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The axiom of choice and the paradoxes of the sphere.

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THE AXIOM OF CHOICE AND THE PARADOXES OF THE SPHERE

by

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CHAPTER I

INTRODUCTION

This thesis primarily deals with the Axiom of Choice, a controversial axiom of Axiomatic Set Theory, and two paradoxes of the sphere that result from the use of the Axiom of Choice.

The main question to be discussed is the following: Should the Axiom of Choice be accepted or rejected? No attempt is made to give a final answer to this question, but a tentative conclusion is made.

The two paradoxes, the Hausdorff Paradox and the Banach and Tarski Paradox are discussed at length in Chapters III and IV respectively. In the presentation of these paradoxes, the specific questions to be kept in mind are the following:

(1) Can the surface of a sphere be decomposed into subsets in such a way that a half and a third of the surface may be congruent to each other?

(2) Can a solid sphere of fixed radius be decomposed into a finite number of pieces and can these pieces be reassembled to form two solid spheres with the given radius?

(3) If the answer to the previous question is in the affirmative , what is the minimum number of pieces required?

An understanding of the basic terms of set theory including proper subset, disjoint, equivalent sets, etc. is presupposed.

In Chapter II the Axiom of Choice is stated and several examples are given to explain the meaning and use of this axiom. Two fundamental terms, namely congruence and equivalence by finite decomposition, are defined and several examples are given. The measure problem is also defined.

In Chapters III and IV a compilation of the work of several authors on the Hausdorff and Banach and Tarski Paradoxes is presented. Particular attention is given to the work of Hausdorff, Banach and Tarski and Robinson, while a brief review of the papers of von Neumann, Sierpinski, Adams, and Dekker and de Groot is also presented. Particular reference is made to the use of the Axiom of Choice in these Paradoxes.

The discussion of the controversy over the Axiom of Choice in Chapter V follows closely the material in Sierpinski [17], Fraenkel and Bar-Hillel [9], and Fraenkel's introduction to Bernays [4], while the applications in Chapter VI come from Sierpinski [17].

CHAPTER II

DEFINITIONS AND EXAMPLES

2.1 The Axiom of Choice.

In 1904 E. Zermelo stated an axiom as a basis for his proof of the well-ordering theorem. This axiom, which he called the Axiom of Choice, reads as follows: For every set Z whose elements are sets A, non-empty and mutually disjoint, there exists at least one set B having one and only one element from each of the sets A belonging to Z.

B. Russell called this axiom the Multiplicative Axiom which, in many ways, is a more appropriate name. This is evident if the axiom is stated in a different form. The following is necessary: Theorem: For every disjointed set Z, there exists the set whose elements are the sets which contain a single element from each element of Z. This set is called the Cartesian product and is denoted by CZ. Then the Axiom of Choice can be given in the following form: For every disjointed set Z for which the null set is not a member of Z, the Cartesian product CZ differs from the null set.

Each element of CZ is called a selection-set of Z since it contains a single element from each element of Z.

The Axiom of Choice is sometimes stated in a third form on the basis of a choice function which can be defined as follows: f is a choice function for Z if, and only if, f is a function whose domain is the collection of non-empty subsets of Z and for every $A \subseteq Z$ with $A \neq 0$, $f(a) \in A$.

The Axiom of Choice may now be formulated: Every set has a choice function.[1]

All of the above forms of the Axiom of Choice are used extensively in Axiomatic Set Theory, but the first form will be the only one referred to throughout the remainder of this paper.

A few examples will now be given to illustrate the meaning and use of the Axiom of Choice. The first of these is a simple non-mathematical illustration.

(1) Let Z be the set of all people living on the earth, and let $A=A_1, A_2, A_3 \ldots A_n$ be the subsets of these people living in the various n countries of the earth, with the assumption that each person lives in one and only one country. From the Axiom of Choice the conclusion is that there exists at least one set B consisting of one and only one resident of each country. This is an example of the trivial case where the set Z consists of a finite sequence of sets.

It is to be noted that the Axiom of Choice states that such a set B exists, but it does not give the rule for finding B or even state that B can be found. It is for this very reason that some mathematicians, especially those of the intuitionist school headed by L. Brouwer, reject the Axiom of Choice. To them, mathematical existence is equivalent to constructibility. If a set cannot be constructed, it does not exist.

However, a larger number of mathematicians accept the Axiom of Choice, although many do so with reservations. If the rules for the construction of a given set cannot be formulated, they, in general, use the Axiom of

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[1]The Axiom of Choice is equivalent to various principles. Some of the most important of these are listed in Appendix I.

Choice to verify the existence of this set. For them, existence and constructibility are not equivalent terms.

(2) B. Russell has given an illustration of this problem of constructibility. Given an infinite collection of pairs of shoes, the Axiom of Choice is not needed to establish the existence of a set of shoes containing one and only one shoe from each pair. This set can be constructed by the simple rule of selecting all left shoes from the collection. However, given an infinite collection of pairs of socks, all socks identical as to size, color, etc., the Axiom of Choice is needed to establish the existence of a set containing exactly one sock from each pair. There is no possibility of giving a rule by which this set may be constructed.[1]

The following are two examples of a more mathematical nature.

(3) Consider the set of all infinite sequences of real numbers. Let us arrange this set into subsets in such a way that each subset will contain all sequences that differ only in the order of their terms. It is impossible to define a set B containing one and only one sequence from each of these subsets. We must use the Axiom of Choice to prove the existence of such a set B.

(4) Consider the set of all convergent sequences of real numbers. Let us include in the same subset all sequences converging to the same limit. Then, without the aid of the Axiom of Choice, it would be possible to define the set B containing one sequence from each subset (e.g., take B to be the set of all infinite sequences $\{x_n+a\}$ (n=1, 2, ...) for real x and real a).[2]

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[1]Russell, pp. 126-127.

[2]Sierpinski, Cardinal and Ordinal Numbers, p. 106.

2.2 Congruence.

The paradoxes mentioned in this paper all deal with congruences between figures, particularly between sets and proper subsets. Two point sets in Euclidean space are congruent if, and only if, one set may be transformed into the other by rigid motion as defined in elementary geometry (i.e., by rotation about a fixed point and translation, one operation following the other or either operation alone; reflections are excluded unless otherwise stated). An appropriate formal definition of congruence is based on the fact that distances remain invariant under rigid motions. It is assumed that to every pair of points (a, b) there corresponds a real number d(a, b), called the distance between the points a and b. Definition: The sets of points A and B are congruent: $A \simeq B$ if there exists a function f, which transforms A into B in a one-to-one manner such that if a_1 and a_2 are two arbitrary points of the set A, then $d(a_1, a_2)=d[f(a_1), f(a_2)]$.[1]

The following examples illustrate congruence between a set and a proper subset.

