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Uniform in time bounds for a stochastic hybrid system with fast periodic sampling and small white-noise

UNIFORM-IN-TIME BOUNDS FOR A STOCHASTIC HYBRID SYSTEM WITH FAST PERIODIC SAMPLING AND SMALL WHITE-NOISE

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Abstract. We study the asymptotic behavior, uniform-in-time, of a non-linear dynamical system under the combined effects of fast periodic sampling with period δ and small white noise of size ε, $0 < \varepsilon, \delta \ll 1$. The dynamics depend on both the current and recent measurements of the state, and as such it is not Markovian. Our main results can be interpreted as Law of Large Numbers (LLN) and Central Limit Theorem (CLT) type results. LLN type result shows that the resulting stochastic process is close to an ordinary differential equation (ODE) uniformly in time as $\varepsilon, \delta \searrow 0$. Further, in regards to CLT, we provide quantitative and uniform-in-time control of the fluctuations process. The interaction of the small parameters provides an additional drift term in the limiting fluctuations, which captures both the sampling and noise effects. As a consequence, we obtain a first-order perturbation expansion of the stochastic process along with time-independent estimates on the remainder. The zeroth- and first-order terms in the expansion are given by an ODE and SDE, respectively. Simulation studies that illustrate and supplement the theoretical results are also provided.

1. Introduction

In this paper, we consider the asymptotic analysis of a controlled stochastic differential equation (SDE)

$$dX_t^{\varepsilon,\delta} = \left\{ f(X_t^{\varepsilon,\delta}) + \kappa(X_{\pi_\delta(t)}^{\varepsilon,\delta}) \right\} dt + \varepsilon\sigma(X_t^{\varepsilon,\delta})dW_t, \quad X_0^{\varepsilon,\delta} = x_0, \tag{1}$$

as $\varepsilon, \delta \searrow 0$ and $t \in [0, \infty)$. Here, the parameters ε and $\delta = \delta(\varepsilon)$, $0 < \varepsilon, \delta \ll 1$, correspond to the size of noise and the rate of state measurements, respectively. The operator $\pi_\delta(t) \triangleq \delta\lfloor t/\delta \rfloor$ transforms the continuous time $t \in [0, \infty)$ to the integer multiple of δ and the process $W = \{W_t : t \geq 0\}$ represents a standard Brownian motion. For the detailed assumptions on the coefficients of (1) see Section 2.1. In our problem set-up, equation (1) can be thought of as a small random perturbation of a non-linear control system

$$dx = \{f(x) + u\}dt, \quad x(0) = x_0 \tag{2}$$

with a feedback control law $u = \kappa(x)$ using its sample-and-hold implementation. In our model, by the sample-and-hold implementation of the control function u, we mean u is updated through state measurements at the time instants $k\delta$ and it remains fixed throughout in the sampling interval $[k\delta, (k+1)\delta)$, $k \in \mathbb{Z}^+$. We note that the instantaneous value of $dX_t^{\varepsilon,\delta}$ in (1) depends on both the current state value $X_t^{\varepsilon,\delta}$ and the most recent measurement $X_{\pi_\delta(t)}^{\varepsilon,\delta}$. Hence, it is not a Markovian process.

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The study of system (1) is important as it appears very frequently in control theory where a system of the form $dx = f(x)dt$ is controlled by a digital computer and is perturbed by a small white noise. In such cases, the state of the system is evolved continuously with time whereas the control function requires the state values (samples) only at certain discrete-time instances. This interaction of state evolution at discrete- and continuous-time makes the system a hybrid dynamical system (HDS)[GST12]. An example of HDS is sampled-data systems [YG14] where the measurements of the state are only available at discrete time instants and control input is updated using a digital computer. Much of the study on these systems focused on stability analysis (e.g., [NTC09, YG14]). To the best of our knowledge, there is very little work on the interaction of sampling and noise (see, [DP21, DP23] for the recent results in this direction). Hence, in this paper, we aim to explore the asymptotic analysis of a HDS under the combined effects of sampling and noise, uniform-in-time.

Part of the contribution of this work is in studying the asymptotic analysis, uniform-in-time (UIT), of the non-linear[1] SDE (1) under the combined influence of fast ($\delta \searrow 0$) periodic sampling and small ($\varepsilon \searrow 0$) state-dependent white-noise. As $\varepsilon, \delta \searrow 0$, we found that the resulting stochastic process $X_t^{\varepsilon,\delta}$ and its rescaled fluctuation process $Z_t^{\varepsilon,\delta} \triangleq \varepsilon^{-1}(X_t^{\varepsilon,\delta} - x_t)$ are close to an ordinary differential equation (ODE) and SDE, respectively. The ODE describing the mean behavior is the closed-loop ODE, regardless of how $\varepsilon, \delta \searrow 0$; however, the fluctuation behavior is found to depend on the relative rates at which $\varepsilon, \delta \searrow 0$. In the fluctuation study, most interesting case is when both the small parameters are comparable in size, i.e., δ/ε tends to a positive constant. In this case, the limiting SDE for the fluctuations has both a diffusive term due to small noise and an extra drift term which captures the sampling effect. More precisely, in our main results (Theorems 2.5 and 2.6), we show that for any natural number $p \geq 1$,

$$\sup_{t\geq 0} \mathbb{E}\left[|X_t^{\varepsilon,\delta} - x_t|^p\right] \leq C\eta(\varepsilon,\delta),$$
$$\sup_{t\geq 0} \mathbb{E}\left[|Z_t^{\varepsilon,\delta} - Z_t|^p\right] \leq C\theta(\varepsilon,\delta),$$

where C is always a *time-independent* positive constant, $\eta(\varepsilon,\delta)$ and $\theta(\varepsilon,\delta)$ are some functions which converge to zero as ε,δ vanish and x_t is the solution of (2). Here, the stochastic process Z_t solves the SDE

$$Z_t = \int_0^t [Df(x_s) + D\kappa(x_s)]Z_s\,ds - \frac{\mathsf{c}}{2}\int_0^t D\kappa(x_s)[f(x_s) + \kappa(x_s)]\,ds + \int_0^t \sigma(x_s)dW_s, \tag{3}$$

where $\mathsf{c} \triangleq \lim_{\varepsilon\searrow 0}\frac{\delta}{\varepsilon}$ (see equation (10)). As a consequence of the CLT (Theorem 2.6), the stochastic process $X_t^{\varepsilon,\delta}$ can be approximated, uniform-in-time, by the process $x_t + \varepsilon Z_t$ as $\varepsilon,\delta \searrow 0$, that is,

$$X_t^{\varepsilon,\delta} \approx x_t + \varepsilon Z_t, \qquad t \in [0,\infty).$$

Indeed, we make this approximation precise by a first-order perturbation expansion of the stochastic process $X_t^{\varepsilon,\delta}$ (i.e., $X_t^{\varepsilon,\delta} = x_t + \varepsilon Z_t + o(\varepsilon^2)$) in the powers of small parameter ε along with the time-independent error bounds on the remainder. Here, the zeroth- and first-order terms are characterized by the ODE $\frac{dx}{dt} = f(x) + \kappa(x)$, $x(0) = x_0$ describing the mean behavior and the SDE (3) which captures the fluctuations about the mean.

The results of this paper are novel mainly in terms of the time-independent convergence bounds and identification of an additional drift term in the fluctuation analysis. The fluctuations allow one to write a first-order perturbation expansion of the stochastic process of interest along with the bounds on the remainder uniform-in-time. The way at which $\epsilon,\delta \searrow 0$ affects the limiting dynamics in a crucial

[1]Equation (1) is classified as a non-linear SDE because the functions f, κ and σ could be any non-linear functions satisfying Assumptions A1, A2 and A3.

way. In addition, our proof of the uniform-in-time CLT shows that what is needed is a control on the behavior of the gradient of the drift of the limiting dynamics (2) via Assumption A3 together with uniform-in-time control of appropriate moments of $X_t^{\varepsilon,\delta}$ and the related uniform-in-time LLN.

The difficulty of the mathematical analysis lies on the careful and diligent analysis and appropriate decomposition needed in order to extract the best possible bounds that are also uniform-in-time. The presence of the sampling term complicates the uniform-in-time analysis of the fluctuations.

In recent decades, the asymptotic behavior of several (multi scale) stochastic systems has been explored by many authors in the form of averaging principle/homogenization. This enables one to obtain simple dynamics of the slowly-varying component by averaging over the quickly varying component. For the fundamental work in this direction, we refer to the seminal papers of Khasminskii [Kha68, Kha66]. Since then, a great amount of work has been developed in the form of different limit theorems. To mention a few, one can see, for example, [FS99, PV01, PV03, PV05, FW12, Spi14, RX21b, RX21a], for the averaging principle and/or fluctuation analysis for SDE with multiple time scales and multiple small parameters in different asymptotic regimes. For the interaction of Large Deviation Principle (which describes the asymptotic behavior of probabilities of rare events in terms of certain rate functions) with stochastic processes, one can see [Ver00, Spi13] and the references therein. Averaging and fluctuation analysis for stochastic partial differential equations are investigated in , e.g., [Cer09, CF09].

The averaging and fluctuation analysis for multiscale systems have typically been considered over finite time horizons (i.e., on time intervals $[0, T]$ for fixed $T < \infty$). Recently, [CDO21, FG20] provided sufficient conditions for *uniform-in-time* convergence of error of the Euler scheme for a SDE. Further, a uniform-in-time averaging principle for a slow-fast fully coupled stochastic system, has been explored in [CDG+22]. For the LDP over infinite time horizons, we refer to a recent article [BZ24].

This paper is organized as follows. In Section 2.1, we present our problem set-up and our main results (Theorems 2.5 and 2.6). In the same section, we mention certain regularity assumptions and a sufficient condition, which enable us to accomplish the uniform time analysis. In addition, we present some potential applications of our results and simulation studies that illustrate and complement the theoretical results. Further, Section 3 is devoted to the proof of Theorem 2.5 through a series of helpful lemmas. In Section 4, we present the proof of our second main result Theorem 2.6. Section 5 has the proofs of a number of supporting lemmas needed in the proof of Theorem 2.6 and mainly related to properly controlling the sampling effect.

Notation and conventions. We list some of the special notations and conventions used throughout this manuscript. The symbol $\triangleq$ is read "is defined to equal." We denote the set of all positive integers, non-negative integers and real numbers by $\mathbb{N}$, $\mathbb{Z}^+$, and $\mathbb{R}$, respectively. For any $m \in \mathbb{N}$, and a given function $h : \mathbb{R} \to \mathbb{R}$, $D^m h(x)$ represents the m^{th}-derivative of h at the point $x \in \mathbb{R}$; of course, in the higher dimensions $Dh(x)$ stands for the Jacobian matrix. Throughout this manuscript, we use $\pi_\delta(t)$ for $\delta\lfloor t/\delta \rfloor$ and $\|\cdot\|_\infty$ for the sup norm. The set $C_b^1(\mathbb{R})$ denotes the collection of all real-valued continuous functions whose first derivatives are bounded. For a random variable Y, which has a normal distribution with parameters μ and ξ^2, we write $Y \sim \mathscr{N}(\mu, \xi^2)$. Throughout the paper, the letter C denotes a positive constant which may depend on various parameters *except for* the time parameter t and the small parameters ε and δ; the value of C may change from line to line. In the statement of results, the constant C will be denoted with a subscript (like, $C_{2.5}$). On many occasions in this paper, we will use the following fundamental inequalities without explicitly mentioning it: for any $n \in \mathbb{N}$, $p > 0$ and positive real numbers $a_1, \cdots a_n$,

$$a_1 a_2 \cdots a_n \le C(a_1^n + \cdots + a_n^n), \quad \text{and} \quad (a_1 + \cdots + a_n)^p \le C(a_1^p + \cdots + a_n^p).$$

2. Problem Statement, Main Results and Application

2.1. **Problem Statement and Assumptions.** We consider a nonlinear dynamical system affine in control $\frac{dx}{dt} = f(x) + u$, $x(0) = x_0$, where $x(t) : [0, \infty) \to \mathbb{R}$ represents the state of the system and $u \in \mathbb{R}$ is a control input. Throughout this analysis, a state-feedback control law $u = \kappa(x)$ is a priori fixed to get the closed-loop system

$$\frac{dx}{dt} = f(x) + \kappa(x), \quad x(0) = x_0. \tag{4}$$

We now use a sample-and-hold implementation of the control law $u = \kappa(x)$. By this we mean, for a fixed $\delta > 0$, the state x of system (4) is measured at the periodic sampling time instants $k\delta$, $k \in \mathbb{Z}^+$ and the control is updated according to the feedback control law $u = \kappa(x_{k\delta})$ and is held fixed in equation (4) over the sampling interval $[k\delta, (k+1)\delta)$. In this case, the dynamics of system (4), in the interval $[k\delta, (k+1)\delta)$, is governed by the differential equation $\frac{dx_t^\delta}{dt} = f(x_t^\delta) + \kappa(x_{k\delta}^\delta)$ with the initial conditions $x_{k\delta}^\delta = x_{k\delta-}^\delta$ and $x_{0-}^\delta = x_0^\delta = x_0$. Further, for the analysis purposes, we rewrite the dynamics of x_t^δ using the time-discretization function $\pi_\delta(t) \triangleq \delta\lfloor t/\delta \rfloor$, $t \in [0, \infty)$ to get

$$\frac{dx_t^\delta}{dt} = f(x_t^\delta) + \kappa(x_{\pi_\delta(t)}^\delta), \quad x_0^\delta = x_0. \tag{5}$$

One can note here that the instantaneous rate $\frac{dx_t^\delta}{dt}$ depends on both the current value x_t^δ and the most recent sample of the state, that is, $x_{\pi_\delta(t)}^\delta$ and we expect as $\delta \searrow 0$, the dynamics of x_t^δ converges to x_t solving (4).

We would now like to explore the situation when the system (5) is subjected to a small white-noise effect. To make things precise, we assume that the process $W = \{W_t : t \geq 0\}$, defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, represents a one-dimensional Brownian motion and the parameter $\varepsilon > 0$ corresponds to the size of noise. The resulting dynamics of system (5) is now described by the stochastic process $X^{\varepsilon,\delta} = \{X_t^{\varepsilon,\delta} : t \geq 0\}$ which solves the SDE

$$dX_t^{\varepsilon,\delta} = \left\{ f(X_t^{\varepsilon,\delta}) + \kappa(X_{\pi_\delta(t)}^{\varepsilon,\delta}) \right\} dt + \varepsilon\sigma(X_t^{\varepsilon,\delta}) dW_t, \quad X_0^{\varepsilon,\delta} = x_0. \tag{6}$$

Throughout this paper, the functions $f : \mathbb{R} \to \mathbb{R}$, $\kappa : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ in model (6) are assumed to satisfy the Assumptions A1, A2, A3 below and that they yield a unique and, at least, weak solution to (6).

Assumption A1. *The mapping f satisfies local polynomial growth Lipschitz continuity condition and the functions κ, σ are assumed globally Lipschitz continuous, i.e., for all $x, y \in \mathbb{R}$, there exist positive numbers q, ξ and μ such that*

$$|f(x) - f(y)| \leq [\xi(|x|^q + |y|^q) + \mu]|x - y|,$$
$$|\kappa(x) - \kappa(y)| \leq L_\kappa|x - y|,$$
$$|\sigma(x) - \sigma(y)| \leq L_\sigma|x - y|,$$

where L_κ, L_σ are the global Lipschitz constants for κ, σ respectively. In addition, we assume f satisfies a contractive Lipschitz continuity condition. In particular, we assume that there is $\lambda > 0$ such that for $x, y \in \mathbb{R}$, we have

$$(x - y) \cdot (f(x) - f(y)) \leq -\lambda|x - y|^2. \tag{7}$$

Furthermore, we assume that the constants λ, L_κ are such that $\frac{\lambda}{2} - L_\kappa > 0$.

An important consequence of (7) is that there exist positive numbers α, β so that

$$x \cdot f(x) \leq -\alpha|x|^2 + \beta \quad \text{for} \quad x \in \mathbb{R}. \tag{8}$$

Assumption A2. *The functions κ, σ are uniformly bounded, i.e., there exists $\gamma < \infty$ such that*

$$|\sigma(x)| + |\kappa(x)| \leq \gamma, \quad x \in \mathbb{R}.$$

Assumption A3. *We assume that f, κ are twice differentiable and that for $m \in \{1, 2\}$, there exists a time-independent positive constant C such that*

$$\sup_{t\geq 0} \int_0^t e^{\int_s^t m[Df(x_u)+D\kappa(x_u)]\,du}\,ds \leq C.$$

The second derivative of κ is assumed to be uniformly bounded whereas the second derivative of f is assumed to grow polynomially.

