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The determination of refractive indices of dynamic gaseous media by a scanning grid (Dissertation)

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A RE-RADIATING ZONE PLATE ANTENNAE SYSTEM

By

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INTRODUCTION

The purpose of this paper is to investigate theoretically the change in the electric field of a short radiating antenna caused by the introduction into the original field of a special radiating antenna. This re-radiating antenna is analagous to the Fresnel zone plate used in physical optics inthat it consists of a series of concentric rings, electrically connected, each a half wave length further from a given point onthe axis of symmetry than the ring of next smaller radius. This is shown in Figure I.

We are ultimately concerned with a result which may show an improvement of signal strength in the electromagnetic propagation through an improvement in the electric and magnetic fields. This would of course be of interest in relaying radio signals.

In the past four years, the commercial progress of television alone would warrant investigating any possibility whose end is an improvement in signal radiation. The expansion of microwave relay circuits provides additional impetus for this investigation.

Since antenna problems are solved through the well known Maxwell equations, which often become unwieldy in application, certain simplifications must be made. These are largely in the geometric forms of the elements of the system to be studied. While these simplifications somewhat limit the generality of the problem; since this problem is solved theoretically, the result will still allow us to gain some insight into the advantages of the proposed arrangement.

While innumerable ingenious radiation configurations have been devised to give various signal strength distributions, excepting wave guides, most of these have been based on the premise that the elements are located within a limited space. This paper concerns a radiating antenna which is extended in space; that is one element of the system is located many wave lengths fromthe other. It is to be realized, of course, that any secondary re-radiating antenna would derive an amount of energy from the signal of the primary radiator. We hope to find a configuration which will change the field of the primary radiator in such a way that the <u>net</u> effect is an increase in field strength at certain points.

In the Review of Research it willbe pointed out that experiments have shown that as the wave length of the radiated signal decreases, properties usually thought of as being associated with light become evident and detectable in these high frequency radiations. It would seem reasonable that the re-radiator could be devised to take advantage of some of these properties. If we think of these optical properties and at the same time wish to re-distribute the energy in the field; some sort of interference method comes to mind. The Fresnel zone plate is used to produce interference in physical optics and, accordingly, we have chosen the zone plate to produce interference in the electromagnetic field.

Now to do this with any rigorousness at all, we could hardly hope to base our arguments on geometry alone as is oftendone in physical optics. It is essential to base our results on the Maxwell equations and introduce the re-radiating zone plate as a boundary condition. In the Review of Research, it will be shown that boundaries of various forms have been studied. On these precedents, and using conventional methods, this problem will also be studied.

It should be understood that this is essentially the type of problem first solved by Dirichlet; that is a primary radiator used in conjunction with a re-radiator in which a current is induced. We have further conformed with Dirichlet's method by assuming the system to be isolated in free space. Indeed, the problem of introducing the earth as a boundary is difficult in itself and will be cited later. Since the simplification of free space is assumed, we can hardly expect good agreement with experiment. Nevertheless, as was previously stated, while these limitations somewhat reduce the scope of the problem; a theoretical incite is still to be expected.

The first section of this paper gives some Maxwell relations and manipulates them into a form which is convenient to use for this antenna study. The field vectors of a short radiator on which an arbitrary signal is impressed are derived. The second section defines the geometry of the Fresnel zone plate and derives the currents induced in this re-radiator. The third section evaluates the composite electric field caused by a short radiator and zone plate together. Finally, some conclusions are given.

REVIEW OF RESEARCH

The methods used in this problem have been well developed in the solution of boundary value problems and diffraction problems of electromagnetic waves. The simplest, of course, is the well known radiation of a simple periodic current in a short antenna[1]. Watson[2] attempted to account for this radiation following the contour of the earth by studying the diffraction of electromagnetic waves. The surface wave he derived was never observed experimentally. This work is cited since diffraction is essentially interference. Epstein[3], Burrows[4], and Wise[5] carried on his work, assuming various electric properties for the earth. It is surprising that only recently Kahan and Eckhart[6] showed that the surface wave predicted earlier cannot really exist since it is an incompatible boundary condition. This will substantiate the earlier statement that introducing the earth into a radiation problem causes great difficulty in itself.

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[1]H. Skilling. Fundamentals of Electric Waves, Wiley and Sons, 1942, p. 134

[2]G. N. Watson. "The Diffraction of Electric Waves by the Earth", Proc. Royal Soc. A, 95, 1918, p. 83

[3]P. S. Epstein. "On the Bending of Electromagnetic Microwaves Below the Horizon", Proc. Nat. Acad. Sci., 21, Jan. 1935, p. 62

[4]C. R. Burrows. "Radio Propagation Over Spherical Earth", Proc. I'R.E., 23, ay 1935, p. 461

[5]W. H. Wise. "The Physical Reality of Zenneck's Surface Wave", Bell Sys. Tech. Jour., 16, Jan. 1937, p. 35

[6]T. Kahan and G. Eckhart. "Somerfelds Surface Wave. Final Solution of a Problem which has remained Undecided for a Long Time", C.R. Acad. Science, 226, ay 1948, p. 1513

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A considerable amount of work on Dirichlet's problem has been done. Keller and Keller[1] have solved the problem of a short radiating dipole on which a simple periodic signal has been impressed, located at the center of curvature of a conducting and grounded shell. Papas and King[2] slightly earlier had evaluated the currents induced in the shell of the previous arrangement but used a quarter wave length antenna. This increases the difficulty of the problem since this radiator does not have a uniform current distribution with respect to time or length. Methods used in these papers are used in part in the problem at hand. A boundary problem worked by Horton[3] determined the diffraction of a plane wave by a conducting sheet of semi-infinite dimensions. Ledinegg[4] has recently worked out the general boundary problem in general functions with discontinuities in the boundaries.

All of these are important since they provide specific methods of which we will make use, especially Ledinegg's general solution which in terms of the general functions used, outlines the steps to the solution.

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[1]H. B. Keller and J. B. Keller. "Reflection and Transmission of Electro-magnetic Waves by a Spherical Shell", J. Appl. Physics, 20, April 1949, p. 393

[2]C. H. Papas and R. King. "Surface Currents on a Conducting Sphere Excited by a Dipole", J. Appl. Physics, 19, Sept 1948, p. 808

[3]C. W. Horton. "On the Diffraction of a Plane Wave by a Semi-infinite Conducting Sheet", Phys. Rev., 75, April 1949, p. 1263

[4]E. Ledinegg. "Boundary and Discontinuity Problems of Maxwell Equations", S.B. Ost. Akad. Wiss. Art., II A, 156, 1948, p 417

[redacted]

All of these papers are of course devoted to the application of the Maxwell Equations to specific propagation problems. Bucholz[1] made an interesting summary of all the work along these lines, especially of boundary value problems in propagation, carried on in Germany during the war. Most of these were concerned with the boundaries of wave guides; another type of problem solved by the Maxwell Equations.

Coates[2] recently performed an interesting experiment. Assuming that microwaves will demonstrate the interference properties common to visible wave lengths and basing his equipment on the geometric considerations of physical optics; he constructed a grating spectrometer. His results show that his assumption was well taken since a spatial distribution of energy analagous to that obtained in optics was obtained.

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[1] H. Bucholz. "Electro-magnetic Waves in Wave Guides and Related Problems", FIAT Review Ger. Sci. (1939-1946) Off. Mil. Govt., Appl. Math., Pt. V, 1947, p 81

[2] R. J. Coates. "A Grating Spectrometer for Millimeter Waves", Rev. Aci. Instr., 19, Sept. 1948, p 586

I. The Short Radiator

Maxwell's equations are well known

$$\nabla \times H = \frac{4\pi J}{c} + \frac{1}{c}\frac{\partial D}{\partial t} \tag{1}$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{2}$$

$$\nabla \cdot D = 4\pi \rho \tag{3}$$

$$\nabla \cdot B = 0 \tag{4}$$

and are usually used in conjunction with

$$J = \sigma E \tag{5}$$

$$W = \frac{1}{4\pi}(E \cdot D + B \cdot H) \tag{6}$$

$$B = \mu H \tag{7}$$

$$D = \varepsilon E \tag{8}$$

It is with these relations that virtually all soluble problems in antenna radiation may be considered. In using them to consider potential distribution, it is common practice to use the splendid method provided by Dirichlet which enables reasonably formed re-radiators to be placed in the field of the source radiator and the net effect to be determined. Specifically, if the potential due to a source radiator is given by

$$\Phi_0 = \Phi_0 (r, \theta, \varphi, t) \tag{9}$$

and the re-radiator is a grounded series resonant conductor such that when t = 0, the potential due to the re-radiator is 0, the potential of the latter is

$$\Phi_1 = \Phi_1 (r, \theta, \phi, t) \tag{10}$$

Since this is a scalar quantity, the additive sum of the two potentials is the net potential at any point in space

$$\Phi = \Phi(r, \theta, \varphi, t) \qquad (11),$$

expressible in terms of the source and re-radiating antennas

$$\Phi = \Phi_0 - \Phi_1 \qquad (12).$$

How this is used in conjunction with the Maxwell equations is easily shown; for while the potential distribution is of interest, in radiation problems we are primarily concerned with the power and electric field strength attenuation in ~~free~~ space. Knowing, then,

$$H = \nabla \times A \qquad (13)$$

and

$$\nabla \times E = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad (14)$$

a substitution results in

$$\nabla \times E = -\frac{1}{c}\nabla \times \frac{\partial A}{\partial t} \qquad (15).$$

We are justified in introducing any scalar quantity of the nature

$$\nabla \times \nabla \Phi = 0 \qquad (16)$$

and finally may simplify (15) to

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla \Phi \qquad (17),$$

where we may say the scalar introduced is to be the potential at any point in the field. We also make use of an earlier relation

$$\nabla \cdot E = 4\pi e \qquad (18)$$

and the Lorentz condition

$$\nabla \cdot A = -\frac{1}{c}\frac{\partial \Phi}{\partial t} \qquad (19)$$

which in conjunction with our earlier relations gives

$$\rho = \frac{1}{4\pi}\left(\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2}\right) \qquad (20).$$

Similarly we can use

$$\nabla \times H = \frac{4\pi J}{c} + \frac{1}{c}\frac{\partial E}{\partial t} \qquad (21).$$

This latter may be transposed into

$$J = \frac{c}{4\pi}\left(\nabla \times H - \frac{1}{c}\frac{\partial E}{\partial t}\right) \qquad .(22)$$

and after substitution of

$$H = \nabla \times A \qquad (23)$$

simplifies directly into

$$J = \frac{c}{4\pi}\left[\nabla \times \nabla \times A + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla\left(\frac{1}{c}\frac{\partial \Phi}{\partial t}\right)\right] \qquad (24)$$

or finally

$$J = -\frac{c}{4\pi}\left(\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}\right) \qquad (25).$$

The similarity between (20) and (25) in form is obvious, and Kirchoff has provided a retarded solution for source problems of this nature.

