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Direct integration of measured viscoelastic relaxation data in time-domain finite element simulations

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Abstract

The current approach to modeling viscoelastic materials in most commercial finite element packages is based on the General Maxwell Model, which views these materials as combinations of spring and dashpot elements. However, the data can be incorporated more directly into a transient finite element study by direct interpolation of the relaxation function. This work explores a linear interpolation scheme to the inclusion of viscoelastic relaxation functions on an example problem. The results show several benefits over the General Maxwell Model for transient studies. Included in the analysis are displacement solutions utilizing both approaches, relaxation function error calculations for both approaches, and parametric runtime studies comparing speed of calculation. The variation in computational flop counts is considered and an argument is made for the preference of the proposed approach.

Keywords: viscoelasticity, constitutive relations, finite elements, interpolation, transient

1. Introduction

Study of viscoelastic material remains commonplace in the literature [1–3]. Indeed, many manufactured materials are viscoelastic and naturally occurring biological materials often possess viscoelastic properties. Evaluating such materials in the time domain with finite elements has traditionally been done by using a constitutive model, which is usually the General Maxwell

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Model. Fitting the parameters of this model can be arduous and remains an area of study [4–6]. To enable the use of the General Maxwell Model with a step-wise integration scheme, the time integral in the statement of stress for a viscoelastic material is split into a long history of all timesteps and a short one encompassing only the current time step. This approach would seem to have been developed in the 1970s, without publication [7, 8]. To the author's knowledge, it is how all major finite element packages function [9–11].

The authors previously outlined a general approach to the problem and put it into the literature for the first time [12]. As part of that effort, a mathematical approach for direct inclusion of discrete measured data was outlined that allowed various interpolation schemes between points. This current effort takes a linear interpolation framework and applies it to a basic problem to evaluate it against a traditional General Maxwell Model formulation.

The results of this analysis show that linear interpolation has substantial benefits over the General Maxwell Model as applied to the time domain finite element formulation. The first benefit is that linear interpolation is faster for the problem studied. Basic analysis of flop counts as they relate to calculation of the relaxation function indicate that it should also be faster for all studies, regardless of the number of degrees of freedom or number of time steps. The second benefit is that linear interpolation is more accurate in expressing relaxation at the given time steps. Linear interpolation must by calculation have the exact value of the discrete data points for that time, while the General Maxwell Model is only a curve fit and so contains error at every data point. The final and most substantial improvement of the linear interpolation scheme is that it requires lower analyst skill. While the General Maxwell Model fit can be improved by more advanced algorithms and techniques like value seeding, linear interpolation is the same for every analyst.

Implementing the General Maxwell Model requires fitting a Prony series to data. While many current researchers may pride themselves on fitting Prony terms effectively, linear interpolation offers a more accurate fit that requires no expertise in viscoelasticity on the part of the analyst. The aim of this effort is to propose a discrete data interpolation approach to replace the General Maxwell Model in commercial finite element packages.

It is important to specify that the contents of this paper concern only linear viscoelastic behavior. To this end, there is no consideration of nonlinear behavior past critical strain, temperature effects or variation in the Poisson ratio for materials. Additionally, where used, strain-displacement relations are assumed to be linear. However, it may be possible to extend this effort in the future.

In section 2, interpolation of discrete relaxation data will be discussed including how the relaxation is calculated in the formulation at each timestep. In section 3, a structural bar containing multiple viscoelastic profiles is proposed and described as the object to be considered by both the General Maxwell Model and linear interpolation finite element approaches. Section 4 contains results from the analysis including a confirmation of the efficacy between approaches, solution time comparison and comparisons in accuracy of representing the relaxation function.

2. Interpolation of measured data

It is best to consider how interpolation may be calculated in place of a constitutive relation by first re-examining the statement of stress for a viscoelastic material. This is well documented in previous literature as [13]

$$\sigma(t) = \epsilon(t)\,Y(0) + \int_{0+}^{t} \epsilon(\tau)\,\frac{\mathrm{d}Y(t-\tau)}{\mathrm{d}(t-\tau)}\,\mathrm{d}\tau. \tag{1}$$

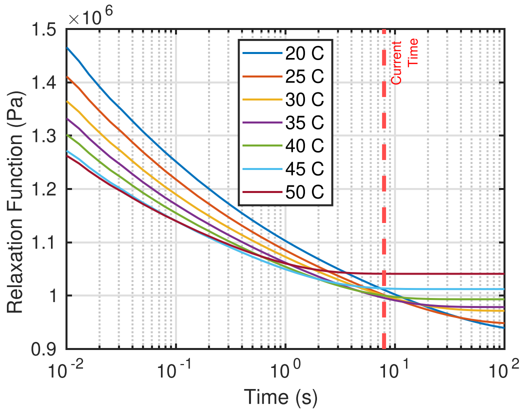

Figure 1. Relaxation modulus for shore 50 rubber as measured in [14].