(1) Consider the horizontal real line L. Let us form two rays on this line in the following manner. One ray, R_1, consists of all points X such that $X \geq a$ where a is an arbitrarily chosen point on the line L. The second ray, R_2, consists of all points X such that $X \geq b$ where $b > a$. Now it is clear that R_2 is a proper subset of R_1 while $R_1 \simeq R_2$ since R_1 can be made to coincide with R_2 by a translation along line L so that a coincides with b. (Note that both rays contain an infinite set of points

[1]Translated from Banach and Tarski, p. 245.

and the above example also illustrates the definition of an infinite set, namely, a set R_1 is said to be infinite if R_1 contains a proper subset which is equivalent to R_1.)[1]

(2) Let us consider on the unit circle the set A of points whose angles formed with the x-axis are $n\theta$, $n=1, 2, \ldots$ where θ is such that θ/n is not a rational number. If B is the subset of A for which $n=2,3,\ldots$ and the set A is rotated by an angle θ, then it will coincide with B. [2]

(3) The above two examples illustrate how a set B may be divided into two parts, $A=A_1+A_2$ such that $A_1\neq 0$, $A_2\neq 0$, $A_1 \cdot A_2=0$ and $A \underline{\sim} A_1$. Now let us consider an example in which these conditions hold and in addition the two subsets are congruent to each other and to the set A, i.e., $A_1 \simeq A_2 \simeq A$.

Let the set A consist of a point O of a plane together with all points in the plane obtainable from O by all possible finite combinations of the following operations: (1) counter-clockwise rotation through one radian about O; (2) translation through one unit in a fixed direction.

These operations are applied to O and to each point obtained from O by their use. As a particular case, let O be the origin of a cartesian coordinate system and consider translations in the direction of the positive x-axis. If the final operation in the sequence of operations yielding a point p of A is a rotation (1), then the point p will be defined as belonging to A_1. Otherwise, it will belong to A_2.

Now $A=A_1+A_2$, $A_1\neq 0$, $A_2\neq 0$. In addition, $A_1 \cdot A_2=0$ since a

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[1]Blumenthal, p. 347.

[2]Figueiredo, p. 1.

point p of A cannot be obtained by two sequences of operations, one ending in a rotation and the other ending in a translation. This is true since $A_1 \cdot A_2 \neq 0$ implies that e^i satisfies an algebraic equation with integral coefficients, which is impossible since e^i is transcendental.

If the entire set A is rotated about 0 through one radian, the set A will coincide with the subset A_1 since the rotation transforms every point p of A into a point of A_1, while every point of A_1 is certainly a point of the rotated set. Therefore $A \simeq A_1$. Similarly, if the entire set A is translated one unit to the right, it will coincide with A_2. Therefore $A_1 \simeq A_2 \simeq A_1 + A_2$. It is noted that the Axiom of Choice is not needed anywhere in this example.[1]

2.3 Measure.

It is not the purpose of this paper to deal at length with the problem of measure. However, in order to clearly understand the Hausdorff Paradox, it is necessary to discuss briefly an historic measure problem. The question reads as follows: Is it possible to define a real function f for all bounded sets A of an Euclidean space such that:

(1) $f(A) \geq 0$ for all A

(2) $f(A_0) > 0$ for some set A_0 in the space

(3) if $A \cdot B = 0$, then $f(A+B) = f(A) + f(B)$

(4) if $A \simeq B$, then $f(A) = f(B)$.

This function f is called a finite measure. Property (3) is known as

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[1] S. Mazurkiewicz and W. Sierpinski, pp. 618-619 as discussed by Blumenthal, pp. 348-349.

finite additivity.[1]

This problem will be discussed further in Chapter III.

2.4 Equivalence by Finite Decomposition.

The definition as originally given by Banach and Tarski will be used.

Definition: Two sets of points, A and B are equivalent by finite decomposition, $A \underset{f}{=} B$, provided sets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ exist with the following properties:

(1) $A=A_1+A_2+\ldots+A_n$ $\qquad B=B_1+B_2+\ldots+B_n$

(2) $A_j \cdot A_k=B_j \cdot B_k=0$, $1 \leq j < k \leq n$

(3) $A_j \simeq B_j$, $1 \leq j \leq n$

There is an analogous definition for the concept of equivalence by denumerable decomposition.[2]

To become better acquainted with the concept of equivalence by finite decomposition, two examples are presented, one in Euclidean two space and one in three space.

(1) Let S_1 be a square with area 100 square inches, and S_2 a second square with area 1 square inch. If these two squares are equivalent by finite decomposition, then they can be cut into the same finite number of parts such that corresponding parts may be made to coincide after appropriate rigid motions. This is impossible as it would appear intuitively and with the aid of a theorem by Banach [2] it can be shown that two polygons are equivalent by finite decomposition if, and only if, they have the same area.

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[1] Figueiredo, p. 2.

[2] Banach and Tarski, p. 246.

(2) Now let C_1 be a cube of volume 100 cubic inches, and C_2 a second cube with volume 1 cubic inch. The meaning of equivalence by finite decomposition is completely analogous to the two dimensional case except that the "parts" as well as the cube itself are of three dimensions. Here, contrary to intuition, the two cubes are equivalent by finite decomposition. This surprising result is due to the work of Banach and Tarski which will be discussed at length in Chapter IV.

CHAPTER III

THE HAUSDORFF PARADOX

3.1 The work of Hausdorff.

In 1923 Banach [2] gave an affirmative answer to the historic measure problem for n=2. That is, he proved that a finite measure could be assigned to all bounded sets in the plane.

Previously, in 1914, Hausdorff [10] gave a negative answer to the measure problem for the surface K of a sphere. He arrived at this answer by first proving what is often called the Hausdorff Paradox, that the surface K of the sphere could be decomposed into four disjoint subsets A, B, C, and Q such that

(1) $K=A+B+C+Q$

(2) $A\cong B\cong C,\ A\cong B+C$

where Q is denumerable.

To solve the measure problem Hausdorff showed that if such a finite measure function f exists, then f(Q) must be equal to zero. This follows from properties (3) and (4) from page 8 and from the fact that Q is denumerable. Congruences (2) imply that f(K) must simultaneously equal 3f(A) and 2f(A). This can be true only if f(K)=0 and then property (2) on page 8 cannot be fulfilled. Therefore it is impossible to define a finitely additive measure for all the subsets of the surface K of the sphere.