The former, then, is soluble as

$$\Phi = \iiint \frac{\rho(t - \frac{r}{c})\, dv}{r} \qquad (26)$$

and the latter as

$$A = \frac{1}{c}\iiint \frac{i(t - \frac{r}{c})\, dv}{r} \qquad (27),$$

using only retarded potential

in which no reference is yet made to boundary conditions.
As previously stated, some means for considering power distribution is of interest, so that since

$$W = \frac{1}{4\pi}\left[E \cdot D + B \cdot H\right] \tag{6}$$

and we will be working in free space, then

$$D = \varepsilon E = E \tag{7}$$

$$B = \mu H = H \tag{8}$$

and finally the energy density at a point in space will be

$$W = \frac{1}{4\pi}\left[E^2 + H^2\right] \tag{28}$$

With the previous relations, we will hope to evaluate the E and H and thereby determine through the potential and Dirichlet's method the Electric field vector at any point and the power distribution.

Let us restate an earlier result as

$$A = \frac{1}{c}\iiint \frac{i\left(t - \frac{r}{c}\right)}{r}\,dv \tag{29}$$

and apply it to a short antenna. A three dimensional cartesian coordinate system is assumed with the radiator of length l, located so that l/2 is at the zero point and runs along the vertical axis, z. l is sufficiently small so that the radius rector[1] to any point distant from it is considered constant. A current of sinusoidal nature passes along the antenna in the z direction alone. We may now restate (29) as

$$A_z = \frac{1}{c}\int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{e^{inw\left(t - \frac{r}{c}\right)}}{r}\,dz \tag{30}$$

[1] Wherein reference is made to the magnitude of the vector only.

Performing the integration along the z axis is readily done and we get the result

$$A_z = \frac{I_0 \ell}{c\,r} e^{inw(t-\frac{r}{c})} + a \qquad (31).$$

Changing from a cartesian coordinate system to a set of spherical coordinates requires projecting (31) into the new coordinate axis and this results in

$$A_r = \frac{I_0 \ell}{c\,r} e^{inw(t-\frac{r}{c})} \cos\theta + b \qquad (32)$$

$$A_\theta = -\frac{I_0 \ell}{c\,r} e^{inw(t-\frac{r}{c})} \sin\theta + d' \qquad (33)$$

$$A_\phi = 0 \qquad (34)$$

and by performing the operation specified in (13)

$$\nabla \times A = H = \begin{vmatrix} \frac{1_r}{r^2 \sin\theta} & \frac{1_\theta}{r \sin\theta} & \frac{1_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \qquad (13),$$

we arrive at the magnetic field vector components,

$$H_r = 0 \qquad (35)$$

$$H_\theta = 0 \qquad (36)$$

$$H_\phi = \frac{I_0 \ell}{c\,r}\left(\frac{1}{r} + \frac{inw}{c}\right) e^{inw(t-\frac{r}{c})} + d' \qquad (37).$$

Now making use of (19) again

$$\nabla \cdot A = -\frac{1}{c}\frac{\partial \Phi}{\partial t} \qquad (19)$$

we have finally the relation

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla^2 \int A\,dt \qquad (38),$$

so we may evaluate all the quantities which would be of interest. The operation in (19) is performed in a relatively simple manner for the case in question and before the specification of the initial condition that when $t = 0$, the potential must be zero, we arrive at

$$\Phi = \frac{I_0 l \cos\theta}{c r}\left[\frac{-1}{c} + \frac{-1}{i n \omega}\right] e^{i n \omega (t - \frac{r}{c})} + \frac{bt}{r} + \frac{d' \operatorname{ctn}\theta}{r} t + k' \qquad (39).$$

Now to this point, our highly restricted case of a radiating antenna of short length was quite formal. Of course in practice, even were a short length antenna in free space used, the signal applied in the antenna would be of a far more complex form. Normally for radiation problems this is of no consequence and one spatial distribution is identical with all others in so far as the Fourier series analysis with respect to time is usually concerned, but since we are to be concerned with a non-homologous signal to be used in conjunction with a re-radiator primarily usable at only one wave length, we shall want to investigate the effect of frequency discrimination. Accordingly, the Fourier series in time of the applied signal will be of considerable importance and using

$$A = \frac{1}{c}\iiint \frac{i(t - \frac{r}{c})\, dv}{r} \qquad (27)$$

we must now specify i more in agreement with practice, so that

$$i = I_0 \sum_n A_n \sin n\omega (t - \tfrac{r}{c}) + B_n \cos n\omega (t - \tfrac{r}{c}) \qquad (40).$$

Assuming the same radiating attennule specified and reverting to the cartesian coordiante system, integration along the z axis is of the form

$$A_z = \frac{1}{c}\int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{I_0 \sum A_n \sin n\omega (t - \frac{r}{c}) + B_n \cos n\omega (t - \frac{r}{c})\, dz}{r} \qquad (41)$$

which after integration is

$$A_z = \frac{I_0 \ell}{c r} \sum A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right) + \alpha \quad (42).$$

Now tranforming to the spherical coordinate system

$$A_r = \frac{I_0 \ell}{c r} \sum \left[A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right] \cos\theta + \beta \quad (43)$$

$$A_\theta = -\frac{I_0 \ell}{c r} \sum \left[A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right] \sin\theta + \delta \quad (44)$$

$$A_\phi = 0 \quad (45).$$

Using (19)

$$\Phi = -c \int \nabla \cdot A \, dt \quad (19)$$

and since the operations are interchangeable, we get the divergence as

$$\nabla \cdot A = \frac{I_0 \ell}{c r} \sum \left[-\frac{A_n \omega n}{c} \cos n\omega\left(t-\frac{r}{c}\right) + \frac{B_n \omega n}{c} \sin n\omega\left(t-\frac{r}{c}\right)\right] \cos\theta$$
$$- \frac{I_0 \ell}{c r^2} \sum \left[A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right] \cos\theta$$
$$+ \frac{\beta}{r} + \frac{\delta \operatorname{ctn}\theta}{r} \quad (46).$$

The integration with respect to time is obtainable as

$$\Phi_0 = I_0 \ell \cos\theta \left\{ \left(\frac{1}{r^2 \omega}\right) \sum \left[\frac{B_n}{n} \sin n\omega\left(t-\frac{r}{c}\right) - \frac{A_n}{n} \cos n\omega\left(t-\frac{r}{c}\right)\right] \right.$$
$$\left. + \frac{1}{c r} \sum \left[A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right] \right\} \quad (47).$$
$$+ \frac{\beta t}{r} + \frac{\delta t \operatorname{ctn}\theta}{r} + k$$

It is this latter relation which we will use in conjunction with the potential of the re-radiating antenna to find the net distribution. We then proceed , after application of initial and boundary conditions in connection with the earlier relations to arrive at the necessary field quantities.

Referring back to

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla \Phi. \qquad (38)$$

and performing the operations indicated yields the components of the electric field.

$$E_r = \frac{2I_0 l}{cr^2}\sum\left[A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right]\cos\theta$$
$$+ \frac{2I_0 l}{r^3}\sum\left[\frac{B_n}{n\omega}\sin n\omega\left(t-\frac{r}{c}\right) - \frac{A_n}{n\omega}\cos n\omega\left(t-\frac{r}{c}\right)\right]\cos\theta \qquad (48)$$
$$+ \frac{\beta t}{r^2} + \frac{\delta t \operatorname{ctn}\theta}{r^2} + m$$

$$E_\theta = \frac{I_0 l}{cr^2}\sum\left[A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right]\sin\theta \qquad (49)$$
$$+ \frac{I_0 l}{\omega r^3}\sum\left[\frac{B_n}{n}\sin n\omega\left(t-\frac{r}{c}\right) - \frac{A_n}{n}\cos n\omega\left(t-\frac{r}{c}\right)\right]\sin\theta$$
$$+ \frac{I_0 l}{c^2 r}\sum\left[A_n \omega n\cos n\omega\left(t-\frac{r}{c}\right) - B_n \sin n\omega\left(t-\frac{r}{c}\right)\right]\sin\theta + \frac{\delta t \operatorname{ctn}\theta}{r} + l \qquad (50).$$

$$E_\phi = 0$$

Insisting on the initial condition that when $t = 0$, the potential is zero, permits a restatement of (47) as

$$0 = (47)_{t=0} \qquad (51).$$

From this the constant is evaluated as

$$k = \frac{I_0 l}{cr}\sum\left(A_n \sin\frac{n\omega r}{c} - B_n \cos\frac{n\omega r}{c}\right)\cos\theta \qquad (52)$$
$$+ \frac{I_0 l}{\omega r^2}\sum\left(\frac{B_n}{n}\sin\frac{n\omega r}{c} + \frac{A_n}{n}\cos\frac{n\omega r}{c}\right)\cos\theta$$

and the unique result is

$$\Phi_0 = \frac{I_0 l}{cr}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\cos\theta + \frac{I_0 l}{r^2}\sum\left(\frac{B_n}{n\omega}\sin n\omega\left(t-\frac{r}{c}\right)\right.$$
$$\left.- \frac{A_n}{n\omega}\cos n\omega\left(t-\frac{r}{c}\right)\right)\cos\theta + \frac{I_0 l}{cr}\sum\left(A_n \sin\frac{n\omega r}{c} - B_n \cos\frac{n\omega r}{c}\right)\cos\theta \qquad (53).$$
$$+ \frac{I_0 l}{\omega r^2}\sum\left(\frac{B_n}{n}\sin\frac{n\omega r}{c} + \frac{A_n}{n}\cos\frac{n\omega r}{c}\right)\cos\theta + \frac{\beta t}{r} + \frac{\delta t \operatorname{ctn}\theta}{r}$$