A few remarks on Assumptions A1, A2 and A3 are in order.

Remark 2.1. Assumptions A1, A2 and A3 on f are satisfied, for example, by the functions $f(x) = -\lambda x$, $f(x) = -x^3 - \lambda x$ and bounded, differentiable functions κ with sufficiently small Lipschitz constant.

We also note that the contractive Lipschitz condition is a consequence of the one-sided Lipschitz condition with negative "one-sided Lipschitz" constant. In particular, the function f such that for $x > y$ the relation holds

$$f(x) - f(y) \leq -\lambda(x - y),$$

will also satisfy (7). Note that for such f that are also differentiable, we have that $\sup_x f'(x) = -\lambda$.

Remark 2.2. Assumption (7) is used in the proof of Theorem 2.5 (which proves the uniform convergence of $X_t^{\varepsilon,\delta}$ to x_t) and is needed in order to control the magnitude of the term $|X_t^{\varepsilon,\delta} - x_t|$ for any given $t \in [0, \infty)$.

Remark 2.3. Relation (8) is used in Lemma 3.1 to prove the uniform moment bound for $X_t^{\varepsilon,\delta}$ and the uniform bound for x_t. We note that a simple inspection of the proof of Lemma 3.1 shows that the claim holds if we only assume (8) to be true for $|x| > R$ for some $R > 0$. We chose to use (8) as a consequence of (7) instead of using this more general condition because, despite our efforts, we could not relax accordingly Assumption (7) that is used for Theorem 2.5. This is due to the fact that in Theorem 2.5, we need to appropriately control the term $(X_t^{\varepsilon,\delta} - x_t)[f(X_t^{\varepsilon,\delta}) - f(x_t)]$ for all $t \in [0, \infty)$ and not to just control the behavior for large $|X_t^{\varepsilon,\delta}|$. In particular, bounding moments of $|X_t^{\varepsilon,\delta} - x_t|$ is not enough, we need those moments to go to zero as $\varepsilon, \delta \searrow 0$.

However, in Subsection 2.5, we demonstrate our principal results via simulation studies also on examples that do not necessarily satisfy all of the assumptions, demonstrating that our theoretical result is expected to hold broader.

Remark 2.4. Assumption A3 is only used in the proof of the uniform-in-time CLT, Theorem 2.6. The proof of Theorem 2.6 makes it clear, that besides Assumption A3, one mainly needs to have the uniform moment bounds and uniform law of large numbers result of Lemma 3.1 and Theorem 2.5 respectively and then the proof of the uniform-in-time CLT goes through. This means that the uniform-in-time CLT will hold under a proper control of the derivative of the drift functions via Assumption A3 and any assumptions that guarantee the validity of Lemma 3.1 and Theorem 2.5. The assumption on the existence of the second derivatives of f and κ stems from the need to control the fluctuations in the technical lemmas of Subsection 5.2, which we accomplish through appropriate Taylor expansions.

2.2. **Main Results.** We now state our first main result.

Theorem 2.5. *(Law of Large Numbers Type Result) Let x_t and $X_t^{\varepsilon,\delta}$ be the solutions of (4) and (6), respectively satisfying Assumptions A1, A2. Then, for any $p \in \mathbb{N}$ and for all $\varepsilon, \delta > 0$ sufficiently small, there exists a time-independent positive constant $C_{2.5}$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left[|X_t^{\varepsilon,\delta} - x_t|^p\right] \leq C_{2.5}(\delta^p + \varepsilon^p + \delta^{\frac{p}{2}}\varepsilon^p). \tag{9}$$

This result essentially gives the rate of convergence of the stochastic process $X_t^{\varepsilon,\delta}$ to its deterministic counterpart x_t uniformly in time as $\varepsilon, \delta \searrow 0$. This theorem can be interpreted as a Law of Large Numbers (LLN) type result. We present the proof of Theorem 2.5 in Section 3.

We next explore the fluctuation analysis of $X_t^{\varepsilon,\delta}$ uniform-in-time about its mean x_t. Here, the fluctuation behavior is found to vary, depending on the relative rates at which the two small parameters ε, δ tend to zero. To make this precise, we define two different regimes:

$$\mathsf{c} \triangleq \lim_{\varepsilon\searrow 0} \delta_\varepsilon/\varepsilon \begin{cases} = 0 & \text{Regime 1,} \\ \in (0,\infty) & \text{Regime 2,} \end{cases} \tag{10}$$

where, we assume $\delta = \delta_\varepsilon$ and $\lim_{\varepsilon\searrow 0} \delta_\varepsilon/\varepsilon$ exists in $[0,\infty)$. For Regimes 1 and 2, we consider the rescaled fluctuation process

$$Z_t^{\varepsilon,\delta} \triangleq \frac{X_t^{\varepsilon,\delta} - x_t}{\varepsilon}. \tag{11}$$

Of course, the fluctuation process $Z_t^{\varepsilon,\delta}$ solves the stochastic integral equation (22). We now present our second main result which can be interpreted as a Central Limit Theorem (CLT) type result. This result focuses on the limiting behavior, uniform-in-time, of the process $Z_t^{\varepsilon,\delta}$ as ε, δ approach to zero.

Theorem 2.6. *(Central Limit Theorem Type Result) Let x_t and $X_t^{\varepsilon,\delta}$ be the solutions of (4) and (6), respectively satisfying Assumptions A1, A2, A3 and the fluctuation process $Z_t^{\varepsilon,\delta}$ be defined in equation (11). Suppose that we are in Regime $i \in \{1,2\}$, i.e., $\lim_{\varepsilon\searrow 0}\delta_\varepsilon/\varepsilon = \mathsf{c} \in [0,\infty)$. Let $Z = \{Z_t : t \geq 0\}$ be the unique solution of*

$$Z_t = \int_0^t [Df(x_s) + D\kappa(x_s)]Z_s\,ds - \frac{\mathsf{c}}{2}\int_0^t D\kappa(x_s)[f(x_s) + \kappa(x_s)]\,ds + \int_0^t \sigma(x_s)dW_s. \tag{12}$$

Then, for any integer $p \geq 1$, there exists a time-independent positive constant $C_{2.6}$ such that for all sufficiently small $\varepsilon, \delta > 0$, we have

$$\sup_{t\geq 0}\mathbb{E}\left[|Z_t^{\varepsilon,\delta} - Z_t|^p\right] = \frac{1}{\varepsilon^p}\sup_{t\geq 0}\mathbb{E}\left[|X_t^{\varepsilon,\delta} - x_t - \varepsilon Z_t|^p\right] \leq C_{2.6}\left[\frac{\delta^{2p}}{\varepsilon^p} + \left|\frac{\delta}{\varepsilon} - \mathsf{c}\right|^p + \left(\frac{\delta^p}{\varepsilon^p} + 1\right)(\delta^p + \varepsilon^p + \delta^{\frac{p}{2}}\varepsilon^p)^{\frac{1}{2}} + \delta^{p/2} + \varepsilon^p\right]. \tag{13}$$

Remark 2.7. (Interpretation) The result (Theorem 2.6) can be interpreted as follows: The expression $\sup_{t\geq 0}\mathbb{E}\left[|X_t^{\varepsilon,\delta} - x_t - \varepsilon Z_t|^p\right]$ tends to zero faster than ε^p does; hence $\sup_{t\geq 0}\mathbb{E}\left[|X_t^{\varepsilon,\delta} - x_t - \varepsilon Z_t|^p\right] = o(\varepsilon^p)$. The latter enables us to approximate in the limit the non-Markovian (due to a memory of length δ) process $X_t^{\varepsilon,\delta}$ by the Markov process $x_t + \varepsilon Z_t$.

2.3. **Assumptions for a generalization of our model.** In this section, we provide the conditions under which our results hold for a generalized version of equation (6) in higher dimensions. We consider the SDE and ODE

$$dX_t^{\varepsilon,\delta} = \left\{ f(X_t^{\varepsilon,\delta}) + g(X_t^{\varepsilon,\delta})\kappa(X_{\pi_\delta(t)}^{\varepsilon,\delta}) \right\} dt + \varepsilon\sigma(X_t^{\varepsilon,\delta})dW_t, \tag{14}$$

$$\frac{dx_t}{dt} = f(x_t) + g(x_t)\kappa(x_t), \tag{15}$$

where, for $d, n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $\kappa : \mathbb{R}^n \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are certain regular functions and W is an n-dimensional Brownian motion.

The matrix-valued function $\Phi(t) \triangleq e^{\int_0^t [Df(x_s)+D(g\kappa)(x_s)]\, ds}$ satisfies the matrix differential equations

$$\frac{d}{dt}\Phi(t) = [Df(x_t) + D(g\kappa)(x_t)]\Phi(t), \quad \Phi(0) = I. \tag{16}$$

In equation (16), x_t solves ODE (15), I represents the identity matrix, and $D(g\kappa)(x_t) \triangleq g(x_t)D\kappa(x_t)+\sum_{i=1}^d Dg_i(x_t)\cdot\kappa_i(x_t)$ for the column vectors $g_i \in \mathbb{R}^n$ of matrix g and the entries $\kappa_i \in \mathbb{R}$ of matrix κ, $i = 1, \cdots, d$.

We now provide the sufficient conditions below under which one can get Theorems 2.5 and 2.6 for model (14), and these conditions can be thought as corresponding to Assumptions A1, A2 and A3. In the following assumptions, let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ represent the standard inner product and Euclidean norm of vectors, respectively. For any matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, $|A|_F \triangleq \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ represents the Frobenious norm.

Assumption (A1)′. The mapping f satisfies the local polynomial growth Lipschitz continuity condition and the functions κ, σ and g are assumed globally Lipschitz continuous, i.e., for all $x, y \in \mathbb{R}^n$, there exist positive numbers q, ξ, L and μ such that

$$|f(x) - f(y)| \leq [\xi(|x|^q + |y|^q) + \mu]\,|x - y|,$$
$$|\kappa(x) - \kappa(y)| + |\sigma(x) - \sigma(y)|_F + |g(x) - g(y)|_F \leq L|x - y|.$$

Furthermore, we assume f satisfies a contractive Lipschitz continuity. In particular, we assume that there exists $0 < \lambda < \infty$ such that for all $x, y \in \mathbb{R}^n$, we have

$$\langle x - y, f(x) - f(y)\rangle \leq -\lambda|x - y|^2.$$

With $L_{g\kappa}$ denoting the Lipschitz constant associated to the function $g\kappa$, we assume that $\frac{\lambda}{2} - L_{g\kappa} > 0$. As in one-dimensional case, a consequence of the contractive Lipschitz conditions is that there exist positive numbers α and β such that

$$\langle x, f(x)\rangle \leq -\alpha|x|^2 + \beta \quad \text{for} \quad x \in \mathbb{R}^n.$$

Assumption (A2)′. The functions κ, σ and g are uniformly bounded, i.e., there exists $\gamma < \infty$ such that

$$|\kappa(x)| + |\sigma\sigma^\top(x)|_F + |g(x)|_F \leq \gamma, \quad x \in \mathbb{R}^n.$$

Assumption (A3)′. For $m \in \{1, 2\}$, there exists a time-independent positive constant C such that

$$\sup_{t\geq 0} \int_0^t \left[|\Phi(t)\Phi(s)^{-1}|_F\right]^m ds \leq C,$$

where, $\Phi(t) \triangleq e^{\int_0^t [Df(x_s)+D(g\kappa)(x_s)]\, ds}$ is the solution of equation (16).

Since the idea of the proof of Theorems 2.5 and 2.6 with Assumptions (A1)$'$, (A2)$'$, (A3)$'$ is same as with Assumptions A1, A2 and A3, for the sake of presentation, we present the proof only in one-dimensional case with $g = 1$ in (14).

2.4. **An application of Theorem 2.6.** We provide an application of our main result (Theorem 2.6) in Corollary 2.8 below. The latter essentially states that the behavior of the stochastic process $X_t^{\varepsilon,\delta}$ is close (uniform-in-time) to a Gaussian process $x_t+\varepsilon Z_t$ with a specified mean and variance (see equation (17)).

Corollary 2.8. *Let $X_t^{\varepsilon,\delta}$ be the solution of (6) and $V_t^\varepsilon \triangleq x_t + \varepsilon Z_t$, where x_t and Z_t solve (4) and (12), respectively. Then, for any $\varphi \in C_b^1(\mathbb{R})$, and integer $p \geq 1$, there exists a positive constant $C_{2.8}$ (time-independent) such that*

$$\sup_{t\geq 0} \mathbb{E}\left[|\varphi(X_t^{\varepsilon,\delta}) - \varphi(V_t^\varepsilon)|^p\right] \leq \varepsilon^p C_{2.8}\left[\frac{\delta^{2p}}{\varepsilon^p} + \left|\frac{\delta}{\varepsilon} - \mathsf{c}\right|^p + \left(\frac{\delta^p}{\varepsilon^p} + 1\right)(\delta^p + \varepsilon^p + \delta^{\frac{p}{2}}\varepsilon^p)^{\frac{1}{2}} + \delta^{p/2} + \varepsilon^p\right].$$

The stochastic process V_t^ε is a Gaussian process with mean μ_t and variance ξ_t^2 given by

$$\begin{aligned} \mu_t &= x_t - \frac{\mathsf{c}\varepsilon}{2}\int_0^t D\kappa(x_s) e^{\int_s^t [Df(x_u) + D\kappa(x_u)]\, du}[f(x_s) + \kappa(x_s)]\, ds, \\ \xi_t^2 &= \varepsilon^2 \int_0^t e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]\, du}\sigma^2(x_s)\, ds. \end{aligned} \tag{17}$$

Proof. Using Itô's formula in equation (12) (i.e., a limiting SDE for the fluctuation process $Z_t^{\varepsilon,\delta}$), we obtain

$$Z_t = -\frac{\mathsf{c}}{2}\int_0^t D\kappa(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]\, du}[f(x_s) + \kappa(x_s)]\, ds + \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]\, du}\sigma(x_s)\, dW_s.$$

Now, let $V_t^\varepsilon \triangleq x_t + \varepsilon Z_t$, then, $V_t^\varepsilon \sim \mathcal{N}(\mu_t, \xi_t^2)$, where, the expressions for the mean μ_t and variance ξ_t^2 are defined as follows:

$$\mu_t = x_t - \frac{\mathsf{c}\varepsilon}{2}\int_0^t D\kappa(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]\, du}[f(x_s)+\kappa(x_s)]\, ds,\ \xi_t^2 = \varepsilon^2\int_0^t e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]\, du}\sigma^2(x_s)\, ds.$$

Next, for any $\varphi \in C_b^1(\mathbb{R})$, a simple algebra followed by Taylor's theorem yields

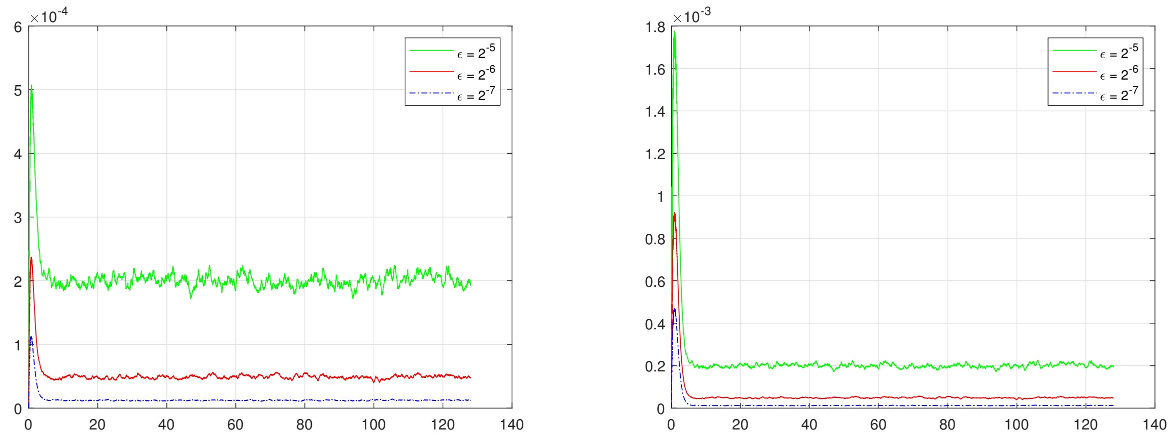
$$\varphi(X_t^{\varepsilon,\delta}) = \varphi\left(\frac{X_t^{\varepsilon,\delta} - x_t}{\varepsilon}\varepsilon + x_t\right) = \varphi\left(\varepsilon Z_t^{\varepsilon,\delta} - \varepsilon Z_t + \varepsilon Z_t + x_t\right) = \varphi(\varepsilon Z_t + x_t) + \varepsilon\varphi'(z)[Z_t^{\varepsilon,\delta} - Z_t],$$

where $z \in \mathbb{R}$ is a point lying on the line segment joining $X_t^{\varepsilon,\delta}$ and $x_t + \varepsilon Z_t$. Hence, for any integer $p \geq 1$, we have

$$\left|\varphi(X_t^{\varepsilon,\delta}) - \varphi(V_t^\varepsilon)\right|^p \leq \varepsilon^p |\varphi'(z)|^p \left|Z_t^{\varepsilon,\delta} - Z_t\right|^p.$$

Finally, using the boundedness of φ' followed by Theorem 2.6, one can obtain the required result. □

2.5. **Numerical Examples and Simulation.** This section is devoted to the numerical illustration of the application of our theoretical result (i.e., Corollary 2.8) in the context of a few simple examples. In Example 1, the functions f and κ satisfy the specified assumptions, however, Examples 2, 3 and 4 demonstrate that our main results can be expected to hold when f satisfies the dissipative condition for large values of x, or the control function κ is not necessarily a bounded function. In this paper, we provide conditions for the generic model under which we could prove the uniform-in-time results. Examples 2, 3 and 4 demonstrate that for concrete models with special structure, one can go beyond the generic assumptions of this paper.