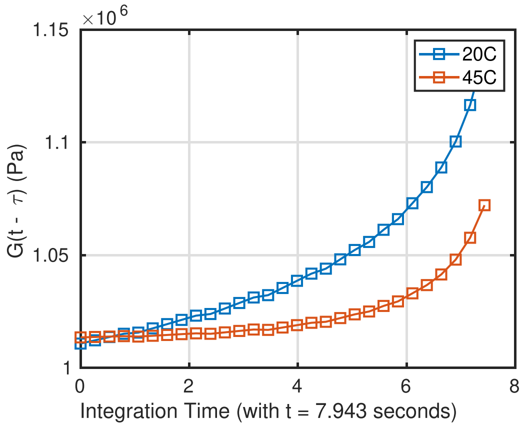

Figure 2. Value of the shear relaxation function for a current time value of $t = 7.943$ s and two selected temperatures.

Here $\sigma(t)$ is stress, $\epsilon(t)$ is strain and $Y(t-\tau)$ is the relaxation function. Additionally, $t-\tau$ will hereafter be called shifted time. What is critical to evaluate by either a constitutive relation or interpolation is the function $\mathrm{d}Y(t-\tau)/\mathrm{d}(t-\tau)$. For a given time t, which is assumed to be fixed, this critical term can be calculated for all τ up to that time. Practically this is done for a set number of time steps that need not be measured data points. To demonstrate this concept, a current time is specified in data available from the literature for various temperatures as shown in figure 1.

The experimentally measured relaxation function in figure 1 is from previous literature for various temperatures of a shore 50 nitrile rubber [14]. The current time chosen is 7.943 s, which is a specific measured data point from that data set. However, the data set is specifically the shear relaxation function and so from this point forward will be listed as G and the derivative G'. Similarly, the bulk relaxation function will be B and the derivative B' throughout this work. Now, a value of $G(t-\tau)$ can be calculated for each time step that came before the current time as shown in figure 2.

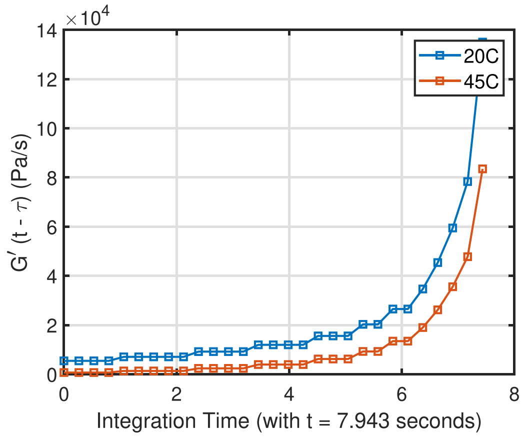

Figure 3. Value of the derivative of the relaxation function for a current time value of $t = 7.943$ s and two selected temperatures.

This is done at each time, t, by subtracting the reduced time from the current time and then interpolating the relaxation function $G(t)$, as the resulting time is unlikely to be at a measured time step. Similarly, the all important derivative of this term can then be expressed by considering how that value changed from each previous time step to the next time step for all time steps before t. This is shown in figure 3.

The values in figure 3 are then the slope of the line in figure 2. Since the ability to interpolate data from a data set is inarguable and the ability to measure the derivative of that data should be equally inarguable, it becomes clear that interpolation is a plausible way to represent the viscoelastic relaxation function.