This is changed into the form

$$\Phi_0 = \frac{I_0 l}{cr}\sum\left[A_n\left(\sin n\omega\left(t-\frac{r}{c}\right) + \sin\frac{n\omega r}{c}\right) + B_n\left(\cos n\omega\left(t-\frac{r}{c}\right) - \cos\frac{n\omega r}{c}\right)\right]\cos\theta$$
$$+ \frac{I_0 l}{\omega r^2}\sum\left[\frac{B_n}{n}\left(\sin n\omega\left(t-\frac{r}{c}\right) + \sin\frac{n\omega r}{c}\right) - \frac{A_n}{n}\left(\cos n\omega\left(t-\frac{r}{c}\right)\right.\right. \qquad (54).$$
$$\left.\left.- \cos\frac{n\omega r}{c}\right)\right]\cos\theta + \frac{\beta t}{r} + \frac{\delta t \operatorname{ctn}\theta}{r}.$$

Applying the same initial condition to the electric field components results in E_r being, when $t = 0$

$$0 = \frac{2I_0\ell}{cr^2}\left(\sum\left[-A_n \sin\frac{n\omega r}{c} + B_n\cos\frac{n\omega r}{c}\right]\right)\cos\theta \qquad (55)$$

$$+\frac{2I_0\ell}{r^3}\left(\sum\left[-\frac{B_n}{n\omega}\sin\frac{n\omega r}{c} - \frac{A_n}{\omega n}\cos\frac{n\omega r}{c}\right]\right)\cos\theta + m$$

so that

$$m = \frac{2I_0\ell}{r^3}\left(\sum\left[\frac{B_n}{n\omega}\sin\frac{n\omega r}{c} + \frac{A_n}{\omega n}\cos\frac{n\omega r}{c}\right]\right)\cos\theta \qquad (56)$$

$$+\frac{2I_0\ell}{cr^2}\left(\sum\left[A_n\sin\frac{n\omega r}{c} - B_n\cos\frac{n\omega r}{c}\right]\right)\cos\theta$$

and the unique result for E_r is

$$E_r = \frac{2I_0\ell}{r^3}\sum\left[\frac{B_n}{n\omega}\left(\sin n\omega(t-\tfrac{r}{c}) + \sin\frac{n\omega r}{c}\right) - \frac{A_n}{\omega n}\left(\cos n\omega(t-\tfrac{r}{c}) - \cos\frac{n\omega r}{c}\right)\right]\cos\theta \qquad (57).$$

$$+\frac{2I_0\ell}{cr^2}\sum\left[A_n\left(\sin n\omega(t-\tfrac{r}{c}) + \sin\frac{n\omega r}{c}\right) + B_n\left(\cos n\omega(t-\tfrac{r}{c}) - \cos\frac{n\omega r}{c}\right)\right]\cos\theta + \frac{\beta t}{r^2} + \frac{\delta t\ \mathrm{ctn}\,\theta}{r^2}$$

Repeating the initial condition for E_θ

$$0 = \frac{I_0\ell}{cr^2}\left(\sum\left[-A_n\sin\frac{n\omega r}{c} + B_n\cos\frac{n\omega r}{c}\right]\right)\sin\theta \qquad (58)$$

$$+\frac{I_0\ell}{\omega r^3}\left(\sum\left[-\frac{B_n}{n}\sin\frac{n\omega r}{c} - \frac{A_n}{n}\cos\frac{n\omega r}{c}\right]\right)\sin\theta + \ell$$

results in

$$E_\theta = \frac{I_0\ell}{cr^2}\left(\sum\left[A_n\left(\sin n\omega(t-\tfrac{r}{c}) + \sin\frac{n\omega r}{c}\right) + B_n\left(\cos n\omega(t-\tfrac{r}{c}) - \cos\frac{n\omega r}{c}\right)\right]\sin\theta \qquad (59)$$

$$+\frac{I_0\ell}{\omega r^3}\left(\sum\left[\frac{B_n}{n}\left(\sin n\omega(t-\tfrac{r}{c}) + \sin\frac{n\omega r}{c}\right) - \frac{A_n}{n}\left(\cos n\omega(t-\tfrac{r}{c}) - \cos\frac{n\omega r}{c}\right)\right]\sin\theta + \frac{\delta t\ \mathrm{ctn}\,\theta}{r}\right.$$

So that finally we have only the last component E_ϕ which is zero from symmetry.

What we have considered up to this point is the small radiating antenna; that is an antenna of short length in comparison with the wave length of the impressed signal. This means that we may consider the distance from any point on the antenna to a given point in space to be the same. This is quite a common practice in antenna theory and allows integrations with respect to coordinates to be made more easily. Referring to equation (41) it is evident that if the r in the denominator were not considered constant, the integrationwould not proceed so readily.

Now this is the first of the geometric limitations imposed as was mentioned in the Introduction. It does limit the generality of the problem somewhat; but we can still expect the result to be of some use. That is, since we are really hoping to compare the field of a short radiator with the field due to the antenna system proposed, it is only the difference between the two fields which is to be determined. Whether the field of the "improved" system proposed is more or less advantageous for applications suggested in the Introduction remains to be seen.

II. The equation for the Fresnel zone plate is given by

$$b^2 + r_n^2 = \left(b + \frac{n'\lambda'}{2}\right)^2 \quad \text{where } \lambda' \text{ is } n\lambda \text{ of radiated signal} \tag{60}$$

This is shown in Figure I in the appendix.

The zone plate, as was mentioned in the Introduction, is a series of concentric rings. In optics the transparent rings alternate with non-transparent rings and each ring is a half wave length further from a point on the axis of symmetry than the ring of next smaller radius. These radii are given by r_n in the relation above. b designates the distance to the point in space on the axis and λ' is the wave length of the light being used with the zone plate. That is, for each different wave length of light used, a new zone plate must be constructed and the dimensions of the rings are given by (60).

We have two critical problems confronting us in adapting this to the mathematics of the electromagnetic theory. First, we must of course construct the theoretical zone plate with properties which will allow an integration without too many mathematical difficulties. Since our short radiator is most conveniently worked mathematically in spherical coordinates, the rings will be considered annular sections of the surface of a sphere. The geometry of equation (60) will then have to be changed to conform with this new geometric form. Second, we must consider that in electromagnetic theory and practice the monochromatic source is often not used. To keep generality in the problem, it is desirable to assume the source to be radiating a signal composed of components of many frequencies. For this purpose, instead of using a simple periodic function, a Fourier series is used as the form of the current in the antennule.

In principle, we can then arrive

at a result which can be used irrespective of the wave form being used to "drive" the antenna.

In keeping with practice, and to keep the mathematics from becoming unwieldy, a few further assumptions are made. The electromagnetic analogue of the zone plate is considered to be made of a metal which has no resistance. This is a reasonable assumption since there are metals of negligible resistivity. This assumption is often made. We must further agree that the reactance can be made zero for the reradiating zone plate. This can be done in practice by adjusting the electric elements in the "ground lead", but at any rate it is valid in theory.

We are now in a position to make some statements about the currents induced in theanalogue zone plate by the radiated signal of the source antennule.

Further back, we had

$$i = -\frac{c}{4\pi}\left(\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}\right) \qquad (24).$$

Use will also be made of the components of the vector potential of the radiator

$$A_r = \frac{I_0 l}{cr}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\cos\theta + \beta \qquad (43)$$

$$A_\theta = -\frac{I_0 l}{cr}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right) + B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta + \gamma \qquad (44)$$

$$A_\phi = 0 \qquad (45)$$

It will now be necessary to evaluate the separate components of the current density induced in the zone plate using (24).

These are

$$\frac{\partial^2 A_r}{\partial t^2} = \frac{I_0 l}{cr}\sum\left(-A_n\omega^2 n^2 \sin n\omega\left(t-\frac{r}{c}\right) - B_n\omega^2 n^2 \cos n\omega\left(t-\frac{r}{c}\right)\right)\cos\theta \qquad (61)$$

$$\frac{\partial^2 A}{\partial t^2}\theta = -\frac{I_o l}{cr}\sum\left(-A_n \omega^2 n^2 \sin n\omega(t-\tfrac{r}{c}) - B_n \omega^2 n^2 \cos n\omega(t-\tfrac{r}{c})\right)\cos\theta \quad (62).$$

$$\nabla = \left(\frac{\partial}{\partial r}\right)_r + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)_\phi + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)_\theta \quad (63).$$

The scalar Del square product (a scalar) which is to operate on the vector quantity of the vector potential is given by

$$\nabla^2 = \left(\frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) + \left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta} + \frac{ctn\,\theta}{r^2}\frac{\partial}{\partial\theta}\Big) \quad (64)$$

so that we may state the final result of the various component operations as

$$\frac{\partial^2 A_r}{\partial\phi^2} = 0 \quad (65)$$

$$\frac{\partial^2 A_\theta}{\partial\varphi^2} = 0 \quad (66)$$

$$\frac{\partial^2 A_\phi}{\partial\phi^2} = 0 \quad (67)$$

$$\frac{\partial A_r}{\partial r} = -\frac{I_o l}{cr^2}\sum\left(A_n \sin n\omega(t-\tfrac{r}{c}) + B_n\cos n\omega(t-\tfrac{r}{c})\right)\cos\theta \quad (68)$$
$$+\frac{I_o l}{cr}\sum\left(-\frac{A_n\omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n\omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\cos\theta$$

$$\frac{\partial A_\theta}{\partial r} = \frac{I_o l}{cr^2}\sum\left(A_n\sin n\omega(t-\tfrac{r}{c}) + B_n\cos n\omega(t-\tfrac{r}{c})\right)\sin\theta \quad (69)$$
$$-\frac{I_o l}{cr}\sum\left(-\frac{\omega n A_n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{\omega n B_n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\sin\theta$$

$$\frac{\partial^2 A_r}{\partial\theta^2} = \frac{-I_o l}{cr}\sum\left(A_n\sin n\omega(t-\tfrac{r}{c}) + B_n\cos n\omega(t-\tfrac{r}{c})\right)\cos\theta \quad (70)$$

$$\frac{\partial^2 A_\theta}{\partial\theta^2} = \frac{I_o l}{cr}\sum\left(A_n\sin n\omega(t-\tfrac{r}{c}) + B_n\cos n\omega(t-\tfrac{r}{c})\right)\sin\theta \quad (71)$$

$$\frac{\partial^2 A_r}{\partial r^2} = \frac{2I_o l}{cr^3}\sum\left(A_n\sin n\omega(t-\tfrac{r}{c}) + B_n\cos n\omega(t-\tfrac{r}{c})\right)\cos\theta$$
$$-\frac{I_o l}{cr^2}\sum\left(-\frac{A_n\omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n n\omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\cos\theta \quad (72)$$
$$-\frac{I_o l}{cr^2}\sum\left(-\frac{A_n\omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n\omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\cos\theta$$
$$+\frac{I_o l}{cr}\sum\left(-\frac{A_n\omega^2 n^2}{c^2}\sin n\omega(t-\tfrac{r}{c}) + \frac{B_n\omega^2 n^2}{c^2}\cos n\omega(t-\tfrac{r}{c})\right)\cos\theta$$

$$\frac{\partial^2 A_\theta}{\partial r^2} = \frac{-2I_o l}{cr^3}\sum\left(A_n \sin n\omega(t-\tfrac{r}{c}) + B_n \cos n\omega(t-\tfrac{r}{c})\right)\sin\theta$$
$$+ \frac{I_o l}{cr^2}\sum\left(-\frac{A_n \omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n \omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\sin\theta$$
$$+ \frac{I_o l}{cr^2}\sum\left(-\frac{\omega n A_n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{\omega n B_n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\sin\theta$$
$$- \frac{I_o l}{cr}\sum\left(-\frac{\omega^2 n^2}{c^2}A_n \sin n\omega(t-\tfrac{r}{c}) - \frac{\omega^2 n^2}{c^2}B_n \cos n\omega(t-\tfrac{r}{c})\right)\sin\theta \quad (73)$$