This negative answer to the measure problem implies a similar negative answer for all bounded sets in $n \geq 3$ dimensional space. It should be noted that the difference in the solution of the measure problem for two and three or more dimensions is closely connected to the fact that a paradox, comparable to those of Hausdorff and Banach and Tarski does not exist in two dimensional space.

To produce the decomposition of the surface K of the sphere Hausdorff introduces only rotations. A and B+C are interchanged by a rotation ϕ of 180° about one axis, and A, B, C are permuted by a rotation ψ of 120° about another suitably chosen axis. A proper choice of these rotation axes is necessary so that ϕ and ψ will be independent except for the following relationship: $\phi^2 = \psi^3 = 1$ (where 1 denotes the identical transformation). Hausdorff constructs the subsets so as to satisfy the following:

$$A\phi = B+C \qquad (B+C)\phi = A \qquad Q\phi = Q$$

$$A\psi = B \qquad B\psi = C \qquad C\psi = A \qquad Q\psi = Q$$

where $A\phi$, for example, denotes the transformation of the point set A by ϕ.

The following is the main argument as presented by Hausdorff. A rotation group G is generated with ϕ, ψ, ψ^2 the simple factors. It appears as follows:

$$(G) \qquad 1 \mid \phi, \psi, \psi^2 \mid \phi\psi, \phi\psi^2, \psi\phi, \psi^2\phi \mid \ldots$$

To verify the independence of ϕ and ψ, except for $\phi^2 = \psi^3 = 1$, it is noted that the products of two or more factors are of one of four forms.

$$\alpha = \phi\psi^{m_1}\phi\psi^{m_2} \ldots \phi\psi^{m_n}$$
$$\beta = \psi^{m_1}\phi\psi^{m_2}\phi \ldots \psi^{m_n}\phi$$
$$\gamma = \phi\psi^{m_1}\phi\psi^{m_2} \ldots \phi\psi^{m_n}\phi$$
$$\delta = \psi^{m_1}\phi\psi^{m_2} \ldots \phi\psi^{m_n}$$

where n is a natural number and $m_1, m_2, \ldots, m_n$ are equal to 1 or 2. A relationship $\rho = \sigma$ between two formally distinct products would result in $\rho\sigma^{-1}=1$. It is shown that this relationship can be assumed to have the form $\alpha = 1$. The problem is then to show that all products α are different from 1 if both rotation axes are chosen properly.

A right angle coordinate system through the center of the sphere is chosen such that the ϕ axis is in the xz plane and the ψ axis is on the z axis. The angle formed by the two axes is labeled $\frac{1}{2}D$ and cos $2/3\,\pi$ is designated by L, while sin $2/3\,\pi$ is designated by M.

The following orthogonal transformations fit the rotations:

$$(\psi)\quad \begin{cases} x'=xL-yM \\ y'=xM+yL \\ z'=z \end{cases}$$

$$(\phi)\quad \begin{cases} x'=-x\cos D+z\sin D \\ y'=-y \\ z'=x\sin D+z\cos D \end{cases}$$

$$(\phi\psi)\quad \begin{cases} x'=-xL\cos D+yM+zL\sin D \\ y'=-xM\cos D-yL+zM\sin D \\ z'=x\sin D \qquad +z\cos D \end{cases}$$

It is noted that ψ^2 will replace ψ if M is interchanged with -M. α denotes a product of n double factors, $\phi\psi$ or $\phi\psi^2$ while $\alpha'=\alpha\phi\psi$ or $\alpha'=\alpha\phi\psi^2$ denotes a product of (n+1) such double factors. The point 0, 0, 1 is transformed by α in x, y, z and by α' in x', y', z' such that the transformation $(\phi\psi)$ or the transformation $(\phi\psi^2)$ results. Since $\frac{x}{\sin D}$ and $\frac{y}{\sin D}$ are polynomials of the (n-1)th degree and z is a polynomial of the nth degree in cos D, it follows that

$$x=\sin D(a \cos D^{n-1}+\ldots)$$

$$y=\sin D(b \cos D^{n-1}+\ldots)$$

$$z=c \cos D^{n}+\ldots$$

For n=1, the point 0, 0, 1 is transformed by $\phi\psi$ or $\phi\psi^2$ into the point L sin D, $\pm$ M sin D, cos D. It is noted that there are only a finite number of values of cos D for which α =1 and it is possible, by avoiding at most the denumerable values, to choose the angle D such that no product α will equal 1.

Accordingly, the rotations (G) are not only formally distinct in pairs but are such that they are distributed into three classes A, B, C in the following manner: one of the two rotations ρ , $\rho\phi$ belongs to A and the other belongs to B+C; one of the three rotations ρ , $\rho\psi$, $\rho\psi^2$, belongs to each of A, B, C. This distribution is possible.

Finally, Q is the denumerable set of fixed points (rotation pole) of the rotations of the group that differ from 1 and K=P+Q. The rotations (G) transform a point x of P into the points
$x, x\phi, x\psi , x\psi^2, \ldots$

These points are distinct in pairs and their set is P_x. Two such sets P_x, P_y either have no points in common or are identical. Exactly one point x is chosen from each set P_x, and thus the set N= $\{x, y, \ldots\}$ is formed. Then $P=N+N\phi+N\psi+N\psi^2\ldots$ In accordance with the above distribution of rotations into classes, P now decomposes into three sets: P=A+B+C where $A=N+N\psi\phi+N\psi^2\phi+N\phi\psi^2+\ldots$

$$B=N\phi+N\psi+\ldots$$

$$C=N\psi^2+N\phi\psi+\ldots$$

and according to the construction $A\phi=B+C$, $A\psi=B$, $A\psi^2=C$. Therefore, the sets A, B, C, B+C are congruent.[1]

It is noted that Hausdorff relies on the Axiom of Choice to complete his proof. The set N is formed by choosing exactly one point from each set P_x. This is possible only with the aid of this axiom. Consequently one cannot define the subsets into which the surface of the sphere is decomposed without the aid of the Axiom of Choice.

An alternative form of the Hausdorff Paradox which answers the question (1) in the introduction is the following: Disregarding a denumerable set, a half and a third of the surface of a sphere may be congruent to each other.

3.2 Subsequent work on the problem.

Robinson observed that the set Q in the Hausdorff decomposition could not be eliminated if only rotations around the center of the sphere were allowed to form congruences. However, he was able to reduce the denumerable set Q to a set consisting of only two points by certain modifications in the methods of rotation. [2]

Finally, Adams, in allowing reflections as well as rotations, presented a corollary to his decomposition theorem that gave the Hausdorff decomposition in which the set Q is entirely eliminated. He showed that the surface K of the sphere may be decomposed into three disjoint point sets, $K=A+B+C$ so that $A\simeq B\simeq C\simeq A+B\simeq B+C\simeq C+A$. [3]

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[1]Translated and condensed from Hausdorff, pp. 469-472.