2.1. Linear interpolation

If the relaxation function can be calculated based on interpolation from a known data set, then a finite element approach incorporating viscoelastic data directly is possible based on work that has already been developed [12]. This analysis considers two dimensional displacements in a layer of constant thickness h. The final equation of motion from [12] is

$$\mathbf{M}\ddot{\mathbf{\Delta}}_{i+1} + \mathbf{C}\dot{\mathbf{\Delta}}_{i+1} + \mathbf{K}\mathbf{\Delta}_{i+1} = \mathbf{f}_{i+1} + \mathbf{Q}_{i+1} + \mathbf{g}_i + \mathbf{b}_i \tag{2}$$

where Δ_{i+1} is the displacement vector for the time step being calculated, $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{Q}_{i+1}$ is a traction force, $\mathbf{f}_{i+1}$ is a body force, $\mathbf{g}_i$ is a viscoelastic internal shear force from all previous time steps, $\mathbf{b}_i$ is a viscoelastic internal bulk force from all previous time steps and $\mathbf{C}$ incorporates the elements $\mathbf{G}_i$ and $\mathbf{B}_i$ and is

$$\mathbf{C} = \mathbf{G}_i + \mathbf{B}_i \tag{3}$$

where $\mathbf{B}_i$ and $\mathbf{G}_i$ are

$$\mathbf{G}_i = h\mathbf{J}_G \int_{t_i}^{t_{i+1}} G'\left(t_{i+1} - \tau\right) \left(\frac{1}{2}\right) \left(\frac{(\tau - t_i)^2}{\Delta t}\right) \mathrm{d}\tau \tag{4}$$

$$\mathbf{B}_i = h\mathbf{J}_B \int_{t_i}^{t_{i+1}} B'(t_{i+1}-\tau)\left(\frac{1}{2}\right)\left(\frac{(\tau-t_i)^2}{\Delta t}\right)\mathrm{d}\tau. \tag{5}$$

The constants $\mathbf{J}_B$ and $\mathbf{J}_G$ are the spatial integrals for a finite element and h is the thickness of a layer. Similarly, $\mathbf{g}_i$ and $\mathbf{b}_i$ are

$$\mathbf{g}_i = -h\mathbf{J}_G \int_{0+}^{t_i} G'(t_{i+1}-\tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau - h\mathbf{J}_G \int_{t_i}^{t_{i+1}} G'(t_{i+1}-\tau) \times \left[\dot{\boldsymbol{\Delta}}_i(\tau-t_i) + \left(\frac{1}{2}\right)\left(\frac{(\tau-t_i)^2}{\Delta t}\right)\left(-\dot{\boldsymbol{\Delta}}_i\right) + \boldsymbol{\Delta}_i\right]\mathrm{d}\tau \tag{6}$$

$$\mathbf{b}_i = -h\mathbf{J}_B \int_{0+}^{t_i} B'(t_{i+1}-\tau)\,\boldsymbol{\Delta}(\tau)\,\mathrm{d}\tau - h\mathbf{J}_B \int_{t_i}^{t_{i+1}} B'(t_{i+1}-\tau) \times \left[\dot{\boldsymbol{\Delta}}_i(\tau-t_i) + \left(\frac{(\tau-t_i)^2}{2\Delta t}\right)\left(-\dot{\boldsymbol{\Delta}}_i\right) + \boldsymbol{\Delta}_i\right]\mathrm{d}\tau. \tag{7}$$

The following assumption from [12] then provides a linear interpolation scheme for viscoelastic finite elements

$$G(t_{i+1}-\tau) = a(t_{i+1}-\tau) + b, \tag{8}$$

$$G'(t_{i+1}-\tau) = a. \tag{9}$$

In this framework the only relaxation input in the formulation of viscoelastic terms is actually the derivative of the relaxation function, $G'(t_{i+1}-\tau)$. This would be true of any interpolation scheme or the General Maxwell Model. In the case of any relaxation fit, a function is only needed to assess slopes in the relaxation curves in between time steps. In the case of the General Maxwell Model, the fit would be one exponential function across all data points. However, in the case of linear interpolation the fits used are effectively smaller and each only holds for the space between the two points it is defined by. In either case, there is no deep impact to the behavior of the solver.

3. Implementation on a 1D bar

To test the implementation of this scheme a notional structure is defined and shown in figure 4. The selection of a bar element is arbitrary, another element, such as a beam or sandwich could have been chosen. For now, the goal is simply to demonstrate the efficacy of the approach using any structure. The end of section 2 highlights the fact that the ultimate role of linear viscoelastic material relations is only in providing values for the relaxation function. It is therefore reasonable to assume the change from the General Maxwell Model to an interpolation scheme does not limit modeling geometry. Section **A** is the shore 50 nitrile rubber presented in [14] at 45 °C, section **B** is a generic aluminum profile, assumed to be elastic, section **C** is also a shore 50 nitrile rubber profile from [14], but at 20 °C. The model computes displacement $u(x,t)$ in the x direction.