(A) $X_0^{\varepsilon,\delta} = -0.07$ (B) $X_0^{\varepsilon,\delta} = 1.5$

FIGURE 1. For the non-linear model (18), varying effect of $\mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths with different values of ε and two initial conditions $X_0^{\varepsilon,\delta} = -0.07, 1.5$. Here, the error $\mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ and time are represented on the vertical and horizontal axes, respectively.

Remark 2.9. It is important to note that our main results are derived under certain assumptions including that the function f satisfies contractive Lipschitz continuity (Equation (7)) and the control function κ is bounded. The significance of these conditions is as follows. Due to the contractive Lipschitz continuity of f, in Theorem 2.5, we are able to demonstrate that the term $|X_s^{\varepsilon,\delta} - x_s|^p$ tends to zero via handling a challenging term $(X_t^{\varepsilon,\delta} - x_t)[f(X_t^{\varepsilon,\delta}) - f(x_t)]$. We refer to Remark 2.3 for more details. Next, the boundedness of κ is crucial as it is used (for the first-time) in the proof of Lemma 3.1. It helps to control the product involving the sampling term $(X_t^{\varepsilon,\delta})^{p-1}|\kappa(X_{\pi_\delta(t)}^{\varepsilon,\delta}|$, $t \in [0,\infty)$ which is difficult to handle without assuming the boundedness of κ in UIT setting.

Example 1. We consider the non-linear system

$$dX_t^{\varepsilon,\delta} = [-(X_t^{\varepsilon,\delta})^3 - X_t^{\varepsilon,\delta}]dt - \frac{1}{1+e^{-X_{\pi_\delta(t)}^{\varepsilon,\delta}}}dt + \varepsilon dW_t, \tag{18}$$

where it is easy to check that the functions $f(x) = -x^3 - x$, $\sigma(x) = 1$ and $\kappa(x) = -\frac{1}{1+e^{-x}}$ satisfy Assumptions A1, A2 and A3. For the Monte Carlo simulation of the quantity $\sup_{t\geq 0}\left|\mathbb{E}\varphi(X_t^{\varepsilon,\delta}) - \mathbb{E}\varphi(V_t^{\varepsilon})\right|^p$ over the 300 sample paths of Brownian motion, we fix the function $\varphi(x) = x$ and the parameters $T = 2^7, X_0^{\varepsilon,\delta} = -0.07, 1.5$ and $\delta = 2\varepsilon$. Figure 1 and Table 1 show that the absolute error decreases as the values of ε get small.

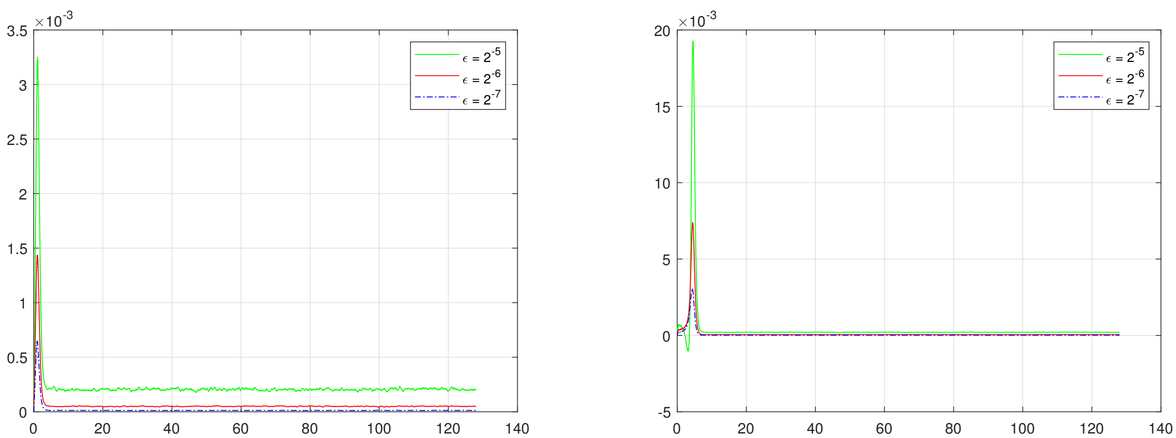
Example 2. We consider the SDE

$$dX_t^{\varepsilon,\delta} = [-(X_t^{\varepsilon,\delta})^3 + X_t^{\varepsilon,\delta}]dt - \frac{1}{1+e^{-X_{\pi_\delta(t)}^{\varepsilon,\delta}}}dt + \varepsilon dW_t. \tag{19}$$

In this case, we illustrate the varying effect of ε (noise size) on the quantity $\sup_{t\geq 0}\left|\mathbb{E}\varphi(X_t^{\varepsilon,\delta}) - \mathbb{E}\varphi(V_t^{\varepsilon})\right|^p$ for the particular function $\varphi(x) = x$, $p = 1$ using Monte Carlo simulation over 300 sample paths of

TABLE 1. The absolute maximum and minimum values of Error $\triangleq \mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths decrease as the values of ε decrease.

Initial condition $X_0^{\varepsilon,\delta} = -0.07$			
ε	$\delta = 2\varepsilon$	|Maximum Error|	|Minimum Error|
2^{-5}	2^{-4}	5.0744×10^{-4}	8.9676×10^{-5}
2^{-6}	2^{-5}	2.3710×10^{-4}	2.5126×10^{-5}
2^{-7}	2^{-6}	1.1243×10^{-4}	6.1564×10^{-6}
Initial condition $X_0^{\varepsilon,\delta} = 1.5$			
2^{-5}	2^{-4}	0.0018	1.7179×10^{-4}
2^{-6}	2^{-5}	9.2053×10^{-4}	3.9899×10^{-5}
2^{-7}	2^{-6}	4.6849×10^{-4}	1.0321×10^{-5}



(A) $X_0^{\varepsilon,\delta} = -0.07$ (B) $X_0^{\varepsilon,\delta} = 1.5$

FIGURE 2. For the non-linear model (19), varying effect of $\mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths with different values of ε and two initial conditions $X_0^{\varepsilon,\delta} = -0.07, 1.5$.

Brownian motion. The process V_t^ε is defined as $x_t + \varepsilon Z_t$. We fix the parameters $T = 2^7, X_0^{\varepsilon,\delta} = -0.07, 1.5$ and $\delta = 2\varepsilon$. For these fixed parameters, Figure 2 and Table 2 demonstrate that as the values of ε decrease, their corresponding errors (which is defined as the mean of $[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over the specified number of sample paths) also decrease.

Example 3. We consider the SDE

$$dX_t^{\varepsilon,\delta} = -3X_t^{\varepsilon,\delta}dt - 0.3166X_{\pi_\delta(t)}^{\varepsilon,\delta}dt + \varepsilon dW_t. \tag{20}$$

Of course, the functions $f(x) = -3x$, $\sigma(x) = 1$ satisfy (7) and Assumption A2 and the function $\kappa(x) = -0.3166x$ has linear growth (not necessarily bounded). In this case, we illustrate the varying effect of ε (noise size) on the quantity $\sup_{t\geq 0}\left|\mathbb{E}\varphi(X_t^{\varepsilon,\delta}) - \mathbb{E}\varphi(V_t^\varepsilon)\right|^p$ for the particular function $\varphi(x) = x$, $p = 1$ using Monte Carlo simulation over 300 sample paths of Brownian motion. The process V_t^ε is defined as $x_t + \varepsilon Z_t$. We fix the parameters $T = 2^7, X_0^{\varepsilon,\delta} = 0.1$ and $\delta = 2\varepsilon$. For these fixed parameters, Figure 3

TABLE 2. The absolute maximum and minimum values of Error $\triangleq \mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths decrease as the values of ε decrease.

Initial condition $X_0^{\varepsilon,\delta} = -0.07$			
ε	$\delta = 2\varepsilon$	|Maximum Error|	|Minimum Error|
2^{-5}	2^{-4}	0.0032	1.346×10^{-4}
2^{-6}	2^{-5}	0.0014	3.366×10^{-5}
2^{-7}	2^{-6}	6.496×10^{-4}	8.4150×10^{-6}
Initial condition $X_0^{\varepsilon,\delta} = 1.5$			
2^{-5}	2^{-4}	0.0193	0.0010
2^{-6}	2^{-5}	0.0074	4.2751×10^{-5}
2^{-7}	2^{-6}	0.0031	1.0880×10^{-5}

and Table 3 demonstrate that as the values of ε decrease, their corresponding errors (which is defined as the mean of $[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over the specified number of sample paths) also decrease.

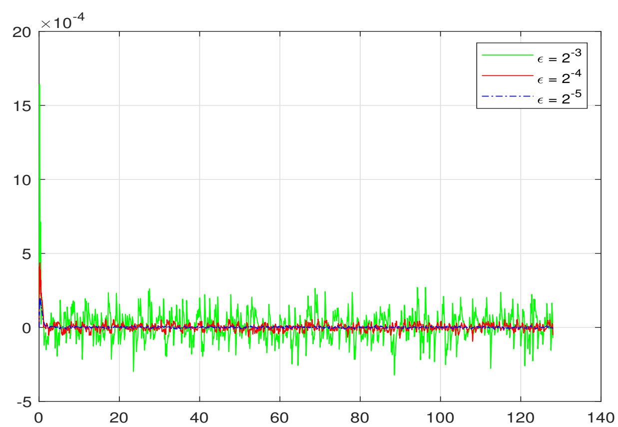


FIGURE 3. For the linear model (20), varying effect of $\mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths with different values of ε. Here, the error $\mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ and time are represented on the vertical and horizontal axes, respectively.

TABLE 3. The absolute maximum and minimum values of Error $\triangleq \mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths decrease as the values of ε decrease.

ε	$\delta = 2\varepsilon$	|Maximum Error|	|Minimum Error|
2^{-3}	2^{-2}	0.00164	3.2175×10^{-4}
2^{-4}	2^{-3}	4.3641×10^{-4}	9.4164×10^{-5}
2^{-5}	2^{-4}	1.9649×10^{-4}	1.8798×10^{-5}

Example 4. We consider the non-linear system

$$dX_t^{\varepsilon,\delta} = \left\{ f(X_t^{\varepsilon,\delta}) + \kappa(X_{\pi_\delta(t)}^{\varepsilon,\delta}) \right\} dt + \varepsilon dW_t, \quad X_0^{\varepsilon,\delta} = 0.1 \tag{21}$$

for the functions $f(x) = \frac{\sin x}{1+x^2} - 3x$, $\sigma(x) = 1$ and $\kappa(x) = \frac{1}{1+e^{-x}} - 5x$. It is easy to check that the mappings f and κ satisfy Assumption A3 as $Df < 0$ and $D\kappa < 0$. For the Monte Carlo simulation of the quantity $\sup_{t\geq 0} \left| \mathbb{E}\varphi(X_t^{\varepsilon,\delta}) - \mathbb{E}\varphi(V_t^\varepsilon) \right|^p$ over the 300 sample paths of Brownian motion, we fix the function $\varphi(x) = x$ and the parameters $T = 2^7, X_0^{\varepsilon,\delta} = 0.1$ and $\delta = 2\varepsilon$. Figure 4 and Table 4 show that the absolute error decreases as the values of ε get small.

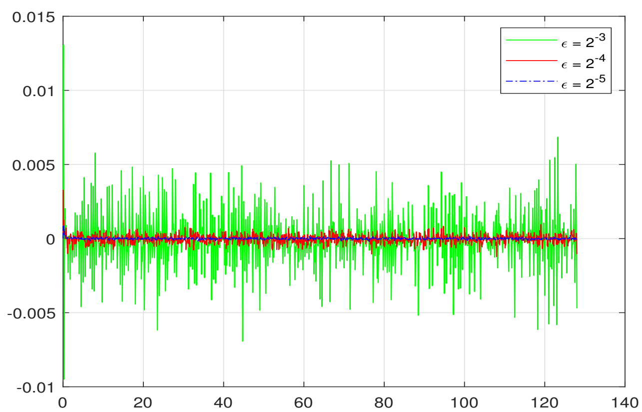


FIGURE 4. For the non-linear model (21), varying effect of $\mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths with different values of ε. Here, the error $\mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ and time are represented on the vertical and horizontal axes, respectively.

TABLE 4. The absolute maximum and minimum values of Error $\triangleq \mathbb{E}[X_T^{\varepsilon,\delta} - x_T - \varepsilon Z_T]$ over 300 sample paths decrease as the values of ε decrease.

ε	$\delta = 2\varepsilon$	|Maximum Error|	|Minimum Error|
2^{-3}	2^{-2}	0.0131	0.0095
2^{-4}	2^{-3}	0.0033	0.0012
2^{-5}	2^{-4}	9.2848×10^{-4}	2.0363×10^{-4}

3. LLN TYPE RESULT: PROOF OF THEOREM 2.5

In this section, we prove Theorem 2.5 (LLN Type Result) under Assumptions A1 and A2. The main building blocks of the proof are Lemmas 3.1 and 3.2 below. Here, Lemma 3.1 gives the uniform-in-time bound for the quantities $\mathbb{E}[|X_t^{\varepsilon,\delta}|^p]$ and x_t. Lemma 3.2 deals with the quantity $\mathbb{E}|X_{\pi_\delta(s)}^{\varepsilon,\delta} - X_s^{\varepsilon,\delta}|^p$. In both the results, the generic constant C is always time-independent.

Lemma 3.1. *Let $X_t^{\varepsilon,\delta}$ and x_t be the solutions of (6) and (4) respectively satisfying Assumptions A1 and A2. Then, for any $p \geq 1$, there exists a positive constant $C_{3.1}$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left[|X_t^{\varepsilon,\delta}|^p\right] + \sup_{t\geq 0} |x_t|^p \leq C_{3.1}.$$

Lemma 3.2. *Suppose that $X_t^{\varepsilon,\delta}$ is the solution of (6) satisfying Assumptions A1 and A2. Then, for any $\varepsilon, \delta > 0$ and even p, there exists a positive constant $C_{3.2}$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left[|X_{\pi_\delta(t)}^{\varepsilon,\delta} - X_t^{\varepsilon,\delta}|^p\right] \leq C_{3.2}[\delta^p + \varepsilon^p \delta^{\frac{p}{2}}].$$

The proofs of these lemmas are provided in Section 3.1. We now prove Theorem 2.5 under Assumptions A1 and A2.