[2]Robinson, pp. 258-260.

[3]Adams, p. 99.

CHAPTER IV

THE BANACH AND TARSKI PARADOX

4.1 The Work of Banach and Tarski.

In 1924 S. Banach and A. Tarski [3] proved the following two theorems, the first of which is known as the Banach and Tarski Paradox:

(1) In any Euclidean space of dimension $n \geq 3$, two arbitrary sets, bounded and containing interior points,[1] are equivalent by finite decomposition.

(2) In any Euclidean space of dimension $n \geq 1$, any two arbitrary sets (bounded or not), with interior points, are equivalent by denumerable decomposition.

The second result will not be dealt with in further detail.

An analogous theorem to (1) for sets lying on the surface of the sphere is proved by Banach and Tarski, but a corresponding theorem applied to one or two dimensions is false.

Two geometric theorems follow from the above results:

A. Two arbitrary polyhedra are equivalent by finite decomposition.

B. Two arbitrary polygons are equivalent by finite decomposition if, and only if, they have the same area.

A startling illustration of theorem (1) is given when two solid

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[1]i.e., there exists at least one point of the set such that a neighborhood of it is entirely contained in the set.

spheres O and E are considered, where O is the size of an orange and E is the size of the earth. Theoretically, according to this theorem, it is possible to decompose the set of points making up the sphere O into a finite number of disjoint subsets such that, by ordinary rigid motions (translations and rotations), these subsets may be made to fill out the entire sphere E. To state it in another way, an orange and the earth may both be divided into a finite number of disjoint parts so that every single part of one is congruent to a unique part of the other, and so that after each part of the orange has been matched with a part of the earth, there is no part of the earth left over. That is, a one-to-one correspondence is set up between the parts of one set and the parts of the other set. This means that, according to Banach and Tarski's results, the earth could be decomposed in such a way that it could be placed in one's hand.

Banach and Tarski prove three important lemmas in the process of arriving at their principal result. These lemmas and the principal theorem are stated here without proof.[1] In each case it is the solid sphere that is considered.

Lemma. The entire sphere S contains two disjoint subsets A_1 and A_2 such that $S \overset{=}{_f} A_1$ and $S \overset{=}{_f} A_2$.

Lemma. S_1 and S_2 being congruent spheres, we have $S_1 \overset{=}{_f} S_1 + S_2$.

Lemma. If the bounded set A, located in a Euclidean space of three dimensions, contains the sphere S, then $A \overset{=}{_f} S$.

Theorem. If two arbitrary sets A and B, lying in a Euclidean space of

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[1]The proofs as given by Banach and Tarski are included in Appendix II, pp. 35-37. See Lemmas 9, 10, 11 and Theorem 12. Additional important theorems are also included.

three dimensions, are bounded and contain interior points, then $A \overset{=}{_F} B$.

It is noted that Banach and Tarski have used an application of Bernstein's Equivalence Theorem and Hausdorff's Theorem, in arriving at their results.[1] As mentioned above, Hausdorff relied on the Axiom of Choice to prove the existence of the sets into which the surface of the sphere was decomposed. In a like manner, the Axiom of Choice must be used for the solid sphere problem. The subsets into which the solid sphere is decomposed cannot be defined otherwise. It is important to note in both Paradoxes where the words "can be decomposed" are used that there is no known rule that tells one how to actually perform the decomposition in question.

4.2 Subsequent work on the problem.

According to the theorem of Banach and Tarski it is possible to cut a solid sphere of fixed radius into a finite number of pieces, and then to reassemble these pieces to form two solid spheres with the given radius. Note that this is an affirmative answer to question (2) in the Introduction. In other words, there exist three decompositions of the sphere into disjoint pieces.

(1) $S=A_1+A_2+\ldots+A_{k+L}$

(2) $S=B_1+B_2+\ldots+B_k$

(3) $S=B_{k+1}+\ldots+B_{k+L}$

such that $A_i \simeq B_i$, $i=1, 2, \ldots, k+L$

Banach and Tarski did not mention the number of pieces required.

[1]See Appendix, Theorem 6, P. 35, and proof of Lemma 9, pp. 35-36, respectively.

In 1929 von Neumann stated without proof that nine pieces (k=4 and L=5) are a solution of the problem.[1] In 1945 Sierpinski [19] proved that eight pieces (k=3 and L=5, or k=2 and L=6) are a solution. In 1947, Robinson [15] gave a complete answer to the question by showing that five is the smallest possible number of pieces (k=2, L=3, and A_5 may be considered a single point). This answers the final question in the introduction.

Sierpinski also proved two interesting theorems:

Theorem. The solid sphere S contains a denumerably infinite number of parts, disjoint and congruent in pairs, of which each is $\underset{5}{=}$S.[2] (Here $\underset{5}{=}$ means equivalent by finite decomposition into five parts).

Theorem. The solid sphere S is a sum of a nondenumerable number of disjoint sets of which each is $\underset{9}{=}$S.[3]

Returning to the "pieces" problem, Robinson showed that the number could not be reduced below five even if reflections were allowed. Previous to his work reflections had not been used.

Robinson first obtained a result for a similar problem for the surface of the sphere. He showed that the surface K can be divided into two pieces, each of which can be divided into two pieces congruent to itself. This is possible on the basis of the main result of Robinson's paper which is as follows:

Theorem. It is possible to decompose the spherical surface K into n mutually disjoint, non-empty pieces A_1, A_2, ..., A_n satisfying a given (finite) system of congruences, each having the form

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[1]von Neumann, p. 77.

[2]Sierpinski, "Sur le Paradoxe de M. M. Banach et Tarski", p. 234.

[3]Sierpinski, "Sur le paradoxe de la sphere", p. 244.

$A_{k_1} + A_{k_2} + \ldots + A_{k_r} \simeq A_{L_1} + A_{L_2} + \ldots + A_{L_s}$ where $0 < r < n$, $0 < s < n$ and $1 \leq k_1 < k_2 < \ldots < k_r \leq n$, $1 \leq L_1 < L_2 < \ldots < L_s \leq n$ if, and only if, none of the given congruences and no congruence obtainable from them by taking complements (in K) or by using transitivity (to derive new congruences from the given system) asserts the congruence of two complementary portions of K. Furthermore, if the decomposition is possible, then each congruence may be effectuated by an independent rotation. (By a rotation is meant a rotation of the three-dimensional space which leaves the origin fixed.)[1]

It should be noted that the above theorem is obtained by specializing a more general decomposition theorem, the proof of which is based on the Axiom of Choice. The Axiom is needed to verify the existence of the rotations necessary to prove the general theorem.

