Lagrangian fibers in Gelfand-Cetlin systems

LAGRANGIAN FIBERS OF GELFAND-CETLIN SYSTEMS

YUNHYUNG CHO, YOOSIK KIM, AND YONG-GEUN OH

ABSTRACT. Motivated by the study of Nishinou-Nohara-Ueda on the Floer thoery of Gelfand-Cetlin systems over complex partial flag manifolds, we provide a complete description of the topology of Gelfand-Cetlin fibers. We prove that all fibers are *smooth* isotropic submanifolds and give a complete description of the fiber to be Lagrangian in terms of combinatorics of Gelfand-Cetlin polytope. Then we study (non-)displaceability of Lagrangian fibers. After a few combinatorial and numercal tests for the displaceability, using the bulk-deformation of Floer cohomology by Schubert cycles, we prove that every full flag manifold $\mathcal{F}(n)$ $(n \geq 3)$ with a monotone Kirillov-Kostant-Souriau symplectic form carries a continuum of non-displaceable Lagrangian tori which degenerates to a non-torus fiber in the Hausdorff limit. In particular, the Lagrangian S^3-fiber in $\mathcal{F}(3)$ is non-displaceable the question of which was raised by Nohara-Ueda who computed its Floer cohomology to be vanishing.

CONTENTS

1. INTRODUCTION

A partial flag manifold $\mathrm{GL}(n, \mathbb{C})/P$ can be defined as the orbit $\mathcal{O}_\lambda$ of the diagonal matrix whose diagonal entries are given by a decreasing sequence λ of integers under the conjugate $U(n)$-action. It comes with a $U(n)$-invariant Kähler form (unique up to scaling), so-called a *Kirillov-Kostant-Souriau symplectic form*. Guillemin-Sternberg [GS2] constructed a completely integrable system on $\mathcal{O}_\lambda$ respecting the circle actions arising from the Cartan subgroups of connected closed subgroups that form a nested sequence of $U(n)$. They named it a *Gelfand-Cetlin system* (*GC system* for short) as the torus actions was used by Gelfand-Cetlin [GC] to decompose the space of holomorphic sections of the $U(n)$-equivariant line bundle corresponding to λ, which is the irreducible representation of the highest weight λ by the Borel-Weil theorem. GC systems are of main interest in this article.

A projective variety admitting a toric degeneration can be studied through the toric variety located at the center. Partial flag manifolds are known to have toric degenerations by works of Gonciulea-Lakshmibai [GL], Caldero [Ca] and Kogan-Miller [KoM]. They have been versatile tools to study algebraic and symplectic geometry of partial flag manifolds for understanding Schubert varieties, studying mirror symmetry, constructing integrable systems, estimating Gromov width, etc by taking advantage of combinatorial obejcts associated with the toric varieties. At the level of completely integrable systems, Nishinou-Nohara-Ueda [NNU1] constructed a toric degeneration of a GC system to a toric moment map. They computed (deformed) Floer cohomology of torus fibers. En route, they constructed a Landau-Ginzburg mirror by classifying holomorphic discs of Maslov index two bounded by a torus fiber on a partial flag manifold. It can be thought as Floer theoretical realization of Landau-Ginzburg models appearing in the context of closed mirror symmetry [EHX, Gi, BCKV].

Just like the toric cases, the image of any GC system is a polytope and the fiber over any point in its interior is a Lagrangian torus. However, contrary to the toric cases, the action of the big torus does not extend to a partial flag variety, which is reflected to an appearance of non-torus Lagrangian fibers. For the purpose of homological mirror symmetry, those Lagrangians cannot be ignored since a certain non-toric Lagrangian is indeed a non-zero object over the Novikov ring over the field of complex numbers according to the work of Nohara-Ueda [NU2]. This feature contrasts with the toric cases where a certain collection of Lagrangian toric fibers together with deformation data can split-generate the Fukaya category of a toric manifold investigated by the third named author with Abouzaid, Fukaya, Ohta, and Ono [AFOOO]. It is also responsible for the fact that the number of critical points of the superpotential is sometimes strictly less than the sum of betti numbers of the ambient variety. By the work of Nohara-Ueda [NU1] and Harada-Kaveh [HaK], one can produce completely integrable systems on a partial flag manifold coming from various toric degenerations, but one still misses critical points having some critical values if toric fibers are only considered. Therefore, non-toric fibers or their (quasi-)equivalent objects should be taken into consideration to do mirror symmetry, which motivates us to classify non-toric Lagrangian fibers in this paper.

Besides the aspect of the Fukaya category of partial flag manifolds, the process revealing the topology of fibers is itself interesting as it involves combinatorics of ladder diagrams and tableaux. In representation theory and Schubert calculus, Young diagrams and tableaux are very effective combinatorial tools to study crystal graphs for representations of quantum groups and the product structure on Schubert cocycles for instance, see [HoK, Ful]. In this paper, we device a combinatorial procedure on ladder diagrams and tableaux, which are analogues to Young diagrams and tableaux, in order to understand the topology of GC fibers and symplectic geometry of partial flag manifolds. The first named author with An and Kim [ACK] found an order-preserving one-to-one correspondence between the faces of a GC polytope and certain subgraphs of the ladder diagram. By playing "3D-TETRIS® " with certain shapes of blocks on the subgraph corresponding to a face of a GC polytope, one can tell the topology of fibers over the relative interior of the face and whether the fibers are Lagrangian or not. Those types of combinatorics on a ladder diagram will be further developed in [CKO] to obtain topological and geometric data on partial flag manifolds.

In the second part of the paper, based on the above detailed description of Lagrangian fibers, we test displaceability and non-displaceability of Lagrangian fibers. In Nishinou-Nohara-Ueda [NNU1], they showed the Lagrangian torus fiber at the center of a GC polytope is non-displaceable. Other than toric fibers, there are not many things to be known about non-displaceability of non-toric fibers. As far as the authors know, the only known non-displaceable non-toric GC fibers are in some limited cases of Grassmannians such as the monotone $U(2)$-fiber in $\mathrm{Gr}(2,4)$ in Nohara-Ueda [NU2] and the monotone $U(n)$-fiber in $\mathrm{Gr}(n,2n)$ in Evans-Lekili [EL2]. In this article, we deal with the case of complete flag manifolds $\mathcal{F}(n)$ for $n \geq 3$ equipped with a monotone Kirillov-Kostant-Souriau symplectic form. We detect several non-displaceable non-toric Lagrangian GC fibers diffeomorphic to $U(m) \times T^s$. For this purpose, we consider line segments connecting the center of the polytope and the centers of certain faces having Lagrangian fibers. Using Lagrangian Floer theory deformed by ambient cycles developed by the third named author with Fukaya, Ohta and Ono [FOOO1], every toric fiber over the line segments is shown to be non-displaceable.

More specifically, for deformation we employ Schubert cycles, which degenerate into unions of toric subvarieties corresponding to facets of a moment polytope according to Kogan, Kogan-Miller [Ko, KoM]. Then, non-displaceability follows because non-toric fibers are realized as the Hausdorff limit of non-displaceable toric fibers. In the particular case for $\mathcal{F}(3)$, we completely classify the non-displaceable GC fibers, answering questions of Nohara-Ueda [NU2] and Pabiniak [Pa].

We hope that our detailed description of the geometry of Lagrangian fibers will have similar effect on the Floer theory and homological mirror symmetry to that of Cho and the third named author's article [CO], especially as the testing ground of homological mirror symmetry for the non-toric Fano manifolds. For instance, it is expected to be used to recover Rietsch's Landau-Ginzburg mirror [Ri] that is preferred in the aspect of closed mirror symmetry. It consists of a partial compactification of an algebraic torus together with a holomorphic function on it. Smoothing faces containing a face admitting Lagrangians leads to a Lagrangian torus fibration with singular fibers. By gluing *formal* deformation spaces of immersed Lagrangians and Lagrangian tori in the scheme of Cho-Hong-Lau [CHL], one can recover the mirror and then the category of matrix factorizations should serve as the mirror of the Fukaya category of partial flag manifolds. It will be invesigated by the second named author with Hong and Lau.

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Part 1. The topology of Gelfand-Cetlin fibers

2. Introduction to Part 1.

A partial flag manifold arises as the orbit $\mathcal{O}_\lambda$ of an element λ in the dual Lie algebra of $\mathfrak{u}(n)$ under the co-adjoint action of the unitary group $U(n)$ equipped with a $U(n)$-invariant Kähler form ω_λ (unique up to scaling), so-called a *Kirillov-Kostant-Souriau symplectic form* (*KKS form* for short). On a co-adjoint orbit, Guillemin and Sternberg [GS2] built a completely integrable system

$$\Phi_\lambda \colon (\mathcal{O}_\lambda, \omega_\lambda) \to \mathbb{R}^{\dim_\mathbb{C} \mathcal{O}_\lambda},$$

which is called a *Gelfand-Cetlin system*, or a *GC system*. The image of $\mathcal{O}_\lambda$ under Φ_λ is a polytope, denoted by Δ_λ. We call it a *Gelfand-Cetlin polytope*, abbreviated as a *GC polytope*.

The first part of this article studies the topology of the fibers of a GC system. By the celebrated Arnold-Liouville theorem, the fiber over any interior point is a Lagrangian torus. Moreover, the fiber over each point not in the interior of any GC polytope turns out to be a smooth isotropic manifold, see Theorem A. It is notable because there is almost no control on the fiber over a non-regular value of a general completely integrable system. Furthermore, a GC fiber not in the interior can be Lagrangian, which shows one marked difference between GC systems and toric moment maps (on toric manifolds). Table 1 summarizes similarities and differences between them.

<table>
<tr><th></th><th>Gelfand-Cetlin fiber</th><th>Toric moment fiber</th></tr>
<tr><td>over any point</td><td colspan="2">isotropic submanifold</td></tr>
<tr><td>over any interior point</td><td colspan="2">Lagrangian torus</td></tr>
<tr><td rowspan="3">over any point in the relative interior of a k-dim face</td><td colspan="2">$\pi_1(\text{fiber}) = \mathbb{Z}^k, \quad \pi_2(\text{fiber}) = 0$</td></tr>
<tr><td>not necessarily torus</td><td>k-dimensional torus</td></tr>
<tr><td>can be Lagrangian</td><td>can *not* be Lagrangian</td></tr>
</table>

TABLE 1. Features of Gefand-Cetlin fibers and toric fibers

To show the above, we will describe each GC fiber as the total space of the iterated bundle constructed by playing a game with various "LEGO® blocks". A game manual will be provided in Section 5 and 6. The first main result of the article, obtained from the game, is stated as follows.

Theorem A (Theorem 5.12)**.** *Let Φ_λ be the Gelfand-Cetlin system on the co-adjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ for $\lambda \in \mathfrak{u}(n)^*$ and let Δ_λ be the corresponding Gelfand-Cetlin polytope. For any point $\mathbf{u} \in \Delta_\lambda$, the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is an isotropic submanifold of $(\mathcal{O}_\lambda, \omega_\lambda)$ and is the total space of an iterated bundle*

$$\Phi_\lambda^{-1}(\mathbf{u}) = E_{n-1} \xrightarrow{p_{n-1}} E_{n-2} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_2} E_1 \xrightarrow{p_1} E_0 = \text{point}$$

such that the fiber at each stage is either a point or a product of odd dimensional spheres. Two fibers $\Phi_\lambda^{-1}(\mathbf{u}_1)$ and $\Phi_\lambda^{-1}(\mathbf{u}_2)$ are diffeomorphic if two points $\mathbf{u}_1$ and $\mathbf{u}_2$ are contained in the relative interior of the same face.

A GC system Φ_λ is known to admit a toric degeneration from Φ_λ to a toric integrable system Φ on a toric variety. On the toric degeneration of algebraic varieties in stages constructed by Kogan and Miller [KoM], Nishinou, Nohara and Ueda constructed the degeneration (see Section 7) of the GC system Φ_λ into the moment map Φ of the (singular) projective toric variety X_λ associated with Δ_λ using Ruan's technique [Ru] of gradient-Hamiltonian flows, see Theorem 1.2 in [NNU1]. Then, we obtain the following commutative diagram

$$\begin{tikzcd} (\mathcal{O}_\lambda, \omega_\lambda) \arrow[rr, "\phi"] \arrow[dr, "\Phi_\lambda"'] & & (X_\lambda, \omega) \arrow[dl, "\Phi"] \\ & \Delta_\lambda & \end{tikzcd} \tag{2.1}$$

where ϕ is a continuous map from $\mathcal{O}_\lambda$ onto X_λ. Moreover, the map ϕ induces a symplectomorphism from $\Phi_\lambda^{-1}(\mathring{\Delta}_\lambda)$ to $\Phi^{-1}(\mathring{\Delta}_\lambda)$ where $\mathring{\Delta}_\lambda$ is the interior of Δ_λ. As a consequence, one can see that each fiber located at $\mathring{\Delta}_\lambda$ is a Lagrangian torus.

According to the work of Batyrev, Ciocan-Fontanine, Kim, and Van Straten [BCKV], this degeneration process given by the map ϕ in (2.1) can be interpreted as a smoothing of conifold staratum, which is (a half of) a *conifold* transition. Namely, through the map ϕ, a partial flag manifold is deformed into a singular toric variety having conifold strata. The following theorem indeed visualizes how each GC fiber degenerates into a toric fiber. Specifically, through ϕ, every odd-dimensional sphere of dimension > 1 appeared in each stage of the iterated bundle $\{E_\bullet\}$ simultaneously contracts to a point and each S^1-factor persists.

Theorem B (Theorem 7.9)**.** *Let $\mathbf{u}$ be a point lying on the relative interior of an r-dimensional face. Then every S^1-factor appeared in any stage of the iterated bundle given in Theorem A comes out as a trivial factor so that*

$$\Phi_\lambda^{-1}(\mathbf{u}) \cong T^r \times B(\mathbf{u})$$

where $B(\mathbf{u})$ is the iterated bundle obtained from the original bundle by removing all S^1-factors. Moreover, the map $\phi \colon \Phi_\lambda^{-1}(\mathbf{u}) \to \Phi^{-1}(\mathbf{u})$ is nothing but the projection $T^r \times B(\mathbf{u}) \to T^r$ on the first factor.

Because of Theorem A, other fibers located in the interior of a face f of Δ_λ are all Lagrangian as soon as the fiber of one single point at the interior of f is Lagrangian. In this sense, it is reasonable to call such a face f a *Lagrangian face*. Since every fiber is isotropic again by Theorem A, in order to show that a face f is Lagrangian, it suffices to check that the dimension of the fiber over a point in the interior of f is exactly the half of the real dimension of $\mathcal{O}_\lambda$. We will classify all Lagrangian faces of a GC polytope by providing a refined combinatorial procedure testing whether a face is Lagrangian or not, see Corollary 5.23.

The remaing parts of Part I is organized as follows. In Section 3, we review a construction and properties of GC systems. Section 4 discusses the face structure of a GC polytope in terms of certain graphs of the ladder diagram associated with the polytope. Section 5 is devoted to introduce combinatorics that will be used to describe the GC fibers and classify all Lagrangian faces. In Section 6, we provide the proof of Theorem A by relating the combinatorics on ladder diagrams to the topology of fibers. Finally, the proof of Theorem B will be given in Section 7.

3. Gelfand-Cetlin systems

In this section, we briefly overview Gelfand-Cetlin systems on partial flag manifolds.

For a positive integer $r \in \mathbb{N}$, let $n_0, n_1, \cdots, n_r, n_{r+1}$ be a finite sequence of non-negative integers such that

$$0 = n_0 < n_1 < n_2 < \cdots < n_r < n_{r+1} = n. \tag{3.1}$$

The *partial flag manifold* $\mathcal{F}(n_1, \cdots, n_r; n)$ is the space of nested sequences of complex vector subspaces whose dimensions are $n_1, \cdots, n_r$, respectively. That is,

$$\mathcal{F}(n_1, \cdots, n_r; n) = \{V_\bullet := 0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n \mid \dim_\mathbb{C} V_i = n_i \text{ for } i = 1, \cdots, r\}.$$

An element $V_\bullet$ of $\mathcal{F}(n_1, \cdots, n_r; n)$ is called a *flag*.

Note that the linear $U(n)$-action on $\mathbb{C}^n$ induces a transitive $U(n)$-action on $\mathcal{F}(n_1, \cdots, n_r; n)$ and each flag has an isotropy subgroup isomorphic to $U(k_1) \times \cdots \times U(k_{r+1})$ where

$$k_i = n_i - n_{i-1} \tag{3.2}$$

for $i = 1, \cdots, r+1$. Thus, $\mathcal{F}(n_1, \cdots, n_r; n)$ is a homogeneous space diffeomorphic to $U(n)/(U(k_1) \times \cdots \times U(k_{r+1}))$. In particular, we have

$$\dim_\mathbb{R} \mathcal{F}(n_1, \cdots, n_r; n) = n^2 - \sum_{i=1}^{r+1} k_i^2. \tag{3.3}$$

For notational simplicity, we denote by $\mathcal{F}(n)$ the *complete flag manifold* $\mathcal{F}(1, 2, \cdots, n-1; n)$.

3.1. Description of $\mathcal{F}(n_1, \cdots, n_r; n)$ as a co-adjoint orbit of $U(n)$.

Consider the conjugate action of $U(n)$ on itself. The action fixes the identity matrix $I_n \in U(n)$, which induces the $U(n)$-action, called the *adjoint action* and denoted by Ad, on the Lie algebra $\mathfrak{u}(\mathfrak{n}) := T_{I_n} U(n)$. Note that $\mathfrak{u}(n)$ is the set of $(n \times n)$ skew-hermitian matrices

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) \mid A^* = -A\}, \quad A^* = \overline{A}^t$$

and the adjoint action can be written as

$$\begin{array}{rccc} Ad\colon & U(n) \times \mathfrak{u}(n) & \to & \mathfrak{u}(n) \\ & (M, A) & \mapsto & MAM^*. \end{array}$$

The *co-adjoint action* Ad^* of $U(n)$ is the action on the dual Lie algebra $\mathfrak{u}(n)^*$ induced by Ad, explicitly given by

$$\begin{array}{rccc} Ad^*\colon & U(n) \times \mathfrak{u}(n)^* & \to & \mathfrak{u}(n)^* \\ & (M, X) & \mapsto & X_M \end{array}$$

where $X_M \in \mathfrak{u}(n)^*$ is defined by $X_M(A) = X(M^*AM)$ for every $A \in \mathfrak{u}(n)$.

Proposition 3.1 (p.51 in [Au]). *Let $\mathcal{H}_n = i\mathfrak{u}(n) \subset M_n(\mathbb{C})$ be the set of $(n \times n)$ hermitian matrices with the conjugate $U(n)$-action. Then there is a $U(n)$-equivariant $\mathbb{R}$-vector space isomorphism $\phi\colon \mathcal{H}_n \to \mathfrak{u}(n)^*$.*

Henceforth, we always think of the co-adjoint action of $U(n)$ on $\mathfrak{u}(n)^*$ as the conjugate $U(n)$-action on $\mathcal{H}_n$. Let $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ be a non-increasing sequence of real numbers such that

$$\lambda_1 = \cdots = \lambda_{n_1} > \lambda_{n_1+1} = \cdots = \lambda_{n_2} > \cdots > \lambda_{n_r+1} = \cdots = \lambda_{n_{r+1}} (= \lambda_n) \tag{3.4}$$

and let $I_\lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) \in \mathcal{H}_n$ be the diagonal matrix whose i-th diagonal entry is λ_i for $i = 1, \cdots, n$. Then the isotropy subgroup of I_λ is given by

$$\begin{pmatrix} U(k_1) & 0 & \cdots & 0 \\ 0 & U(k_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U(k_{r+1}) \end{pmatrix} \subset U(n).$$

If we denote by $\mathcal{O}_\lambda$ the $U(n)$-orbit of I_λ, then we have

$$\mathcal{O}_\lambda \cong U(n)/\left(U(k_1) \times \cdots \times U(k_{r+1})\right),$$

which is diffeomorphic to $\mathcal{F}(n_1, \cdots, n_r; n)$. We call $\mathcal{O}_\lambda$ the *co-adjoint orbit associated with eigenvalue pattern* λ.

Remark 3.2. Any two similar matrices have the same eigenvalues with the same multiplicities and any hermitian matrix is unitarily diagonalizable. Thus the co-adjoint orbit $\mathcal{O}_\lambda$ is the set of all hermitian matrices having the eigenvalue pattern λ respecting the multiplicities.

3.2. **Symplectic structure on a co-adjoint orbit $\mathcal{O}_\lambda$.**

For any compact Lie group G with the Lie algebra $\mathfrak{g}$ and for any dual element $\lambda \in \mathfrak{g}^*$, there is a *canonical* G-invariant symplectic form ω_λ, called the *Kirillov-Kostant-Souriau symplectic form* (*KKS form* shortly), on the orbit $\mathcal{O}_\lambda$. Furthermore, $\mathcal{O}_\lambda$ admits a unique G-invariant Kähler metric compatible with ω_λ, and therefore $(\mathcal{O}_\lambda, \omega_\lambda)$ forms a Kähler manifold. We refer the reader to [Br, p.150] for more details.

The KKS form ω_λ can be described more explicitly in the case where $G = U(n)$ as below. For each $h \in \mathcal{H}_n$, we define a real-valued skew-symmetric bilinear form $\widetilde{\omega}_h$ on $\mathfrak{u}(n) = i\mathcal{H}_n$ by

$$\widetilde{\omega}_h(X, Y) := \mathrm{tr}(ih[X, Y]) = \mathrm{tr}(iY[X, h]), \quad X, Y \in \mathfrak{u}(n).$$

The kernel of $\widetilde{\omega}_h$ is then

$$\ker \widetilde{\omega}_h = \{X \in \mathfrak{u}(n) \mid [X, h] = 0\}.$$

Since $\mathcal{O}_\lambda$ is a homogeneous $U(n)$-space, we may express each tangent space $T_h\mathcal{O}_\lambda$ at a point $h \in \mathcal{O}_\lambda$ as

$$T_h\mathcal{O}_\lambda = \{[X, h] \in T_h\mathcal{H}_n = \mathcal{H}_n \mid X \in \mathfrak{u}(n)\}.$$

Then we get a non-degenerate two form ω_λ on $\mathcal{O}_\lambda$ defined by

$$(\omega_\lambda)_h([X, h], [Y, h]) := \widetilde{\omega}_h(X, Y), \quad h \in \mathcal{O}_\lambda, \quad X, Y \in \mathfrak{u}(n).$$

The closedness of ω_λ then follows from the Jacobi identity on $\mathfrak{u}(n)$, see [Au, p.52] for instance.

Remark 3.3. The diffeomorphism type of $\mathcal{O}_\lambda$ does not depend on the choice of λ but on $k_\bullet$'s. However, the symplectic form ω_λ depends on the choice of λ. For instance, two co-adjoint orbits $\mathcal{O}_\lambda$ and $\mathcal{O}_{\lambda'}$ have $k_1 = k_2 = 1$ when $\lambda = \{1, -1\}$ and $\lambda' = \{1, 0\}$ and both orbits are diffeomorphic to $U(2)/\left(U(1) \times U(1)\right) \cong \mathbb{P}^1$. However, the symplectic area of $(\mathcal{O}_\lambda, \omega_\lambda)$ and $(\mathcal{O}_{\lambda'}, \omega_{\lambda'})$ are one and two, respectively.

Any partial flag manifold is known to be a Fano manifold and hence it admits a monotone Kähler form. The following proposition gives a complete description of the monotonicity of ω_λ.

Proposition 3.4 (p.653-654 in [NNU1])**.** *The symplectic form ω_λ on $\mathcal{O}_\lambda$ satisfies*

$$c_1(T\mathcal{O}_\lambda) = [\omega_\lambda]$$

if and only if

$$\lambda = (\underbrace{n - n_1, \cdots}_{k_1}, \underbrace{n - n_1 - n_2, \cdots}_{k_2}, \cdots, \underbrace{n - n_{r-1} - n_r, \cdots}_{k_r}, \underbrace{-n_r, \cdots, -n_r}_{k_{r+1}}) + (\underbrace{m, \cdots, m}_{n}),$$

for some $m \in \mathbb{R}$.

3.3. Completely integrable system on $\mathcal{O}_\lambda$.

We adorn a co-adjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ with a completely integrable system, called the Gelfand-Cetlin system. We recall a standard definition of a completely integrable system.

Definition 3.5. A *completely integrable system* on a $2n$-dimensional symplectic manifold (M, ω) is an n-tuple of smooth functions

$$\Phi := (\Phi_1, \cdots, \Phi_n) \colon M \to \mathbb{R}^n$$

such that

(1) $\{\Phi_i, \Phi_j\} = 0$ for every $1 \leq i, j \leq n$ and
(2) $d\Phi_1, \cdots, d\Phi_n$ are linearly independent on an open dense subset of M.

If Φ is a proper map, the Arnold-Liouville theorem states that for any regular value $\mathbf{u} \in \mathbb{R}^n$ of Φ the preimage $\Phi^{-1}(\mathbf{u})$ is a Lagrangian torus. However, if $\mathbf{u}$ is a critical value, the fiber might not be a manifold in general.

Harada and Kaveh [HaK] proved that any smooth projective variety equips a completely integrable system whenever it admits a flat toric degeneration. (See Section 7 for more details.) However, the terminology "completely integrable system" used in [HaK] is a weakened version of Definition 3.5 in the following sense.

Definition 3.6. A *(continuous) completely integrable system* on a $2n$-dimensional symplectic manifold (M, ω) is a collection of n *continuous* functions

$$\Phi := \{\Phi_1, \cdots, \Phi_n\} \colon M \to \mathbb{R}^n$$

such that there exists an open dense subset $\mathcal{U}$ of M on which

(1) each Φ_i is smooth,
(2) $\{\Phi_i, \Phi_j\} = 0$ for every $1 \leq i, j \leq n$, and
(3) $d\Phi_1, \cdots, d\Phi_n$ are linearly independent.

For any co-adjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$, Guillemin and Sternberg [GS2] constructed a completely integrable system (in the sense of Definition 3.6)

$$\Phi_\lambda \colon \mathcal{O}_\lambda \to \mathbb{R}^{\dim_{\mathbb{C}} \mathcal{O}_\lambda},$$

called *the Gelfand-Cetlin system on* $\mathcal{O}_\lambda$ (*GC system* for short) with respect to the KKS symplectic form ω_λ. The GC system on $\mathcal{O}_\lambda$ is in general continuous but not smooth. From now on, a completely integrable system will be meant to be a conitnuous completely integrable system in Definition 3.6 unless mentioned.

We briefly recall a construction of the GC system on $\mathcal{O}_\lambda$ as follows. (See also [NNU1, p.7-9].) Let $n_\bullet$'s and $k_\bullet$'s be given in (3.1) and (3.2), respectively, and let λ be a non-increasing sequence satisfying (3.4). From (3.3), it follows that

$$\dim_{\mathbb{R}} \mathcal{O}_\lambda = 2 \dim_{\mathbb{C}} \mathcal{O}_\lambda = n^2 - \sum_{i=1}^{r+1} k_i^2.$$

For any $x \in \mathcal{O}_\lambda \subset \mathcal{H}_n$, let $x^{(k)}$ be the $(k \times k)$ leading principal minor of x for each $k = 1, \cdots, n-1$. Since $x^{(k)}$ is also a hermitian matrix, the eigenvalues are all real. Let

$$\mathcal{I} = \{(i,j) \mid i, j \in \mathbb{N}, \ i + j \leq n + 1\} \tag{3.5}$$

be an index set. We then define the real-valued function

$$\Phi_\lambda^{i,j} \colon \mathcal{O}_\lambda \to \mathbb{R}, \quad (i,j) \in \mathcal{I}$$

where $\Phi_\lambda^{i,j}(x)$ is assigned to be the i-th largest eigenvalue of $x^{(i+j-1)}$. Note that the eigenvalues of $x^{(k)}$ are arranged by

$$\Phi_\lambda^{1,k}(x) \geq \Phi_\lambda^{2,k-1}(x) \geq \cdots \geq \Phi_\lambda^{k,1}(x)$$

in the descending order. Collecting all $\Phi_\lambda^{i,j}$'s for $(i,j) \in \mathcal{I}$, we obtain Φ_λ.

Definition 3.7. Let λ be given in (3.4). The *Gelfand-Cetlin system* Φ_λ *associated with* λ is defined by the collection of real-valued functions

$$\Phi_\lambda := \left(\Phi_\lambda^{i,j}\right)_{(i,j)\in\mathcal{I}} : \mathcal{O}_\lambda \to \mathbb{R}^{\frac{n(n+1)}{2}}. \tag{3.6}$$

Remark 3.8. We label each component of Φ_λ by multi-index $(i,j) \in \mathcal{I}$ such that $\Phi_\lambda^{i,j}$ corresponds to the lattice point $(i,j) \in \mathbb{Z}^2$ in a ladder diagram in Definition 4.1. (See also Figure 9.) Notice that the labeling of components of Φ_λ used in [NNU1] is different from ours.

Now, we consider the coordinate system of $\mathbb{R}^{n(n+1)/2}$

$$\left\{(u_{i,j}) \in \mathbb{R}^{n(n+1)/2} \mid (i,j) \in \mathcal{I}\right\} \tag{3.7}$$

where each component $u_{i,j}$ records the values of $\Phi_\lambda^{i,j}$. Fix $x \in \mathcal{O}_\lambda$ and let $(u_{i,j})_{(i,j)\in\mathcal{I}} = \Phi_\lambda(x)$, that is, $u_{i,j} = \Phi_\lambda^{i,j}(x)$ for each $(i,j) \in \mathcal{I}$. By the min-max principle, the components $u_{i,j}$'s satisfy the following pattern:

$$\begin{array}{ccccccccccc}
\lambda_1 & & & & \lambda_2 & & & \lambda_3 & \cdots & \lambda_{n-1} & & & \lambda_n \\
 & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\geq$} & & & & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\geq$} & \\
 & & u_{1,n-1} & & & u_{2,n-2} & & & & & & u_{n-1,1} & \\
 & & & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\geq$} & & & & & & \rotatebox{45}{$\geq$} & & \\
 & & & & u_{1,n-2} & & & & & u_{n-2,1} & & & \\
 & & & & & \rotatebox{-45}{$\geq$} & & & \rotatebox{45}{$\geq$} & & & & \\
 & & & & & & \ddots & \iddots & & & & & \\
 & & & & & & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\geq$} & & & & & \\
 & & & & & & & u_{1,1} & & & & &
\end{array} \tag{3.8}$$

Note that each $\Phi_\lambda^{j,n+1-j}$ assigns the j-th largest eigenvalue of $x^{(n)} = x$, and therefore $\Phi_\lambda^{j,n+1-j}(x) = \lambda_j$ for every $x \in \mathcal{O}_\lambda$ and $j = 1, \cdots, n$. Furthermore, by our assumption (3.4), we have $\lambda_{n_{i-1}+1} = \cdots = \lambda_{n_i}$ for every $i = 1, \cdots, r+1$. Then it is an immediate consequence of (3.8) that for all $j = n_{i-1}+1, \cdots, n_i - 1$ and each $k = j, \cdots, n_i - 1$, we have

$$\Phi_\lambda^{j,n+1-k}(x) = \lambda_{n_i}$$

for all $x \in \mathcal{O}_\lambda$. Thus, there are exactly $\frac{1}{2}(n^2 - \sum_{i=1}^{r+1} k_i^2)$ non-constant functions among $\{\Phi_\lambda^{j,k}\}_{(j,k)\in\mathcal{I}}$ on $\mathcal{O}_\lambda$. Let

$$\mathcal{I}_\lambda := \{(i,j) \in \mathcal{I} \mid \Phi_\lambda^{i,j} \text{ is } \textit{not} \text{ a constant function on } \mathcal{O}_\lambda\}. \tag{3.9}$$

Collecting all non-constant components of Φ_λ, we rename

$$\Phi_\lambda = \left(\Phi_\lambda^{i,j}\right)_{(i,j)\in\mathcal{I}_\lambda} : \mathcal{O}_\lambda \to \mathbb{R}^{|\mathcal{I}_\lambda|}, \quad |\mathcal{I}_\lambda| = \dim_\mathbb{C} \mathcal{O}_\lambda = \frac{1}{2}(n^2 - \sum_{i=1}^{r+1} k_i^2) \tag{3.10}$$

as the *Gelfand-Cetlin system*. By abuse of notation, the collection is still denoted by Φ_λ. Guillemin and Sternberg [GS2] prove that Φ_λ satisfies all properties given in Definition 3.6, and hence it is a completely integrable system on $\mathcal{O}_\lambda$ [1]. We will not distinguish two notations (3.6) and (3.10) unless any confusion arises.

Definition 3.9. The *Gelfand-Cetlin polytope*[2] Δ_λ is the collection of points $(u_{i,j})$ satisfying (3.8). A Gefland-Cetlin polytope will be abbreviated to a *GC polytope* for simplicity.

Indeed, we have another description of a GC polytope.

Proposition 3.10 ([GS2])**.** *The Gelfand-Cetlin polytope* Δ_λ *coincides with the image of* $\mathcal{O}_\lambda$ *under* Φ_λ.

---

[1] In general, the Gelfand-Cetlin system is never smooth on the whole space $\mathcal{O}_\lambda$ unless $\mathcal{O}_\lambda$ is a projective space.

[2] It is straightforward to see that Δ_λ is a convex polytope, since Δ_λ is the intersection of half-spaces defined by inequalities in (3.8).

3.4. Smoothness of Φ_λ.

Let λ be given in (3.4) and let Φ_λ be the GC system on $(\mathcal{O}_\lambda, \omega_\lambda)$. In general, Φ_λ is *not* smooth on the whole $\mathcal{O}_\lambda$. However, the following proposition due to Guillemin-Sternberg states that Φ_λ is smooth on $\mathcal{O}_\lambda$ *almost everywhere*.

Proposition 3.11 (Proposition 5.3, p.113, and p.122 in [GS2]). *For each $(i,j) \in \mathcal{I}_\lambda$, the component $\Phi_\lambda^{i,j}$ is smooth at $z \in \mathcal{O}_\lambda$ if*

$$\Phi_\lambda^{i+1,j}(z) < \Phi_\lambda^{i,j}(z) < \Phi_\lambda^{i,j+1}(z). \tag{3.11}$$

In particular, Φ_λ is smooth on the open dense subset $\Phi_\lambda^{-1}(\mathring{\Delta}_\lambda)$ of $\mathcal{O}_\lambda$ where $\mathring{\Delta}_\lambda$ is the interior of Δ_λ. Furthermore, $d\Phi_\lambda^{i,j}(z) \neq 0$ for every point z satisfying (3.11).

One important remark is that a Hamiltonian trajectory of each $\Phi_\lambda^{i,j}$ passing through a point $z \in \mathcal{O}_\lambda$ satisfying (3.11) is periodic with integer period. Therefore, each $\Phi_\lambda^{i,j}$ generates a Hamiltonian circle action on the subset of $\mathcal{O}_\lambda$ on which $\Phi_\lambda^{i,j}$ is smooth. See [GS2, Theorem 3.4 and Section 5] for more details.

4. Ladder diagram and its face structure

In order to visualize a GC polytope, it is convenient to employ an alternative description of its face structure in terms of certain graphs in the ladder diagram provided by the first named author with An and Kim in [ACK]. The goal of this section is to review the description of the face structure.

We begin by the definition of a ladder diagram. Let $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ be given in (3.4). Then λ uniquely determines $n_\bullet$'s and $k_\bullet$'s in (3.1) and (3.2), respectively.

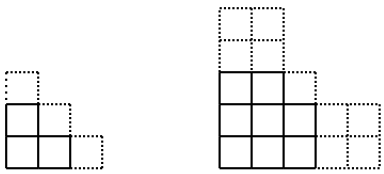
Definition 4.1 ([BCKV], [NNU1]). Let $\Gamma_{\mathbb{Z}^2} \subset \mathbb{R}^2$ be the square grid graph satisfying

1. its vertex set is $\mathbb{Z}^2 \subset \mathbb{R}^2$ and
2. each vertex $(a,b) \in \mathbb{Z}^2$ connects to exactly four vertices $(a, b\pm 1)$ and $(a\pm 1, b)$.

The *ladder diagram* $\Gamma(n_1, \cdots, n_r; n)$ is defined as the induced subgraph of $\Gamma_{\mathbb{Z}^2}$ that is formed from the set $V_{\Gamma(n_1,\cdots,n_r;n)}$ of vertices given by

$$V_{\Gamma(n_1,\cdots,n_r;n)} := \bigcup_{j=0}^{r} \left\{ (a,b) \in \mathbb{Z}^2 \;\middle|\; (a,b) \in [n_j, n_{j+1}] \times [0, n - n_{j+1}] \right\}.$$

As λ determines $n_\bullet$'s, we simply denote $\Gamma(n_1, \cdots, n_r; n)$ by Γ_λ. We call Γ_λ the *ladder diagram associated with* λ.



$\Gamma(1,2,3)$ $\Gamma(2,3,5)$

FIGURE 1. Ladder diagrams $\Gamma(1,2;3)$ and $\Gamma(2,3;5)$.

We call the vertex of Γ_λ located at $(0,0)$ *the origin*. Also, we call $v \in V_\Gamma$ a *top vertex* if v is a farthest vertex from the origin with respect to the taxicab metric. Equivalently, a vertex $v = (a,b) \in V_\Gamma$ is a top vertex if $a+b = n$.

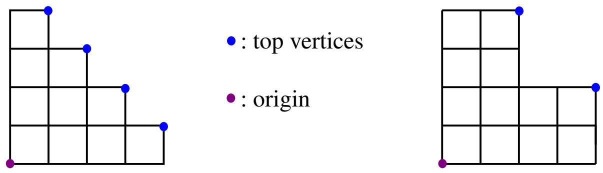


FIGURE 2. Top vertices for $\Gamma(1,2,3,4;5)$ and $\Gamma(2,4;6)$.

Definition 4.2 (Definition 2.2 in [BCKV]). A *positive path* on a ladder diagram Γ_λ is a shortest path from the origin to some top vertex in Γ_λ.

Now, we define the face structure of Γ_λ as follows.

Definition 4.3 (Definition 1.5 in [ACK]). Let Γ_λ be a ladder diagram.

- A subgraph γ of Γ_λ is called a *face* of Γ_λ if
  (1) γ contains all top vertices of Γ_λ,
  (2) γ can be represented by a union of positive paths.
- For two faces γ and γ' of Γ_λ, γ is said to be a *face* of γ' if $\gamma \subset \gamma'$.
- The *dimension* of a face γ is defined by

$$\dim \gamma := \operatorname{rank}_{\mathbb{Z}} H_1(\gamma; \mathbb{Z}),$$

  regarding γ as a 1-dimensional CW-complex. In other words, $\dim \gamma$ is the number of minimal cycles in γ.

It is straightforward from Definition 4.3 that for any two faces γ and γ' of Γ_λ, $\gamma \cup \gamma'$ is also a face of Γ_λ, which is the smallest face containing γ and γ'.

We now characterize the face structure of the GC polytope Δ_λ in terms of the face structure of the ladder diagram Γ_λ.

Theorem 4.4 (Theorem 1.9 in [ACK]). *For a sequence λ of real numbers given in (3.4), let Γ_λ be the ladder diagram and Δ_λ be the Gelfand-Cetlin polytope. Then there exists a bijective map*

$$\{\textit{faces of } \Gamma_\lambda\} \xrightarrow{\Psi} \{\textit{faces of } \Delta_\lambda\}$$

such that for faces γ and γ' of Γ_λ

- *(Order-preserving) $\gamma \subset \gamma'$ if and only if $\Psi(\gamma) \subset \Psi(\gamma')$,*
- *(Dimension) $\dim \Psi(\gamma) = \dim \gamma$.*

Example 4.5. Let $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_1 > \lambda_2 > \lambda_3$. The co-adjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ is diffeomorphic to the complete flag manifold $\mathcal{F}(3)$. Let Γ_λ be the ladder diagram associated with λ as in Figure 3. Here, the blue dots are top vertices and the purple dot is the origin of Γ_λ.

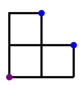


FIGURE 3. Ladder diagram Γ_λ.

The zero, one, two, and three-dimensional faces of Γ_λ are respectively listed in Figure 4, 5, 6, and 7. Here, v_i denotes a vertex for $i \in \{1, \cdots, 7\}$, e_{ij} is the edge containing v_i and v_j, f_I is the facet containing all v_i's for $i \in I$, and $I_{1234567}$ is the three dimensional face, i.e., the whole Γ_λ.

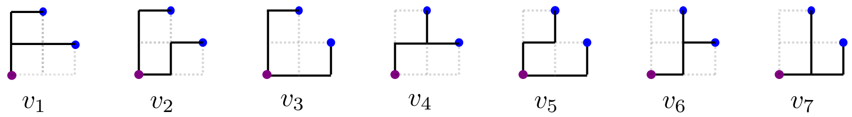


FIGURE 4. The zero-dimensional faces of Γ_λ.

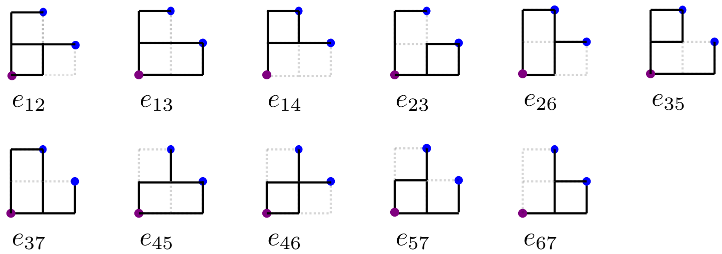


FIGURE 5. The one-dimensional faces of Γ_λ.

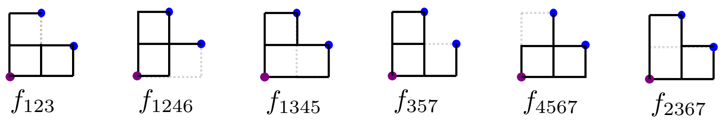


FIGURE 6. The two-dimensional faces of Γ_λ.

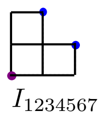


FIGURE 7. The three-dimensional face of Γ_λ.

The GC polytope Δ_λ is given in Figure 8. (See Figure 5 in [Ko] or Figure 4 in [NNU1].) We can easily see that the correspondence $\Psi(v_i) = w_i$ of vertices naturally extends to the set of faces of Γ_λ, satisfying (1) and (2) in Theorem 4.4.

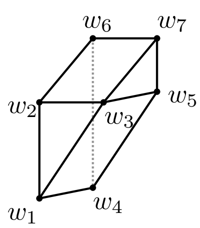


FIGURE 8. The GC polytope Δ_λ for $\lambda = \{\lambda_1 > \lambda_2 > \lambda_3\}$.

For our convenience, we describe each point in Δ_λ by using Γ_λ *with a filling*, putting each component $u_{i,j}$ of the coordinate system (3.7) of $\mathbb{R}^{|\mathcal{I}|} = \mathbb{R}^{\frac{n(n+1)}{2}}$ into the unit box whose top-right vertex is (i,j) of Γ_λ. The GC pattern (3.8) implies that $\{u_{i,j}\}_{(i,j)\in\mathcal{I}}$ is

$$\begin{cases} (1) \ \textit{increasing} \text{ along the columns of } \Gamma_\lambda \text{ , } \textit{and} \\ (2) \ \textit{decreasing} \text{ along the rows of } \Gamma_\lambda \end{cases} \tag{4.1}$$

allowing repetitions (cf. a Young tableau in [Ful]).

Example 4.6. Let $\mathcal{O}_\lambda \simeq \mathcal{F}(3)$ be the co-adjoint orbit from Example 4.5. Recall that the pattern (3.8) consists of the following inequalities:

$$u_{1,2} \geq u_{1,1}, \ \ u_{1,1} \geq u_{2,1}, \ \ \lambda_1 \geq u_{1,2}, \ \ u_{1,2} \geq \lambda_2, \ \ \lambda_2 \geq u_{2,1}, \ \ u_{2,1} \geq \lambda_3.$$

The ladder diagram Γ_λ with a filling by variables $u_{i,j}$'s is as in Figure 9.

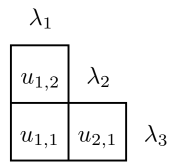


FIGURE 9. The ladder diagram Γ_λ with $u_{i,j}$'s variables.

Now, we explain how the map Ψ in Theorem 4.4 is in general defined. For a given face γ of Γ_λ, consider γ with a filling by the coordinate system $\{u_{i,j}\}$. The image of γ under Ψ is the intersection of facets supported by the hyperplanes that are given by equating two adjacent variables $u_{i,j}$'s *not* divided by any positive paths.

Example 4.7. Suppose that $\lambda = \{4, 4, 3, 2, 1\}$ and let γ be a face given as in Figure 10. Then, the corresponding face $\Psi(\gamma)$ in Δ_λ is defined by

$$\Psi(\gamma) = \Delta_\lambda \cap \{u_{2,1} = u_{1,1}\} \cap \{u_{1,1} = u_{1,2}\} \cap \{u_{2,2} = u_{1,2}\} \cap \{u_{2,1} = u_{2,2}\} \cap \{u_{2,2} = u_{2,3}\} \cap \{u_{3,1} = 4\}.$$

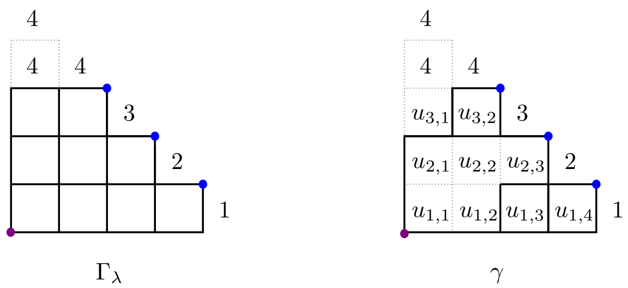


FIGURE 10. The bijection Ψ

5. CLASSIFICATION OF LAGRANGIAN FIBERS

Our first main theorem A, which will be proven in Section 6, states that every fiber of the GC system Φ_λ on a co-adjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ is an isotropic submanifold and is the total space of certain iterated bundle

$$E_{n-1} \stackrel{p_{n-1}}{\longrightarrow} E_{n-2} \stackrel{p_{n-2}}{\longrightarrow} \cdots \stackrel{p_2}{\longrightarrow} E_1 \stackrel{p_1}{\longrightarrow} E_0 = \text{point} \tag{5.1}$$

such that the fiber at each stage is either a point or a product of odd dimensional spheres. In this section, we provide a combinatorial way of "reading off" the topology of the fiber of each projection map p_i from the ladder diagram (Theorem 5.12). Furthermore, we classify all positions of Lagrangian fibers in the Gelfand-Cetlin polytope (Corollary 5.23).

We first consider a $2n$-dimensional compact symplectic toric manifold (M, ω) with a moment map $\Phi\colon M \to \mathbb{R}^n$. It is a smooth completely integrable system on M in the sense of Definition 3.5 and the Atiyah-Guillemin-Sternberg convexity theorem [At, GS1] yields that the image $\Delta := \Phi(M)$ is an n-dimensional convex polytope. It is well-known that for any k-dimensional face f of Δ and a point $\mathbf{u} \in \mathring{f}$ in its relative interior $\mathring{f}$ of f. the fiber $\Phi^{-1}(\mathbf{u})$ is a k-dimensional isotropic torus. In particular, a fiber $\Phi^{-1}(\mathbf{u})$ is Lagrangian if and only if $\mathbf{u} \in \mathring{\Delta}$. .

In contrast, in the GC system case the preimage of a point in the inteior of a k-th dimensional face of the GC polytope Δ_λ might have the dimension greater than k as the torus action does not extend to the whole space. In particular, Φ_λ might admit a Lagrangian fiber over a point not contained in the interior of Δ_λ.

Definition 5.1. We call a face f of Δ_λ *Lagrangian* if it contains a point $\mathbf{u}$ in its relative interior $\mathring{f}$ such that the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is Lagrangian. Also, we call a face γ of Γ_λ *Lagrangian* if the corresponding face $f_\gamma := \Psi(\gamma)$ of Δ_λ is Lagrangian where Ψ is given in Theorem 4.4.

Remark 5.2. We will see later that if $\mathbf{u}$ and $\mathbf{u}'$ are contained in the relative interior of a same face of Δ_λ, then $\Phi_\lambda^{-1}(\mathbf{u})$ and $\Phi_\lambda^{-1}(\mathbf{u}')$ are diffeomorphic. In particular if f is a Lagrangian face of Δ_λ, then every fiber over any point in $\mathring{f}$ is Lagrangian, see Corollary 5.23.

Example 5.3 ([Ko, NNU1])**.** For a complete flag manifold $\mathcal{F}(3) \simeq \mathcal{O}_\lambda$ of complex dimension three from Example 4.5, the fiber $\Phi_\lambda^{-1}(w_3)$ is a Lagrangian submanifold diffeomorphic to S^3 where the vertex w_3 is in Figure 8. See Example 3.8 in [NNU1].

From now on, we tacitly identify faces of the ladder diagram Γ_λ with faces of the Gelfand-Cetlin polytope Δ_λ via the map Ψ in Theorem 4.4. For example, "a point r in a face γ" of Γ_λ means a point r in the face $\Psi(\gamma) = f_\gamma$ of Δ_λ.

5.1. W-shaped blocks and M-shaped blocks.

For each $(a,b) \in \mathbb{Z}^2$, let $\square^{(a,b)}$ be the simple closed region bounded by the unit square in $\mathbb{R}^2$ whose vertices are lattice points in $\mathbb{Z}^2$ and the top-right vertex is (a,b). The region $\square^{(a,b)}$ is simply said to be the *box* at (a,b).

Definition 5.4. For each integer $k \geq 1$, a *k-th W-shaped block* denoted by W_k, or simply *a W_k-block*, is defined by

$$W_k := \bigcup_{(a,b)} \square^{(a,b)}$$

where the union is taken over all (a,b)'s in $(\mathbb{Z}_{\geq 1})^2$ such that $k+1 \leq a+b \leq k+2$. A lattice point closest from the origin (with respect to the taxicab metric) in the W_k-block is called a *bottom vertex*.

The following figures illustrate W_1, W_2, and W_3 where the red dots in each figure indicate the vertices over which the union is taken in Definition 5.4. The purple dots are bottom vertices of each W-blocks.