Proof of Theorem 2.5. We start by noting that it suffices to prove the statement for even powers p. For $p = 1$, the required bound follows from Hölder's inequality. So, let us take p to be an even integer. Using the integral representation for the process $X_t^{\varepsilon,\delta}$ and x_t, we get

$$X_t^{\varepsilon,\delta} - x_t = \int_0^t [f(X_s^{\varepsilon,\delta}) - f(x_s)]ds + \int_0^t [\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - \kappa(x_s)]ds + \varepsilon \int_0^t \sigma(X_s^{\varepsilon,\delta})dW_s.$$

Applying Itô's formula to the function $\varphi(t,x) = e^{\frac{tp\lambda}{2}}|x|^p$, we have

$$\begin{aligned} e^{\frac{tp\lambda}{2}}|X_t^{\varepsilon,\delta} - x_t|^p = &\int_0^t \frac{p\lambda}{2} e^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^p\, ds + p\int_0^t e^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^{p-1} \varepsilon\sigma(X_s^{\varepsilon,\delta})\, dW_s \\ &+ \int_0^t pe^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^{p-1} \left[f(X_s^{\varepsilon,\delta}) - f(x_s)\right] ds \\ &+ \int_0^t pe^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^{p-1} \left[\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - \kappa(x_s)\right] ds \\ &+ \frac{p(p-1)}{2} \int_0^t e^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^{p-2} \varepsilon^2 \sigma^2(X_s^{\varepsilon,\delta})\, ds. \end{aligned}$$

Next, using Assumption A1 (in particular, Equation (7)), we obtain

$$\begin{aligned} e^{\frac{tp\lambda}{2}}|X_t^{\varepsilon,\delta} - x_t|^p \leq &\int_0^t \frac{p\lambda}{2} e^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^p\, ds + p\int_0^t e^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^{p-1} \varepsilon\sigma(X_s^{\varepsilon,\delta})\, dW_s \\ &- \lambda \int_0^t pe^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^p\, ds + \int_0^t pe^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^{p-1} \left[\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - \kappa(x_s)\right] ds \\ &+ \frac{p(p-1)}{2} \int_0^t e^{\frac{sp\lambda}{2}} (X_s^{\varepsilon,\delta} - x_s)^{p-2} \varepsilon^2 \sigma^2(X_s^{\varepsilon,\delta})\, ds. \end{aligned}$$

We further take expectation (the stochastic integral is zero due to Lemma 3.1) and use Young's inequality to the term $(X_s^{\varepsilon,\delta} - x_s)^{p-2}\varepsilon^2$ (with the Hölder conjugates $\frac{p}{p-2}$, $\frac{p}{2}$) to get a sufficiently small $\xi > 0$ and boundedness of σ to obtain

$$\begin{aligned} \mathbb{E}\left[e^{\frac{tp\lambda}{2}}|X_t^{\varepsilon,\delta} - x_t|^p\right] \leq &\left(\frac{p\lambda}{2} - \lambda p + \xi \frac{p(p-1)}{2} \frac{(p-2)}{p} \|\sigma^2\|_\infty\right) \int_0^t e^{\frac{sp\lambda}{2}} \mathbb{E}(X_s^{\varepsilon,\delta} - x_s)^p\, ds \\ &+ \int_0^t pe^{\frac{sp\lambda}{2}} \mathbb{E}\left[(X_s^{\varepsilon,\delta} - x_s)^{p-1}[\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - \kappa(x_s)]\right] ds + \varepsilon^p \frac{p(p-1)}{p} \frac{\|\sigma^2\|_\infty}{\xi^{\frac{p-2}{p}}} \int_0^t e^{\frac{sp\lambda}{2}} ds. \end{aligned}$$

Now, in the above expression, first writing $\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)})-\kappa(x_s)$ as $\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)})-\kappa(X^{\varepsilon,\delta}_s)+\kappa(X^{\varepsilon,\delta}_s)-\kappa(x_s)$, then using $(X^{\varepsilon,\delta}_s-x_s)^{p-1}[\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)})-\kappa(x_s)]\le |X^{\varepsilon,\delta}_s-x_s|^{p-1}|\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)})-\kappa(x_s)|$ followed by Lipschitz continuity of κ, we obtain

$$\mathbb{E}\left[e^{\frac{tp\lambda}{2}}|X^{\varepsilon,\delta}_t-x_t|^p\right]\le p\left(-\left(\frac{\lambda}{2}-L_\kappa\right)+\xi\frac{(p-1)(p-2)}{2p}\|\sigma^2\|_\infty\right)\int_0^t e^{\frac{sp\lambda}{2}}\mathbb{E}(X^{\varepsilon,\delta}_s-x_s)^p\,ds$$
$$+L_\kappa\int_0^t pe^{\frac{sp\lambda}{2}}\mathbb{E}\left[|X^{\varepsilon,\delta}_s-x_s|^{p-1}|X^{\varepsilon,\delta}_{\pi_\delta(s)}-X^{\varepsilon,\delta}_s|\right]ds+\frac{\varepsilon^p(p-1)\|\sigma^2\|_\infty}{\xi^{\frac{p-2}{p}}}\int_0^t e^{\frac{sp\lambda}{2}}\,ds.$$

Using the generalized Young's inequality $ab\le \frac{\eta}{r_1}a^{r_1}+\frac{1}{\eta^{r_2/r_1}r_2}b^{r_2}$ with the Hölder conjugates $r_1=\frac{p}{p-1}$, $r_2=p$, any $\eta>0$ and Lemma 3.2, we obtain for $\eta>0$ and $\xi>0$ to be chosen

$$\mathbb{E}\left[e^{\frac{tp\lambda}{2}}|X^{\varepsilon,\delta}_t-x_t|^p\right]\le -p\left(\frac{\lambda}{2}-\left(1+\eta\frac{p-1}{p}\right)L_\kappa-\xi\frac{(p-1)(p-2)}{2p}\|\sigma^2\|_\infty\right)\int_0^t e^{\frac{sp\lambda}{2}}\mathbb{E}[|X^{\varepsilon,\delta}_s-x_s|^p]\,ds$$
$$+C[\varepsilon^p+\delta^p+\varepsilon^p\delta^{\frac{p}{2}}]\int_0^t e^{\frac{sp\lambda}{2}}\,ds,$$

for some constant $C<\infty$ that depends on $p,\eta,\xi,L_\kappa,\|\sigma^2\|_\infty$. Given that we have assumed $\frac{\lambda}{2}-L_\kappa>0$, we can choose $\eta,\xi>0$ both sufficiently small such that one has

$$\frac{\lambda}{2}-\left(1+\eta\frac{p-1}{p}\right)L_\kappa-\xi\frac{(p-1)(p-2)}{2p}\|\sigma^2\|_\infty>0.$$

This then yields

$$\mathbb{E}\left[e^{\frac{tp\lambda}{2}}|X^{\varepsilon,\delta}_t-x_t|^p\right]\le C[\varepsilon^p+\delta^p+\varepsilon^p\delta^{\frac{p}{2}}]\int_0^t e^{\frac{sp\lambda}{2}}\,ds,$$

yielding the uniform-in-time bound

$$\mathbb{E}\left[|X^{\varepsilon,\delta}_t-x_t|^p\right]\le C[\varepsilon^p+\delta^p+\varepsilon^p\delta^{\frac{p}{2}}]e^{-\frac{tp\lambda}{2}}\int_0^t e^{\frac{sp\lambda}{2}}\,ds\le C[\varepsilon^p+\delta^p+\varepsilon^p\delta^{\frac{p}{2}}],$$

with a potentially different constant $C<\infty$ that is independent of time $t\in\mathbb{R}_+$. This completes the proof of the theorem. □

3.1. **Proof of Lemmas 3.1 and 3.2.**

Proof of Lemma 3.1. We will only prove the bound for $\sup_{t\ge0}\mathbb{E}[|X^{\varepsilon,\delta}_t|^p]$ as the bound for $\sup_{t\ge0}|x_t|$ (and consequently for $\sup_{t\ge0}|x_t|^p$) follows by the exact same argument and is simpler.

We note that it is just sufficient to prove the statement for even powers p. We start by using Itô's formula to the function $\varphi(t,x)=e^{\frac{tp\alpha}{2}}|x|^p$, and noting $e^{\frac{tp\alpha}{2}}|x|^p=e^{\frac{tp\alpha}{2}}x^p$ to yield

$$e^{\frac{tp\alpha}{2}}|X^{\varepsilon,\delta}_t|^p=|X^{\varepsilon,\delta}_0|^p+\int_0^t\frac{p\alpha}{2}e^{\frac{sp\alpha}{2}}(X^{\varepsilon,\delta}_s)^p\,ds+\int_0^t pe^{\frac{sp\alpha}{2}}(X^{\varepsilon,\delta}_s)^{p-1}\left[f(X^{\varepsilon,\delta}_s)+\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)})\right]ds$$
$$+\frac{p(p-1)}{2}\int_0^t e^{\frac{sp\alpha}{2}}(X^{\varepsilon,\delta}_s)^{p-2}\varepsilon^2\sigma^2(X^{\varepsilon,\delta}_s)\,ds+p\int_0^t e^{\frac{sp\alpha}{2}}(X^{\varepsilon,\delta}_s)^{p-1}\varepsilon\sigma(X^{\varepsilon,\delta}_s)\,dW_s.$$

Therefore,

$$e^{\frac{tp\alpha}{2}}|X_t^{\varepsilon,\delta}|^p \leq |X_0^{\varepsilon,\delta}|^p + \int_0^t pe^{\frac{sp\alpha}{2}}\left[\frac{\alpha}{2}|X_s^{\varepsilon,\delta}|^2 + X_s^{\varepsilon,\delta}\cdot f(X_s^{\varepsilon,\delta})\right]|X_s^{\varepsilon,\delta}|^{p-2}ds + \int_0^t pe^{\frac{sp\alpha}{2}}(X_s^{\varepsilon,\delta})^{p-1}|\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta}|\,ds + \frac{p(p-1)}{2}\int_0^t e^{\frac{sp\alpha}{2}}|X_s^{\varepsilon,\delta}|^{p-2}\varepsilon^2|\sigma(X_s^{\varepsilon,\delta})|^2\,ds + p\int_0^t e^{\frac{sp\alpha}{2}}(X_s^{\varepsilon,\delta})^{p-1}\varepsilon\sigma(X_s^{\varepsilon,\delta})\,dW_s.$$

Next, using (8) and inequality $(X_t^{\varepsilon,\delta})^{p-1} \leq |X_t^{\varepsilon,\delta}|^{p-1}$ followed by taking expectation, we get for some time-independent constant $C < \infty$

$$e^{\frac{tp\alpha}{2}}\mathbb{E}[|X_t^{\varepsilon,\delta}|^p] \leq \mathbb{E}|X_0^{\varepsilon,\delta}|^p + \int_0^t -\frac{p\alpha}{2}\mathbb{E}\left[e^{\frac{sp\alpha}{2}}|X_s^{\varepsilon,\delta}|^p\right]ds + \int_0^t p\|\kappa\|_\infty\mathbb{E}\left[e^{\frac{sp\alpha}{2}}|X_s^{\varepsilon,\delta}|^{p-1}\right]ds + \left(p\beta + \varepsilon^2\frac{p(p-1)\|\sigma^2\|_\infty}{2}\right)\int_0^t \mathbb{E}\left[e^{\frac{sp\alpha}{2}}|X_s^{\varepsilon,\delta}|^{p-2}\right]ds.$$

We further employ the generalized Young's inequality to the terms $(p\|\kappa\|_\infty)|X_s^{\varepsilon,\delta}|^{p-1}$ (with the Hölder conjugates $\frac{p}{p-1}$, p) and $\left(p\beta + \varepsilon^2\frac{p(p-1)\|\sigma^2\|_\infty}{2}\right)|X_s^{\varepsilon,\delta}|^{p-2}$ (with the Hölder conjugates $\frac{p}{p-2}$, $\frac{p}{2}$) obtaining for some positive constants η_1, η_2 to be chosen that

$$e^{\frac{tp\alpha}{2}}\mathbb{E}[|X_t^{\varepsilon,\delta}|^p] \leq -\left(\frac{p\alpha}{2} - \eta_1(p-1)\|\kappa\|_\infty - \eta_2\frac{p-2}{p}\left(p\beta + \varepsilon^2\frac{p(p-1)\|\sigma^2\|_\infty}{2}\right)\right)\int_0^t \mathbb{E}\left[e^{\frac{sp\alpha}{2}}|X_s^{\varepsilon,\delta}|^p\right]ds + \mathbb{E}|X_0^{\varepsilon,\delta}|^p.$$

Choosing now $\eta_1, \eta_2 > 0$ sufficiently small so that

$$\left(\frac{p\alpha}{2} - \eta_1(p-1)\|\kappa\|_\infty - \eta_2\frac{p-2}{p}\left(p\beta + \varepsilon^2\frac{p(p-1)\|\sigma^2\|_\infty}{2}\right)\right) > 0$$

yields the bound $\mathbb{E}[|X_t^{\varepsilon,\delta}|^p] \leq e^{-\frac{tp\alpha}{2}}|x_0|^p < C$ for some time-independent constant $C < \infty$ and for all $\varepsilon, \delta > 0$. This completes the proof of lemma. □

Proof of Lemma 3.2. From the integral representation of $X_t^{\varepsilon,\delta}$, we get

$$\left|X_{\pi_\delta(s)}^{\varepsilon,\delta} - X_s^{\varepsilon,\delta}\right|^p \leq C\left|\int_{\pi_\delta(s)}^s \left\{f(X_u^{\varepsilon,\delta}) + \kappa(X_{\pi_\delta(u)}^{\varepsilon,\delta})\right\}du\right|^p + C\left|\int_{\pi_\delta(s)}^s \varepsilon\sigma(X_u^{\varepsilon,\delta})dW_u\right|^p.$$

For some $q \in \mathbb{N}$, the boundedness of κ, Assumption A1 and the martingale moment inequalities [KS91, Proposition 3.26] yield

$$\mathbb{E}\left[|X_{\pi_\delta(s)}^{\varepsilon,\delta} - X_s^{\varepsilon,\delta}|^p\right] \leq C\int_{[\pi_\delta(s),s]^p}\sum_{i=1}^p\left\{1 + \mathbb{E}|X_{r_i}^{\varepsilon,\delta}|^q\right\}dr_1\cdots dr_p + C\varepsilon^p\mathbb{E}\left\{\int_{[\pi_\delta(s),s]}\sigma^2(X_u^{\varepsilon,\delta})\,du\right\}^{\frac{p}{2}}ds.$$

Finally, using boundedness of σ from Assumption A2, Lemma 3.1 and the fact $s - \pi_\delta(s) < \delta$, we obtain the required bound. □

4. CLT Type Result: Proof of Theorem 2.6

We now prove Theorem 2.6 with Assumptions A1, A2, A3. An organization of this section is as follows. In Section 4.1, we prove Theorem 2.6 through its main building blocks Propositions 4.1, 4.2 and 4.3. The proofs of Propositions 4.1 through 4.3 are provided in Section 4.2. Proposition 4.1 plays a key role in the proof of Theorem 2.6 and is proved through a series of helpful lemmas. The proofs of these lemmas are deferred to Section 4.3.

4.1. **Proof of Theorem 2.6.** Before proving Theorem 2.6, let us get more insights into the fluctuation process $Z^{\varepsilon,\delta}_t$ and its limiting process Z_t solving (12). Recalling the integral representation for the process $X^{\varepsilon,\delta}_t$ and x_t from equations (6) and (4), respectively, followed by Taylor's theorem, we have

(22)
$$\begin{aligned}Z^{\varepsilon,\delta}_t &= \int_0^t \frac{f(X^{\varepsilon,\delta}_s)-f(x_s)}{\varepsilon}ds + \int_0^t \frac{[\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)})-\kappa(X^{\varepsilon,\delta}_s)]}{\varepsilon}ds + \int_0^t \frac{[\kappa(X^{\varepsilon,\delta}_s)-\kappa(x_s)]}{\varepsilon}ds \\ &\qquad + \int_0^t \sigma(X^{\varepsilon,\delta}_s)dW_s \\ &= \int_0^t [Df(x_s)+D\kappa(x_s)]Z^{\varepsilon,\delta}_s\,ds - \int_0^t D\kappa(x_s)\frac{X^{\varepsilon,\delta}_s - X^{\varepsilon,\delta}_{\pi_\delta(s)}}{\varepsilon}\,ds + \int_0^t \sigma(X^{\varepsilon,\delta}_s)dW_s + \int_0^t \mathsf{R}^{\varepsilon,\delta}_s\,ds,\end{aligned}$$

where,

(23)
$$\begin{aligned}\mathsf{R}^{\varepsilon,\delta}_s \triangleq \sum_{i=1}^3 \mathsf{R}^{\varepsilon,\delta}_i(s) &\triangleq \left[\frac{f(X^{\varepsilon,\delta}_s)-f(x_s)}{\varepsilon} - Df(x_s)Z^{\varepsilon,\delta}_s\right] + \left[\frac{\kappa(X^{\varepsilon,\delta}_s)-\kappa(x_s)}{\varepsilon} - D\kappa(x_s)Z^{\varepsilon,\delta}_s\right] \\ &\qquad + \left[\frac{\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)})-\kappa(X^{\varepsilon,\delta}_s)}{\varepsilon} - D\kappa(x_s)\frac{X^{\varepsilon,\delta}_{\pi_\delta(s)}-X^{\varepsilon,\delta}_s}{\varepsilon}\right].\end{aligned}$$

One can now identify the limiting SDE (that is, equation (12)) for the process Z_t from equation (22) as follows. This identification is essentially obtained by replacing $Z^{\varepsilon,\delta}_t$ by Z_t, $\int_0^t D\kappa(x_s)\frac{X^{\varepsilon,\delta}_s - X^{\varepsilon,\delta}_{\pi_\delta(s)}}{\varepsilon}\,ds$ by $\frac{\mathsf{c}}{2}\int_0^t D\kappa(x_s)[f(x_s)+\kappa(x_s)]\,ds$, $X^{\varepsilon,\delta}_t$ by x_t in the stochastic integral $\int_0^t \sigma(X^{\varepsilon,\delta}_s)dW_s$, and finally showing the remainder term $\int_0^t \mathsf{R}^{\varepsilon,\delta}_s\,ds$ is small. These approximations are made precise in Propositions 4.1, 4.2 and 4.3 stated below.

