4)} = \frac{6079232}{307546875}$

- $G_2^{1,(2)} = \frac{1}{6834375\Delta(x)^{10}}(-371936x^{20} + 4803226x^{19} - 21523757x^{18} + 38988643x^{17} - 35114069x^{16} + 161440552x^{15} + 107735247x^{14} - 1577876619x^{13} + 2268678786x^{12} + 2942948147x^{11} - 7615329394x^{10} + 2942948147x^9 + 2268678786x^8 - 1577876619x^7 + 107735247x^6 + 161440552x^5 - 35114069x^4 + 38988643x^3 - 21523757x^2 + 4803226x - 371936)$

- $G_4^{1,(0)} = \frac{1}{32805000\Delta(x)^{12}}(85072x^{24} - 2099108x^{23} + 13455756x^{22} - 12921090x^{21} + 147170281x^{20} - 698849930x^{19} + 3524017584x^{18} + 2230020768x^{17} - 34162937438x^{16} + 79999579944x^{15} - 26758597924x^{14} - 155011998246x^{13} + 270985521139x^{12} - 155011998246x^{11} - 26758597924x^{10} + 79999579944x^9 - 34162937438x^8 + 2230020768x^7 + 3524017584x^6 - 698849930x^5 + 147170281x^4 - 12921090x^3 + 13455756x^2 - 2099108x + 85072)$

- $G_0^{2,(2)} = \frac{27617936}{922640625}$

- $G_2^{2,(0)} = \frac{1}{246037500\Delta(x)^{12}}(-3130976x^{24} + 46265003x^{23} - 253351628x^{22} + 658791068x^{21} - 539768196x^{20} + 587781222x^{19} + 10305154244x^{18} - 11500914149x^{17} - 81171452536x^{16} + 358909614117x^{15} - 342465804764x^{14} - 440636552938x^{13} + 1045466945108x^{12} - 440636552938x^{11} - 342465804764x^{10} + 358909614117x^9 - 81171452536x^8 - 11500914149x^7 + 10305154244x^6 + 587781222x^5 - 539768196x^4 + 658791068x^3 - 253351628x^2 + 46265003x - 3130976)$

For the knot L^2R, let

$$\Delta(x) := \sqrt{x^4 - 2x^3 - 5x^2 - 2x + 1}.$$

We find for $\chi = 3, 4$:

- $G_1^{0,(4)} = \frac{1}{28812\Delta(x)^5}(-405x^{10} + 2509x^9 + 1829x^8 - 31810x^7 + 17655x^6 + 103891x^5 + 17655x^4 - 31810x^3 + 1829x^2 + 2509x - 405)$

- $G_3^{0,(2)} = \frac{1}{98784\Delta(x)^7}(405x^{14} - 15543x^{13} + 85620x^{12} + 205355x^{11} - 848819x^{10} + 2602758x^9 + 3614417x^8 - 2326433x^7 + 3614417x^6 + 2602758x^5 - 848819x^4 + 205355x^3 + 85620x^2 - 15543x + 405)$

- $G_5^{0,(0)} = \frac{x}{120\Delta(x)^{9/2}}(5x^{16} + 330x^{14} + 345x^{13} + 923x^{12} + 5863x^{11} + 1742x^{10} + 2213x^9 + 12970x^8 + 2213x^7 + 1742x^6 + 5863x^5 + 923x^4 + 345x^3 + 330x^2 + 5)$

- $G_1^{1,(2)} = \frac{1}{76832\Delta(x)^7}(-1629x^{14} + 16263x^{13} - 32316x^{12} - 96499x^{11} + 235411x^{10} + 1561162x^9 - 602369x^8 - 3948791x^7 - 602369x^6 + 1561162x^5 + 235411x^4 - 96499x^3 - 32316x^2 + 16263x - 1629)$