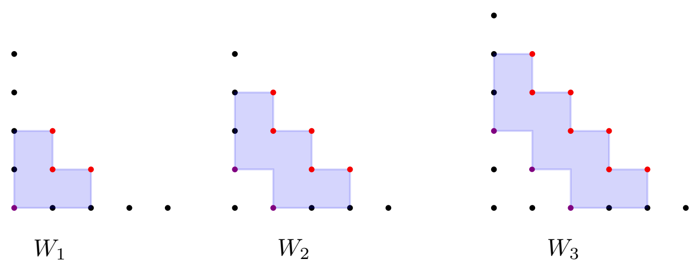


FIGURE 11. W-shaped blocks

For a given diagram Γ_λ, the set of edges of each face γ divides W_k into several pieces of simple closed regions.

Definition 5.5. For the W_k-block and a face γ of Γ_λ, we denote by $W_k(\gamma)$ the W_k-block with 'walls', where a wall is an edge of γ lying on the interior of the W_k-block.

Example 5.6. In Example 4.5, we consider the vertex v_3 of Γ_λ in Figure 4. There are no edges of v_3 inside W_1 and hence $W_1(v_3) = W_1$ with no walls. The W_2-block is divided by v_3 into three pieces of simple closed regions so that $W_2(v_3)$ is W_2 with two walls indicated with red line segments in Figure 12.

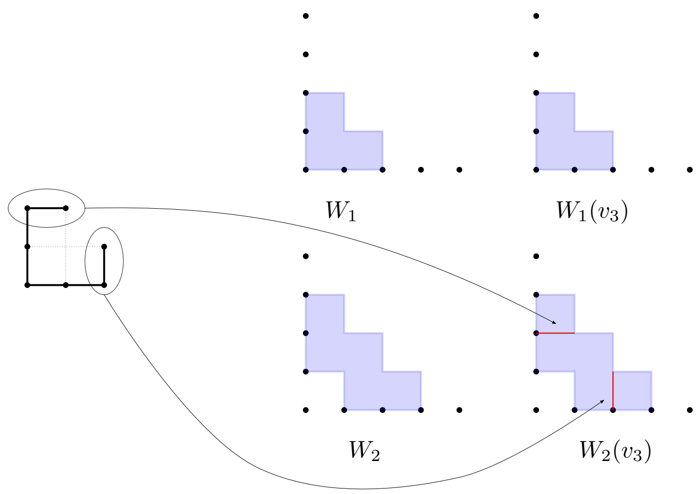


FIGURE 12. $W_i(v_3)$-blocks

Next, we introduce the notion of *M-shaped blocks*.

Definition 5.7. For each positive integer $k \geq 1$, a *k-th M-shaped block* denoted by M_k, or simply an *M_k-block*, is defined, *up to translation in* $\mathbb{R}^2$, by

$$M_k := \bigcup_{(a,b)} \square^{(a,b)}$$

where the union is taken over all (a, b)'s in $(\mathbb{Z}_{\geq 1})^2$ such that

- $k + 1 \leq a + b \leq k + 2$,
- $(a, b) \neq (k + 1, 1)$, and
- $(a, b) \neq (1, k + 1)$.

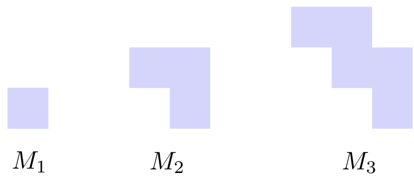


FIGURE 13. M-shaped blocks.

Remark 5.8. Note that M_k can be obtained from W_k by deleting two boxes $\square^{(k+1,1)}$ and $\square^{(1,k+1)}$. The reader should keep in mind that each W-shaped block W_k is located at the specific position, but M_k is not since it is defined up to translation in $\mathbb{R}^2$.

For each divided simple closed region $\mathcal{D}$ in $W_k(\gamma)$, we assign a topological space to $S_k(\mathcal{D})$ by the following rule :

$$S_k(\mathcal{D}) = \begin{cases} S^{2\ell-1} & \text{if } \mathcal{D} \text{ is the } M_\ell\text{-block and } \mathcal{D} \text{ contains at least one bottom vertex of the } W_k\text{-block,} \\ \text{point} & \text{otherwise.} \end{cases}$$

We then put

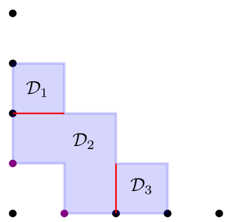
$$S_k(\gamma) := \prod_{\mathcal{D} \subset W_k(\gamma)} S_k(\mathcal{D}) \tag{5.2}$$

where the product is taken over all simple closed regions in $W_k(\gamma)$ distinguished by walls coming from edges of γ.

Example 5.9. Again, consider the vertex v_3 of Γ_λ in Example 5.6. Note that $S_1(v_3) = \text{pt}$ since $W_1(v_3)$ consists of one simple closed region W_1 which is not an M-shaped block. In contrast, $W_2(v_3)$ consists of three simple closed regions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ as in the figure below. (See also Figure 12.) Even if $\mathcal{D}_1$ and $\mathcal{D}_3$ are M_1-blocks, they do not contain any bottom vertices so that $S_2(\mathcal{D}_1) = \text{point}$ and $S_2(\mathcal{D}_3) = \text{point}$. Observe that $\mathcal{D}_2$ is an M_2-block containing a bottom vertex of W_2. Therefore, we have

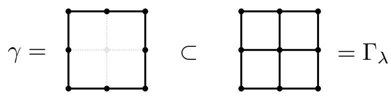
$$S_2(v_3) = S_2(\mathcal{D}_1) \times S_2(\mathcal{D}_2) \times S_2(\mathcal{D}_3) \cong S^3.$$

For $k > 2$, $W_k(v_3)$ has no walls and hence $W_k(v_3)$ consists of one simple closed region W_k which is never an M-shaped block. Thus $S_k(v_3) = \text{point}$ for every positive integer $k > 2$.

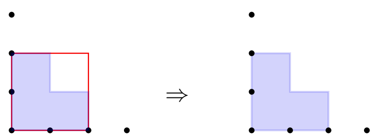


$W_2(v_3)$

Example 5.10. Let $\lambda = \{t, t, 0, 0\}$ with $t > 0$. Then, the co-adjoint orbit $\mathcal{O}_\lambda$ is a complex Grassmannian $\mathrm{Gr}(2, 4)$. Let γ be the one-dimensional face of Γ_λ given by

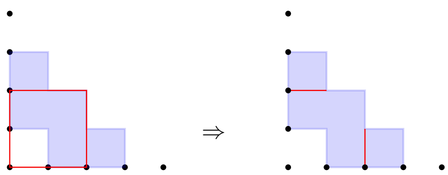


As we see in the following figure, $W_1(\gamma)$ consists of one simple closed region that is not an M-shaped block. Thus we have $S_1(\gamma) = \text{point}$.



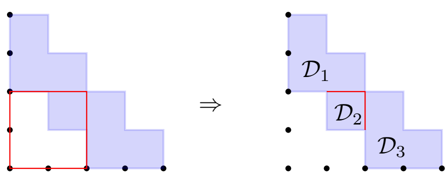
$W_1(\gamma)$

On the other hand, $W_2(\gamma)$ has two walls (red line segments in the figure below) and is exactly the same as $W_2(v_3)$ in Example 5.9. Thus we have $S_2(\gamma) = S^3$.



$W_2(\gamma)$

Finally, $W_3(\gamma)$ has two walls as we see below, and therefore there are three simple closed regions, $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, in $W_3(\gamma)$.



$W_3(\gamma)$

Since $\mathcal{D}_1$ and $\mathcal{D}_3$ are not M-shaped blocks, we have $S_3(\mathcal{D}_1) = S_3(\mathcal{D}_3) = \text{point}$. Since $\mathcal{D}_2 = M_1$ and contains a botton vertex of W_3, we have $S_3(\mathcal{D}_2) = S^1$. Therefore, we have

$$S_3(\gamma) = S_3(\mathcal{D}_1) \times S_3(\mathcal{D}_2) \times S_3(\mathcal{D}_3) \cong S^1.$$

For $k > 3$, $W_k(\gamma)$ consists only one simple closed region, the W_k-block itself, and it is never an M-shaped block. Thus $S_k(\gamma) = \text{point}$ for $k > 3$.

Proposition 5.11. *Let λ be a sequence of real numbers satisfying (3.4) and γ be a face of the ladder diagram Γ_λ. For each $i \geq 1$, let $\{\mathcal{D}_1, \cdots, \mathcal{D}_{m_i}\}$ be simple closed regions in $W_i(\gamma)$ such that $S_i(\mathcal{D}_j) = S^1$ for every $j = 1, \cdots, m_i$. Then we have*

$$\dim \gamma = \sum_{i=1}^{n-1} m_i.$$

Proof. Note that $\dim \gamma$ is the number of minimal cycles in γ by Definition 4.3. Also, each minimal cycle σ in γ can be represented by the union of two shortest paths connecting the bottom-left vertex and the top-right vertex of σ. We denote by v_σ the top-right vertex of σ. Note that $\square^{v_\sigma}$ (blue-colored region in Figure 14) is contained in the simple closed region bounded by σ. Therefore, if we denote by $\Sigma := \{\sigma_1, \cdots, \sigma_m\}$ the set of minimal cycles in γ, then there is a one-to-one correspondence between Σ and $\{v_{\sigma_i}\}_{1 \leq i \leq m}$.

On the other hand, observe that $\square^{v_\sigma}$ is appeared as an M_1-block in $W_i(\gamma)$ where

$$i + 1 = a + b, \quad v_\sigma = (a, b)$$

for each $\sigma \in \Sigma$. Also, every M_1-block appeared in $W_i(\gamma)$ for some i should be one of such $\square^{v_\sigma}$'s. Consequently, there is a one-to-one correspondence between Σ and M_1-blocks appeared in $W_i(\gamma)$ for $i \geq 1$. Since $|\Sigma| = \dim \gamma$ by definition, this completes the proof. $\square$

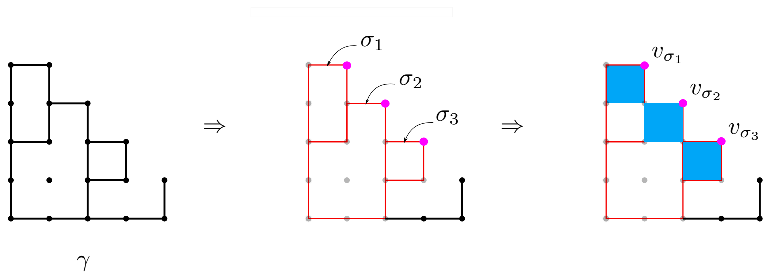


FIGURE 14. Correspondence between minimal cycles and M_1-blocks

Now, we state one of our main theorem which characterizes the topology of each fiber of Φ_λ, where the proof will be given in Section 6.

Theorem 5.12. *Let $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ be a non-increasing sequence of real numbers satisfying (3.4). Let γ be a face of Γ_λ and $f_\gamma = \Psi(\gamma)$ be the corresponding face of Δ_λ described in Theorem 4.4. For any point $\mathbf{u}$ in the relative interior of f_γ, the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is an isotropic submanifold of $(\mathcal{O}_\lambda, \omega_\lambda)$ and is the total space of an iterated bundle*

$$\Phi_\lambda^{-1}(\mathbf{u}) \cong \overline{S_{n-1}}(\gamma) \xrightarrow{p_{n-1}} \overline{S_{n-2}}(\gamma) \to \cdots \xrightarrow{p_2} \overline{S_1}(\gamma) \xrightarrow{p_1} \overline{S_0}(\gamma) := \text{point} \tag{5.3}$$

where $p_k \colon \overline{S_k}(\gamma) \to \overline{S_{k-1}}(\gamma)$ is an $S_k(\gamma)$-bundle over $\overline{S_{k-1}}(\gamma)$ for $k = 1, \cdots, n-1$. In particular, the dimension of $\Phi_\lambda^{-1}(\mathbf{u})$ is given by

$$\dim \Phi_\lambda^{-1}(\mathbf{u}) = \sum_{k=1}^{n-1} \dim S_k(\gamma).$$

We illustrate Theorem 5.12 by the following examples.

Example 5.13. Let λ be given in Example 4.5. Using Theorem 5.12, we will compute the fiber $\Phi_\lambda^{-1}(v_i)$ of each vertex v_i given in Figure 4 for $i = 1, \cdots, 7$.

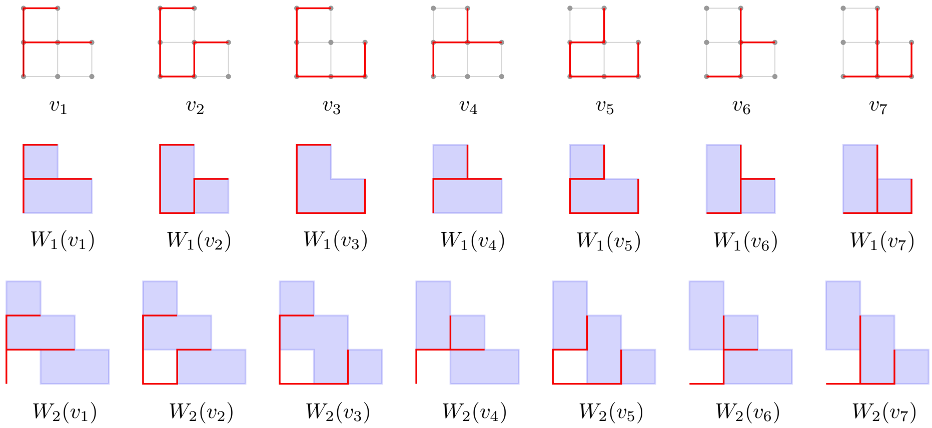


FIGURE 15. $W_1(v_i)$'s and $W_2(v_i)$'s for $\mathcal{F}(3)$

Figure 15 shows that $S_1(v_i) = \text{pt}$ for every $i = 1, \cdots, 7$ since each $W_1(v_i)$ does not contain any M-shaped block containing a bottom vertex of W_1, that is, the origin $(0,0)$. Also, we can easily check that $S_2(v_i) = \text{point}$ unless $i = 3$. When $i = 3$, there is one M-shaped block M_2 inside $W_2(v_3)$ containing a bottom vertex of W_2. Thus we have

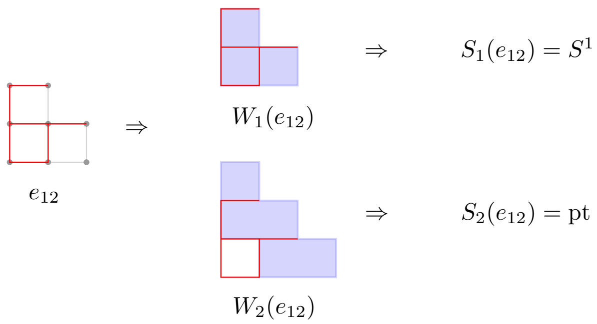
$$S_2(v_3) = \text{point} \times S^3 \times \text{point} \cong S^3.$$

Since $\Phi_\lambda^{-1}(v_i)$ is an $S_2(v_i)$-bundle over $S_1(v_i)$ and $S_1(v_i)$ is a point for every $i = 1, \cdots, 7$, by Theorem 5.12, we obtain

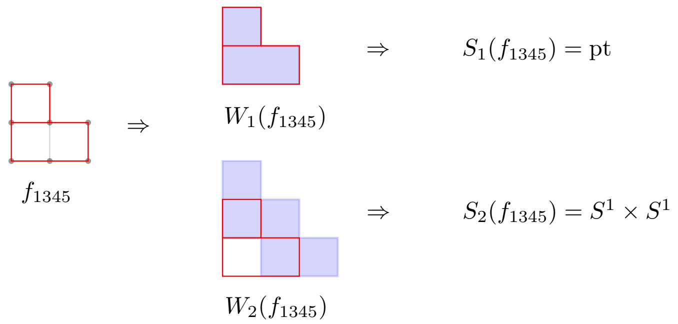
$$\Phi_\lambda^{-1}(v_i) = \begin{cases} S^3 & \text{if } i = 3 \\ \text{point} & \text{otherwise.} \end{cases}$$

Example 5.14. Again, we consider Example 4.5 and compute the fibers over the points in the relative interior of some higher dimensional face of Δ_λ as follows.

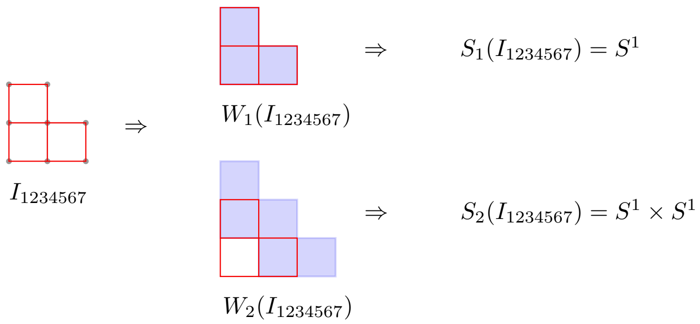
Consider the edge $e = e_{12}$ in Figure 5 and let $\mathbf{u} \in \mathring{e}_{12}$. By Theorem 5.12, $\Phi_\lambda^{-1}(\mathbf{u})$ is an $S_2(e_{12})$-bundle over $S_1(e_{12})$ so that it is diffeomorphic to S^1. See the figure below. For any other edge e of Γ_λ, we can show that $\Phi_\lambda^{-1}(\mathbf{u}) \cong S^1$ for every point $\mathbf{u} \in \mathring{e}$ in a similar way.



For a two-dimensional face of Γ_λ, we first consider the face f_{1345} of Δ_λ described in Figure 6. Again by Theorem 5.12, we have $\Phi_\lambda^{-1}(\mathbf{u}) \cong T^2$, an $S_2(f_{1345})$-bundle over $S_1(f_{1345})$, for every point $\mathbf{u} \in \mathring{f}_{1345}$. Similarly, we obtain $\Phi_\lambda^{-1}(\mathbf{u}) \cong T^2$ for every interior point $\mathbf{u}$ of any two-dimensional face of Δ_λ.



Finally, consider $I_{1234567}$, the improper face of Δ_λ. Then Theorem 5.12 tells us that $\Phi_\lambda^{-1}(\mathbf{u})$ is an $S^1 \times S^1$-bundle over S^1 for every interior point $\mathbf{u}$ of Γ_λ. In fact, the Arnold-Liouville theorem implies that the bundle is trivial, i.e., $\Phi_\lambda^{-1}(\mathbf{u})$ is a torus T^3 for every $\mathbf{u} \in \mathring{\Delta}_\lambda$, see also Theorem 7.9.



Consequently, a Lagrangian fiber of the GC system on $(\mathcal{O}_\lambda, \omega_\lambda)$ is diffeomorphic to either T^3 (a fiber over an interior point of Δ_λ) or S^3 (a fiber over v_3). Other fibers are isotropic but not Lagrangian submanifolds of $(\mathcal{O}_\lambda, \omega_\lambda)$ for dimensional reasons.

Remark 5.15. In general, one should *not* expect that every iterated bundle in (5.3) is trivial. Namely, $\Phi_\lambda^{-1}(\mathbf{u})$ might not be the product space $\prod_{k=1}^{n-1} S_k(\gamma)$. For instance, consider the co-adjoint orbit $\mathcal{O}_\lambda \simeq \mathcal{F}(2,3;5)$ associated with $\lambda = (3,3,0,-3,-3)$ as in Figure 16. By Theorem 5.12, the GC fiber $\Phi_\lambda^{-1}(\mathbf{0})$ over the origin is an S^3-bundle over S^5. Meanwhile, Proposition 2.7 in [NU2] implies that $\Phi_\lambda^{-1}(\mathbf{0})$ is $SU(3)$. It is however well-known that $SU(3)$ is not homotopy equivalent to $S^5 \times S^3$.

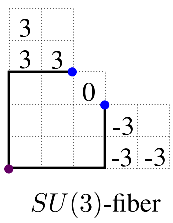


FIGURE 16. $SU(3)$-fiber

5.2. Classification of Lagrangian faces.

Recall that Theorem 5.12 implies that a face γ is Lagrangian if and only if the GC fiber over an interior point of γ is of dimension $\dim_{\mathbb{C}} \mathcal{O}_\lambda$. Therefore to determine whether γ is Lagrangian or not, it is sufficient to check that

$$\sum_{k=1}^{n-1} \dim S_k(\gamma) = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{O}_\lambda.$$

In this section, we present an efficient way of checking whether a given face of Γ_λ is Lagrangian or not by using so called "L-shaped blocks".

Definition 5.16. For each positive integer $k \in \mathbb{Z}_{\geq 1}$ and every lattice point $(p,q) \in \mathbb{Z}^2 \subset \mathbb{R}^2$, *a k-th L-shaped block at* (p,q), or simply *an L_k-block at* (p,q), is the closed region defined by

$$L_k(p,q) := \bigcup_{(a,b)} \square^{(a,b)}$$

where the union is taken over all (a,b)'s in $\mathbb{Z}^2$ such that

- $(a,b) = (p, q+i)$ for $0 \leq i \leq k-1$, and
- $(a,b) = (p+i, q)$ for $0 \leq i \leq k-1$.

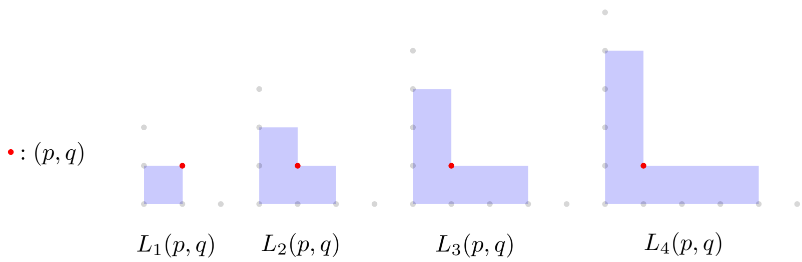


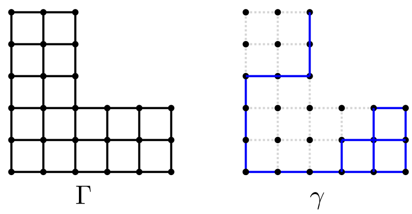
FIGURE 17. L-shaped blocks

Remark 5.17. Note that GC patterns described in (3.8) are linearly ordered on any of W-shaped, M-shaped, and L-shaped blocks in the direction from the right or bottom most block to the left or top most block.

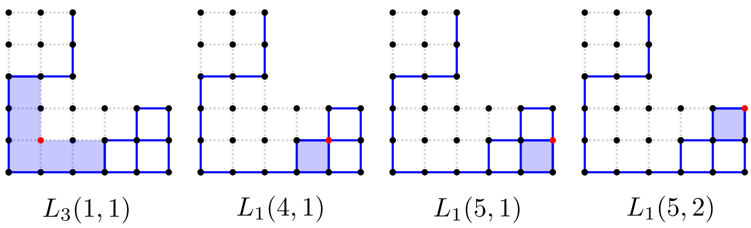
Definition 5.18. Let γ be a face of Γ_λ. For a given positive integer $k \in \mathbb{Z}_{\geq 1}$ and a lattice point $(p,q) \in \mathbb{Z}^2$, we say that $L_k(p,q)$ is *rigid in* γ if

(1) the interior of $L_k(p,q)$ does not contain an edge of γ and
(2) the rightmost edge and the top edge of $L_k(p,q)$ should be edges of γ.

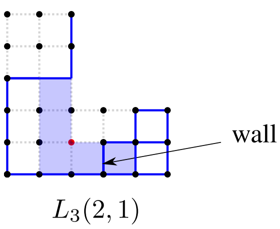
Example 5.19. Let us consider $\Gamma = \Gamma(2,5;7)$ and let γ be given as follows.



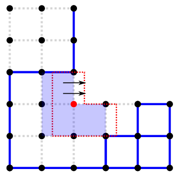
In this example, there are exactly four rigid L_k-blocks : $L_3(1,1)$, $L_1(4,1)$, $L_1(5,1)$, and $L_1(5,2)$.



We can check that any other L-blocks are *not* rigid. For instance, $L_3(2,1)$ is not rigid since its interior contains an edge of γ.



Also, $L_2(2,2)$ is not rigid because its rightmost edge is *not* an edge of γ violating the condition (2) in Definition 5.18.



$L_2(2,2)$

The following lemma follows from the min-max principle of GC pattern (3.8), or more specifically from (4.1).

Lemma 5.20. *Suppose that $L_k(a,b)$ is rigid in a face γ of Γ_λ. Let $Q_k(a,b)$ be the closed region defined by*

$$Q_k(a,b) := \bigcup_{0 \leq i,j \leq k-1} \square^{(a+i,b+j)},$$

i.e., $Q_k(a,b)$ is the square of size $(k \times k)$ that contains $L_k(a,b)$. Then there are no edges of γ in the interior of $Q_k(a,b)$.

Proof. If $k = 1$, then $L_1(a,b) = Q_1(a,b)$ and it has no edge of γ in its interior so that there is nothing to prove. Now, assume that $k \geq 2$ and suppose that there is an edge $e = [v_0 v_1]$ of γ contained in the interior of $Q_k(a,b)$. Then, without loss of generality, we may assume that $v_0 = (a_0, b_0)$ is in the interior of $Q_k(a,b)$ so that $a \leq a_0 < a + k - 1$ and $b \leq b_0 < b + k - 1$.

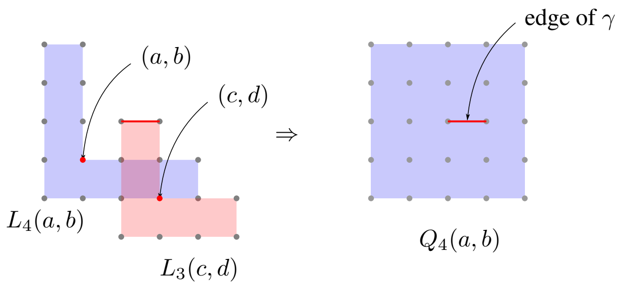
By Definition 4.3, there exists a positive path δ contained in γ passing through v_0. Then δ should pass through the interior of $L_k(a,b)$ since δ contains a shortest path from the origin of Γ_λ and v_0, which intersects the interior of $L_k(a,b)$. This contradicts the rigidity (1) in Definition 5.18. $\square$

Lemma 5.21. *If two different L-blocks $L_i(a,b)$ and $L_j(c,d)$ are rigid in the same face γ, then they cannot be overlapped, i.e.,*

$$\mathring{L}_i(a,b) \cap \mathring{L}_j(c,d) = \emptyset$$

where $\mathring{L}_i(a,b)$ and $\mathring{L}_j(c,d)$ denote the interior of $\mathring{L}_i(a,b)$ and $\mathring{L}_j(c,d)$, respectively.

Proof. If $(a,b) = (c,d)$ and $i \neq j$, then both $L_i(a,b)$ and $L_i(c,d)$ cannot be rigid since at least one of these two blocks violates (1) in Definition 5.18. Suppose that $(a,b) \neq (c,d)$ and $\mathring{L}_i(a,b) \cap \mathring{L}_j(c,d) \neq \emptyset$. Without loss of generality, we may assume that $j \geq i$. Then it is straightforward that either the top edge or the rightmost edge of $L_i(a,b)$ lies on the interior of $Q_j(c,d)$. It leads to a contradiction to Lemma 5.20 since the top edge and the rightmost edge of $L_i(a,b)$ are edges of γ by Definition 5.16.



$\square$

Proposition 5.22. *Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be given in (3.4). For a given face γ of Γ_λ, let $\mathcal{L}(\gamma)$ be the set of all rigid L-shaped blocks in γ. Then, for any point $\mathbf{u}$ in the relative interior of the face $f_\gamma = \Psi(\gamma)$ of Δ_λ, we have*

$$\dim \Phi_\lambda^{-1}(\mathbf{u}) = \sum_{L_k(a,b) \in \mathcal{L}(\gamma)} |L_k(a,b)| = \sum_{L_k(a,b) \in \mathcal{L}(\gamma)} (2k-1)$$

where $|L_k(a,b)| = 2k - 1$ is the are of $L_k(a,b)$.

Proof. By Theorem 5.12, it is enough to show that

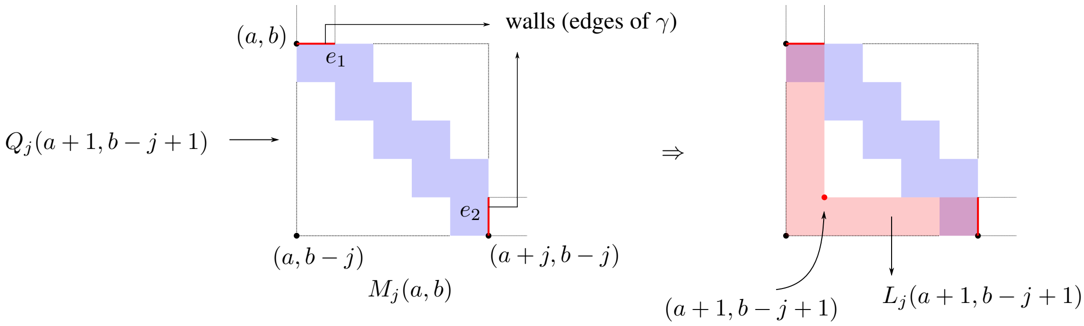
$$\sum_{k=1}^{n-1} \dim S_k(\gamma) = \sum_{L_k(a,b) \in \mathcal{L}(\gamma)} |L_k(a,b)|$$

where

$$S_k(\gamma) = \prod_{\mathcal{D} \subset W_k(\gamma)} S_k(\mathcal{D})$$

as defined in (5.2).

Let $\mathcal{D}$ be a simple closed region in $W_k(\gamma)$. Suppose that $\mathcal{D}$ contains a bottom vertex of W_k and $\mathcal{D} = M_j(a,b)$ for some $j \geq 1$ where $M_j(a,b)$ denotes the j-th M-shaped block whose top-left vertex is (a,b). Then there are two edges e_1 and e_2 of γ on the boundary of M_j as we see below. By (4.1), there is no edge of γ in the interior of



$Q_j(a+1,b-j+1)$. Thus $L_j(a+1,b-j+1)$ is a rigid L_j-block in γ.

Similarly, for any rigid L-shaped block $L_j(a+1,b-j+1)$ in γ (for some $j \in \mathbb{Z}_{\geq 0}$ and $(a,b) \in (\mathbb{Z}_{\geq 1})^2$), we can find a closed region $\mathcal{D}$ in some $W_k(\gamma)$ such that $\mathcal{D}$ is $M_j(a,b)$ that contains a bottom vertex of W_k. Therefore, there is a one-to-one correspondence between the set of rigid L-shaped blocks in γ and the set of simple closed regions $\mathcal{D}$ appeared in some $W_k(\gamma)$ such that such that $S_k(\mathcal{D}) = S^{2j-1}$. In other words,

$$\begin{aligned} \bigcup_{j=1} \{\text{rigid } L_j\text{-blocks in } \gamma\} \quad &\Leftrightarrow \quad \bigcup_{k=1}^{n-1} \bigcup_{j=1} \{\mathcal{D} \subset W_k(\gamma) \mid S_k(\mathcal{D}) = S^{2j-1}\} \\ L_j(a+1,b-j+1) \quad &\leftrightarrow \quad M_j(a,b). \end{aligned}$$

Moreover, we have $|M_j| = |L_j| = \dim S_k(\mathcal{D}) = 2j-1$, and hence it completes the proof. ☐

The following corollary is derived from Theorem 5.12, Lemma 5.21 and Proposition 5.22.

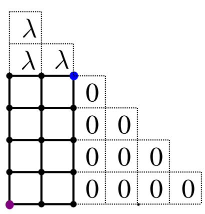
Corollary 5.23. *Let γ and f_γ be as in Proposition 5.22. Then the followings are equivalent.*

(1) *For an interior point $\mathbf{u}$ of f_γ, the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is a Lagrangian submanifold of $(\mathcal{O}_\lambda, \omega)$.*
(2) *For any interior point $\mathbf{u}$ of f_γ, the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is a Lagrangian submanifold of $(\mathcal{O}_\lambda, \omega)$.*
(3) *The set of rigid L-shaped blocks in γ covers the whole Γ_λ.*

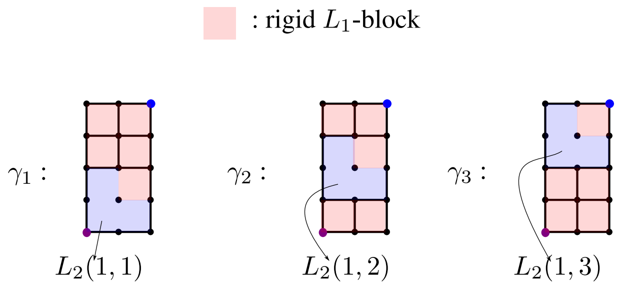
Also, we have the following corollary which follows from Corollary 5.23.

Corollary 5.24. *A Gelfand-Cetlin system $\Phi_\lambda \colon \mathcal{O}_\lambda \to \Delta_\lambda$ on a co-adjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ always possesses a non-torus Lagrangian fiber unless $\mathcal{O}_\lambda$ is a projective space.*

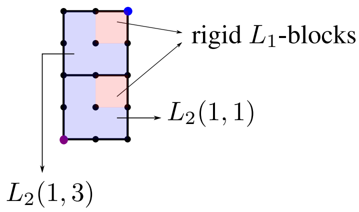
Example 5.25. Let $\lambda = \{t,t,0,0,0,0\}$ with $t > 0$. The co-adjoint orbit $\mathcal{O}_\lambda$ is a complex Grassmannian $\mathrm{Gr}(2,6)$ of two planes in $\mathbb{C}^6$ and the corresponding ladder diagram Γ_λ is given as below.



Observe that any faces of Γ_λ do not admit rigid L_k-blocks of $k > 2$. Also, note that there are three Lagrangian faces γ_1, γ_2 and γ_3 of Γ_λ which have only one rigid L_2-block as follows.

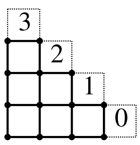


Finally, there is exactly one Lagrangian face γ_4 which has two rigid L_2-blocks as below.

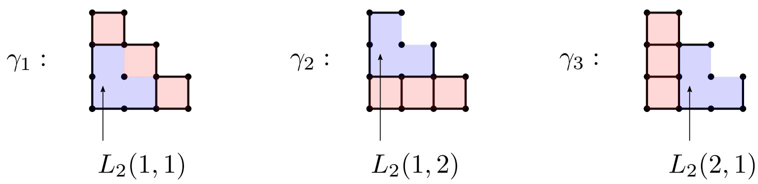


Thus there are exactly four proper Lagrangian faces $\gamma_1, \gamma_2, \gamma_3$ and γ_4 of Γ_λ.

Example 5.26. Let $\lambda = \{3, 2, 1, 0\}$. Then the co-adjoint orbit $\mathcal{O}_\lambda$ is a complete flag manifold $\mathcal{F}(4)$ and the corresponding ladder diagram Γ_λ is as follows.



We can easily see that any face of Γ_λ does not have a rigid L_k-block for $k \geq 3$. There are exactly three Lagrangian faces of Γ_λ containing one rigid L_2-block as below.



Also, it is not hard to see that there is no Lagrangian face that contains more than one L_2-block. Thus γ_1, γ_2, and γ_3 are the only proper Lagrangian faces of Γ_λ.

6. Iterated bundle structures on Gelfand-Cetlin fibers

In this section, for each point $\mathbf{u}$ in a GC polytope Δ_λ, we will construct the iterated bundle $E_\bullet$ described in Section 5 whose total space is the fiber $\Phi_\lambda^{-1}(\mathbf{u})$. Using this construction, we complete the proof of Theorem 5.12 by showing that each $\Phi_\lambda^{-1}(\mathbf{u})$ is an isotropic submanifold of $\mathcal{O}_\lambda$.

For a fixed integer $k \geq 1$, consider sequences $\mathfrak{a} = (a_1, \cdots, a_{k+1})$ and $\mathfrak{b} = (b_1, \cdots, b_k)$ of real numbers satisfying

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq a_k \geq b_k \geq a_{k+1}. \tag{6.1}$$

Denoting by $\mathcal{H}_\ell$ the set of $(\ell \times \ell)$ hermitian matrices for $\ell \geq 1$ and by $\mathrm{sp}(x)$ the spectrum of x, we let

$$\mathcal{O}_\mathfrak{a} = \left\{ x \in \mathcal{H}_{k+1} \,\middle|\, \mathrm{sp}(x) = \{a_1, \cdots, a_{k+1}\} \right\}$$

be the co-adjoint $U(k+1)$-orbit of the diagonal matrix $I_{\mathfrak{a}} := \mathrm{diag}(a_1, \cdots, a_{k+1})$ in $\mathcal{H}_{k+1} \cong \mathfrak{u}(k+1)^*$.

Consider a subspace

$$\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b}) = \left\{ x \in \mathcal{H}_{k+1} \;\middle|\; \mathrm{sp}(x) = \{a_1, \cdots, a_{k+1}\},\ \mathrm{sp}(x^{(k)}) = \{b_1, \cdots, b_k\} \right\}$$

where $x^{(k)}$ denotes the $(k \times k)$ leading principal minor submatrix of x. It naturally comes with the projection map

$$\begin{array}{cccc} \rho_{k+1} \colon & \mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b}) & \to & \mathcal{O}_{\mathfrak{b}} \\ & x & \mapsto & x^{(k)} \end{array}$$

from $\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ to the co-adjoint $U(k)$-orbit $\mathcal{O}_{\mathfrak{b}}$ of the diagonal matrix $I_{\mathfrak{b}}$ in $\mathcal{H}_k \cong \mathfrak{u}(k)^*$.

Let $W_k(\mathfrak{a}, \mathfrak{b})$ be the k-th W-shaped block W_k together with walls defined by the equalities of a_i's and b_j's as in Figure 18. By comparing the divided regions by the walls on $W_k(\mathfrak{a}, \mathfrak{b})$ with M-shaped blocks as in (5.2), we define a topological space $S_k(\mathfrak{a}, \mathfrak{b})$, which is either a single point or a product space of odd dimensional spheres.

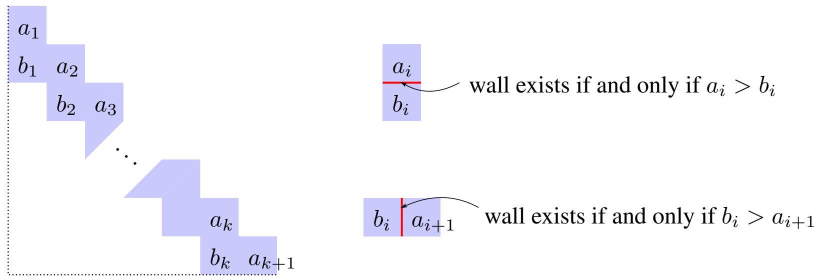
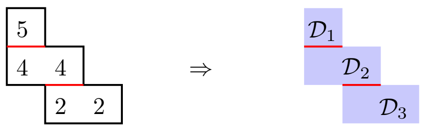


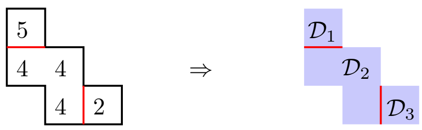
FIGURE 18. $W_k(\mathfrak{a}, \mathfrak{b})$

Example 6.1. (1) For $\mathfrak{a} = (a_1, a_2, a_3) = (5, 4, 2)$ and $\mathfrak{b} = (b_1, b_2) = (4, 2)$, $W_2(\mathfrak{a}, \mathfrak{b})$ is divided by three simply closed regions $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. Since $\mathcal{D}_1$ does not contain any bottom vertices and neither $\mathcal{D}_2$ nor $\mathcal{D}_3$ match with M-shaped blocks, $S_2(\mathfrak{a}, \mathfrak{b}) = S_2(\mathcal{D}_1) \times S_2(\mathcal{D}_2) \times S_2(\mathcal{D}_3) \cong$ point.



$W_2(\mathfrak{a}, \mathfrak{b})$

(2) For $\mathfrak{a} = (5, 4, 2)$ and $\mathfrak{b} = (4, 4)$, $W_2(\mathfrak{a}, \mathfrak{b})$ is divided by three simply closed regions $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. Observe that $\mathcal{D}_1$ and $\mathcal{D}_3$ do not contain any bottom vertices and $\mathcal{D}_2$ is an M_2-block containing bottom vertices of W_2. Therefore, we have $S_2(\mathfrak{a}, \mathfrak{b}) = \text{point} \times S^3 \times \text{point} \cong S^3$.



$W_2(\mathfrak{a}, \mathfrak{b})$

Proposition 6.2. *With the notations above, $\rho_{k+1} \colon \mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b}) \to \mathcal{O}_{\mathfrak{b}}$ is an $S_k(\mathfrak{a}, \mathfrak{b})$-bundle over $\mathcal{O}_{\mathfrak{b}}$.*

Before starting the proof of Proposition 6.2, as preliminaries, we introduce some notations and prove lemmas. We denote by $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b})$ the set of matrices in $\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ whose $(k \times k)$ leading principal minor is the diagonal matrix $I_{\mathfrak{b}}$. So, a matrix in $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b})$ is of the form

$$Z_{(\mathfrak{a}, \mathfrak{b})}(z) = \begin{pmatrix} b_1 & & 0 & \overline{z}_1 \\ & \ddots & & \vdots \\ 0 & & b_k & \overline{z}_k \\ z_1 & \cdots & z_k & z_{k+1} \end{pmatrix}$$

for $z = (z_1, \cdots, z_k) \in \mathbb{C}^k$. Since $Z_{(\mathfrak{a},\mathfrak{b})}(z)$ has the eigenvalues $\mathfrak{a} = \{a_1, \cdots, a_{k+1}\}$, the $(k+1, k+1)$-entry of $Z_{(\mathfrak{a},\mathfrak{b})}(z)$ is to be constant

$$z_{k+1} = \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k} b_i \in \mathbb{R}$$

by computing the trace of $Z_{(\mathfrak{a},\mathfrak{b})}(z)$. The characteristic polynomial of $Z_{(\mathfrak{a},\mathfrak{b})}(z)$ is expressed as

$$\det(xI - Z_{(\mathfrak{a},\mathfrak{b})}(z)) = (x - z_{k+1}) \cdot \prod_{i=1}^{k}(x - b_i) - \sum_{j=1}^{k} \left(\frac{|z_j|^2}{x - b_j} \cdot \prod_{i=1}^{k}(x - b_i) \right) = 0, \tag{6.2}$$

whose zeros are $x = a_1, \cdots, a_{k+1}$ by our assumption. By inserting $x = a_1, \cdots, a_{k+1}$ into (6.2), we obtain the system of $(k+1)$ equations of real coefficients, which are linear with respect to $(|z_1|^2, \cdots, |z_k|^2) \in (\mathbb{R}_{\geq 0})^k$. We sometimes regard an element $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$ as an element $\mathbb{C}^k$ under the identification $Z_{(\mathfrak{a},\mathfrak{b})}(z) \mapsto z$. The following lemma implies that the solution space is *never* empty as long as $(\mathfrak{a},\mathfrak{b})$ obeys (6.1).

Lemma 6.3 (Lemma 3.5 in [NNU1]). *Let $a_1, b_1, \cdots, a_k, b_k$, and a_{k+1} be real numbers satisfying (6.1). Then there exists $z_1, \ldots, z_k \in \mathbb{C}$ and $z_{k+1} \in \mathbb{R}$ such that*

$$\begin{pmatrix} b_1 & & 0 & \overline{z}_1 \\ & \ddots & & \vdots \\ 0 & & b_k & \overline{z}_k \\ z_1 & \ldots & z_k & z_{k+1} \end{pmatrix}$$

has eigenvalues $a_1, \ldots, a_{k+1}$.

Lemma 6.4. *Suppose in addition that $a_1, b_1, \cdots, a_k, b_k$, and a_{k+1} in Lemma 6.3 are all distinct. Then there exist positive numbers $\delta_1, \cdots, \delta_k$ such that*

$$\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b}) = \left\{ \begin{pmatrix} b_1 & & 0 & \overline{z}_1 \\ & \ddots & & \vdots \\ 0 & & b_k & \overline{z}_k \\ z_1 & \ldots & z_k & z_{k+1} \end{pmatrix} \in \mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b}) \ : \ (z_1, \cdots, z_k) \in \mathbb{C}^k, \ |z_i|^2 = \delta_i \text{ for } i = 1, \cdots, k \right\}$$

where $z_{k+1} = \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k} b_i$. In particular, we have $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b}) \cong T^k$.

Proof. We first note that if $|z_j|^2 = 0$ for some j, then the equation (6.2) (with respect to x) has a solution $x = b_j$. It implies that $b_j \in \{a_1, \cdots, a_{k+1}\}$, which contradicts to our assumption that a_i's and b_j's are all distinct. Thus, it is enough to show existence and uniqueness of a solution $(|z_1|^2, \cdots, |z_k|^2)$ of the system of linear equations in (6.2).

The existence immediately follows from Lemma 6.3. Let

$$\{|z_1|^2 = \delta_1 > 0, \cdots, |z_k|^2 = \delta_k > 0\}$$

be one of solutions of (6.2) so that $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$ contains a real k-torus

$$T^k = \{(z_1, \cdots, z_k) \in \mathbb{C}^k \mid |z_i|^2 = \delta_i, \ i = 1, \cdots, k\},$$

which yields that $\dim_{\mathbb{R}} \widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b}) \geq k$. Since (6.2) is a system of non-homogeneous *linear* equations with respect to the variables $|z_1|^2, \cdots, |z_k|^2$, the set

$$\left\{ (|z_1|^2, \cdots, |z_k|^2) \mid (z_1, \cdots, z_k) \in \widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b}) \right\}$$

is an affine subspace of $\mathbb{R}^k$. Therefore, $\dim_{\mathbb{R}} \widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b}) = k$ if and only if the equations (6.2) has a unique solution $(|z_1|^2, \cdots, |z_k|^2) = (\delta_1, \cdots, \delta_k)$. It suffices to show that $\dim_{\mathbb{R}} \widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b}) = k$.

Note that $\mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b})$ is an $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$-bundle over $\mathcal{O}_{\mathfrak{b}}$ whose projection map is

$$\begin{array}{rccc} \rho_{k+1}: & \mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b}) & \to & \mathcal{O}_{\mathfrak{b}} \\ & x & \mapsto & x^{(k)}. \end{array}$$

More precisely, for each element $y \in \mathcal{O}_{\mathfrak{b}}$, there exists a unitary matrix $g_y \in U(k)$ (depending on y) such that

$$g_y y g_y^{-1} = I_{\mathfrak{b}}$$

where $I_{\mathfrak{b}}$ is the diagonal matrix $\mathrm{diag}(b_1, \cdots, b_k)$. Then the preimage $\rho_{k+1}^{-1}(y)$ of y can be identified with $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b})$ via

$$\begin{array}{rcl} \rho_{k+1}^{-1}(y) & \longrightarrow & \widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b}) \\ Y = \begin{pmatrix} y & * \\ *^t & z_{k+1} \end{pmatrix} & \mapsto & \begin{pmatrix} g_y & 0 \\ 0 & 1 \end{pmatrix} \cdot Y \cdot \begin{pmatrix} g_y^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ & & = \begin{pmatrix} g_y \cdot y \cdot g_y^{-1} & g_y \cdot * \\ *^t \cdot g_y^{-1} & z_{k+1} \end{pmatrix} \\ & & = \begin{pmatrix} I_{\mathfrak{b}} & g_y \cdot * \\ *^t \cdot g_y^{-1} & z_{k+1} \end{pmatrix} \end{array}$$

so that $\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ is an $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b})$-bundle over $\mathcal{O}_{\mathfrak{b}}$ via ρ_{k+1}.

Now, we consider a sequence of real numbers $\mathfrak{c} = \{c_1, \cdots, c_{k-1}\}$ such that

$$b_1 > c_1 > \cdots > b_{k-1} > c_{k-1} > b_k.$$

Restricting the fibration ρ_{k+1} to $\mathcal{A}_k(\mathfrak{b}, \mathfrak{c})$, we similary have an $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b})$-bundle over $\mathcal{A}_k(\mathfrak{b}, \mathfrak{c})$. Note that its total space is the collection of $(k+1) \times (k+1)$ hermitian matrices such that the spectra, the spectra of the $(k \times k)$ leading principal minor and the spectra of the $(k-1) \times (k-1)$ leading principal minor are resepectively $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$.

By a similar way described as above, we see that $\mathcal{A}_k(\mathfrak{b}, \mathfrak{c})$ is an $\widetilde{\mathcal{A}}_k(\mathfrak{b}, \mathfrak{c})$-bundle over $\mathcal{O}_{\mathfrak{c}}$ with the projection map $\rho_k \colon \mathcal{A}_k(\mathfrak{b}, \mathfrak{c}) \to \mathcal{O}_{\mathfrak{c}}$ such that $\dim_{\mathbb{R}} \widetilde{\mathcal{A}}_k(\mathfrak{b}, \mathfrak{c}) \geq k-1$. Taking a sequence of real numbers $\mathfrak{d} = \{d_1, \cdots, d_{k-2}\}$ so that

$$c_1 > d_1 > \cdots > c_{k-2} > d_{k-2} > c_{k-1},$$

the restriction of ρ_k to $\mathcal{A}_{k-1}(\mathfrak{c}, \mathfrak{d})$ induces an $\widetilde{\mathcal{A}}_k(\mathfrak{b}, \mathfrak{c})$-bundle over $\mathcal{A}_{k-1}(\mathfrak{c}, \mathfrak{d})$.