The bar is assumed to be fixed at $x = 0$ and have a square cross-sectional profile. The selection of two separate viscoelastic profiles and one elastic profile is intentional. In the implementation of the forcing function in the equations of motion, a structure containing only a single

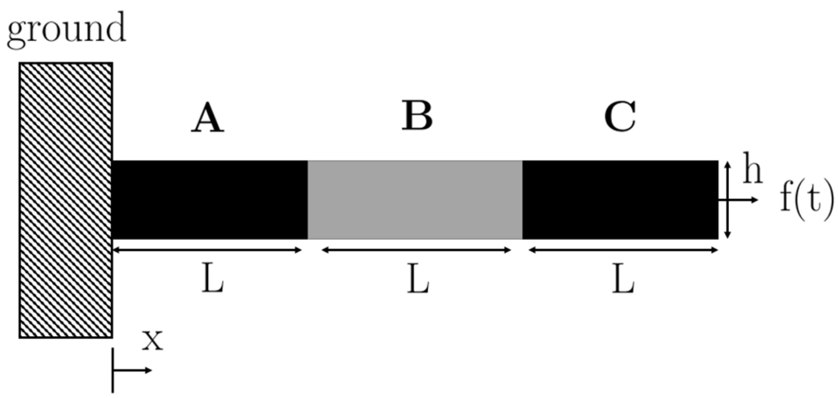

Figure 4. Diagram of test problem incorporating two different materials and two different viscoelastic profiles for the same material due to temperature variation.

Table 1. Properties of the bar test structure.

Property	Description	Value
L	Section length	1 m
h	Bar height	0.5 m
b	Bar width	0.5 m
ρ_r	Rubber density	$1300\,\mathrm{kg\,m^{-3}}$
ν_r	Rubber Poisson ratio	.47
ρ_a	Aluminum density	$2.710\,\mathrm{kg\,m^{-3}}$
E	Aluminum Youngs Modulus	68 GPa
ν_a	Aluminum Poisson ratio	.33
x_0	Initial Displacement	0
$\dot{x}_0$	Initial Velocity	0

viscoelastic profile presents opportunities for simplification that are not representative of a generalized scheme consistent with a commercial finite element package. Equally, an elastic aluminum profile is included to illustrate that any proposed code be generalized to include both viscoelastic and elastic profiles. The selection of two different linear viscoelastic profiles by temperature is arbitrary and should not be associated with a model intended to account for variations in behavior under extreme temperature changes.

Table 1 presents the relevant properties of the materials, the dimensions of the bar and the initial conditions. Although the rubber used in two sections is the same, the temperature in each section is different, causing the two different viscoelastic profiles.

The forcing function, $f(t)$ is given as

$$f(t) = 5\sin(15t), (N) \tag{10}$$

3.1. Comparison with the generalized Maxwell model

To support the proposal of a discrete data interpolation approach to replace the General Maxwell Model, the model fit from the original paper presenting the data will be used [14]. Values of fitting parameters given in the paper were used. This fit is shown in figure 5.

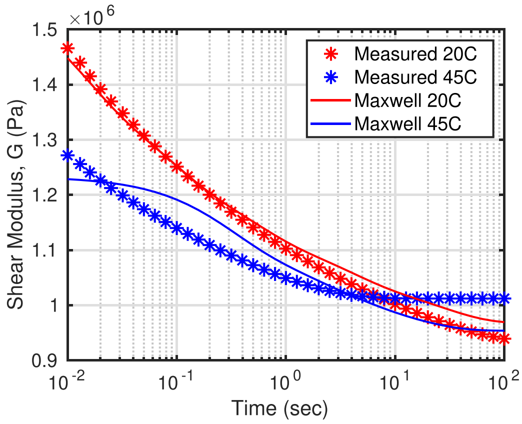

Figure 5. Comparison of Maxwell model fit versus discrete data for shear modulus of nitrile rubber at two different temperatures [14].