- $G_3^{1,(0)} = \frac{1}{1778112\Delta(x)^{9/2}}(2403x^{18} - 122715x^{17} + 979587x^{16} + 6670552x^{15} + 3536346x^{14} + 85677342x^{13} + 209713378x^{12} - 94163736x^{11} + 43538733x^{10} + 570479611x^9 + 43538733x^8 - 94163736x^7 + 209713378x^6 + 85677342x^5 + 3536346x^4 + 6670552x^3 + 979587x^2 - 122715x + 2403)$

- $G_1^{2,(0)} = \frac{1}{13829760\Delta(x)^{9/2}}(-50175x^{18} + 573019x^{17} - 945107x^{16} - 5002380x^{15} + 43143278x^{14} + 222424934x^{13} + 80209986x^{12} - 626011936x^{11} + 338200043x^{10} + 1765057977x^9 + 338200043x^8 - 626011936x^7 + 80209986x^6 + 222424934x^5 + 43143278x^4 - 5002380x^3 - 945107x^2 + 573019x - 50175)$

- $G_0^{0,(6)} = \frac{9153}{4705960}$

- $G_2^{0,(4)} = \frac{1}{4840416\Delta(x)^4}(-35235x^{16} + 468576x^{15} - 1040588x^{14} - 8404896x^{13} + 28367282x^{12} + 60969152x^{11} - 203134224x^{10} - 4676800x^9 + 482817047x^8 - 4676800x^7 - 203134224x^6 + 60969152x^5 + 28367282x^4 - 8404896x^3 - 1040588x^2 + 468576x - 35235)$

- $G_4^{0,(2)} = \frac{1}{8297856\Delta(x)^5}(14013x^{20} - 649818x^{19} + 4602051x^{18} + 10477662x^{17} - 93881621x^{16} + 671713524x^{15} + 641665002x^{14} - 1837161956x^{13} + 5685855609x^{12} + 5582126130x^{11} - 5373374195x^{10} + 5582126130x^9 + 5685855609x^8 - 1837161956x^7 + 641665002x^6 + 671713524x^5 - 93881621x^4 + 10477662x^3 + 4602051x^2 - 649818x + 14013)$

- $G_6^{0,(0)} = \frac{1}{11520\Delta(x)^6}(-x^{24} + 108x^{23} + 162x^{22} + 20224x^{21} + 48651x^{20} + 193992x^{19} + 1360390x^{18} + 753036x^{17} + 2395590x^{16} + 10112724x^{15} + 1162890x^{14} + 2199528x^{13} + 17809139x^{12} + 2199528x^{11} + 1162890x^{10} + 10112724x^9 + 2395590x^8 + 753036x^7 + 1360390x^6 + 193992x^5 + 48651x^4 + 20224x^3 + 162x^2 + 108x - 1)$

- $G_0^{1,(4)} = \frac{62217}{7529536}$

- $G_2^{1,(2)} = \frac{1}{2765952\Delta(x)^5}(-45009x^{20} + 716538x^{19} - 2766655x^{18} - 5389822x^{17} + 40664361x^{16} + 240587788x^{15} - 395122322x^{14} - 309050076x^{13} + 3991287235x^{12} + 303836206x^{11} - 7021681713x^{10} + 303836206x^9 + 3991287235x^8 - 309050076x^7 - 395122322x^6 + 240587788x^5 + 40664361x^4 - 5389822x^3 - 2766655x^2 + 716538x - 45009)$

- $G_4^{1,(0)} = \frac{1}{398297088\Delta(x)^6}(302247x^{24} - 13531428x^{23} + 115638930x^{22} + 1251445824x^{21} + 3537434803x^{20} + 49918941352x^{19} + 159322479190x^{18} + 33538177532x^{17} + 678466880918x^{16} + 1483815568612x^{15} - 850173440710x^{14} - 14701670200x^{13} + 3241900251163x^{12} - 14701670200x^{11} - 850173440710x^{10} + 1483815568612x^9 + 678466880918x^8 + 33538177532x^7 + 159322479190x^6 + 49918941352x^5 + 3537434803x^4 + 1251445824x^3 + 115638930x^2 - 13531428x + 302247)$

- $G_0^{2,(2)} = \frac{3393836431381}{94743488888832}$

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