Proceeding this procedure inductively, we end up obtaining a tower of bundles such that the total space E is a generic fiber of the GC system of $\mathcal{O}_{\mathfrak{a}}$. Namely, E is the preimage of a point in the interior of the GC polytope $\Delta_{\mathfrak{a}}$. By Proposition 3.11, E is a smooth manifold of dimension

$$\dim_{\mathbb{R}} E = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{O}_{\mathfrak{a}} = \frac{k(k+1)}{2}.$$

On the other hand, by our construction, the dimension of E is the sum of dimensions of all fibers of ρ_i's for $i = 2, \cdots, k+1$ so that

$$\dim E = \dim \widetilde{\mathcal{A}}_{k+1} + \dim \widetilde{\mathcal{A}}_k + \cdots + \dim \widetilde{\mathcal{A}}_2.$$

Since $\dim \widetilde{\mathcal{A}}_{i+1} \geq i$ for each i, we get $\dim \widetilde{\mathcal{A}}_{i+1} = i$ for every $i = 1, \cdots, k$. Lemma 6.4 is established. □

Note that Lemma 6.4 deals with the case where $a_j \notin \{b_1, \cdots, b_k\}$ for $j = 1, \cdots, k+1$. Now, let us consider the case where $a_{j+1} \in \{b_1, \cdots, b_k\}$ for some $j \in \{0, 1, \cdots, k\}$. Denoting the multiplicity of a_{j+1} in $\mathfrak{a}$ by ℓ, without loss of generality, we assume that $a_j > a_{j+1} = \cdots = a_{j+\ell} > a_{j+\ell+1}$. Then either $a_{j+1} = b_j$ or $a_{j+1} = b_{j+1}$. For the first case, there are two possible cases:

$$\begin{cases} (1)\ a_j > b_j = a_{j+1} = \cdots = a_{j+\ell} > b_{j+\ell}, \text{ or} \\ (2)\ a_j > b_j = a_{j+1} = \cdots = a_{j+\ell} = b_{j+\ell} > a_{j+\ell+1}. \end{cases} \tag{6.3}$$

For the second case, we have two possible cases too:

$$\begin{cases} (3)\ b_j > a_{j+1} = b_{j+1} = \cdots = a_{j+\ell} = b_{j+\ell} > a_{j+\ell+1}, \text{ or} \\ (4)\ b_j > a_{j+1} = b_{j+1} = \cdots = a_{j+\ell} > b_{j+\ell}, \end{cases} \tag{6.4}$$

Note that only the case (2) above have more multiplicity of b's than a's, i.e., multiplicity $\ell+1$ and ℓ respectively: The cases (1) and (3) have the same multiplicities of both a and b while in the case (4) a has multiplicity ℓ and b has multiplicity $\ell-1$.

We start with the first inequality of (6.4).

Lemma 6.5 (case (3) of (6.4))**.** *Suppose that $b_j > a_{j+1} = b_{j+1} = \cdots = a_{j+\ell} = b_{j+\ell} > a_{j+\ell+1}$. Then, every solution $(z_1, \cdots, z_k) \in \mathbb{C}^k$ of the equation (6.2) satisfies*

$$z_{j+1} = \cdots = z_{j+\ell} = 0.$$

Proof. Observe that each term of the equation (6.2)

$$\det(xI - Z_{(\mathfrak{a},\mathfrak{b})}(z)) = (x - z_{k+1}) \cdot \prod_{i=1}^{k}(x - b_i) - \sum_{i=1}^{k} \left(\frac{|z_i|^2}{x - b_i} \cdot \prod_{m=1}^{k} (x - b_m) \right) = 0$$

is divisible by $(x - b_{j+1})^{\ell-1}$ by our assumption. In particular, the first term of the equation

$$(x - z_{k+1}) \cdot \prod_{i=1}^{k}(x - b_i)$$

is divisible by $(x - b_{j+1})^{\ell}$. For each $i \notin \{j+1, \cdots, j+\ell\}$, so is

$$\frac{|z_i|^2}{x - b_i} \cdot \prod_{m=1}^{k} (x - b_m)$$

since $b_{j+1} = \cdots = b_{j+\ell}$. Taking

$$g(x) := \det(xI - Z_{(\mathfrak{a},\mathfrak{b})}(z))/(x - b_{j+1})^{\ell-1},$$

we have $g(b_{j+1}) = g(a_{j+1}) = 0$ because $x = a_{j+1} = b_{j+1}$ is a solution of (6.2) with multiplicity ℓ. It yields

$$\left(|z_{j+1}|^2 + \cdots + |z_{j+\ell}|^2\right) \cdot \prod_{\substack{m=1 \\ m \notin \{j+1, \cdots, j+\ell\}}}^{k} (b_{j+1} - b_m) = 0.$$

Since

$$\prod_{\substack{m=1 \\ m \notin \{j+1, \cdots, j+\ell\}}}^{k} (b_{j+1} - b_m) \neq 0,$$

we deduce that $|z_{j+1}|^2 + \cdots + |z_{j+\ell}|^2 = 0$ and hence $z_{j+1} = \cdots = z_{j+\ell} = 0$. ☐

Therefore, under the assumption (case (3) in (6.4)) on $\mathfrak{a}$ and $\mathfrak{b}$, the equation (6.2) is written by

$$\det(xI - Z_{(\mathfrak{a},\mathfrak{b})}(z)) = (x - b_{j+1})^{\ell} \cdot \left\{ (x - w_{k+1-\ell}) \cdot \prod_{i=1}^{k-\ell}(x - b'_i) - \sum_{i=1}^{k-\ell} \left(\frac{|w_i|^2}{x - b'_i} \cdot \prod_{m=1}^{k-\ell} (x - b'_m) \right) \right\} = 0$$

where

- $(b'_1. \cdots, b'_{k-\ell}) = (b_1, \cdots, b_j, \widehat{b_{j+1}}, \cdots, \widehat{b_{j+\ell}}, b_{j+\ell+1}, \cdots, b_k)$,
- $(w_1. \cdots, w_{k-\ell}) = (z_1, \cdots, z_j, \widehat{z_{j+1}}, \cdots, \widehat{z_{j+\ell}}, z_{j+\ell+1}, \cdots, z_k)$, and
- $w_{k-\ell+1} = z_{k+1}$.

Observe that the equation

$$\det(xI - Z_{(\mathfrak{a},\mathfrak{b})}(z))/(x - b_{j+1})^{\ell} = 0$$

is same as the equation $\det(xI - Z_{(\mathfrak{a}',\mathfrak{b}')}(w)) = 0$ where

- $\mathfrak{a}' = (a'_1, \cdots, a'_{k-\ell+1}) = (a_1, \cdots, a_j, \widehat{a_{j+1}}, \cdots, \widehat{a_{j+\ell}}, a_{j+\ell+1}, \cdots, a_{k+1})$ and
- $\mathfrak{b}' = (b'_1, \cdots, b'_{k-\ell})$.

Thus, $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$ can be identified with $\widetilde{\mathcal{A}}_{k+1-\ell}(\mathfrak{a}',\mathfrak{b}')$ via

$$\begin{array}{ccc} \widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b}) & \to & \widetilde{\mathcal{A}}_{k+1-\ell}(\mathfrak{a}',\mathfrak{b}') \\ (z_1,\cdots,z_j,0,\cdots,0,z_{j+\ell+1},\cdots,z_k) & \mapsto & (z_1,\cdots,z_j,\widehat{z_{j+1}},\cdots,\widehat{z_{j+\ell}},z_{j+\ell+1},\cdots,z_k). \end{array} \tag{6.5}$$

Therefore, we obtain the following corollary.

Corollary 6.6. *For sequences $\mathfrak{a}=(a_1,\cdots,a_{k+1})$ and $\mathfrak{b}=(b_1,\cdots,b_k)$ of real numbers obeying (6.1), suppose that there exist $j,\ell\in\mathbb{Z}_{>0}$ such that*

$$b_j > a_{j+1} = b_{j+1} = \cdots = a_{j+\ell} = b_{j+\ell} > a_{j+\ell+1}.$$

Setting $\mathfrak{a}'$ (respectively $\mathfrak{b}'$) to be the sequence of real numbers obtained by deleting $a_{j+1},\cdots,a_{j+\ell}$ (respectively $b_{j+1},\cdots,b_{j+\ell}$), $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$ can be identified with $\widetilde{\mathcal{A}}_{k+1-\ell}(\mathfrak{a}',\mathfrak{b}')$ under (6.5).

The following two lemmas below are about the cases of (4) in (6.4) and (1) of (6.3). Since they can be proven by using exactly same method of the proof of Lemma 6.5, we omit the proofs.

Lemma 6.7 (case (1) of (6.3))**.** *Suppose that $a_j > b_j = a_{j+1} = \cdots = a_{j+\ell} > b_{j+\ell}$. Then $(z_1,\cdots,z_k)\in\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$ if and only if*

- *$z_j = \cdots = z_{j+\ell-1} = 0$, and*
- *$(z_1,\cdots,z_{j-1},z_{j+\ell},\cdots,z_k)\in\widetilde{\mathcal{A}}_{k-\ell+1}(\mathfrak{a}',\mathfrak{b}')$*

where $\mathfrak{a}'$ is obtained by deleting $\{a_{j+1},\cdots,a_{j+\ell}\}$ from $\mathfrak{a}$ and $\mathfrak{b}'=\{b'_1,\cdots,b'_{k-\ell}\}$ is obtained by deleting $\{b_j,\cdots,b_{j+\ell-1}\}$ from $\mathfrak{b}$.

Lemma 6.8 (case (4) of (6.4))**.** *Suppose that $b_j > a_{j+1} = \cdots = a_{j+\ell} > b_{j+\ell}$. Then $(z_1,\cdots,z_k)\in\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$ if and only if*

- *$z_{j+1} = \cdots = z_{j+\ell-1} = 0$, and*
- *$(z_1,\cdots,z_j,z_{j+\ell},\cdots,z_k)\in\widetilde{\mathcal{A}}_{k-\ell+2}(\mathfrak{a}',\mathfrak{b}')$*

where $\mathfrak{a}'=\{a'_1,\cdots,a'_{k-\ell+2}\}$ is obtained by deleting $\{a_{j+2},\cdots,a_{j+\ell}\}$ from $\mathfrak{a}$ and $\mathfrak{b}'=\{b'_1,\cdots,b'_{k-\ell+1}\}$ is obtained by deleting $\{b_{j+1},\cdots,b_{j+\ell-1}\}$ from $\mathfrak{b}$.

It remains to take care of the case (2) of (6.3).

Lemma 6.9 (case (2) of (6.3))**.** *Suppose that*

$$a_j > b_j = a_{j+1} = \cdots = a_{j+\ell} = b_{j+\ell} > a_{j+\ell+1}$$

Then there exists a unique positive real number $C_j > 0$ such that

$$|z_j|^2 + \cdots + |z_{j+\ell}|^2 = C_j.$$

for any $(z_1,\cdots,z_k)\in\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$.

Proof. By Lemma 6.5, 6.7, and 6.8, we may reduce $\mathfrak{a}=\{a_1,\cdots,a_{k+1}\}$ and $\mathfrak{b}=\{b_1,\cdots,b_k\}$ to

$$\mathfrak{a}'=\{a'_1,\cdots,a'_{r+1}\},\quad \text{and} \quad \mathfrak{b}'=\{b'_1,\cdots,b'_r\}$$

for some $r>0$ so that there are no subsequences of type (3), (4) of (6.4) or (1) of (6.3) in $(a'_1\geq b'_1\geq\cdots\geq a'_r\geq b'_r\geq a'_{r+1})$. Also, the above series of lemmas implies that $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$ is identified with $\widetilde{\mathcal{A}}_{r+1}(\mathfrak{a}',\mathfrak{b}')$ under the identification of $w=(w_1,\cdots,w_r)$ with suitable sub-coordinates $(z_{i_1},\cdots,z_{i_r})$ of $(z_1,\cdots,z_{k+1})$. Therefore, it is enough to prove Lemma 6.9 in the case where $(a_1,b_1,\cdots,a_k,b_k,a_{k+1})$ does not contain any pattern of type (3), (4) of (6.4) or (1) of (6.3).

We temporarily assume that

$$a_j > b_j = a_{j+1} = \cdots = a_{j+\ell} = b_{j+\ell} > a_{j+\ell+1}.$$

is the unique pattern of type (2) of (6.3) in $(a_1,b_1,\cdots,a_k,b_k,a_{k+1})$. Then the equation (6.2) is written as

$$\det(xI-Z) = (x-b_j)^\ell\cdot g(x)$$

where

$$g(x) = (x - z_{k+1}) \prod_{\substack{i=1 \\ i \notin \{j+1,\cdots,j+\ell\}}}^{k} (x - b_i) - \sum_{i=1}^{k} \left(\frac{|z_i|^2}{x - b_i} \cdot \prod_{\substack{m=1 \\ m \notin \{j+1,\cdots,j+\ell\}}}^{k} (x - b_m) \right)$$

is a polynomial of degree $(k - \ell + 1)$ with respect to x. For the sake of simplicity, we denote by

$$B(x) := \prod_{\substack{m=1 \\ m \notin \{j+1,\cdots,j+\ell\}}}^{k} (x - b_m).$$

Since our assumption says that $\dfrac{1}{x - b_j} = \cdots = \dfrac{1}{x - b_{j+\ell}}$, the second part of $g(x)$ can be written by

$$\sum_{i=1}^{k} \left(\frac{|z_i|^2}{x - b_i} \cdot B(x) \right) = \left(\frac{(|z_j|^2 + \cdots + |z_{j+\ell}|^2)}{x - b_j} + \sum_{\substack{i=1 \\ i \notin \{j,\cdots,j+\ell\}}}^{k} \frac{|z_i|^2}{x - b_i} \right) \cdot B(x)$$

By substituting $\mathfrak{a}' = \{a_1' \cdots, a_{k+1-\ell}'\}$ and $\mathfrak{b}' = \{b_1', \cdots, b_{k-\ell}'\}$ where

- $a_i' = a_i$ and $b_i' = b_i$ for $1 \leq i \leq j$,
- $a_i' = a_{i+\ell}$ for $j + 1 \leq i \leq k - \ell + 1$, and
- $b_i' = b_{i+\ell}$ for $j + 1 \leq i \leq k - \ell$,

$\mathfrak{a}'$ and $\mathfrak{b}'$ satisfies

$$a_1' > b_1' > \cdots > a_{k-\ell}' > b_{k-\ell}' > a_{k+1-\ell}'.$$

Then we have $g(x) = \det(xI - Z_{(\mathfrak{a}',\mathfrak{b}')}(w))$ where

- $|w_i|^2 = |z_i|^2$ for $1 \leq i \leq j - 1$,
- $|w_j|^2 = |z_j|^2 + \cdots + |z_{j+\ell}|^2$,
- $|w_i|^2 = |z_{i+\ell}|^2$ for $j + 1 \leq i \leq k - \ell$, and
- $w_{k-\ell+1} = \sum_{i=1}^{k-\ell+1} a_i' - \sum_{i=1}^{k-\ell} b_i' = \sum_{i=1}^{k+1} a_i - \sum_{i=1}^{k} b_i = z_{k+1}$.

Thus Lemma 6.4 implies that there exist positive constants $C_1, \cdots, C_{k-\ell}$ such that

$$\widetilde{\mathcal{A}}_{k-\ell+1}(\mathfrak{a}', \mathfrak{b}') = \left\{ (w_1, \cdots, w_{k-\ell}) \in \mathbb{C}^{k-\ell} \mid |w_j|^2 = C_j, \; j = 1, \cdots, k - \ell \right\}.$$

In particular, we have $|w_j|^2 = |z_j|^2 + \cdots + |z_{j+\ell}|^2 = C_j$.

It remains to prove the case where $(a_1, b_1, \cdots, a_k, b_k, a_{k+1})$ contains more than one pattern of type (2) of (6.3). However, since all patterns of type (2) of (6.3) are disjoint from one another, we can apply the same argument to each pattern inductively. This completes the proof. □

Now, we are ready to prove Proposition 6.2.

Proof of Proposition 6.2. For a given sequence $a_1 \geq b_1 \geq \cdots \geq a_k \geq b_k \geq a_{k+1}$, let us consider the W-shaped block $W_k(a, b)$ with walls defined by strict inequalities $a_j > b_j$ or $b_j > a_{j+1}$ for each $j = 1, \cdots, k$. (See Figure 18.) Note that each pattern of type (2) in (6.3) corresponds to an M-shaped block inside of $W_k(\mathfrak{a}, \mathfrak{b})$. More specifically, if

$$a_j > b_j = a_{j+1} = \cdots = a_{j+\ell} = b_{j+\ell} > a_{j+\ell+1}.$$

is one of patterns of type (2) in (6.3) for some j, then it corresponds to a simple closed region which is an M-shaped block $M_{\ell+1}$. In particular, we have

$$|M_{\ell+1}| = 2\ell + 1 = \dim \left\{ (z_j, \cdots, z_{j+\ell}) \in \mathbb{C}^{\ell+1} \mid |z_j|^2 + \cdots + |z_{j+\ell}|^2 = C_j \right\} = \dim S^{2\ell+1}.$$

for a positive real number C_j. Combining the series of Lemma 6.5, 6.7, 6.8, and 6.9, we see $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b}) \cong S_k(\mathfrak{a}, \mathfrak{b})$. Note that $\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ is an $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b})$-bundle over $\mathcal{O}_{\mathfrak{b}}$ via

$$\begin{array}{rccc} \rho_{k+1}: & \mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b}) & \to & \mathcal{O}_{\mathfrak{b}} \\ & x & \mapsto & x^{(k)}. \end{array} \tag{6.6}$$

Thus $\mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b})$ is an $S_k(\mathfrak{a},\mathfrak{b})$-bundle over $\mathcal{O}_{\mathfrak{b}}$. $\square$

Corollary 6.10. *Let f be a face of the Gelfand-Cetlin polytope Δ_λ and γ be the face of the ladder diagram Γ_λ corresponding to f. For any point $\mathbf{u}$ in the interior of f, the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ has an iterated bundle structure given by*

$$\Phi_\lambda^{-1}(\mathbf{u}) = \overline{S_{n-1}(\gamma)} \xrightarrow{p_{n-2}} \overline{S_{n-2}(\gamma)} \to \cdots \xrightarrow{p_1} \overline{S_1(\gamma)} = S_1(\gamma)$$

where $p_{k-1} : \overline{S_k(\gamma)} \to \overline{S_{k-1}(\gamma)}$ is an $S_k(\gamma)$-bundle over $\overline{S_{k-1}(\gamma)}$ for $k = 1, \cdots, n-1$. In particular, $\Phi_\lambda^{-1}(\mathbf{u})$ is of dimension

$$\dim \Phi_\lambda^{-1}(\mathbf{u}) = \sum_{k=1}^{n-1} \dim S_k(\gamma).$$

Proof. For each $(i,j) \in \mathbb{Z}^2_{\geq 1}$, we denote by $\Phi_\lambda^{i,j} \colon \mathcal{O}_\lambda \to \mathbb{R}$ be the component of Φ_λ which corresponds to the unit box $\square^{(i,j)}$ of Γ_λ whose top-right vertex is located at (i,j) in Γ_λ. For each $k \in \mathbb{Z}_{>1}$ and $1 \leq i \leq k$, let us define

$$a_i(k) := \Phi_\lambda^{i,k+1-i}(\mathbf{u}) \quad \text{and} \quad b_i(k) := a_i(k-1).$$

provided with $a_1(1) := \Phi_\lambda^{1,1}(\mathbf{u})$. Let $\mathfrak{a}(k) := (a_1(k), \cdots, a_k(k))$ and $\mathfrak{b}(k) := (b_1(k), \cdots, b_{k-1}(k))$. By applying Proposition 6.2 repeatedly and observing that $S_k(\gamma) = S_k(\mathfrak{a}(k+1), \mathfrak{b}(k+1))$, we describe the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ as the total space of an iterated bundle as in (6.7). The dimension formula immediately follows. $\square$

(6.7)
$$\begin{tikzcd}
\overline{S_{n-1}(\gamma)} \arrow[r] \arrow[d, "p_{n-1}"] & \cdots \arrow[r] \arrow[d] & \iota^*_{(n-1)}(\mathcal{A}(\mathfrak{a}(n),\mathfrak{b}(n))) \arrow[r] \arrow[d, "\iota^*_{(n-1)}\rho_n"] & \mathcal{A}(\mathfrak{a}(n),\mathfrak{b}(n)) \arrow[r, hook, "\iota_{(n)}"] \arrow[d, "\rho_n"] & \mathcal{O}_{\mathfrak{a}(n)} \\
\overline{S_{n-2}(\gamma)} \arrow[r] \arrow[d, "p_{n-2}"] & \cdots \arrow[r] \arrow[d] & \mathcal{A}(\mathfrak{a}(n-1),\mathfrak{b}(n-1)) \arrow[r, hook, "\iota_{(n-1)}"] \arrow[d, "\rho_{n-1}"] & \mathcal{O}_{\mathfrak{a}(n-1)} & \\
\vdots \arrow[r] \arrow[d] & \cdots \arrow[r, hook] \arrow[d] & \mathcal{O}_{\mathfrak{a}(n-2)} & & \\
\overline{S_1(\gamma)} = \mathcal{A}(\mathfrak{a}(2),\mathfrak{b}(2)) \arrow[r, hook, "\iota_{(2)}"] \arrow[d, "p_1=\rho_2"] & \cdots & & & \\
\overline{S_0(\gamma)} = \mathcal{O}_{\mathfrak{a}(1)} & & & &
\end{tikzcd}$$

To complete the proof of Theorem 5.12, it remains to verify that $\Phi_\lambda^{-1}(\mathbf{u})$ is an isotropic submanifold of $(\mathcal{O}_\lambda, \omega_\lambda)$ for every $\mathbf{u} \in \Delta_\lambda$. Recall the definition of the KKS from Section 3.2. For a fixed positive integer $k > 1$, let $\mathfrak{a} = (a_1, \cdots, a_{k+1})$ and $\mathfrak{b} = (b_1, \cdots, b_k)$ be sequences of real numbers satisfying (6.1) and let $\rho_{k+1} \colon \mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b}) \to \mathcal{O}_{\mathfrak{b}}$ be the map defined by $\rho_{k+1}(x) = x^{(k)}$. Then ρ_{k+1} makes $\mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b})$ into a $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a},\mathfrak{b})$-bundle over $\mathcal{O}_{\mathfrak{b}}$. See Proposition 6.2.

For any $x \in \mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b}) \subset \mathcal{O}_{\mathfrak{a}} \subset \mathcal{H}_{k+1}$, let $V_x \subset T_x\mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b})$ be the vertical tangent space at x with respect to ρ_{k+1} and let H_x be the subspace of $T_x\mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b})$ generated by $U(k)$-action where $U(k)$ acts on $\mathcal{A}_{k+1}(\mathfrak{a},\mathfrak{b})$ as a subgroup of $U(k+1)$ via the embedding

$$\begin{array}{cccc} i_k \colon & U(k) & \hookrightarrow & U(k+1) \\ & A & \mapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{array}$$

Then we can see that

$$(\rho_{k+1})_*|_{H_x} \colon H_x \to T_{\rho_{k+1}(x)}\mathcal{O}_{\mathfrak{b}}$$

is surjective since ρ_{k+1} is $U(k)$-invariant and the $U(k)$-action on $\mathcal{O}_{\mathfrak{b}}$ is transitive. Let $(i_k)_* \colon \mathfrak{u}(k) \to \mathfrak{u}(k+1)$ be the induced Lie algebra monomorphism. Then the kernel of $\ker(\rho_{k+1})_*|_{H_x}$ is given by

$$\begin{aligned} \ker(\rho_{k+1})_*|_{H_x} &= \left\{[(i_k)_*(X), x] \mid X \in \mathfrak{u}(k),\ [X, x^{(k)}] = 0\right\} \\ &= \{[(i_k)_*(X), x] \mid X \in T_e U(k)_{x^{(k)}}\} \end{aligned}$$

where $x^{(k)}$ is the $(k \times k)$ leading principal minor of x and $U(k)_{x^{(k)}}$ is the stabilizer of $x^{(k)} \in \mathcal{O}_{\mathfrak{b}}$ for the $U(k)$-action.

From now on, we assume that $U(k)$ acts on $\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ via i_k, unless stated otherwise.

Lemma 6.11. *$U(k)$ acts transitively on $\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$.*

Proof. Since any element $x \in \mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ is conjugate to an element of the following form

$$Z_{(\mathfrak{a},\mathfrak{b})}(z) = \begin{pmatrix} b_1 & & 0 & \overline{z}_1 \\ & \ddots & & \vdots \\ 0 & & b_k & \overline{z}_k \\ z_1 & \dots & z_k & z_{k+1} \end{pmatrix} \in \widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b}) \subset \mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$$

with respect to the $U(k)$-action, it is enough to show that the isotropy subgroup $U(k)_{I_b}$ of $I_{\mathfrak{b}}$ acts on $\widetilde{\mathcal{A}}_{k+1}(\mathfrak{a}, \mathfrak{b})$ transitively where $I_{\mathfrak{b}}$ is the diagonal matrix whose (i,i)-th entry is b_i for $i = 1, \cdots, k$.

Now, let us assume that

$$b_{i_0} := b_1 = \cdots = b_{i_1} > b_{i_1+1} = \cdots = b_{i_2} > \cdots > b_{i_{r-1}+1} = \cdots = b_{i_r} := b_k.$$

for some $r \geq 1$ provided with $i_0 = 0$ and $i_r = k$. Then it is not hard to show that $U(k)_{I_b} = U(k_1) \times \cdots \times U(k_r)$ where $k_j = i_j - i_{j-1}$ for $j = 1, \cdots, r$. For each j, we know that each $(z_{i_j+1}, \cdots, z_{i_{j+1}}) \in \mathbb{C}^{k_{j+1}}$ satisfies either

- $|z_{i_j+1}|^2 + \cdots + |z_{i_{j+1}}|^2 = 0$, or
- $|z_{i_j+1}|^2 + \cdots + |z_{i_{j+1}}|^2 = C_{j+1}$ for some positive constant $C_{j+1} \in \mathbb{R}_{>0}$

by Lemma 6.5, 6.7, 6.8, and 6.9. In the latter case, $U(k)_{I_b}$-action is written as

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} I_b & \overline{z}^t \\ z & z_{k+1} \end{pmatrix} \cdot \begin{pmatrix} g^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_b & g\overline{z}^t \\ zg^{-1} & z_{k+1} \end{pmatrix}$$

for every $g \in U(k)_{I_b}$ and $z = (z_1, \cdots, z_k)$. Note that every $g \in U(k)_{I_b}$ is of the form

$$g = \begin{pmatrix} g_1 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & g_r \end{pmatrix}$$

where $g_i \in U(k_i)$ for $i = 1, \cdots, r$. Thus each $g \in U(k)_{I_b}$ acts on $(z)_{j+1} := (z_{i_j+1}, \cdots, z_{i_{j+1}}) \in \mathbb{C}^{k_{j+1}}$ by $(z)_{j+1} \cdot g_{j+1}^{-1}$ which is equivalent to the standard linear $U(k_{j+1})$-action on the sphere $S^{2k_{j+1}-1} \subset \mathbb{C}^{k_j}$ of radius $\sqrt{C_{j+1}}$. Therefore, the action is transitive. □

Lemma 6.12. *For each $x \in \mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ and any $\xi, \eta \in T_x\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$, we have*

$$(\omega_{\mathfrak{a}})_x(\xi, \eta) = (\omega_{\mathfrak{b}})_{\rho_{k+1}(x)}((\rho_{k+1})_*\xi, (\rho_{k+1})_*\eta).$$

In particular, the vertical tangent space $V_x \subset T_x\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ of ρ_{k+1} is contained in $\ker(\omega_{\mathfrak{a}})_x$.

Proof. Note that Lemma 6.11 implies that any tangent vector in $T_x\mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$ can be written as $[(i_k)_*(X), x]$ for some $X \in \mathfrak{u}(k)$ where

$$(i_k)_*(X) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{u}(k+1).$$

Thus for any $\xi, \eta \in T_x \mathcal{A}_{k+1}(\mathfrak{a}, \mathfrak{b})$, there exist $X, Y \in \mathfrak{u}(k)$ such that

$$\xi = [(i_k)_*(X), x], \quad \eta = [(i_k)_*(Y), x].$$

Therefore, we have

$$(\omega_{\mathfrak{a}})_x(\xi, \eta) = \mathrm{tr}(ix[(i_k)_*(X), (i_k)_*(Y)]) = \mathrm{tr}(ix^{(k)}[X, Y]) = (\omega_{\mathfrak{b}})_{x^{(k)}}([X, x^{(k)}], [Y, x^{(k)}])$$

since the $(k+1, k+1)$-th entry of the matrix $x[(i_k)_*(X), (i_k)_*(Y)]$ is zero by direct computation. Since ρ_{k+1} is $U(k)$-invariant, we obtain that $[X, x^{(k)}] = (\rho_{k+1})_*(\xi)$ and $[Y, x^{(k)}] = (\rho_{k+1})_*(\eta)$. This completes the proof. ☐

Proposition 6.13. *For any* $\mathbf{u} \in \Delta_\lambda$*, the fiber* $\Phi_\lambda^{-1}(\mathbf{u})$ *is an isotropic submanifold of* $(\mathcal{O}_\lambda, \omega_\lambda)$*, i.e.,* $\omega|_{\Phi_\lambda^{-1}(\mathbf{u})} = 0$.

Proof. Suppose that γ is a face of Γ_λ such that the corresponding face f_γ contains $\mathbf{u}$ in its interior. Let $x \in \Phi_\lambda^{-1}(\mathbf{u})$ and let $\xi, \eta \in T_x \Phi_\lambda^{-1}(\mathbf{u})$. Then Corollary 6.10 says that $\Phi_\lambda^{-1}(\mathbf{u})$ is the total space of an iterated bundle

$$\Phi_\lambda^{-1}(\gamma) = \overline{S_{n-1}}(\gamma) \xrightarrow{p_{n-2}} \overline{S_{n-2}}(\gamma) \to \cdots \xrightarrow{p_1} \overline{S_1}(\gamma) = S_1(\gamma).$$

described in (6.7).

For each integer k with $1 < k \leq n$ and $1 \leq i \leq k$, let us define

$$a_i(k) := \Phi_\lambda^{i, k+1-i}(\mathbf{u}).$$

provided with $a_1(1) := \Phi_\lambda^{1,1}(\mathbf{u})$ and let $\mathfrak{a}(k) := (a_1(k), \cdots, a_k(k))$. In particular, we have $\mathfrak{a}(n) = \lambda = \{\lambda_1, \cdots, \lambda_n\}$. Then Lemma 6.12 implies that

$$(\omega_{\mathfrak{a}(n)})_x(\xi, \eta) = (\omega_{\mathfrak{a}(n-1)})_{\rho_n(x)}((\rho_n)_*(\xi), (\rho_n)_*(\eta)).$$

Since p_{n-2} is the restriction of ρ_n to $\overline{S}_{n-1}(\gamma) \subset \mathcal{A}(\mathfrak{a}(n), \mathfrak{a}(n-1))$, both $(\rho_n)_*(\xi)$ and $(\rho_n)_*(\eta)$ are lying on $T_{\pi_{n-2}(x)} \overline{S}_{n-2}(\gamma) \subset T_{p_{n-2}(x)} \mathcal{A}(\mathfrak{a}(n-1), \mathfrak{a}(n-2))$.

Thus we can apply Lemma 6.12 inductively so that we have

$$\begin{aligned}
(\omega_{\mathfrak{a}(n)})_x(\xi, \eta) &= (\omega_{\mathfrak{a}(n-1)})_{\rho_n(x)}((\rho_n)_*(\xi), (\rho_n)_*(\eta)) \\
&= (\omega_{\mathfrak{a}(n-2)})_{\rho_{n-1} \circ \rho_n(x)}((\rho_{n-1} \circ \rho_n)_*(\xi), (\rho_{n-1} \circ \rho_n)_*(\eta)) \\
&= \cdots \\
&= (\omega_{\mathfrak{a}(2)})_{\rho_2 \circ \cdots \circ \rho_n(x)}((\rho_2 \circ \cdots \circ \rho_n)_*(\xi), (\rho_2 \circ \cdots \circ \rho_n)_*(\eta)) \\
&= 0.
\end{aligned}$$

The last equality follows from $\rho_2 \colon \mathcal{A}(\mathfrak{a}(2), \mathfrak{a}(1)) \to \{a_1(1) = \Phi_\lambda^{1,1}(\mathbf{u})\} = \text{point}$. ☐

Proof of Theorem 5.12. It follows from Corollary 6.10 and Proposition 6.13. ☐

7. DEGENERATIONS OF FIBERS TO TORI

In this section, we study the topology of GC fibers via toric degenerations and describe how each fiber of a GC system degenerates to a torus fiber of a toric moment map.

Let λ be given in (3.8) and let f be a face of the GC polytope Δ_λ of dimension r. It is shown in Section 6 that a fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is the total space of an iterated bundle

$$\Phi_\lambda^{-1}(\mathbf{u}) \cong \overline{S_{n-1}}(\gamma_f) \xrightarrow{p_{n-1}} \overline{S_{n-2}}(\gamma_f) \to \cdots \xrightarrow{p_2} \overline{S_1}(\gamma_f) \xrightarrow{p_1} \overline{S_0}(\gamma_f) := \text{point} \tag{7.1}$$

where γ_f is the face of Γ_λ corresponding to f, see Corollary 6.10. The main theorem of this section further claims that every S^1-factor appeared in any stage of the iterated bundle in (7.1) comes out as a trivial factor. Namely, letting $\overline{S_i}(\gamma)$ be $(S^1)^{m_i} \times Y_i$ such that $\sum_{i=1}^{n-1} m_i = r$, the fiber is written as the product

$$\Phi_\lambda^{-1}(\mathbf{u}) = (S^1)^r \times Y(\mathbf{u})$$

where $Y(\mathbf{u})$ is the total space of an iterated bundle whose fibers at stages are $Y_\bullet$'s, see Theorem 7.9. Indeed, $Y(\mathbf{u})$ shrinks to a point through a toric degeneration, which illustrates how fibers degenerate into toric fibers. As an application, it provides a more concrete description of the GC fiber. Furthermore, we compute the fundamental group and the second homotopy group of $\Phi_\lambda^{-1}(\mathbf{u})$.

Recall that for a given Kähler manifold (X, ω), a flat family $\pi\colon \mathcal{X} \to \mathbb{C}$ of algebraic varieties is called a *toric degeneration* (X, ω) if there exists an algebraic embedding $i\colon \mathcal{X} \hookrightarrow \mathbb{P}^N \times \mathbb{C}$ such that

(1) the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & \mathbb{P}^N \times \mathbb{C} \arrow[d, "q"] \\ & \mathbb{C} \end{tikzcd} \tag{7.2}$$

commutes where $q\colon \mathbb{P}^N \times \mathbb{C} \to \mathbb{C}$ is the projection to the second factor,

(2) $\pi^{-1}(\mathbb{C}^*)$ is isomorphic to $X \times \mathbb{C}^*$ as a complex variety, and

(3) For the product Kähler form $\widetilde{\omega} := \omega_{\mathrm{FS}} \oplus \omega_{\mathrm{std}}$ on $\mathbb{P}^N \times \mathbb{C}$ where ω_{FS} is a multiple of the Fubini-Study form on $\mathbb{P}^N$ and ω_{std} is the standard Kähler form on $\mathbb{C}$,
   - $(X_1, \widetilde{\omega}|_{X_1})$ is symplectomorphic to (X, ω), and
   - X_0 is a projective toric variety (in $\mathbb{P}^N$) and $\widetilde{\omega}|_{X_0}$ is a torus invariant Kähler form denoted by ω_0

   where $X_t := \pi^{-1}(t) \cong i(\pi^{-1}(t)) \subset \mathbb{P}^N \times \{t\}$ is a projective variety for every $t \in \mathbb{C}$.

Let $\mathcal{X}^{\mathrm{sm}} \subset \mathcal{X}$ be the smooth locus of $\mathcal{X}$. The Hamiltonian vector field, denoted by $\widetilde{V}_\pi$, for the imaginary part $\mathrm{Im}(\pi)$ of π is defined on $\mathcal{X}^{\mathrm{sm}}$. By the holomorphicity, π satisfies the Cauchy-Riemann equation which induces $\nabla \mathrm{Re}(\pi) = -\widetilde{V}_\pi$ where ∇ denotes the gradient vector field with respect to a Kähler metric associated with $\widetilde{\omega}$. We call

$$V_\pi := \widetilde{V}_\pi / ||\widetilde{V}_\pi||^2$$

the *gradient-Hamiltonian vector field of* π, see Ruan [Ru]. Then the one-parameter subgroup generated by V_π induces a symplectomorphism

$$\phi\colon (\mathcal{U}, \omega) \to (\phi(\mathcal{U}), \omega_0) \tag{7.3}$$

on an open dense subset $\mathcal{U}$ of X ($\cong X_1$) such that $\phi(\mathcal{U}) = X_0^{\mathrm{sm}}$ and it is extended to a surjective continuous map

$$\phi\colon X \to X_0$$

defined on the whole X, see Harada-Kaveh [HaK, Theorem A] for more details.

We may also consider a toric degeneration of a Kähler manifold *"equipped with a completely integrable system"* as follows. Consider a triple (X, ω, Θ) where $\Theta = (\Theta_\alpha)_{\alpha \in \mathcal{I}}$ is a (continuous) completely integrable system on (X, ω) and $\mathcal{I}$ is the index set for Θ such that $|\mathcal{I}| = \dim_{\mathbb{C}} X_0$. We call $\pi\colon \mathcal{X} \to \mathbb{C}$ a *toric degeneration* of (X, ω, Θ) if π is a toric degeneration of (X, ω) and $\Theta = \Phi \circ \phi$ where $\Phi\colon X_0 \to \mathbb{R}^{|\mathcal{I}|}$ is a toric moment map on (X_0, ω_0), see [NNU1, Definition 1.1].

$$\begin{tikzcd} (X_1, \omega_1) \cong (X, \omega) \arrow[rr, "\phi"] \arrow[dr, "\Theta"'] & & (X_0, \omega_0) \subset \mathbb{P}^N \times \{0\} \arrow[dl, "\Phi"] \\ & \Delta_0 & \end{tikzcd} \tag{7.4}$$

The Hamiltonian vector field of each component Φ_α of Φ ($= \{\Phi_\alpha\}_{\alpha \in \mathcal{I}}$) is globally defined on X_0, even though X_0 is singular, by the following reason. Note that $X_0 \subset \mathbb{P}^N \times \{0\}$ is a projective toric variety, which is the Zariski closure of the single $(\mathbb{C}^*)^{|\mathcal{I}|}$-orbit on X_0. The $(\mathbb{C}^*)^{|\mathcal{I}|}$-action on X_0 extends to the linear Hamiltonian action on $\mathbb{P}^N$ with respect to ω_{FS}. We denote by $(S^1)^{|\mathcal{I}|}$ the maximal compact subgroup of $(\mathbb{C}^*)^{|\mathcal{I}|}$, by $\widetilde{\Phi} = (\widetilde{\Phi}_\alpha)_{\alpha \in \mathcal{I}}$ a moment map for the $(S^1)^{|\mathcal{I}|}$-action on $\mathbb{P}^N$, and by ξ_α the fundamental vector field of $\widetilde{\Phi}_\alpha$ on $\mathbb{P}^N$ for each $\alpha \in \mathcal{I}$. Then each component Φ_α coincides with the restriction of $\widetilde{\Phi}_\alpha$ to X_0. Since X_0 is $T^{|\mathcal{I}|}$-invariant, the trajectory of the flow of ξ_α passing through any point of X_0 should be on X_0. In other words, the restriction $\xi_\alpha|_{X_0}$ should be tangent[3] to X_0, and therefore the Hamiltonian vector field of Φ_α is defined on the whole X_0.

[3] Every toric variety is a stratified space [LS] so that each point in X_0 is contained in an open smooth stratum and each vector field ξ_α is tangent to the stratum.

Now, let $\mathcal{V}_\alpha$ be the open dense subset of X on which Θ_α is smooth. Then the Hamiltonian vector field, denoted by ζ_α, of Θ_α is defined on $\mathcal{V}_\alpha$. For any subset $\mathcal{I}' \subset \mathcal{I}$, we let

$$\mathcal{V}_{\mathcal{I}'} := \bigcap_{\alpha \in \mathcal{I}'} \mathcal{V}_\alpha \tag{7.5}$$

so that the Hamiltonian vector field of Θ_α is defined on $\mathcal{V}_{\mathcal{I}'}$ for every $\alpha \in \mathcal{I}'$. If Θ_α is a periodic Hamiltonian, i.e., if Θ_α generates a circle action for every $\alpha \in \mathcal{I}'$, then $\mathcal{V}_{\mathcal{I}'}$ admits the $T^{|\mathcal{I}'|}$-action generated by $\{\zeta_\alpha\}_{\alpha \in \mathcal{I}'}$. Note that $\mathcal{V}_{\mathcal{I}'}$ is open dense in X so that $\mathcal{U} \cap \mathcal{V}_{\mathcal{I}'}$ is also open dense in X where $\mathcal{U}$ is in (7.3).

Lemma 7.1. *For any $\alpha \in \mathcal{I}$ and $p \in \mathcal{V}_\alpha$, we have*

$$\phi(\exp(t\,\zeta_\alpha) \cdot p) = \exp(t\,\xi_\alpha) \cdot \phi(p)$$

for every $t \in \mathbb{R}$.

Proof. Fix $\alpha \in \mathcal{I}$. From the fact that

- $\phi^*\omega_0 = \omega$ on $\mathcal{U} \cap \mathcal{V}_\alpha$, and
- $\Theta = \Phi \circ \phi$,

it follows that

$$\omega_0(\phi_*(\zeta_\alpha), \phi_*(\cdot)) = \omega(\zeta_\alpha, \cdot) = d\Theta_\alpha(\cdot) = d\Phi_\alpha \circ d\phi(\cdot) = \omega_0(\xi_\alpha, \phi_*(\cdot))$$

so that $\phi_*(\zeta_\alpha) = \xi_\alpha$ on $\mathcal{U} \cap \mathcal{V}_\alpha$. Since $\mathcal{U} \cap \mathcal{V}_\alpha$ is open dense in $\mathcal{V}_\alpha$ and ξ_α is defined on whole X_0, the equality

$$\phi_*(\zeta_\alpha) = \xi_\alpha$$

holds on $\mathcal{V}_\alpha$. This completes the proof by the uniqueness of a solution of first-order ODE's. $\square$

Let $\mathcal{I}' \subset \mathcal{I}$ and suppose that Θ_α is a periodic Hamiltonian on $\mathcal{V}_{\mathcal{I}'}$ for every $\alpha \in \mathcal{I}'$. Since Φ_α is also a periodic Hamiltonian on X_0, we deduce the following immediately from Lemma 7.1.

Corollary 7.2. *Let $\mathcal{I}' \subset \mathcal{I}$ such that $\{\Theta_\alpha\}_{\alpha \in \mathcal{I}'}$ are periodic Hamiltonians on $\mathcal{V}_{\mathcal{I}'}$. Then ϕ is $T^{|\mathcal{I}'|}$-equivariant on $\mathcal{V}_{\mathcal{I}'}$.*

We will apply Corollary 7.2 to GC systems. Recall that for any λ given in (3.8), Nishinou-Nohara-Ueda [NNU1] built a toric degeneration of the GC system Φ_λ on a partial flag manifold $(\mathcal{O}_\lambda, \omega_\lambda, \Phi_\lambda)$. We first describe their toric degeneration of $(\mathcal{O}_\lambda, \omega_\lambda, \Phi_\lambda)$ and a continuous map $\phi \colon \mathcal{O}_\lambda \to X_0$ given in (7.3) obtained by the gradient-Hamiltonian flow, where X_0 is the central fiber of the toric degeneration. (We also refer the reader to [KoM] or [NNU1] for more details.)

In [KoM], Kogan and Miller constructed an $(n-1)$-parameter family $F \colon \mathcal{X} \to \mathbb{C}^{n-1}$, called a *toric degeneration in stage*, of projective varieties that can be factored as

$$F = q \circ \iota, \qquad \mathcal{X} \overset{\iota}{\hookrightarrow} P \times \mathbb{C}^{n-1} \overset{q}{\to} \mathbb{C}^{n-1}, \quad P = \prod_{k=1}^{r} \mathbb{P}_{n_k}{}^{4} \tag{7.6}$$

where ι is an algebraic embedding with a Kähler form $\widetilde{\omega}$ on $P \times \mathbb{C}^{n-1}$ such that

- $(F^{-1}(1, \cdots, 1), \widetilde{\omega}|_{F^{-1}(1,\cdots,1)}) \cong (\mathcal{O}_\lambda, \omega_\lambda)$ and
- $F^{-1}(0, \cdots, 0)$ is isomorphic to the GC toric variety X_0 whose moment map image is Δ_λ with the torus-invaraint Kähler form $\widetilde{\omega}|_{F^{-1}(0,\cdots,0)}$ on X_0.

See [NNU1, Section 5 and Remark 5.2] for more details. Following [NNU1], we denote the coordinates of $\mathbb{C}^{n-1}$ by $(t_2, \cdots, t_n)$ and $F^{-1}(1, \cdots, 1, t = t_k, 0, \cdots, 0)$ by $X_{k,t}$ for $2 \leq k \leq n$ and $t \in \mathbb{C}$. Then the set

$$\{X_{k,t}\}_{2 \leq k \leq n, t \in \mathbb{C}}$$

can be regarded as a family of algebraic varieties in P via the embedding ι where $X_{n,1} \subset P$ is the image of the Plücker embedding of $\mathcal{O}_\lambda$ and $X_{2,0} \subset P$ is the toric variety X_0 associated with Δ_λ.

[4] For $m \in \mathbb{Z}_+$, we denote by $\mathbb{P}_m := \mathbb{P}(\wedge^m \mathbb{C}^n) = \mathbb{P}^{\binom{n}{m}-1}$.

Let $T^{\frac{n(n-1)}{2}}$ be the compact subtorus of $(\mathbb{C}^*)^{\frac{n(n-1)}{2}} \subset X_0$ and consider a decomposition

$$T^{\frac{n(n-1)}{2}} \cong T^1 \times T^2 \times \cdots \times T^{n-1}.$$

For each $k = 1, \cdots, n-1$, we denote the i-th coordinate of T^k by $\tau_{i,j}$ [5] where $i + j = k + 1$. Then, each S^1-action on X_0 corresponding to $\tau_{i,j}$ can be extended to the linear Hamiltonian S^1-action on P and we denote a corresponding moment map by

$$\Phi^{i,j} : P \to \mathbb{R}. \tag{7.7}$$

On the other hand, recall that $U(n)$ acts on P in a Hamiltonian fashion with a moment map $\mu^{(n)} : P \to \mathfrak{u}(n)^*$ and consider the sequence of subgroups of $U(n)$

$$U(1) \subset U(2) \subset \cdots \subset U(n-1) \subset U(n), \quad U(k) := \begin{pmatrix} U(k) & 0 \\ 0 & I_{n-k} \end{pmatrix}.$$

Then each $U(k)$ also acts on P in a Hamiltonian fashion and the moment map induced from $\mu^{(n)}$ is given by

$$\mu^{(k)} \colon P \to \mathfrak{u}(k)^* \cong \{(k \times k)\text{-Hermitian matrices}\}$$

for $k = 1, \cdots, n-1$. For each pair $(i,j) \in (\mathbb{Z}_+)^2$ with $i + j = k + 1$, define

$$\Phi_\lambda^{i,j} \colon P \to \mathbb{R} \tag{7.8}$$

which assigns the i-th largest eigenvalue of $\mu^{(k)}(p)$ for every $p \in P$.

Remark 7.3. If one follows the notations used in [NNU1, Section 5], then we may express as

$$\tau_{i,j} = \tau_i^{(j)}, \quad \Phi^{i,j} = v_i^{(k)}, \quad \Phi_\lambda^{i,j} = \lambda_i^{(k)}, \quad i + j = k + 1.$$

An important fact is that a fiber of F in (7.6) is not invariant neither under the $U(n)$-action nor under the $T^1 \times T^2 \times \cdots \times T^{n-1}$-action on P, but $X_{k,t}$ is invariant under the $U(k-1)$ action and the $T^k \times \cdots \times T^{n-1}$ action for every $k \geq 2$ and $t \in \mathbb{C}$. The following theorem states that the maps $\Phi_\lambda^{i,j}$'s in (7.8) and $\Phi^{i,j}$'s in (7.7) defined on P induces a completely integrable system on $X_{k,t}$ and how the GC system Φ_λ on $X_{n,1} \cong \mathcal{O}_\lambda$ degenerates into the toric moment map Φ on $X_{2,0} \cong X_0$ *in stages*. See also Section 5 and Section 7 of [NNU1].

Theorem 7.4 (Theorem 6.1 in [NNU1]). *For every $k \geq 2$ and $t \in \mathbb{C}$, the map*

$$\Phi_{k,t} := \left. (\underbrace{\Phi_\lambda^{1,1}}_{1}, \cdots, \underbrace{\Phi_\lambda^{1,k-1}, \cdots, \Phi_\lambda^{k-1,1}}_{k-1}, \underbrace{\Phi^{1,k}, \cdots, \Phi^{k,1}}_{k}, \cdots, \underbrace{\Phi^{1,n-1}, \cdots, \Phi^{n-1,1}}_{n-1}) \right|_{X_{k,t}} : X_{k,t} \to \mathbb{R}^{\dim \Delta_\lambda}$$

is a completely integrable system on $X_{k,t}$ in the sense of Definition 3.6. Moreover, we have $\Phi_\lambda^{i,k-i} = \Phi^{i,k-i}$ on $X_{k,0} = X_{k-1,1}$ for every $i = 1, \cdots, k-1$.

Note that $\Phi_{k,t}$'s are related to one another via the *gradient-Hamiltonian flows* introduced by Wei-Dong Ruan [Ru]. For each $m = 1, \cdots, n-1$, let F_m be the m-th component of F in (7.6) and let $\widetilde{V}_m$ be the Hamiltonian vector field of $\mathrm{Im}(F_m)$ on the smooth locus $\mathcal{X}^{\mathrm{sm}}$ of $\mathcal{X}$. Then the gradient-Hamiltonian vector field is defined by

$$V_m := \widetilde{V}_m / ||\widetilde{V}_m||^2.$$

The flow of V_m, which we denote by $\phi_{m,t}$ where t is a time parameter, preserves the fiberwise symplectic form and so $\phi_{m,t}$ induces a symplectomorphism on an open subset of each fiber on which $\phi_{m,t}$ is smooth. As a corollary, we have the following.

[5] For the consistency of (3.7), we use the index (i,j).

Corollary 7.5 (Corollary 7.3 in [NNU1]). *The gradient-Hamiltonian vector field V_k gives a deformation of $X_{k,t}$ preserving the structure of completely integrable systems. In particular, we have the following commuting diagram for every $t \geq 0$:*

(7.9)
$$\begin{tikzcd} X_{k,1} \arrow[rr, "\phi_{k,1-t}"] \arrow[dr, "\Phi_{k,1}"'] & & X_{k,t} \arrow[dl, "\Phi_{k,t}"] \\ & \Delta_\lambda & \end{tikzcd}$$

By Corollary 7.5, we obtain a continuous map

(7.10)
$$\phi\colon \mathcal{O}_\lambda = X_{n,1} \to X_0 = X_{2,0}$$

where $\phi = \phi_{2,1} \circ \cdots \circ \phi_{n,1}$ and it satisfies $\Phi \circ \phi = \Phi_\lambda$. Note that

$$\phi\colon \Phi_\lambda^{-1}(\mathring{\Delta}_\lambda) \to \Phi^{-1}(\mathring{\Delta}_\lambda)$$

is a symplectomorphism.