Combining all of these components into a final expression to give the vector quantity expressing the current density induced in the re-radiator positioned in the radiating field, we arrive at the somewhat formidable and lengthy expression

$$\bar{i} = \frac{-c}{4\pi}\Big[-\frac{2I_o l}{cr^3}\sum\left(A_n \sin n\omega(t-\tfrac{r}{c}) + B_n \cos n\omega(t-\tfrac{r}{c})\right)\cos\theta_{(r)} \quad (74)$$
$$+ \frac{2I_o l}{cr^2}\sum\left(-\frac{A_n \omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n \omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\cos\theta_{(r)}$$
$$+ \frac{2I_o l}{cr^3}\sum\left(A_n \sin n\omega(t-\tfrac{r}{c}) + B_n \cos n\omega(t-\tfrac{r}{c})\right)\sin\theta_{(\theta)}$$
$$- \frac{2I_o l}{cr^2}\sum\left(-\frac{\omega n A_n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{\omega n B_n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\sin\theta_{(\theta)}$$
$$+ \frac{2I_o l}{cr^3}\sum\left(A_n \sin n\omega(t-\tfrac{r}{c}) + B_n \cos n\omega(t-\tfrac{r}{c})\right)\cos\theta_{(r)}$$
$$- \frac{I_o l}{cr^2}\sum\left(-\frac{A_n \omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n \omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\cos\theta_{(r)}$$
$$- \frac{I_o l}{cr^2}\sum\left(-\frac{A_n \omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n \omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\cos\theta_{(r)}$$
$$+ \frac{I_o l}{cr}\sum\left(-\frac{A_n \omega^2 n^2}{c^2}\sin n\omega(t-\tfrac{r}{c}) + \frac{B_n \omega^2 n^2}{c^2}\cos n\omega(t-\tfrac{r}{c})\right)\cos\theta_{(r)}$$
$$- \frac{2I_o l}{cr^3}\sum\left(A_n \sin n\omega(t-\tfrac{r}{c}) + B_n \cos n\omega(t-\tfrac{r}{c})\right)\sin\theta_{(\theta)}$$
$$+ \frac{I_o l}{cr^2}\sum\left(-\frac{A_n \omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n \omega n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\sin\theta_{(\theta)}$$
$$+ \frac{I_o l}{cr^2}\sum\left(-\frac{\omega n A_n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{\omega n B_n}{c}\sin n\omega(t-\tfrac{r}{c})\right)\sin\theta_{(\theta)}$$
$$+ \frac{I_o l}{cr}\sum\left(\frac{\omega^2 n^2}{c^2}A_n \sin n\omega(t-\tfrac{r}{c}) + \frac{\omega^2 n^2}{c^2}B_n \cos n\omega(t-\tfrac{r}{c})\right)\sin\theta_{(\theta)}$$

(over)

(continued)

$$- \frac{I_o l}{c r^3} \sum \left(A_n \sin n\omega (t - \tfrac{r}{c}) + B_n \cos n\omega (t - \tfrac{r}{c}) \right) \cos \theta_{(\theta)}$$

$$+ \frac{I_o l}{c r^3} \sum \left(A_n \sin n\omega (t - \tfrac{r}{c}) + B_n \cos n\omega (t - \tfrac{r}{c}) \right) \sin \theta_{(r)}$$

$$- \frac{I_o l}{c r^3} \sum A_n \sin n\omega (t - \tfrac{r}{c}) + B_n \cos n\omega (t - \tfrac{r}{c}) \Big) \cos \theta_{(r)}$$

$$- \frac{I_o l}{c r^3} \sum \left(A_n \sin n\omega (t - \tfrac{r}{c}) + B_n \cos n\omega (t - \tfrac{r}{c}) \right) \cos \theta \operatorname{ctn} \theta_{(\theta)}$$

$$- \frac{I_o l}{c^3 r} \sum \left(-A_n \omega^2 n^2 \sin n\omega (t - \tfrac{r}{c}) - B_n \omega^2 n^2 \cos n\omega (t - \tfrac{r}{c}) \right) \cos \theta_{(r)}$$

$$+ \frac{I_o l}{c^3 r} \sum \left(-A_n \omega^2 n^2 \sin n\omega (t - \tfrac{r}{c}) - B_n \omega^2 n^2 \cos n\omega (t - \tfrac{r}{c}) \right) \cos \theta_{(\theta)} \Big]$$

Now certain simplifications are permissible. We prescribe the re-radiator to be infinitely thin so that no current is permitted along the radial component, thereby, eliminating this component from the whole expression. A symmetry in phi exists from the outset by virtue of the short dipole used and so we may similarly disregard this component and presume that the component in theta alone will have the necessary spatical variation in which we are concerned. This results in the slightly simplified form

$$
\begin{aligned}
i = -\frac{I_0 l}{4\pi}\Big[& \frac{2}{\rho^3}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right)+B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \\
& -\frac{2}{\rho^2}\sum\left(\frac{-\omega n A_n}{c}\cos n\omega\left(t-\frac{r}{c}\right)+\frac{\omega n B_n}{c}\sin n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \\
& -\frac{2}{\rho^3}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right)+B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \qquad (75) \\
& +\frac{1}{\rho^2}\sum\left(\frac{A_n\omega n}{c}\cos n\omega\left(t-\frac{r}{c}\right)+\frac{B_n\omega n}{c}\sin n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \\
& +\frac{1}{\rho^2}\sum\left(\frac{-\omega n A_n}{c}\cos n\omega\left(t-\frac{r}{c}\right)+\frac{\omega n B_n}{c}\sin n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \\
& +\frac{1}{\rho}\sum\left(\frac{\omega^2 n^2}{c n}A_n \sin n\omega\left(t-\frac{r}{c}\right)+\frac{\omega^2 n^2}{c n}B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \\
& -\frac{1}{\rho^3}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right)+B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\cos\theta \\
& -\frac{1}{\rho^3}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right)+B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\cos\theta \operatorname{ctn}\theta \\
& +\frac{1}{\rho}\sum\left(-A_n \omega^2 n^2 \sin n\omega\left(t-\frac{r}{c}\right)-B_n \omega^2 n^2 \cos n\omega\left(t-\frac{r}{c}\right)\cos\theta\right)\Big]
\end{aligned}
$$

which can be algebraically manipulated into

$$
\begin{aligned}
i_\theta = -\frac{I_0 l}{4\pi}\Big[& \frac{1}{\rho^2}\sum\left(\frac{+\omega n A_n}{c}\cos n\omega\left(t-\frac{r}{c}\right)-\frac{\omega n}{c}B_n \sin n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \\
& +\frac{1}{\rho}\sum\left(\frac{\omega^2 n^2}{c\rho}A_n \sin n\omega\left(t-\frac{r}{c}\right)+\frac{\omega^2 n^2}{c\rho}B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \\
& +\frac{1}{\rho^2}\sum\left(\frac{-A_n\omega n}{c}\cos n\omega\left(t-\frac{r}{c}\right)+\frac{B_n\omega n}{c}\sin n\omega\left(t-\frac{r}{c}\right)\right)\sin\theta \qquad (76) \\
& -\frac{1}{\rho^3}\sum\left(A_n \sin n\omega\left(t-\frac{r}{c}\right)+B_n \cos n\omega\left(t-\frac{r}{c}\right)\right)\left(\cos\theta \operatorname{ctn}\theta+\cos\theta-2\sin\theta\right) \\
& +\frac{1}{c^2\rho}\sum\left(-A_n \omega^2 n^2 \sin n\omega\left(t-\frac{r}{c}\right)-B_n \omega^2 n^2 \cos n\omega\left(t-\frac{r}{c}\right)\cos\theta\right)\Big]
\end{aligned}
$$

It is now necessary to investigate the fields set up by the zone plate by virtue of the current induced in it. In keeping with Dirichlet's method, the components of these fields are then added to the fields originated by the source antennule to get the resultant distribution in space of the fields of the system as a whole.

We must now study the geometry of the radiating zone plate. Making use of Figure II, we can see that

$$\delta\theta = \frac{\text{surface length}}{\rho} \tag{77}$$

and that the area of each radiating zone is proportional to n

$$n \cong \frac{n'\lambda'}{2} - \frac{(n'-1)'\lambda'}{2} = \frac{\lambda'}{2} \tag{78}.$$

As was mentioned, the geometry of the Fresnel Zone plate in optics is not of much use here. The optical zone plate is flat and this analogue has been given a spherical surface. Referring again to Figure II, we could state from the law of Cosines for triangles

$$s = \left[\left(b + \frac{n'\lambda'}{2}\right)^2 + \left(b + \frac{(n-1)'\lambda'}{2}\right)^2 - 2\left(b+\lambda' n'\right)\left(b + \frac{(n-1)'\lambda'}{2}\right)(1)\right]^{1/2} \tag{79}$$

For this to be true, s must be a straight segment. Now since we are studying the radiation field, which by definition is the field at great distances from a radiator; b is large.