Using the specialized theorem Robinson obtains a result for the surface problem in the following manner: The spherical surface K may be decomposed into four disjoint pieces,

$K = A_1 + A_2 + A_3 + A_4$ such that $A_1 \simeq A_2 \simeq A_1 + A_2$, $A_3 \simeq A_4 \simeq A_3 + A_4$.

A_1 and A_3 may be rotated in such a way as to exactly fit together to form K, and similarly for A_2 and A_4. Thus K may be cut into four pieces and they may be reassembled in pairs to form two copies of K. The minimum number of pieces is four since a copy of K cannot be formed out of a single piece which is not all of K.

From the surface problem Robinson turns to the solid sphere S. He produces a decomposition of S into six disjoint parts, $S = A_1 + A_2 + A_3 + A_4 + O + P$ where O is the center of the sphere and P is another single point. Using

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[1]Robinson, p. 252.

four independent rotations he reduces this to five disjoint pieces, A_1, A_2, A_3+O, A_4, and P, of which A_1 and A_3+O can be fitted to form one copy of S, and A_2, A_4, and P can be fitted to form a second copy. In all cases rotations about the origin are used except a translation is used to take P into O. Robinson concludes with a proof that five is the minimum number of pieces possible.[1]

Adams [1] modified the theorem of Robinson's above by allowing reflections as well as rotations. He stated and proved the following:

Theorem. Being given any set X of congruences such as

$A_{i_1} + A_{i_2} + \cdots + A_{i_r} \cong A_{j_1} + A_{j_2} + \cdots + A_{j_s}$, on the subsets A_k($k=1, 2, \ldots n$, and $0 < r < n$, $0 < s < n$), one may decompose K, the sphere surface, into disjoint subsets A_k satisfying the congruences X.

As stated previously a special case of this theorem allowed Adams to eliminate the denumerable set Q in Hausdorff's decomposition.

Dekker and de Groot [6] sharpened and extended Robinson's Decomposition Theorem. The three important extensions are the following:

(1) They eliminated the assumption of finiteness. Any cardinal $\leq C$, the cardinal of the set of real numbers, can be used.

(2) They reduced the necessary and sufficient conditions to the empty set, i.e., any system of congruences can be satisfied.

(3) If the number of pieces and the number of congruences are both $< C$, the pieces can be chosen so they are connected and locally connected. This shows that the pieces are really pieces and not some kind of scattered sets.

The theorem as stated by Dekker and de Groot is as follows:

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[1] Robinson, p. 257-258.

<u>Theorem</u>. The spherical surface K may be decomposed into α mutually disjoint, non-empty pieces-- α being any cardinal $\leq$ C--satisfying any given number $\beta \leq$ C of congruences between non-empty and non-complete (i.e., whose sum does not equal K)--but otherwise arbitrary, finite or infinite--sums of the pieces mentioned. Moreover, if α , $\beta <$ C, all pieces can be chosen in such a way that they are connected and locally connected.

The proof of this theorem is difficult and requires the use of the Axiom of Choice.[1]

The following statement regarding the solid sphere can be proved as the result of this theorem: The solid sphere S is for any infinite cardinal $\alpha \leq$ C the sum of α mutually disjoint sets, each of which is equivalent by finite decomposition to S.[2]

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[1] Dekker and de Groot, pp. 187-193.

[2] see Sierpinski [18].

CHAPTER V

CONTROVERSY REGARDING THE AXIOM OF CHOICE

In Chapter II the controversy regarding the Axiom of Choice was briefly mentioned. The issue will now be discussed at length.[1] Here again is the axiom: For every set Z whose elements are sets A, nonempty and mutually disjoint, there exists at least one set B having one and only one element from each of the sets A belonging to Z.

The intuitionists do not accept the axiom because for them existence and constructibility are equivalent terms. The Axiom of Choice does not say how the sought-after set can be constructed, but states only that it exists. For members of this school of thought there is no debate. The Axiom of Choice is rejected completely.

Excluding the intuitionists there are many mathematicians who are skeptical about the Axiom of Choice for various reasons and reject it under certain circumstances. For some, it is perhaps the name itself that has been at least part of the cause of rejection. As mentioned above, the axiom does not say anything about the possibility of choosing one element from each of the sets A. Zermelo, himself, stated in a letter, ". . . the name 'axiom of choice' concerns only the psychological method of presentation, while the axiom, as its wording, by the way, makes sufficiently clear, should be regarded as a pure axiom of

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[1]Much of this section is based on material found in Sierpinski [17], Fraenkel and Bar-Hillel [9], and Fraenkel's introduction to Bernays [4].

existence."[1]

Much of the controversy over the Axiom of Choice is due to the fact that it has been understood in different ways. Therefore it is important that we clarify the meaning of this axiom.

The main cause for rejection is the case where the set Z consists of an infinite sequence of sets $A_1, A_2, A_3, \ldots$ Even those who reject it in the general form do not question the axiom in the case of a finite number of sets. Here it is readily seen that the axiom is true by accepting the case when Z consists of a single set A and by induction. It is certainly true for the single set A because if the set A is not empty, then there must exist at least one element of the set A. Then by induction it is true for every finite set of sets. For example, assume that A_1 and A_2 are two non-empty, disjoint sets. There exist elements $e_1 \in A_1$ and $e_2 \in A_2$ and the set $\{e_1, e_2\}$ contains one and only one element from each of the sets A_1 and A_2.[2]

In the infinite case, however, the problem is much more difficult. The problem is to find a rule according to which to each of the sets there corresponds a certain element of the set. The following theorem verifies the existence of this correspondence but, of course, does not define it: Theorem: For every set there exists a correspondence such that to every non-empty subset of that set corresponds a certain element of that set.[3]

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[1] Sierpinski, Cardinal and Ordinal Numbers, p. 92.

[2] Ibid., pp. 92-93.

[3] Ibid., p. 94.

Some mathematicians distinguish between a denumerable and a non-denumerable number of choices. Some are more inclined to accept the denumerable situation while others believe the problem is equally difficult for both cases.

N. Lusin considered the use of the Axiom of Choice as very limited. For him a proof by means of the axiom was of value only to point out the uselessness of attempting to prove a given theorem false.[1]

Fraenkel points out that much of the opposition to the Axiom of Choice is due to its consequences, particularly the well-ordering theorem which states that every set can be well-ordered. Some mathematicians were unwilling to accept the well-ordering theorem and so were quick to deny the Axiom of Choice without much further investigation.[2]

Of course there were other results proved by means of the Axiom of Choice, particularly geometrical statements which, because of their paradoxical nature, aroused skepticism among many mathematicians. In fact, even those who are inclined to accept the Axiom of Choice, cannot help but be skeptical when it comes to these paradoxes. The two prime examples are the Hausdorff and Banach and Tarski Paradoxes studied at length in this paper.