Now, for a given $\mathbf{u} \in \Delta_\lambda$, we investigate which component $\Phi_\lambda^{i,j}$ is smooth on $\Phi_\lambda^{-1}(\mathbf{u})$. Let γ be an r-dimensional face of Γ_λ and let f_γ be the corresponding face of Δ_λ. Let

(7.11)
$$\mathcal{I}_{\mathcal{C}(\gamma)} := \{(i,j) \mid (i,j) = v_\sigma \text{ for some minimal cycle } \sigma \text{ of } \gamma\}$$

so that $|\mathcal{I}_{\mathcal{C}(\gamma)}| = r$. (See Figure 14.)

Lemma 7.6. *Each $\Phi_\lambda^{i,j}$ is smooth on $\Phi_\lambda^{-1}(\mathbf{u})$ for every $(i,j) \in \mathcal{I}_{\mathcal{C}(\gamma)}$. In particular, $\Phi_\lambda^{i,j}$ is smooth on $\Phi_\lambda^{-1}(\mathring{f}_\gamma)$. Furthermore, $\{\Phi_\lambda^{i,j}\}_{(i,j)\in\mathcal{I}_{\mathcal{C}(\gamma)}}$ generates a smooth fiberwise T^r-action on $\Phi_\lambda^{-1}(\mathbf{u})$ for each $\mathbf{u} \in \mathring{f}_\gamma$.*

Proof. The smoothness of each $\Phi_\lambda^{i,j}$ on $\Phi_\lambda^{-1}(\mathring{f}_\gamma)$ follows from the condition $(i,j) = v_\sigma$ and Proposition 3.11. Also, we have seen in Section 3.4 that each $\Phi_\lambda^{i,j}$ generates a smooth circle action on an open dense subset of $\mathcal{O}_\lambda$ on which $\Phi_\lambda^{i,j}$ is smooth. Since all components of Φ_λ Poisson-commute with each other, it finishes the proof. □

We recall the following well-known fact on toric varieties. Let $\Delta \subset \mathbb{R}^N \cong \mathfrak{t}^*$ be a convex polytope of dimension N and let X_Δ be the corresponding projective toric variety where $\mathfrak{t}$ is the Lie algebra of the maximal compact torus T^N in $(\mathbb{C}^*)^N \subset X_0$ and $\mathfrak{t}^*$ is the dual of $\mathfrak{t}$. Let f be an r-dimensional face of Δ and suppose that $F_1, \cdots, F_m$ are facets of Δ such that $f = \cap_{i=1}^m F_i$. Also, we let $v_i \in \mathfrak{t}$ be the inward primitive integral normal vector of F_i for $i = 1, \cdots, m$

Lemma 7.7 (Exercise 12.4.7.(d) in [CLS]). *Let $\xi_1, \cdots, \xi_r$ be primitive integral vectors in $\mathfrak{t}$ which generates an r-dimensional subtorus T^r of T^N. Then the T^r acts on $\Phi^{-1}(\mathring{f})$ freely if $\mathbb{Z}^N \cong \langle v_1, \cdots, v_m, \xi_1, \cdots, \xi_r\rangle$.*

We then obtain the following.

Proposition 7.8. *Fix $\mathbf{u} \in \mathring{f}_\gamma$ and consider the T^r-action on $\Phi_\lambda^{-1}(\mathbf{u})$ generated by $\{\Phi_\lambda^{i,j}\}_{(i,j)\in\mathcal{I}_{\mathcal{C}(\gamma)}}$ as given in Lemma 7.6. Then the T^r-action on $\Phi_\lambda^{-1}(\mathbf{u})$ is free. Furthermore, $\Phi_\lambda^{-1}(\mathbf{u})$ becomes a trivial principal bundle over $\Phi_\lambda^{-1}(\mathbf{u})/T^r$, that is,*

$$\Phi_\lambda^{-1}(\mathbf{u}) \cong T^r \times \Phi_\lambda^{-1}(\mathbf{u})/T^r.$$

Proof. We first show that the T^r-action is free on $\Phi^{-1}(\mathbf{u})$. For each $(i,j) \in \mathcal{I}$[6], we denote by

$$\xi_{i,j} := (e_{k,l}) \in \mathbb{R}^{\frac{n(n-1)}{2}}, \quad \begin{cases} e_{k,l} = 1 \text{ if } (k,l) = (i,j) \text{ and } e_{k,l} = 0 \text{ otherwise} & \text{if } (i,j) \in \mathcal{I}_\lambda \\ e_{k,l} = 0 \text{ for every } (k,l) & \text{if } (i,j) \notin \mathcal{I}_\lambda. \end{cases}$$

By the min-max principle (3.8) and the dimension formula given in Definition 4.3, an inward primitive integral normal vector v_F for any facet F of Δ_λ is either

$$v_{vert}^{i,j} := \xi_{i,j+1} - \xi_{i,j}, \quad \text{or} \quad v_{hor}^{i,j} := -\xi_{i+1,j} + \xi_{i,j}$$

[6] See (3.5) and (3.9) for the definition of $\mathcal{I}$ and $\mathcal{I}_\lambda$, respectively.

for some $(i,j) \in \mathcal{I}_\lambda$. In particular, if F contains f_γ, then v_F is either

$$\begin{cases} v^{i,j}_{vert}, & \square^{i,j} \text{ and } \square^{i,j+1} \text{ are in the same simple region of } \gamma, \text{ or} \\ v^{i,j}_{hor}, & \square^{i,j} \text{ and } \square^{i+1,j} \text{ are in the same simple region of } \gamma \end{cases}$$

for some $(i,j) \in \mathcal{I}_\lambda$. Then, it is not hard to see that

$$\{v_F\}_{f_\gamma \subset F} \cup \{\xi_{i,j}\}_{(i,j)\in \mathcal{I}_{\mathcal{C}(\gamma)}}$$

generates the full lattice $\mathbb{Z}^N$ where $N = \dim \Delta_\lambda$. Therefore the T^r-action generated by $\{\Phi^{i,j}\}_{(i,j)\in\mathcal{I}_{\mathcal{C}(\gamma)}}$ on $\Phi^{-1}(\mathring{f}_\gamma)$ is free by Lemma 7.7, and hence the T^r-action on each fiber $\Phi^{-1}(\mathbf{u})$ is free for every $\mathbf{u} \in \mathring{f}_\gamma$. Since

$$\phi_{\mathbf{u}} := \phi|_{\Phi_\lambda^{-1}(\mathbf{u})} \colon \Phi_\lambda^{-1}(\mathbf{u}) \to \Phi^{-1}(\mathbf{u}) \cong T^r.$$

is T^r-equivariant by Corollary 7.2, we see that the T^r-action on $\Phi_\lambda^{-1}(\mathbf{u})$ is also free.

The freeness of the T^r-action on $\Phi_\lambda^{-1}(\mathbf{u})$ implies that the map $\phi_{\mathbf{u}} := \phi|_{\Phi_\lambda^{-1}(\mathbf{u})}$ becomes a principal bundle map such that

$$\begin{tikzcd} \Phi_\lambda^{-1}(\mathbf{u}) \arrow[r, "\phi_{\mathbf{u}}"] \arrow[d, "/T^r"'] & \Phi^{-1}(\mathbf{u}) \cong T^r \arrow[d, "/T^r"] \\ \Phi_\lambda^{-1}(\mathbf{u})/T^r \arrow[r] & \text{point} \end{tikzcd}$$

commutes. In particular, $\Phi_\lambda^{-1}(\mathbf{u})$ is a pull-back bundle of the trivial T^r-bundle over a point so that $\Phi_\lambda^{-1}(\mathbf{u})$ is a trivial T^r-bundle as desired. $\square$

To sum up, we can describe how each fiber of a GC system deforms into a torus fiber of a moment map of X_0 via a toric degeneration as follows.

Theorem 7.9. *Let γ be a face of the ladder diagram Γ_λ of dimension r and let f_γ be the corresponding face of the Gelfand-Cetlin polytope Δ_λ. For each point $\mathbf{u}$ in the relative interior $\mathring{f}_\gamma$, all S^1-factors that appeared in each stage of the iterated bundle structure $\overline{S_\bullet}(\gamma)$ of $\Phi_\lambda^{-1}(\mathbf{u})$ given in Theorem 5.12 are factored out. That is,*

$$\Phi_\lambda^{-1}(\mathbf{u}) \cong T^r \times \overline{S_\bullet}(\gamma)'$$

where $\overline{S_\bullet}(\gamma)'$ is the total space of the iterated bundle which can be obtained by the construction of $\overline{S_\bullet}(\gamma)$ ignoring all S^1-factors appeared in each stage. Furthermore, the continuous map ϕ in (7.10) on each fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is the projection map

$$\Phi_\lambda^{-1}(\mathbf{u}) \cong T^r \times \overline{S_\bullet}(\gamma)' \xrightarrow{\phi} T^r \cong \Phi^{-1}(\mathbf{u}).$$

Proof. Consider the iterated bundle structure of $\Phi_\lambda^{-1}(\mathbf{u})$ given in Theorem 5.12 :

$$\begin{array}{ccccccccc} \Phi_\lambda^{-1}(\mathbf{u}) \cong \overline{S_{n-1}}(\gamma) & \xrightarrow{p_{n-1}} & \overline{S_{n-2}}(\gamma) & \to & \cdots & \xrightarrow{p_2} & \overline{S_1}(\gamma) & \xrightarrow{p_1} & \overline{S_0}(\gamma) := \text{point} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ S_{n-1}(\gamma) & & S_{n-2}(\gamma) & & \cdots & & S_1(\gamma) & & \end{array} \tag{7.12}$$

where $S_k(\gamma)$ is the fiber of $p_k \colon \overline{S_k}(\gamma) \to \overline{S_{k-1}}(\gamma)$ at the k-th stage defined in (5.2). Each $S_k(\gamma)$ can be factorized into $S_k(\gamma) = (S^1)^{r_k} \times Y_k$ where Y_k is either a point or a product of odd-dimensional spheres without any S^1-factors. (See the proof of Proposition 6.2.) Then we claim that

(1) there is a one-to-one correspondence between the S^1-factors that appeared in each stage and the elements in $\mathcal{I}_{\mathcal{C}(\gamma)}$,
(2) $(S^1)^{r_k}$ acts on $\overline{S_k}(\gamma)$ fiberwise with respect to $p_k \colon \overline{S_k}(\gamma) \to \overline{S_{k-1}}(\gamma)$, and
(3) the torus action on $\Phi_\lambda^{-1}(\mathbf{u}) \cong \overline{S_{n-1}}(\gamma)$ generated by $\{\Phi_\lambda^{i,j}\}_{(i,j)\in\mathcal{I}_{\mathcal{C}(\gamma)}, i+j-1=k}$ is an extension of the $(S^1)^{r_k}$-action on $\overline{S_k}(\gamma)$ given in (2).

The first statement (1) is straightforward since each $(i,j) \in \mathcal{I}_{\mathcal{C}(\gamma)}$ corresponds to an M_1-block in $W_{i+j-1}(\gamma)$ containing a bottom vertex of W_{i+j-1} so that each $(i,j) \in \mathcal{I}_{\mathcal{C}(\gamma)}$ assigns an S^1-factor in $S_{i+j-1}(\gamma)$. See Section 5.1. The third statement (3) is also clear since each $\Phi_\lambda^{i,j}$ with $i+j-1=k$ descends to a function $\Phi_{\lambda_k}^{i,j}$ on $\overline{S_k}(\gamma)$ where

$$\lambda_k = (\Phi_\lambda^{1,k}(\mathbf{u}), \cdots, \Phi_\lambda^{k,1}(\mathbf{u})).$$

For the second statement (2), fix $k \geq 1$ and consider the k-th stage

$$\begin{array}{ccc} S_k(\gamma) = (S^1)^{r_k} \times Y_k & \hookrightarrow & \overline{S_k}(\gamma) \subset (\mathcal{O}_{\mathfrak{a}}, \omega_{\mathfrak{a}}) \\ & & \downarrow p_k \\ & & \overline{S_{k-1}}(\gamma) \subset (\mathcal{O}_{\mathfrak{b}}, \omega_{\mathfrak{b}}) \end{array} \tag{7.13}$$

of $\overline{S_\bullet}(\gamma)$. As we have seen in the diagram (6.7), $\overline{S_k}(\gamma)$ is a subset of $\mathcal{A}(\mathfrak{a}, \mathfrak{b})$ where $\mathfrak{a} = (a_1, \cdots, a_{k+1})$ and $\mathfrak{b} = (b_1, \cdots, b_k)$ with

- $a_i = \Phi_\lambda^{i,k+1-i}(\mathbf{u})$ for $1 \leq i \leq k+1$ and
- $b_j = \Phi_\lambda^{j,k-j}(\mathbf{u})$ for $1 \leq j \leq k$.

Note that $\Phi_{\mathfrak{a}}^{i,k-i}$ generates a circle action on $\mathcal{O}_{\mathfrak{a}}$ whenever $a_i < b_i < a_{i+1}$, see Section 3.4. For any smooth functions f on $\mathcal{O}_{\mathfrak{b}}$ and $\widehat{f} = f \circ p_k$ on $\mathcal{O}_{\mathfrak{a}}$, we denote by ξ_f and $\xi_{\widehat{f}}$ the Hamiltonian vector fields for f and $\widehat{f}$, respectively. Then it follows that $\xi_{\widehat{f}}$ is projectable under p_k and it satisfies $(p_k)_* \xi_{\widehat{f}} = \xi_f$, i.e., $dp_k(\xi_{\widehat{f}})(x) = \xi_f(p_k(x))$ for every $x \in \mathcal{O}_{\mathfrak{b}}$ since

$$\begin{aligned} \omega_{\mathfrak{b}}((p_k)_* \xi_{\widehat{f}}, (p_k)_*(\cdot)) &= (p_k)^* \omega_{\mathfrak{b}}(\xi_{\widehat{f}}, \cdot) \\ &= \omega_{\mathfrak{a}}(\xi_{\widehat{f}}, \cdot) \\ &= d\widehat{f}(\cdot) = df((p_k)_*(\cdot)) \\ &= \omega_{\mathfrak{b}}(\xi_f, (p_k)_*(\cdot)) \end{aligned} \tag{7.14}$$

where the second equality comes from Lemma 6.12. Also, note that $\Phi_{\mathfrak{b}}^{i,k-i}$ is a constant function on $\mathcal{O}_{\mathfrak{b}}$ and $\Phi_{\mathfrak{a}}^{i,k-i} = \Phi_{\mathfrak{b}}^{i,k-i} \circ p_k$. By applying (7.14), we can see that the Hamiltonian flow generated by $\Phi_{\mathfrak{a}}^{i,k-i}$ preserves the $(k \times k)$ leading principal minor, and therefore its Hamiltonian vector field is tangent to the vertical direction of p_k.

Once (1), (2), and (3) are satisfied, its iterated bundle in (7.12) descends to

$$\begin{array}{ccccccccc} \Phi_\lambda^{-1}(\mathbf{u})/(S^1)^r \cong \overline{S_{n-1}}(\gamma)' & \xrightarrow{p'_{n-1}} & \overline{S_{n-2}}(\gamma)' & \to & \cdots & \xrightarrow{p'_2} & \overline{S_1}(\gamma)' & \xrightarrow{p'_1} & \overline{S_0}(\gamma)' := \text{point} \\ \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\ Y_{n-1} & & Y_{n-2} & & \cdots & & Y_1 & & \end{array} \tag{7.15}$$

where $\overline{S_k}(\gamma)' = \overline{S_k}(\gamma)/(S^1)^{r_k + \cdots + r_1}$ and the $(S^1)^{r_k+\cdots+r_1}$-action is generated by $\{\Phi_\lambda^{i,j}\}_{(i,j) \in \mathcal{I}_{\mathcal{C}(\gamma)}, i+j-1 \leq k}$. Since $\Phi_\lambda^{-1}(\mathbf{u}) \cong T^r \times Y(\mathbf{u})$ for some $Y(\mathbf{u})$ by Proposition 7.8, we have $Y(\mathbf{u}) \cong \Phi_\lambda^{-1}(\mathbf{u})/T^r \cong \overline{S_\bullet}(\gamma)'$, which completes the proof. $\square$

As an application of Theorem 7.9, one can provide a more explicit description of GC fibers. As mentioned in Remark 5.15, an iterated bundle in Theorem 5.12 is in general *not* trivial. Generally speaking, a torus bundle over a torus might be non-trivial, e.g. Kodaira-Thurston example, a 2-torus bundle over a 2-torus whose first betti number is 3, see [Th]. Yet, Theorem 7.9 guarantees that all torus factors in the iterated bundle can be taken out from $\Phi_\lambda^{-1}(\mathbf{u})$. Using this observation, in some case, the iterated bundle can be characterized explicitly.

Example 7.10. (1) Let $\mathcal{O}_\lambda \simeq \mathcal{F}(6)$ be the co-adjoint orbit associated with $\lambda = (5,3,1,-1,-3,-5)$. Consider the face γ_1 defined in Figure 19. The diffeomorphic type of the fiber over a point $\mathbf{u}$ in the relative interior of γ_1 is the product of $(S^1)^7$ and $Y(\mathbf{u})$ by Theorem 7.9. Here, $Y(\mathbf{u})$ is diffeomorphic to $SU(3)$ because $Y(\mathbf{u})$ is the total space of the S^3-bundle over S^5 from Remark 5.15. In sum,

$$\Phi_\lambda^{-1}(\mathbf{u}) \simeq (S^1)^7 \times SU(3).$$

(2) Let $\lambda = (3,3,3,-3,-3,-3)$. Then, the co-adjoint orbit $\mathcal{O}_\lambda$ is $\mathrm{Gr}(3,6)$. Consider the face γ_2 defined in Figure 19. We claim that the diffeomorphic type of the fiber over a point in the relative interior of γ_2 is

$$\Phi_\lambda^{-1}(\mathbf{u}) \simeq (S^1)^3 \times (S^3)^2.$$

Because of Theorem 7.9, the fiber is of the form $(S^1)^3 \times Y(\mathbf{u})$ where $Y(\mathbf{u})$ is an S^3-bundle over S^3. Since every S^3-bundle over S^3 is trivial (see Steenrod [St]), we have $Y(\mathbf{u}) \simeq (S^3)^2$.

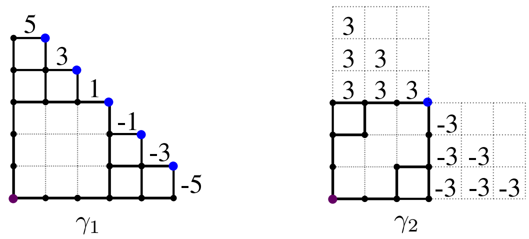


FIGURE 19. Gelfand-Cetlin fibers.

Another application of Theorem 7.9 is to compute the first and the second homotopy groups of each $\Phi_\lambda^{-1}(\mathbf{u})$ as follows. Let $\mathbf{u} \in \Delta_\lambda$ and let f be the face of Δ_λ containing $\mathbf{u}$ in its relative interior. Also, let γ be the face of Γ_λ corresponding to f. For each $k = 1, \cdots, n-1$, the fibration (7.13) induces the long exact sequence of homotopy groups given by

$$\begin{aligned} \cdots &\to \pi_2(S_k(\gamma)) \to \pi_2(\overline{S_k}(\gamma)) \to \pi_2(\overline{S_{k-1}}(\gamma)) \\ &\to \pi_1(S_k(\gamma)) \to \pi_1(\overline{S_k}(\gamma)) \to \pi_1(\overline{S_{k-1}}(\gamma)) \to \pi_0(S_k(\gamma)) \to \cdots \end{aligned} \tag{7.16}$$

Proposition 7.11. *Let $\mathbf{u} \in \Delta_\lambda$. Then the followings hold.*

- $\pi_2(\Phi_\lambda^{-1}(\mathbf{u})) = 0.$
- *If $\mathbf{u}$ is a point in the relative interior of an r-dimensional face of Δ_λ, then*

$$\pi_1(\Phi_\lambda^{-1}(\mathbf{u})) \cong \mathbb{Z}^r.$$

Proof. Since $S_k(\gamma)$ in (7.13) is a point or a product space of odd dimensional spheres, we have $\pi_2(S_k(\gamma)) = 0$ for every $k = 1, \cdots, n-1$. Note that $\pi_2(\overline{S_1}(\gamma)) = \pi_2(S_1(\gamma)) = 0$. Therefore, it easily follows that $\pi_2(\overline{S_k}(\gamma)) = 0$ for every k by the induction on k. The second statement is deduced from Theorem 7.9, since $\overline{S_\bullet}(\gamma)'$ is simply connected. ☐

Corollary 7.12. *For a point $\mathbf{u} \in \Delta_\lambda$, the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is a Lagrangian torus if and only if $\mathbf{u}$ is an interior point of Δ_λ*

Proof. The "if" statement follows immediately from Theorem 7.9, and the "only if" part follows from Proposition 7.11. ☐

Part 2. Non-displaceability of Lagrangian fibers

8. Introduction of Part 2

In the second part of this article, we focus on detecting displaceable and non-displaceable Lagrangian [7] GC fibers.

There has been research on displaceability and non-displaceability of GC fibers on partial flag manifolds. Nishinou, Nohara, and Ueda showed that the Lagrangian GC torus fiber at the center is non-displaceable. To show it, they calculated the potential function after constructing a toric degeneration from a GC system to a toric moment map and found a weak bounding cochain such that the deformed Floer cohomology is non-vanishing.

Theorem 8.1 (Theorem 12.1 in [NNU1])**.** *For any non-increasing sequence λ of real numbers, the Gelfand-Cetlin system $\Phi_\lambda \colon (\mathcal{O}_\lambda, \omega_\lambda) \to \Delta_\lambda$ admits a non-displaceable Lagrangian torus fiber $\Phi_\lambda^{-1}(\mathbf{u}_0)$ over the center $\mathbf{u}_0$ of the Gelfand-Cetlin polytope Δ_λ.*

[7]See Definition 9.1 for (non-)displacealbe Lagrangian submanifolds

Over the Novikov ring over the field of complex numbers, Nohara and Ueda [NU2] found bounding cochains making deformed Floer cohomology of $U(2)$-fiber non-zero in $\mathcal{O}_\lambda \cong \mathrm{Gr}(2,4)$ for

$$\lambda = \{\lambda_1 = \lambda_2 = 2 > \lambda_3 = \lambda_4 = -2\}. \tag{8.1}$$

The authors explicitly classified moduli spaces of holomorphic discs bounded by the fiber over the center of γ, using the homogeneity of the fiber in $\mathrm{Gr}(2,4)$. Here, the notion of homogeneous Lagrangians was introduced by Evans and Lekili, see [EL, Definition 1.1.1].

Theorem 8.2 (Theorem 1.2 in [NU2])**.** *For the sequence λ in (8.1), let γ be the one-dimensional Lagrangian face of Γ_λ in Example 5.10. Then, the Gelfand-Cetlin $U(2)$-fiber over the center of the face γ is non-displaceable. Moreover, any other Gelfand-Cetlin $U(2)$-fibers located at its interior are displaceable.*

For a sequence

$$\lambda = (\lambda_1 = \cdots = \lambda_n = n > \lambda_{n+1} = \cdots = \lambda_{2n} = -n), \tag{8.2}$$

the co-adjoint orbit $\mathcal{O}_\lambda$ gives rise to $\mathrm{Gr}(n, 2n)$ and the $(n \times n)$-square γ in Γ_λ corresponds to the one-dimensional face whose fiber at the interior is diffeomorphic to $U(n)$. According to the result of the third named author in [Oh], any Lagangian submanifold that is a fixed point set of an anti-symplectic involution of a monotone Hermitian symmetric space has non-zero Floer cohomology over the field of characteristic two. By Iriyeh, Sakai, and Tasaki in [IST], the $U(n)$-fiber is the fixed point set of an anti-symplectic involution on $\mathrm{Gr}(n, 2n)$, a Hermitian symmetric space. Using those results, Evans and Lekili [EL2] observed that the $U(n)$-fiber at the center of the face γ has non-zero Floer cohomology over the field in characteristic two.

Theorem 8.3 (Corollary 7.4.2 in [EL2])**.** *For the sequence λ in (8.2), the Lagrangian $U(n)$-fiber at the center of one-dimensional edge γ is non-displaceable. Moreover, the Lagrangian $U(N)$-fiber at the center of γ split-generates the Fukaya category of $\mathrm{Gr}(N, 2N)$ in characteristic 2 for $N = 2^s$ with a positive integer s .*

In this paper, we discuss non-displaceable GC fibers on complete flag manifolds equipped with a monotone KKS form. In contrast to the case of the $U(n)$-fiber in $\mathrm{Gr}(n, 2n)$, a non-toric fiber can not be realized as an orbit of a global holomorphic action in general because it has torus factors from torus actions not extending to the ambient manifold. Because of this feature, it seems that the arguments in Theorem 8.2 and Theorem 8.3 cannot be applied to this case.

Let

$$\lambda = \{\lambda_1 > \lambda_2 > \cdots > \lambda_n\}$$

be a decreasing sequence of real numbers so that the co-adjoint orbit $\mathcal{O}_\lambda$ is diffeomorphic to a complete flag manifold. When the KKS form ω_λ is monotone, we will show non-displaceability of toric fibers and non-toric fibers as well. To show it, we first locate a half-open line segment $[\mathbf{u}_0, \mathbf{u}_\gamma) \subset \Delta_\lambda$ where $\mathbf{u}_0$ is the center [8] of Δ_λ and $\mathbf{u}_\gamma$ is a point in the relative interior of some Lagrangian face γ of Δ_λ. Then, in some cases, we are able to show that each torus fiber $\Phi_\lambda^{-1}(\mathbf{u})$ ($\mathbf{u} \in [\mathbf{u}_0, \mathbf{u}_\gamma)$) is non-displaceable by showing that a Lagrangian Floer cohomology (with a certain bulk-deformation parameter and a weak bounding cochain) is non-zero. Since the non-toric Lagrangian submanifold $\Phi_\lambda^{-1}(\mathbf{u}_\gamma)$ is realized as the "limit" of non-displaceable Lagrangian tori, we deduce that the fiber $\Phi_\lambda^{-1}(\mathbf{u}_\gamma)$ is also non-displaceable, see Proposition 10.13 for the precise statement.

We describe line segments over which the fibers are non-displaceable in the GC polytope Δ_λ explicitly. If the symplectic form ω_λ is monotone, the center of the polytope Δ_λ is expressed as follows. Recall that the form ω_λ is monotone if and only if $\lambda_n - \lambda_{n-1} = \lambda_{n-1} - \lambda_{n-2} = \cdots = \lambda_2 - \lambda_1$ by Proposition 3.4. By scaling ω_λ if necessary, we may assume that

$$\lambda = \{\lambda_i := n - 2i + 1 : \, i = 1, \cdots, n\}, \tag{8.3}$$

[8] For any moment polytope Δ, Fukaya, Oh, Ohta, and Ono [FOOO3, Proposition 9.1] described a unique interior point (which they denoted by $\mathbf{u}_0$ or P_K) and we call it the *center* of Δ. The center $\mathbf{u}_0$ of Δ is a point over which the corresponding toric fiber is non-displaceable, see Theorem 1.5 and Proposition 4.7 in [FOOO3] for more details. Notice that the center is *not* meant to be the barycenter of a polytope. The center and the barycenter are in general different.

which is the case where $[\omega_\lambda] = c_1(T\mathcal{O}_\lambda)$. The polytope Δ_λ is of dimension $\frac{n(n-1)}{2}$. It turns out that Δ_λ is a reflexive [9] polytope, see [BCKV, Corollary 2.2.4] or [NNU1, Lemma 3.12]. One well-known fact on a reflexive polytope is that there exists a unique lattice point in its interior, that is exactly the center of the polytope.

We start from the simplest case where the co-adjoint orbit $\mathcal{O}_\lambda$ of a sequence $\lambda = \{\lambda_1 > \lambda_2 > \lambda_3\}$. In this case, Pabiniak investigated displaceable GC fibers.

Theorem 8.4 ([Pa]). *For $\lambda = \{\lambda_1 > \lambda_2 > \lambda_3\}$, let $\mathcal{O}_\lambda$ be the co-adjoint orbit, which is a complete flag manifold $\mathcal{F}(3)$ equipped with ω_λ.*

(1) *If ω_λ is not monotone, i.e. $(\lambda_1 - \lambda_2) \neq (\lambda_2 - \lambda_3)$, then all Gelfand-Cetlin fibers but one over the center are displaceable.*
(2) *If ω_λ is monotone i.e. $(\lambda_1 - \lambda_2) = (\lambda_2 - \lambda_3)$, then all Gelfand-Cetlin fibers but the fibers over the line segment*

$$I := \{(u_{1,1}, u_{1,2}, u_{2,1}) = (0, a-t, -a+t) \in \mathbb{R}^3 : 0 \le t \le a\} \tag{8.4}$$

are displaceable where $2a = \lambda_1 - \lambda_2$. Observe that the line segment I is the red line in Figure 20.

Note that the line segment I connects the center $(0, a, -a)$ of Δ_λ and the position $(0,0,0)$ of the Lagrangian 3-sphere. Combining with Theorem 8.1, Theorem 8.4 finishes the classification of displaceable and non-displaceable fibers of the GC system Φ_λ for the non-monotone case. The following theorem asserts every GC fiber in the family $\{\Phi_\lambda^{-1}(\mathbf{u}) \mid \mathbf{u} \in I\}$ is non-displaceable. Thus, together with Theorem 8.4, our result provides the complete classification of displaceable and non-displaceable Lagrangian GC fibers when ω_λ is monotone. We postpone its proof until Section 10.3.

Theorem C (Theorem 10.12). *Let $\lambda = \{\lambda_1 = 2 > \lambda_2 = 0 > \lambda_3 = -2\}$. Consider the co-adjoint orbit $\mathcal{O}_\lambda$, a complete flag manifold $\mathcal{F}(3)$ equipped with the monotone Kirillov-Kostant-Souriau symplectic form ω_λ, Then the Gelfand-Cetlin fiber over a point $\mathbf{u} \in \Delta_\lambda$ is non-displaceable if and only if $\mathbf{u} \in I$ where*

$$I := \{(u_{1,1}, u_{1,2}, u_{2,1}) = (0, 1-t, -1+t) \in \mathbb{R}^3 \mid 0 \le t \le 1\} \tag{8.5}$$

In particular, the Lagrangian 3-sphere $\Phi_\lambda^{-1}(0,0,0)$ is non-displaceable.

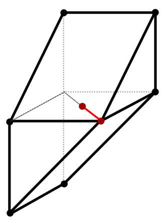


FIGURE 20. The positions of non-displaceable GC Lagrangian fibers in $\mathcal{F}(3)$.

Remark 8.5. The fiber over the center $(0,1,-1)$ of Δ_λ is known to be non-displaceable by Theorem 8.1. Also, Nohara and Ueda [NU2] calculated a Floer cohomology of the Lagrangian 3-sphere $\Phi_\lambda^{-1}(0,0,0)$, which turns out to be zero over the Novikov field Λ. Nevertheless, Theorem C says that it is non-displaceable.

Next, we deal with a general case for an arbitrary positive integer $n \ge 4$ where λ is given as in (8.3). In this case, the GC polytope Δ_λ is a reflexive polytope whose center is

$$(u_{i,j} := j - i : i + j \le n) \in \Delta_\lambda \subset \mathbb{R}^{n(n-1)/2}.$$

Consider the face f_m of Δ_λ defined by

$$\{u_{i,j} = u_{i,j+1} : 1 \le i \le m, 1 \le j \le m-1\} \cup \{u_{i+1,j} = u_{i,j} : 1 \le i \le m-1, 1 \le j \le m\}.$$

[9] A convex lattice polytope $\mathcal{P}$ is called *reflexive* if its dual polytope $\mathcal{P}^*$ is also a lattice polytope.

for any integer m satisfying $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Note that there are $\left(\lfloor \frac{n}{2} \rfloor - 1\right)$ such faces in Δ_λ. For instance, we have two faces f_2 and f_3 for the case where $n = 7$ (see Figure 21), and three faces f_2, f_3 and f_4 for $n = 8$ (see Figure 22). Furthermore, those faces can be filled by L-shaped blocks so that they are Lagrangian by Corollary 5.23. Regarding f_m as a polytope, the center of f_m admits a unique lattice point in its interior, whose components are given by

$$u_{i,j} := \begin{cases} 0 & \text{if } \max(i,j) \leq m \\ j - i & \text{if } \max(i,j) > m. \end{cases} \tag{8.6}$$

Candidates for non-displaceable Lagrangian fibers are the fibers over the line segment $I_m \subset \Delta_\lambda$ connecting the center of Δ_λ and the center of f_m for each $m \geq 2$. Explicitly, the line segment I_m is parameterized by $\{I_m(t) \in \Delta_\lambda : 0 \leq t \leq 1\}$ where

$$I_m(t) := \begin{cases} u_{i,j}(t) = (j-i) - (j-i)\,t & \text{if } \max(i,j) \leq m \\ u_{i,j}(t) = (j-i) & \text{if } \max(i,j) > m. \end{cases} \tag{8.7}$$

We denote by $L_m(t)$ the Lagrangian GC fiber over the point $I_m(t)$, that is $L_m(t) := \Phi_\lambda^{-1}(I_m(t))$. Now, we state our second main theorem in this section. Again its proof will be postponed to Section 11 and Section 12.

Theorem D (Theorem 12.16)**.** *Let $\lambda = \{\lambda_i := n - 2i + 1 \,|\, i = 1, \cdots, n\}$ be an n-tuple of real numbers for an arbitrary integer $n \geq 4$. Consider the co-adjoint orbit $\mathcal{O}_\lambda$, a complete flag manifold $\mathcal{F}(n)$ equipped with the monotone Kirillov-Kostant-Souriau symplectic form ω_λ. Then each Gelfand-Cetlin fiber $L_m(t)$ is non-displaceable Lagrangian for every $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. In particular, there exists a family of non-displaceable non-torus Lagrangian fibers*

$$\left\{ L_m(1) : 2 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

of the Gelfand-Cetlin system Φ_λ where $L_m(1)$ is diffeomorphic to $U(m) \times T^{\frac{n(n-1)}{2} - m^2}$.

Example 8.6. (1) A monotone complete flag manifold $\mathcal{F}(7)$ admits (at least) two line segments I_2 and I_3 in the GC polytope over which the fibers are non-displaceable, see Figure 21. Particularly, it has non-displaceable Lagrangian submanifolds diffeomorphic to $U(2) \times T^{17}$ and $U(3) \times T^{12}$.

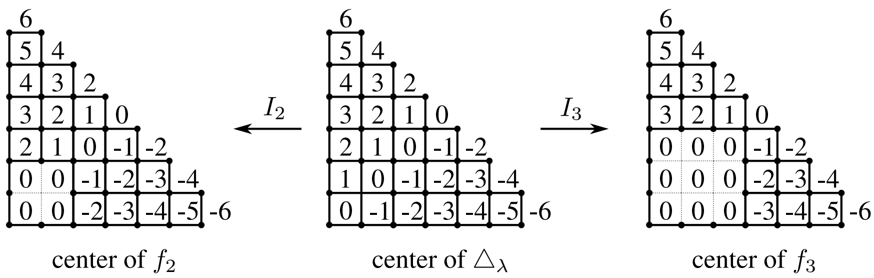


FIGURE 21. Positions of non-displaceable Lagrangian GC fibers in $\mathcal{F}(7)$.

(2) A monotone complete flag manifold $\mathcal{F}(8)$ admits (at least) three line segments I_2, I_3 and I_4 in the GC polytope over which the fibers are non-displaceable, see Figure 22. Particularly, it has non-displaceable Lagrangian submanifolds diffeomorphic to $U(2) \times T^{24}$, $U(3) \times T^{19}$ and $U(4) \times T^{12}$.

Remark 8.7. The third named author with Fukaya, Ohta, and Ono [FOOO4] found a continuum of non-displaceable toric fibers on some compact toric manifolds including a non-monotone toric blowup of $\mathbb{C}P^2$ at two points, see also Woodward [Wo]. Using the degeneration models, they also produced a continuum of non-displaceable Lagrangian tori on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the cubic surface respectively in [FOOO5] and [FOOO8]. Vianna [Vi] also showed a continuum of non-displaceable tori in $(\mathbb{C}P^1)^{2n}$. In the case of toric orbifolds, dealing with more restrictive classes of Hamiltonian isotopies, non-displaceable toric fibers usually exist in abundance, see Woodward [Wo], Wilson-Woodward [WW], and Cho-Poddar [CP].

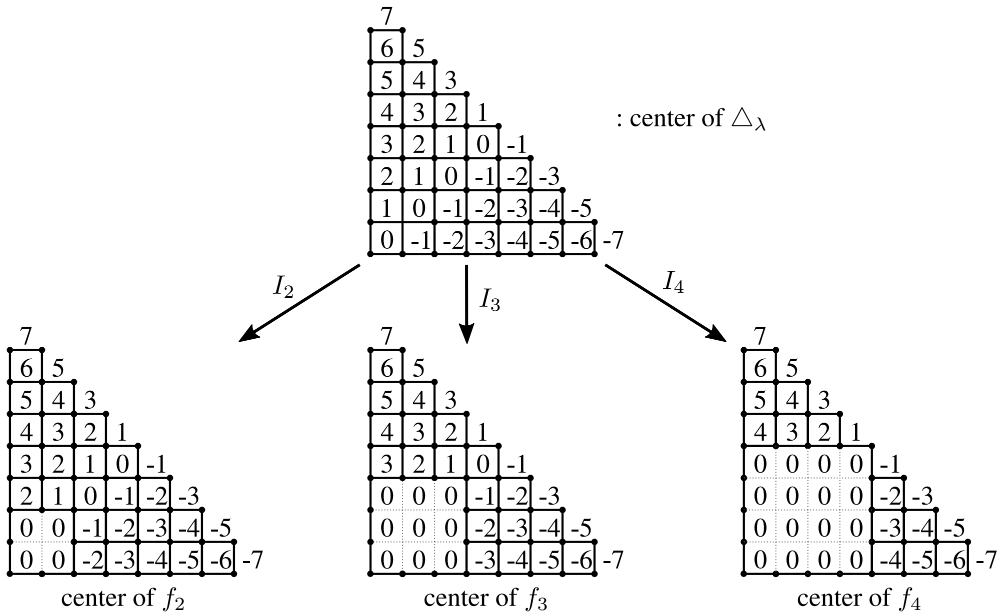


FIGURE 22. Positions of non-displaceable Lagrangian GC fibers in $\mathcal{F}(8)$.

Before dealing with non-displaceable GC fibers, we study displaceability of fibers in GC systems in Section 9. In the toric case, McDuff [McD] and Abreu-Borman-McDuff [ABM] developed the method of *probes* to detect displaceable toric fibers. The probe method can be also applied to Lagrangian *torus* fibers in GC systems because they are related to the toric fibers via a (local) symplectomorphism in (2.1), however, it is not applicable to non-torus fibers. In the case where a co-adjoint orbit is $\mathcal{F}(3)$, Pabiniak [Pa] investigated displaceable fibers.

We develop several numerical and combinatorial criteria for detecting displaceable GC fibers which can be applied to *both* torus fibers and non-torus fibers in GC systems. Even though our criteria are not exhaustive to classify all displaceable fibers, it is enough to detect almost all displaceable fibers in that the non-displaceable fibers should be located over a measure zero set of the polytope Δ_λ. In particular, we are able to displace all non-torus fibers in some cases.

Theorem E (Corollary 9.14)**.** *Let p be a prime number. Then every non-torus Lagrangian Gelfand-Cetlin fiber of the complex Grassmannian* $\mathrm{Gr}(2,p)$ *is displaceable.*

The rest of the paper is organized as follows. In Section 9, we introduce several numerical criteria testing displaceability of GC fibers and the proof of Theorem E will be discussed. Section 10 is devoted to review Lagrangian Floer theory and to explain Theorem C. To prove Theorem D, the split leading term equation is introduced and its solvability is discussed in Section 11 and 12. Section 13 focuses on calculation of the potential function deformed Schubert cycles.

9. CRITERIA FOR DISPLACEABLILITY OF GELFAND-CETLIN FIBERS

For a given moment polytope, McDuff [McD] and Abreu-Borman-McDuff [ABM] developed the techniques (using *probes*) to detect positions of displaceable toric moment fibers by using combinatorial data of the polytope. In contrast to the toric cases, any partial flag manifold always possesses a non-torus Lagrangian GC fiber unless it is diffeomorphic to a projective space, see Corollary 5.24. The probe method also can be applied to some extent to the case of GC systems, see [Pa] for example. However, it works only for *torus* fibers over interior points of a GC polytope. In this section, we provide several numerical and combinatorial criteria for displaceability of both torus and non-torus Lagrangian fibers of GC systems. (See Proposition 9.4, 9.10, 9.12.)

Definition 9.1. Let (M, ω) be a symplectic manifold and let Y be a subset. We say that Y is *displaceable* if there exists a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$ such that

$$\phi(Y) \cap \overline{Y} = \emptyset.$$

If there is no such a diffeomorphism, then we say that Y is *non-displaceable*.

Definition 9.2. Let λ be a sequence of non-increasing real numbers given in (3.4). We say that a face γ is *displaceable* if $\Phi_\lambda^{-1}(\mathbf{u})$ is displaceable for every $\mathbf{u} \in \mathring{\gamma}$.

9.1. Testing by diagonal entries.

For a sequence $\lambda = \{\lambda_1, \cdots, \lambda_n\}$ in (3.4), let $\Phi_\lambda \colon \mathcal{O}_\lambda \to \Delta_\lambda \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ be the GC system. For each lattice point $(i,j) \in (\mathbb{Z}_{\geq 1})^2$ with $2 \leq i+j \leq n$, we denote by $\Phi_\lambda^{i,j} \colon \mathcal{O}_\lambda \to \mathbb{R}$ the (i,j)-th component of Φ_λ. We denote the coordinate system of $\mathbb{R}^{\frac{n(n-1)}{2}}$ by $(u_{i,j})_{2 \leq i+j \leq n}$ as described in Figure 9.

Note that the symmetric group $\mathfrak{S}_n$ can be regarded as a subgroup of $U(n) \subset \mathrm{Ham}(\mathcal{O}_\lambda, \omega_\lambda)$. Namely, each element $w \in \mathfrak{S}_n$ is represented by the $(n \times n)$ elementary matrix, which we still denote by w, whose $(i, w(i))$-th entry is 1 for each $i = 1, \cdots, n$ and other entries are zero. Then each $x = (x_{i,j})_{1 \leq i,j \leq n} \in \mathcal{O}_\lambda \subset \mathcal{H}_n$ and $w \in \mathfrak{S}_n$ satisfies

$$(w \cdot x)_{ij} = (wxw^{-1})_{ij} = x_{w(i),w(j)}$$

for every $1 \leq i,j \leq n$. The following lemma says the diagonal entries of any element $x \in \Phi_\lambda^{-1}(\mathbf{u})$ are completely determined by $\mathbf{u}$.

Lemma 9.3. *Let $\mathbf{u} = (u_{i,j})_{i,j \geq 1} \in \Delta_\lambda$. Then for any $x \in \Phi_\lambda^{-1}(\mathbf{u})$, we have $x_{1,1} = u_{1,1}$ and*

$$x_{k,k} = \sum_{i+j=k+1} u_{i,j} - \sum_{i+j=k} u_{i,j}.$$

for $1 < k \leq n$ where

$$u_{i,j} := \lambda_i \quad \text{for} \quad i+j = n+1.$$

Proof. It is straightforward from the fact that

$$\mathrm{tr}(x^{(k)}) = x_{1,1} + \cdots + x_{k,k} = \sum_{i+j=k+1} u_{i,j}$$

for every $k = 1, \cdots, n$ where $x^{(k)}$ denotes the $(k \times k)$ leading principal minor matrix of x. ☐

For the sake of simplicity, let $d_1(\mathbf{u}) := u_{1,1}$ and

$$d_k(\mathbf{u}) := \sum_{i+j=k+1} u_{i,j} - \sum_{i+j=k} u_{i,j} \tag{9.1}$$

for $\mathbf{u} \in \Delta_\lambda$ and $1 < k \leq n$ so that $x_{k,k} = d_k(\mathbf{u})$ for every $x \in \Phi_\lambda^{-1}(\mathbf{u})$ by Lemma 9.3. We then state a numerical criterion for displaceable fibers.

Proposition 9.4. *Let $\mathbf{u}$ be a point in the Gelfand-Cetlin polytope Δ_λ. If the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is non-displaceable, then*

$$d_1(\mathbf{u}) = \cdots = d_n(\mathbf{u}).$$

Proof. Note that $\Phi_\lambda^{1,1}(x) = x_{1,1}$ and $\Phi_\lambda^{1,1}(w \cdot x) = x_{w(1),w(1)}$ for $w \in \mathfrak{S}_n$. If $d_1(\mathbf{u}) \neq d_k(\mathbf{u})$ for some k with $1 \leq k \leq n$, then

$$\Phi_\lambda^{1,1}(x) = x_{1,1} = d_1(\mathbf{u}) \neq d_k(\mathbf{u}) = x_{k,k} = x_{w(1),w(1)} = \Phi_\lambda^{1,1}(w \cdot x)$$

for every $x \in \Phi_\lambda^{-1}(\mathbf{u})$ where w is the transposition $(1,k)$. Thus,

$$w \cdot \Phi_\lambda^{-1}(\mathbf{u}) \cap \Phi_\lambda^{-1}(\mathbf{u}) = \emptyset$$

and hence $\Phi_\lambda^{-1}(\mathbf{u})$ is displaceable. This completes the proof. ☐

In a GC system, Proposition 9.4 implies that almost all fibers are displaceable.

Corollary 9.5. *For any Gelfand-Cetlin system Φ_λ, there exists a dense open subset $\mathcal{U}$ of the Gelfand-Cetlin polytope Δ_λ such that for each $\mathbf{u} \in \mathcal{U}$ the fiber $\Phi_\lambda^{-1}(\mathbf{u})$ is displaceable.*

Remark 9.6. This contrasts to the case of toric orbifolds. There is a symplectic toric orbifold which contains an open subset $\mathcal{U}$ of a moment polytope such that every fiber over a point in $\mathcal{U}$ is non-displaceable, see Wilson-Woodward [WW] and Cho-Poddar [CP].

Example 9.7. We demonstrate how to apply Proposition 9.4 to detect displaceable fibers.

(1) (Complex projective spaces) For $\lambda = \{\lambda_1 = n, \ \lambda_2 = \cdots = \lambda_n = 0\}$, the co-adjoint oribt $(\mathcal{O}_\lambda, \omega_\lambda)$ is a complex projective space $\mathbb{C}P^{n-1}$ equipped with a (multiple of) the Fubini-Study form ω_λ. In this case, every component $\Phi_\lambda^{i,j}$ of the GC system Φ_λ is a *constant* function unless $i = 1$ so that Δ_λ is an $(n-1)$-dimensional simplex. For any $\mathbf{u} \in \Delta_\lambda$ and $x \in \Phi_\lambda^{-1}(\mathbf{u})$, we have

$$\begin{cases} x_{1,1} = d_1(\mathbf{u}) = u_{1,1}, \\ x_{k,k} = d_k(\mathbf{u}) = u_{1,k} - u_{1,k-1} \text{for } 1 \leq k \leq n-1, \\ x_{n,n} = d_n(\mathbf{u}) = u_{1,n} - u_{1,n-1} = n - u_{1,n-1}. \end{cases}$$

Then $d_1(\mathbf{u}) = \cdots = d_n(\mathbf{u})$ implies that $d_k(\mathbf{u}) = 1$ for every $1 \leq k \leq n$. Therefore, by Proposition 9.4, if $d_k(\mathbf{u}) \neq 1$ for some k, then $\Phi_\lambda^{-1}(\mathbf{u})$ is displaceable. The only possible candidate for a non-displaceable fiber is $\Phi^{-1}(\mathbf{u}_0)$ where $\mathbf{u}_0$ is the center of Δ_λ, i.e., $\mathbf{u}_0 = (u_{i,j})$ with $u_{1,j} = j$ for $j = 1, \cdots, n-1$. Note that it has been shown by Cho [Cho1] that $\Phi_\lambda^{-1}(\mathbf{u}_0)$ is non-displaceable. Therefore, $\Phi_\lambda^{-1}(\mathbf{u}_0)$ is a *stem*.[10]

(2) (Complete flag manifold $\mathcal{F}(3)$) Let $\lambda = \{\lambda_1, 0, -\lambda_2\}$ for $\lambda_1, \lambda_2 > 0$ so that $(\mathcal{O}_\lambda, \omega_\lambda)$ is a complete flag manifold $\mathcal{F}(3)$ and it admits a unique proper Lagrangian face, a vertex v_3, of Δ_λ in Example 4.5, see Figure 23 below. For any $x \in \Phi_\lambda^{-1}(v_3)$, we can easily see that $x_{1,1} = x_{1,2} = x_{2,1} = x_{2,2} = 0$, and $x_{3,3} = \lambda_1 - \lambda_2$. By Proposition 9.4, we can conclude that $\Phi^{-1}(v_3)$ is displaceable whenever $\lambda_1 \neq \lambda_2$. We have reproduced the result of Pabiniak [Pa] on displaceability of a (unique) non-torus GC Lagrangian fiber in $\mathcal{F}(3)$. In Section 10.3, $\Phi^{-1}(v_3)$ will be shown to be non-displaceable when $\lambda_1 = \lambda_2$.

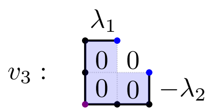


FIGURE 23. The Lagrangian face of Γ_λ in $\mathcal{F}(3)$

(3) (Complex Grassmannian $\mathrm{Gr}(2,4)$) Let $\lambda = \{t, t, 0, 0\}$ for $t > 0$. By Corollary 5.23, the edge e in Figure 24 of Δ_λ is the unique proper Lagrangian face of Γ_λ. For a positive real number a with $0 \leq a \leq t$, we consider the point $r_a \in e$ given by $u_{1,1} = u_{1,2} = u_{2,1} = u_{2,2} = a$. Nohara-Ueda [NU2] proved that the every fiber over the edge except the fiber $\Phi_\lambda^{-1}(r_{t/2})$ is displaceable and moreover $\Phi_\lambda^{-1}(r_{t/2})$ is non-displaceable. Our combinatorial test (Proposition 9.4) easily tells us that $\Phi_\lambda^{-1}(r_a)$ is displaceable unless $2a = t$.