Since s is necessarily small in comparison with b, the difference in b between s considered curved or flat is negligible, and the equation used (79) is a reasonably valid assumption. Furthermore, considering s curved would result in a geometric relation which is considerably more complex and of course makes any result that much less appealing to the "physical sense". Combining (78) and (79) gives

$$\delta\theta = \frac{(79)}{\rho} \tag{80}$$

Now the general expression for the vector potential at a point b distance along the axis, measured from the re-radiator is given by

$$A = \frac{1}{c}\iiint \frac{i(t - r/c)}{b}\, dV \tag{81}$$

or specifically in the spherical coordinates which are being used

$$A = \frac{1}{c}\iint \frac{i(t - r/c)}{b}\, \rho^2\, d\phi\, d\theta\, \sin\theta \tag{82}.$$

The volume element in this expression is reduced to a double integral by the assumption that the radial thickness of the zone plate is negligible. We are going to integrate over the limits of each segment whose magnitude is given by(79) and sum the individual components (in this case theta alone) over the total number of such zones in the zone plate. This is

$$A = \sum_{2h'-1} \int_0^{\pi}\int_0^{(79)} \frac{i(t - r/c)}{b}\, \rho^2\, d\theta\, d\phi\, \sin\theta \tag{83}$$

in which we may algebraically reduce the upper limit and get a form which is more useful

$$A = \sum_{2h'-1} \int_0^{\pi}\int_0^{[(h-1)'\lambda']^2 + h'\lambda'^2} \frac{i(t - r/c)}{b}\, \rho^2\, d\theta\, d\phi\, \sin\theta \tag{84}.$$

To perform this integration, which is quite a difficult operation, b may be considered to be a constant. This is reasonable since the re-radiator is of small dimensions compared with the distance b from this re-radiator to the point in space being studied. Furthermore, since b was taken as being large further back, there will be very little change in the magnitude of b measured from this point to the outermost ring of the plate. This means simply that the values of b measured to the point from the extremities of the surface do not vary appreciably from the distance along the axis to these points. Accordingly, without introduction of much error and in a manner used for the short dipole, we will ignore the radial dependence and call this distance constant substantially simplifying the integration. We get after integration over symmetric phi,

$$A_\theta = \sum_{2n'-1} \int_0^{[(n-1)'\lambda']^2+n'\lambda'^2} -\frac{I_0 l}{b}\Big[\sum\Big(\frac{\omega n A_n}{c}\cos n\omega(t-\tfrac{r}{c}) - \frac{\omega n}{c}B_n \sin n\omega(t-\tfrac{r}{c})\Big)\sin\theta$$
$$+\sum\Big(\frac{\omega^2 n^2}{c}A_n \sin n\omega(t-\tfrac{r}{c}) + \frac{\omega^2 n^2}{c}B_n\cos n\omega(t-\tfrac{r}{c})\Big)\sin\theta \qquad (85).$$
$$-\frac{1}{\rho}\sum\Big(A_n \sin n\omega(t-\tfrac{r}{c}) + B_n \cos n\omega(t-\tfrac{r}{c})\Big)(\cos\theta \operatorname{ctn}\theta + \cos\theta - 2\theta\sin\theta)$$
$$+\frac{\rho}{c^2}\sum\Big(-A_n\omega^2 n^2 \sin n\omega(t-\tfrac{r}{c}) - B_n\omega^2 n^2\cos n\omega(t-\tfrac{r}{c})\Big)\cos\theta$$
$$+\sum\Big(-\frac{A_n\omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n\omega n}{c}\sin n\omega(t-\tfrac{r}{c})\Big) + c_\phi\Big]\sin\theta\, d\theta$$

The individual integrations to be performed are quite simple and are set forth as

$$\int_0^{\{[(n-1)'\lambda']^2+n'\lambda'^2\}57.3} \sin^2\theta\, d\theta = \Big[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin^2\theta\Big]_0^{\{[\ \ ]+\ \ \}57.3} \qquad (86)$$

$$\int_0^{\{[(n-1)'\lambda']^2+n'\lambda'^2\}57.3} \sin\theta\cos\theta\, d\theta = \Big[\tfrac{1}{2}\sin^2\theta\Big]^{\{[\ \ ]+\}57.3} \qquad (87)$$

$$\int_0^{\{[(n-1)'\lambda']^2+n'\lambda'^2\}57.3} \cos^2\theta\, d\theta = \Big[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin^2\theta\Big]_0^{\{[\ \ ]+\ \ \}57.3} \qquad (88)$$

$$\int_0^{\{[(n-1)'\lambda']^2+n'\lambda'^2\}57.3} \sin\theta\, d\theta = \Big[-\cos\theta\Big]_0^{\{[\ \ ]+\ \ \}57.3} \qquad (89)$$

in which the limits for our problem are included.

Finally, we have the theta component of the vector potential as

$$A_\theta = \sum_{2n'-1} -\frac{I_0 l}{4b}\Big\{\Big(\sum_n \frac{\omega n A_n}{c}\cos n\omega(t-\tfrac{r}{c}) - \frac{\omega n}{c}B_n\sin n\omega(t-\tfrac{r}{c})\Big)$$
$$\Big(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\Big)_0^{\{[\ \ ]+\}57.3} + \sum\Big(-\frac{A_n\omega n}{c}\cos n\omega(t-\tfrac{r}{c}) + \frac{B_n\omega n}{c}\sin n\omega(t-\tfrac{r}{c})\Big) \qquad (90).$$
$$(-\cos\theta)_0^{\{[\ \ ]+\}57.3} + \sum\frac{\omega^2 n^2}{c}A_n\sin n\omega(t-\tfrac{r}{c}) + \frac{\omega^2 n^2}{c}B_n\cos n\omega(t-\tfrac{r}{c})\Big)$$
$$\Big[\frac{\theta}{2} - \frac{\sin^2\theta}{4}\Big]_0^{\{[\ \ ]+\ \ \}57.3} - \frac{1}{\rho}\sum\Big(A_n\sin n\omega(t-\tfrac{r}{c}) + B_n\cos n\omega(t-\tfrac{r}{c})\Big)$$
$$\Big[\frac{\theta}{2} + \frac{\sin^2\theta}{4} + \frac{\sin^2\theta}{2} - \theta + \frac{\sin^2\theta}{2}\Big]_0^{\{[\ \ ]+\}57.3} + \frac{\rho}{c^2}\sum\Big(-A_n\omega^2 n^2\sin n\omega(t-\tfrac{r}{c})$$
$$-B_n\omega^2 n^2\cos n\omega(t-\tfrac{r}{c})\Big)\Big[\frac{\sin^2\theta}{2}\Big]_0^{\{[\ \ ]+\}57.3} + \Big[-c_\phi\cos\theta\Big]^{\{[\ \ ]+\}57.3}\Big\} + k_\theta$$

So we have here the field of vector potential contributed alone by the re-radiator as a result of its induced current due to its position in the field of radiation due to the primary dipole. Since this is a vector quantity, although only one component is present, it is essential to add it vectorially to the primary radiation due to the dipole alone, this latter having two components. The resulting expression is then considered in light of the initial and field boundary conditions and finally used to arrive at the increment or diminution in total field strength at various portions of the field at which the radiation as a result of both together is being investigated.

The total radiated field along the radial axis is

$$A_r = \frac{I_0 l}{cr} \sum \left(A_n \sin n\omega\left(t - \frac{r}{c}\right) + B_n \cos n\omega\left(t - \frac{r}{c}\right)\right) \cos\theta \qquad (91)$$

and long the theta component

$$A_\theta = (\text{same as equation } 90). \qquad (92).$$

Because of symmetry, of course, we still have

$$A_\phi = 0 \qquad (93).$$

III. The field components for the primary radiating source have been completely determined. From the vector potential at which we have just arrived, it is possible through

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla^2 \int A\,dt \tag{38}$$

or in this particular case

$$E_\theta = -\frac{1}{c}\frac{\partial A_\theta}{\partial t} - \nabla^2 \int A_\theta\,dt \tag{94}$$

to arrive at the field strength components for the re-radiator alone. This of course is the field set up by the re-radiator as a result of induced currents in the zone plate as a result of its being in a field of a radiator.

It is necessary to separately determine the elements in the equation given by (94). One of these is

$$\begin{aligned}
\frac{\partial A_{\theta'}}{\partial t} = \sum_{2n'-1} \frac{-I_0 l}{4b} \Bigg[& \Big(\sum_n -\frac{\omega^2 n^2}{c} A_n \sin n\omega\left(t-\tfrac{r}{c}\right) - \frac{\omega^2 n^2}{c} B_n \cos n\omega\left(t-\tfrac{r}{c}\right)\Big) \\
& \Big| \frac{\theta}{2} - \frac{1}{4}\sin^2\theta \Big|_0^{\{[\ ]+\}57.3} - \frac{1}{\rho}\sum\Big(A_n \omega n \cos n\omega\left(t-\tfrac{r}{c}\right) \\
& - B_n \omega n \sin n\omega\left(t-\tfrac{r}{c}\right)\Big) \Big| \frac{\sin^2\theta}{4} - \frac{\theta}{2} \Big|_0^{\{[\ ]+\}57.3} \\
& + \frac{\rho}{c^2}\sum\Big(-A_n \omega^3 n^3 \cos n\omega\left(t-\tfrac{r}{c}\right) + B_n \omega^3 n^3 \sin n\omega\left(t-\tfrac{r}{c}\right)\Big| \sin^2\theta \Big|_0^{\{[\ ]+\}57.3} \\
& + \sum\Big(\frac{A_n \omega^2 n^2}{c} \sin n\omega\left(t-\tfrac{r}{c}\right) + B_n \omega^2 n^2 \cos n\omega\left(t-\tfrac{r}{c}\right)\Big) \Big| -\cos\theta \Big|_0^{\{[\ ]+\}57.3} \\
& + \sum \frac{\omega^3 n^3}{c} A_n \cos n\omega\left(t-\tfrac{r}{c}\right) - \frac{\omega^3 n^3}{c} B_n \sin n\omega\left(t-\tfrac{r}{c}\right)\Big) \Big| \frac{\theta}{2} - \frac{\sin^2\theta}{4} \Big|_0^{\{[\ ]+\}57.3^\circ} \Bigg]
\end{aligned} \tag{95}$$