Turning to a more favorable view toward the Axiom of Choice, it is noted that if the axiom is rejected, problems arise in analysis that are no less paradoxical than the Banach and Tarski Theorem.[3] In addition, there are numerous applications of the axiom in many branches of

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[1]Sierpinski, Cardinal and Ordinal Numbers, p. 95.

[2]Fraenkel and Bar-Hillel, pp. 75-76.

[3]Ibid., p.77.

mathematics, especially in set theory and topology, as well as in analysis.

A. Fraenkel points out that the Axiom of Choice has been obtained as other axioms by the logical analysis of known processes of reasoning. In this manner the axiom of parallels was developed, and yet, even though it has been proved independent, nobody suggests that it be rejected nor that the parts of Euclidean geometry that depend on it be renounced. Similarly, it is not justifiable to reject those branches of mathematics that depend on the Axiom of Choice.[1] He tempers these thoughts somewhat by writing that at the same time one should "examine what results can be obtained without the axiom and avoid it whenever possible. Hereby one learns to distinguish the domains of mathematics which are independent of the existential principles of choice and well-ordering from those where they are indispensable."[2]

The Axiom of Choice is valuable as a heuristic tool. It offers a means for discovering new theorems, the proofs of which can then be sought without the axiom.

As a brief summary of the evidence in favor of the Axiom of Choice, Sierpinski lists the following four points:

> (1) a large number of particular cases of this axiom are true (which has been proved independently of it); (2) from the axiom of choice a great many conclusions have been drawn of which none so far has led to a contradiction; (3) the axiom of choice simplifies considerably various parts of the Theory of Sets and of the Calculus and is indispensable for the proof of many important theorems of those theories.
>
> Finally . . . in 1938 K. Godel proved that the Axiom of Choice

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[1] Fraenkel and Bar-Hillel, p. 79.

[2] Ibid., pp. 79-80.

is consistent with other generally accepted axioms of the Theory of Sets, provided they are consistent with one another.[1]

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[1]Sierpinski, Cardinal and Ordinal Numbers, p. 88.

CHAPTER VI

APPLICATIONS OF THE AXIOM OF CHOICE

The applications of the axiom are most numerous in analysis, topology and set theory. In arithmetic the axiom is used in connection with the concept of finite set or number. In set theory the most important statements equivalent to the Axiom of Choice are the well-ordering theorem and the trichotomy theorem for cardinal numbers. (In general, the cardinal number of a set indicates the number of elements of the set.) The latter states that every two cardinal numbers can be joined by one of the three signs: $<, =, >$. The axiom is also equivalent to Zorn's theorem which states that in every closed family of sets there is at least one set not contained in any other set of that family.

Numerous theorems on cardinal numbers are equivalent to the Axiom of Choice, a few of which are now listed without proof.

Theorem. A non-finite cardinal number is not the sum of two cardinal numbers smaller than that number.

Theorem. Every non-finite cardinal number is a prime number (i.e., it is not the product of two cardinal numbers smaller than that number.)

Theorem. The difference $m-n$ exists for every cardinal number m and every cardinal number $n < m$.

Theorem. For any cardinal numbers m, n, m_1 and n_1 the inequalities $m < n$ and $m_1 < n_1$ imply the inequalities (1) $m+m_1 < n+n_1$ (2) $mm_1 < nn_1$.

Theorem. For every cardinal number m that is not finite the following formula holds: $m^2=m$.

Theorem. For cardinal numbers m and n the equality $m^2=n^2$ implies the equality $m=n$.

Theorem. For every cardinal number m there exists a cardinal number n such that (1) $m<n$ and (2) $m<p$ implies $n\leq p$.[1]

The following are some applications where in each case the theorem is proved with the aid of the Axiom of Choice. The proofs of the theorems are omitted.[2]

Theorem. If we decompose any set A into disjoint non-empty subsets, then the set of all those subsets is of power $\leq$ than the power of the set A.

Theorem. If we decompose a set of the power of the continuum into two subsets, then at least one of them will be of the power of the continuum.

Theorem. Every infinite set contains a denumerable subset.

Theorem. If the set A is non-denumerable and the set B is finite or denumerable, then $A-B$ is equivalent to A.

Theorem. The sum of an infinite series of disjoint non-empty sets, finite or denumerable, is a denumerable set.

Theorem. If we decompose a set of the power of the continuum into an infinite series of subsets, then at least one of them will be of the power of the continuum.

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[1]Sierpinski, Cardinal and Ordinal Numbers, pp. 414-424.

[2]For proofs, ibid., pp. 109-125.

CHAPTER VII

RESULTS

The specific questions listed in the Introduction have been answered as follows: With the aid of the Axiom of Choice (1) the surface of a sphere can be decomposed into subsets in such a way that a half and a third of the surface may be congruent to each other; (2) a solid sphere of fixed radius can be decomposed into a finite number of pieces and these pieces can be reassembled to form two solid spheres of the given radius; (3) the minimum number of pieces required in (2) is five. Since the proof of these paradoxes requires the Axiom of Choice the answer to these questions is final, providing the axiom is accepted.

As to the general question, "Should the Axiom of Choice be accepted or rejected", it would appear, unless one is an intuitionist, that the question is at present an unanswerable one. The problem of existence and the paradoxes that result from the axiom are major arguments against its use. However, the axiom simplifies many parts of set theory, analysis, and topology. The fact that Godel has proved the Axiom of Choice consistent with other generally accepted axioms of set theory, provided they are consistent with one another, is a second major point in its favor.

This consistency proof would mean that if any results from the Axiom of Choice and the other axioms caused a contradiction, the latter axioms are inconsistent. Of course, the possibility of the present axioms being replaced someday by other axioms that are not consistent with the Axiom of Choice cannot be ignored.

Perhaps a solution to two major problems, the question of independence of the Axiom of Choice and the Continuum Hypothesis, will shed further light on the subject. Meanwhile it would seem a proper attitude would be to accept and use the Axiom of Choice when there is no other apparent solution, but otherwise to avoid its use whenever possible. The ultimate role of the Axiom of Choice will be decided when the full character of modern analysis, topology and axiomatic set theory has been developed.

APPENDIX I

STATEMENTS EQUIVALENT TO THE AXIOM OF CHOICE

The following material comes from McShane and Botts [13]. The definitions are given first for clarification.