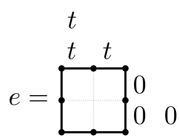


FIGURE 24. Lagrangian face of Γ_λ in $\mathrm{Gr}(2,4)$

(4) (Complex Grassmannian $\mathrm{Gr}(2,6)$) Let $\lambda = \{6, 6, 0, 0, 0, 0\}$. By Example 5.25, there are exactly four proper Lagrangian faces γ_2, γ_3, γ_4, and $\gamma_{2,4}$ of Γ_λ as follows.

[10] A fiber of a moment map is called a *stem* if all other fibers are displaceable. See [EP].

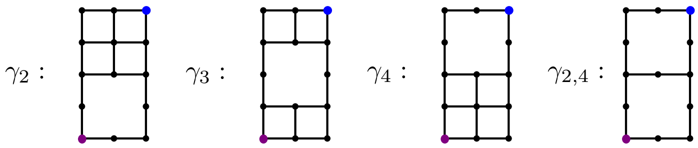


FIGURE 25. Four proper Lagrangian faces of Γ_λ

First, consider two faces γ_3 and $\gamma_{2,4}$. By Proposition 9.4, we can easily check that every Lagrangian fiber over a point in γ_3 (resp. $\gamma_{2,4}$) is displaceable except for the fiber at $\mathbf{u}_3$ (resp. $\mathbf{u}_{2,4}$) described in Figure 26.

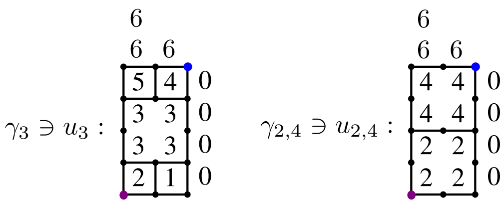


FIGURE 26. Lagrangian faces γ_3 and $\gamma_{2,4}$

For the other faces γ_2 and γ_4, again by Proposition 9.4, every fiber is displaceable except for the one-parameter families of Lagrangian fibers $\mathbf{u}_2(t)$ and $\mathbf{u}_4(t)$ $(-1 < t < 1)$ described in Figure 27 in γ_2 and γ_4, respectively.

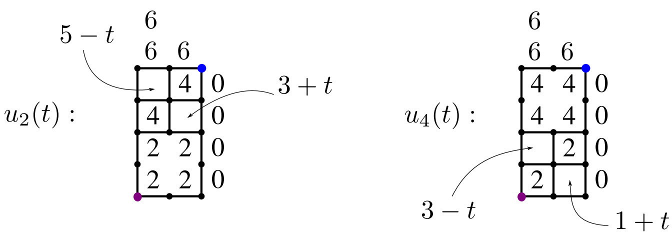


FIGURE 27. Lagrangian faces γ_2 and γ_4

In Section 9.2, we will give another method for detecting displaceability of fibers and show that γ_2 and γ_4 are indeed displaceable. Consequently, there are exactly two non-torus Lagrangian fiber $\Phi_\lambda^{-1}(\mathbf{u}_3)$ and $\Phi_\lambda^{-1}(\mathbf{u}_{2,4})$ that might be non-displaceable.

9.2. **Symmetry on Γ_λ (complex Grassmannian cases).**

In this section, we study displaceability of non-torus Lagrangian fibers of GC systems on complex Grassmannians. Let $\mathrm{Gr}(k,n)$ be the Grassmannian of complex k-dimensional subspaces of $\mathbb{C}^n$. Since $\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n)$, without loss of generality, we assume that $n-k \geq k$. Let $\lambda = \{\underbrace{t,\cdots,t}_{k \text{ times}}, \underbrace{0,\cdots,0}_{n-k \text{ times}}\}$ for $t > 0$ so that

$$\mathrm{Gr}(k,n) \cong \mathcal{O}_\lambda.$$

We start with the following series of algebraic lemmas which seems to be well-known.

Lemma 9.8. *Let A be any complex $(n \times k)$ matrix. Then AA^* and A^*A have the same non-zero eigenvalues with the same multiplicities.*

Proof. Recall that a singular value decomposition yields

$$A = U\Sigma V^*$$

where U is an $(n \times n)$ unitary matrix, Σ is an $(n \times k)$ matrix such that $\Sigma_{i,j} = 0$ unless $i = j$, and V is a $(k \times k)$ unitary matrix. Then $AA^* = U\Sigma\Sigma^* U^*$ and $A^*A = V\Sigma^*\Sigma V^*$ are unitary diagonalizations of AA^* and A^*A respectively. Since $\Sigma\Sigma^*$ and $\Sigma^*\Sigma$ have the same nonzero eigenvalues with the same multiplicities, this completes the proof. $\square$

Lemma 9.9. *Any element $x \in \mathcal{O}_\lambda$ can be expressed by*

$$x = XX^*$$

for some $(n \times k)$ matrix $X = [v_1, \cdots, v_k]$ such that

- $|v_i|^2 = t$ *for every* $i = 1, \cdots, k$, *and*
- $\langle v_i, v_j \rangle = 0$ *for every* $i \neq j$

where $\langle \cdot, \cdot \rangle$ means the standard hermitian inner product on $\mathbb{C}^n$. In particular, we have

$$X^*X = tI_k$$

where I_k is the $(k \times k)$ identity matrix.

Proof. Let $x \in \mathcal{O}_\lambda$. Since x is semi-positive definite by our choice of λ, there exists an $(n \times n)$ lower triangular matrix L having non-negative diagonal entries such that

$$x = LL^*. \tag{9.2}$$

The expression (9.2) is called a *Cholesky factorization of* x, see [HJ, Corollary 7.2.9]. Let $L = U\Sigma V^*$ be a singular value decomposition of L. In this case, the matrices U, Σ, and V are all $(n \times n)$ matrices. Then we have

$$x = LL^* = U\Sigma\Sigma^*U^* \quad \text{where} \quad \Sigma\Sigma^* = \begin{pmatrix} tI_k & 0 \\ 0 & O_k \end{pmatrix}.$$

Let $D = D^* = \sqrt{\Sigma\Sigma^*}$ so that $x = UDD^*U^*$ and denote by $U = [u_1, \cdots, u_n]$. Since $UD = \sqrt{t}\cdot[u_1, \cdots, u_k, 0, \cdots, 0]$, we have $UDD^*U^* = UD(UD)^* = XX^*$ where X is taken to be the $(n \times k)$ matrix

$$X = [v_1, \cdots, v_k] = \sqrt{t} \cdot [u_1, \cdots, u_k].$$

This finishes the proof. $\square$

Now, we consider Lagrangian faces of particular types in a ladder diagram Γ_λ described as follows. Let ${}^{(0,i)}\square_k$ denote the $(k \times k)$ square whose upper-left corner is $(0, i)$ so that the vertices of ${}^{(0,i)}\square_k$ are $(0, i), (k, i), (k, i - k)$ and $(0, i - k)$. Let γ_i be the graph drawn by all positive paths not passing through the interior of the square ${}^{(0,i)}\square_k$. Hence, γ_i contains exactly one $(k \times k)$-sized simple closed region ${}^{(0,i)}\square_k$ and the other simple closed regions in γ_i are unit squares, see Figure 28. Note that γ_i is a Lagrangian face.

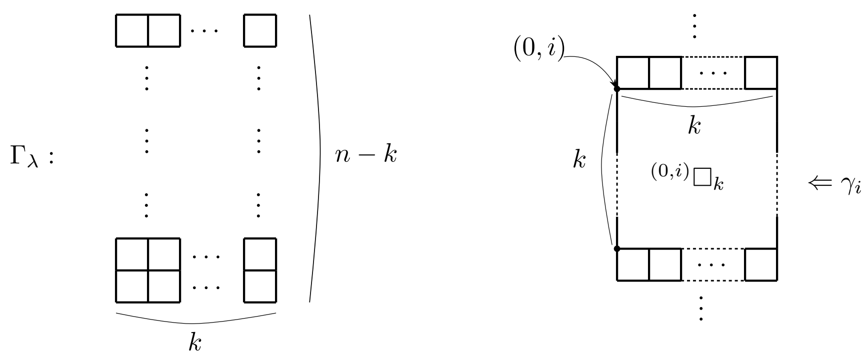


FIGURE 28. Ladder diagram Γ_λ and Lagrangian faces $\{\gamma_i\}_{k \leq i \leq n-k}$

Proposition 9.10. *For each $k \leq i \leq n-k$, there exists a permutation matrix $w \in U(n)$ such that*

$$w \cdot \Phi_\lambda^{-1}(\mathring{\gamma}_i) \subset \Phi_\lambda^{-1}(\gamma_{n-i}).$$

In particular,

$$w \cdot \Phi_\lambda^{-1}(\mathring{\gamma}_i) \cap \Phi_\lambda^{-1}(\mathring{\gamma}_i) = \emptyset.$$

unless $n = 2i$. Consequently, the face γ_i is displaceable provided that $n \neq 2i$.

Proof. Let u be any point in $\mathring{\gamma}_i$. Then we have

$$u_{1,i} = u_{2,i-1} = \cdots = u_{k,i-k+1},$$

which implies that for every $x \in \Phi_\lambda^{-1}(\mathbf{u})$, the i-th leading principal minor $x^{(i)}$ of x has eigenvalues $u_{1,i}$ with multiplicity k and 0 with multiplicity $i-k$.

Now, choose any $x \in \Phi_\lambda^{-1}(\mathbf{u})$. Then Lemma 9.9 implies that there exists an $(n \times k)$ matrix

$$X = (v_1, \cdots, v_k) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

such that $x = XX^*$ and it satisfies

- $|v_j|^2 = t$ for every $j = 1, \cdots, k$, and
- $\langle v_j, v_{j'} \rangle = 0$ for every $j \neq j'$.

Now, we divide X into two submatrices $X^{(i)}$ and $\check{X}^{(i)}$ where

$$X^{(i)} := \left(v_1^{(i)}, \cdots, v_k^{(i)}\right) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_i \end{pmatrix}, \qquad \check{X}^{(i)} := \left(\check{v}_1^{(i)}, \cdots, \check{v}_k^{(i)}\right) = \begin{pmatrix} w_{i+1} \\ \vdots \\ w_n \end{pmatrix}.$$

In other words, $X^{(i)}$ (resp. $\check{X}^{(i)}$) is the $(i \times k)$ (resp. $((n-i) \times k)$) submatrix of X obtained by deleting all ℓ-th rows of X for $\ell > i$ (resp. $\ell \leq i$). Thus we have

$$x^{(i)} = X^{(i)}(X^{(i)})^*.$$

Then Lemma 9.8 implies that $x^{(i)} = X^{(i)}(X^{(i)})^*$ and $(X^{(i)})^* X^{(i)}$ have the same non-zero eigenvalue $u_{1,i}$ with the same multiplicity k. Since $(X^{(i)})^* X^{(i)}$ is a $(k \times k)$ matrix, we get

$$(X^{(i)})^* X^{(i)} = u_{1,i} \cdot I_k.$$

In particular, we have

- $|v_j^{(i)}|^2 = u_{1,i}$ for every $j = 1, \cdots, k$, and
- $\langle v_j^{(i)}, v_{j'}^{(i)} \rangle = 0$ for every $j \neq j'$,

i.e., the columns of $X^{(i)}$ are orthogonal to each other and have the same square norm equal to $u_{1,i}$. This implies that $\check{X}^{(i)} = [\check{v}_1^{(i)}, \cdots, \check{v}_k^{(i)}]$ satisfies

- $|\check{v_j}^{(i)}|^2 = t - u_{1,i}$ for every $j = 1, \cdots, k$, and
- $\langle \check{v}_j^{(i)}, \check{v}_{j'}^{(i)} \rangle = 0$ for every $j \neq j'$

so that we have

$$(\check{X}^{(i)})^* \check{X}^{(i)} = (t - u_{1,i}) \cdot I_k.$$

Now, let $w \in \mathfrak{S}_n \subset U(n)$ be any permutation satisfying $w(i+j) = j$ for $j = 1, \cdots, n-i$. Then $(wX)^{(n-i)} = \check{X}^{(i)}$ and the following matrix

$$w \cdot x = wxw^{-1} = wXX^*w^* = (wX)(wX)^*.$$

has the $(n-i)$-th leading principal minor

$$(w \cdot x)^{(n-i)} = (wX)^{(n-i)}((wX)^{(n-i)})^* = (\check{X}^{(i)})(\check{X}^{(i)})^*.$$

Again by Lemma 9.8, the nonzero eigenvalue of $(w\cdot x)^{(n-i)}$ is $t-u_{1,i}$ with multiplicity k and zero with multiplicity $n-i-k$. Thus we have

$$\Phi_\lambda^{1,n-i}(w\cdot x)=\Phi_\lambda^{2,n-i-1}(w\cdot x)=\cdots=\Phi_\lambda^{k,n-i-k+1}(w\cdot x)=t-u_{1,i}.$$

Therefore, we have $\Phi_\lambda(w\cdot x)\in\gamma_{n-i}$. In particular if $n-i\neq i$, then $\Phi_\lambda(w\cdot x)$ is not in $\mathring{\gamma}_i$ since $\mathring{\gamma}_i\cap\gamma_{n-i}=\emptyset$. $\square$

Example 9.11. Let us reconsider the Lagrangian faces γ_2 and γ_4 in Example 9.7 (4), which is the case where $n=6$ and $k=2$. Then

$$n=6\neq 2\cdot 2=2i\ \text{ for } i=2,\ \text{ and } n=6\neq 2\cdot 4=2i\ \text{ for } i=4.$$

Thus the faces γ_2 and γ_4 are displaceable by Proposition 9.10. For the face γ_3 in Example 9.7 (4), however, we have

$$n=6=2\cdot 3=2i\quad\text{and}\quad i=3$$

so that we cannot determine displaceability of γ_3 by using Proposition 9.10.

Now, we extend Proposition 9.10 to Lagrangian faces containing multiple squares of type ${}^{(0,\bullet)}\square_k$ in Γ_λ. For a sequence $(i_1,\cdots,i_r)$ satisfying

$$\begin{cases} k\leq i_1<\cdots<i_r\leq n-k,\\ i_{s+1}-i_s\geq k \text{ for each } s,\end{cases}\tag{9.3}$$

let $\gamma_{i_1,\cdots,i_r}$ be the Lagrangian face of Γ_λ which contains r simple closed regions $\{C_1\cdots,C_r\}$ where C_s is the square ${}^{(0,i_s)}\square_k$ and the other simple closed regions of $\gamma_{i_1,\cdots,i_r}$ are of size 1×1, see $\gamma_{2,4}$ in Figure 25 for example.

Proposition 9.12. *Suppose $\{i_1,\cdots,i_r\}$ is given satisfying (9.3). Then there is a permutation $w\in\mathfrak{S}_n$ such that*

$$w\cdot\Phi_\lambda^{-1}(\mathring{\gamma}_{i_1,\cdots,i_r})\cap\Phi_\lambda^{-1}(\mathring{\gamma}_{i_1,\cdots,i_r})=\emptyset\tag{9.4}$$

unless $i_s=s\cdot i_1$ for every $s=1,\cdots,r+1$ provided that $i_{r+1}:=n$.

Remark 9.13. Proposition 9.10 can be obtained by Proposition 9.12 by taking $r=1$.

Proof. Let $\mathbf{u}\in\mathring{\gamma}_{i_1,\cdots,i_r}$. Then $\mathbf{u}$ satisfies

$$u_{1,i_s}=u_{2,i_s-1}=\cdots=u_{k,i_s-k+1}$$

for every $s=1,\cdots,r$.

For any $x\in\Phi_\lambda^{-1}(\mathbf{u})$, Lemma 9.9 implies that there exists an $(n\times k)$ matrix $X=[v_1,\cdots,v_k]$ such that

- $x=XX^*$,
- $|v_j|^2=t$ for every $j=1,\cdots,k$, and
- $\langle v_j,v_{j'}\rangle=0$ for $j\neq j'$.

Then we can divide X into $r+1$ submatrices $X_{(i_0)}^{(i_1)},X_{(i_1)}^{(i_2)},\cdots,X_{(i_{r-1})}^{(i_r)},X_{(i_r)}^{(i_{r+1})}$ of X (provided that $i_0:=0$ and $i_{r+1}:=n$) where

$$X_{(i_s)}^{(i_{s+1})}=\left(v_1{}_{(i_s)}^{(i_{s+1})},\cdots,v_k{}_{(i_s)}^{(i_{s+1})}\right):=\begin{pmatrix} w_{i_s+1}\\ \vdots\\ w_{i_{s+1}}\end{pmatrix}$$

for each $s=0,\cdots,r$. In other words, $X_{(i_s)}^{(i_{s+1})}$ is the $((i_{s+1}-i_s)\times k)$ submatrix of X obtained by deleting all ℓ-th rows of X for $\ell>i_{s+1}$ and $\ell\leq i_s$. By using Lemma 9.8 and Lemma 9.9, it is not hard to show that

- $|v_j{}_{(i_s)}^{(i_{s+1})}|^2=u_{1,i_{s+1}}-u_{1,i_s}$ for every $j=1,\cdots,k$ and $s=0,\cdots,r$, and
- $\langle v_j{}_{(i_s)}^{(i_{s+1})},v_{j'}{}_{(i_s)}^{(i_{s+1})}\rangle=0$ for $j\neq j'$.

Now, suppose that

$$w \cdot \Phi_\lambda^{-1}(\mathring{\gamma}_{i_1,\cdots,i_r}) \cap \Phi_\lambda^{-1}(\mathring{\gamma}_{i_1,\cdots,i_r}) \neq \emptyset \tag{9.5}$$

for every $w \in \mathfrak{S}_n$. We claim that

$$i_s = s \cdot i_1 \quad \text{for every } s = 1, \cdots, r+1 \quad \text{where} \quad i_{r+1} := n.$$

To show this, consider $w := w(s, s') \in \mathfrak{S}_n \subset U(n)$ for any $s, s' \in \{1, \cdots, r+1\}$ with $s < s'$ defined by

$$\begin{array}{c} \{1, \cdots, i_s, \underline{i_s + 1, \cdots, i_{s+1}}, i_{s+1}+1, \cdots, i_{s'}, \underline{i_{s'}+1, \cdots, i_{s'+1}}, i_{s'+1}+1, \cdots, n\} \\ w \Big\downarrow \\ \{1, \cdots, i_s, \underline{i_{s'} + 1, \cdots, i_{s'+1}}, i_{s+1}+1, \cdots, i_{s'}, \underline{i_s+1, \cdots, i_{s+1}}, i_{s'+1}+1, \cdots, n\}. \end{array}$$

Then we can easily see that the permutation w swaps the position of $X^{(i_{s+1})}_{(i_s)}$ with $X^{(i_{s'+1})}_{(i_{s'})}$ in X so that

$$\Phi_\lambda(w \cdot x) \in \gamma_{i'_1,\cdots,i'_r}$$

for some $k \leq i'_1 < \cdots < i'_r \leq n - k$ where

$$i'_{s+1} = i_s + (i_{s'+1} - i_{s'}).$$

By our assumption (9.5), we have $\gamma_{i'_1,\cdots,i'_r} \cap \mathring{\gamma}_{i_1,\cdots,i_r} \neq \emptyset$. Since two faces $\gamma_{i'_1,\cdots,i'_r}$ and $\gamma_{i_1,\cdots,i_r}$ have the same dimensions, they must coincide and hence

$$i'_{s+1} = i_s + (i_{s'+1} - i_{s'}) = i_{s+1}.$$

Therefore, we may deduce that

$$i_{r+1} - i_r = i_r - i_{r-1} = \cdots = i_2 - i_1 = i_1 - i_0 = i_1,$$

which completes the proof. □

Corollary 9.14 (Theorem E)**.** *Let $\lambda = \{t, t, 0, \cdots, 0\}$ so that $\mathcal{O}_\lambda \cong \mathrm{Gr}(2, n)$. If n is prime, then every proper Lagrangian face of the Gelfand-Cetlin system on $(\mathcal{O}_\lambda, \omega_\lambda)$ is displaceable.*

Proof. Note that every proper Lagrangian face of Γ_λ is of the form $\gamma_{i_1,\cdots,i_r}$ for some $i_1, \cdots, i_r$. By Proposition 9.12, i_1 divides n where $2 \leq i_1 \leq n - 2$ which is impossible since n is prime. Thus there is no proper non-displaceable Lagrangian face of Γ_λ. □

10. Lagrangian Floer theory on Gelfand-Cetlin systems

The aim of this section is to review Lagrangian Floer theory which will be used to prove the results in Section 8. After briefly recalling Lagrangian Floer theory and its deformation developed by the third named author with Fukaya, Ohta, and Ono in a general context, we review work of Nishinou, Nohara, and Ueda about the calculation of the potential function of a GC system. Then, using the combinatorial description of Schubert cycles in complete flag manifolds by Kogan, we will express the potential function deformed by a combination of Schubert cycles of codimension two as a Laurent series. Finally, combining those ingredients, we provide the proof of Theorem C.

10.1. **Potential functions of Gelfand-Cetlin systems.**

Let Λ be the Novikov field over the field of complex numbers, defined by

$$\Lambda := \left\{ \sum_{j=1}^{\infty} a_j T^{\lambda_j} \,\middle|\, a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \to \infty} \lambda_j = \infty \right\}. \tag{10.1}$$

It is algebraically closed by Lemma A.1 in [FOOO3]. It comes with the valuation

$$\mathfrak{v}_T : \Lambda \backslash \{0\} \to \mathbb{R}, \quad \mathfrak{v}_T\left(\sum_{j=1}^{\infty} a_j T^{\lambda_j}\right) := \inf_j \{\lambda_j : a_j \neq 0\}.$$

We also play with two subrings of Λ given by

$$\Lambda_0 := \mathfrak{v}_T^{-1}[0,\infty) \cup \{0\} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \,\middle|\, \lambda_i \geq 0 \right\}$$

$$\Lambda_+ := \mathfrak{v}_T^{-1}(0,\infty) \cup \{0\} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda \,\middle|\, \lambda_i > 0 \right\}.$$

Let Λ_U be the collection of unitary elements of Λ. That is,

$$\Lambda_U := \Lambda_0 \backslash \Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda_0 \,\middle|\, \mathfrak{v}_T \left(\sum_{i=1}^{\infty} a_i T^{\lambda_i} \right) = 0 \right\}.$$

For a compact relatively spin Lagrangian submanifold L in a compact symplectic manifold (X, ω), thanks to the work of Fukaya [Fuk], one can associate a sequence of A_∞-structure maps $\{\mathfrak{m}_k\}_{k \geq 0}$ on the Λ_0-valued de Rham complex of L, which comes from moduli spaces of holomorphic discs bounded by L. Following the procedure in [FOOO2] for instance, the A_∞-algebra can be converted into the canonical model on $H^\bullet(L; \Lambda_0)$. By an abuse of notation, the structure maps of the canonical model are still denoted by $\mathfrak{m}_k$'s because the canonical model is only dealt with from now on.

A solution $b \in H^1(L; \Lambda_+)$ of the (weak) Maurer-Cartan equation

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) \equiv 0 \mod \mathrm{PD}[L]$$

is said to be a *(weak) bounding cochain*. The value of the *potential function* $\mathfrak{PO}$ at a bounding cochain b is assigned to be the multiple of the Poincaré dual $\mathrm{PD}[L]$ of L. Namely,

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) = \mathfrak{PO}(b) \cdot \mathrm{PD}[L].$$

Since $\mathrm{PD}[L]$ is the strict unit of the A_∞-algebra, the deformed map

$$\mathfrak{m}_1^b(h) = \sum_{l,k} \mathfrak{m}_{l+k+1}(b^{\otimes l}, h, b^{\otimes k})$$

becomes a differential and thus the Floer cohomology (deformed by b) over Λ_0 can be defined by

$$HF((L,b); \Lambda_0) := \mathrm{Ker}(\mathfrak{m}_1^b) \,/\, \mathrm{Im}(\mathfrak{m}_1^b).$$

The Floer cohomology (deformed by b) over Λ is defined by

$$HF((L,b); \Lambda) := HF((L,b); \Lambda_0) \otimes_{\Lambda_0} \Lambda.$$

The reader is referred to [FOOO1, FOOO3, FOOO4, FOOO7] for details.

Now, we specialize to the case of a Lagrangian GC torus fiber $\Phi_\lambda^{-1}(\mathbf{u})$ in a co-adjoint orbit $\mathcal{O}_\lambda$. As a deformation of the action of the Borel subgroup, Kogan and Miller [KoM] realized the toric degeneration of a flag manifold given by Gonciulea-Lakshmibai [GL] and Batyrev et al. [BCKV]. Using the degeneration of a (partial) flag manifold to the GC toric variety (in stages), Nishinou-Nohara-Ueda [NNU1] constructed a (not-in-stages) degeneration of the GC system of $\mathcal{O}_\lambda$ to the moment map of the toric variety.

Theorem 10.1 (Theorem 1.2 in [NNU1]). *For any non-increasing sequence $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_n\}$, there exists a toric degeneration of the Gelfand-Cetlin system Φ_λ on the co-adjoint orbit $(\mathcal{O}_\lambda, \omega_\lambda)$ in the following sense.*

(1) *There is a flat family $f \colon \mathcal{X} \to I = [0,1]$ of algebraic varieties and a symplectic form $\widetilde{\omega}$ on $\mathcal{X}$ such that*
   (a) *$X_0 := f^{-1}(0)$ is the toric variety associated with the Gelfand-Cetlin polytope Δ_λ and $\omega_0 := \widetilde{\omega}|_{X_0}$ is a torus-invariant Kähler form.*
   (b) *$X_1 := f^{-1}(1)$ is the co-adjoint orbit $\mathcal{O}_\lambda$ and $\omega_1 = \widetilde{\omega}|_{X_1}$ is the Kirillov-Kostant-Souriau symplectic form ω_λ.*

(2) *There is a family $\{\Phi_t \colon X_t \to \Delta_\lambda\}_{0 \leq t \leq 1}$ of completely integrable systems such that Φ_0 is the moment map for the torus action on X_0 and Φ_1 is the Gelfand-Cetlin system.*

(3) *Let* $\Delta_\lambda^{\rm sm} := \Delta_\lambda \backslash \Phi_0\,(\mathrm{Sing}(X_0))$ *and* $X_t^{\rm sm} := \Phi_t^{-1}(\Delta_\lambda^{\rm sm})$ *where* $\mathrm{Sing}(X_0)$ *is the set of singular points of* X_0*. Then, there exists a flow* ϕ_t *on* $\mathcal{X}$ *such that for each* $0 \leq t \leq s$*, the restricted flow* $\phi_t|_{X_s^{\rm sm}} : X_s^{\rm sm} \to X_{s-t}^{\rm sm}$ *respects the symplectic structures and the complete integrable systems:*

$$\begin{tikzcd} (X_s^{\rm sm}, \omega_s) \arrow[rr, "\phi_t|_{X_s^{\rm sm}}"] \arrow[dr, "\Phi_s"'] & & (X_{s-t}^{\rm sm}, \omega_{s-t}) \arrow[dl, "\Phi_{s-t}"] \\ & \Delta_\lambda^{\rm sm} & \end{tikzcd}$$

Let $\phi'_s \colon X_s \to X_0$ be a (continuous) extension of the flow $\phi_s \colon X_s^{\rm sm} \to X_0^{\rm sm}$ in Theorem 10.1 ([NNU1, Section 8]). The extended map ϕ'_s effectively transports data for Floer theory from the toric moment map to a nearby integrable system. As the deformation in Theorem 10.1 is through Fano varieties and the GC toric variety admits a small resolution at the singular loci, any holomorphic discs bounded by L intersecting the loci collapsing to the singular loci of X_0 must have the Maslov index strictly greater than two so that such discs do *not* contribute to the potential function. Furthermore, because the Fredholm regularity is an open condition, the holomorphic discs of Maslov index two intersecting the toric divisor in the toric variety X_0 give rise to regular holomorphic discs at X_s for sufficiently small s. Set $X := X_s$ and let L be any Lagrangian torus fiber of $\Phi_s \colon X_s \to \Delta_\lambda$. Combining those with the results of Cho-Oh [CO], Nishinou-Nohara-Ueda proved the followings.

(1) Each Lagrangian torus fiber L does not bound any non-constant holomorphic discs whose classes are of Maslov index less than or equal to zero. [NNU1, Lemma 9.20]
(2) Every class $\beta \in \pi_2(X, L)$ of Maslov index two is Fredholm regular. [NNU1, Proposition 9.17]
(3) There is a one-to-one correspondence between the holomorphic discs of Maslov index two bounded by a Lagrangian Gelfand-Cetlin torus fiber and the facets of Gelfand-Cetlin polytope. [NNU1, Lemma 9.22]
(4) For each class β of Maslov index two, the open Gromov-Witten invariant n_β, which counts the holomorphic discs passing through a generic point in L and representing β, is 1. [NNU1, Proposition 9.16]

The above conditions imply that every 1-cochain in $H^1(L;\Lambda_0)$ is a weak bounding cochain and hence the potential function can be defined on $H^1(L;\Lambda_0)$. To extend deformation space from $H^1(L;\Lambda_+)$ to $H^1(L;\Lambda_0)$, one should consider Floer theory twisted with flat non-unitary line bundles, see Cho [Cho2]. Moreover, the potential function can be written as

$$\mathfrak{PO}\,(L; b) = \sum_\beta n_\beta \cdot \exp(\partial\beta \cap b)\, T^{\omega(\beta)/2\pi} \tag{10.2}$$

where the summation is taken over all homotopy classes in $\pi_2(X, L)$ of Maslov index two. Setting

$$\Gamma(n) = \{(i,j) : 2 \leq i + j \leq n\},$$

we fix the basis $\{\gamma_{i,j} : (i,j) \in \Gamma(n)\}$ for $H^1(L;\mathbb{Z})$ dual to the basis for $H_1(L,\mathbb{Z})$ which consists of the circles generated by $\{u_{i,j} : (i,j) \in \Gamma(n)\}$. Take the exponential variables [11]

$$y_{i,j} := e^{x_{i,j}} \tag{10.3}$$

where b is expressed as the linear combination $\sum_{(i,j)\in\Gamma(n)} x_{i,j} \cdot \gamma_{i,j}$. Then, the potential function can be expressed as a Laurent polynomial $\mathfrak{PO}(\mathbf{y})$ with respect to $\{y_{i,j} : (i,j) \in \Gamma(n)\}$. Setting $u_{i,n+1-i} := \lambda_i$, keep in mind that Δ_λ is defined by

$$\left\{(u_{i,j}) \in \mathbb{R}^{n(n-1)/2} : u_{i,j+1} - u_{i,j} \geq 0,\, u_{i,j} - u_{i+1,j} \geq 0\right\}.$$

(see Theorem 4.4 for instance.)

[11] In [NNU1], Nishinou, Nohara, and Ueda set

$$y_{i,j} := e^{x_{i,j}} T^{u_{i,j}}$$

so that it is different from our $y_{i,j}$ in (10.3). To keep track of the valuations of holomorphic discs, we prefer to take $y_{i,j}$ as the expotential variable without weights.

Theorem 10.2 (Theorem 10.1 in [NNU1]). *Setting*

$$y_{i,n+1-i} := 1 \quad u_{i,n+1-i} := \lambda_i,$$

we consider the Lagrangian torus L over a point $\{u_{i,j} : (i,j) \in \Gamma(n)\}$. Then, the potential function on L is written as

$$\mathfrak{PO}(L; \mathbf{y}) = \sum_{(i,j)} \left(\frac{y_{i,j+1}}{y_{i,j}} T^{u_{i,j+1} - u_{i,j}} + \frac{y_{i,j}}{y_{i+1,j}} T^{u_{i,j} - u_{i+1,j}} \right) \tag{10.4}$$

where the summation is taken over $\Gamma(n) = \{(i,j) : 2 \leq i + j \leq n\}$.

Remark 10.3. As we are mainly concerned with non-displaceability, it is enough to show that L_s is non-displaceable because of the homotopy invariance of A_∞-structures.

For t with $0 \leq t < 1$, the potential function of the Lagrangian torus $L_m(t)$ over $I_m(t)$ is arranged as follows:

(10.5)

$$\mathfrak{PO}(L_m(t); \mathbf{y}) = \left(\sum_{i=1}^{m} \sum_{j=1}^{m-1} \frac{y_{i,j+1}}{y_{i,j}} + \sum_{i=1}^{m-1} \sum_{j=1}^{m} \frac{y_{i,j}}{y_{i+1,j}} \right) T^{1-t} + \left(\sum_{\max(i,j) \geq m+1} \left(\frac{y_{i,j+1}}{y_{i,j}} + \frac{y_{i,j}}{y_{i+1,j}} \right) \right) T^1$$
$$+ \left(\sum_{i=0}^{m-1} \left(\frac{y_{m-i,m+1}}{y_{m-i,m}} + \frac{y_{m,m-i}}{y_{m+1,m-i}} \right) T^{1+it} \right)$$

For simplicity, we frequently omit $L_m(t)$ in $\mathfrak{PO}(L_m(t); \mathbf{y})$ if $L_m(t)$ is clear in the context. The logarithmic derivative with respect to $y_{i,j}$ is denoted by

$$\partial(i,j)(\mathbf{y}) := y_{i,j} \frac{\partial \mathfrak{PO}}{\partial y_{i,j}}. \tag{10.6}$$

Example 10.4. In the complete flag manifold $\mathcal{F}(5) \simeq \mathcal{O}_\lambda$ where $\lambda = \{4, 2, 0, -2, -4\}$, some logarithmic derivatives of $\mathfrak{PO}(L_2(t); \mathbf{y})$ are as follows.

$$\begin{aligned}
\partial(1,1)(\mathbf{y}) &= \left(-\frac{y_{1,2}}{y_{1,1}} + \frac{y_{1,1}}{y_{2,1}} \right) T^{1-t}, \\
\partial(2,2)(\mathbf{y}) &= \left(-\frac{y_{1,2}}{y_{2,2}} + \frac{y_{2,2}}{y_{2,1}} \right) T^{1-t} + \left(-\frac{y_{2,3}}{y_{2,2}} + \frac{y_{2,2}}{y_{3,2}} \right) T^1, \\
\partial(2,3)(\mathbf{y}) &= \left(-\frac{1}{y_{2,3}} - \frac{y_{1,3}}{y_{2,3}} + y_{2,3} + \frac{y_{2,3}}{y_{2,2}} \right) T^1.
\end{aligned}$$

10.2. **Bulk-deformations by Schubert cycles.**

We will apply Lagrangian Floer theory deformed by cycles of an ambient symplectic manifold, developed by the third named author with Fukaya, Ohta, and Ono in [FOOO1, FOOO4, FOOO7], in order to show non-displaceability. For a Lagrangian torus L from Section 10.1, we deform the underlying A_∞-algebra by empolying moduli spaces of holomorphic discs with interior marked points passing through a combination of designated ambient cycles at the image of interior marked points. We are particularly interested in a combination of cycles $\mathscr{D}_j$ of degree two not intersecting L

$$\mathfrak{b} := \sum_{j=1}^{B} \mathfrak{b}_j \cdot \mathscr{D}_j,$$

which is called a *bulk-deformation parameter*. As the A_∞-algebra is deformed, the potential function is also deformed. The deformed potential function is denoted by $\mathfrak{PO}^{\mathfrak{b}}$.

Remark 10.5. For the purpose of the present paper, we use only cycles of degree two to deform Lagrangian Floer theory because the deformed potential function is computable. It also simplifies construction of virtual fundamental cycles needed for the definition of open Gromov-Witten invariants [FOOO4, Definition 6.7], denoted by $n_\beta(\mathbf{p})$.

We first recall the formula for the potential function of a toric fiber L deformed by a combination of toric divisors

$$\mathfrak{b} := \sum_{j=1}^{B} \mathfrak{b}_j \cdot \mathscr{D}_j,$$

in a compact toric manifold X from [FOOO4].

Theorem 10.6 ([FOOO4])**.** *The bulk-deformed potential function, also called the potential function with bulk, is written as*

$$\mathfrak{PO}^{\mathfrak{b}}(L;b) = \sum_{\beta} n_\beta \cdot \exp\left(\sum_{j=1}^{B} (\beta \cap \mathscr{D}_j)\, \mathfrak{b}_j\right) \exp(\partial\beta \cap b)\, T^{\omega(\beta)/2\pi}. \tag{10.7}$$

where the summantion is taken over all homotopy classes in $\pi_2(X,L)$ of Maslov index two.

In the derivation of this in [FOOO4], the properties that the relevant ambient cycles are smooth and T^n-invariant are used. Since our Schubert cycles are neither smooth nor T^n-invariant in general, we will provide details of the proof of this theorem for the current GC case modifying the arguments used in the proof of [FOOO4, Proposition 4.7] similarly as done in [NNU1, Section 9], see Section 13. The upshot is that we still have the same formula for the potential function with bulk in the current GC case, see (13.10). Again by taking the system of exponential coordinates in (10.3), $\mathfrak{PO}^{\mathfrak{b}}$ in (13.10) becomes a Laurent series with respect to $\{y_{i,j} : (i,j) \in \Gamma(n)\}$.

Theorem 10.7 (Section 8 in [FOOO4])**.** *If the bulk-deformed potential function $\mathfrak{PO}^{\mathfrak{b}}(L;\mathbf{y})$ admits a critical point* $\mathbf{y}$ *whose components are in Λ_U, then L is non-displaceable.*

In our case, we will employ Schubert cycles to deform the potential function $\mathfrak{PO}$. In his thesis [Ko], Kogan found an expression of a Schubert cycle in terms of a certain union of the inverse images of faces in the GC system of a complete flag manifold, see also Kogan-Miller [KoM]. As we have seen in Section 5, due to presence of non-torus fibers, the inverse image of a face might have boundary so that it does *not* represent a cycle. What he proved is that a certain combination of faces can represent a cycle because the boundaries are cancelled out.

We review the result in terms of ladder diagrams. A facet in a GC polytope is said to be *horizontal* (resp. *vertical*) if it is given by $u_{i,j} = u_{i+1,j}$ (resp. $u_{i,j+1} = u_{i,j}$). Let $P^{\text{hor}}_{i,i+1}$ (resp. $P^{\text{ver}}_{j+1,j}$) be the union of horizontal (resp. vertical) facets between the i-th column and the $(i+1)$-th column (resp. the $(j+1)$-th row and the j-th row) of the ladder diagram. That is,

$$P^{\text{hor}}_{i,i+1} := \bigcup_{s=1}^{n-i} \{u_{i,s} = u_{i+1,s}\}, \quad P^{\text{ver}}_{j+1,j} := \bigcup_{r=1}^{n-j} \{u_{r,j+1} = u_{r,j}\}$$

for $1 \leq i,j \leq n-1$ where $\{u_{\bullet,\bullet} = u_{\bullet,\bullet}\}$ denotes the facet given by the equation inside. Let

$$\mathscr{D}^{\text{hor}}_{i,i+1} := \Phi_\lambda^{-1}\left(P^{\text{hor}}_{i,i+1}\right), \quad \mathscr{D}^{\text{ver}}_{j+1,j} := \Phi_\lambda^{-1}\left(P^{\text{ver}}_{j+1,j}\right),$$

which are respectively called a *horizontal* and *vertical Schubert cycle* (of degree two) because of Theorem 10.9.

Example 10.8. Consider the co-adjoint orbit $\mathcal{O}_\lambda \simeq \mathcal{F}(6)$ where $\lambda = (5,3,1,-1,-3,-5)$.

(1) $P^{\text{hor}}_{4,5}$ is the union of two horizontal facets

$$P^{\text{hor}}_{4,5} = \{u_{4,2} = -3\} \cup \{u_{4,1} = u_{5,1}\}$$

as in Figure 29.

(2) $P^{\text{ver}}_{4,3}$ is the union of three vertical facets

$$P^{\text{ver}}_{4,3} = \{1 = u_{3,3}\} \cup \{u_{2,4} = u_{2,3}\} \cup \{u_{1,4} = u_{1,3}\}$$

as in Figure 30.

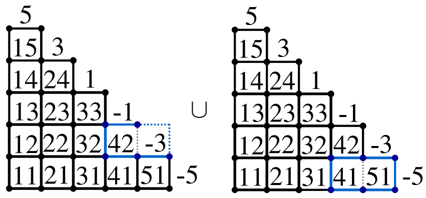


FIGURE 29. $P^{\text{hor}}_{4,5}$ in $\mathcal{F}(6)$.

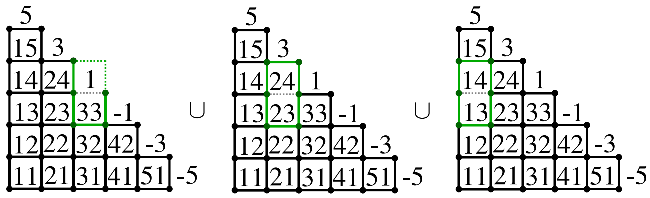


FIGURE 30. $P^{\text{ver}}_{4,3}$ in $\mathcal{F}(6)$.

By following the combinatorial process playing with *reduced pipe dreams* in [Ko, KoM], observe that the Schubert varieties associated with the adjacent transpositions [12], having a complex codimension one, are corresponding to either unions of horizontal facets or unions of vertical facets. The opposite Schubert varieties are corresponding to the other, see Remark 9 in [KoM].

Theorem 10.9 (Theorem 2.3.1 in [Ko], Theorem 8 in [KoM])**.** *The inverse image $\mathscr{D}^{\text{hor}}_{\bullet,\bullet+1}$ (or $\mathscr{D}^{\text{ver}}_{\bullet+1,\bullet}$) represents a Schubert cycle of degree two. For any (opposite) Schubert divisor X_w, there exists either $\mathscr{D}^{\text{hor}}_{\bullet,\bullet+1}$ or $\mathscr{D}^{\text{ver}}_{\bullet+1,\bullet}$ such that it represents the cycle $[X_w]$ of the divisor.*

Now, we apply (13.10), which is a counterpart of (10.6), to calculate the bulk-deformed potential function. Because of the condition (3) in Section 10.1, let

$$\beta^{i,j}_{i+1,j} \quad \left(\text{resp. } \beta^{i,j+1}_{i,j}\right)$$

be the homotopy class in $\pi_2(\mathcal{O}_\lambda, L)$ represented by a holomorphic disc of Maslov index 2 corresponding to $u_{i,j} = u_{i+1,j}$ (resp. $u_{i,j+1} = u_{i,j}$).

Lemma 10.10. *Let $\mathscr{D}$ be either a horizontal or a vertical Schubert cylce in X_s. Then, we have*

$$\beta^{i,j}_{i+1,j} \cap \mathscr{D} = \begin{cases} 1 & \text{if } \mathscr{D} = \mathscr{D}^{\text{hor}}_{i,i+1} \\ 0 & \text{otherwise,} \end{cases} \qquad \beta^{i,j+1}_{i,j} \cap \mathscr{D} = \begin{cases} 1 & \text{if } \mathscr{D} = \mathscr{D}^{\text{ver}}_{j+1,j} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\phi'_s \colon X_s \to X_0$ be a (continuous) extension of the flow $\phi_s \colon X^{\text{sm}}_s \to X^{\text{sm}}_0$ in Theorem 10.1, see [NNU1, Section 8]. Let $\varphi \colon (\mathbb{D}, \partial\mathbb{D}) \to (X_s, L_s)$ be a holomorphic disc in the class $\beta^{i,j}_{i+1,j}$ of Maslov index two for example. Then, we have a (topological) disc $\phi'_s \circ \varphi \colon (\mathbb{D}, \partial\mathbb{D}) \to (X_s, L_s) \to (X_0, L_0)$, representing $(\phi'_s)_* \beta^{i,j}_{i+1,j}$. Note that there exists a holomorphic disc φ_0 by [CO] in the class $(\phi'_s)_* \beta^{i,j}_{i+1,j} = [\phi'_s \circ \varphi]$. Meanwhile, by our choice of $\mathscr{D}$, $\phi'_s(\mathscr{D})$ is the union of the components over either $P^{\text{hor}}_{i,i+1}$ or $P^{\text{ver}}_{j+1,j}$. Since the flow ϕ'_s gives rise to a symplectomorphism from X^{sm}_s to X^{sm}_0 and the image of the disc φ is contained in X^{sm}_s, the (local) intersection number should be preserved through the flow ϕ'_s. To calculate the intersection number, we consider a small resolution $p \colon \widetilde{X}_0 \to X_0$. Because the intersection happens only outside the singular loci at X_0, we can lift the divisor and the disc φ_0 to $\widetilde{\mathscr{D}}_0$ and $\widetilde{\varphi}_0$ in $\widetilde{X}_0$ without any change of the intersection number. Then, we have

$$\beta^{i,j}_{i+1,j} \cap \mathscr{D} = [\varphi] \cap \mathscr{D} = [\widetilde{\varphi}_0] \cap \widetilde{\mathscr{D}}_0,$$

which completes the proof. □

[12] An adjacent transposition is a transposition of the form $(i, i+1)$.

We take

$$\mathfrak{b} := \sum_i \mathfrak{b}^{\text{hor}}_{i,i+1} \cdot \varphi'_{1-s}\left(\mathscr{D}^{\text{hor}}_{i,i+1}\right) + \sum_j \mathfrak{b}^{\text{ver}}_{j+1,j} \cdot \varphi'_{1-s}\left(\mathscr{D}^{\text{ver}}_{j+1,j}\right) \tag{10.8}$$

where $\mathfrak{b}^{\text{hor}}_{i,i+1}, \mathfrak{b}^{\text{ver}}_{j+1,j} \in \Lambda_0$ and $\varphi'_{1-s} \colon X_1 \to X_s$. By abuse of notation for simplicity, we denote $\varphi'_{1-s}\left(\mathscr{D}^{\text{hor}}_{i,i+1}\right)$ (resp. $\varphi'_{1-s}\left(\mathscr{D}^{\text{ver}}_{j+1,j}\right)$) by $\mathscr{D}^{\text{hor}}_{i,i+1}$ (resp. $\mathscr{D}^{\text{ver}}_{j+1,j}$). By the homotopy invariance of the A_∞-structures, we may calculate the (bulk-deformed) Floer cohomology of $L_{m,s}(t)$ in X_s for s sufficiently close to 0. In particular, non-displaceability of $L_m(t)$ can be achieved as long as the Floer cohomology of $L_{m,s}(t)$ is non-zero. Whenever turning on a bulk-deformation, this process passing to $L_{m,s}(t)$ and $\varphi'_{1-s}(\mathscr{D})$ in X_s will be taken into consideration. Also, depending on the position t of a Lagrangian torus $L_m(\,\cdot\,)$, we need to consider different $\mathfrak{b}^{\text{hor}}_{i,i+1}$ and $\mathfrak{b}^{\text{ver}}_{j+1,j}$.

Corollary 10.11. *Taking a bulk-deformation parameter as in (10.8), the deformed potential function is expressed as*

$$\mathfrak{PO}^{\mathfrak{b}}(L; \mathbf{y}) = \sum_{(i,j)} \left(\exp\left(\mathfrak{b}^{\text{hor}}_{i,i+1}\right) \frac{y_{i,j}}{y_{i+1,j}} T^{u_{i,j}-u_{i+1,j}} + \exp\left(\mathfrak{b}^{\text{ver}}_{j+1,j}\right) \frac{y_{i,j+1}}{y_{i,j}} T^{u_{i,j+1}-u_{i,j}} \right). \tag{10.9}$$

Note that the gradient of the bulk-deformed potential is given as follows:

$$\begin{aligned} \partial^{\mathfrak{b}}(i,j)(\mathbf{y}) := y_{i,j} \frac{\partial \mathfrak{PO}^{\mathfrak{b}}}{\partial y_{i,j}}(\mathbf{y}) = &-c^{\text{ver}}_{j+1,j} \frac{y_{i,j+1}}{y_{i,j}} T^{u_{i,j+1}-u_{i,j}} - c^{\text{hor}}_{i-1,i} \frac{y_{i-1,j}}{y_{i,j}} T^{u_{i-1,j}-u_{i,j}} \\ &+ c^{\text{hor}}_{i,i+1} \frac{y_{i,j}}{y_{i+1,j}} T^{u_{i,j}-u_{i+1,j}} + c^{\text{ver}}_{j,j-1} \cdot \frac{y_{i,j}}{y_{i,j-1}} T^{u_{i,j}-u_{i,j-1}} \end{aligned} \tag{10.10}$$

where $c^{\text{hor}}_{i,i+1} = \exp(\mathfrak{b}^{\text{hor}}_{i,i+1})$ and $c^{\text{ver}}_{j+1,j} = \exp(\mathfrak{b}^{\text{ver}}_{j+1,j})$.

10.3. **The non-displaceable Gelfand-Cetlin fibers in $\mathcal{F}(3)$.**

In this section, the case of $\mathcal{F}(3)$ will be discussed in details. The following theorem will be proven.

Theorem 10.12 (Theorem C)**.** *Let $\lambda = \{\lambda_1 = 2 > \lambda_2 = 0 > \lambda_3 = -2\}$. Consider the co-adjoint orbit $\mathcal{O}_\lambda$, a complete flag manifold $\mathcal{F}(3)$ equipped with the monotone Kirillov-Kostant-Souriau symplectic form ω_λ. Then the Gelfand-Cetlin fiber over a point $\mathbf{u} \in \Delta_\lambda$ is non-displaceable if and only if $\mathbf{u} \in I$ where*

$$I := \left\{ (u_{1,1}, u_{1,2}, u_{2,1}) = (0, 1-t, -1+t) \in \mathbb{R}^3 \mid 0 \leq t \leq 1 \right\} \tag{10.11}$$

In particular, the Lagrangian 3-sphere $\Phi_\lambda^{-1}(0,0,0)$ is non-displaceable.