In Regimes 1 and 2, we see that rescaled process $Z^{\varepsilon,\delta}_t \triangleq \frac{X^{\varepsilon,\delta}_t - x_t}{\varepsilon}$ and the effective process Z_t solve equations (22) and (12), respectively. To show the convergence of $Z^{\varepsilon,\delta}_t$ to Z_t, as $\varepsilon,\delta \searrow 0$, one first need to demonstrate the following convergence in an appropriate sense.

(24)
$$\begin{aligned}\lim_{\substack{\varepsilon,\delta\searrow 0\\ \delta/\varepsilon\to\mathsf{c}}} \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} D\kappa(x_s)\frac{X^{\varepsilon,\delta}_s - X^{\varepsilon,\delta}_{\pi_\delta(s)}}{\varepsilon}\,ds \\ = \frac{\mathsf{c}}{2}\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} D\kappa(x_s)[f(x_s)+\kappa(x_s)]\,ds.\end{aligned}$$

In the fluctuation study, equation (24) enables us to identify the extra drift term capturing both the fast sampling and small noise effects. In Proposition 4.1 below, we show the convergence of

$\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} D\kappa(x_s) \frac{X_s^{\varepsilon,\delta}-X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon}\, ds$ to the term $\frac{\mathsf{c}}{2}\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} D\kappa(x_s)[f(x_s)+\kappa(x_s)]\, ds$, as $\varepsilon, \delta \searrow 0$. In Propositions 4.1 through 4.3 below, the constant C is always time-independent.

Proposition 4.1. *Let $X_t^{\varepsilon,\delta}$ be the solution of* SDE (6) *and*

$$I_1^{\varepsilon,\delta}(t) \triangleq \left| \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \left[\frac{\mathsf{c}}{2} D\kappa(x_s)[f(x_s)+\kappa(x_s)] - D\kappa(x_s) \frac{X_s^{\varepsilon,\delta}-X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon} \right] ds \right|^p,$$

where, $p \geq 1$ is an integer. Then, for all sufficiently small $\varepsilon, \delta > 0$, there exists a positive constant $C_{4.1}$ such that

$$\sup_{t\geq 0} \mathbb{E}\left[I_1^{\varepsilon,\delta}(t) \right] \leq C_{4.1} \left[\frac{\delta^{2p}}{\varepsilon^p} + \left| \frac{\delta}{\varepsilon} - \mathsf{c} \right|^p + \left(\frac{\delta^{2p}+\delta^p}{\varepsilon^p} \right) (\delta^p + \varepsilon^p + \delta^{\frac{p}{2}} \varepsilon^p) + \delta^{p/2} \right].$$

In Propositions 4.2 and 4.3 stated below, we show that the terms $I_2^{\varepsilon,\delta}(t)$ and $I_3^{\varepsilon,\delta}(t)$ are small in an appropriate sense as ε, δ vanish.

Proposition 4.2. *Let $X_t^{\varepsilon,\delta}$ be the solution of* SDE (6) *and*

$$I_2^{\varepsilon,\delta}(t) \triangleq \left| \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \left[\sigma(X_s^{\varepsilon,\delta}) - \sigma(x_s) \right] dW_s \right|^p,$$

where, $p \geq 1$ is an integer. Then, for all sufficiently small $\varepsilon, \delta > 0$, there exists a positive constant $C_{4.2}$ such that

$$\sup_{t\geq 0} \mathbb{E}\left[I_2^{\varepsilon,\delta}(t) \right] \leq C_{4.2} \left(\delta^{2p} + \varepsilon^{2p} + \delta^p \varepsilon^{2p} \right)^{\frac{1}{2}}.$$

Proposition 4.3. *Let $X_t^{\varepsilon,\delta}$ be the solution of* SDE (6), *$\mathsf{R}_t^{\varepsilon,\delta}$ be defined as in equation (23), and*

$$I_3^{\varepsilon,\delta}(t) \triangleq \left| \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \mathsf{R}_s^{\varepsilon,\delta}\, ds \right|^p, \tag{25}$$

where, $p \geq 1$ is an integer. Then, for all sufficiently small $\varepsilon, \delta > 0$, there exists a positive constant $C_{4.3}$ such that

$$\sup_{t\geq 0} \mathbb{E}\left[I_3^{\varepsilon,\delta}(t) \right] \leq \frac{C_{4.3}}{\varepsilon^p} (\varepsilon^{2p} + \delta^{2p} + \varepsilon^{2p}\delta^p).$$

We are now ready to prove Theorem 2.6.

Proof of Theorem 2.6. Using the integral representations for the stochastic processes $Z_t^{\varepsilon,\delta}$ and Z_t given by equations (22) and (12), respectively, we have

$$\begin{aligned} Z_t^{\varepsilon,\delta} - Z_t = \int_0^t [Df(x_s)+D\kappa(x_s)](Z_s^{\varepsilon,\delta} - Z_s)\, ds + \int_0^t \left[\sigma(X_s^{\varepsilon,\delta}) - \sigma(x_s) \right] dW_s + \int_0^t \mathsf{R}_s^{\varepsilon,\delta}\, ds \\ + \left[\frac{\mathsf{c}}{2} \int_0^t D\kappa(x_s)[f(x_s)+\kappa(x_s)]\, ds - \int_0^t D\kappa(x_s) \frac{X_s^{\varepsilon,\delta}-X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon}\, ds \right]. \end{aligned}$$

We now employ Itô's formula to the function $\varphi(t,x) = e^{-\int_0^t [Df(x_u)+D\kappa(x_u)]du} x$ to get

$$Z_t^{\varepsilon,\delta} - Z_t = \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \left[\frac{\mathsf{c}}{2} D\kappa(x_s)[f(x_s)+\kappa(x_s)] - D\kappa(x_s) \frac{X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon} \right] ds$$
$$+ \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \left[\sigma(X_s^{\varepsilon,\delta}) - \sigma(x_s) \right] dW_s + \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \mathsf{R}_s^{\varepsilon,\delta}\, ds.$$

Hence, for any integer $p \geq 1$,

(26)
$$|Z_t^{\varepsilon,\delta} - Z_t|^p \leq C \left| \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \left[\frac{\mathsf{c}}{2} D\kappa(x_s)[f(x_s)+\kappa(x_s)] - D\kappa(x_s) \frac{X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon} \right] ds \right|^p$$
$$+ C \left| \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \left[\sigma(X_s^{\varepsilon,\delta}) - \sigma(x_s) \right] dW_s \right|^p + C \left| \int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \mathsf{R}_s^{\varepsilon,\delta}\, ds \right|^p$$
$$\triangleq C \left[I_1^{\varepsilon,\delta}(t) + I_2^{\varepsilon,\delta}(t) + I_3^{\varepsilon,\delta}(t) \right].$$

Finally, putting Propositions 4.1, 4.2 and 4.3 together in equation (26), we get

$$\sup_{t\geq 0} \mathbb{E}\left[|Z_t^{\varepsilon,\delta} - Z_t|^p \right] \leq C \left[\frac{\delta^{2p}}{\varepsilon^p} + \left| \frac{\delta}{\varepsilon} - \mathsf{c} \right|^p + \left(\frac{\delta^p}{\varepsilon^p} + 1 \right) (\delta^p + \varepsilon^p + \delta^{\frac{p}{2}} \varepsilon^p)^{\frac{1}{2}} + \delta^{p/2} + \varepsilon^p \right].$$

□

4.2. **Proof of Propositions 4.1 through 4.3.** In this section, we provide the proofs of Propositions 4.1 through 4.3. To prove Proposition 4.1, we first state (without proof) a series of auxiliary Lemmas 4.4 to 4.8 below. The proofs of the aforementioned lemmas are postponed to Section 4.3.

Moving in the direction of the proof of Proposition 4.1, Lemma 4.4 gives the decomposition of the term $(1/\varepsilon)D\kappa(x_s)\left[X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right]$ which appears in the dynamics of fluctuations given by (22).

Lemma 4.4. *Let $X_t^{\varepsilon,\delta}$ be the solution of* SDE (6)*. Then,*

$$(1/\varepsilon)D\kappa(x_s)\left[X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right] = \sum_{i=1}^4 \mathsf{M}_i^{\varepsilon,\delta}(s), \quad \textit{where}$$

(27)
$$\mathsf{M}_1^{\varepsilon,\delta}(s) \triangleq (1/\varepsilon)D\kappa(x_s)f(x_{\pi_\delta(s)})[s-\pi_\delta(s)], \qquad \mathsf{M}_2^{\varepsilon,\delta}(s) \triangleq (1/\varepsilon)D\kappa(x_s)\kappa(X_{\pi_\delta(r)}^{\varepsilon,\delta})[s-\pi_\delta(s)],$$
$$\mathsf{M}_3^{\varepsilon,\delta}(s) \triangleq \frac{1}{\varepsilon}D\kappa(x_s)\int_{\pi_\delta(s)}^s \{f(X_r^{\varepsilon,\delta}) - f(X_{\pi_\delta(r)}^{\varepsilon,\delta})\}dr + \frac{1}{\varepsilon}D\kappa(x_s)[f(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - f(x_{\pi_\delta(s)})][s-\pi_\delta(s)],$$
$$\mathsf{M}_4^{\varepsilon,\delta}(s) \triangleq D\kappa(x_s)\int_{\pi_\delta(s)}^s \sigma(X_u^{\varepsilon,\delta})dW_u.$$

Now, Lemma 4.5 stated below deals with the deterministic term $\mathsf{M}_1^{\varepsilon,\delta}(t)$ and shows that $\mathsf{M}_1^{\varepsilon,\delta}(t)$ is close, uniform-in-time, to the first part of the effective drift term (i.e, $\frac{\mathsf{c}}{2}D\kappa(x_t)f(x_t)$, see (24)).

Lemma 4.5. *Let $\mathsf{M}_1^{\varepsilon,\delta}(t)$ be defined as in equation (27) and $p \geq 1$ be an integer. Then, there exists a positive constant $C_{4.5}$ such that*

$$\sup_{t\geq 0}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_1^{\varepsilon,\delta}(s)-\frac{\mathsf{c}}{2}D\kappa(x_s)f(x_s)\right]ds\right|^p \leq C_{4.5}\left|\frac{\delta}{\varepsilon}-\mathsf{c}\right|^p+\frac{\delta^{2p}}{\varepsilon^p}C_{4.5}.$$

Next, Lemma 4.6 handles the asymptotic analysis, uniform-in-time, of the process $\mathsf{M}_2^{\varepsilon,\delta}(t)$ as ε,δ tend to zero and shows that $\mathsf{M}_2^{\varepsilon,\delta}(t)$ is close to the second part of the effective drift term (i.e, $\frac{\mathsf{c}}{2}D\kappa(x_t)\kappa(x_t)$, see equation (24)).

Lemma 4.6. *Let $\mathsf{M}_2^{\varepsilon,\delta}(t)$ be defined as in equation (27) and $p \geq 1$ be an integer. Then, for all sufficiently small $\varepsilon,\delta > 0$, there exists a positive constant $C_{4.6}$ such that*

$$\sup_{t\geq 0}\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_2^{\varepsilon,\delta}(s)-\frac{\mathsf{c}}{2}D\kappa(x_s)\kappa(x_s)\right]ds\right|^p$$
$$\leq C_{4.6}\left[\frac{\delta^{2p}}{\varepsilon^p}+\left|\frac{\delta}{\varepsilon}-\mathsf{c}\right|^p+\left(\frac{\delta^{2p}+\delta^p}{\varepsilon^p}\right)(\delta^p+\varepsilon^p+\delta^{\frac{p}{2}}\varepsilon^p)\right].$$

Finally, Lemmas 4.7 and 4.8 deal with the processes $\mathsf{M}_3^{\varepsilon,\delta}(t)$ and $\mathsf{M}_4^{\varepsilon,\delta}(t)$, respectively and show that the terms of interest are small in uniform time setting.

Lemma 4.7. *Let $\mathsf{M}_3^{\varepsilon,\delta}(t)$ be defined as in equation (27) and $p \geq 1$ be an integer. Then, for all sufficiently small $\varepsilon,\delta > 0$, there exists a positive constant $C_{4.7}$ such that*

$$\sup_{t\geq 0}\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{M}_3^{\varepsilon,\delta}(s)\,ds\right|^p \leq C_{4.7}\frac{\delta^p}{\varepsilon^p}(\delta^p+\varepsilon^p+\delta^{\frac{p}{2}}\varepsilon^p)+C_{4.7}\frac{\delta^{2p}}{\varepsilon^p}.$$

Lemma 4.8. *Let $\mathsf{M}_4^{\varepsilon,\delta}(t)$ be defined as in equation (27) and $p \geq 1$ be an integer. Then, there exists a positive constant $C_{4.8}$ such that*

$$\sup_{t\geq 0}\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{M}_4^{\varepsilon,\delta}(s)\,ds\right|^p \leq C_{4.8}\delta^{\frac{p}{2}}.$$

We now prove Proposition 4.1.

Proof of Proposition 4.1. For any integer $p \geq 1$, using Lemma 4.4 for the decomposition of the process $D\kappa(x_s)\frac{X_s^{\varepsilon,\delta}-X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon}$, we have

$$\sup_{t\geq 0}\mathbb{E}\left[I_1^{\varepsilon,\delta}(t)\right] \leq C\sup_{t\geq 0}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_1^{\varepsilon,\delta}(s)-\frac{\mathsf{c}}{2}D\kappa(x_s)f(x_s)\right]ds\right|^p$$
$$+C\sup_{t\geq 0}\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_2^{\varepsilon,\delta}(s)-\frac{\mathsf{c}}{2}D\kappa(x_s)\kappa(x_s)\right]ds\right|^p$$
$$+C\sup_{t\geq 0}\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{M}_3^{\varepsilon,\delta}(s)\,ds\right|^p$$
$$+C\sup_{t\geq 0}\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{M}_4^{\varepsilon,\delta}(s)\,ds\right|^p.$$

Now, employing Lemmas 4.5, 4.6, 4.7 and 4.8 for the terms on the right hand side of the above equation, we get the required bound. □

Next, we present the proof of Proposition 4.2.