Another element is givenby

$$\int A_\theta\, dt = \sum_{2n'-1} -\frac{I_0 l}{4b}\Big[\Big(\sum_n \frac{A_n}{c}\sin n\omega\left(t-\tfrac{r}{c}\right)+\frac{B_n}{c}\cos n\omega\left(t-\tfrac{r}{c}\right)\Big)$$

$$\Big|\tfrac{1}{2}\theta-\tfrac{1}{4}\sin^2\theta\Big|_0^{\{[\;]+\}57.3} - \frac{1}{\rho}\sum\Big(-\frac{A_n}{\omega n}\cos n\omega\left(t-\tfrac{r}{c}\right) \qquad (96).$$

$$+\frac{B_n}{\omega n}\sin n\omega\left(t-\tfrac{r}{c}\right)\Big)\Big|\frac{\sin^2\theta}{4}-\frac{\theta}{2}\Big|_0^{\{[\;]+\}57.3} + \sum\Big(-\frac{A_n}{c}\sin n\omega\left(t-\tfrac{r}{c}\right)$$

$$-\frac{B_n}{c}\cos n\omega\left(t-\tfrac{r}{c}\right)\Big)\Big|-\cos\theta\Big|_0^{\{[\;]+\}57.3} + \sum -\frac{\omega n}{c}A_n\cos n\omega\left(t-\tfrac{r}{c}\right)$$

$$+\frac{\omega n}{c}B_n\sin n\omega\left(t-\tfrac{r}{c}\right)\Big)\Big|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3} + \frac{\rho}{c^2}\sum\Big(A_n\omega_n\cos n\omega\left(t-\tfrac{r}{c}\right)$$

$$-B_n\omega n\sin n\omega\left(t-\tfrac{r}{c}\right)\Big)\Big|\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3}\Big| - c_\phi\cos\theta(t)\Big|_0^{\{[\;]+\}57.3}$$

$$+K_{\theta t}+k_t$$

There remains the operation of the scalar del square. Since we are investigating in terms of r , we can substitute the variable in terms of the radius and position of the zone plate and the distance from the zone plate measured along the axis to an arbitrary point under consideration. This is applied to the theta component alone and is given by

$$\nabla^2 = \left(\frac{2}{\rho+b}\;\frac{\partial}{\partial b}+\frac{\partial^2}{\partial b^2}\right)\left(\frac{1}{(\rho+b)^2}\;\frac{\partial^2}{\partial\theta^2}+\frac{\operatorname{ctn}\theta}{(\rho+b)^2}\;\frac{\partial}{\partial\theta}\right)+(0)_\phi \qquad (97)$$

and the differentiations used are shown valid since

$$\frac{\partial}{\partial b} = \frac{\partial}{\partial r}\;\frac{\partial r}{\partial b} \qquad (98)$$

$$\frac{\partial}{\partial\theta} = c\phi\sin\theta t \qquad (98a)$$

It is convenient first to rewrite (96) in terms of the substituted variable, then perform the operations indicated accordingly,

$$\int A_\theta dt = \sum_{2n'-1} \frac{I_0 l}{4b}\Big[\Big(\sum_n \frac{A_n}{c}\sin n\omega(t-\tfrac{\rho+b}{c}) + \frac{B_n}{c}\cos n\omega(t-\tfrac{\rho+b}{c})\Big)$$
$$\Big|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3} - \frac{1}{(r-b)}\sum_n\Big(-\frac{A_n}{\omega n}\cos n\omega(t-\tfrac{\rho+b}{c})$$
$$+\frac{B_n}{\omega n}\sin n\omega(t-\tfrac{\rho+b}{c})\Big)\Big|\frac{\sin^2\theta}{4}-\frac{\theta}{2}\Big|_0^{\{[\;]+\}57.3} \quad (99).$$
$$+\sum -\frac{A_n}{c}\sin n\omega(t-\tfrac{\rho+b}{c}) - \frac{B_n}{c}\cos n\omega(t-\tfrac{\rho+b}{c})\Big)\Big|-\cos\theta\Big|_0^{\{[\;]+\}57.3}$$
$$+\sum -\frac{\omega n}{c}A_n\cos n\omega(t-\tfrac{\rho+b}{c}) + \frac{\omega n}{c}B_n\sin n\omega(t-\tfrac{\rho+b}{c})$$
$$\Big|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3} + \frac{(r-b)}{c^2}\sum_n\Big(A_n\omega n\cos n\omega(t-\tfrac{\rho+b}{c})$$
$$-B_n\omega n\sin n\omega(t-\tfrac{\rho+b}{c})\Big)\Big|\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3}\Big|-c_\phi\cos\theta t\Big|_0^{\{[\;]+\}57.3}\Big]$$
$$+K_{\theta t}+k_t$$

The differentiation is performed first with respect to the distance along the axis measured from the zone plate into the re-radiated field giving

$$\frac{\partial\int A_\theta dt}{\partial b} = \sum_{2n'-1}\frac{-I_0 l}{4b^2}\Big[\Big(\sum\frac{A_n}{c}\sin n\omega(t-\tfrac{\rho+b}{c}) + \frac{B_n}{c}\cos n\omega(t-\tfrac{\rho+b}{c})\Big)$$
$$\Big|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3} - \frac{1}{(r-b)}\sum_n\Big(-\frac{A_n}{\omega n}\cos n\omega(t-\tfrac{\rho+b}{c}) \quad (100)$$
$$+\frac{B_n}{\omega n}\sin n\omega(t-\tfrac{\rho+b}{c})\Big)\Big|\frac{\sin^2\theta}{4}-\frac{\theta}{2}\Big|_0^{\{[\;]+\}57.3} + \sum -\frac{A_n}{c}\sin n\omega(t-\tfrac{\rho+b}{c})$$
$$-\frac{B_n}{c}\cos n\omega(t-\tfrac{\rho+b}{c})\Big)\Big|-\cos\theta\Big|_0^{\{[\;]+\}57.3} + \sum -\frac{\omega n A_n}{c}\cos n\omega(t-\tfrac{\rho+b}{c})$$
$$+\frac{\omega n}{c}B_n\sin n\omega(t-\tfrac{\rho+b}{c})\Big|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3} + \frac{(r-b)}{c^2}\sum_n\Big(A_n\omega n\cos n\omega(t-\tfrac{\rho+b}{c})$$
$$-B_n\omega n\sin n\omega(t-\tfrac{\rho+b}{c})\Big)\Big|\frac{\theta}{2}+\sin^2\theta\Big|_0^{\{[\;]+\}57.3} + \Big|-c_\varphi\cos\theta t\Big|_0^{\{[\;]+\}57.3}$$
$$+K_{\theta t}+k_t\Big] + \sum_{2n'-1}\frac{I_0 l}{4b}\Big[\Big(\sum_n -\frac{n\omega A_n}{c^2}\cos n\omega(t-\tfrac{\rho+b}{c})$$
$$+\frac{n\omega B_n}{c^2}\sin n\omega(t-\tfrac{\rho+b}{c})\Big)\Big|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\Big|_0^{\{[\;]+\}57.3}$$
$$-\frac{1}{(r-b)}\sum_n\Big(-\frac{A_n}{c}\sin n\omega(t-\tfrac{\rho+b}{c}) - \frac{B_n}{c}\cos n\omega(t-\tfrac{\rho+b}{c})\Big)$$

(over)

(continued)

$$\left| \frac{\sin^2\theta}{4} - \frac{\theta}{2} \right|_0^{\{[\ \ ]+\}\,57.3} - \frac{1}{(r-b)^2} \sum_n \left(-\frac{A_n}{wn} \cos nw\left(t - \frac{\rho+b}{c}\right) \right.$$

$$\left. + \frac{B_n}{wn} \sin nw\left(t - \frac{\rho+b}{c}\right) \right) \left| \frac{\sin^2\theta}{4} - \frac{\theta}{2} \right|_0^{\{[\ \ ]+\}\,57.3}$$

$$+ \sum \left(\frac{A_n}{c^2} wn \cos nw\left(t - \frac{\rho+b}{c}\right) - \frac{B_n}{c^2} wn \sin nw\left(t - \frac{\rho+b}{c}\right) \right)$$

$$\left| -\cos\theta \right|_0^{\{[\ \ ]+\}\,57.3} + \sum \frac{w^2 n^2}{c^2} A_n \sin nw\left(t - \frac{\rho+b}{c}\right)$$

$$- \frac{w^2 n^2}{c^2} B_n \cos nw\left(t - \frac{\rho+b}{c}\right) \left| \frac{\theta}{2} - \frac{\sin^2\theta}{4} \right|_0^{\{[\ \ ]+\}\,57.3}$$

$$+ \frac{(r-b)}{c^2} \sum_n \left(\frac{A_n w^2 n^2}{c} \sin nw\left(t - \frac{\rho+b}{c}\right) + \frac{B_n w^2 n^2}{c} \cos nw\left(t - \frac{\rho+b}{c}\right) \right)$$

$$\left| \frac{\sin^2\theta}{4} \right|_0^{\{[\ \ ]+\}\,57.3} - \frac{1}{c^2} \sum_n \left(A_n wn \cos nw\left(t - \frac{\rho+b}{c}\right) \right.$$

$$\left. + B_n wn \sin nw\left(t - \frac{\rho+b}{c}\right) \right) \left| \frac{\sin^2\theta}{4} \right|_0^{\{[\ \ ]+\}\,57.3} \Bigg]$$

The second differentiation with respect to the same variable, to which we were eventually headed is

$$\frac{\partial^2 \int A_\theta dt}{\partial b^2} = \sum_{2n'-1} \frac{I_0 l_2}{4b^3}\left[\text{terms}\right] - \frac{I_0 l}{4b^2}\left[\text{terms}\right]'$$

$$- \sum_{2n'-1} \frac{I_0 l}{4b^2}\left[\text{terms}\right] + \sum_{2n'-1} \frac{I_0 l}{4b}\left[\left(\left(\sum_n -\frac{n^2\omega^2}{c^3} A_n \cos n\omega\left(t - \frac{\rho+b}{c}\right)\right.\right.\right. \quad (101)$$

$$\left.\left.+ \frac{n^2\omega^2 B_n}{c^2} \sin n\omega\left(t - \frac{\rho+b}{c}\right)\right)\right) \Big| \frac{\theta}{2} - \frac{\sin^2\theta}{4} \Big|_0^{\{[\ \ ]+\}\ 57.3}$$

$$+ \frac{1}{(r-b)^2}\left[\text{terms}\right] - \frac{1}{r-b} \sum_n \frac{A_n \omega n}{c^2} \cos n\omega\left(t - \frac{\rho+b}{c}\right)$$

$$- \frac{B_n \omega n}{c} \sin n\omega\left(t - \frac{\rho+b}{c}\right) \Big| \frac{\sin^2\theta}{4} - \frac{\theta}{2} \Big|_0^{\{[\ \ ]+\}\ 57.3}$$

$$+ \frac{2}{(r-b)^3}\left[\text{terms}\right] - \frac{1}{(r-b)^2}\left[\text{terms}\right]'$$

$$+ \sum\left(\frac{A_n \omega^2 n^2}{c^3} \omega n \cos n\omega\left(t - \frac{\rho+b}{c}\right) + \frac{B_n \omega^2 n^2}{c^3} \cos n\omega\left(t - \frac{\rho+b}{c}\right)\right)$$

$$\Big| -\cos\theta \Big|_0^{\{[\ \ ]+\}\ 57.3} + \sum -\frac{\omega^3 n^3}{c^3} A_n \cos n\omega\left(t - \frac{\rho+b}{c}\right)$$

$$- \frac{\omega^3 n^3}{c^3} B_n \sin n\omega\left(t - \frac{\rho+b}{c}\right)\Big) \Big| \frac{\theta}{2} - \frac{\sin^2\theta}{4} \Big|_0^{\{[\ \ ]+\}\ 57.3}$$

$$+ \frac{(r-b)}{c^2}\left[\text{terms}\right] - \frac{1}{c^2}\left[\text{terms}\right] - \frac{1}{c^2}\left[\text{terms}\right]$$