Before starting our proof, we explain an alternative description for the fibers over the red line in Figure 20 following Chan-Pomerleano-Ueda [CPU]. They proved the homological mirror symmetry for the conifold by realizing Strominger-Yau-Zaslow(SYZ) mirror symmetry on the smoothing Y_ε of the conifold where

$$Y_\varepsilon = \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 : u_1 v_1 - u_2 v_2 + \varepsilon = 0\}$$

It can be embedded into

$$\mathring{Y}_\varepsilon = \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^5 : u_1 v_1 = z - a, u_2 v_2 = z - b\}$$

where a and b are positive real numbers such that $\varepsilon = b - a$. For the purpose of doing Strominger-Yau-Zaslow(SYZ) mirror symmetry, they came up with a double conic fibration on the complement of the anticanonical divisor $\{z = 0\}$ in $\mathring{Y}_\varepsilon$. Consider the projection to the z-variable $\mathring{Y}_\varepsilon \to \mathbb{C}$. Note that we have singular fibers over two points $z = a$ and $z = b$. Also, it admits the following fiberwise T^2-action

$$\begin{cases} \theta_1 * (u_1, v_1) = (e^{\sqrt{-1}\theta} u_1, e^{-\sqrt{-1}\theta} v_1) \\ \theta_2 * (u_2, v_2) = (e^{\sqrt{-1}\theta} u_2, e^{-\sqrt{-1}\theta} v_2). \end{cases}$$

Any T^2-orbit satisfying $|u_i| = |v_i|$ for $i = 1, 2$ is called an *equator*. By collecting the equators over a circle centered at the origin in the z-plane, we obtain a (special) Lagrangian torus fibration (with respect to the holomorphic volume form $d\log z \wedge d\log u_1 \wedge d\log u_2$), see Figure 31. The fibration carries the exactly two walls at $z = a$ and $z = b$. Also, collecting the equators over the line segment connecting a and b, we obatin two solid tori that forms Lagrangian S^3.

This picture will serve as a local model for a Lagrangian torus fibration on $\mathcal{F}(3)$ equipped with a (non-standard) symplectic form and the tori consisting of equators over simply closed curve containing a, b and 0 in the z-plane will be in our interest.

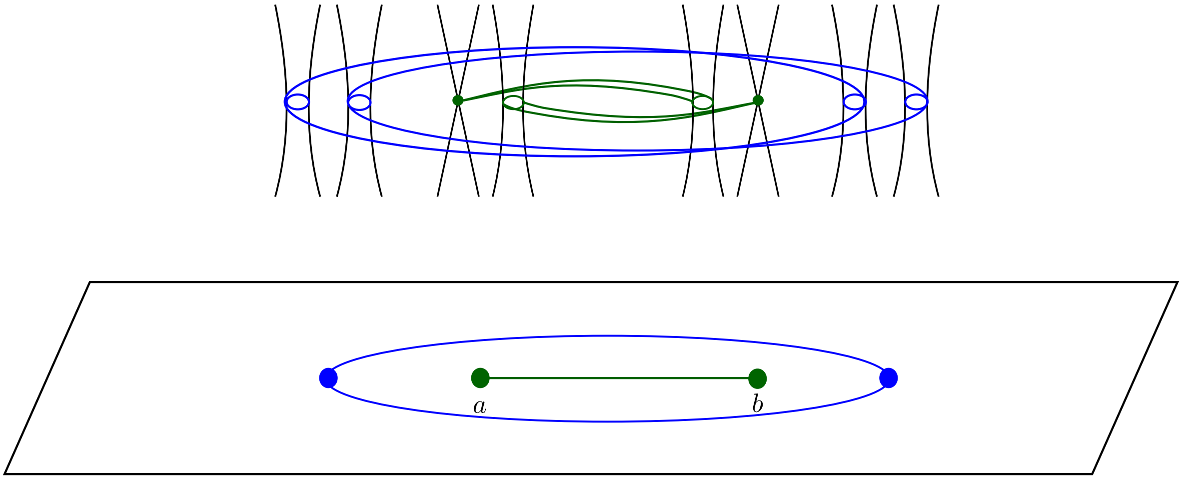


FIGURE 31. Double conic fibration.

Returning to the case of $\mathcal{F}(3)$, we have a toric degeneration of algebraic varieties consisting of

$$X_\varepsilon = \{([Z_1 : Z_2 : Z_3], [Z_{12} : Z_{23} : Z_{13}]) \in \mathbb{C}P^2 \times \mathbb{C}P^2 : \varepsilon Z_1 Z_{23} - Z_2 Z_{13} + Z_3 Z_{12} = 0\}.$$ [13]

For any $\varepsilon \neq 0$, X_ε is diffeomorphic to a complete flag manifold $\mathcal{F}(3)$. When $\varepsilon = 0$, we have a toric variety X_0 at the center. A generic one X_ε ($\varepsilon \neq 0$) can be viewed as a compactification of Y_ε. By getting rid of divisors $Z_1 = 0$ and $Z_{23} = 0$ in X_ε and setting

$$u_1 := \frac{Z_3}{Z_1}, v_1 := \frac{Z_{12}}{Z_{23}}, u_2 := \frac{Z_2}{Z_1}, v_2 := \frac{Z_{13}}{Z_{23}},$$

we obtain a torus fibration on an open subvariety of X_ε by following the above recipe with respect to the symplectic form induced from X_ε.

As we are concerned with Lagrangians, we may set $a = 0$ and $b = \varepsilon$ and then fix a family $\{\gamma(t) : 0 < t \leq 1\}$ of simply closed curves enclosing a and b at the interior and converging to the line segment $[a, b]$. The torus fibers consisting of the equators over the simple curves in the family degenerates into the Lagrangian S^3. We would like to show that those tori are non-displaceable.

As $\varepsilon \to 0$, X_ε degenerates into a toric variety X_0. A torus over the simple closed curve enclosing a and b is Lagrangian isotopic to a torus over a simple closed curve centered at the origin, which comes from the toric fibers. In this case, according to Knutson-Miller [KnM, Section 1.3], the (opposite) Schubert divisors on X_ε can be defined as matrix Schubert variety. In terms of the Plücker coordinates, their results can be written as

$$X_{132} = \{Z_{12} = 0\} = \{v_1 = 0\},\ X_{213} = \{Z_1 = 0\} = \{u_1 = 0\}$$

where X_w is the Schubert variety associated with w in the symmetric group S_3. By [KoM], these correspond to

$$\mathscr{D}^{\text{ver}}_{2,1} = X_{132},\ \mathscr{D}^{\text{ver}}_{3,2} = X_{213} \tag{10.12}$$

By taking the involution where,

$$Z_1 \leftrightarrow \overline{Z}_{23}, Z_2 \leftrightarrow \overline{Z}_{13}, Z_3 \leftrightarrow \overline{Z}_{12}$$

we also have

$$\mathscr{D}^{\text{hor}}_{1,2} = \{Z_3 = 0\},\ \mathscr{D}^{\text{hor}}_{2,3} = \{Z_{23} = 0\}. \tag{10.13}$$

[13] The conventions constructing toric degenerations on [NNU1] and [KoM] are different. In [NNU1], the diagonal term in the Plücker embedding survives, whereas in [KoM] the anti-diagonal term does. Here, we are following the convention of [KoM].

To deform the Floer theory, we employ a combination of the vertical and horizontal divisors in (10.12) and (10.13). Note that $\mathscr{D}^{\mathrm{ver}}_{2,1}$ and $\mathscr{D}^{\mathrm{hor}}_{1,2}$ are over $z = 0$ and $\mathscr{D}^{\mathrm{ver}}_{3,2}$ and $\mathscr{D}^{\mathrm{hor}}_{2,3}$ are at ∞. Therefore, any of the divisors do *not* intersect the torus fibers.

$$\mathfrak{b} = \mathfrak{b}^{\mathrm{ver}}_{2,1} \cdot \mathscr{D}^{\mathrm{ver}}_{2,1} + \mathfrak{b}^{\mathrm{hor}}_{1,2} \cdot \mathscr{D}^{\mathrm{hor}}_{1,2} + \mathfrak{b}^{\mathrm{ver}}_{3,2} \cdot \mathscr{D}^{\mathrm{ver}}_{3,2} + \mathfrak{b}^{\mathrm{hor}}_{2,3} \cdot \mathscr{D}^{\mathrm{hor}}_{2,3} \tag{10.14}$$

For the proof of Theorem 10.12, we need the following proposition.

Proposition 10.13. *Let $\Phi\colon X \to \Delta \subset \mathbb{R}^d$ be a completely integrable system (Definition 3.6) such that Φ is proper. If there exists a sequence $\{\mathbf{u}_i : i \in \mathbb{N}\}$ such that*

1. *Each $\Phi^{-1}(\mathbf{u}_i)$ is non-displaceable.*
2. *The sequence $\mathbf{u}_i$ converges to some point $\mathbf{u}_\infty$ in Δ.*

then $\Phi^{-1}(\mathbf{u}_\infty)$ is also non-displaceable.

Proof. For a contradiction, suppose that $\Phi^{-1}(\mathbf{u}_\infty)$ is displaceable. There is a (time-dependent) Hamiltonian diffeomorphism ϕ and an open set U containing $\Phi^{-1}(\mathbf{u}_\infty)$ in X such that

$$\phi(U) \cap U = \emptyset.$$

For each i, there exists a point $x_i \in \Phi^{-1}(\mathbf{u}_i)$ such that $x_i \notin U$ since $\Phi^{-1}(\mathbf{u}_i)$ is non-displaceable. It implies that any subsequence of $\{x_i\}$ cannot converge to a point in U. On the other hand, passing to a subsequence, we may assume that x_i converges to x_∞ for some $x_\infty \in X$ since Φ is proper. By the continuity of Φ, we then have

$$\mathbf{u}_\infty = \lim_{i\to\infty} \mathbf{u}_i = \lim_{i\to\infty} \Phi(x_i) = \Phi(x_\infty).$$

It leads to a contradiction that $x_\infty \in \Phi^{-1}(\mathbf{u}_\infty) \subset U$. ∎

We now start the proof of Theorem 10.12. Since

Proof of Theorem 10.12. For any fixed t with $0 \leq t < 1$, let $L(t)$ be the Lagrangian torus fiber over $(0, 1 - t, -1 + t) \in I$ in (10.11). Let $L_s(t)$ be the fiber corresponding to $L(t)$ in X_s via a toric degeneration of completely integrable systems in Theorem 10.1. By taking $s > 0$ sufficiently close to 0, the potential function of $L_s(t)$ can be arranged as

$$\mathfrak{PO}(\mathbf{y}) = \left(\frac{y_{1,2}}{y_{1,1}} + \frac{y_{1,1}}{y_{2,1}} + y_{1,2} + \frac{1}{y_{2,1}}\right) T^{1-t} + \left(\frac{1}{y_{1,2}} + y_{2,1}\right) T^{1+t}.$$

We use a combination of Schubert cycles in (10.8) (or equivalently (10.14)) to deform the potential function. A strategy we take is to postpone determining bulk-deformation parameters. Namely, we start with a tentative a parameter, determine solutions for $\mathbf{y}$ first, and then adjust the parameter to make the chosen $\mathbf{y}$ a critical point.

Take a *tentative* bulk-parameter $\mathfrak{b}' := \mathfrak{b}^{\mathrm{ver}}_{2,1} \cdot \mathscr{D}^{\mathrm{ver}}_{2,1}$ such that $\exp(\mathfrak{b}^{\mathrm{ver}}_{2,1}) = 1 + T^{2t}$, i.e.,

$$\mathfrak{b}^{\mathrm{ver}}_{2,1} = T^{2t} - \frac{1}{2} T^{4t} + \cdots \in \Lambda_+.$$

By Corollary 10.11, the potential function is deformed into

$$\mathfrak{PO}^{\mathfrak{b}'}(\mathbf{y}) = \left(\frac{y_{1,2}}{y_{1,1}} + \frac{y_{1,1}}{y_{2,1}} + y_{1,2} + \frac{1}{y_{2,1}}\right) T^{1-t} + \left(\frac{y_{1,2}}{y_{1,1}} + \frac{1}{y_{2,1}} + \frac{1}{y_{1,2}} + y_{2,1}\right) T^{1+t},$$

whose logarithmic derivatives are

$$\begin{cases} y_{1,1} \dfrac{\partial \mathfrak{PO}^{\mathfrak{b}'}}{\partial y_{1,1}}(\mathbf{y}) = \left(-\dfrac{y_{1,2}}{y_{1,1}} + \dfrac{y_{1,1}}{y_{2,1}}\right) T^{1-t} + \left(-\dfrac{y_{1,2}}{y_{1,1}}\right) T^{1+t} \\[2ex] y_{1,2} \dfrac{\partial \mathfrak{PO}^{\mathfrak{b}'}}{\partial y_{1,2}}(\mathbf{y}) = \left(\dfrac{y_{1,2}}{y_{1,1}} + y_{1,2}\right) T^{1-t} + \left(\dfrac{y_{1,2}}{y_{1,1}} - \dfrac{1}{y_{1,2}}\right) T^{1+t} \\[2ex] y_{2,1} \dfrac{\partial \mathfrak{PO}^{\mathfrak{b}'}}{\partial y_{2,1}}(\mathbf{y}) = \left(-\dfrac{y_{1,1}}{y_{2,1}} - \dfrac{1}{y_{2,1}}\right) T^{1-t} + \left(-\dfrac{1}{y_{2,1}} + y_{2,1}\right) T^{1+t}. \end{cases}$$

We set $y_{1,2} = 1, y_{2,1} = 1$ and take $y_{1,1}$ as the solution of $(y_{1,1})^2 = 1 + T^{2t}$ satisfying $y_{1,1} \equiv -1 \mod T^{>0}$. It is easy to see that $y_{1,1} \frac{\partial \mathfrak{PO}^{\mathfrak{b}'}}{\partial y_{1,1}}(\mathbf{y}) = 0$. Note that $y_{1,1}$ is of the form

$$y_{1,1} \equiv -1 - \frac{1}{2} T^{2t} \mod T^{>2t}.$$

We now adjust a bulk-deformation parameter from $\mathfrak{b}'$ to $\mathfrak{b}$ in order for the chosen $(y_{1,1}, y_{1,2}, y_{2,1})$ to be a critical point of $\mathfrak{PO}^{\mathfrak{b}}$. Let

$$\mathfrak{b} := \mathfrak{b}' + \mathfrak{b}^{\text{ver}}_{3,2} \cdot \mathscr{D}^{\text{ver}}_{3,2} + \mathfrak{b}^{\text{hor}}_{2,3} \cdot \mathscr{D}^{\text{hor}}_{2,3}.$$

Since $\mathscr{D}^{\text{ver}}_{3,2}$ and $\mathscr{D}^{\text{hor}}_{2,3}$ do *not* intersect with the homotopy classes corresponding to $\mathscr{D}^{\text{ver}}_{2,1}$ in $\pi_2(\mathcal{O}_\lambda, \Phi_\lambda^{-1}(t))$, we still have

$$y_{1,1} \frac{\partial \mathfrak{PO}^{\mathfrak{b}}}{\partial y_{1,1}}(\mathbf{y}) = y_{1,1} \frac{\partial \mathfrak{PO}^{\mathfrak{b}'}}{\partial y_{1,1}}(\mathbf{y}) = 0.$$

Plugging the chosen $y_{i,j}$'s, we have

$$\begin{cases} y_{1,2} \dfrac{\partial \mathfrak{PO}^{\mathfrak{b}}}{\partial y_{1,2}}(\mathbf{y}) = \left(-\dfrac{1}{2} - \exp(\mathfrak{b}^{\text{ver}}_{3,2}) \right) T^{1+t} + \mathfrak{P}^{(1,2)} \cdot T^{1+t} \\ \\ y_{2,1} \dfrac{\partial \mathfrak{PO}^{\mathfrak{b}}}{\partial y_{2,1}}(\mathbf{y}) = \left(-\dfrac{1}{2} + \exp(\mathfrak{b}^{\text{hor}}_{2,3}) \right) T^{1+t} + \mathfrak{P}^{(2,1)} \cdot T^{1+t}. \end{cases}$$

for some constant $\mathfrak{P}^{(1,2)}, \mathfrak{P}^{(2,1)} \in \Lambda_+$. By choosing $\mathfrak{b}^{\text{ver}}_{3,2}, \mathfrak{b}^{\text{hor}}_{2,3} \in \Lambda_0$ so that

$$\begin{cases} \exp(\mathfrak{b}^{\text{ver}}_{3,2}) = -\dfrac{1}{2} + \mathfrak{B}^{(1,2)} \\ \\ \exp(\mathfrak{b}^{\text{hor}}_{2,3}) = \dfrac{1}{2} - \mathfrak{B}^{(2,1)}. \end{cases}$$

we can make $\mathfrak{PO}^{\mathfrak{b}}(\mathbf{y})$ admit a critical point. By Theorem 10.7, $L_\varepsilon(t)$ has a non-vanishing (bulk-)deformed Floer cohomology. By the hamiltonian invariance of A_∞-structures, so does $L(t)$ and therefore it is non-displaceable.

In sum, each torus fiber over the line segment in (10.11) is non-displaceable. Furthermore, Proposition 10.13 yields non-displaceability of the Lagrangain 3-sphere. ☐

11. Decompositions of the gradient of potential function

In this section, in order to prove Theorem D, we introduce the split leading term equation of the potential function in (10.5), which is the analogue of the leading term equation in [FOOO3, FOOO4]. We discuss the relation between its solvability and non-triviality of Floer cohomology under a certain bulk-deformation.

11.1. **Outline of Section 11 and Section 12.**

Due to Theorem 10.7, in order to show that the GC torus fiber $L_m(t)$ for each t with $0 \leq t < 1$ is non-displaceable, it suffices to find a bulk-deformation parameter $\mathfrak{b}$ such that $\mathfrak{PO}^{\mathfrak{b}}$ admits a critical point $\mathbf{y}$. Section 11 and Section 12 will be occupied to discuss how to determine them.

Before giving the outline, we explain why this process takes so long by pointing out the differences from the case of toric fibers in a symplectic toric manifold. In the toric case, the (generalized) leading term equation was introduced to detect non-displaceable toric fibers effectively in [FOOO4, Section 11]. Roughly speaking, it consists of the initial terms of the gradient of a (bulk-deformed) potential function with respect to a suitable choice of exponential variables. It is proven therein that there always exists a bulk-deformation parameter $\mathfrak{b}$ so that the complex solution becomes a critical point of the bulk-deformed potential function $\mathfrak{PO}^{\mathfrak{b}}$ as soon as the leading term equation admits a solution whose components are in $\mathbb{C}\backslash\{0\}$. Indeed, the positions where the leading term equation is solvable are characterized by the intersection of certain tropicalizations in [KLS]. The key features for proving the above statements are in order. First, there is a one-to-one correspondence between the *honest* holomorphic discs bounded by a toric fiber of Maslov index 2 and the facets of the moment polytope. Second, the preimage of each facet represents a cycle of degree 2. Therefore, all terms corresponding to the facets can be independently controlled, yielding a parameter $\mathfrak{b}$ and providing a solution of the equation.

In a GC system, however, the inverse image of a facet may not represent a cycle of degree 2 so that the terms of $\mathfrak{PO}$ cannot be independently controlled. [14] Thus, the above statements are not expected to hold anymore. But for a family of Lagrangian tori $L_m(t)$ in $\mathcal{O}_\lambda \simeq \mathcal{F}(n)$ with the monotone symplectic form ω_λ, we will show the existence of a bulk-deformation parameter $\mathfrak{b}$ and a critical point $\mathbf{y}$. In Section 11, we define the *split leading term equation* (See Definition 11.3), which replaces the role of the leading term equation in the toric case. We then demonstrate how to determine a bulk-deformation parameter and extend a solution of the split leading term equation to a critical point of the bulk-deformed potential function. In Section 12, we show that the split leading term equation always admits a solution. In general, finding a solution of a general system of multi-variable equations is not simple at all even with the aid of a computer. Yet, in this case, we are able to find a solution, guided by ladder diagrams regarding as the containers of exponential variables.

Let $B(m)$ be the sub-diagram consisting of $(m \times m)$ lower-left unit boxes in the ladder diagram $\Gamma(n) := \Gamma(1, \cdots, n)$ of $\mathcal{F}(n)$. The diagrams $\Gamma(n)$ and $B(m)$ are often regarded as collections of double indices as follows:

$$\begin{aligned} \Gamma(n) &= \{(i,j) : 2 \leq i + j \leq n\} \\ B(m) &= \{(i,j) : 1 \leq i, j \leq m\}. \end{aligned} \tag{11.1}$$

Recalling (10.5), the potential function of $L_m(t)$ is arranged as several groups with repsect to the energy levels. Observe that the valuation of $\partial(i,j)(\mathbf{y})$ for $(i,j) \in B(m)$ is $(1-t)$ and that of $\partial(i,j)(\mathbf{y})$ for $(i,j) \in \Gamma(n) \backslash B(m)$ is 1.

We will decompose the gradient of the potential function deformed by $\mathfrak{b}$ in (10.8) into two pieces along the boundary of $B(m)$. We are planning to determine a critical point in the following steps.

(1) Find a solution $y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\}$ of the system consisting of the equations $\partial^{\mathfrak{b}}(i,j)(\mathbf{y}) \equiv 0 \mod T^{>1}$ in $\Gamma(n) \backslash B(m) \cup \{(m,m)\}$ and equations relating the variables adjacent to $B(m)$ in Section 12.
(2) Find a solution $y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\}$ of $\partial^{\mathfrak{b}}(i,j)(\mathbf{y}) \equiv 0 \mod T^{>1-t}$ in $B(m)$ in Section 11.3.
(3) Determine a solution $y_{i,j} \in \Lambda_U$ of $\partial^{\mathfrak{b}}(i,j)(\mathbf{y}) = 0$ in $B(m)$ such that $y_{i,j} \equiv y^{\mathbb{C}}_{i,j} \mod T^{>0}$ in Section 11.4.
(4) Determine a solution $y_{i,j} \in \Lambda_U$ of $\partial^{\mathfrak{b}}(i,j)(\mathbf{y}) = 0$ in $\Gamma(n) \backslash B(m)$ such that $y_{i,j} \equiv y^{\mathbb{C}}_{i,j} \mod T^{>0}$ in Section 11.5.

The *split leading term equation* (See Definition 11.3) arises in the first step (1). In this section, assuming that the split leading term equation is solvable, we explain how to complete the remaing steps $(2), (3)$ and (4). The next section focuses on solving the split leading term equation.

Example 11.1. In the co-adjoint orbit $\mathcal{O}_\lambda$ of a sequence $\lambda = \{5, 3, 1, -1, -3, -5\}$ for instance, the potential function of $L_2(t)$ is arranged as follows:

$$\mathfrak{PO}(L_2(t); \mathbf{y}) = \left(\frac{y_{1,2}}{y_{1,1}} + \frac{y_{1,1}}{y_{2,1}} + \frac{y_{1,2}}{y_{2,2}} + \frac{y_{2,2}}{y_{2,1}} \right) T^{1-t} + \left(\frac{y_{1,4}}{y_{1,3}} + \frac{y_{1,3}}{y_{2,3}} + \cdots \right) T^1 + \left(\frac{y_{1,3}}{y_{1,2}} + \frac{y_{2,1}}{y_{3,1}} \right) T^{1+t}.$$

In this example, the valuation of partial derivatives of $\mathfrak{PO}$ jumps along the red line in Figure 32.

Turning on bulk-deformation, according to (10.10), A complex number $y_{i,j} \in \mathbb{C}\backslash\{0\}$ has to satisfy

$$\begin{cases} -c^{\text{ver}}_{2,1} \cdot \frac{y_{1,2}}{y_{1,1}} + c^{\text{hor}}_{1,2} \cdot \frac{y_{1,1}}{y_{2,1}} = 0, \; c^{\text{ver}}_{2,1} \cdot \frac{y_{1,2}}{y_{1,1}} + c^{\text{hor}}_{1,2} \cdot \frac{y_{1,2}}{y_{2,2}} = 0, \\ \\ -c^{\text{hor}}_{1,2} \cdot \frac{y_{1,1}}{y_{2,1}} - c^{\text{ver}}_{2,1} \cdot \frac{y_{2,2}}{y_{2,1}} = 0, \; -c^{\text{hor}}_{1,2} \cdot \frac{y_{1,2}}{y_{2,2}} + c^{\text{ver}}_{2,1} \cdot \frac{y_{2,2}}{y_{2,1}} = 0, \end{cases} \tag{11.2}$$

[14] Because of this feature, Bernstein-Kushnirenko theorem [Be, Ku] cannot be applied in our situation.

which comes from the initial parts of the partial derivatives inside $B(2)$, and

(11.3)
$$\begin{cases} -c^{\text{ver}}_{6,5} \cdot \frac{1}{y_{1,5}} + c^{\text{hor}}_{1,2} \cdot y_{1,5} + c^{\text{ver}}_{5,4} \cdot \frac{y_{1,5}}{y_{1,4}} = 0, \ -c^{\text{ver}}_{5,4} \cdot \frac{1}{y_{2,4}} - c^{\text{hor}}_{1,2} \cdot \frac{y_{1,4}}{y_{2,4}} + c^{\text{hor}}_{2,3} \cdot y_{2,4} + c^{\text{ver}}_{4,3} \cdot \frac{y_{2,4}}{y_{2,3}} = 0, \cdots \\ -c^{\text{ver}}_{5,4} \cdot \frac{y_{1,5}}{y_{1,4}} + c^{\text{hor}}_{1,2} \cdot \frac{y_{1,4}}{y_{2,4}} + c^{\text{ver}}_{4,3} \cdot \frac{y_{1,4}}{y_{1,3}} = 0, \ -c^{\text{ver}}_{4,3} \cdot \frac{y_{2,4}}{y_{2,3}} - c^{\text{hor}}_{1,2} \cdot \frac{y_{1,3}}{y_{2,3}} + c^{\text{hor}}_{2,3} \cdot \frac{y_{2,3}}{y_{3,3}} + c^{\text{ver}}_{3,2} \cdot \frac{y_{2,3}}{y_{2,2}} = 0, \cdots \\ -c^{\text{ver}}_{4,3} \cdot \frac{y_{1,4}}{y_{1,3}} + c^{\text{hor}}_{1,2} \cdot \frac{y_{1,3}}{y_{2,3}} = 0, \ -c^{\text{ver}}_{3,2} \cdot \frac{y_{2,3}}{y_{2,2}} + c^{\text{hor}}_{2,3} \cdot \frac{y_{2,2}}{y_{3,2}} = 0, \ -c^{\text{ver}}_{2,1} \cdot \frac{y_{3,2}}{y_{3,1}} + c^{\text{hor}}_{3,4} \cdot \frac{y_{3,1}}{y_{4,1}} = 0, \end{cases}$$

which comes from the initial parts of the partial derivatives inside $\Gamma(6) \backslash B(2) \cup \{(2,2)\}$, see Figure 32. Solving the first system (11.2) is related to the step (2) and solving the second system (11.3) is related to the step (1).

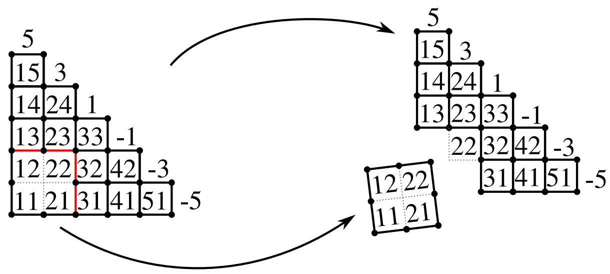


FIGURE 32. Decomposition of the gradient of the potential function in $\mathcal{F}(6)$.

Remark 11.2. More generally, one might consider the fibers over the line segment connecting the center of Δ_λ and the center of another Lagrangian face γ. Depending on γ in the ladder diagram $\Gamma(n)$ in some cases, one might decompose the potential function into several pieces along the simply connected regions that are not unit-sized blocks. The reader is referred to [CKO] in order to consult a geometric implication of the potential function *inside* the decomposed blocks. Also, it is discussed therein when the potential function has a critical point.

The split leading term equation would be formed by the system from the outside of the decomposed blocks, providing the complex part of a critical point with a suitable choice of a complex solution within the decomposed blocks. This intuition motivates us to name it the "split" leading term equation. In this article, the general form will not be discussed as the split leading term equation from $I_m(t)$ in (8.7) is only dealt with.

11.2. **Split leading term equation.**

We now define the split leading term equation arising from the potential function of $L_m(t)$. In this case, it suffices to take a bulk-deformation parameter

$$\mathfrak{b} := \sum_{i \geq k} \mathfrak{b}^{\text{hor}}_{i,i+1} \cdot \mathscr{D}^{\text{hor}}_{i,i+1} + \sum_{j \geq k} \mathfrak{b}^{\text{ver}}_{j+1,j} \cdot \mathscr{D}^{\text{ver}}_{j+1,j} \tag{11.4}$$

instead of the form (10.8). Therefore, we should set

$$\begin{cases} c^{\text{hor}}_{i,i+1} := \exp\left(\mathfrak{b}^{\text{hor}}_{i,i+1} = 0\right) = 1 \quad \text{for } i < k, \\ c^{\text{ver}}_{j+1,j} := \exp\left(\mathfrak{b}^{\text{ver}}_{j+1,j} = 0\right) = 1 \quad \text{for } j < k. \end{cases}$$

In particular, we see that $\partial^{\mathfrak{b}}(i,j)$ in (10.10) coincides with $\partial(i,j)$ in (10.6) for all $1 \leq i, j < k$.

Definition 11.3. Let $k = \lceil n/2 \rceil$, that is $n = 2k-1$ or $2k$. We set

$$\begin{cases} c^{\text{hor}}_{i,i+1} := 1 \quad \text{for } i < k, \quad c^{\text{ver}}_{j+1,j} := 1 \quad \text{for } j < k \\ y_{i,m} := \infty \quad \text{for } i < m, \quad y_{m,j} := 0 \quad \text{for } j < m \\ y_{\bullet,0} := \infty, \ y_{0,\bullet} := 0, \ y_{i,n+1-i} := 1 \quad \text{for } 1 \leq i \leq n \end{cases} \tag{11.5}$$

The *split leading term equation of* $\Gamma(n)$ *associated with* $B(m)$ is the system of the following equations:

$$\begin{cases} \partial^{\mathfrak{b}}_m(i,j)(\mathbf{y}) = 0 \\ \partial_m(l)(\mathbf{y}) = 0 \end{cases} \tag{11.6}$$

for all $(i,j) \in \Gamma(n) \backslash B(m) \cup \{(m,m)\}$ and all l with $1 \leq l < m$. Here,

$$\partial^{\mathfrak{b}}_m(i,j)(\mathbf{y}) := -c^{\text{ver}}_{j+1,j} \cdot \frac{y_{i,j+1}}{y_{i,j}} - c^{\text{hor}}_{i-1,i} \cdot \frac{y_{i-1,j}}{y_{i,j}} + c^{\text{hor}}_{i,i+1} \cdot \frac{y_{i,j}}{y_{i+1,j}} + c^{\text{ver}}_{j,j-1} \cdot \frac{y_{i,j}}{y_{i,j-1}} \tag{11.7}$$

$$\partial_m(l)(\mathbf{y}) := (-1)^{m+1-l} \cdot \frac{y_{l,m+1}}{y_{m,m}} + \frac{y_{m,m}}{y_{m+1,l}}, \tag{11.8}$$

and $c^{\text{hor}}_{i,i+1}$'s and $c^{\text{ver}}_{j+1,j}$'s for $i,j \geq k$ are non-zero complex numbers.

We explain how the split leading term equation of $\Gamma(n)$ associated with $B(m)$ can be written. Cutting the boxes $B(m) \backslash \{(m,m)\}$ off from the diagram $\Gamma(n)$, (11.7) comes from $\partial^{\mathfrak{b}}(i,j)(\mathbf{y})$ for (i,j)'s on the cut diagram as explained in (11.3) for the case $\mathcal{F}(6)$. In addition to them, we impose (11.8) to relate the variables adjacent to $B(m)$. It will be explained in (11.23) and (11.27) why (11.8) appears.

Remark 11.4. Furthermore, we may take $\mathfrak{b}^{\text{hor}}_{k,k+1} = 0$ so that $c^{\text{hor}}_{k,k+1} = 1$ if $n = 2k$. See Remark 11.18 to see why.

Example 11.5. The split leading term equation of $\Gamma(5)$ associated with $B(2)$ consists of

$$\begin{cases} -c^{\text{ver}}_{5,4} \cdot \dfrac{1}{y_{1,4}} + y_{1,4} + c^{\text{ver}}_{4,3} \cdot \dfrac{y_{1,4}}{y_{1,3}} = 0, \; -c^{\text{ver}}_{4,3} \cdot \dfrac{1}{y_{2,3}} - \dfrac{y_{1,3}}{y_{2,3}} + y_{2,3} + \dfrac{y_{2,3}}{y_{2,2}} = 0, \\ \\ -\dfrac{1}{y_{3,2}} - \dfrac{y_{2,2}}{y_{3,2}} + c^{\text{hor}}_{3,4} \cdot y_{3,2} + \dfrac{y_{3,2}}{y_{3,1}} = 0, -\dfrac{1}{y_{4,1}} - c^{\text{hor}}_{3,4} \cdot \dfrac{y_{3,1}}{y_{4,1}} + c^{\text{hor}}_{4,5} \cdot y_{4,1} = 0, \\ \\ -c^{\text{ver}}_{4,3} \cdot \dfrac{y_{1,4}}{y_{1,3}} + \dfrac{y_{1,3}}{y_{2,3}} = 0, \; -\dfrac{y_{3,2}}{y_{3,1}} + c^{\text{hor}}_{3,4} \cdot \dfrac{y_{3,1}}{y_{4,1}} = 0, \; -\dfrac{y_{2,3}}{y_{2,2}} + \dfrac{y_{2,2}}{y_{3,2}} = 0, \; \dfrac{y_{1,3}}{y_{2,2}} + \dfrac{y_{2,2}}{y_{3,1}} = 0. \end{cases}$$

Remark 11.6. We would like to address that the split leading term equation is *not* same as the initial part of the gradient of the potential function. Note that $\partial^{\mathfrak{b}}(2,2) = 0$ is of the form

$$\left(-\frac{y_{1,2}}{y_{2,2}} + \frac{y_{2,2}}{y_{2,1}} \right) T^{1-t} + \left(-\frac{y_{2,3}}{y_{2,2}} + \frac{y_{2,2}}{y_{3,2}} \right) T^1 = 0.$$

Its initial part will be used to obtain a critical point of the potential function within $B(2)$ in Section 11.3. As we have seen in Example 11.5, the split leading term equation captures the terms with the second energy level

$$-\frac{y_{2,3}}{y_{2,2}} + \frac{y_{2,2}}{y_{3,2}} = 0$$

as well.

The main theorem of this section is as follows.

Theorem 11.7. *Let* $\lambda = \{\lambda_i := n - 2i + 1 : i = 1, \cdots, n\}$ *be an* n*-tuple of real numbers for an arbitrary integer* $n \geq 4$. *Consider the co-adjoint orbit* $\mathcal{O}_\lambda$, *a complete flag manifold* $\mathcal{F}(n)$ *equipped with the monotone form* ω_λ. *Fix one Lagrangian Gelfand-Cetlin torus* $L_m(t)$ *over* $I_m(t)$ *for* $0 \leq t < 1$ *in* $\mathcal{O}_\lambda$. *If the split leading term equation* (11.6) *admits a solution* $\{y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} : (i,j) \in \Gamma(n)\backslash B(m) \cup \{(m,m)\}\}$ *each component of which is a non-zero complex number for some nonzero complex numbers* $c^{\text{hor},\mathbb{C}}_{i,i+1}$*'s and* $c^{\text{ver},\mathbb{C}}_{j+1,j}$*'s* $(i,j \geq k)$, *then there exists a bulk-deformation parameter* $\mathfrak{b}$ *(depending on* m *and* t*) of the form* (10.8) *such that*

(1) *The bulk-deformed potential function* $\mathfrak{PO}^{\mathfrak{b}}(\mathbf{y})$ *has a critical point* $\{y_{i,j} \in \Lambda_U : (i,j) \in \Gamma(n)\}$ *satisfying*

$$y^{\mathbb{C}}_{i,j} \equiv y_{i,j} \mod T^{>0} \quad \textit{for } (i,j) \in \Gamma(n)\backslash B(m) \cup \{(m,m)\}.$$

(2) *Also,* $c^{\text{hor},\mathbb{C}}_{i,i+1} \equiv \exp(\mathfrak{b}^{\text{hor}}_{i,i+1})$, $c^{\text{ver},\mathbb{C}}_{j+1,j} \equiv \exp(\mathfrak{b}^{\text{ver}}_{j+1,j}) \mod T^{>0}$.

The existence of a solution for the split leading term equation (11.6) implies that the assumption of the following lemma is satisfied. We will repeatedly employ it in order to extend a solution in $\mathbb{C}\backslash\{0\}$ to that in Λ_U of the gradient of the (bulk-deformed) potential function.

Lemma 11.8. *Let $c_{j+1,j}, c_{i-1,i}, c_{i,i+1}$ and $c_{j,j-1}$ be elements in Λ_U. Suppose that we are given $y_{i-1,j} \in \Lambda_U \cup \{0\}$, $y_{i,j-1} \in \Lambda_U \cup \{\infty\}$ and $y_{i+1,j}, y_{i,j}$ (resp. $y_{i,j}, y_{i,j+1}$) $\in \Lambda_U$. If there is a* non-zero *complex solution $y^{\mathbb{C}}_{i,j+1}$ (resp. $y^{\mathbb{C}}_{i+1,j}$) for*

$$- c^{\mathbb{C}}_{j+1,j} \cdot \frac{y^{\mathbb{C}}_{i,j+1}}{y^{\mathbb{C}}_{i,j}} - c^{\mathbb{C}}_{i-1,i} \cdot \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} + c^{\mathbb{C}}_{i,i+1} \cdot \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} + c^{\mathbb{C}}_{j,j-1} \cdot \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i,j-1}} = 0, \tag{11.9}$$

then there exists a unique element $y_{i,j+1}$ (resp. $y_{i+1,j}$) $\in \Lambda_U$ that solves

$$- c_{j+1,j} \cdot \frac{y_{i,j+1}}{y_{i,j}} - c_{i-1,i} \cdot \frac{y_{i-1,j}}{y_{i,j}} + c_{i,i+1} \cdot \frac{y_{i,j}}{y_{i+1,j}} + c_{j,j-1} \cdot \frac{y_{i,j}}{y_{i,j-1}} = 0. \tag{11.10}$$

Here, for $y \in \Lambda_U$, $y^{\mathbb{C}}$ denotes a unique complex number such that $y^{\mathbb{C}} \equiv y \mod T^{>0}$.

Furthermore, assume in addition that $c_{\bullet,\bullet}$'s are non-zero complex numbers and

$$\mathfrak{v}_T \left(y_{i,j} - y^{\mathbb{C}}_{i,j}\right) > \lambda, \mathfrak{v}_T \left(y_{i-1,j} - y^{\mathbb{C}}_{i-1,j}\right) > \lambda \text{ and } \mathfrak{v}_T \left(y_{i,j-1} - y^{\mathbb{C}}_{i,j-1}\right) > \lambda.$$

Then, $\mathfrak{v}_T(y_{i+1,j}) = \lambda$ if and only if $\mathfrak{v}_T(y_{i,j+1}) = \lambda$.

Proof. The proof immediately follows from the observation that $y_{i+1,j}$ (or $y_{i,j+1}$) in (11.10) can be expressed as a rational function in terms of the other variables. $\square$

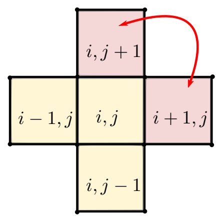


FIGURE 33. Graphical description of Lemma 11.8.

11.3. Symmetric complex solutions within $B(m)$.

In this section, we focus on the system of equations

$$\partial(i,j)(\mathbf{y}) := y_{i,j} \frac{\partial \mathfrak{PO}}{\partial y_{i,j}}(\mathbf{y}) = 0 \quad \text{for all } (i,j) \in B(m) \tag{11.11}$$

within $B(m)$. For an index $(i,j) \in B(m)$, ignoring the variables outside of $B(m)$, the initial part of $\partial(i,j)(\mathbf{y})$ is denoted by $\partial_m(i,j)(\mathbf{y})$. By (10.5), we obtain

$$\partial_m(i,j)(\mathbf{y}) = -\frac{y_{i,j+1}}{y_{i,j}} - \frac{y_{i-1,j}}{y_{i,j}} + \frac{y_{i,j}}{y_{i+1,j}} + \frac{y_{i,j}}{y_{i,j-1}} \tag{11.12}$$

where $y_{0,\bullet}, y_{\bullet,m+1,}, y_{\bullet,0}$ and $y_{m+1,\bullet}$ are respectively set to be $0, 0, \infty$ and ∞. The goal of the section is to find a "symmetric" complex solution for (11.12), see Proposition 11.11.

Lemma 11.9. *Let c be a non-zero complex number. If there is a solution $\{y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} : (i,j) \in B(m)\}$ of the system of equations*

$$\partial_m(i,j)(\mathbf{y}) = 0 \quad \text{for } (i,j) \in B(m), \tag{11.13}$$

then

$$\widetilde{y^{\mathbb{C}}_{i,j}} := c \cdot y^{\mathbb{C}}_{i,j} \quad \text{for } (i,j) \in B(m)$$

also forms a solution of (11.13).

Proof. It follows from $\partial_m(i,j)(\mathbf{y}) = \partial_m(i,j)(c \cdot \mathbf{y})$. $\square$

Lemma 11.10. *There exists a solution $\{y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} : i+j \leq m+1\}$ of the system of equations*

$$\partial_m(i,j)(\mathbf{y}) = 0 \quad \text{for } i+j \leq m \tag{11.14}$$

such that

$$y^{\mathbb{C}}_{i,j} = (y^{\mathbb{C}}_{j,i})^{-1}. \tag{11.15}$$

Proof. We claim that

$$y^{\mathbb{C}}_{i,j} := \begin{cases} 1 & \text{for } i=j \\ \displaystyle\prod_{r=0}^{j-i-1}(2i+2r) & \text{for } i<j \\ \displaystyle\prod_{r=0}^{i-j-1}(2j+2r)^{-1} & \text{for } i>j \end{cases} \tag{11.16}$$

forms a solution for (11.14) satisfying (11.15).

For the case where $i<j$, we see

$$y^{\mathbb{C}}_{i,j} = \prod_{r=0}^{j-i-1}(2i+2r) = \left(\prod_{r=0}^{j-i-1}(2i+2r)^{-1}\right)^{-1} = \left(y^{\mathbb{C}}_{j,i}\right)^{-1}$$

and

$$\begin{aligned}
\partial_m(i,j)(\mathbf{y}) &= -\frac{y^{\mathbb{C}}_{i,j+1}}{y^{\mathbb{C}}_{i,j}} - \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i,j-1}} \\
&= -\frac{\prod_{r=0}^{j-i}(2i+2r)}{\prod_{r=0}^{j-i-1}(2i+2r)} - \frac{\prod_{r=0}^{j-i}(2(i-1)+2r)}{\prod_{r=0}^{j-i-1}(2i+2r)} + \frac{\prod_{r=0}^{j-i-1}(2i+2r)}{\prod_{r=0}^{j-i-2}(2(i+1)+2r)} + \frac{\prod_{r=0}^{j-i-1}(2i+2r)}{\prod_{r=0}^{j-i-2}(2i+2r)} \\
&= -2j - (2i-2) + 2i + (2j-2) = 0.
\end{aligned}$$

The case for $i>j$ follows from $\partial_m(i,j)(\mathbf{y}) = -\partial_m(j,i)(\mathbf{y})$. When $i=j$,

$$\partial_m(i,i)(\mathbf{y}) = -\frac{y^{\mathbb{C}}_{i,i+1}}{y^{\mathbb{C}}_{i,i}} - \frac{y^{\mathbb{C}}_{i-1,i}}{y^{\mathbb{C}}_{i,i}} + \frac{y^{\mathbb{C}}_{i,i}}{y^{\mathbb{C}}_{i+1,i}} + \frac{y^{\mathbb{C}}_{i,i}}{y^{\mathbb{C}}_{i,i-1}} = -y^{\mathbb{C}}_{i,i+1} - y^{\mathbb{C}}_{i-1,i} + \frac{1}{y^{\mathbb{C}}_{i+1,i}} + \frac{1}{y^{\mathbb{C}}_{i,i-1}} = 0.$$

□

We are ready to prove the existence of a symmetric solution in the sense of (11.15).

Proposition 11.11. *There exists a solution $\{y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} : (i,j) \in B(m)\}$ of the system (11.13) of equations such that (11.15) holds for $(i,j) \in B(m)$.*

Proof. We start with a solution $\{y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} : i+j \leq m+1\}$ from Lemma 11.10. For an index (i,j) with $i+j > m+1$, we take

$$y^{\mathbb{C}}_{i,j} := (-1)^{i+j-m-1}\, y^{\mathbb{C}}_{m+1-j,m+1-i}. \tag{11.17}$$

We show that (11.17) forms a solution for (11.13). For $(i,j) \in B(m)$ with $i+j \geq m+2$, it is straightforward to see

$$\partial_m(i,j)(\mathbf{y}) = -\partial_m(m+1-j, m+1-i)(\mathbf{y}) = 0$$

by Lemma 11.10. For an index (i,j) with $i+j = m+1$,

$$\partial_m(i,j)(\mathbf{y}) = -\frac{y^{\mathbb{C}}_{i,j+1}}{y^{\mathbb{C}}_{i,j}} - \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i,j-1}} = \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} - \frac{y^{\mathbb{C}}_{i-1,j}}{y^{\mathbb{C}}_{i,j}} + \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} - \frac{y^{\mathbb{C}}_{i,j}}{y^{\mathbb{C}}_{i+1,j}} = 0.$$

From (11.15) for (i,j) with $i+j \leq m+1$, it follows (11.15) for (i,j) with $i+j > m+1$ because

$$y^{\mathbb{C}}_{i,j} = (-1)^{i+j-m-1}\, y^{\mathbb{C}}_{m+1-j,m+1-i} = (-1)^{i+j-m-1}\, (y^{\mathbb{C}}_{m+1-i,m+1-j})^{-1} = (y^{\mathbb{C}}_{j,i})^{-1}.$$

Thus, we have just found a symmetric solution $\{y^{\mathbb{C}}_{i,j} : (i,j) \in B(m)\}$ such that $y_{i,i} = \pm 1$. □

Corollary 11.12. *For any non-zero complex number c, there exists a solution $\{y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} \,:\, (i,j) \in B(m)\}$ of the system (11.13) of equations such that*

(1) $y^{\mathbb{C}}_{i,j} \cdot y^{\mathbb{C}}_{j,i} = c^2$.
(2) $y^{\mathbb{C}}_{m,m} = c$
(3) $y^{\mathbb{C}}_{i,i} = \pm c$ *for any* $1 \leq i < m$.

Proof. The component $y^{\mathbb{C}}_{m,m}$ of a solution from Proposition 11.11 is either 1 or -1. By multiplying by $\pm c$, because of Lemma 11.9, we have another solution satisfying (1), (2) and (3). □

11.4. **Inside of $B(m)$.**

Assume that the split leading term equation of $\Gamma(n)$ associated with $B(m)$ has a solution for some non-zero complex numbers $c^{\text{hor},\mathbb{C}}_{i,i+1}$'s and $c^{\text{ver},\mathbb{C}}_{j+1,j}$'s for $i, j \geq k$. Let

$$y^{\mathbb{C}}_{m,m},\, y^{\mathbb{C}}_{1,m+1}, \cdots, y^{\mathbb{C}}_{m,m+1} \tag{11.18}$$

where $y^{\mathbb{C}}_{i,j}$ is the (i,j)-th component of a solution of the split leading term equation, which is a non-zero complex number. By Corollary 11.12, we obtain a symmetric complex solution such that c becomes the (m,m)-component $y^{\mathbb{C}}_{m,m}$ of the solution. In order to emphasize that (11.18) is pre-determined, let

$$d_{i,j} := y^{\mathbb{C}}_{i,j}.$$

Setting it as the initial part for a solution and using Lemma 11.8, we extend it to a solution of (11.11) in Λ_U. For a pictorial outline of Section 11.4, see Figure 34.

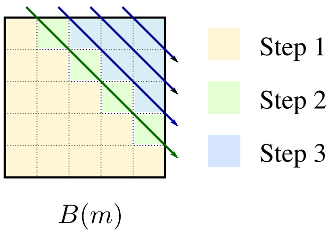


FIGURE 34. Pictorial outline of Section 11.4.

Step 1. $(i,j) \in B(m)$ **with** $i + j \leq m + 1$

We begin by taking $y_{i,j} := y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} \subset \Lambda_U$ where $\{y^{\mathbb{C}}_{i,j} : (i,j) \in B(m)\}$ is a solution satisfying $y^{\mathbb{C}}_{m,m} = d_{m,m}$ from Corollary 11.12 for all indices (i,j)'s with $i + j \leq m + 1$. Then, the equations $\partial(i,j)(\mathbf{y}) = 0$ for $i + j \leq m$ hold because $\partial(i,j)(\mathbf{y}) = \partial_m(i,j)(\mathbf{y})\, T^{1-t}$ by (10.6).

Step 2. $(i,j) \in B(m)$ **with** $i + j = m + 2$

Next, we determine all entries $y_{i,j}$'s of the anti-diagonal given by $i + j = m + 2$ within $B(m)$. For this purpose, we decompose the equation $\partial(1,m)(\mathbf{y}) = 0$ in the system (10.6) into two pieces as follows:

$$\begin{aligned}
\partial(1,m)(\mathbf{y}) &= \left(\frac{y_{1,m}}{y_{2,m}} + \frac{y_{1,m}}{y_{1,m-1}}\right) T^{1-t} + \left(-\frac{y_{1,m+1}}{y_{1,m}}\right) T^{1+(m-1)t} \\
&= \left(\frac{y_{1,m}}{y_{2,m}} + \frac{y_{1,m}}{y_{1,m-1}}\right) T^{1-t} + \left(-\frac{y_{1,m+1}}{y_{1,m}} + a_1 \frac{y_{1,m}}{y_{1,m-1}} - a_1 \frac{y_{1,m}}{y_{1,m-1}}\right) T^{1+(m-1)t} \\
&= \left(\frac{y_{1,m}}{y_{2,m}} + \frac{y_{1,m}}{y_{1,m-1}} - a_1 \frac{y_{1,m}}{y_{1,m-1}} T^{mt}\right) T^{1-t} + \left(-\frac{y_{1,m+1}}{y_{1,m}} + a_1 \frac{y_{1,m}}{y_{1,m-1}}\right) T^{1+(m-1)t}
\end{aligned}$$

where a_1 will be determined shortly.