Proof of Proposition 4.2. Using the Burkholder-Davis-Gundy inequalities followed by Jensen's inequality[2] for concave functions, we have

$$\mathbb{E}\left[I_2^{\varepsilon,\delta}(t)\right] \leq C\mathbb{E}\left[\left(\int_0^t e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]du}|\sigma(X_s^{\varepsilon,\delta}) - \sigma(x_s)|^2 ds\right)^{\frac{p}{2}}\right]$$

$$\leq C\left[\mathbb{E}\left(\int_0^t e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]du}|\sigma(X_s^{\varepsilon,\delta}) - \sigma(x_s)|^2 ds\right)^p\right]^{\frac{1}{2}}.$$

Now, employing the Lipschitz continuity of σ, and Assumption A3, we obtain from the above equation

$$\begin{aligned}
\mathbb{E}\left[I_2^{\varepsilon,\delta}(t)\right] &\leq C\left[\mathbb{E}\left(\int_0^t e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]du}|X_s^{\varepsilon,\delta} - x_s|^2 ds\right)^p\right]^{\frac{1}{2}} \\
&= C\left[\mathbb{E}\int_{[0,t]^p}\left(\prod_{i=1}^p e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]du}|X_{s_i}^{\varepsilon,\delta} - x_{s_i}|^2\right) ds_1\cdots ds_p\right]^{\frac{1}{2}} \\
&\leq C\left[\mathbb{E}\int_{[0,t]^p}\left(\prod_{i=1}^p e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right)\left(\sum_{i=1}^p |X_{s_i}^{\varepsilon,\delta} - x_{s_i}|^{2p}\right) ds_1\cdots ds_p\right]^{\frac{1}{2}} \\
&\leq C\left[\int_{[0,t]^p}\left(\prod_{i=1}^p e^{2\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right)\left(\sum_{i=1}^p \mathbb{E}|X_{s_i}^{\varepsilon,\delta} - x_{s_i}|^{2p}\right) ds_1\cdots ds_p\right]^{\frac{1}{2}}.
\end{aligned}$$

After using a simple algebra and Theorem 2.5, we obtain the desired result. □

Finally, we prove Proposition 4.3.

Proof of Proposition 4.3. For any integer $p \geq 1$, recalling the definition of the process $I_3^{\varepsilon,\delta}(t)$ from equation (25), we have

$$\sup_{t\geq 0}\mathbb{E}\left[I_3^{\varepsilon,\delta}(t)\right] \leq \sup_{t\geq 0}\mathbb{E}\left[\mathsf{R}_1^{\varepsilon,\delta}(t)\right] + \sup_{t\geq 0}\mathbb{E}\left[\mathsf{R}_2^{\varepsilon,\delta}(t)\right] + \sup_{t\geq 0}\mathbb{E}\left[\mathsf{R}_3^{\varepsilon,\delta}(t)\right],$$

where $\mathsf{R}_i^{\varepsilon,\delta}(s)$, $i = 1, 2, 3$ are defined in equation (23). Using Proposition 5.9 for the terms $\sup_{t\geq 0}\mathbb{E}\left[\mathsf{R}_1^{\varepsilon,\delta}(t)\right]$ and $\sup_{t\geq 0}\mathbb{E}\left[\mathsf{R}_2^{\varepsilon,\delta}(t)\right]$ and Proposition 5.11 for the term $\sup_{t\geq 0}\mathbb{E}\left[\mathsf{R}_3^{\varepsilon,\delta}(t)\right]$, we obtain the required bound. □

4.3. **Proof of Lemmas 4.4 through 4.8.**

Proof of Lemma 4.4. For any $s \geq 0$, exploiting the integral representation for $X_s^{\varepsilon,\delta}$ from (6) followed by a simple algebra, we get

$$\begin{aligned}
\frac{X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon} &= \int_{\pi_\delta(s)}^s \frac{f(X_r^{\varepsilon,\delta}) + \kappa(X_{\pi_\delta(r)}^{\varepsilon,\delta})}{\varepsilon} dr + \int_{\pi_\delta(s)}^s \sigma(X_u^{\varepsilon,\delta})dW_u \\
&= \int_{\pi_\delta(s)}^s \frac{f(X_r^{\varepsilon,\delta}) - f(X_{\pi_\delta(r)}^{\varepsilon,\delta})}{\varepsilon} dr + \int_{\pi_\delta(s)}^s \frac{f(X_{\pi_\delta(r)}^{\varepsilon,\delta}) + \kappa(X_{\pi_\delta(r)}^{\varepsilon,\delta})}{\varepsilon} dr \\
&\quad + \int_{\pi_\delta(s)}^s \sigma(X_u^{\varepsilon,\delta})dW_u.
\end{aligned}$$

[2]For a random variable Y and a concave function ψ with $\mathbb{E}|Y|, \mathbb{E}|\psi(Y)| < \infty$, we have $\mathbb{E}[\psi(Y)] \leq \psi(\mathbb{E}[Y])$.

Writing $f(X^{\varepsilon,\delta}_{\pi_\delta(r)}) + \kappa(X^{\varepsilon,\delta}_{\pi_\delta(r)})$ in the middle term of the above equation as $f(X^{\varepsilon,\delta}_{\pi_\delta(r)}) - f(x_{\pi_\delta(s)}) + \kappa(X^{\varepsilon,\delta}_{\pi_\delta(r)}) + f(x_{\pi_\delta(s)})$, we see that

$$\begin{aligned}
D\kappa(x_s)\frac{X^{\varepsilon,\delta}_s - X^{\varepsilon,\delta}_{\pi_\delta(s)}}{\varepsilon} &= D\kappa(x_s) f(x_{\pi_\delta(s)})\frac{s-\pi_\delta(s)}{\varepsilon} + D\kappa(x_s)\int_{\pi_\delta(s)}^s \frac{\kappa(X^{\varepsilon,\delta}_{\pi_\delta(r)})}{\varepsilon}\,dr \\
&\quad + D\kappa(x_s)\int_{\pi_\delta(s)}^s \frac{f(X^{\varepsilon,\delta}_r) - f(X^{\varepsilon,\delta}_{\pi_\delta(r)})}{\varepsilon}\,dr + D\kappa(x_s)\frac{f(X^{\varepsilon,\delta}_{\pi_\delta(s)}) - f(x_{\pi_\delta(s)})}{\varepsilon}(s-\pi_\delta(s)) \\
&\quad + D\kappa(x_s)\int_{\pi_\delta(s)}^s \sigma(X^{\varepsilon,\delta}_u)dW_u.
\end{aligned}$$

Defining right hand side of the above equation as $\sum_{i=1}^4 \mathsf{M}^{\varepsilon,\delta}_i(t)$, we finish the proof of the lemma. □

Proof of Lemma 4.5. We recall the definition of $\mathsf{M}^{\varepsilon,\delta}_1(s)$ from equation (27). For any $s \geq 0$, writing $\mathsf{M}^{\varepsilon,\delta}_1(s)$ as $\mathsf{M}^{\varepsilon,\delta}_1(s) + (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)]$, we get

$$\begin{aligned}
\mathsf{M}^{\varepsilon,\delta}_1(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)f(x_s) &= (1/\varepsilon)\{D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})\}f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] \\
&\quad + (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\mathsf{c}}{2}D\kappa(x_s)f(x_s) \\
&= (1/\varepsilon)\{D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})\}f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] \\
&\quad + (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)f(x_s) \\
&\quad + \frac{1}{2}\left(\frac{\delta}{\varepsilon} - \mathsf{c}\right)D\kappa(x_s)f(x_s).
\end{aligned}$$

Now, multiplying both sides of the above equation by $e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}$, we have

$$\begin{aligned}
&e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}^{\varepsilon,\delta}_1(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)f(x_s)\right] \\
&\quad = (1/\varepsilon)\{D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})\}f(x_{\pi_\delta(s)})[s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\qquad + \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)f(x_s)\right]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\qquad + \frac{1}{2}\left(\frac{\delta}{\varepsilon} - \mathsf{c}\right)D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}.
\end{aligned}$$

Next, for any integer $p \geq 1$, we use Lemma 5.2 to handle the term $D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})$, Assumption A3 and the hypothesis that f is of polynomial growth and $\sup_{t\geq 0}|x_t| < \infty$ (Lemma 3.1) to get

$$\begin{aligned}
&\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_1^{\varepsilon,\delta}(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)f(x_s)\right]ds\right|^p \\
&\leq \frac{\delta^{2p}}{\varepsilon^p}C\left(\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right)^p + \frac{1}{2^p}C\left|\frac{\delta}{\varepsilon} - \mathsf{c}\right|^p\left(\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right)^p \\
&\qquad + C\left|\int_0^t \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)f(x_s)\right]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right|^p \\
&\leq C\left(\frac{\delta^{2p}}{\varepsilon^p} + \left|\frac{\delta}{\varepsilon} - \mathsf{c}\right|^p\right) \\
&\qquad + C\left|\int_0^t \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)f(x_s)\right]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right|^p.
\end{aligned}$$

The proof can be concluded by employing Lemma 5.6 for the last term on the right side. □

Proof of Lemma 4.6. Recalling the definition of the process $\mathsf{M}_2^{\varepsilon,\delta}(s)$ from equation (27) and using a simple algebra, we obtain

$$\begin{aligned}
\mathsf{M}_2^{\varepsilon,\delta}(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)\kappa(x_s) &= (1/\varepsilon)\{D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})\}\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)] \\
&\quad + (1/\varepsilon)\{D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})\}\left[\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)}) - \kappa(x_{\pi_\delta(s)})\right][s-\pi_\delta(s)] \\
&\quad + (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})\left[\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)}) - \kappa(x_{\pi_\delta(s)})\right][s-\pi_\delta(s)] \\
&\quad + (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)\kappa(x_s) \\
&\quad + \frac{1}{2}\left(\frac{\delta}{\varepsilon} - \mathsf{c}\right)D\kappa(x_s)\kappa(x_s).
\end{aligned}$$

Now, multiplying both sides of the above equation by $e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}$, we have

$$\begin{aligned}
&e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_2^{\varepsilon,\delta}(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)\kappa(x_s)\right] \\
&\quad = (1/\varepsilon)\{D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})\}\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\quad + \frac{1}{\varepsilon}\{D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})\}\left[\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)}) - \kappa(x_{\pi_\delta(s)})\right][s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\quad + (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})\left[\kappa(X^{\varepsilon,\delta}_{\pi_\delta(s)}) - \kappa(x_{\pi_\delta(s)})\right][s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\quad + \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)\kappa(x_s)\right]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\quad + \frac{1}{2}\left(\frac{\delta}{\varepsilon} - \mathsf{c}\right)D\kappa(x_s)\kappa(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}.
\end{aligned}$$

Using the Lipschitz continuity and boundedness of the function κ and its derivatives, the fact $s-\pi_\delta(s) < \delta$, for any $\delta > 0$ and $\sup_{t\geq 0}|x_t| < \infty$ (Lemma 3.1), we have

$$\begin{aligned}
&\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_2^{\varepsilon,\delta}(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)\kappa(x_s)\right]ds\right|^p \\
&\qquad \leq C\frac{\delta^p}{\varepsilon^p}\left[\int_0^t |D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \\
&\qquad + C\left|\int_0^t \left[\frac{1}{\varepsilon}D\kappa(x_{\pi_\delta(s)})\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)\kappa(x_s)\right] e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right|^p \\
&\qquad + C\frac{\delta^p}{\varepsilon^p}\left[\int_0^t |D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})|\left|X^{\varepsilon,\delta}_{\pi_\delta(s)} - x_{\pi_\delta(s)}\right| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \\
&\qquad\qquad + C\frac{\delta^p}{\varepsilon^p}\left[\int_0^t \left|X^{\varepsilon,\delta}_{\pi_\delta(s)} - x_{\pi_\delta(s)}\right| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \\
&\qquad\qquad + C\left|\frac{\delta}{\varepsilon} - \mathsf{c}\right|^p\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p.
\end{aligned}$$

Taking expectation on both sides, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_2^{\varepsilon,\delta}(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)\kappa(x_s)\right]ds\right|^p\right] \\
&\qquad \leq C\frac{\delta^p}{\varepsilon^p}\left[\int_0^t |D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \\
&\qquad + C\left|\int_0^t \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)\kappa(x_s)\right] e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right|^p \\
&\qquad + C\frac{\delta^p}{\varepsilon^p}\mathbb{E}\left[\int_0^t |D\kappa(x_s) - D\kappa(x_{\pi_\delta(s)})|\left|X^{\varepsilon,\delta}_{\pi_\delta(s)} - x_{\pi_\delta(s)}\right| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \\
&\qquad\qquad + C\frac{\delta^p}{\varepsilon^p}\mathbb{E}\left[\int_0^t \left|X^{\varepsilon,\delta}_{\pi_\delta(s)} - x_{\pi_\delta(s)}\right| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \\
&\qquad\qquad + C\left|\frac{\delta}{\varepsilon} - \mathsf{c}\right|^p\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p.
\end{aligned}$$

Employing Lemma 5.2 to handle the term $\sup_{s\geq 0}|D\kappa(x_u) - D\kappa(x_{\pi_\delta(u)})|$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left[\mathsf{M}_2^{\varepsilon,\delta}(s) - \frac{\mathsf{c}}{2}D\kappa(x_s)\kappa(x_s)\right]ds\right|^p\right] \\
&\qquad \leq C\frac{\delta^{2p}}{\varepsilon^p}\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p + C\left|\frac{\delta}{\varepsilon} - \mathsf{c}\right|^p\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \\
&\qquad + C\left|\int_0^t \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)\kappa(x_s)\right] e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right|^p \\
&\qquad\qquad + C\left(\frac{\delta^{2p}+\delta^p}{\varepsilon^p}\right)\mathbb{E}\left[\int_0^t \left|X^{\varepsilon,\delta}_{\pi_\delta(s)} - x_{\pi_\delta(s)}\right| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p.
\end{aligned}$$

Finally, we use Assumption A3 to handle the term $\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\,ds$ and Lemmas 5.7 and 5.8 for the last two terms in the above equation to complete the proof. □

Proof of Lemma 4.7. Recalling the definition of the process $\mathsf{M}_3^{\varepsilon,\delta}(s)$ from equation (27), we have

$$\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{M}_3^{\varepsilon,\delta}(s)\,ds$$
$$= \int_0^t \frac{1}{\varepsilon} e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} D\kappa(x_s)\left(\int_{\pi_\delta(s)}^s \{f(X_r^{\varepsilon,\delta}) - f(X_{\pi_\delta(r)}^{\varepsilon,\delta})\}\,dr\right)ds$$
$$+ \int_0^t \frac{1}{\varepsilon} e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} D\kappa(x_s)\left[f(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - f(x_{\pi_\delta(s)})\right][s-\pi_\delta(s)]\,ds.$$

Now, for any $p \geq 1$, using the boundedness of the first-order derivatives of κ, the fact $s - \pi_\delta(s) < \delta$, for any $\delta > 0$, and a simple algebra, we have after taking expectation

$$\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{M}_3^{\varepsilon,\delta}(s)\,ds\right|^p \leq \frac{C}{\varepsilon^p}\mathbb{E}\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\int_{\pi_\delta(s)}^s \left|f(X_r^{\varepsilon,\delta}) - f(X_{\pi_\delta(r)}^{\varepsilon,\delta})\right| dr\,ds\right]^p$$
$$+ C\frac{\delta^p}{\varepsilon^p}\mathbb{E}\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left|f(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - f(x_{\pi_\delta(s)})\right| ds\right]^p$$
$$\leq \frac{C}{\varepsilon^p}\mathbb{E}\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\int_{\pi_\delta(s)}^s |f(X_r^{\varepsilon,\delta}) - f(x_r)|dr\,ds\right]^p$$
$$+ \frac{C}{\varepsilon^p}\mathbb{E}\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\int_{\pi_\delta(s)}^s |f(X_{\pi_\delta(r)}^{\varepsilon,\delta}) - f(x_{\pi_\delta(r)})|dr\,ds\right]^p$$
$$+ \frac{C}{\varepsilon^p}\mathbb{E}\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\int_{\pi_\delta(s)}^s \left|f(x_r) - f(x_{\pi_\delta(r)})\right| dr\,ds\right]^p$$
$$+ C\frac{\delta^p}{\varepsilon^p}\mathbb{E}\left[\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left|f(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - f(x_{\pi_\delta(s)})\right| ds\right]^p.$$

The next step is to use the local Lipschitz continuity of the mapping f by Assumption A1 together with Hölder's inequality, Theorem 2.5 and Lemmas 3.1 and 5.1. We omit the details for brevity. □

Proof of Lemma 4.8. Let us define for notational convenience

$$J^{\varepsilon,\delta}(t) \triangleq \left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} D\kappa(x_s)\int_{\pi_\delta(s)}^s \sigma(X_u^{\varepsilon,\delta})\,dW_u\,ds\right|^p.$$

Using the basic integral inequality for Lebesgue integrals $|\int \cdot| \leq \int |\cdot|$, we have

$$J^{\varepsilon,\delta}(t) \leq C\left(\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left|\int_{\pi_\delta(s)}^s \sigma(X_u^{\varepsilon,\delta})\,dW_u\right| ds\right)^p$$
$$= C\left[\int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\times\left(\prod_{i=1}^p\left|\int_{[\pi_\delta(s_i),s_i]}\sigma(X_{u_i}^{\varepsilon,\delta})\,dW_{u_i}\right|\right)ds_1\cdots ds_p\right]$$
$$\leq C\left[\int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\times\left(\sum_{i=1}^p\left|\int_{[\pi_\delta(s_i),s_i]}\sigma(X_{u_i}^{\varepsilon,\delta})\,dW_{u_i}\right|^p\right)ds_1\cdots ds_p\right].$$

Taking expectation followed by martingale moment inequalities [KS91, Proposition 3.26], we obtain

$$\mathbb{E}\left[J^{\varepsilon,\delta}(t)\right] \le C\left[\int_{[0,t]^p}\left(\prod_{i=1}^{p} e^{\int_{s_i}^{t}[Df(x_u)+D\kappa(x_u)]du}\right)\left(\sum_{i=1}^{p}\mathbb{E}\left\{\int_{[\pi_\delta(s_i),s_i]}\sigma^2(X_{u_i}^{\varepsilon,\delta})\,du_i\right\}^{\frac{p}{2}}\right)ds_1\cdots ds_p\right].$$

Employing the boundedness of σ, Assumption A2, the fact $s-\pi_\delta(s)\le\delta$, and a simple algebra, we obtain the desired estimate.