Definition. A relation $\succ$ in a given set P partially orders P, or is a partial order in P, if it is both (1) transitive: whenever x, y, z are in P and $x \succ y$ and $y \succ z$, then $x \succ z$; and (2) antisymmetric: whenever x and y are in P and $x \succ y$ and $y \succ x$, then $x=y$. A partially ordered set means a pair $(P, \succ)$ consisting of a set P and a partial order $\succ$ in P.

Definition. If $(P, \succ)$ is a partially ordered set, E is a subset of P, and z is an element of P, then z is termed an upper bound for E if for every x in E, $z \succeq x$.

Definition. An element m of a partially ordered set P is termed maximal in P if and only if there is no x in P for which $x \neq m$ and $x \succ m$.

Definition. A collection F of subsets of a given set S is said to be of finite character if and only if it satisfies the following requirement: for every subset T of S, $T \in F$ if and only if $T_1 \in F$ for every finite subset T_1 of T.

Definition. A chain in a partially ordered set P means a subset C of P such that whenever x and y are distinct elements of C, either $x \succ y$ or $y \succ x$.

Definition. A given partially ordered set $(W, \succ)$ is called well-ordered if and only if each non-empty subset A of W has a first element--i.e., an element a such that for every other element b of A, $a \prec b$.

Listed below are four of the most important statements that are equivalent to the Axiom of Choice:

Hausdorff Maximality Principle. For every partially ordered set P, the collection $\mathscr{C}$ of all chains in P has a maximal member, with respect to the partial ordering of $\mathscr{C}$ by the inclusion relation $\supset$.

Tukey Maximality Principle. If a given non-empty collection F of subsets of a given set S is of finite character, then F has a maximal member, with respect to the partial ordering of F by the inclusion relation $\supset$.

Zorn's Lemma. If in a given partially ordered set P every chain has an upper bound, then P has a maximal element.

Zermelo's Well-Ordering Theorem. Every set W can be well-ordered. That is, for every set W, there is a partial order $\succ$ for W, such that $(W, \succ)$ is a well-ordered set.[1]

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[1]McShane and Botts, pp. 27-30, 251-253.

APPENDIX II

IMPORTANT THEOREMS OF BANACH AND TARSKI

Certain theorems are involved directly in the proof of the main result of Banach and Tarski's paper (pp. 246-264). These theorems have been translated and are listed here. The first eight theorems and corollaries are listed without proof. Lemmas 9, 10, and 11 and Theorem 12 are given with proof since the lemmas are key statements and theorem 12 is the principal result.

Theorem 1. If $A=B$ or else $A \cong B$, then $A \underset{f}{=} B$

Theorem 2. If $A \underset{f}{=} B$ and $B \underset{f}{=} C$, then $A \underset{f}{=} C$

Theorem 3. If the sets A and B can be decomposed into some disjoint subsets:

$$A=\sum_{k=1}^{n} A_k \qquad B=\sum_{k=1}^{n} B_k$$

so that $A_k \underset{f}{=} B_k$ for $1 \leq k \leq n$

then $A \underset{f}{=} B$

Theorem 4. If $A \underset{f}{=} B$, there exists a function ϕ defined for all points of the set A and fulfilling the following conditions:

I. The function ϕ transforms A into B in a one-to-one manner.

II. C being an arbitrary subset of A, we have $C \underset{f}{=} \phi(C)$

Corollary 5. If $A \underset{f}{=} B$, to every subset C of A there corresponds a subset D of B subject to the following conditions:

I. $C \underset{f}{=} D$

II. If $C \neq A$, then $D \neq B$

Theorem 6. If $A_1 \subset A$, $B_1 \subset B$, $A \overset{=}{_f} B_1$ and $B \overset{=}{_f} A_1$, then $A \overset{=}{_f} B$

Corollary 7. If $A \supset B \supset C$ and $A \overset{=}{_f} C$, then $A \overset{=}{_f} B$ and $B \overset{=}{_f} C$

Theorem 8. If $A \overset{=}{_f} A+B_k$ for $1 \leq k \leq n$, then $A \overset{=}{_f} A + \sum_{k=1}^{n} B_k$

Lemmas 9, 10, 11 and Theorem 12 apply in particular to Euclidean space of three dimensions. In order to extend the results to space of $n > 3$ dimensions, it will merely be necessary to consider the sets of all the points $(x_1, x_2, \ldots x_n)$, of which the coordinates satisfy the following conditions: $(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2=d^2$, $b \leq x_k \leq c$ for $3 < k \leq n$, a_1, a_2, a_3, d, b and c being constants.

Lemma 9. The entire sphere (solid) S contains two disjoint subsets A_1 and A_2 such that $S \overset{=}{_f} A_1$ and $S \overset{=}{_f} A_2$.

Proof. From Hausdorff's theorem,[1] we can decompose the surface of the sphere S into four disjoint subsets: B', C', D' and E', of which E' is a denumerable set and the sets B', C' and D' verify the formulas: $B' \cong C'+D'$, $B' \cong C' \cong D'$.

Let p be the center of the sphere S. Indicate by B, C, D and E the sum-sets of all radii of the sphere S, the center p excluded, of which the exterior points belong respectively to B', C', D', and E'.

One clearly obtains in this fashion the decomposition of the sphere into five disjoint parts: (1) $S=B+C+D+E+(p)$ [2], subject to the conditions: (2) $B \cong C+D$, (3) $B \cong C \cong D$.

The following property, indicated by Hausdorff, will be used

[1]Hausdorff, p. 469.

[2]The symbol (p) designates the set composed of a single element p.

concerning the set E: (4) there exists a subset F of B+C+D such that $E \overset{=}{} F$ by a suitable rotation of the sphere S around one of its axes.

From (2) and (3) we easily obtain: $B \overset{=}{_f} B+C$, $B+C \overset{=}{_f} B+C+D$ whence from theorem 2 (5) $B \overset{=}{_f} B+C+D$.

We will set: (6) $A_1=B+E+(p)$, then formulas (1), (5) and (6) will give, following theorem 3: (7) $S \overset{=}{_f} A_1$.

On the other hand, from (3) and (5) there also results:

(8) $C \overset{=}{_f} B+C+D$ and (9) $D \overset{=}{_f} B+C+D$.

In applying Corollary 5, we deduce from (4) and (8) the existence of a set G, which verifies the formulas:

(10) $F \overset{=}{_f} G$, whence $E \overset{=}{_f} G$ and (11) $G \subset C$ and $G \neq C$.