In order to solve $\partial(1,m)(\mathbf{y}) = 0$, it suffices to find a solution of the following system

$$\begin{cases} \partial(1,m)^{(1)}(\mathbf{y}) := \dfrac{y_{1,m}}{y_{2,m}} + \dfrac{y_{1,m}}{y_{1,m-1}} - a_1 \dfrac{y_{1,m}}{y_{1,m-1}} T^{mt} = 0 \\ \\ \partial(1,m)^{(2)}(\mathbf{y}) := -\dfrac{y_{1,m+1}}{y_{1,m}} + a_1 \dfrac{y_{1,m}}{y_{1,m-1}} = 0 \end{cases} \tag{11.19}$$

By our choice of $y_{i,j}$'s so far, we have

$$\frac{y_{1,m}}{y_{1,m-1}} = \frac{y^{\mathbb{C}}_{1,m}}{y^{\mathbb{C}}_{1,m-1}} \neq 0.$$

Then, $y_{1,m+1} = d_{1,m+1}$ uniquely determines the value $a_1 \in \mathbb{C}\backslash\{0\}$ from $\partial(1,m)^{(2)}(\mathbf{y}) = 0$. Then, there exists a unique $y_{2,m} \in \Lambda_U$ such that $\partial(1,m)^{(1)}(\mathbf{y}) = 0$.

By applying Lemma 11.8 succesively, we can determine the remaining entries of the anti-diagonal containing $y_{2,m}$ within $B(m)$. Namely, from a pre-determined $y_{i+1,m-i+1} \in \Lambda_U$ with $\mathfrak{v}_T(y_{i+1,m-i+1} - y^{\mathbb{C}}_{i+1,m-i+1}) = mt$, we determine a solution $y_{i+2,m-i} \in \Lambda_U$ of $\partial(i+1, m-i)(\mathbf{y}) = 0$ so that

$$\mathfrak{v}_T \left(y_{i+2,m-i} - y^{\mathbb{C}}_{i+2,m-i} \right) = mt. \tag{11.20}$$

Then, all anti-diagonal entries $y_{i+2,m-i}$'s inside $B(m)$ are chosen to obey

$$\begin{cases} \partial(1,m)^{(1)}(\mathbf{y}) = 0 \\ \partial(i+1, m-i)(\mathbf{y}) = 0 \quad \text{for } 1 \leq i \leq m-2. \end{cases}$$

Plugging the previously determined $y_{i,j}$'s into $\partial(m,1)(\mathbf{y}) = 0$, we convert $\partial(m,1)(\mathbf{y}) = 0$ into $\partial(m,1)^{(2)}(\mathbf{y}) = 0$ of the form (11.22). By solving $\partial(m,1)^{(2)}(\mathbf{y}) = 0$, we obtain $y_{m+1,1}$.

We would like to find a sufficient condition that $y_{m+1,1}$ exists in Λ_U such that $y^{\mathbb{C}}_{m+1,1} \equiv y_{m+1,1} \mod T^{>0}$. Because of (11.20), we put

$$y_{i+2,m-i} \equiv y^{\mathbb{C}}_{i+2,m-i} + A_i \cdot T^{mt} \mod T^{>mt} \tag{11.21}$$

where $A_i \in \mathbb{C}\backslash\{0\}$. A straightforward calculation via consideration of $\partial(i+1, m-i)(\mathbf{y}) = 0$ gives us the following lemma.

Lemma 11.13. *A recurrence relation for A_i's is*

$$\begin{cases} A_0 &= -a_1 \cdot y^{\mathbb{C}}_{1,m-1} \\ A_i &= -\frac{(y^{\mathbb{C}}_{i+2,m-i})^2}{(y^{\mathbb{C}}_{i+1,m-i})^2} A_{i-1}. \end{cases}$$

From (11.21) and Lemma 11.13, it follows

$$\begin{aligned} \partial(m,1)(\mathbf{y}) T^{t-1} &= -\frac{y_{m-1,1}}{y_{m,1}} - \frac{y_{m,2}}{y_{m,1}} + \frac{y_{m,1}}{y_{m+1,1}} T^{mt} \\ &\equiv \left(-\frac{y^{\mathbb{C}}_{m-1,1}}{y^{\mathbb{C}}_{m,1}} - \frac{y^{\mathbb{C}}_{m,2}}{y^{\mathbb{C}}_{m,1}} \right) + \left(-\frac{A_{m-2}}{y^{\mathbb{C}}_{m,1}} + \frac{y^{\mathbb{C}}_{m,1}}{y_{m+1,1}} \right) T^{mt} && \mod T^{>mt} \\ &\equiv \left((-1)^m a_1 \frac{y^{\mathbb{C}}_{1,m-1}}{y^{\mathbb{C}}_{m,1}} \left(\prod_{i=1}^{m-2} \frac{(y^{\mathbb{C}}_{i+2,m-i})^2}{(y^{\mathbb{C}}_{i+1,m-i})^2} \right) + \frac{y^{\mathbb{C}}_{m,1}}{y_{m+1,1}} \right) T^{mt} && \mod T^{>mt}. \end{aligned}$$

By Corollary 11.12, we have

- $y^{\mathbb{C}}_{i,j} \cdot y^{\mathbb{C}}_{j,i} = (d_{m,m})^2$, $(y^{\mathbb{C}}_{i,i})^2 = (d_{m,m})^2$
- $y^{\mathbb{C}}_{1,m-1} + y^{\mathbb{C}}_{2,m} = 0$
- $y^{\mathbb{C}}_{1,m-1} \cdot y^{\mathbb{C}}_{m-1,1} = (d_{m,m})^2$,

Using them, we simplify the above expression as follows.

$$
\begin{aligned}
\partial(m,1)(\mathbf{y})T^{t-1} &\equiv \left((-1)^m \, a_1 \frac{y^{\mathbb{C}}_{1,m-1}}{y^{\mathbb{C}}_{m,1}} \frac{(d_{m,m})^2}{(y^{\mathbb{C}}_{2,m})^2} + \frac{y^{\mathbb{C}}_{m,1}}{y_{m+1,1}} \right) T^{mt} && \mod T^{>mt} \\
&\equiv \left((-1)^m \, a_1 \frac{1}{y^{\mathbb{C}}_{m,1}} \frac{(d_{m,m})^2}{y^{\mathbb{C}}_{1,m-1}} + \frac{y^{\mathbb{C}}_{m,1}}{y_{m+1,1}} \right) T^{mt} && \mod T^{>mt} \\
&\equiv \left((-1)^m a_1 \frac{y^{\mathbb{C}}_{m-1,1}}{y^{\mathbb{C}}_{m,1}} + \frac{y^{\mathbb{C}}_{m,1}}{y_{m+1,1}} \right) T^{mt} && \mod T^{>mt}.
\end{aligned}
$$

Thus, $\partial(m,1)(\mathbf{y}) = 0$ yields

$$
\partial(m,1)^{(2)}(\mathbf{y}) := (-1)^m a_1 \frac{y_{m-1,1}}{y_{m,1}} + \frac{y_{m,1}}{y_{m+1,1}} + \mathfrak{P}(m,1)^{(2)}(a_1) = 0 \tag{11.22}
$$

for some constant $\mathfrak{P}(m,1)^{(2)}(a_1) \in \Lambda_+$ (depending on a_1). Then, $y_{m+1,1} \in \Lambda_U$ can be determined so that $\partial(m,1)^{(2)}(\mathbf{y}) = 0$ holds. Because of Corollary 11.12 and (11.19), we observe

$$
y_{m+1,1} \equiv (-1)^{m+1} \frac{1}{a_1} \frac{(y_{m,1})^2}{y_{m-1,1}} \equiv (-1)^{m+1} \frac{(d_{m,m})^2}{a_1} \frac{y_{1,m-1}}{(y_{1,m})^2} \equiv (-1)^{m+1} \frac{(d_{m,m})^2}{y_{1,m+1}} \mod T^{>0}, \tag{11.23}
$$

which explains why the equation $\partial_m(1)(\mathbf{y}) = 0$ in the system (11.6) appears. In other words, (11.8) provides a sufficient condition to solve $y_{m+1,1}$ over Λ_U in (11.22) such that $y^{\mathbb{C}}_{m+1,1} \equiv y_{m+1,1} \mod T^{>0}$.

Step 3. $(i,j) \in B(m)$ **with** $m+2 < i+j \leq 2m$

Now, we determine all elements $y_{i,j}$'s for $(i,j) \in B(m)$ satisfying $m+2 < i+j < 2m$. For an index j with $2 < j < m$, we decompose $\partial(j,m)(\mathbf{y})$ as follows:

$$
\begin{aligned}
\partial(j,m)(\mathbf{y}) &= \left(-\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}} \right) T^{1-t} + \left(-\frac{y_{j,m+1}}{y_{j,m}} \right) T^{1+(m-j)t} \\
&= \left(\left(-\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}} \right) + \left(-a_j \frac{y_{j,m}}{y_{j+1,m}} - a_j \frac{y_{j,m}}{y_{j,m-1}} \right) T^{(m-j+1)t} \right) T^{1-t} \\
&\quad + \left(-\frac{y_{j,m+1}}{y_{j,m}} + a_j \frac{y_{j,m}}{y_{j+1,m}} + a_j \frac{y_{j,m}}{y_{j,m-1}} \right) T^{1+(m-j)t}
\end{aligned}
$$

In order to have a solution of $\partial(j,m)(\mathbf{y}) = 0$, we decompose it into the following equations.

$$
\begin{cases}
\partial(j,m)^{(1)}(\mathbf{y}) := \left(-\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}} \right) - a_j \left(\frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}} \right) T^{(m-j+1)t} = 0 \\[2ex]
\partial(j,m)^{(2)}(\mathbf{y}) := -\frac{y_{j,m+1}}{y_{j,m}} + a_j \left(\frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}} \right) = 0.
\end{cases}
$$

Due to the following lemma, it is enough to find a solution of the following system to solve $\partial(j,m)(\mathbf{y}) = 0$.

Lemma 11.14. *A solution of the system*

$$
\begin{cases}
\partial(j,m)^{(1)}(\mathbf{y}) = 0 \\
\partial(j,m)^{(2)}(\mathbf{y}) - a_j \cdot \partial(j,m)(\mathbf{y})\, T^{t-1} = -\frac{y_{j,m+1}}{y_{j,m}} + a_j \left(\frac{y_{j-1,m}}{y_{j,m}} + \frac{y_{j,m+1}}{y_{j,m}} T^{(m-j+1)t} \right) = 0
\end{cases} \tag{11.24}
$$

is also a solution of $\partial(j,m)(\mathbf{y}) = 0$.

Proof. Note that

$$
\partial(j,m)(\mathbf{y}) = \partial(j,m)^{(1)}(\mathbf{y})\, T^{1-t} + \partial(j,m)^{(2)}(\mathbf{y})\, T^{1+(m-j)t}.
$$

A solution of (11.24) satisfies

$$
\partial(j,m)(\mathbf{y}) = \partial(j,m)^{(2)}(\mathbf{y})\, T^{1+(m-j)t} = a_j \cdot \partial(j,m)(\mathbf{y})\, T^{(m-j+1)t},
$$

which gives rise to $\partial(j,m)(\mathbf{y}) = 0$. ∎

Suppose that we are given a solution

$$\{y_{r,s} \in \Lambda_U \,:(r,s) \in B(m),\, r+s \leq m+j\}$$

of $\partial(r,s)(\mathbf{y}) = 0$ for all $(r,s) \in B(m)$ and $r+s < m+j$ such that

$$\mathfrak{v}_T(y_{r,s} - y^{\mathbb{C}}_{r,s}) \geq (m-j+2)t$$

as the induction hypothesis. Since each $y^{\mathbb{C}}_{r,s}$ is non-zero by our choice, we then obtain $y^{\mathbb{C}}_{j-1,m}/y^{\mathbb{C}}_{j,m} \neq 0$ so that $\partial(j,m)^{(2)}(\mathbf{y}) - a_j \cdot \partial(j,m)(\mathbf{y})T^{t-1} = 0$ determines a unique value $a_j \in \Lambda_U$ from $y_{j,m+1} = d_{j,m+1}$. Then, the equation $\partial(j,m)^{(1)}(\mathbf{y}) = 0$ yields

$$\frac{y^{\mathbb{C}}_{j,m}}{y_{j+1,m}}\left(1 - a_j\, T^{(m-j+1)t}\right) \equiv \left(\frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{j,m}} - \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j,m-1}}\right) + a_j\,\frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{j,m}}\,T^{(m-j+1)t} \quad \mod T^{>(m-j+1)t}.$$

Keeping in mind that $y^{\mathbb{C}}_{r,s}$'s from Corollary 11.12 satisfy

$$\partial_m(j,m)(\mathbf{y}) = -\frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{j,m}} + \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j+1,m}} + \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j,m-1}} = 0,$$

we obtain

$$\frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j+1,m}} \neq 0 \;\text{ and }\; \frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{j,m}} - \frac{y^{\mathbb{C}}_{j,m}}{y^{\mathbb{C}}_{j,m-1}} \neq 0$$

since $y^{\mathbb{C}}_{j,m}$ is non-zero. Hence $y_{j+1,m} \in \Lambda_U$ with $\mathfrak{v}_T(y_{j+1,m} - y^{\mathbb{C}}_{j+1,m}) = (m-j+1)t$.

Suppose that

$$\{y_{r,s} \in \Lambda_U : (r,s) \in B(m),\, r+s \leq m+j\} \cup \{y_{r,s} \in \Lambda_U : (r,s) \in B(m),\, r+s = m+j+1,\, r > m-i\}$$

are given and $\mathfrak{v}_T(y_{i+j,m-i+1} - y^{\mathbb{C}}_{i+j,m-i+1}) = (m-j+1)t$. By Lemma 11.8, the equation $\partial(i+j, m-i) = 0$ determines $y_{i+j+1,m-i} \in \Lambda_U$ such that

$$\mathfrak{v}_T(y_{i+j+1,m-i} - y^{\mathbb{C}}_{i+j+1,m-i}) = (m-j+1)t. \tag{11.25}$$

In order to find $y_{m+1,j}$, we convert $\partial(m,j)(\mathbf{y}) = 0$ into $\partial(m,j)^{(2)}(\mathbf{y}) = 0$ by inserting the previously determined $y_{i,j}$'s. For $0 \leq i \leq m-j-1$, due to (11.25), we may set

$$y_{i+j+1,m-i} \equiv y^{\mathbb{C}}_{i+j+1,m-i} + A_i \cdot T^{(m-j+1)t} \quad \mod T^{>(m-j+1)t}$$

where $A_i \in \mathbb{C}\backslash\{0\}$. As in Lemma 11.13, we derive the following lemma.

Lemma 11.15. *A recurrence relation for A_i's is*

$$\begin{cases} A_0 &= -a_j \cdot \frac{(y^{\mathbb{C}}_{j+1,m})^2}{(y^{\mathbb{C}}_{j,m})^2} \cdot y^{\mathbb{C}}_{j-1,m} \\ A_i &= -\frac{\left(y^{\mathbb{C}}_{i+j+1,m-i}\right)^2}{\left(y^{\mathbb{C}}_{i+j,m-i}\right)^2}\, A_{i-1}. \end{cases}$$

By Lemma 11.15 and Corollary 11.12,

$$\begin{aligned} \partial(m,j)(\mathbf{y})T^{t-1} &= \left(-\frac{y_{m,j+1}}{y_{m,j}} - \frac{y_{m-1,j}}{y_{m,j}} + \frac{y_{m,j}}{y_{m,j-1}}\right) + \frac{y_{m,j}}{y_{m+1,j}} T^{(m-j+1)t} \\ &\equiv \left(-\frac{y^{\mathbb{C}}_{m,j+1}}{y^{\mathbb{C}}_{m,j}} - \frac{y^{\mathbb{C}}_{m-1,j}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y^{\mathbb{C}}_{m,j-1}}\right) + \left(-\frac{A_{m-j-1}}{y^{\mathbb{C}}_{m,j}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}}\right) T^{(m-j+1)t} && \mod T^{>(m-j+1)t} \\ &\equiv \left((-1)^{m-j+1} a_j\, \frac{y^{\mathbb{C}}_{j-1,m}}{y^{\mathbb{C}}_{m,j}} \left(\prod_{i=0}^{m-j-1} \frac{(y^{\mathbb{C}}_{i+j+1,m-i})^2}{(y^{\mathbb{C}}_{i+j,m-i})^2}\right) + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}}\right) T^{(m-j+1)t} && \mod T^{>(m-j+1)t} \\ &\equiv \left((-1)^{m-j+1} a_j\, \frac{1}{y^{\mathbb{C}}_{m,j}} \frac{(d_{m,m})^2}{y^{\mathbb{C}}_{m,j-1}} \frac{(y^{\mathbb{C}}_{m,j})^2}{(d_{m,m})^2} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}}\right) T^{(m-j+1)t} && \mod T^{>(m-j+1)t} \\ &\equiv \left((-1)^{m-j+1} a_j\, \frac{y^{\mathbb{C}}_{m,j}}{y^{\mathbb{C}}_{m,j-1}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}}\right) T^{(m-j+1)t} && \mod T^{>(m-j+1)t} \end{aligned}$$

which yields

$$\partial(m,j)^{(2)}(\mathbf{y}) = \left((-1)^{m-j+1} a_j \frac{y^{\mathbb{C}}_{m,j}}{y^{\mathbb{C}}_{m,j-1}} + \frac{y^{\mathbb{C}}_{m,j}}{y_{m+1,j}} \right) + \mathfrak{P}(m,j)^{(2)}(a_1, \cdots, a_j) = 0. \tag{11.26}$$

for some $\mathfrak{P}(m,j)^{(2)}(a_1, \cdots, a_j) \in \Lambda_+$. We then have

$$y_{m+1,j} \equiv (-1)^{m-j} \frac{y_{m,j-1}}{a_j} \equiv (-1)^{m-j} \frac{(d_{m,m})^2}{a_j \cdot y_{j-1,m}} \equiv (-1)^{m-j} \frac{(d_{m,m})^2}{y_{j,m+1}} \mod T^{>0}. \tag{11.27}$$

which explains why the equation $\partial_m(j)(\mathbf{y}) = 0$ in the system (11.6) appears. In other words, (11.8) provides a sufficient condition to solve $y_{m+1,j}$ over Λ_U in (11.26) such that $y_{m+1,j} = y^{\mathbb{C}}_{m+1,j}$.

Finally, we convert $\partial(m,m)(\mathbf{y}) = 0$ into $\partial(m,m)^{(2)}(\mathbf{y}) = 0$ as follows. Inserting $y_{m-1,m}, y_{m,m}, y_{m,m-1}$ and $y_{m,m+1} = d_{m,m+1}$ into $\partial(m,m)(\mathbf{y}) = 0$, we derive

$$\partial(m,m)^{(2)}(\mathbf{y}) = \left(-\frac{y_{m,m+1}}{y^{\mathbb{C}}_{m,m}} + \frac{y^{\mathbb{C}}_{m,m}}{y_{m+1,m}} \right) + \mathfrak{P}(m,m)^{(2)}(\mathbf{a}) = 0 \tag{11.28}$$

for some $\mathfrak{P}(m,m)^{(2)}(\mathbf{a}) \in \Lambda_+$. We obtain

$$y_{m+1,m} \equiv \frac{(y_{m,m})^2}{y_{m,m+1}} \mod T^{>0}$$

and determine $y_{m+1,m}$ in Λ_U.

In summary, the above discussion is summarized as follows.

Proposition 11.16. *For any tuple $(d_{m,m}, d_{1,m+1}, \cdots, d_{m,m+1})$ of non-zero complex numbers, there exist*

- $y_{i,j} \in \Lambda_U$ *for* $(i,j) \in B(m)$,
- $y_{i,m+1} \in \Lambda_U$ *for* $1 \leq i \leq m$,
- $y_{m+1,j} \in \Lambda_U$ *for* $1 \leq j \leq m$

satisfying

(1) $y_{m,m} \equiv d_{m,m} \mod T^{>0}$,
(2) $y_{i,m+1} = d_{i,m+1}$ *for each* $i = 1, \cdots, m$,
(3) $\partial(i,j)(\mathbf{y}) = 0$ *for* $(i,j) \in B(m)$,
(4) $(-1)^{m+1-l} \dfrac{y_{l,m+1}}{y_{m,m}} + \dfrac{y_{m,m}}{y_{m+1,l}} \equiv 0 \mod T^{>0}$.

11.5. Outside of $B(m)$.

Suppose that we are given a complex solution

$$\{y^{\mathbb{C}}_{i,j} \in \mathbb{C}\backslash\{0\} : (i,j) \in \Gamma(n)\backslash B(m) \cup \{(m,m)\}\}$$

for (11.6) together with non-zero complex numbers $c^{\mathrm{ver},\mathbb{C}}_{i+1,i}$'s and $c^{\mathrm{hor},\mathbb{C}}_{j,j+1}$'s, which is the hypothesis of Theorem 11.7. In this section, we discuss how to determine a bulk-deformation parameter $\mathfrak{b}$ in (11.4) from $c^{\mathrm{ver},\mathbb{C}}_{i+1,i}$'s and $c^{\mathrm{hor},\mathbb{C}}_{j,j+1}$'s and how to extend it to a solution in Λ_U from $y^{\mathbb{C}}_{i,j}$'s for $\partial^{\mathfrak{b}}(i,j)(\mathbf{y}) = 0$. Assume that $m < k = \lceil n/2 \rceil$. For the case $m = k$, see Remark 11.18. Here is a pictorial outline of the section.

Step 1. $(i,j) \in B(m) \cup \mathcal{I}_{\mathbf{seed}}$

Let

$$\mathcal{I}_{\mathrm{seed}} := \left\{ (m,m), (1,m+1), \cdots, (m,m+1), (m+1,m+1), (m+1,m+2), \cdots, \left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil \right) \right\} \tag{11.29}$$

and

$$\mathcal{I}_{\mathrm{initial}} := \mathcal{I}_{\mathrm{seed}} \backslash \{(m,m)\}. \tag{11.30}$$

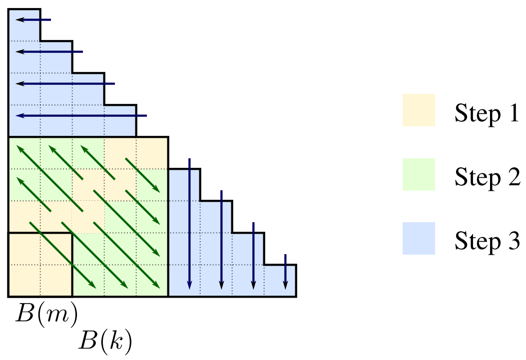


FIGURE 35. Pictorial outline of Section 11.5.

Remark 11.17. We will define a *seed* in Definition 12.1 to generate a candidate for a solution for the split leading term equation. The set (11.29) is the collection of indices where the corresponding variables will be generically chosen as the initial step.

We start to take $y_{i,j} := y^{\mathbb{C}}_{i,j}$ for $(i,j) \in \mathcal{I}_{\text{initial}}$. Then, we fix a complex solution in $B(m)$ from Corollary 11.12 such that $c = y^{\mathbb{C}}_{m,m}$.

Step 2. $(i,j) \in B(k) \backslash B(m)$

By following Section 11.4, the chosen element $y_{1,m+1} \in \mathbb{C}\backslash\{0\}$ determines $y_{i+1,m-i+1}$'s in Λ_U for $1 \leq i \leq m-1$. Moreover, we have $y_{m+1,1} \in \Lambda_U$. Again by Section 11.4, we also find $y_{i,j}$'s in Λ_U for (i,j) with $i \leq m+1$, $j \leq m+1$ and $i+j = m+3$ satisfying (11.27). If $m+1 = k$, then proceed to the next anti-diagonal. If $m+1 < k$, then it remains to determine $y_{1,m+2}$ and $y_{m+2,1}$ in this anti-diagonal within $B(k)\backslash B(m)$. Since the hypothesis of Lemma 11.8 is fulfilled at the equations $\partial(1, m+1)(\mathbf{y}) = 0$ and $\partial(m+1, 1)(\mathbf{y}) = 0$ by our standing assumption, they are determined in Λ_U. Proceeding inductively, we fill up all $y_{i,j}$'s for $(i,j) \in B(k)\backslash B(m)$ obeying

(1) $y_{i,j} \in \Lambda_U$ such that $y_{i,j} \equiv y^{\mathbb{C}}_{i,j} \mod T^{>0}$,
(2) $\partial(i,j)(\mathbf{y}) = 0$ for $(i,j) \in B(k-1)$.

Step 3. $(i,j) \in \Gamma(n)\backslash B(k)$ **and** $\mathfrak{b}$

We determine $y_{i,j}$'s for $(i,j) \in \Gamma(n)\backslash B(k)$ and $\mathfrak{b}^{\text{hor}}_{i,i+1}$'s and $\mathfrak{b}^{\text{ver}}_{j+1,j}$'s in (11.4) over Λ_U. Notice that

$$\partial(i,j)(\mathbf{y}) = \partial^{\mathfrak{b}}(i,j)(\mathbf{y}) \text{ for } (i,j) \in B(k-1)$$

because of our choice of $\mathfrak{b}$, and thus we may keep $\{y_{i,j} \in \Lambda_U : (i,j) \in B(k)\}$ as a solution of $\partial^{\mathfrak{b}}(i,j)(\mathbf{y}) = 0$.

From now on, we focus only on the case where $n = 2k-1$ because the case $n = 2k$ can be similarly dealt with. In this case, there are $(k^2 - 1)$ variables in $B(k)$. As all variables $y_{k-1,k}, y_{k-2,k}$ and $y_{k-1,k-1}$ in

$$\partial(k-1,k)(\mathbf{y}) = -\frac{1}{y_{k-1,k}} - \frac{y_{k-2,k}}{y_{k-1,k}} + y_{k-1,k} + \frac{y_{k-1,k}}{y_{k-1,k-1}}$$

have been already determined by previous inductive steps, we do *not* have any extra variables to make $\partial(k-1,k)(\mathbf{y}) = 0$ hold. It is time to adjust the equation $\partial(k-1,k)(\mathbf{y}) = 0$ by selecting $\mathfrak{b}^{\text{ver}}_{k+1,k}$ suitably. By (10.10), we have

$$\partial^{\mathfrak{b}}(k-1,k)(\mathbf{y}) := -\exp(\mathfrak{b}^{\text{ver}}_{k+1,k}) \cdot \frac{1}{y_{k-1,k}} - \frac{y_{k-2,k}}{y_{k-1,k}} + y_{k-1,k} + \frac{y_{k-1,k}}{y_{k-1,k-1}}.$$

From the following equation

$$-c^{\text{ver},\mathbb{C}}_{k+1,k} \cdot \frac{1}{y^{\mathbb{C}}_{k-1,k}} - \frac{y^{\mathbb{C}}_{k-2,k}}{y^{\mathbb{C}}_{k-1,k}} + y^{\mathbb{C}}_{k-1,k} + \frac{y^{\mathbb{C}}_{k-1,k}}{y^{\mathbb{C}}_{k-1,k-1}} = 0,$$

one equation in the split leading term equation, it follows that

$$-\frac{y^{\mathbb{C}}_{k-2,k}}{y^{\mathbb{C}}_{k-1,k}} + y^{\mathbb{C}}_{k-1,k} + \frac{y^{\mathbb{C}}_{k-1,k}}{y^{\mathbb{C}}_{k-1,k-1}} \neq 0,$$

otherwise $-c^{\text{ver},\mathbb{C}}_{k+1,k} = 0$. Since

$$-\frac{y_{k-2,k}}{y_{k-1,k}} + y_{k-1,k} + \frac{y_{k-1,k}}{y_{k-1,k-1}} \equiv -\frac{y^{\mathbb{C}}_{k-2,k}}{y^{\mathbb{C}}_{k-1,k}} + y^{\mathbb{C}}_{k-1,k} + \frac{y^{\mathbb{C}}_{k-1,k}}{y^{\mathbb{C}}_{k-1,k-1}} \neq 0 \mod T^{>0},$$

there exists a unique bulk-deformation parameter $\mathfrak{b}^{\text{ver}}_{k+1,k} \in \Lambda_0$ such that

(1) $\partial^{\mathfrak{b}}(k-1,k)(\mathbf{y}) = 0$
(2) $\exp(\mathfrak{b}^{\text{ver}}_{k+1,k}) \equiv c^{\text{ver},\mathbb{C}}_{k+1,k} \mod T^{>0}$

Notice that $\mathfrak{b}^{\text{ver}}_{k+1,k}$ does not only deforms $\frac{1}{y_{k-1,k}}$, but also deforms $\frac{y_{j,k+1}}{y_{j,k}}$ into $\exp(\mathfrak{b}^{\text{ver}}_{k+1,k}) \cdot \frac{y_{j,k+1}}{y_{j,k}}$ for all j with $1 \leq j < k-1$ as in Corollary 10.11. Therefore, we need to solve the deformed equation

$$\partial^{\mathfrak{b}}(j,k)(\mathbf{y}) := -\exp(\mathfrak{b}^{\text{ver}}_{k+1,k}) \cdot \frac{y_{j,k+1}}{y_{j,k}} - \frac{y_{j-1,k}}{y_{j,k}} + \frac{y_{j,k}}{y_{j+1,k}} + \frac{y_{j,k}}{y_{j,k-1}} = 0$$

in order to decide $y_{\bullet,k+1} \in \Lambda_U$.

For the induction hypothesis, assume that $y_{r,s}$'s for $s \leq j$ and $\mathfrak{b}^{\text{ver}}_{s,s-1}$'s for $s \leq j$ are determined. We pick a bulk-deformation parameter $\mathfrak{b}^{\text{ver}}_{j+1,j} \in \Lambda_0$ so that

(1) $\partial^{\mathfrak{b}}(n-j,j)(\mathbf{y}) = 0$
(2) $\exp(\mathfrak{b}^{\text{ver}}_{j+1,j}) \equiv c^{\text{ver},\mathbb{C}}_{j+1,j} \mod T^{>0}$.

After fixing $\mathfrak{b}^{\text{ver}}_{j+1,j}$, we determine $y_{\bullet,j+1} \in \Lambda_U$. Hence, all entries above $B(k)$ together with $\mathfrak{b}^{\text{ver}}_{j+1,j}$'s are determined in this way. Symmetrically, we can choose $\mathfrak{b}^{\text{hor}}_{i,i+1}$'s and fill up the other part of $\Gamma(n)\backslash B(k)$. Hence, Theorem 11.7 is now verified.

Remark 11.18. We outline the proof of Theorem 11.7 when $n = 2k$ and $m = k$. In this case, taking $c = 1$ for $y^{\mathbb{C}}_{m,m}$, Corollary 11.12 will give us the initial parts $y^{\mathbb{C}}_{i,j}$'s of $y_{i,j}$'s for $(i,j) \in B(m)$. We then follow Section 11.4 to extend to $y_{i,j}$'s in Λ_U. If one uses both $\mathfrak{b}^{\text{ver}}_{m+1,m}$ and $\mathfrak{b}^{\text{hor}}_{m,m+1}$ to deform $\partial(m,m) = 0$, then we have two extra variables $c^{\text{ver}}_{m+1,m}$ and $c^{\text{hor}}_{m,m+1}$ in $\partial^{\mathfrak{b}}(m,m) = 0$. For our convenience, recall that we have chosen $\mathfrak{b}^{\text{hor}}_{m,m+1} = 0$ in Remark 11.4. Now, we need to take $\mathfrak{b}^{\text{ver}}_{m+1,m}$ so that $\partial^{\mathfrak{b}}(m,m) = 0$. Since $y^{\mathbb{C}}_{m,m} = 1$, we get $\mathfrak{b}^{\text{ver}}_{m+1,m} \in \Lambda_+$, which yields that $1 = c^{\text{ver},\mathbb{C}}_{m+1,m} = \exp(\mathfrak{b}^{\text{ver}}_{m+1,m}) \mod T^{>0}$. After fixing $\mathfrak{b}^{\text{ver}}_{m+1,m}$, we solve $y_{\bullet,m+1}$ by solving $\partial^{\mathfrak{b}}(\bullet,m)^{(2)} = 0$ where

$$\partial^{\mathfrak{b}}(1,m)^{(2)}(\mathbf{y}) := -\exp(\mathfrak{b}^{\text{ver}}_{m+1,m})\,\frac{y_{1,m+1}}{y_{1,m}} + a_1 \frac{y_{1,m}}{y_{1,m-1}} = 0$$

$$\partial^{\mathfrak{b}}(j,m)^{(2)}(\mathbf{y}) := -\exp(\mathfrak{b}^{\text{ver}}_{m+1,m})\,\frac{y_{j,m+1}}{y_{j,m}} + a_j \left(\frac{y_{j,m}}{y_{j+1,m}} + \frac{y_{j,m}}{y_{j,m-1}}\right) = 0 \quad \text{for } j > 2.$$

The remaining steps are similar to the case for $m < k$ in Section 11.5.

12. Solvability of split leading term equation

This section aims to verify the assumption for Theorem 11.7 when the split leading term equation (11.6) comes from the line segment $I_m \subset \Delta_\lambda$ in (8.7). To find its solution, we introduce a *seed* generating a candidate for a solution and prove that there exists a "good" choice of seeds such that the candidate is indeed a solution.

12.1. Seeds.

We begin by the definition of a seed. Recall the notations $\Gamma(n)$ and $B(m)$ in (11.1).

Definition 12.1. A *seed* of $\Gamma(n)$ *associated with* $B(m)$ consists of the two data $(\mathbf{d}, \mathcal{I})$.

- An $(n-m)$-tuple $\mathbf{d}$ of elements in Λ_U

$$\mathbf{d} = (d_1, \cdots, d_{n-m})$$

- An $(n-m)$-tuple $\mathcal{I}$ of double indices

$$\mathcal{I} = \{(m,m), (i_1,j_1), \cdots, (i_{n-m-1}, j_{n-m-1})\} \subset \{(m,m)\} \cup (\Gamma(n)\backslash B(m))$$

satisfying

(1) the first index is (m,m)

(2) the remaining indices are contained in $\Gamma(n)\backslash B(m)$ such that any two indices must not come from the same anti-diagonal of $\Gamma(n)\backslash B(m)$.

We are particularly interested in seeds $(\mathbf{d}, \mathcal{I})$ of the form

- $\mathbf{d}$ is a tuple of *non-zero real* numbers.
- $\mathcal{I} := \mathcal{I}_{\text{seed}}$ in (11.29).

Let $\mathbf{y}_{\mathcal{I}}$ denote the components of $\mathbf{y}$ associated with the set $\mathcal{I}$ of indices. Namely,

$$\mathbf{y}_{\mathcal{I}} := \left(y_{m,m}, y_{i_1,j_1}, \cdots, y_{i_{n-m-1},j_{n-m-1}}\right).$$

Then, as the initial step, we take

$$\mathbf{y}_{\mathcal{I}} := \mathbf{d}.$$

So, the double indices designate the places in which the components of $\mathbf{d}$ are plugged. Since $\mathcal{I}$ is always taken to be $\mathcal{I}_{\text{seed}}$, $\mathcal{I}$ will be often omitted from now on. We instead set $d_{i,j}$ to denote the component of $\mathbf{d}$ corresponding to (i,j). For instance, we have $d_1 = d_{m,m}$.

Following the procedure in Section 11.5, see Figure 35, we generate the other $y_{i,j}$'s such that $\mathbf{y}$ satisfies the split leading term equation with a suitable choice of complex numbers

$$\mathbf{c} := \left(c^{\text{hor}}_{k,k+1}, \cdots, c^{\text{hor}}_{n-1,n}, c^{\text{ver}}_{k+1,k}, \cdots, c^{\text{ver}}_{n,n-1}\right).$$

Namely, by isolating one undetermined variable and plugging the determined variables in one equation of the split leading term equation, we can solve the remaining $y_{i,j}$'s and $\mathbf{c}$ inductively. However, the undetermined variable might be zero or undefined when generating a candidate from a seed. A *good* choice of seed, we call a *generic* seed, must avoid the issue.

We would like to find a condition for generic seeds. In the setup of (11.5) and Remark 11.4, we put

$$\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{y}) := \begin{cases} -c^{\text{ver}}_{j,j-1} \cdot \dfrac{1}{y_{i,j-1}} + \dfrac{1}{(y_{i,j})^2}\left(c^{\text{ver}}_{j+1,j} \cdot y_{i,j+1} + c^{\text{hor}}_{i-1,i} \cdot y_{i-1,j}\right) & \text{if } i \geq j \\[2ex] -c^{\text{hor}}_{i-1,i} \cdot y_{i-1,j} + (y_{i,j})^2 \left(c^{\text{hor}}_{i,i+1} \cdot \dfrac{1}{y_{i+1,j}} + c^{\text{ver}}_{j,j-1} \cdot \dfrac{1}{y_{i,j-1}}\right) & \text{if } i < j. \end{cases} \tag{12.1}$$

Note that $\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{y})$ is achieved by isolating $c^{\text{hor}}_{i,i+1} \cdot (y_{i+1,j})^{-1}$ and $c^{\text{ver}}_{j+1,j} \cdot y_{i,j+1}$ in $\partial^{\mathfrak{b}}_m(i,j)(\mathbf{y}) = 0$, see (11.7). Moreover, (12.1) appears when isolating the undetermined variable so that the expression is required to be non-zero.

Definition 12.2. A seed $\mathbf{d}$ is called *generic* if the candidate generated by $\mathbf{y}_{\mathcal{I}} = \mathbf{d}$ satisfies

$$\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{y}) \neq 0 \mod T^{>0} \tag{12.2}$$

for all (i,j)'s.

Example 12.3. A straightforward calculation asserts that the tuples

(1) $\mathbf{d} = (-1, 1, 1, -1, 1)$
(2) $\mathcal{I} = \mathcal{I}_{\text{seed}} = ((2,2), (1,3), (2,3), (3,3), (3,4))$

form a generic seed of $\Gamma(7)$ to $B(2)$. The tuples

(1) $\mathbf{d} = (-1, 1, 1, 1, 1)$
(2) $\mathcal{I} = \mathcal{I}_{\text{seed}} = ((2,2), (1,3), (2,3), (3,3), (3,4))$

form a seed of $\Gamma(7)$ to $B(2)$, but not a generic seed because $\widetilde{\partial^{\mathfrak{b}}_2}(1,5)(\mathbf{y}) = 0$.

The main proposition of this section is the existence of a generic seed, which will be proven throughout this section.

Proposition 12.4. *For each integer m where $2 \leq m \leq k = \lceil n/2 \rceil$, a generic seed of $\Gamma(n)$ to $B(m)$ exists.*

As a corollary, we assert solvability of the split leading term equation.

Corollary 12.5. *The split leading term equation of $\Gamma(n)$ associated with $B(m)$ has a solution each component of which is a non-zero complex number.*

Proof. Once a seed has the property (12.2), the remaining $y_{i,j}$'s and a sequence $\mathbf{c}$ are (uniquely) determined to be in $\mathbb{C}\backslash\{0\}$ by the exactly same process in Section 11.5. $\square$

12.2. **Pre-generic elements.**

We now introduce a coordinate system $\{z_{i,j} : (i,j) \in \Gamma(n)\backslash B(m) \cup \{(m,m)\}\}$ with respect to which the system of equations

$$\partial_m^{\mathfrak{b}}(i,j)(\mathbf{y}) = 0, \quad \text{for } i+j<n$$

does *not* depend on the choice of a bulk-deformation parameter $\mathfrak{b}$. We define

$$(12.3)\qquad \begin{cases} z_{i+1,\bullet} := \left(\prod_{r=k}^{i} c^{\text{hor}}_{r,r+1}\right)^{-1} y_{i+1,\bullet} & \text{if } i \geq k \\ z_{\bullet,j+1} := \left(\prod_{r=k}^{j} c^{\text{ver}}_{r+1,r}\right) y_{\bullet,j+1} & \text{if } j \geq k \\ z_{i,j} := y_{i,j} & \text{otherwise.} \end{cases}$$

Under this coordinate system, we convert $\partial_m^{\mathfrak{b}}(i,j)(\mathbf{y})$ in (11.7) into

$$(12.4)\quad \partial_m^{\mathfrak{b}}(i,j)(\mathbf{z}) := \begin{cases} -\dfrac{z_{i,j+1}}{z_{i,j}} - \dfrac{z_{i-1,j}}{z_{i,j}} + \dfrac{z_{i,j}}{z_{i+1,j}} + \dfrac{z_{i,j}}{z_{i,j-1}} & \text{if } i+j<n \\ -\left(\prod_{r=k}^{i-1} c^{\text{hor}}_{r,r+1}\right)^{-1} \dfrac{1}{z_{i,j}} - \dfrac{z_{i-1,j}}{z_{i,j}} + \left(\prod_{r=k}^{i} c^{\text{hor}}_{r,r+1}\right) z_{i,j} + \dfrac{z_{i,j}}{z_{i,j-1}} & \text{if } i \geq j,\ i+j=n \\ -\left(\prod_{r=k}^{j} c^{\text{ver}}_{r+1,r}\right) \dfrac{1}{z_{i,j}} - \dfrac{z_{i-1,j}}{z_{i,j}} + \left(\prod_{r=k}^{j-1} c^{\text{ver}}_{r+1,r}\right)^{-1} z_{i,j} + \dfrac{z_{i,j}}{z_{i,j-1}} & \text{if } i<j,\ i+j=n. \end{cases}$$

Here, one should interpret that the product over the empty set is 1. For example,

$$\prod_{r=k}^{k-1} c^{\text{ver}}_{r+1,r} = 1.$$

We set

(12.5)

$$\widetilde{\partial_m^{\mathfrak{b}}}(i,j)(\mathbf{z}) := \begin{cases} -\dfrac{1}{z_{i,j-1}} + \dfrac{1}{(z_{i,j})^2}\left(z_{i,j+1} + z_{i-1,j}\right) \left(= \dfrac{1}{z_{i+1,j}}\right) & \text{if } i \geq j,\ i+j<n \\ -z_{i-1,j} + (z_{i,j})^2 \left(\dfrac{1}{z_{i+1,j}} + \dfrac{1}{z_{i,j-1}}\right) (= z_{i,j+1}) & \text{if } i<j,\ i+j<n \\ -\dfrac{1}{z_{i,j-1}} + \dfrac{1}{(z_{i,j})^2}\left(\left(\prod_{r=k}^{i-1} c^{\text{hor}}_{r,r+1}\right)^{-1} + z_{i-1,j}\right) \left(= \left(\prod_{r=k}^{i} c^{\text{hor}}_{r,r+1}\right)\right) & \text{if } i \geq j,\ i+j=n \\ -z_{i-1,j} + (z_{i,j})^2 \left(\left(\prod_{r=k}^{j-1} c^{\text{ver}}_{r+1,r}\right)^{-1} + \dfrac{1}{z_{i,j-1}}\right) \left(= \left(\prod_{r=k}^{j} c^{\text{ver}}_{r+1,r}\right)\right) & \text{if } i<j,\ i+j=n, \end{cases}$$

where $\widetilde{\partial_m^{\mathfrak{b}}}(i,j)(\mathbf{z})$ is obtained from isolating the expression in the parentheses in $\partial_m^{\mathfrak{b}}(i,j)(\mathbf{z})$.

We then have the following lemma, which says it suffices to check $\widetilde{\partial_m^{\mathfrak{b}}}(i,j)(\mathbf{z}) \neq 0$ to show $\widetilde{\partial_m^{\mathfrak{b}}}(i,j)(\mathbf{y}) \neq 0$.

Lemma 12.6. *$\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{z}) \neq 0$ for all indices $(i,j) \in \Gamma(n)\backslash B(m)$ if and only if $\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{y}) \neq 0$ for all indices $(i,j) \in \Gamma(n)\backslash B(m)$.*

Proof. Under the coordinate change (12.3), (12.1) is converted into (12.5). □

To show that $\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{z}) \neq 0$, we now start to solve (12.4) from $\mathbf{y}_{\mathcal{I}} := \mathbf{d}$ by isolating the undetermined variable in (12.4). When $m < k = \lceil n/2 \rceil$, since $\mathcal{I} \subset B(k)$ and $\mathbf{y}_{\mathcal{I}} = \mathbf{z}_{\mathcal{I}}$ because of (12.3), we may insert $\mathbf{d}$ into $\mathbf{z}_{\mathcal{I}}$ as the starting point. For the case $m = k$, we take $c^{\text{hor}}_{m,m+1} = 1$ and $c^{\text{ver}}_{m+1,m} = 1$ (see Remark 11.18) and hence $\mathbf{z}_{\mathcal{I}} = \mathbf{y}_{\mathcal{I}} = \mathbf{d}$ as well. For simplicity, we set

$$\mathbf{z}_{(l \backslash m)} := \{ z_{i,j} \in \mathbb{C}\backslash\{0\} : (i,j) \in \Gamma(l) \backslash B(m) \cup \{(m,m)\} \}. \tag{12.6}$$

Choosing the component in an anti-diagonal generically, we can easily make the first two equations of (12.5) non-zero because of the following lemma.

Lemma 12.7. *Suppose that the set $\mathbf{z}_{(r+s-1\backslash m)}$ is determined. Each variable $z_{r-i,s+i}$ can be expressed as a* non-constant *rational function with respect to $z_{r,s}$.*

Proof. We only show the case for $i > 0$ since the case where $i < 0$ can be similarly proven. Let

$$X(i) := z_{r-i,s+i}.$$

By (12.4), a recurrence relation for $X(i)$'s is

$$X(i) = [i] + \frac{[i,i-1]}{X(i-1)} \tag{12.7}$$

where

$$[i] := -z_{r-i-1,s+i} + \frac{(z_{r-i,s+i-1})^2}{z_{r-i,s+i-2}}, \;\; [i,i-1] := (z_{r-i,s+i-1})^2.$$

Composing (12.7) several times, $X(i)$ is expressed as a continued fraction in terms of $X(0)$. Letting $A(0) = 1$ and $B(0) = 0$, it becomes

$$X(i) = \frac{A(i) \cdot X(0) + B(i)}{A(i-1) \cdot X(0) + B(i-1)} \tag{12.8}$$

for some constants $A(i)$'s and $B(i)$'s determined by the given set $\mathbf{z}_{(r+s-1\backslash m)}$. Thus, $X(i)$ is a rational function with respect to $X(0)$.

To show that every $X(i)$ is *non-constant* with respect to $X(0)$, we investigate properties of $A(i)$'s and $B(i)$'s. By induction, we can show that the terms of $A(i)$ correspond to the partitions of $\{i, i-1, \cdots, 1\}$ into one single number or two consecutive numbers. Also, the terms of $B(i)$ correspond to the partitions of $\{i, i-1, \cdots, 1, 0\}$ into one single or two consecutive numbers containing the subset $[1,0]$. For instance, $A(3)$ and $B(3)$ are expressed as

$$A(3) = [3][2][1] + [3][2,1] + [3,2][1],$$
$$B(3) = [3][2][1,0] + [3,2][1,0].$$

It then follows that

$$A(i) = [i] \cdot A(i-1) + [i,i-1] \cdot A(i-2)$$
$$B(i) = [i] \cdot B(i-1) + [i,i-1] \cdot B(i-2).$$

Note that $X(0)$ and $X(1)$ are non-constant functions with respect to $X(0)$. Suppose to the contrary that $X(i)$ is a constant function with the value C and all $X(j)$'s for all $j < i$ are non-constant rational functions with respect to $X(0)$. Let

$$X(i) := \frac{A(i) \cdot X(0) + B(i)}{A(i-1) \cdot X(0) + B(i-1)} = C.$$

We then obtain

$$C \cdot A(i-1) = A(i) = [i] \cdot A(i-1) + [i,i-1] \cdot A(i-2)$$
$$C \cdot B(i-1) = B(i) = [i] \cdot B(i-1) + [i,i-1] \cdot B(i-2).$$

We claim that $C - [i] \neq 0$. Otherwise, $A(i-2) = B(i-2) = 0$ because $[i, i-1] = (z_{r-i,s+i-1})^2 \neq 0$. It yields that $X(i-2) \equiv 0$, contradicting to the assumption that $X(i-2)$ is not constant.

We then have

$$A(i-1) = C' \cdot A(i-2)$$
$$B(i-1) = C' \cdot B(i-2)$$

where $C' = [i, i-1]/(C - [i])$. So, we deduce a contradiction that

$$X(i-1) = \frac{A(i-1) \cdot X(0) + B(i-1)}{A(i-2) \cdot X(0) + B(i-2)} = C'$$

is constant. Hence, every $X(i)$ has to be a non-constant rational function. □

Corollary 12.8. *Suppose that the set $\mathbf{z}_{(r+s-1\backslash m)}$ is determined. There exists a non-zero real number $d_{r,s}$ such that if we set $z_{r,s} = d_{r,s}$*

$$\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{z}) \neq 0 \tag{12.9}$$

for all (i,j)'s obeying $i + j = r + s - 1$.

Proof. Since each $z_{r-i,s+i}$ is a non-constant rational function with respect to $z_{r,s}$, there are only finitely many $z_{r,s}$'s so that $z_{r-i,s+i}$ is zero or is not defined. Avoid these values when choosing $d_{r,s}$. □

Definition 12.9. Suppose the set $\mathbf{z}_{(r+s-1\backslash m)}$ is given. For an index $(r,s) \in \Gamma(n)\backslash B(m)$, an element $d_{r,s}$ is said to be *pre-generic with respect to* $\mathbf{z}_{(r+s-1\backslash m)}$ if (12.9) holds for any $(i,j) \in \Gamma(r+s)\backslash(B(m) \cup \Gamma(r+s-1))$.

For the later purpose, we prove the following property of the the pre-generic elements.