These fits are consistent with the form of the ANSYS viscoelastic curve fit tool given as

$$G(t) = G_0 \left[\alpha_0 + \sum_{n=1}^{N} \alpha_n e^{(-t/\tau_n)} \right]. \tag{11}$$

For the 20 °C and 45 °C fit, the authors assumed an α_0 value of .66 and .75 respectively.

In equation (11), G_0, α_0, α_n, and τ_n are all traditionally assumed to be positive [15]. When all these terms are held to be positive, the resulting relaxation function is guaranteed to be monotonically decreasing. This is a key difference betweeen the two schemes considered here, since many interpolation schemes may only be monotonically decreasing when the data itself is monotonically decreasing. Additional work is underway to determine what impact this difference may have on solution stability, but also to consider in which such behavior may be desirable to model.

In the General Maxwell Model, the exponential terms decay at different rates. In early time, all terms contribute to the total value of the relaxation function. As terms decay to zero they stop contributing to the value of the function and only terms with longer time constants continue to play a major role. Eventually the terms effectively decay entirely and a fixed value holds into infinite time.

3.2. The Newmark-Beta method

To evaluate the solution, a fixed time step, Δt was used and displacement, velocity and acceleration were calculated at each of these new time step. A fixed time step offers speed benefits as compared to a variable time step, as the left hand damping matrix does not need to be recomputed. Time step integration was performed using the Newmark-Beta method as set forward in [16]. The assigned γ value was $1/2$ and the assigned β value was $1/4$. The Newmark-Beta method is assumed to be the most representative approach of current commercial finite element packages.

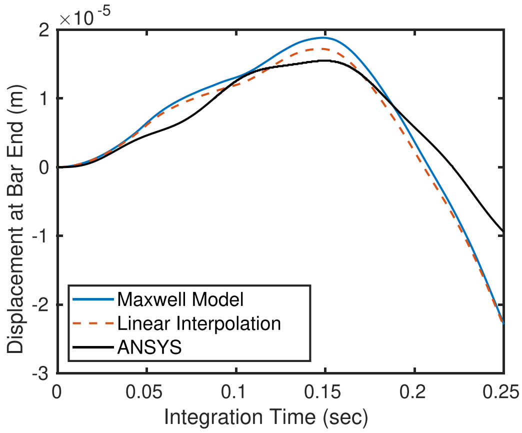

Figure 6. Comparison of displacement solutions for the end of a bar.

4. Results

Presented now are results for various aspects of the study, including a comparison of the displacement results between the two models alongside a commercial benchmark. The follows a comparison of calculation times and relaxation error.

4.1. Displacement solution

The displacement solution in figure 6 is limited to the first 0.25 s to show the difference between constitutive relations. The slope of a relaxation function is greatest at low integration times, so it is convenient for showing disparities.

It is clear that the relaxation function is not identical between the Maxwell model and linear interpolation, which is expected. An ANSYS benchmark is provided for the same model. Some difference is visible which is likely a product of the analysis choices within ANSYS. While the code generated in MATLAB has very few assumptions and reductions, even a basic bar study, has numerous numerical options in ANSYS. The author used link/truss elements, with a moderate speed assumption, with all other settings allowed to be program controlled.

4.2. Comparison of calculation times

Two parameter sweeps were conducted while calculating run time. In the first the number of elements in the bar was increased while holding Δ_t constant. In the second, the reverse was true, a constant 30 elements were used while Δ_t was varied. Figure 7 shows the variation in calculation with element count, while figure 8 shows the variation in calculation time with inverse time step size. Runtime was evaluated on a 14 core laptop with Intel i9 processors.

As expected, as time step decreased total run time increased as more time steps needed to be evaluated for a given study time. Similarly as element counts and thus degree of freedom count increased, run time increased. Linear interpolation produced a faster result than the General Maxwell Model of 8%–10% consistently. Profiling of the solution indicated that this calculation time change was based on the calculation of the parameters g_i and b_i with the other viscoelastic terms composing C being fixed for a constant time step and thus of little impact to solution time.