We come now to what is an important and necessary simplification. As was mentioned further back, we are concerned here with the "radiation" portion of the electromagnetic field. It is customary in problems with antennaes to consider the field as two separate parts. These are the induction field and the radiation field. The radiation field, as defined further back, is the portion of the field at a great distance from the antenna. The induction field, on the other hand, is that portion immediately close to the radiating system. More precisely if in the field relations we omit all terms which have negligible values at great distance, we have the radiation field left. Conversely, omitting terms which are of small value close to the antenna gives the induction field. Now usually the whole electric field expression contains the distance from the system to points in space in the denominator. When the distance is involved as a square power or higher in the denominator, omitting the terms containing this quantity gives the radiation portion. Conversely, if the distance appears to the first power, omitting terms containing these quantities leaves the induction field close to the antenna.

Now while (101) is a measure of the field at any point in space, it is so lengthy and involves so many terms that we do not have much help in framing a geometric picture. Thatis, it is hard to see exactly what effect on the relation the various quantities involved have.

The assumptions made presumed that we would be concerned with points located at quite some distance from the re-radiator. Therefore, to be consistent and insure accuracy, we should concern ourselves mainly with the radiation field at great distances from the re-radiator.

We perform the essential operation then of considering the properties only of the radiation field and to this end neglect all terms of square powers and higher in b and thus [illegible] the substituted variable. Our azimuthal angle component with which we have been working takes the form

$$E_\theta = \sum \frac{I_0 l}{cb}\left[terms\right] - \sum_{2n'-1} \frac{I_0 l_2}{4b_3}\left[terms\right] + \frac{I_0 l}{4b^2}\left[terms\right]'$$

$$+ \sum \frac{I_0 l}{4b^2}\left[terms\right] - \sum_{2n'-1} \frac{I_0 l}{4b}\Big[\Big(\sum_n -\frac{n^2\omega^2}{c^3} A_n \cos n\omega\left(t - \frac{\rho + b}{c}\right)$$

$$+ \frac{n^2\omega^2 B_n}{c^2} \sin n\omega\left(t - \frac{\rho+b}{c}\right)\Big)\Big| \frac{\theta}{2} - \frac{\sin^2\theta}{4}\Big|_0^{\{[\ \ ]+\}57.3} \quad (102)$$

$$- \frac{1}{(r-b)^2}\left[terms\right] + \frac{1}{r-b}\sum_n \frac{A_n \omega n}{c^2} \cos n\omega\left(t - \frac{\rho+b}{c}\right)$$

$$- \frac{B_n \omega n}{c} \sin n\omega\left(t - \frac{\rho+b}{c}\right)\Big)\Big| \frac{\sin^2\theta}{4} - \frac{\theta}{2}\Big|_0^{\{[\ \ ]+\}57.3}$$

$$- \frac{2}{(r-b)^3}\left[terms\right] + \frac{1}{(r-b)^2}\left[terms\right]$$

$$- \sum \left(\frac{A_n \omega^2 n^2}{c^3} \cos n\omega\left(t - \frac{\rho+b}{c}\right) + \frac{B_n \omega^2 n^2}{c^3} \sin n\omega\left(t - \frac{\rho+b}{c}\right)\right)$$

$$\Big| -\cos\theta \Big|_0^{\{[\ \ ]+\}57.3} + \sum \frac{\omega^3 n^3}{c^3} A_n \cos n\omega\left(t - \frac{\rho+b}{c}\right)$$

$$+ \frac{\omega^3 n^3}{c^3} B_n \sin n\omega\left(t - \frac{\rho+b}{c}\right)\Big)\Big| \frac{\theta}{2} - \frac{\sin^2\theta}{4}\Big|_0^{\{[\ \ ]+\}57.3}$$

$$+ \frac{(r-b)}{c^2}\left[terms\right] - \frac{1}{c^2}\left[terms\right] - \frac{1}{c^2}\left[terms\right]$$

Neglecting these same induction field components in the azimuthal component of the primary radiator, we have

$$E_\theta = \frac{ct_n'\,\theta\, rt}{r} - \frac{1}{r^2\omega}\sum\left(\frac{B_n}{n}\sin n\omega\left(t-\frac{r}{c}\right)\right. \qquad (103)$$

$$\left.-\frac{A_n}{n}\cos n\omega\left(t-\frac{r}{c}\right)\right)$$

The arithmetic sum of (103) and (102) provides us with the total azimuthal electric field strength

$$\sum E_\theta = \frac{ct_n'\,\theta}{r}\,\gamma t - \frac{1}{r^2\omega}\sum \frac{B_n}{n}\sin n\omega\left(t-\frac{r}{c}\right)$$

$$-\frac{A_n}{n}\cos n\omega\left(t-\frac{r}{c}\right)$$

$$+\sum_{2n'-1}\frac{I_0 l^n}{4b}\left[\sum_n\left(\frac{n^2\omega^2}{c^3}A_n\cos n\omega\left(t-\frac{\rho+b}{c}\right)\right.\right. \qquad (104)$$

$$\left.-\frac{n^2\omega^2}{c^2}B_n\sin n\omega\left(t-\frac{\rho+b}{c}\right)\right)\left|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\right|_0^{\{[\ ]+\}\,57.3}$$

$$+\frac{1}{r-b}\sum_n \frac{A_n\omega n}{c^2}\cos n\omega\left(t-\frac{\rho+b}{c}\right)$$

$$-\frac{B_n\omega n}{c}\sin n\omega\left(t-\frac{\rho+b}{c}\right)\Big)\left|\frac{\sin^2\theta}{4}-\frac{\theta}{2}\right|_0^{\{[\ ]+\}\,57.3}$$

$$-\sum\left(\frac{A_n\omega^2n^2}{c^3}\cos n\omega\left(t-\frac{\rho+b}{c}\right)+\frac{B_n\omega^2n^2}{c^3}\sin n\omega\left(t-\frac{\rho+b}{c}\right)\right)$$

$$\Big|-\cos\theta^{\{[\ ]+\}\,57.3} + \sum\frac{\omega^3n^3}{c^3}A_n\cos n\omega\left(t-\frac{\rho+b}{c}\right)$$

$$+\frac{\omega^3n^3}{c^3}B_n\sin n\omega\left(t-\frac{\rho+b}{c}\right)\Big)\left|\frac{\theta}{2}-\frac{\sin^2\theta}{4}\right|_0^{\{[\ ]+\}\,57.3}$$

$$E_r = \text{same as radiated field} \qquad (105)$$

$$E_\varphi = 0 \qquad (106)$$

The geometric aspects of the re-radiator, it was pointed out, do not materially affect the radial or other angular component of the field and these are the same as that of the primary radiator alone. We have determined completely the electric field vector for the whole antenna system.

CONCLUSION

Relations (104), (105) and (106) are the result of this investigation. Referring to (106) we see that the electric field component in the phi direction is zero. This is not surprising since early in the paper, symmetry with the respect to this coordinate was prescribed in the theoretical construction of the radiating antennule and the re-radiating zone plate.

Referring to (105), we see that the field for the whole system is the same as the field for the antennule alone. When we consider that the problem was worked using an infinitely thin zone plate, this is not surprising.

Equation (103) is the relation which gives us the magnitude of the electric field in the theta direction. This is, of course, the magnitude of the distant radiation field. It is to be noted that this expression is not the same for the whole system as for the radiating antennule alone.

By inserting the zone plate, we have "rotated" the electric field. The basis for this statement requires that we investigate slightly the nature of vectors. Since we have used spherical coordinates here, which are orthogonal, the magnitude of any vector is simply the square root of $\sum(\text{components})^2$. Now if the theta component has changed in magnitude, its square will change in magnitude and accordingly the vector magnitude will change in magnitude .

Now the direction of a vector is given by specifying an angle in the coordinate system in each of three directions. This angle is determined usually in terms of the tangent of this angle; being defined as the magnitude of the component in a particular direction divided by the magnitude of the vector. Now since two of the components were shown unchanged by inserting the re-radiating zone plate, and yet the vector was shown changed in magnitude; it is obvious that the angles in the direction of these components will change. In other words, the numerators remain fixed while the denominator changes. This, of course, changes the value of the angle's tangent and so the direction of the vector has changed.

This is precisely what has happened in this case with the electric field vector whose components are given in (104), (105) and (106). Having changed its angle, it has been rotated as was mentioned.

Electromotive force may be defined as the scalar or dot product of the electric field and the line element of the conductor in which the electromotive force is being developed. This dot product contains a cosine term, so that when the cosine is unity, the electromotive force is aa maximum. This means the angle corresponding to the cosine can be 0 degrees; and in this case, it is the angle between the line element and the electric field vector.

If we are "receiving" the signal of the short antenna in an antennule, we would presumably orient it to gain the maximum electromotive force that we can. Now if the re-radiator is inserted in the system, the electric field is rotated and if we still wish to develop the maximum electromotive force, we must rotate the receiving antennule. To gain maximum electromotive force, we must again make the cosine of the angle between the antennule and the electric field unity, or the angle 0 degrees. This means, of course, that we must rotate through an angle exactly equal to that through which the electric field was rotated. Noting (104) again we can arrive at some conclusions of further interest. The terms caused by the introduction ofthe re-radiator into the system are all included under a summation sign. This summation corresponds to the number of zones used in constructing the theoretical zone plate. That is, the integration of the re-radiated field was performed for each half wave zone in the zone plate. These were then summed over the number of zones which we choose to use. As the relation is given, the choice in the number of zones is perfectly arbitrary.. We may choose to use no zones, which amounts to not having any re-radiating zone plate in the system. We should expect to have the equations representing the electric field of the whole system degenerate back to the field of the radiating antennule along. Since all of the terms contributed by the re-radiator are contained within this summation, as was mentioned, using no zones and summing over zero eliminates the field due to the zone plate. Thus, we are left with the original field.