Let, according to (11): (12) $q \in C-G$. Place: (13) $A_2=D+G+(q)$

The sets C and D being disjoint, one concludes from (11) and (12) that the sets D, G and (q) are disjoint also. Now, the sets (p) and (q) clearly being congruent, one concludes according to (1), (9), (10), and (13): (14) $S \overset{=}{_f} A_2$.

Finally one easily obtains:

(15) $A_1+A_2 \subset S$ and $A_1 \times A_2=0$.

Formulas (7), (14), and (15) prove that A_1 and A_2 are the desired sets.

<u>Lemma 10</u>. S_1 and S_2 being congruent spheres, we have $S_1 \overset{=}{_f} S_1+S_2$.

Proof. According to Lemma 9, let there be two sets A_1 and A_2 subject to the conditions: (1) $S_1 \overset{=}{_f} A_1$, $S_1 \overset{=}{_f} A_2$ and (2) $A_1+A_2 \subset S_1$ and $A_1 \times A_2=0$. From (1) and the hypothesis of the lemma we have $S_2 \overset{=}{_f} A_2$. Corollary 5 implies then the existence of a set B such that (3) $B \subset A_2$ and (4) $B \overset{=}{_f} S_2-S_1$.

Following theorem 3, we easily conclude from (1)-(4):

(5) $A_1+B \overset{=}{_f} S_1+(S_2-S_1)=S_1+S_2$ and (6) $A_1+B \subset S_1 \subset S_1+S_2$.

Formulas (5) and (6) give immediately, by virtue of corollary 7: $S_1 \overset{=}{\underset{f}{}} S_1+S_2$.

Lemma 11. If the bounded set A, located in a Euclidean space of three dimensions, contains the sphere S, then $A \overset{=}{\underset{f}{}} S$.

Proof. A being a bounded set, clearly we can decompose it into n subsets (not necessarily disjoint):

(1) $A=\sum_{k=1}^{n} B_k$, which fulfill the following condition:

(2) the entire set B_k, $1\leq k\leq n$, is contained in a sphere S_k congruent to S.

By virtue of Lemma 10, we have $S \overset{=}{\underset{f}{}} S+S_k$ for $1\leq k\leq n$, whence, according to Theorem 8, (3) $S \overset{=}{\underset{f}{}} S+\sum_{k=1}^{n} S_k$.

On the other hand, we obtain according to (1), (2) and the hypothesis of the lemma: (4) $S\subset A\subset S+\sum_{k=1}^{n} S_k$.

By reason of corollary 7, formulas (3) and (4) imply directly: $A \overset{=}{\underset{f}{}} S$.

We are now ready to establish the following theorem:

Theorem 12. If two arbitrary sets A and B, lying in a Euclidean space of three dimensions, are bounded and are not frontier-sets[1], then $A \overset{=}{\underset{f}{}} B$.

Proof. Let there be two spheres S_1 and S_2 contained in A and B respectively; one can certainly assume that (1) $S_1 \cong S_2$. By virtue of lemma 11 one obtains: (2) $A \overset{=}{\underset{f}{}} S_1$ and $B \overset{=}{\underset{f}{}} S_2$.

Following theorems 1 and 2, one concludes immediately from (1) and (2): $A \overset{=}{\underset{f}{}} B$.

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[1]The set (of points) A is said to be a frontier set if it does not contain any interior point.

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ABSTRACT

THE AXIOM OF CHOICE AND THE PARADOXES OF THE SPHERE

BY DAVID MILTON CARGILL

The Axiom of Choice is stated in the following form: For every set Z whose elements are sets A, non-empty and mutually disjoint, there exists at least one set B having one and only one element from each of the sets A belonging to Z. Examples are given to show the use of the Axiom of Choice and also to show when it is not needed.

Two other fundamental terms are defined, namely "congruence" and "equivalence by finite decomposition", and examples are given. Congruence is defined as follows: The sets of points A and B are congruent: A B, if there exists a function f, which transforms A into B in a one-to-one manner such that if a_1 and a_2 are two arbitrary points of the set A, then $d(a_1, a_2)=d\left[f(a_1), f(a_2)\right]$; d(a, b) is a real number called the distance between the points a and b. The following definition of equivalence by finite decomposition is given: Two sets of points, A and B are equivalent by finite decomposition, $A\underset{F}{=}B$, provided sets $A_1, A_2, \dots,$ A_n and $B_1, B_2, \dots, B_n$ exist with the following properties:

(1) $A=A_1+A_2+\dots+A_n$ $\qquad B=B_1+B_2+\dots+B_n$

(2) $A_j \cdot A_k=B_j \cdot B_k=0$ $\qquad 1\leq j<k\leq n$

(3) $A_j \cong B_j$ $\qquad 1\leq j\leq n$

An historic measure problem is discussed briefly.

Two paradoxes of the sphere, the Hausdorff Paradox and the Banach and

Tarski Paradox are stated and discussed in detail. The Hausdorff Paradox reads as follows: The surface K of the sphere can be decomposed into four disjoint subsets A, B, C, and Q such that (1) $K=A+B+C+Q$ and (2) $A \cong B \cong C$, $A \cong B+C$ where Q is denumerable. A refinement of this Paradox is introduced in which the denumerable set Q is eliminated.

The Banach and Tarski Paradox states that in any Euclidean space of dimension $n \geq 3$, two arbitrary sets, bounded and containing interior points, are equivalent by finite decomposition. Various refinements of this paradox are noted. It is observed that the proofs of both paradoxes require the aid of the Axiom of Choice.

The controversy over the Axiom of Choice is discussed at length. A wide range of viewpoints is studied, ranging from total rejection by the intuitionists to practically complete acceptance of the axiom.

Seven theorems on cardinal numbers that are equivalent to the Axiom of Choice are listed. Six examples of theorems which require the aid of the Axiom of Choice in their proof are given.

Based on the results of Hausdorff, Banach and Tarski, and Robinson, three specific questions are answered as follows: With the aid of the Axiom of Choice (1) the surface of a sphere can be decomposed into subsets in such a way that a half and a third of the surface may be congruent to each other. (2) A solid sphere of fixed radius can be decomposed into a finite number of pieces and these pieces can be reassembled to form two solid spheres of the given radius. (3) The minimum number of pieces required in the above problem is five.

It is concluded that the general question, "Should the Axiom of Choice be accepted or rejected" is unanswerable at the present time.

It is pointed out that the problem of existence and the paradoxes that result from the axiom are major arguments against its use. However, the axiom simplifies many parts of set theory, analysis, and topology. The fact that Godel has proved the Axiom of Choice consistent with other generally accepted axioms of set theory, provided they are consistent with one another, is a second major point in its favor.

Finally, Appendix I contains some statements equivalent to the Axiom of Choice, and Appendix II contains some important theorems of Banach and Tarski.