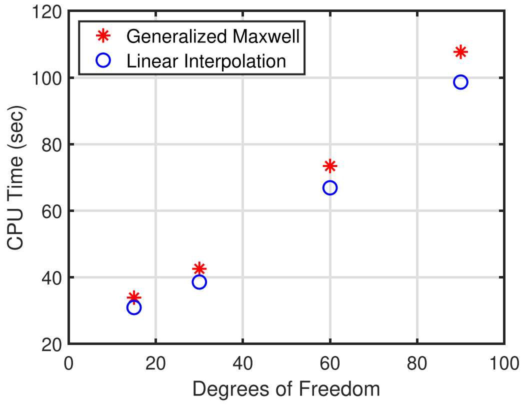

Figure 7. Comparison of Generalized Maxwell and linear interpolation runtimes for varying DOF in a 10 s analysis.

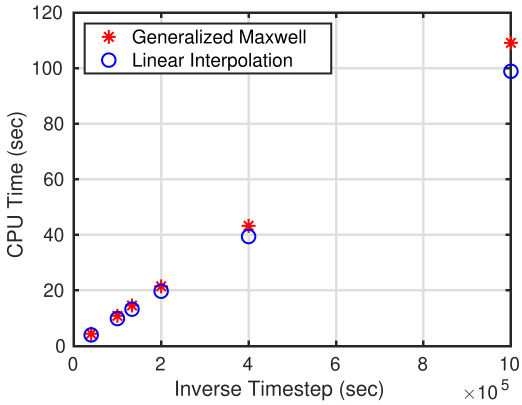

Figure 8. Comparison of Generalized Maxwell and linear interpolation runtimes for varying time step size in a 5 s analysis.

The values presented in figures 7 and 8 are based on an average of five calculations at each data point. Table 2 presents the standard deviation of these CPU time calculations against the mean difference in CPU time between the two methodologies. In all cases, the highest standard deviation was less than 20% of the mean difference in calculation time. This indicates that the performance increase is real and not a product of transient computer behavior.

A close consideration of [12] indicates that this run time improvement should always be present. This is because a complete reduction of the General Maxwell Prony terms to an equivalent value a is possible and the only divide comes from the calculation of exponential functions in Prony terms. Table 3 highlights the flop count for calculations of G' for various Prony term counts.

The calculation in table 3 is based on an assumption of a numerical integration based approach to the solution of exponential for a 1×1 matrix per [17]. Since for a system of two materials, B' and G' would need to be represented at least once per material per time step, the minimum total flop cost per time step is four times as great. In practice, this number is

Table 2. Difference in CPU calculation time across studies based on 5 data point averages.

Figure	Study parameter	CPU time difference	Standard deviation (GMM/Interpolation)
7	15 DOF	2.975 s	0.273/0.485 s
7	30 DOF	3.978 s	0.412/0.791 s
7	60 DOF	6.547 s	0.471/0.447 s
7	90 DOF	9.0875 s	1.806/.0667 s
8	$2.50 * 10^4$ 1/s	0.440 s	.062/.019 s
8	$1.00 * 10^5$ 1/s	0.978 s	.042/.030 s
8	$1.33 * 10^5$ 1/s	1.314 s	.027/.049 s
8	$2.00 * 10^5$ 1/s	1.76 s	.069/.064 s
8	$4.00 * 10^5$ 1/s	3.952 s	.063/.050 s
8	$1.00 * 10^6$ 1/s	10.238 s	.200/.198 s

Table 3. Minimum flops needed for representation of G' or B' per material per time step by number of Prony terms.

Prony terms	Flops to calculate G'
5	120
10	240
15	360
20	480
25	600

even greater still as multiple representations of each in different sequencing arises from calculation of multiple forcing terms. In comparison, calculation of the slope a should require only 20 total flops per time step it is calculated, which need only be a minority of time steps. The result of this difference is that the linear interpolation scheme should always be computationally cheaper than calculations using a General Maxwell Model. The difference should become more minimal in large degree of freedom systems with few viscoelastic profiles and be greatest in low degree of freedom systems with many viscoelastic profiles.

4.3. *Error in interpreting relaxation*

While the runtime decrease for a linear interpolation scheme is beneficial for modern finite element tools, the primary benefit of this approach is the ease and accuracy in interpreting relaxation data. To demonstrate this benefit, two General Maxwell model fits were created using ANSYS again, this time independent of those used in the analysis of the 1D bar structure. In figure 9, these two fits are shown. In the unseeded approach the shear data was fed into ANSYS without any additional analyst operation and a fit to absolute error was used. In the seeded fit, the time elements from [14] were used to generate seed time values along with other seed time guesses, fitting the assumption of a skilled analyst.