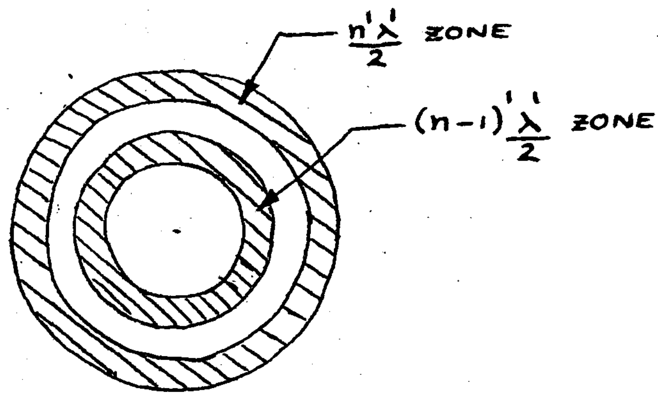
We should also expect that as we go toward infinity the whole electric field should approach zero. Since all of the terms in the result have a distance in the denominator, if the distance approaches infinity all of the terms would approach zero and the electric field thus disappear. In terms of our problem the distances used are either the distance from the re-radiator or from the radiating antennule. We have retained both to more easily identify the terms that correspond to the radiator and those that correspond to the zone plate. However, as we approach infinity both of these distances approach infinity and since they do appear in the denominator terms throughout, the corresponding terms approach zero. Therefore, at infinity the electric field degenerates. This is important in working any Dirichlet problem such as this, since if a true representation of a physical situation is to be given theoretically, we must at least expect that in thelimiting cases, we can get agreement with known experience. It is,of course, assumed that we know the further we get from the electromagnetic radiation of a system such as this; the weaker the radiation becomes. Presumably, then, at infinity, we would expect no radiation from the system to be present.

This situationin the limiting case is mentioned for a specific reason. Like all Dirichlet problems, the system is considered isolated in free space as was mentioned. This is a situation which we can hardly expect to achieve in performing an experiment to verify the results. That is, we can hardly expect any experiment to show much agreement with the theoretical result but this does not particularly decrease the value of the result. We have set out to find out what effect the analogue of a fresnel zone plate would have on the electric field established by a radiating antennule. We have done this and what is probably most important to the physicist, wehave widened the number of problems to which the Dirichlet method may be applied. This, of course, makes for a better unity in physics, which is one of the ultimate goals; ie the solution of as many different problems as possible with as few methods as possible.

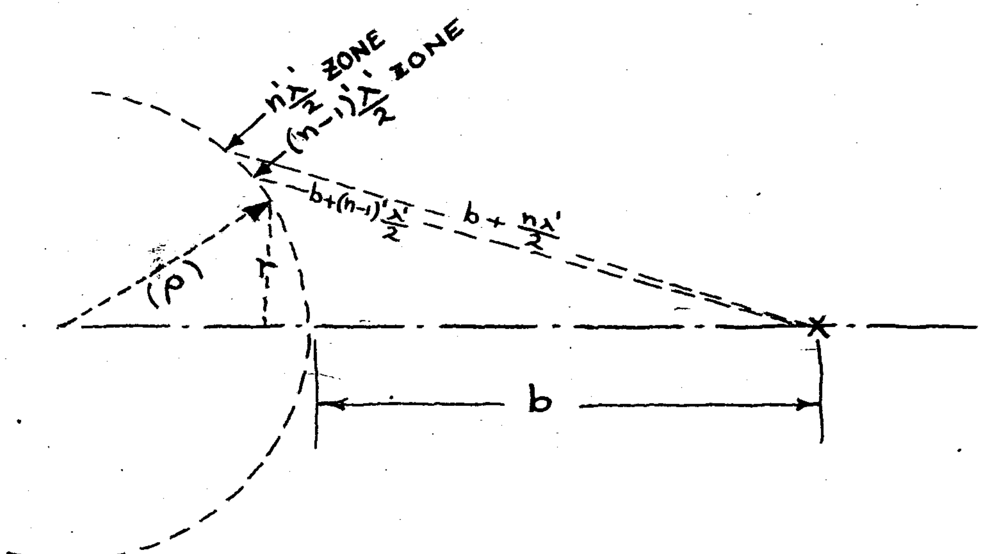
As a suggestion to those interested, some possible additional problems are suggested. These are a few of the many possible. It would be of interest to work the problem of zone plates in series. This implies that we could locate these zone plates concentrically putting the first at a given distance from the antennule, the second at a greater distance and so on.

The use of a reflecting "horn" in conjunction with a sone plate would also be of interest. The "horn" presumably would be placed some distance on one side of the antennule and the zone plate arranged so that the horn islocated at the radius of curvature of and directed at the zone plate.

Problems of this sort, as well as the one contained in this paper, widen further the applications of Dirichlet's method.



DIMENSIONED ELEVATION OF THE
RE-RADIATING ZONE PLATE



CROSS-SECTIONAL ILLUSTRATION OF
ZONAL SEGMENT GEOMETRY

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ABSTRACT OF THE DISSERTATION

In this paper we are concerned with the possibility of improving the field strength of a radiated electromagnetic disturbance. The disturbance is propagated by an antenna of short length which is excited by a signal of purely arbitrary wave form. In an effort to improve the distribution in space of the radiated signal, a secondary re-radiating antenna is introduced into the field of the primary short antenna. This secondary antenna is of special design . Use is made of Fresnel's zone plate from the physical optics.

The Fresnel zone plate is a series of concentric circular rings of varying area, each area alternately transparent and opaque. The result in optics is the strengthening of light intensity at certain points in space at the expense of other points. The mathematical representation of this new distribution of light energy is well known. We hope to make use of the electromagnetic analogue of the zone plate , but instead of making use of geometric concepts only, make a rigorous mathematical definition of the new field distribution in space. This is done with the help of the well known Maxwell Equations.

The radiation pattern of the potential is derived and is given by:

$$\Phi_0 = I_0 \, l \cos\theta \left\{ \left(\frac{1}{r^2 \omega}\right) \sum \left[\frac{B_n}{n} \sin n\omega\left(t - \frac{r}{c}\right) - \frac{A_n}{n} \cos n\omega\left(t - \frac{r}{c}\right) \right] + \frac{1}{cr} \sum \left[A_n \sin n\omega\left(t - \frac{r}{c}\right) + B_n \cos n\omega\left(t - \frac{r}{c}\right) \right] \right\} + \frac{\beta t}{r} + \frac{\delta t \operatorname{ctn}\theta}{r} + k$$

I_0 represents the amplitude of the current of the exciting signal,

l the length of the short primary antenna, and θ the azimuthal angle. This relation is given in spherical coordinates since most of the spatial distributions of the fields are most easily derived in these orthogonal coordinates. Use is made only of Kirchoff's retarded potential solution and the advanced potential is not considered and is dismissed as being physically unreal. By definition,this short primary antenna has the same distance from any point on the antenna to a given point in space.

The re-radiating antenna is introduced at an arbitrary distance from the antenna and its surface is prescribed as spherical. The radius of this sphere is taken equal to the distance from the short radiating antenna to the re-radiator alone and measured along the axis joining their centers. The surface is actually made up of concentric rings of varying area alternating with rings of free space and the former rings are electrically conducting and connected. This is really , as was mentioned, the electromagnetic analogue of the Fresnel Zone Plate.

The currents induced in the re-radiator by the short antenna are then investigated. The radial current component is neglected by making the zone plate negligibly thin. The electric field set up by the currents induced in this re-radiator is then determined through the Maxwell equations after the latter have been manipulated into a form which is more convenient to use in this problem. Finally, the electric field components are summed arithmetically;with the corresponding electric field component of the short antenna being added to that caused by the introduction of the re-radiator.

Some generality is sacrificed to make the problem workable and more appealing to the geometric sense. This results mainly from assuming the re-radiator negligibly thin and the primary radiator short. In principle, however, the method used allows a solution for a radiator of any length and a zone plate of arbitrary thickness. Since this problem is in essence a Dirichlet problem, the usual assumption is tacitly made that all of the elements in the system being studied are located in free space.

The solution presented is that of the radiation field alone. This is the field at large distances from the system which is determined by ignoring terms in the solution which involve an inverse square term or higher of this distance. Practically ,we are interested in improving the signal strength of an antenna at great distances and so it is the radiation field in which we are really interested.

Some conclusions can be drawn from the solution. The electric field vector is rotated and changed in magnitude by the introduction of the re-radiator. This is shown to be dependent upon the number of zones used in the re-radiator. When the number of zones is made zero, which amounts to removing the re-radiator from the system;the electric field vector returns to its original orientation and magnitude as is to be expected. Furthermore, as the distance from the system grows larger, the electric field vector magnitude approaches zero which is in keeping with our physical sense of the problem.

No effort is made to study the field near the system, as was mentioned, since this induction field has little practical use in this problem.

Several further investigations suggest themselves immediately. The effect of stacking re-radiators,concentric with the primary radiator, would probably result in a further alteration of the original field. It would also be of interest to evaluate the field of the system used in this paper when the primary radiator is a quarter-wave antenna or is used in conjunction with a horn or reflector. All of these can be solved in principle with the methods used here and are all of great practical interest in light of the current need for efficient and inexpensive signal relay methods.

AUTOBIOGRAPHY OF THE CANDIDATE

The candidate was born in New York City on August 16, 1924, son of Col. Tobias and Hazel Schumann Klein. Elementary education was obtained in the public school system of that city and secondary training was obtained at the James Madison High School in New York. In 1940, the undergraduate program was undertaken at the Pennsylvania State College which culminated in graduation in October, 1943. At this time, graduate study was commenced at Stanford University, but was soon interrupted for enlistment into the United States Navy. After enlistment in June, 1944, the candidate was placed in the service of the National Advisory Committee for Aeronautics Section of the Committee until July, 1946. In March, 1948, graduate study was recommenced at Boston University leading to the granting of the degree of Master of Arts in Physics in June, 1949.

At the Pennsylvania State College, the candidate was elected an undergraduate teaching assistant in the program of the United States Air Forces during 1943 and was appointed Graduate Teaching Assistant at Stanford University in August of that year. This position was resigned in enlistment in June, 1944.