Lemma 12.10. *Assume that $m < k$. Suppose that we have $d_{s,m+1}$'s for s with $1 \leq s \leq m$ such that for each s, $d_{s,m+1}$ is pre-generic with respect to the previously determined $\mathbf{z}_{(s+m\backslash m)}$ by one choice of $d_{m,m}$ and $d_{1,m+1}, \cdots, d_{s-1,m+1}$. Then, regardless of a choice of $d_{m,m} \in \mathbb{C}\backslash\{0\}$, $d_{s,m+1}$ is pre-generic as long as we do not change $d_{1,m+1}, \cdots, d_{s-1,m+1}$.*

Proof. If $m < k$, we see $y_{i,j} = z_{i,j}$ for $(i,j) \in B(m)$ by (12.3). We claim that

$$z_{j,i+1} = (-1)^{i+j} \cdot \frac{(d_{m,m})^2}{z_{i+1,j}}.$$

Recall from (11.8) that

$$z_{m+1,i} = (-1)^{i+(m+1)-1} \cdot \frac{(d_{m,m})^2}{z_{i,m+1}},$$

which provides the initial step for the induction. Next, by the induction hypothesis, we observe

$$\begin{aligned} 0 &= -\frac{z_{i,j+1}}{z_{i,j}} - \frac{z_{i-1,j}}{z_{i,j}} + \frac{z_{i,j}}{z_{i+1,j}} + \frac{z_{i,j}}{z_{i,j-1}} \\ &= \frac{z_{j,i}}{z_{j+1,i}} + \frac{z_{j,i}}{z_{j,i-1}} + (-1)^{i+j-1} \frac{(d_{m,m})^2}{z_{i+1,j} \cdot z_{j,i}} - \frac{z_{j-1,i}}{z_{j,i}} \\ &= \frac{z_{j,i+1}}{z_{j,i}} + (-1)^{i+j-1} \frac{(d_{m,m})^2}{z_{i+1,j}\, z_{j,i}}. \end{aligned}$$

Thus, we obtain

$$z_{j,i+1} = (-1)^{i+j} \cdot \frac{(d_{m,m})^2}{z_{i+1,j}}.$$

Therefore, $\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{z}) \neq 0$ as long as $\widetilde{\partial^{\mathfrak{b}}_m}(j,i)(\mathbf{z}) \neq 0$. □

12.3. **Generic seeds.**

Applying Corollary 12.8, we make the first two expressions in (12.5) non-zero by taking one entry of an anti-diagonal generically. To make the last two equations non-zero, we need to select the previous ones more carefully. We deal with the three cases separately.

Case 1. $n = 2k - 1$**.**

We need several lemmas.

Lemma 12.11. *Assume that* $\mathbf{z}_{(n-2\backslash m)}$ *is given. Suppose that either* $d_{k-1,k-1} = -1$ *is pre-generic or* $k - 1 = m$*. Then, there is a real number* $d_{k-1,k-1}$ *(sufficiently close to* -1*) and a non-zero real number* $d_{k-1,k}$ *such that if* $z_{k-1,k-1} = d_{k-1,k-1}$ *and* $z_{k-1,k} = d_{k-1,k}$*,*

$$\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{z}) \neq 0 \mod T^{>0} \tag{12.10}$$

for all (i,j) *with* $i + j = n - 1$ *and* $i + j = n$*.*

Note that $\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{z})$'s for (i,j) with $i + j = n$ provide the last two expressions of (12.5).

Proof. Assuming that $d_{k-1,k-1} = -1$ is pre-generic, by definition, every $z_{i,n-1-i}$ is defined and becomes non-zero if we set $z_{k-1,k-1} = d_{k-1,k-1} = -1$. By Corollary 12.8, we can choose and fix a pre-generic element $d_{k-1,k}$ for $z_{k-1,k}$ so that the entries $z_{i,n-i}$'s are also determined.

We would like to emphasize that $d_{k-1,k-1} = -1$ is never being a component of a generic seed because of the following reason. Recall that the equations $\partial^{\mathfrak{b}}_m(i,j)(\mathbf{z}) = 0$'s in (12.4) for (i,j)'s with $i + j \geq n$ and $i < j$ read

$$\begin{cases} \prod_{r=k}^{k} c^{\text{ver}}_{r+1,r} & = -z_{k-2,k} + (z_{k-1,k})^2 \left(1 + \frac{1}{z_{k-1,k-1}}\right), \\ \prod_{r=k}^{k+1} c^{\text{ver}}_{r+1,r} & = -z_{k-3,k+1} + (z_{k-2,k+1})^2 \left(\left(\prod_{r=k}^{k} c^{\text{ver}}_{r+1,r}\right)^{-1} + \frac{1}{z_{k-2,k}}\right), \\ & \cdots \\ \prod_{r=k}^{n-2} c^{\text{ver}}_{r+1,r} & = -z_{1,n-2} + (z_{2,n-2})^2 \left(\left(\prod_{r=k}^{n-3} c^{\text{ver}}_{r+1,r}\right)^{-1} + \frac{1}{z_{2,n-3}}\right), \\ \prod_{r=k}^{n-1} c^{\text{ver}}_{r+1,r} & = -(z_{1,n-1})^2 \left(\left(\prod_{r=k}^{n-2} c^{\text{ver}}_{r+1,r}\right)^{-1} + \frac{1}{z_{1,n-2}}\right). \end{cases} \tag{12.11}$$

If one chooses $z_{k-1,k-1} = d_{k-1,k-1} = -1$, then from (12.11) we obtain

$$\prod_{r=k}^{j} c^{\text{ver}}_{r+1,r} = -z_{n-j-1,j}$$

for $j = k, k+1, \cdots, n-2$ and

$$\widetilde{\partial^{\mathfrak{b}}_m}(1, n-1)(\mathbf{z}) = \prod_{r=k}^{n-1} c^{\text{ver}}_{r+1,r} = 0. \tag{12.12}$$

Thus, a seed $\mathbf{d}$ is *not* generic [15]. Nevertheless, we claim that there exists a choice of $d_{k-1,k-1}$ not equal to -1 but close to -1 so that (12.10) is satisfied.

Note that the fixed $d_{k,k-1}$ remains to be pre-generic even if we perturb the value $z_{k-1,k-1}$ from -1 with sufficiently small amount. This is because the expression $\widetilde{\partial^{\mathfrak{b}}_m}(i,j)(\mathbf{z})$ for each index (i,j) with $i + j = n - 1$ is a continuous function with respect to $z_{k-1,k-1}$ at -1 after inserting $d_{k,k-1}$ into $z_{k,k-1}$. Also, by Lemma 12.7, there exists a dense set of pre-generic elements for $d_{k-1,k-1}$. Therefore, (12.10) is satisfied for $i + j = n - 1$.

Also, we observe that as $z_{k-1,k-1} \to -1$, because of (12.5) and (12.11), $\widetilde{\partial^{\mathfrak{b}}_m}(n-j,j)(\mathbf{z}) \to -z_{n-j-1,j}$ when $j \geq k$. Because $-z_{n-j-1,j} \neq 0$ for j with $k \leq j < n-1$, we still have $\widetilde{\partial^{\mathfrak{b}}_m}(n-j,j)(\mathbf{z}) \neq 0$ for j with $k \leq j < n-1$ if $d_{k-1,k-1}$ is sufficiently close to -1. Finally, we claim that $\widetilde{\partial^{\mathfrak{b}}_m}(1, n-1)(\mathbf{z}) \neq 0$ as soon as $z_{k-1,k-1} \neq -1$ so that the problem in (12.12) is solved. From $\partial^{\mathfrak{b}}_m(k-1,k)(\mathbf{z}) = 0$ and $z_{k-1,k-1} \neq -1$, it follows that

$$c^{\text{ver}}_{k+1,k} \neq -z_{k-2,k}.$$

[15] We also have $\widetilde{\partial^{\mathfrak{b}}_m}(n-1,1)(\mathbf{z}) = 0$ if taking $z_{k-1,k-1} = d_{k-1,k-1} = -1$.

Combining it with $\partial_m^{\mathfrak{b}}(k-2,k+1)(\mathbf{z}) = 0$, we obtain

$$\prod_{r=k}^{k+1} c_{r+1,r}^{\text{ver}} \neq -z_{k-3,k+1}.$$

Proceeding inductively, we deduce $\widetilde{\partial_m^{\mathfrak{b}}}(1,n-1)(\mathbf{z}) \neq 0$. The discussion on the part where $j < k$ is omitted because the argument is symmetrical.

Consequently, we may choose a generic $d_{k-1,k-1}$ sufficiently close to -1 so that (12.10) holds for all (i,j) with $i+j = n-1$ and $i+j = n$.

It remains to take care of the case where $k-1 = m$. The index $(k-1,k-1) = (m,m)$ is contained in the box $B(m)$ so that $d_{k-1,k-1}$ can be freely chosen by Corollary 11.12. Thus, we can apply the exactly same argument as above. □

By applying a similar argument, we can prove the following lemma.

Lemma 12.12. *Suppose that either $d_{i,i} = \pm 1$ is pre-generic for $i > m$ or $i = m$. There is a real number $d_{i,i}$ (sufficiently close to ± 1) and a non-zero real number $d_{i,i+1}$ so that $d_{i+1,i+1} = \mp 1$ becomes pre-generic.*

We now ready to start the proof of Proposition 12.4 for the case where $n = 2k-1$ and $m < k := \lceil n/2 \rceil$.

Proof of Proposition 12.4. We start with a tentative choice of $d_{m,m} = \pm 1$. Choosing pre-generic elements from $d_{1,m+1} := z_{1,m+1}$ to $d_{m-1,m+1} := z_{m-1,m+1}$, we find $\mathbf{z}_{(2m\backslash m)}$ so that that (12.10) is satisfied for each index (i,j) with $i+j \leq 2m-1$. Due to Lemma 12.12, we may select $d_{m,m}$ sufficiently close to ± 1 and $d_{m,m+1}$ so that $d_{m+1,m+1} = \mp 1$ becomes pre-generic. Because of Lemma 12.10, note that $d_{1,m+1}, \cdots, d_{m,m+1}$ remain to be pre-generic even if we choose another $d_{m,m}$. Moreover, applying Lemma 12.12 repeatedly, we assert that $d_{k-1,k-1} = -1$ is also pre-generic by suitably choosing $d_{\bullet,\bullet}$. Hence, we have (12.10) for all indices (i,j)'s with $i+j \leq n-2$. Finally, Lemma 12.11 says that there is $d_{k-1,k-1}$ and $d_{k,k-1}$ such that (12.10) holds for $i+j = n-1, n$. Thus, we have just found a generic seed. □

Case 2. $n = 2k$ **and** $m < k$**.**

Modifying the proofs of Lemma 12.11 and Lemma 12.12, we can prove the following lemma.

Lemma 12.13. *Assume that $\mathbf{z}_{(n-2\backslash m)}$ is given. Suppose that $d_{k-1,k} = -1$ is pre-generic. Then, there is a real number $d_{k-1,k}$ (sufficiently close to -1) and a non-zero real number $d_{k,k}$ such that if $z_{k-1,k} = d_{k-1,k}$ and $z_{k,k} = d_{k,k}$,*

$$\widetilde{\partial_m^{\mathfrak{b}}}(i,j)(\mathbf{z}) \neq 0 \mod T^{>0}$$

for all (i,j) with $i+j = n-1$ and $i+j = n$.

Suppose that $d_{i-1,i} = \pm 1$ is pre-generic for $i \geq m+1$. There is a real number $d_{i-1,i}$ (sufficiently close to ± 1) and a non-zero real number $d_{i,i}$ so that $d_{i,i+1} = \mp 1$ becomes pre-generic.

Also, we need the lemma, which serves as the starting point to obtain the desired $d_{\bullet,\bullet}$'s.

Lemma 12.14. *$d_{m,m+1} = \pm 1$ can be pre-generic.*

Proof. By Lemma 11.10,

$$\widetilde{z}_{i,m+j} := \begin{cases} 1 & \text{for } i = j \\ \displaystyle\prod_{r=0}^{j-i-1} (2i+2r) & \text{for } i < j \\ \displaystyle\prod_{r=0}^{i-j-1} (2j+2r)^{-1} & \text{for } i > j \end{cases} \tag{12.13}$$

is a solution of $\partial_m^{\mathfrak{b}}(i,m+j)(\mathbf{z}) = 0$ in (12.4) for $m+i+j < n$. Furthermore, by Lemma 11.9, so does

$$z_{i,m+j} := a \cdot \widetilde{z}_{i,m+j} \tag{12.14}$$

for any non-zero complex number a. Selecting

$$a := \prod_{r=0}^{m-2} (2+2r),$$

$d_{m,m+1} = z_{m,m+1}$ becomes 1. Because of Lemma 12.10, no matter what we choose any non-zero complex number $d_{m,m}$, $d_{m,m+1}$ is pre-generic (with respect to the previous determined $\mathbf{z}_{(2m \backslash m)}$). ☐

Proof of Proposition 12.4 (continued). Combining Lemma 12.13 and Lemma 12.14, we conclude Proposition 12.4 for the case where $n = 2k$ and $m < k$. ☐

Case 3. $n = 2k$ **and** $m = k$**.**

In this case, we take $d_{m,m} = 1$, see Remark 11.18. Because of Lemma 11.10, note that

$$\widetilde{z}_{i,m+j} := \begin{cases} 1 & \text{for } i = j \\ \displaystyle\prod_{r=0}^{j-i-1} (2i+2r) & \text{for } i < j \\ \displaystyle\prod_{r=0}^{i-j-1} (2j+2r)^{-1} & \text{for } i > j \end{cases}, \quad \widetilde{z}_{m+i,j} := \begin{cases} (-1)^{m+i+j-1} & \text{for } i = j \\ (-1)^{m+i+j-1} \displaystyle\prod_{r=0}^{i-j-1} (2j+2r)^{-1} & \text{for } i > j \\ (-1)^{m+i+j-1} \displaystyle\prod_{r=0}^{j-i-1} (2i+2r) & \text{for } i < j \end{cases} \tag{12.15}$$

respectively form a solution of $\partial^{\mathfrak{b}}_m(i, m+j)(\mathbf{z}) = 0$ and $\partial^{\mathfrak{b}}_m(m+i, j)(\mathbf{z}) = 0$ in (12.4) for $m + i + j < n$. Also, our choice makes $\partial_m(l)(\mathbf{y}) = \partial_m(l)(\mathbf{z}) = 0$ in (11.8) because $c^{\mathrm{ver}}_{m+1,m} = 1$ and $c^{\mathrm{hor}}_{m,m+1} = 1$, see Remark 11.4 and Remark 11.18.

Furthermore,

$$z_{i,m+j} := a \cdot \widetilde{z}_{i,m+j}, \quad z_{m+i,j} := a^{-1} \cdot \widetilde{z}_{m+i,j} \tag{12.16}$$

are also solutions of (12.4) and (11.8) for any non-zero complex number a. Thus, we have a one-parameter family of solutions. Then, the expressions $\partial^{\mathfrak{b}}_m(i, m+j)(\mathbf{z})$ and $\partial^{\mathfrak{b}}_m(m+i, j)(\mathbf{z})$ for (i, j) with $i + j = n$ can be considered as a function with respect to a.

Lemma 12.15. *There exists a choice of the variable a such that*

$$\widetilde{\partial^{\mathfrak{b}}_m}(m-i, m+i)(\mathbf{z}) \neq 0 \text{ and } \widetilde{\partial^{\mathfrak{b}}_m}(m+i, m-i)(\mathbf{z}) \neq 0 \mod T^{>0} \tag{12.17}$$

in (12.5)*.*

Proof. We claim that $\widetilde{\partial^{\mathfrak{b}}_m}(m-i, m+i)(\mathbf{z})/z_{m-i,m+i}$ is a non-constant rational function with respect to a. For $i \geq 1$, we observe that

$$\frac{\widetilde{\partial^{\mathfrak{b}}_m}(m-1, m+1)(\mathbf{z})}{z_{m-1,m+1}} = -\frac{z_{m-2,m+1}}{z_{m-1,m+1}} + z_{m-1,m+1} = -(2m-4) + a \cdot \left(\prod_{r=0}^{m-3} (2+2r)^{-1} \right)$$

is a non-constant linear function with respect to a. By the induction hypothesis, assume that

$$\frac{\widetilde{\partial^{\mathfrak{b}}_m}(m-i, m+i)(\mathbf{z})}{z_{m-i,m+i}} := \frac{P_i(a)}{Q_i(a)}$$

is a non-constant rational function with respect to a. Then, we see

$$\begin{aligned} \frac{\widetilde{\partial^{\mathfrak{b}}_m}(m-i-1, m+i+1)(\mathbf{z})}{z_{m-i-1,m+i+1}} &= \left(-\frac{z_{m-i-2,m+i+1}}{z_{m-i-1,m+i+1}} + \frac{z_{m-i-1,m+i+1}}{z_{m-i-1,m+i}} \right) + \frac{z_{m-i-1,m+i+1}}{\widetilde{\partial^{\mathfrak{b}}_m}(m-i, m+i)(\mathbf{z})} \\ &= \left(-\frac{\widetilde{z}_{m-i-2,m+i+1}}{\widetilde{z}_{m-i-1,m+i+1}} + \frac{\widetilde{z}_{m-i-1,m+i+1}}{\widetilde{z}_{m-i-1,m+i}} \right) + \frac{\widetilde{z}_{m-i-1,m+i+1}}{\widetilde{z}_{m-i,m+i}} \cdot \frac{Q_i(a)}{P_i(a)}, \end{aligned}$$

which is also a non-constant rational function. Similarly, one can see that $\widetilde{\partial^{\mathfrak{b}}_m}(m+i, m-i)(\mathbf{z})$ is also a non-constant rational function for $i \geq 1$. Thus, (12.17) is established if choosing a generically. ☐

We are ready to prove Proposition 12.4 for the case where $n = 2k$ and $m = k := \lceil n/2 \rceil$.

Proof of Proposition 12.4 (continued). By Lemma 12.15, we choose $d_{i,m+1} := a \cdot \widetilde{z}_{i,m+1}$ from (12.16) as a generic seed. It complete the proof. □

12.4. **Proof of Theorem D.**

Finally, we are ready to prove the following theorem.

Theorem 12.16 (Theorem D). *Let $\lambda = \{\lambda_i := n - 2i + 1 \,|\, i = 1, \cdots, n\}$ be an n-tuple of real numbers for an arbitrary integer $n \geq 4$. Consider the co-adjoint orbit $\mathcal{O}_\lambda$, a complete flag manifold $\mathcal{F}(n)$ equipped with the monotone Kirillov-Kostant-Souriau symplectic form ω_λ. Then each Gelfand-Cetlin fiber $L_m(t)$ is non-displaceable Lagrangian for every $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. In particular, there exists a family of non-displaceable non-torus Lagrangian fibers*

$$\left\{ L_m(1) : 2 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

of the Gelfand-Cetlin system Φ_λ where $L_m(1)$ is diffeomorphic to $U(m) \times T^{\frac{n(n-1)}{2} - m^2}$.

Proof. By Corollary 12.5, the split leading term equation (11.6) has a desired solution for some nonzero complex numbers $c^{\mathrm{ver},\mathbb{C}}_{i+1,i}$'s and $c^{\mathrm{hor},\mathbb{C}}_{j,j+1}$'s $(i, j \geq k)$. Theorem 11.7 convinces us that for each Lagrangian torus $L_m(t)$ $(0 \leq t < 1)$, there exists a suitable bulk-deformation parameter $\mathfrak{b}$ of the form (10.8) so that the bulk-deformed potential function admits a critical point. By Theorem 10.7, each GC torus fiber $L_m(t)$ for $0 \leq t < 1$ is non-displaceable. Furthermore, Corollary 5.23 and Lemma 10.13 imply that $L_m(1)$ is Lagrangian and non-displaceable. Finally, $L_m(1)$ is diffeomorphic to $U(m) \times T^{\frac{n(n-1)}{2} - m^2}$ because of Theorem 7.9. This finishes the proof of Theorem 12.16. □

13. Calculation of potential function deformed by Schubert cycles

The potential function *with bulk* we use in the present paper was first constructed in [FOOO1, Section 3.8.5, 3.8.6] and was explicitly computed in [FOOO4, Section 3] for the toric case. Since the main steps of the derivation of (13.10) are the same as that of the proof of [FOOO4, Proposition 4.7] given in Section 7 therein, we will only explain modifications we need to make to apply them to the current GC case. Also for the purpose of proving the counter part of Theorem 10.6 in the present paper, the facts that a Fano manifold X_ϵ has a toric degeneration and that we have only to consider codimension two cycles also help us to simplify the study of holomorphic discs contributing to the potential functions. We closely follow [NNU1, Section 9].

Let L be a Lagrangian submanifold in a symplectic manifold X. Let $\mathcal{M}_{k+1;\ell}(X, L; \beta)$ denote the moduli space of stable maps in the class $\beta \in \pi_2(X, L)$ from a bordered Riemann surface Σ of genus zero with $(k+1)$ marked points $\{z_s\}_{s=0}^k$ on the boundary $\partial\Sigma$ respecting the counter-clockwise orientation and ℓ marked points $\{z_r^+\}_{r=1}^\ell$ at the interior of Σ. It naturally comes with two types of evaluation maps, at the i-th boundary marked point

$$\mathrm{ev}_i \colon \mathcal{M}_{k+1;\ell}(X, L; \beta) \to L; \quad \mathrm{ev}_i\left([\varphi \colon \Sigma \to X, \{z_s\}_{s=0}^{k+1}, \{z_r^+\}_{r=1}^\ell]\right) = \varphi(z_i) \tag{13.1}$$

and at the j-th interior marked point

$$\mathrm{ev}_j^{\mathrm{int}} \colon \mathcal{M}_{k+1;\ell}(X, L; \beta) \to X; \quad \mathrm{ev}_j^{\mathrm{int}}\left([\varphi \colon \Sigma \to X, \{z_s\}_{s=0}^{k+1}, \{z_r^+\}_{r=1}^\ell]\right) = \varphi(z_j^+). \tag{13.2}$$

Set $\mathcal{M}_{k+1}(X, L; \beta) := \mathcal{M}_{k+1;\ell=0}(X, L; \beta)$, the moduli space without interior marked points and let

$$\mathbf{ev}_+ := (\mathrm{ev}_1, \cdots, \mathrm{ev}_k).$$

Recall that an A_∞-structure with the operators

$$\mathfrak{m}_k = \sum_\beta \mathfrak{m}_{k,\beta} \cdot T^{\omega(\beta)/2\pi}, \quad \mathfrak{m}_{k,\beta}(b_1, \cdots, b_k) := (\mathrm{ev}_0)!(\mathbf{ev}_+)^*(\pi_1^* b_1 \otimes \cdots \otimes \pi_k^* b_k)$$

on the de Rham complex $\Omega(L)$ is defined via a smooth correspondence.

$$\begin{tikzcd} & \mathcal{M}_{k+1}(X, L; \beta) \arrow[dl, "\mathbf{ev}_+"'] \arrow[dr, "\mathrm{ev}_0"] & \\ L^k & & L \end{tikzcd} \tag{13.3}$$

where $\pi_i \colon L^k \to L$ denotes the projection to the i-th copy of L. For a general symplectic manifold, one should choose a system of compatible Kuranishi structures and CF-perturbations on $\mathcal{M}_{k+1,l}(X, L; \beta)$'s in order to apply the above smooth correspondence, see [Fuk, FOOO6] for the details of construction. For a Lagrangian toric fiber L in a $2n$-dimensional toric manifold, by constructing a system of compatible T^n-equivariant Kuranishi structures and multi-sections, the smooth correspondence can be applied without adding an auxiliary space for perturbing multi-sections to make it submersive, see [FOOO4, Section 12]. It is because the map ev_0 automatically becomes submersive by the T^n-equivariance.

We now recall Nishinou-Nohara-Ueda's computation of the potential function of a torus fiber $L_\varepsilon \subset X_\varepsilon$ in [NNU1]. They were able to exploit the presence of toric degeneration of X_ϵ to X_0 in their computation the explanation of which is now in order. For the study of holomorphic discs in X_0 which is not smooth, they used the following notion in Nishinou-Siebert [NS].

Definition 13.1 (Definition 4.1 in [NS]). A holomorphic curve in a toric variety X is called *torically transverse* if it is disjoint from all toric strata of codimension greater than one. A stable map $\varphi \colon \Sigma \to X$ is *torically transverse* if $\varphi(\Sigma) \subset X$ is torically transverse and $\varphi^{-1}(\mathrm{Int}\, X) \subset \Sigma$ is dense. Here, $\mathrm{Int}\, X$ is the complement of the toric divisors in X.

We denote by $S_0 := \mathrm{Sing}(X_0)$ the singular locus of X_0. Using the classification result [CO] of holomorphic discs attached to a Lagranigian toric fiber in a smooth toric manifold and the property of the small resolution, Nishinou-Nohara-Ueda [NNU1] proved the following.

Lemma 13.2 (Proposition 9.5 and Lemma 9.15 in [NNU1]). *Any holomorphic disc $\varphi \colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (X_0, L_0)$ can be deformed into a holomorphic disc with the same boundary condition that is torically transverse. Furthermore the moduli space $\mathcal{M}_1(X_0, L_0; \beta)$ is empty if the Maslov index of β is less than two.*

Lemma 13.3 (Lemma 9.9 in [NNU1]). *There is a small neighborhood W_0 of the singular locus $S_0 \subset X_0$ such that no holomorphic discs of Maslov index two intersect W_0.*

Now let $\phi'_\epsilon \colon X_\epsilon \to X_0$ be a (continuous) extension of the flow $\phi_\epsilon \colon X_\epsilon^{\mathrm{sm}} \to X_0^{\mathrm{sm}}$ given in Theorem 10.1 ([NNU1, Section 8]). The following is the key proposition which relates the above mentioned holomorphic discs in (X_0, L_0) to those of (X_ϵ, L_ϵ).

Proposition 13.4 (Proposition 9.16 in [NNU1]). *For any $\beta \in \pi_2(X_0, L_0)$ of Maslov index two, there is a positive real numbers $0 < \varepsilon \leq 1$ and a diffeomorphism*

$$\psi \colon \mathcal{M}_1(X_0, L_0; \beta) \to \mathcal{M}_1(X_\varepsilon, L_\varepsilon; \beta)$$

such that the diagram

$$\begin{tikzcd} H_*(\mathcal{M}_1(X_0, L_0; \beta)) \arrow[r, "(\mathrm{ev}_0)_*"] \arrow[d, "\psi_*"'] & H_*(L_0) \arrow[d, "(\phi_\varepsilon)_*^{-1}"] \\ H_*(\mathcal{M}_1(X_\varepsilon, L_\varepsilon; \beta)) \arrow[r, "(\mathrm{ev}_0)_*"] & H_*(L_\varepsilon) \end{tikzcd}$$

is commutative.

Lemma 13.5 (Lemma 9.22 in [NNU1]). *Let $W_\varepsilon := (\phi'_\varepsilon)^{-1}(W_0)$. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, any holomorphic curve bounded by L_ε in a class of Maslov index two does not intersect W_ε.*

We now combine the diffeomorphism $\psi \colon \mathcal{M}_1(X_0, L_0; \beta) \to \mathcal{M}_1(X_\varepsilon, L_\varepsilon; \beta)$ and $\phi'_\varepsilon \colon X_0 \to X_\varepsilon$ to define an isomorphism between the correspondence

(13.4)
$$\begin{tikzcd} & \mathcal{M}_{k+1}(X_0, L_0; \beta) \arrow[dl, "\mathrm{ev}_+"'] \arrow[dr, "\mathrm{ev}_0"] & \\ L_0^k & & L_0 \end{tikzcd}$$

and the following correspondence

$$
\begin{tikzcd}
 & \mathcal{M}_{k+1}(X_\varepsilon, L_\varepsilon; \beta) \arrow[dl, "\mathbf{ev}_+"'] \arrow[dr, "ev_0"] & \\
L_\varepsilon^k & & L_\varepsilon
\end{tikzcd}
\tag{13.5}
$$

Although they did not explicitly mention a choice of compatible systems of Kuranishi structures or perturbations, Nishinou-Nohara-Ueda [NNU1] essentially constructed an A_∞-structure on $L_\varepsilon \subset X_\epsilon$ and computed its potential function in the same way as on a Fano toric manifolds [CO, FOOO3] using Proposition 13.4 and Lemma 13.5. Thus, they were able to take advantage of properties of T^n-equivariant perturbation in a toric manifold, a open submanifold of a toric variety X_0. We denote the corresponding compatible system of multi-sections by $\mathfrak{s} = \mathfrak{s}_{k+1,\beta}$, see [FOOO3, FOOO4] for the meaning of this notation.

Next we need to involve bulk deformations for our purpose of a construction of continuum of non-displaceable Lagrangian tori in X, whose construction is now in order.

Denote $\mathscr{A}^2_{GS}(\mathbb{Z})$ be the free abelian group generated by the horizontal and vertical Schubert cycles of real codimension two

$$\{\mathscr{D}^{\text{hor}}_{i,i+1} : 1 \leq i \leq n-1\} \cup \{\mathscr{D}^{\text{ver}}_{j+1,j} : 1 \leq j \leq n-1\}. \tag{13.6}$$

We recall

$$L \cap \mathscr{D}^{\text{hor}}_{i,i+1} = \emptyset = L \cap \mathscr{D}^{\text{ver}}_{j+1,j} \tag{13.7}$$

for any i, j and so the cap product of $\beta \in \pi_2(X, L)$ with any element thereof is well-defined. Putting $\mathscr{A}^2_{GS}(\Lambda_0) := \mathscr{A}^2_{GS}(\mathbb{Z}) \otimes \Lambda_0$, any element $\mathfrak{b} \in \mathscr{A}^2_{GS}(\Lambda_0)$ can be expressed as

$$\mathfrak{b} = \sum_{i=1}^{n-1} \mathfrak{b}^{\text{hor}}_{i,i+1} \mathscr{D}^{\text{hor}}_{i,i+1} + \sum_{j=1}^{n-1} \mathfrak{b}^{\text{ver}}_{j+1,j} \mathscr{D}^{\text{ver}}_{j+1,j} \tag{13.8}$$

where $\mathfrak{b}^{\text{hor}}_{i,i+1}, \mathfrak{b}^{\text{ver}}_{j+1,j} \in \Lambda_0$. We formally denote

$$\beta \cap \mathfrak{b} = \sum_{i=1}^{n-1} \mathfrak{b}^{\text{hor}}_{i,i+1} \left(\beta \cap \mathscr{D}^{\text{hor}}_{i,i+1}\right) + \sum_{j=1}^{n-1} \mathfrak{b}^{\text{ver}}_{j+1,j} \left(\beta \cap \mathscr{D}^{\text{ver}}_{j+1,j}\right). \tag{13.9}$$

For simplicity, let us fix an enumeration $\{\mathscr{D}_j \mid j = 1, \cdots, B\}$ of the elements in (13.6) where $B = 2(n-2)$ and set $\mathfrak{b}_j$ to be the coefficient corresponding to $\mathscr{D}_j$ in (13.8).

The following is the statement of the counterpart of Theorem 10.6.

Theorem 13.6. *Let $\mathfrak{b} \in \mathscr{A}^2_{GS}(\Lambda_0)$ and let L_ε be a torus Lagrangian fiber in X_ε. Then, the bulk-deformed potential function is written as*

$$\mathfrak{PO}^{\mathfrak{b}}\left(L_\varepsilon; b\right) = \sum_\beta n_\beta \cdot \exp\left(\beta \cap \mathfrak{b}\right) \cdot \exp(\partial\beta \cap b)\, T^{\omega(\beta)/2\pi}. \tag{13.10}$$

where the summation is taken over all homotopy classes in $\pi_2(X_\varepsilon, L_\varepsilon)$ of Maslov index two.

The remaing part of this section is reserved for the proof of Theorem 13.6.

For a Lagrangian submanifold L of X, denote by

$$ev_i^{\text{int}} \colon \mathcal{M}^{\text{main}}_{k+1;\ell}(L, \beta) \to X$$

the evaluation map at the i-th interior marked point for $i = 1, \ldots, \ell$. We put $\underline{B} = \{1, \ldots, B\}$ and denote the set of all maps $\mathbf{p} : \{1, \ldots, \ell\} \to \underline{B}$ by $Map(\ell, \underline{B})$. We write $|\mathbf{p}| = \ell$ if $\mathbf{p} \in Map(\ell, \underline{B})$. We define a fiber product

$$\mathcal{M}^{\text{main}}_{k+1;\ell}(L, \beta; \mathbf{p}) = \mathcal{M}^{\text{main}}_{k+1;\ell}(L, \beta)_{(ev_1^{\text{int}}, \ldots, ev_\ell^{\text{int}})} \times_{X^\ell} \prod_{i=1}^{\ell} \mathscr{D}_{\mathbf{p}(i)} \tag{13.11}$$

and consider the evaluation maps

$$ev_i \colon \mathcal{M}^{\text{main}}_{k+1;\ell}(L, \beta) \to L$$

by

$$ev_i((\Sigma, \varphi, \{z_i^+\}, \{z_i\})) = \varphi(z_i).$$

It induces

$$ev_i \colon \mathcal{M}^{\text{main}}_{k+1;\ell}(L, \beta; \mathbf{p}) \to L.$$

Note the image of a Schubert horizontal or vertical cycle is (a union of) components of the toric divisor via the map ϕ'_ε so that $\mathscr{D}_j$ can be regarded as the (union of) components of toric divisors corresponding to Schubert horizontal or vertical cycles . Also, by Proposition 13.4 and Lemma 13.5, the holomorphic discs of Maslov index two intersect at the smooth locus of divisors. For a toric fiber L_0 in X_0, there exists a system $\mathfrak{s} = \{\mathfrak{s}_{k+1,\beta;\mathbf{p}}\}$ of T^n-equivariant multi-sections on the moduli spaces $\mathcal{M}_{k+1,l}(X_0, L_0; \beta; \mathbf{p})$ for all classes β with $\mu(\beta) = 2$, see Lemma 6.5 in [FOOO4].

As in the correspondence between (13.4) and (13.5), applying a smooth correspondence in to

$$\begin{tikzcd} & \mathcal{M}_{k+1,l}(X_\varepsilon, L_\varepsilon; \beta; \mathbf{p}) \arrow[dl, "\mathbf{ev}_+"'] \arrow[dr, "\mathrm{ev}_0"] & \\ L^k & & L \end{tikzcd} \tag{13.12}$$

we define

$$\mathfrak{q}_{k,\ell;\beta}\left(\mathbf{p}; b^{\otimes k}\right) := (\mathrm{ev}_0)! \left(\mathbf{ev}_+^*(\pi_1^* b \otimes \cdots \otimes \pi_k^* b)\right).$$

Since X_ε is Fano, Lemma 13.2 and (13.7) allows us to remain $\mathcal{M}_{k+1;\ell}(X_\varepsilon, L_\varepsilon, \beta; \mathbf{p})$ empty if one of the followings is satisfied.

$$\begin{cases} (1)\ \mu(\beta) < 0, \\ (2)\ \mu(\beta) = 0 \text{ and } \beta \neq 0, \\ (3)\ \beta = 0 \text{ and } l > 0. \end{cases} \tag{13.13}$$

Because of the compatibility of the forgetful map forgetting the boundary marked points, see [Fuk, Section 5],

$$\mathfrak{q}_{k,\ell;\beta}\left(\mathbf{p}; b^{\otimes k}\right) = \frac{1}{k!}(\partial\beta \cap b)^k \cdot \mathfrak{q}_{0,\ell;\beta}\left(\mathbf{p}; 1\right). \tag{13.14}$$

Since any moduli spaces satisfying one of the conditions in (13.13) are empty, $\mathfrak{q}_{0,\ell;\beta}(\mathbf{p}; 1)$ represents a cycle, which yields that the Lagrangian L_ε is weakly unobstructed with respect to $\mathfrak{b}$ in (13.8). Passing to the canonical model [FOOO1, FOOO2], we obtain that

$$\mathfrak{q}_{0,\ell;\beta}(\mathbf{p}; 1) = n_\beta(\mathbf{p}) \cdot \mathrm{PD}[L] \tag{13.15}$$

for some $n_\beta(\mathbf{p}) \in \mathbb{Q}$. As a consequence, we obtain that every 1-cochain is a weak bounding cochain with respect to $\mathfrak{b}$. In particular, the potential function with bulk is defined on $H^1(L_\varepsilon; \Lambda_+)$.

Under our situation, $n_\beta(\mathbf{p})$ is well-defined. Especially when $\dim \mathscr{D}_\bullet = 2n-2$, $\mu(\beta) = 2$, we recall that this is precisely the situation where the divisor axiom of the Gromov-Witten theory applies, see [CK, p.193] and [FOOO4, Lemma 9.2]. In particualr, we can calculate $n_\beta(\mathbf{p})$ in the homology level and therefore

$$n_\beta(\mathbf{p}) = n_\beta \cdot \prod_{i=1}^{|\mathbf{p}|} \left(\beta \cap \mathscr{D}_{\mathbf{p}(i)}\right). \tag{13.16}$$

Following [FOOO4, Section 7] and using (13.14), (13.15), (13.16) and (13.9), we obtain that

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathfrak{m}_k^{\mathfrak{b}}\left(b^{\otimes k}\right) &:= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \sum_{\beta;\mu(\beta)=2} \frac{1}{\ell!}\, \mathfrak{q}_{k,\ell;\beta}\left(\mathfrak{b}^{\otimes \ell}; b^{\otimes k}\right) T^{\omega(\beta)/2\pi} \\
&= \sum_{\ell=0}^{\infty} \sum_{\mathbf{p};|\mathrm{p}|=\ell} \sum_{\beta;\mu(\beta)=2} \exp\left(\partial\beta \cap b\right) \cdot \frac{1}{\ell!}\, \mathfrak{b}^{\mathbf{p}}\, \mathfrak{q}_{0,\ell;\beta}\left(\mathbf{p};1\right) T^{\omega(\beta)/2\pi} \\
&= \sum_{\beta;\mu(\beta)=2} \left(\sum_{\ell=0}^{\infty} \sum_{\mathbf{p};|\mathrm{p}|=\ell} \exp\left(\partial\beta \cap b\right) \cdot \frac{1}{\ell!}\, \mathfrak{b}^{\mathbf{p}}\, n_\beta(\mathbf{p}) \right) T^{\omega(\beta)/2\pi} \cdot PD[L] \\
&= \sum_{\beta;\mu(\beta)=2} n_\beta \cdot \left(\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{\mathbf{p};|\mathrm{p}|=\ell} \prod_{i=1}^{|\mathbf{p}|} \mathfrak{b}_{\mathbf{p}(i)} \left(\beta \cap \mathscr{D}_{\mathbf{p}(i)}\right) \right) \cdot \exp(\partial\beta \cap b) \cdot T^{\omega(\beta)/2\pi} \cdot PD[L] \\
&= \sum_{\beta;\mu(\beta)=2} n_\beta \cdot \exp\left(\beta \cap \mathfrak{b}\right) \cdot \exp(\partial\beta \cap b) \cdot T^{\omega(\beta)/2\pi} \cdot PD[L]
\end{aligned}$$

where $\mathfrak{b}^{\mathbf{p}} = \prod_{i=1}^{\ell} \mathfrak{b}_{\mathbf{p}(i)}$. Finally, incorporating with deformation of non-unitary flat line bundle by Cho [Cho2], we can extend the domain of the bulk-deformed potential function to $H^1(L_\varepsilon;\Lambda_0)$. This completes the proof of Theorem 13.6.

REFERENCES

[AFOOO] Mohammed Abouzaid, Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, in preparation. 2

[ABM] Miguel Abreu, Matthew Borman, Dusa McDuff, *Displacing Lagrangian toric fibers by extended probes.* Algebr. Geom. Topol. **14** (2014), no. 2, 687–752. 42

[ACK] Byung Hee An, Yunhyung Cho, Jang Soo Kim, *On the f-vectors of Gelfand-Cetlin polytopes.* European J. Combin. **67** (2018), 61–77. 2, 9, 10

[At] Michael Atiyah, *Convexity and commuting Hamiltonians.* Bull. London Math. Soc. **14** (1982), 1–15. 12

[Au] Michéle Audin, *Topology of Torus actions on symplectic manifolds Second revised edition.* Progress in Mathematics, **93**. Birkhäuser Verlag, Basel (2004). 5, 6

[BCKV] Victor Batyrev, Ionuţ Ciocan-Fontanine, Bumsig Kim, Duco van Straten, *Mirror symmetry and toric degenerations of partial flag manifolds.* Acta Math., **184** (1) (2000), 1-39. 2, 4, 9, 10, 40, 50

[Be] D. N. Bernstein, *The number of roots of a system of equations.* Functional Anal. Appl. **9** (1975), no. 3, (1976) 183–185. 59

[Br] Robert Bryant, *An introduction to Lie groups and symplectic geometry Geometry and quantum field theory (Park City, UT, 1991).* 5–181, IAS/Park City Math. Ser., **1**, Amer. Math. Soc., Providence, RI, (1995). 6

[Ca] Philippe Caldero, *Toric degenerations of Schubert varieties.* Transform. Groups **7** (2002), no. 1, 51–60. 2

[CPU] Kwokwai Chan, Daniel Pomerleano, Kazushi Ueda, *Lagrangian torus fibrations and homological mirror symmetry for the conifold.* Comm. Math. Phys. **341** (2016), no. 1, 135–178. 55

[Cho1] Cheol-Hyun Cho, *Holomorphic discs, spin structures and the Floer cohomology of the Clifford torus.* Int. Math. Res. Not. **35** (2004), 1803-1843. 44

[Cho2] Cheol-Hyun Cho, *Non-displaceable Lagrangian submanifolds and Floer cohomology with non-unitary line bundle.* J. Geom. Phys. **58** (2008), no. 11, 1465–1476. 51, 82

[CHL] Cheol-Hyun Cho, Hansol Hong, Siu-Cheong Lau, in preparation. 3

[CO] Cheol-Hyun Cho, Yong-Geun Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), no. 4, 773–814. 3, 51, 54, 79, 80

[CP] Cheol-Hyun Cho, Mainak Poddar, *Holomorphic orbi-discs and Lagrangian Floer cohomology of symplectic toric orbifolds.* J. Differential Geom. **98** (2014), no. 1, 21–116. 41, 44

[CKO] Yunhyung Cho, Yoosik Kim, Yong-Geun Oh, *Monotone Lagrangians in cotangent bundles.* preprint. 2, 60

[CK] David Cox, Sheldon Katz, *Mirror Symmetry and algebraic geometry.* Math. Surveys and Monographs vol. **68**, Amer. Math. Soc. (1999). 81

[CLS] David Cox, John Little, and Henry Schenck, *Toric varieties.* Graduate Studies in Mathematics, **124**, American Mathematical Society, Providence, RI, (2011). 35

[EHX] Tohru Eguchi, Kentaro Hori, Chuan-Sheng Xiong, *Gravitational Quantum Cohomology.* Internat. J. Modern Phys. A **12** (1997), no. 9, 1743-1782. 2

[EL] Jonathan Evans, Yanki Lekili, *Floer cohomology of the Chiang Lagrangian.* Selecta Math. (N.S.) **21** (2015), no. 4, 1361–1404. 39

[EL2] Jonathan Evans, Yanki Lekili, *Generating the Fukaya categories of Hamiltonian G-manifolds.* preprint(2015), arXiv:1507.05842. 2, 39

[EP] Michael Entov, Leonid Polterovich, *Rigid subsets of symplectic manifolds.* Compos. Math. **145** (2009), no. 3, 773–826. 44

[Ful] William Fulton, *Young Tableaux*, London Math. Soc. Stud. Texts **35**, Cambridge Univ. Press, Cambridge, (1997). 2, 11

[Fuk] Kenji Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory.* Kyoto J. Math. **50** (2010), no. 3, 521–590. 50, 79, 81

[FOOO1] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I & Part II.* AMS/IP Studies in Advanced Math., **46**, Amer. Math. Soc., Providence, RI; International Press, Somerville, MA, (2009). 2, 50, 52, 78, 81

[FOOO2] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Canonical models of filtered A_∞-algebras and Morse complexes.* New perspectives and challenges in symplectic field theory, 201–227, CRM Proc. Lecture Notes, **49**, Amer. Math. Soc., Providence, RI, (2009). 50, 81

[FOOO3] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Lagrangian Floer theory on compact toric manifolds. I.* Duke Math. J. **151** (2010), no. 1, 23–174. 39, 49, 50, 58, 80

[FOOO4] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Lagrangian Floer theory on compact toric manifolds II: bulk deformations.* Selecta Math. (N.S.) **17** (2011), no. 3, 609–711. 41, 50, 52, 53, 58, 78, 79, 80, 81, 82

[FOOO5] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Toric degeneration and nondisplaceable Lagrangian tori in $S^2 \times S^2$.* Int. Math. Res. Not. IMRN (2012), **13**, 2942–2993. 41

[FOOO6] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Kuranishi structure, pseudoholomophric curve, and virtual fundamental chain: Part 1*, preprint(2015), arXiv:1503.5123. 79

[FOOO7] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds.* Astérisque **376** (2016). 50, 52

[FOOO8] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, Kaoru Ono, *Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory.* preprint(2011) arXiv:1105.5123, to appear in Memoirs of the AMS. 41

[GC] I. Gel'fand, M. Cetlin, *Finite-dimensional representations of the group of unimodular matrices. (Russian)* Doklady Akad. Nauk SSSR (N.S.) **71**, (1950). 825–828. 1

[Gi] Alexander Givental, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture. Topics in singularity theory,* 103-115, Amer. Math. Soc. Transl. Ser. 2, 180, Adv. Math. Sci., **34**, Amer. Math. Soc., Providence, RI, 1997. 2

[GL] Nicolae Gonciulea, Venkatramani Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties.* Transform. Groups **1** (1996), no. 3, 215–248. 2, 50

[GS1] Victor Guillemin, Shlomo Sternberg, *Convexity properties of the moment mapping.* Invent. Math. **67** (1982), 491–513. 12

[GS2] Victor Guillemin, Shlomo Sternberg, *The Gel'fand-Cetlin system and quantization of the complex flag manifolds.* J. Funct. Anal. **52** (1983) no. 1, 106–128. 1, 3, 7, 8, 9

[HJ] Roger Horn, Charles Johnson, *Matrix analysis. Second edition.* Cambridge University Press, Cambridge (2013). 46

[HaK] Megumi Harada, Kiumars Kaveh, *Integrable systems, toric degenerations and Okounkov bodies.* Invent. Math. **202** (2015), no. 3, 927–985. 2, 7, 32

[HoK] Jin Hong, Seok-Jin Kang *Introduction to quantum groups and crystal bases.* Graduate Studies in Mathematics, **42**, American Mathematical Society, Providence, RI, (2002). 2

[IST] Hiroshi Iriyeh, Takashi Sakai, Hiroyuki Tasaki, *Lagrangian Floer homology of a pair of real forms in Hermitian symmetric spaces of compact type.* J. Math. Soc. Japan **65** (2013), no. 4, 1135–1151. 39

[KLS] Yoosik Kim, Jaeho Lee, Fumihiko Sanda, *Detecting non-displaceable toric fibers on compact toric manifolds via tropicalizations*, preprint (2015), arXiv:1507.06132. 58

[Ko] Mikhail Kogan, *Schubert geometry of flag varieties and Gelfand-Cetlin theory.* Thesis (Ph.D.) Massachusetts Institute of Technology. 2000. 3, 11, 13, 53, 54

[KnM] Allen Knutson, Ezra Miller, *Gröbner geometry of Schubert polynomials.* Ann. of Math. (2) **161** (2005), no. 3, 1245–1318. 56

[KoM] Mikhail Kogan, Ezra Miller, *Toric degeneration of Schubert varieties and Gelfand-Tsetlin polytopes.* Adv. Math. **193** (2005), no. 1, 1–17. 2, 3, 4, 33, 50, 53, 54, 56

[Ku] A. G. Kušnirenko, *Newton polyhedra and Bezout's theorem.* Functional Anal. Appl. **10** (1976), no. 3, 233–235. 59

[LS] Eugene Lerman, Reyer Sjamaar, *Stratified symplectic spaces and reduction.* Ann. of Math. (2) **134** (1991), no. 2, 375–422. 32

[McD] Dusa McDuff, *Displacing Lagrangian toric fibers via probes.*, Low-dimensional and symplectic topology, 131–160, Proc. Sympos. Pure Math., **82**, Amer. Math. Soc., Providence, RI, (2011). 42

[NNU1] Takeo Nishinou, Yuichi Nohara, Kazushi Ueda, *Toric degenerations of Gelfand-Cetlin systems and potential functions.* Adv. Math. **224** (2) (2010), 648–706. 2, 4, 6, 7, 8, 9, 11, 13, 24, 32, 33, 34, 35, 38, 40, 50, 51, 52, 53, 54, 56, 78, 79, 80

[NS] Takeo Nishinou, Bernd Siebert, *Toric degenerations of toric varieties and tropical curves.* Duke Math. J. **135** (1) (2006), 1–51. 79

[NU1] Yuichi Nohara, Kazushi Ueda, *Toric degenerations of integrable systems on Grassmannians and polygon spaces.* Nagoya Math. J. **214** (2014), 125-168. 2

[NU2] Yuichi Nohara, Kazushi Ueda, *Floer cohomologies of non-torus fibers of the Gelfand-Cetlin system.* J. Symplectic. Geom. **14** (2016), No. 4, 1251–1293. 2, 3, 18, 39, 40, 44

[Oh] Yong-Geun Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. III. Arnold-Givental conjecture.* The Floer memorial volume, 555–573, Progr. Math., **133**, Birkhäuser, Basel, (1995). 39

[Pa] Milena Pabiniak, *Displacing (Lagrangian) submanifolds in the manifolds of full flags.* Adv. Geom. **15** (2015), no. 1, 101–108. 3, 40, 42, 44

[Ri] Konstanze Rietsch, *A mirror symmetric construction of $qH*_T(G/P)_{(q)}$*, Adv. Math. **217** (2008), no. 6, 2401–2442. 3

[Ru] Wei-Dong Ruan, *Lagrangian torus fibration of quintic Calabi-Yau hypersurfaces. II. Technical results on gradient flow construction.* J. Symplectic Geom. **1** (2002), no. 3, 435–521. 4, 32, 34

[St] Norman Steenrod, *The topology of fibre bundles. Reprint of the 1957 edition.* Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ, (1999). 38

[Th] William Thurston, *Some simple examples of symplectic manifolds.* Proc. Amer. Math. Soc. **55** (1976), no. 2, 467–468. 37

[Vi] Renato Vianna, *Continuum families of non-displaceable Lagrangian tori in $(\mathbb{C}P^1)^{2m}$.* preprint(2016), arXiv:1603.02006. 41

[WW] Glen Wilson, Chris Woodward, *Quasimap Floer cohomology for varying symplectic quotients.* Canad. J. Math. **65** (2013), no. 2, 467–480. 41, 44

[Wo] Chris Woodward, *Gauged Floer theory of toric moment fibers.* Geom. Funct. Anal. **21** (2011), no. 3, 680–749. 41

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