In both cases, the fit produces error but the seeded fit is far better, implying that even within a commercial tool, the generation of prony terms is currently dependent on operator skill. In contrast, linear interpolation requires no operator skill or effort. Furthermore, the accuracy of this fit is likely always more accurate for well sampled discrete data. In figures 10 and 11

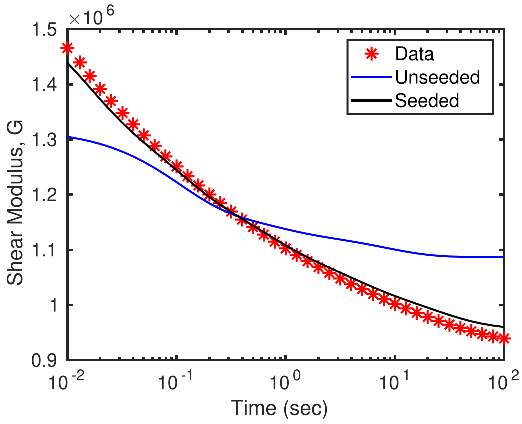

Figure 9. Comparison of seeded and unseeded Prony term fits in ANSYS.

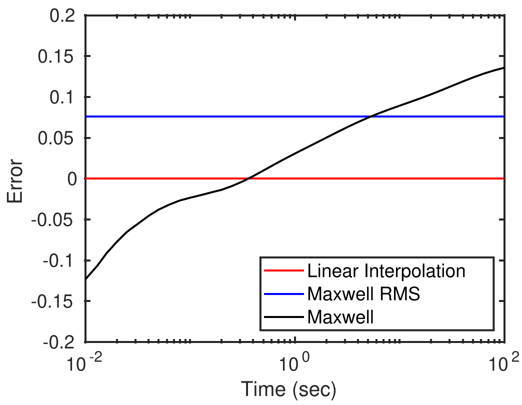

Figure 10. Comparison of maximum and RMS error between a linear interpolation and an unseeded ANSYS Prony term fit as measured at known data points.

relative accuracy is calculated for the General Maxwell fit and linear interpolation as measured at known points.

Throughout this discussion of relaxation error, it is assumed that accuracy in representing discrete measured data points produces a more accurate displacement solution. It is clear from these figures that the linear interpolation scheme produces less error at known points. One fair critique of this calculation may be considering the time intervals between known data points. If the decay of relaxation functions is expected to follow a decaying exponential as Prony terms suggest, then there could be less accuracy betweens points. However, this is a nebulous issue, since the data does not exist and the points bounding the interval likely already contain error under the General Maxwell Scheme. Additionally, the exact shape of the decay between points is distorted by the combination and interplay of Prony terms.

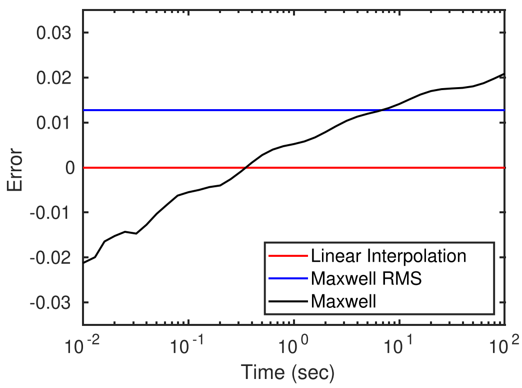

Figure 11. Comparison of maximum and RMS error between a linear interpolation and a seeded ANSYS Prony term fit as measured at known data points.

5. Conclusion

An implementation of a linear interpolation scheme for viscoelastic relaxation has been presented for the case of a 1D bar problem with mutliple viscoelastic profiles. Further analysis of run time was performed with parameter sweeps of time step and element count. Additionally, an analysis of comparative error in calculating relaxation functions has been performed. In all cases the General Maxwell Model was used as a baseline. The results demonstrate three key benefits. The first benefit is the reduction in run time for transient finite element studies. The second is an improvement in the accuracy of representing the material relaxation function. The third is a reduction in the required level of skill and understanding of viscoelastic calculations needed to conduct a relevant finite element study.

In the future this work can be extended to accommodate other viscoelastic interpolation schemes, such as a quadratic interpolation. Also, the same principle of a more direct inclusion of discrete material data may also be beneficial in other calculations in the transient domain, such as hyperelasticity.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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