□

5. Proof of more supporting results: Lemmas 5.1 to 5.8 and Propositions 5.9,5.11

5.1. Proof of Lemmas 5.1 through 5.8.

Lemma 5.1. *Let x_t be the solution of (4). Then, for any integer $p\ge 1$, there exists a positive constant $C_{5.1}$ such that*

$$\sup_{t\ge 0}|x_t - x_{\pi_\delta(t)}|^p \le \delta^p C_{5.1}.$$

Proof. Recalling the integral representation of x_t from equation (4) and using the polynomial growth of the mappings f and boundedness of κ, we have for some $q<\infty$

$$|x_t - x_{\pi_\delta(t)}| \le C\int_{\pi_\delta(t)}^{t}\left(1+\sup_{u\ge 0}|x_u|^q\right)ds.$$

Using the fact $t-\pi_\delta(t)<\delta$ for any $\delta>0$ and $\sup_{u\ge0}|x_u|<\infty$ (Lemma 3.1), we get the required estimate. □

Lemma 5.2. *Let x_t be the solution of equation (4). Then, for any $t\ge 0$, there exists a positive constant $C_{5.2}$ such that*

$$\sup_{t\ge 0}\left|D\kappa(x_t) - D\kappa(x_{\pi_\delta(t)})\right| \le \delta C_{5.2}.$$

Proof. Using Taylor's theorem, we have

$$\left|D\kappa(x_t) - D\kappa(x_{\pi_\delta(t)})\right| \le |D^2\kappa(z)||x_t - x_{\pi_\delta(t)}|,$$

where $z\in\mathbb{R}$ is a point lying on the line segment joining $x_{\pi_\delta(t)}$ and x_t. Now, boundedness of the second-order derivatives of κ (Assumption A2) and Lemma 5.1 for the term $|x_s - x_{\pi_\delta(s)}|^p$ complete the proof of the lemma. □

We now start the preparation of proving Lemma 5.6 through the helpful results: Lemmas 5.3, 5.4, and 5.5 mentioned below. Lemma 5.6 was used in the proof of Lemma 4.5.

Lemma 5.3. *Let x_t be the solution of equation (4) and*

$$\mathscr{R}_1^\delta(t) \triangleq \sum_{i=0}^{\lfloor\frac{t}{\delta}\rfloor-1}\int_{i\delta}^{(i+1)\delta} D\kappa(x_{i\delta})f(x_{i\delta})\int_{s}^{(i+1)\delta}\frac{d}{du}e^{\int_u^t[Df(x_r)+D\kappa(x_r)]\,dr}\,du\,ds.$$

Then, for any integer $p\ge 1$, there exists a positive constant $C_{5.3}$ such that

$$\sup_{t\ge 0}|\mathscr{R}_1^\delta(t)|^p \le \delta^p C_{5.3}.$$

Proof. Using the boundedness of the derivatives of κ, polynomial growth of the function f (Assumptions A1, A2), and $\frac{d}{du}e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr} = -[Df(x_u)+D\kappa(x_u)]e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr}$, we have

(28)
$$\begin{aligned}|\mathscr{R}_1^\delta(t)|^p &\le C\left(\sup_{t\ge 0}|x_t|^q+1\right)^p\left[\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}\int_s^{(i+1)\delta}|Df(x_u)+D\kappa(x_u)|\,e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr}\,du\,ds\right]^p\\ &\le C\left(\sup_{t\ge 0}|x_t|^q+1\right)^{2p}\left[\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}\int_s^{(i+1)\delta}e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr}\,du\,ds\right]^p.\end{aligned}$$

We now focus to get an estimate on the term $\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}\int_s^{(i+1)\delta}e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr}\,du\,ds$ in the above equation. To do this, we note $i\delta \le s\le u\le (i+1)\delta\le t$, for any $i\in\{0,\cdots,\lfloor \frac{t}{\delta}\rfloor -1\}$ and $e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr} = e^{\int_s^t [Df(x_r)+D\kappa(x_r)]\,dr}e^{-\int_s^u [Df(x_r)+D\kappa(x_r)]\,dr}$. Hence,

(29)
$$\begin{aligned}&\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}\int_s^{(i+1)\delta}e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr}\,du\,ds\\ &\qquad= \sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}e^{\int_s^t [Df(x_r)+D\kappa(x_r)]\,dr}\int_s^{(i+1)\delta}e^{-\int_s^u [Df(x_r)+D\kappa(x_r)]\,dr}\,du\,ds.\end{aligned}$$

Now, if for any $r\in[s,(i+1)\delta]$, $[Df(x_r)+D\kappa(x_r)]\ge 0$, then we have $e^{-\int_s^u [Df(x_r)+D\kappa(x_r)]\,dr}\le 1$; again if for any $r\in[s,(i+1)\delta]$, $[Df(x_r)+D\kappa(x_r)]\le 0$, we have $e^{-\int_s^u [Df(x_r)+D\kappa(x_r)]\,dr}\le C$. Indeed, for the latter, $e^{-\int_s^u [Df(x_r)+D\kappa(x_r)]\,dr}\le e^{\int_s^u C dr}\le e^{C\delta}$ as $0\le u-s\le\delta$. Hence, $e^{-\int_s^u [Df(x_r)+D\kappa(x_r)]\,dr}\le \max\{1,C\}$. Returning to equation (29) and using $(i+1)\delta - s\le\delta$, we obtain

(30)
$$\begin{aligned}\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}\int_s^{(i+1)\delta}e^{\int_u^t [Df(x_r)+D\kappa(x_r)]\,dr}\,du\,ds &\le C\delta\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}e^{\int_s^t [Df(x_r)+D\kappa(x_r)]\,dr}\,ds\\ &\le C\delta\int_0^t e^{\int_s^t [Df(x_r)+D\kappa(x_r)]\,dr}\,ds.\end{aligned}$$

Finally, combining the equations (28), (30) followed by the Assumption A3, we obtain the required result. ☐

Lemma 5.4. *Let x_t be the solution of equation (4) and*

$$\mathscr{R}_2^\delta(t)\triangleq\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}\left[D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}-D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right]ds,$$

$$\mathscr{R}_3^\delta(t)\triangleq\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta}\left[D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}-D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right]ds.$$

Then, for any integer $p\ge 1$, there exists a positive constant $C_{5.4}$ such that

$$\sup_{t\ge 0}|\mathscr{R}_2^\delta(t)|^p+\sup_{t\ge 0}|\mathscr{R}_3^\delta(t)|^p\le \delta^p C_{5.4}.$$

Proof. Using Lemma 5.2 to handle term $|D\kappa(x_{i\delta}) - D\kappa(x_s)|$, local Lipschitz continuity of f, boundedness of the derivative of κ, Assumption A3 and Lemma 5.1 for the term $|x_s - x_{\pi_\delta(s)}|^p$, we obtain

$$|\mathscr{R}_2^\delta(t)|^p \le C\left(1+\sup_{t\ge 0}|x_t|\right)^p \left[\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta} |D\kappa(x_{i\delta}) - D\kappa(x_s)| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right]^p \le \delta^p C,$$

$$|\mathscr{R}_3^\delta(t)|^p \le C\left[\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1}\int_{i\delta}^{(i+1)\delta} |f(x_{i\delta}) - f(x_s)| e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right]^p \le \delta^p C.$$

Putting these estimates together, we obtain the required result. □

Lemma 5.5. *Let x_t be the solution of equation (4) and*

$$\mathscr{R}_t^{\varepsilon,\delta} \triangleq \frac{\delta}{2\varepsilon}\left[\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \delta D\kappa(x_{i\delta}) f(x_{i\delta}) e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - \int_0^t D\kappa(x_s) f(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right].$$

Then, for any integer $p \ge 1$, there exists a positive constant $C_{5.5}$ such that

$$\sup_{t\ge 0} |\mathscr{R}_t^{\varepsilon,\delta}|^p \le \frac{\delta^{2p}}{\varepsilon^p} C_{5.5}.$$

Proof. In the expression of $\mathscr{R}_t^{\varepsilon,\delta}$ above, writing the integral $\int_0^t D\kappa(x_s) f(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds$ in the Riemann sum form, we have

$$\begin{aligned}
&\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \delta D\kappa(x_{i\delta}) f(x_{i\delta}) e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - \int_0^t D\kappa(x_s) f(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds \\
&= \sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \int_{i\delta}^{(i+1)\delta} \left[D\kappa(x_{i\delta}) f(x_{i\delta}) e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s) f(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right] ds \\
&\qquad - \int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t D\kappa(x_s) f(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds.
\end{aligned} \tag{31}$$

Now, for each $i \in \{0, 1, \cdots, \lfloor \frac{t}{\delta}\rfloor - 1\}$, using a simple algebra for the integrand of the first term on the right-hand side of the above equation, we get

$$\begin{aligned}
&D\kappa(x_{i\delta}) f(x_{i\delta}) e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s) f(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\quad = D\kappa(x_{i\delta}) f(x_{i\delta}) e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s) f(x_{i\delta}) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\qquad + D\kappa(x_s) f(x_{i\delta}) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s) f(x_s) e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\quad = D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\qquad + D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\qquad + D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\quad = D\kappa(x_{i\delta})f(x_{i\delta})\int_s^{(i+1)\delta} \frac{d}{du} e^{\int_u^t [Df(x_r)+D\kappa(x_r)]dr}\, du \\
&\qquad + D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} \\
&\qquad + D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}.
\end{aligned} \tag{32}
$$

Hence, for any integer $p \geq 1$, from (31) and (32), we get

$$
\begin{aligned}
|\mathscr{R}_t^{\varepsilon,\delta}|^p &\leq \frac{\delta^p}{\varepsilon^p}\left|\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \delta D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - \int_0^t D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right|^p \\
&\leq C\left|\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \int_{i\delta}^{(i+1)\delta} \left[D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right] ds\right|^p \\
&\qquad + C\left|\int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right|^p.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\mathscr{R}_t^{\varepsilon,\delta}|^p &\leq C\left|\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \int_{i\delta}^{(i+1)\delta} D\kappa(x_{i\delta})f(x_{i\delta})\int_s^{(i+1)\delta} \frac{d}{du} e^{\int_u^t [Df(x_r)+D\kappa(x_r)]dr}\, du\, ds\right|^p \\
&\quad + C\left|\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \int_{i\delta}^{(i+1)\delta} \left[D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right] ds\right|^p \\
&\quad + C\left|\sum_{i=0}^{\lfloor \frac{t}{\delta}\rfloor -1} \int_{i\delta}^{(i+1)\delta} \left[D\kappa(x_s)f(x_{i\delta})e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du} - D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\right] ds\right|^p \\
&\qquad + C\left|\int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right|^p \\
&\triangleq C\left(\left|\mathscr{R}_1^\delta(t)\right|^p + \left|\mathscr{R}_2^\delta(t)\right|^p + \left|\mathscr{R}_3^\delta(t)\right|^p + \left|\int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right|^p\right).
\end{aligned}
$$

We now use the condition $|\int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds| < \delta$, boundedness of the derivative of κ and polynomial growth of the function f for the last term in the above equation. Lemmas 5.3, 5.4, Assumption A3 and the fact that $t - \delta\lfloor \frac{t}{\delta}\rfloor < \delta$ for any $\delta > 0$ yield the required result. □

We are now in the position to prove Lemma 5.6.

Lemma 5.6. *Let x_t be the solution of equation* (4) *and*

$$\mathscr{S}_1^{\varepsilon,\delta}(t) \triangleq \int_0^t \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)f(x_s)\right] e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds.$$

Then, for any integer $p \geq 1$, we have a positive constant $C_{5.6}$ such that

$$\sup_{t\geq 0} |\mathscr{S}_1^{\varepsilon,\delta}(t)|^p \leq \frac{\delta^{2p}}{\varepsilon^p} C_{5.6}.$$

Proof. We recall

$$\mathscr{S}_1^{\varepsilon,\delta}(t) = \int_0^t \left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)f(x_s)\right] e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds.$$

Moving in the direction of getting an estimate for the quantity $\sup_{t\geq 0} |\mathscr{S}_1^{\varepsilon,\delta}(t)|^p$, we first write the integral (below) in the Riemann sum form followed by using integration by parts formula to obtain

$$\begin{aligned}
&\int_0^t (1/\varepsilon)D\kappa(x_{\pi_\delta(s)})f(x_{\pi_\delta(s)})[s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds \\
&\qquad = \sum_{i=0}^{\lfloor \frac{t}{\delta} \rfloor -1} \frac{1}{\varepsilon} D\kappa(x_{i\delta})f(x_{i\delta}) \int_{i\delta}^{(i+1)\delta} (s-i\delta)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds \\
&\qquad\qquad + \frac{1}{\varepsilon}D\kappa(x_{\pi_\delta(t)})f(x_{\pi_\delta(t)}) \int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t [s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds \\
&\qquad = \sum_{i=0}^{\lfloor \frac{t}{\delta} \rfloor -1} \frac{1}{\varepsilon} D\kappa(x_{i\delta})f(x_{i\delta}) \frac{\delta^2}{2} e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} \\
&\qquad + \sum_{i=0}^{\lfloor \frac{t}{\delta} \rfloor -1} \frac{1}{2\varepsilon} D\kappa(x_{i\delta})f(x_{i\delta}) \int_{i\delta}^{(i+1)\delta} (s-i\delta)^2 [Df(x_s)+D\kappa(x_s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds \\
&\qquad\qquad + \frac{1}{\varepsilon}D\kappa(x_{\pi_\delta(t)})f(x_{\pi_\delta(t)}) \int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t [s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{S}_1^{\varepsilon,\delta}(t) &= \frac{\delta}{2\varepsilon}\left[\sum_{i=0}^{\lfloor \frac{t}{\delta} \rfloor -1} \delta D\kappa(x_{i\delta})f(x_{i\delta})e^{\int_{(i+1)\delta}^t [Df(x_u)+D\kappa(x_u)]du} - \int_0^t D\kappa(x_s)f(x_s)e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds\right] \\
&\quad + \sum_{i=0}^{\lfloor \frac{t}{\delta} \rfloor -1} \frac{1}{2\varepsilon} D\kappa(x_{i\delta})f(x_{i\delta}) \int_{i\delta}^{(i+1)\delta} (s-i\delta)^2 [Df(x_s)+D\kappa(x_s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds \\
&\qquad + \frac{1}{\varepsilon}D\kappa(x_{\pi_\delta(t)})f(x_{\pi_\delta(t)}) \int_{\delta\lfloor \frac{t}{\delta}\rfloor}^t [s-\pi_\delta(s)]e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\, ds.
\end{aligned}$$

Now, recalling the definition of $\mathscr{R}_t^{\varepsilon,\delta}$ from Lemma 5.5, we have

$$\left|\mathscr{S}_1^{\varepsilon,\delta}(t)\right|^p \leq C|\mathscr{R}_t^{\varepsilon,\delta}|^p + \frac{\delta^p}{\varepsilon^p}C\left(1+\sup_{t\geq 0}|x_t|^q\right)\delta^p\left(\int_0^{\delta\lfloor\frac{t}{\delta}\rfloor}|Df(x_s)+D\kappa(x_s)|e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds\right)^p + \frac{\delta^p}{\varepsilon^p}C\left(1+\sup_{t\geq 0}|x_t|^q\right)\left(\int_{\delta\lfloor\frac{t}{\delta}\rfloor}^t e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds\right)^p.$$

The last equation is obtained by using the boundedness of derivatives of κ and the polynomial growth of f. Employing Lemma 5.5 for the term $|\mathscr{R}_t^{\varepsilon,\delta}|^p$, Assumption A3 for the term $\int_0^t e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds$, assuming the condition $|\int_{\delta\lfloor\frac{t}{\delta}\rfloor}^t e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds| < \delta$ with a note $t-\delta\lfloor\frac{t}{\delta}\rfloor < \delta$ for any $\delta > 0$, we obtain the required estimate. □

Next, we state and prove two supporting Lemmas 5.7 and 5.8; these results were used in the proof of Lemma 4.6.

Lemma 5.7. *Let x_t be the solution of equation (4) and*

$$\mathscr{S}_2^{\varepsilon,\delta}(t) \triangleq \int_0^t\left[(1/\varepsilon)D\kappa(x_{\pi_\delta(s)})\kappa(x_{\pi_\delta(s)})[s-\pi_\delta(s)] - \frac{\delta}{2\varepsilon}D\kappa(x_s)\kappa(x_s)\right]e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds.$$

Then, for any integer $p\geq 1$, we have a positive constant $C_{5.7}$ such that

$$\sup_{t\geq 0}|\mathscr{S}_2^{\varepsilon,\delta}(t)|^p \leq \frac{\delta^{2p}}{\varepsilon^p}C_{5.7}.$$

Proof. The proof follows by similar calculations to the proof of Lemma 5.6. □

Lemma 5.8. *Let x_t and $X_t^{\varepsilon,\delta}$ solve (4) and (6), respectively and $p\geq 1$ be an integer, then, for all sufficiently small $\varepsilon,\delta > 0$, there exists a positive constant $C_{5.8}$ such that*

$$\sup_{t\geq 0}\mathbb{E}\left[\int_0^t\left|X_{\pi_\delta(s)}^{\varepsilon,\delta} - x_{\pi_\delta(s)}\right|e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p \leq C_{5.8}(\delta^p+\varepsilon^p+\delta^{\frac{p}{2}}\varepsilon^p).$$

Proof. For any integer $p\geq 1$, we start by noting

$$\begin{aligned}\left[\int_0^t\left|X_{\pi_\delta(s)}^{\varepsilon,\delta} - x_{\pi_\delta(s)}\right|e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p &= \int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t[Df(x_u)+D\kappa(x_u)]du}\right)\left(\prod_{i=1}^p\left|X_{\pi_\delta(s_i)}^{\varepsilon,\delta} - x_{\pi_\delta(s_i)}\right|\right)ds_1\cdots ds_p\\ &\leq C\int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t[Df(x_u)+D\kappa(x_u)]du}\right)\left(\sum_{i=1}^p\left|X_{\pi_\delta(s_i)}^{\varepsilon,\delta} - x_{\pi_\delta(s_i)}\right|^p\right)ds_1\cdots ds_p.\end{aligned}$$

Taking expectation on both sides of the above equation followed by Theorem 2.5, we get

$$\begin{aligned}\mathbb{E}\left[\int_0^t\left|X_{\pi_\delta(s)}^{\varepsilon,\delta} - x_{\pi_\delta(s)}\right|e^{\int_s^t[Df(x_u)+D\kappa(x_u)]du}\,ds\right]^p &\leq C\int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t[Df(x_u)+D\kappa(x_u)]du}\right)\sum_{i=1}^p\mathbb{E}\left|X_{\pi_\delta(s_i)}^{\varepsilon,\delta} - x_{\pi_\delta(s_i)}\right|^p ds_1\cdots ds_p\\ &\leq C(\delta^p+\varepsilon^p+\delta^{\frac{p}{2}}\varepsilon^p)\int_{[0,t]^p}\prod_{i=1}^p e^{\int_{s_i}^t[Df(x_u)+D\kappa(x_u)]du}ds_1\cdots ds_p.\end{aligned}$$

A simplification of the last integral along with Assumption A3 gives the required bound. □

5.2. **Proof of Propositions 5.9 and 5.11.** This section is devoted to the proofs of Propositions 5.9 and 5.11. These results were used in the proof of Proposition 4.3.

Proposition 5.9. *Let $\mathsf{R}_1^{\varepsilon,\delta}(t)$ and $\mathsf{R}_2^{\varepsilon,\delta}(t)$ be defined as in equation (23) and $p \geq 1$ be an integer. Then, for all sufficiently small $\varepsilon, \delta > 0$, there exists a positive constant $C_{5.9}$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{R}_1^{\varepsilon,\delta}(s)ds\right|^p \leq \frac{C_{5.9}}{\varepsilon^p}(\delta^{2p}+\varepsilon^{2p}+\delta^p\varepsilon^{2p}),$$

$$\sup_{t\geq 0} \mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{R}_2^{\varepsilon,\delta}(s)ds\right|^p \leq \frac{C_{5.9}}{\varepsilon^p}(\delta^{2p}+\varepsilon^{2p}+\delta^p\varepsilon^{2p}).$$

Proof. Using Taylor's theorem and the polynomial growth of the function f, we have

$$|\mathsf{R}_1^{\varepsilon,\delta}(s)|^p \triangleq \left|\frac{f(X_s^{\varepsilon,\delta})-f(x_s)}{\varepsilon} - Df(x_s)Z_s^{\varepsilon,\delta}\right|^p = \frac{1}{2^p\varepsilon^p}\left|D^2f(z)(X_s^{\varepsilon,\delta}-x_s)^2\right|^p, \tag{33}$$

where $z \in \mathbb{R}$ is a (random) point lying on the line segment joining x_s and $X_s^{\varepsilon,\delta}$. Hence, there is some integer $r < \infty$ so that Hölder's inequality together with the bounds from Lemma 3.1 give

$$\mathbb{E}|\mathsf{R}_1^{\varepsilon,\delta}(s)|^p \leq \frac{C}{\varepsilon^p}\left(\mathbb{E}\,|z|^{4rp}+1\right)^{1/2}\left(\mathbb{E}\left|X_s^{\varepsilon,\delta}-x_s\right|^{4p}\right)^{1/2} \leq \frac{C}{\varepsilon^p}\left(\mathbb{E}\left|X_s^{\varepsilon,\delta}-x_s\right|^{4p}\right)^{1/2},$$

for some time-independent constant $C < \infty$. Next, for any integer $p \geq 1$, we have

$$\begin{aligned}
\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{R}_1^{\varepsilon,\delta}(s)ds\right|^p &\leq \left(\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left|\mathsf{R}_1^{\varepsilon,\delta}(s)\right|ds\right)^p \\
&\leq \int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\left(\prod_{i=1}^p\left|\mathsf{R}_1^{\varepsilon,\delta}(s_i)\right|\right)ds_1\cdots ds_p \\
&\leq \int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\left(\sum_{i=1}^p\left|\mathsf{R}_1^{\varepsilon,\delta}(s_i)\right|^p\right)ds_1\cdots ds_p.
\end{aligned}$$

Now, taking expectation, using equation (33), the local Lipschitz condition A1 of f and Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{R}_1^{\varepsilon,\delta}(s)ds\right|^p\right] \\
&\leq \int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\sum_{i=1}^p\mathbb{E}\left[\left|\mathsf{R}_1^{\varepsilon,\delta}(s_i)\right|^p\right]ds_1\cdots ds_p \\
&\leq \frac{C}{\varepsilon^p}\int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\sum_{i=1}^p\left(\mathbb{E}\left[\left|X_{s_i}^{\varepsilon,\delta}-x_{s_i}\right|^{4p}\right]^{1/2}\right)ds_1\cdots ds_p.
\end{aligned}$$

Employing Theorem 2.5, Lemma 3.1 and Assumption A3 in the above equation, we get the required result. Similarly, following the same steps and recalling the definition of $\mathsf{R}_2^{\varepsilon,\delta}(t)$ from equation (23), we have

$$\sup_{t\geq 0} \mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{R}_2^{\varepsilon,\delta}(s)ds\right|^p \leq \frac{C}{\varepsilon^p}(\delta^{2p}+\varepsilon^{2p}+\delta^p\varepsilon^{2p}).$$

□

Before proceeding with the proof of Proposition 5.11, we find it convenient to state and prove Lemma 5.10 below.

Lemma 5.10. *Let x_t and $X_t^{\varepsilon,\delta}$ solve (4) and (6), respectively. For any integer $p \geq 1$ and all sufficiently small $\varepsilon, \delta > 0$, there exists a positive constant $C_{5.10}$ such that*

$$\sup_{u\geq 0} \mathbb{E}\left|\left(D\kappa(x_u) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)})\right)\left(X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right)\right|^p \leq (\varepsilon^{2p} + \delta^{2p} + \varepsilon^{2p}\delta^p)C_{5.10},$$
$$\sup_{u\geq 0} \mathbb{E}\left|X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^{2p} \leq (\varepsilon^{2p} + \delta^{2p} + \varepsilon^{2p}\delta^p)C_{5.10}. \tag{34}$$

Proof. We first prove the first line of equation (34). For this, we first get an estimate for the term $D\kappa(x_u) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)})$. Using Taylor's theorem and boundedness of the second-order derivatives of κ, we have

$$\left|D\kappa(X_u^{\varepsilon,\delta}) - D\kappa(x_u)\right| \leq |D^2\kappa(z)||X_u^{\varepsilon,\delta} - x_u| \leq C|X_u^{\varepsilon,\delta} - x_u|,$$

where $z \in \mathbb{R}$ is a point (random) lying on the line segment joining $X_u^{\varepsilon,\delta}$ and x_u. By the similar arguments, we have $\left|D\kappa(X_u^{\varepsilon,\delta}) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)})\right| \leq C|X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}|$. Further, writing $D\kappa(x_u) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)})$ as $D\kappa(x_u) - D\kappa(X_u^{\varepsilon,\delta}) + D\kappa(X_u^{\varepsilon,\delta}) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)})$ followed by the triangle inequality, we obtain

$$D\kappa(x_u) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)}) \leq C\left(|X_u^{\varepsilon,\delta} - x_u| + |X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}|\right). \tag{35}$$

Now, for the first line in equation (34), employing equation (35) and using a simple algebra, we have

(36)
$$\begin{aligned}
\left|\left[D\kappa(x_u) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)})\right]\left(X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right)\right|^p &\leq C\left(|X_u^{\varepsilon,\delta} - x_u|^p + \left|X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^p\right)\left|X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^p \\
&\leq C\left(|X_u^{\varepsilon,\delta} - x_u|^p\left|X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^p + \left|X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^{2p}\right) \\
&\leq C\left(|X_u^{\varepsilon,\delta} - x_u|^{2p} + |X_u^{\varepsilon,\delta} - x_u|^p\left|x_u - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^p + \left|X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^{2p}\right) \\
&\leq C\left(\left|X_u^{\varepsilon,\delta} - x_u\right|^{2p} + |x_u - x_{\pi_\delta(u)}|^{2p} + \left|x_{\pi_\delta(u)} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right|^{2p}\right).
\end{aligned}$$

Finally, using Theorem 2.5 and Lemma 5.1 for the term $\left|x_u - x_{\pi_\delta(u)}\right|^{2p}$, we get

$$\sup_{u\geq 0} \mathbb{E}\left[\left|\left(D\kappa(x_u) - D\kappa(X^{\varepsilon,\delta}_{\pi_\delta(u)})\right)\left(X_u^{\varepsilon,\delta} - X^{\varepsilon,\delta}_{\pi_\delta(u)}\right)\right|^p\right] \leq (\varepsilon^{2p} + \delta^{2p} + \varepsilon^{2p}\delta^p)C.$$

The second line of equation (34) is obtained by using Theorem 2.5 and Lemma 5.1. □

Proposition 5.11. *Let $\mathsf{R}_3^{\varepsilon,\delta}(t)$ be defined as in equation (23) and $p \in \mathbb{N}$. Then, for all sufficiently small $\varepsilon, \delta > 0$, there exists a positive constant $C_{5.11}$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u) + D\kappa(x_u)]du}\mathsf{R}_3^{\varepsilon,\delta}(s)ds\right|^p \leq \frac{C_{5.11}}{\varepsilon^p}(\varepsilon^{2p} + \delta^{2p} + \varepsilon^{2p}\delta^p).$$

Proof. Using Taylor's theorem followed by a simple algebra, we obtain

$$\kappa(X_s^{\varepsilon,\delta}) = \kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta}) + \left[D\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta}) - D\kappa(x_s)\right]\left(X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right) + D\kappa(x_s)\left(X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right) + \frac{1}{2}D^2\kappa(z)\left(X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right)^2,$$

where $z \in \mathbb{R}$ is a point (random) lying on the line segment joining $X_{\pi_\delta(s)}^{\varepsilon,\delta}$ and $X_s^{\varepsilon,\delta}$. Now, using the boundedness of the second-order derivatives of κ, we get for any integer $p \geq 1$

$$\begin{aligned}\left|\mathsf{R}_3^{\varepsilon,\delta}(s)\right|^p &\triangleq \left|\frac{\kappa(X_s^{\varepsilon,\delta}) - \kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta})}{\varepsilon} - D\kappa(x_s)\frac{X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}}{\varepsilon}\right|^p \\ &\leq \frac{C}{\varepsilon^p}\left|\left[D\kappa(x_s) - D\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta})\right]\left(X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right)\right|^p \\ &\qquad + \frac{C}{\varepsilon^p}\left|D^2\kappa(z)\left(X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right)^2\right|^p \\ &\leq \frac{C}{\varepsilon^p}\left|\left(D\kappa(x_s) - D\kappa(X_{\pi_\delta(s)}^{\varepsilon,\delta})\right)\left(X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right)\right|^p + \frac{C}{\varepsilon^p}\left|X_s^{\varepsilon,\delta} - X_{\pi_\delta(s)}^{\varepsilon,\delta}\right|^{2p}.\end{aligned} \tag{37}$$

Next, for any integer $p \geq 1$, we have

$$\begin{aligned}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{R}_3^{\varepsilon,\delta}(s)ds\right|^p &\leq \left(\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\left|\mathsf{R}_3^{\varepsilon,\delta}(s)\right| ds\right)^p \\ &\leq \int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\left(\sum_{i=1}^p \left|\mathsf{R}_3^{\varepsilon,\delta}(s_i)\right|^p\right) ds_1\cdots ds_p.\end{aligned}$$

Taking expectation and using equation (37), we have

$$\begin{aligned}&\mathbb{E}\left|\int_0^t e^{\int_s^t [Df(x_u)+D\kappa(x_u)]du}\mathsf{R}_3^{\varepsilon,\delta}(s)ds\right|^p \\ &\leq \int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\sum_{i=1}^p \mathbb{E}\left[\left|\mathsf{R}_3^{\varepsilon,\delta}(s_i)\right|^p\right] ds_1\cdots ds_p \\ &\leq \frac{C}{\varepsilon^p}\int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right)\sum_{i=1}^p \mathbb{E}\left[\left|X_{s_i}^{\varepsilon,\delta} - X_{\pi_\delta(s_i)}^{\varepsilon,\delta}\right|^{2p}\right] ds_1\cdots ds_p \\ &\quad + \frac{C}{\varepsilon^p}\int_{[0,t]^p}\left(\prod_{i=1}^p e^{\int_{s_i}^t [Df(x_u)+D\kappa(x_u)]du}\right) \times \\ &\qquad \sum_{i=1}^p \mathbb{E}\left[\left|\left(D\kappa(x_{s_i}) - D\kappa(X_{\pi_\delta(s_i)}^{\varepsilon,\delta})\right)\left(X_{s_i}^{\varepsilon,\delta} - X_{\pi_\delta(s_i)}^{\varepsilon,\delta}\right)\right|^p\right] ds_1\cdots ds_p.\end{aligned}$$

Finally, using Lemma 5.10 and Assumption A3, we obtain the required bound. ☐

Conclusions

In this paper, we studied the asymptotic behavior of a controlled non-Markovian dynamical system under the combined effects of fast periodic sampling and small white noise. Our key contribution is to provide uniform-in-time control of its limiting behavior and of its fluctuations. The limiting stochastic dynamical system describing the fluctuations captures both the sampling and noise effects. We approximate uniformly-in-time the pre-limit non-Markovian process by a simpler limiting Markovian process with time-independent bounds on the remainder.

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